1

Fourier Series

DEVANANTH V - EP20BTECH11004

1

3

CONTENTS

1	Periodic	Function	
1	I CI IUUIC	I UIICUUII	

- 2 Fourier Series 1
- **3** Fourier Transform
- **4** Filter 4
- 5 Filter Design 5

Abstract—This manual provides a simple introduction to Fourier Series

1 Periodic Function

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \tag{1.1}$$

1.1 Plot x(t).

Solution:

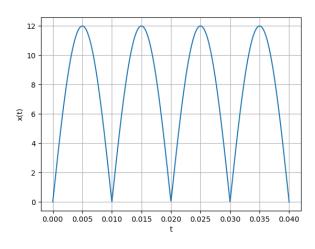


Fig. 1.1

https://github.com/ep20btech11004/lssp/blob/main/assignment-3/codes/1 1.py

1.2 Show that x(t) is periodic and find its period. **Solution:** From Fig. (1.1), we see that x(t) is periodic. Further,

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left| \sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right) \right|$$
 (1.2)

$$= A_0 \left| \sin \left(2\pi f_0 t + \pi \right) \right| \quad (1.3)$$

$$= A_0 |\sin(2\pi f_0 t)| \quad (1.4)$$

$$\therefore$$
 period of $x(t)$ is $\frac{1}{2f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$
 (2.1)

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-J2\pi k f_0 t} dt \qquad (2.2)$$

Solution:

Multiplying both sides by $e^{-J2\pi kf_0t}$ Integrating w.r.t t from $-\frac{1}{2f_0}$ to $\frac{1}{2f_0}$:

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi kf_0t}dt \qquad (2.3)$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \left(\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} \right) e^{-j2\pi k f_0 t} dt \qquad (2.4)$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(n-k)f_0 t} dt \qquad (2.5)$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{\delta(n-k)}{f_0} = \frac{c_k}{f_0}$$
 (2.6)

$$\therefore c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \qquad (2.7)$$

2.2 Find c_k for (1.1) **Solution:**

$$c_{k} = \int_{-\frac{1}{2f_{0}}}^{\frac{1}{2f_{0}}} A_{0} |\sin(2\pi f_{0}t)| e^{-j2\pi k f_{0}t} dt \qquad (2.8)$$

$$= f_{0} \int_{-\frac{1}{2f_{0}}}^{\frac{1}{2f_{0}}} A_{0} |\sin(2\pi f_{0}t)| \cos(2\pi n f_{0}t) dt$$

$$+ Jf_{0} \int_{-\frac{1}{2f_{0}}}^{\frac{1}{2f_{0}}} A_{0} |\sin(2\pi f_{0}t)| \sin(2\pi n f_{0}t) dt$$

$$(2.9)$$

$$= 2f_{0} \int_{0}^{\frac{1}{2f_{0}}} A_{0} \sin(2\pi f_{0}t) \cos(2\pi n f_{0}t) dt$$

$$(2.10)$$

$$= f_{0}A_{0} \int_{0}^{\frac{1}{2f_{0}}} (\sin(2\pi (n+1) f_{0}t)) dt \qquad (2.11)$$

$$= A_{0} \frac{1 + (-1)^{n}}{2\pi} \left(\frac{1}{n+1} - \frac{1}{n-1}\right) \qquad (2.12)$$

$$= \begin{cases} \frac{2A_{0}}{\pi (1-n^{2})} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad (2.13)$$

2.3 Verify (1.1) using python.

https://github.com/ep20btech11004/lssp/blob/ main/assignment-3/codes/2 3.py

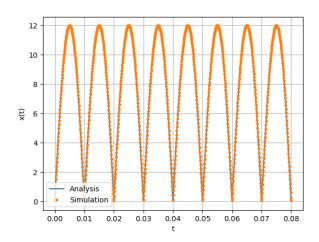


Fig. 2.3

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos j 2\pi k f_0 t + b_k \sin j 2\pi k f_0 t)$$
(2.14)

and obtain the formulae for a_k and b_k . **Solution:**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$
 (2.15)

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t}$$
 (2.16)

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$

$$+\sum_{k=0}^{\infty} (c_k - c_{-k}) \sin(2\pi k f_0 t)$$
 (2.17)

$$\implies a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases}$$
 (2.18)

$$b_k = c_k - c_{-k} (2.19)$$

2.5 Find a_k and b_k for (1.1)

Solution:

From (1.1):

$$x(-t) = x(t)$$
 (2.20)

$$\implies \sum_{k=-\infty}^{\infty} c_{-k} e^{j2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.21)$$

$$\implies c_k = c_{-k} \quad (2.22)$$

$$a_{k} = \begin{cases} \frac{2A_{0}}{\pi(1-n^{2})} & n = 0\\ \frac{4A_{0}}{\pi(1-n^{2})} & n \text{ is even}\\ 0 & \text{otherwise} \end{cases}$$
 (2.22)

$$b_k = 0$$
 (2.24)

2.6 Verify (2.14) using python.

https://github.com/ep20btech11004/lssp/blob/ main/assignment-3/codes/2_6.py

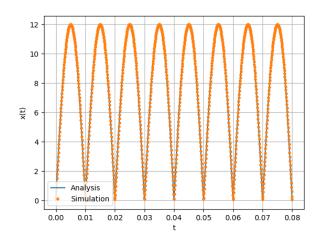


Fig. 2.6: Fourier Expansion of x(t)

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of g(t) is

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} dt \qquad (3.3)$$

3.3 Show that

$$g(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi ft_0}$$
 (3.4)

Solution:

Put $t - t_0 = t'$,

$$g(t - t_0) = g(t') \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t') e^{-j2\pi f t} dt \quad (3.5)$$

$$\int_{-\infty}^{\infty} g(t')e^{-j2\pi f(t'+t_0)} dt' \quad (3.6)$$

$$= G(f)e^{-j2\pi ft_0}$$
 (3.7)

3.4 Show that

$$G(t) \stackrel{\mathcal{F}}{\longleftrightarrow} g(-f)$$
 (3.8)

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft} df = \int_{-\infty}^{\infty} G(f')e^{j2\pi f't} df'$$
(3.9)

Replace t with -f and f' with t and df' with

dt,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{-j2\pi ft} dt \qquad (3.10)$$

$$\implies G(t) \stackrel{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.11}$$

3.5 $\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} ?$

Solution:

$$\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt$$
 (3.12)

$$= e^{j2\pi(0)f} = 1 \tag{3.13}$$

3.6 $e^{-j2\pi f_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} ?$

Solution:

$$e^{-j2\pi f_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t} dt$$
 (3.14)

$$\int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t} dt = \delta(f+f_0)$$
 (3.15)

 $3.7 \cos(2\pi f_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} ?$

Solution:

Using the linearity of the Fourier Transform and (3.15),

$$\cos(2\pi f_0 t) = \frac{1}{2} \left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \right)$$

$$\stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2} \left(\delta \left(f + f_0 \right) + \delta \left(f - f_0 \right) \right)$$
(3.16)

3.8 Find the Fourier Transform of x(t) and plot it. Verify using python.

Solution: From (2.1) and (2.13)

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta(f + kf_0)$$
 (3.17)

$$X(f) = \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f + 2kf_0)}{1 - 4k^2}$$
 (3.18)

Python code used to verify (3.18):

https://github.com/ep20btech11004/lssp/blob/main/assignment-3/codes/3 8.py

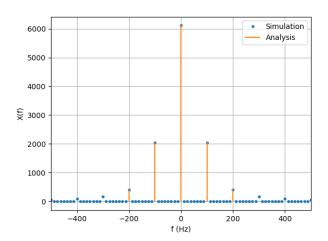


Fig. 3.8

3.9 Show that

$$rect(t) \stackrel{\mathcal{F}}{\longleftrightarrow} sinc(t)$$
 (3.19)

Verify using python.

Solution:

$$\operatorname{rect}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \operatorname{rect}(t) e^{-j2\pi f t} dt \qquad (3.20)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft} dt \qquad (3.21)$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin \pi f}{\pi f} = \text{sinc}(f)$$
 (3.22)

https://github.com/ep20btech11004/lssp/blob/main/assignment-3/codes/3 9.py

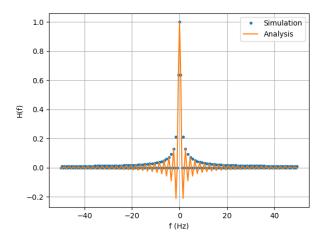


Fig. 3.9

3.10 $\operatorname{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} ?$. Verify using python.

Solution: From (3.11)

$$\operatorname{sinc}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \operatorname{rect}(-f) = \operatorname{rect}(f)$$
 (3.23)

https://github.com/ep20btech11004/lssp/blob/main/assignment-3/codes/3 10.py

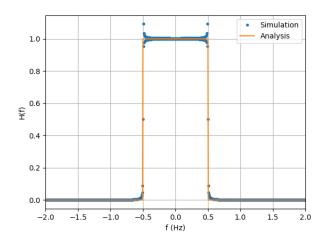


Fig. 3.10

4 Filter

4.1 Find H(f) which transforms x(t) to DC 5V. **Solution:**

H(f) is a *Low-pass* filter which allows only the zeroth harmonic and filters the rest.

If f_0 is the cut-off frequency then an ideal Lowpass filter is described by:

$$H(f) = \operatorname{rect}\left(\frac{f}{2f_0}\right) = \begin{cases} 1 & \text{if } |f| < f_0 \\ 0 & \text{otherwise} \end{cases}$$
 (4.1)

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0} \operatorname{rect}\left(\frac{f}{2f_0}\right) \tag{4.2}$$

where $V_0 = 5$ V.

4.2 Find h(t).

Solution:

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}df \qquad (4.3)$$

$$= \frac{\pi V_0}{2A_0} \int_{-\infty}^{\infty} \text{rect}\left(\frac{f}{2f_0}\right) e^{j2\pi ft} df \tag{4.4}$$

$$= \frac{\pi V_0}{2A_0} \int_{-f_0}^{f_0} e^{j2\pi ft} df$$
 (4.5)

$$= \frac{\pi V_0}{2A_0} \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{j2\pi t}$$
(4.6)

$$= \frac{\pi V_0 J \sin(2\pi f_0 t)}{A_0 J^2 \pi t}$$
 (4.7)

$$= \frac{\pi V_0}{A_0} f_0 \text{sinc} (2f_0 t)$$
 (4.8)

4.3 Verify your result using through convolution. **Solution:**

https://github.com/ep20btech11004/lssp/blob/main/assignment-3/codes/4 3.py

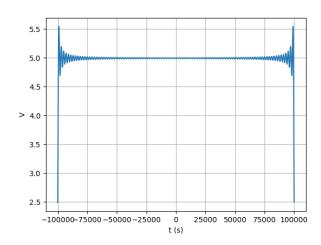


Fig. 4.3

5 Filter Design

5.1 Design a Butterworth filter for H(f). **Solution:**

For an n^{th} order Butterworth filter: The Gain response G(f) is given by

$$G^{2}(f) = |H_{n}(f)|^{2} = \frac{1}{1 + \left(\frac{f}{f_{0}}\right)^{2n}}$$
 (5.1)

Where f_0 is the cutoff frequency. Gain at a frequency in dB is given as

$$A = -10\log_{10}(G^2(f)) = -20\log_{10}|H_n(f)|$$
(5.2)

Assumptions for the lowpass analog Butterworth filter:

- a) Passband edge, $f_p = 50 \text{ Hz}$
- b) Stopband edge, $f_s = 100 \text{ Hz}$
- c) Passband attenuation, $A_p = -1$ dB
- d) Stopband attenuation, $A_s = -20 \text{ dB}$

We are required to find a desriable order n and cutoff frequency f_0 for the filter. From (5.2),

$$A_p = -10\log_{10} \left[1 + \left(\frac{f_p}{f_0} \right)^{2n} \right]$$
 (5.3)

$$A_s = -10\log_{10} \left[1 + \left(\frac{f_s}{f_0} \right)^{2n} \right]$$
 (5.4)

Thus.

$$\left(\frac{f_p}{f_0}\right)^{2n} = 10^{-\frac{A_p}{10}} - 1\tag{5.5}$$

$$\left(\frac{f_s}{f_0}\right)^{2n} = 10^{-\frac{A_s}{10}} - 1\tag{5.6}$$

Therefore, on dividing the above equations and solving for n,

$$n = \frac{\log\left(10^{-\frac{A_s}{10}} - 1\right) - \log\left(10^{-\frac{A_p}{10}} - 1\right)}{2\left(\log f_s - \log f_p\right)}$$
 (5.7)

In this case, making appropriate substitutions gives n = 4.29. Hence, we take n = 5. Solving for f_0 in (5.5) and (5.6),

$$f_{c1} = f_p \left[10^{-\frac{A_p}{10}} - 1 \right]^{-\frac{1}{2n}} = 57.23Hz$$
 (5.8)

$$f_{c2} = f_s \left[10^{-\frac{A_s}{10}} - 1 \right]^{-\frac{1}{2n}} = 63.16Hz$$
 (5.9)

Hence, we take $f_0 = \sqrt{f_{c1}f_{c2}} = 60Hz$ approximately.

5.2 Design a Chebyshev filter for H(f). Solution: The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \epsilon^2 T_n^2 \left(\frac{f}{f_0}\right)\right)}$$
 (5.10)

where

- a) n is the order of the filter
- b) ϵ is the ripple
- c) f_0 is the cutoff frequency
- d) $T_n = \cosh^{-1}(n \cosh x)$ denotes the nth order Chebyshev polynomial, given by

$$T_n(x) = \begin{cases} \cos\left(n\cos^{-1}x\right) & |x| \le 1\\ \cosh\left(n\cosh^{-1}x\right) & \text{otherwise} \end{cases}$$
(5.11)

We are given the following specifications:

- a) Passband edge (which is equal to cutoff frequency), $f_p = f_0$
- b) Stopband edge, f_s
- c) Attenuation at stopband edge, A_s
- d) Peak-to-peak ripple δ in the passband. It is given in dB and is related to ϵ as

$$\delta = 10\log_{10}\left(1 + \epsilon^2\right) \tag{5.12}$$

and we must find a suitable n and ϵ . From (5.12),

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \tag{5.13}$$

At $f_s > f_p = f_0$, using (5.11), A_s is given by

$$A_s = -10\log_{10}\left[1 + \epsilon^2 c_n^2 \left(\frac{f_s}{f_p}\right)\right]$$
 (5.14)

$$\implies c_n \left(\frac{f_s}{f_p} \right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \tag{5.15}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_p}\right)}$$
 (5.16)

We consider the following specifications:

- a) Passband edge/cutoff frequency, $f_p = f_0 = 60Hz$.
- b) Stopband edge, $f_s = 100Hz$.
- c) Passband ripple, $\delta = 0.5dB$
- d) Stopband attenuation, $A_s = -20dB$ $\epsilon = 0.35$ and n = 3.68. Hence, we take n = 4as the order of the Chebyshev filter.
- 5.3 Design a circuit for your Butterworth filter. **Solution:** Looking at the table of normalized element values L_k , C_k , of the Butterworth filter for order 5, and noting that de-normalized

values L'_k and C'_k are given by

$$C_k' = \frac{C_k}{\omega_c} \qquad L_k' = \frac{L_k}{\omega_c} \tag{5.17}$$

De-normalizing these values, taking $f_0 = 60$ Hz,

$$C_1' = C_5' = 1.64mF$$
 (5.18)

$$L_2' = L_4' = 4.29mH \tag{5.19}$$

$$C_3' = 5.31mF (5.20)$$

(5.21)

The L-C network is shown in Fig. 5.3.

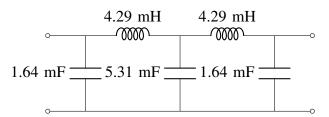


Fig. 5.3: L-C Butterworth Filter

Python code to compare the amplitude response of the simulated circuit with the theoretical expression.

https://github.com/ep20btech11004/lssp/blob/main/assignment-3/codes/5_3.py

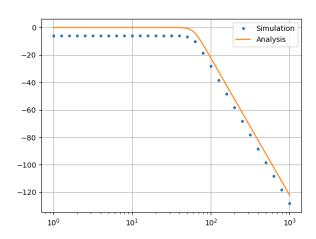


Fig. 5.3: Amplitude Response of Butterworth filter.

5.4 Design a circuit for your Chebyshev filter. **Solution:** Looking at the table of normalized element values of the Chebyshev filter for order 3 and 0.5 dB ripple, and de-nommalizing those values, taking $f_0 = 60Hz$,

$$C_1' = 4.43mF (5.22)$$

$$L_2' = 3.16mH \tag{5.23}$$

$$C_3' = 6.28mF (5.24)$$

$$L_4' = 2.23mH \tag{5.25}$$

The L-C network is shown in Fig. 5.4.

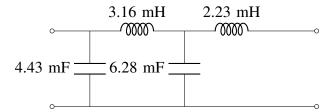


Fig. 5.4: L-C Chebyshev Filter

Python code to compare the amplitude response of the simulated circuit with the theoretical expression.

https://github.com/ep20btech11004/lssp/blob/main/assignment-3/codes/5_4.py

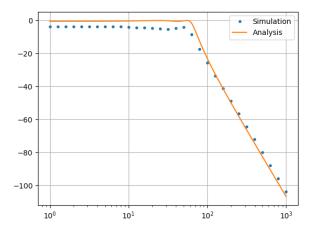


Fig. 5.4: Amplitude Response of Chebyshev filter.