# Advanced Options Theory

Coursework

MSc Risk Management and Financial Engineering 2023-2024

MSc Finance 2023-2024

MSc Investment and Wealth Management 2023-2024

MSc Financial Technology 2023-2024

May 20, 2024

*Attempt all questions. Answers to be submitted no later than 12:00pm on 04/06/2024.*

**QUESTION 1**

Let $V(X_\tau, \tau)$ be the value, at time $t \in [0, T]$, of an European digital put option written on a dividend-paying stock, where $\tau = T - t$. At the maturity time $\tau = 0$, the payoff of the option is:

$$V_0 = V(X_0, 0) = \begin{cases} 1, \text{if} \quad X_0 \leq E, \\ 0, \text{if} \quad X_0 > E, \end{cases}$$

where $E > 0$ is the exercise price of the option. The stock price is assumed to follow the lognormal diffusion process:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t,$$

where $\mu$ is the drift rate, $\mathrm{d}W_t$ is the increment of a Brownian motion, $r > 0$ is the risk-free interest rate, $\delta > 0$ is the continuous dividend-yield paid by the stock, and $\sigma > 0$ is the volatility of the asset price.

(a) Following the arbitrage pricing methodology, derive the partial differential equation (PDE) governing the price process of the digital option.

(b) Use the Laplace transform of the option's PDE from part (a) to find the solution to the digital option's price in the Laplace domain (you do not need to invert this).

*Hint: One of the boundary conditions is* $V(0, \tau) = \mathrm{e}^{-r\tau}$.

(c) The solution of the European digital option price is $V(X_\tau, \tau) = \mathrm{e}^{-r\tau}(1 - \mathcal{N}(d))$, where $\mathcal{N}(\cdot)$ is the cumulative standard normal distribution function $\mathcal{N}(v) = \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2} \mathrm{d}z$, and $d := \frac{\log(\frac{X_\tau}{E}) + (r - \delta - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$. An option's PDE can be written in terms of the option's 'Greeks', delta ($\Delta$), gamma ($\Gamma$) and theta ($\Theta$):

$$\Delta = \frac{\partial V}{\partial X}, \qquad \Gamma = \frac{\partial^2 V}{\partial X^2}, \qquad \Theta = \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}.$$

(i) Rewrite the PDE from part (a) in terms of the Greeks.

(ii) Show that:

$$\Theta = rV + \mathrm{e}^{-r\tau}\phi(d)\left(\frac{r - \delta - \frac{1}{2}\sigma^2}{\sigma\sqrt{\tau}} - \frac{d}{2\tau}\right).$$

**QUESTION 2**

Let $V(S_t, t)$ be the value of an up-and-out barrier European Call option written on a stock, whose price is denoted by $S_t$. The dynamics of the stock price are given by:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma(t, S_t) S_t \mathrm{d}W_t,$$

where $\mu$ is the drift rate, $\mathrm{d}W_t$ is the increment of a Wiener process and the local volatility surface has been calibrated with the following form:

$$\sigma(t, S_t) = 0.13\mathrm{e}^{-t}\left(\frac{100}{S_t}\right)^{\alpha}.$$

The exercise price and maturity date of the option are $K$ and $T$ respectively, and the up-and-out barrier level is $D$.

At the time of initiation of the option contract, $t = 0$, the stock price is $S_0 < D$. Provided that, at no time during the life of the option, the barrier is hit, *i.e.*:

$$S_t < D, \quad \forall t \in [0, T],$$

the behaviour and payoff of the option are identical to that of a Vanilla European Call option *i.e.* it satisfies the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{\sigma(t, S_t)^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t \frac{\partial V}{\partial S} - rV = 0, \quad S_t < D,$$

where $r$ is the constant risk-free interest rate, and with terminal condition:

$$V_T = V(S_T, T) = \begin{cases} \max(S_T - K, 0), \text{ if } \max\limits_{0 \le t \le T} S_t < D, \\ 0, \text{ otherwise.} \end{cases}$$

(a) Use the following: spot price $S_0 = \$100$, strike price $K = \$100$, risk-free rate $r = 1.8\%$, level of the barrier $D = \$118$, maturity $T = 0.5$ years and $\alpha = 0.24$ in the local volatility function, to solve the Black-Scholes PDE numerically to price an up-and-out call option by means on an Explicit finite difference scheme.

(b) Discuss the boundary conditions you should use in the explicit finite difference scheme based on the properties of the up-and-out call option.

(c) Demonstrate that the explicit scheme is conditionally stable and find the condition under which the scheme would be stable.

*Code developer documentation: present the structure of the code, available functions, main variables and any other information to help understand the code; also test runs of your code. Use Python as the programming language.*

(d) Describe briefly any advantages and/or disadvantages the up-and-out barrier European call option has over its Vanilla counterpart to its holder.

(e) Explain why a down-and-out put option is worth zero when $D \geq K$ and an up-and-out call option is worth zero when $D \leq K$.

**QUESTION 3**

In this question we aim to explore the European Call option price under the Heston model. The underlying stock price, $S$, follows a geometric Brownian motion, but with a stochastic variance $\nu$ that follows a mean reverting square root process. The SDEs for underlying stock price and variance are:

$$\begin{cases} \mathrm{d}S_t = \mu S_t \mathrm{d}t + \sqrt{\nu_t} S_t \mathrm{d}W_{1,t}, \\ \mathrm{d}\nu_t = \kappa(\theta - \nu_t)\mathrm{d}t + \sigma\sqrt{\nu_t}\mathrm{d}W_{2,t}, \end{cases}$$

with $\mathrm{d}W_{1,t}^{\mathbb{Q}} \cdot \mathrm{d}W_{2,t}^{\mathbb{Q}} = \rho \mathrm{d}t$.

The price of the proposed Heston European call option could be written as:

$$C_t = S_t P_1 - K\mathrm{e}^{-r\tau} P_2,$$

where $P_1$ and $P_2$ represent the probability of the Call expiring in the money.

(a) *(Heston PDE)* In the Black-Scholes model, we could construct a fully hedged portfolio with respect to the call option with a cash bond and then underlying stock. In the Heston model, another instrument is used to hedge the volatility risk and, thus, render the portfolio riskless. The portfolio therefore consists of one option $V(S,\nu,t)$, $\Delta$ units of the stock and $\phi$ units of another call option for the volatility hedge; then the portfolio is $\Pi = V + \Delta S + \phi U$.

(i) Find $\Delta$ and $\phi$ which renders the portfolio riskless and verify the following relationship

$$\frac{\left[\frac{\partial V}{\partial t} + \frac{\nu S^2}{2}\frac{\partial^2 V}{\partial S^2} + \rho\sigma\nu S\frac{\partial^2 V}{\partial \nu \partial S} + \frac{\sigma^2\nu}{2}\frac{\partial^2 V}{\delta \nu^2}\right] - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial \nu}} = \frac{\left[\frac{\partial U}{\partial t} + \frac{\nu S^2}{2}\frac{\partial^2 U}{\partial S^2} + \rho\sigma\nu S\frac{\partial^2 U}{\partial \nu \partial S} + \frac{\sigma^2\nu}{2}\frac{\partial^2 U}{\delta \nu^2}\right] - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial \nu}}.$$

(ii) The left-hand side (LHS) is a function of $V$ only, and the right-hand side (RHS) is a function of $U$ only. The only way that this can be is for both sides could be written as some function $f$ of independent variables $S$, $\nu$ and $t$. Without the loss of generality, the choice of $f(S,\nu,t)$ is arbitrary. As in Heston (1993), we specify the function as $f(S,\nu,t) = -\kappa(\theta-\nu) + \lambda(S,\nu,t)$, where $\lambda(S,\nu,t)$ is the market price of volatility (variance) risk. Substitute the LHS with $f(S,\nu,t)$ to obtain the PDE.

(b) *(Heston price)* The boundary conditions for the Heston PDE are:

$$\begin{cases} U(S,\nu,T) = max(0, S-K), \\ U(0,\nu,t) = 0, \\ \dfrac{\partial U}{\partial S}(\infty,\nu,t) = 1, \\ U(S,\infty,t) = S. \end{cases}$$

(i) Here we define the log price $x = \log S$. Show the derivation of the following PDE:

$$\frac{\partial U}{\partial t} + \frac{\nu}{2}\frac{\partial^2 U}{\partial x^2} + \left(r - \frac{\nu}{2}\right)\frac{\partial U}{\partial x} + \rho\sigma\nu\frac{\partial^2 U}{\partial x \partial \nu} + \frac{\sigma^2 \nu}{2}\frac{\partial^2 U}{\partial \nu^2} - rU + \left[\kappa(\theta-\nu) - \lambda(S,\nu,t)\right]\frac{\partial U}{\partial \nu} = 0.$$

(ii) We assume the market price of volatility (variance) risk is a linear function of variance $\lambda(S,\nu,t) = \lambda\nu$. Using $x = x_t = \log S_t$, $C_t = \mathrm{e}^x P_1 - K\mathrm{e}^{-r\tau} P_2$.

$$\frac{\partial C}{\partial t} + \frac{\nu}{2}\frac{\partial^2 C}{\partial x^2} + \left(r - \frac{\nu}{2}\right)\frac{\partial C}{\partial x} + \rho\sigma\nu\frac{\partial^2 C}{\partial x \partial \nu} + \frac{\sigma^2 \nu}{2}\frac{\partial^2 C}{\partial \nu^2} - rC + [\kappa(\theta - \nu) - \lambda(S,\nu,t)]\frac{\partial C}{\partial \nu} = 0.$$

(1) Derive $\frac{\partial C}{\partial t}$, $\frac{\partial C}{\partial x}$, $\frac{\partial C}{\partial \nu}$, $\frac{\partial^2 C}{\partial x^2}$, $\frac{\partial^2 C}{\partial \nu^2}$ and $\frac{\partial^2 C}{\partial x \partial \nu}$.

(2) Substitute these terms above into the PDE and derive the two new PDEs for $P_1$ and $P_2$, respectively.

For notational convenience, these two PDEs could be combined into a single equation:

$$\frac{\partial P_j}{\partial t} + \rho\sigma\nu\frac{\partial^2 P_j}{\partial x \partial \nu} + \frac{\nu}{2}\frac{\partial^2 P_j}{\partial x^2} + \frac{\sigma^2 \nu}{2}\frac{\partial^2 P_j}{\partial \nu^2} + (r + u_j \nu)\frac{\partial P_j}{\partial x} + (a - b_j \nu)\frac{\partial P_j}{\partial \nu} = 0, \tag{1}$$

for $j = 1, 2$ and where $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa + \lambda - \rho\sigma$ and $b_2 = \kappa + \lambda$.

(iii) We now come to the core part of the Heston model. The computation of $P_1$ and $P_2$ in the Heston model is much more difficult than that of their counterparts in the Black-Scholes model. The reason for this is that the probability density function of the Heston Model is not available. However, its mirror image in the frequency domain, i.e. the characteristic function, has a nice expression. The probability of a call option expiring in the money can be represented by:

$$P_j = \mathbb{P}(\log S_T > \log K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{\mathrm{e}^{-i\phi \log K} f_j(x,\nu,T;\phi)}{i\phi}\right)\mathrm{d}\phi,$$

where $\phi$ denotes the frequency and $j = 1, 2$. Moreover, $f_1(x,\nu,T;\phi)$ and $f_2(x,\nu,T;\phi)$ stand for the characteristic functions of $P_1$ and $P_2$, respectively. Heston (1993) postulates that the characteristic function of the logarithm of the terminal stock price, $x_T = \log S_T$, is of the log linear form $f_j(x,\nu,T;\phi) = \mathrm{e}^{i\phi x_T}$, and the characteristic function at time $t$ is

$$f_j(x,\nu,t;\phi) = \exp\left(C_j(\tau,\phi) + D_j(\tau,\phi)\nu_t + i\phi x_t\right).$$

The characteristic function $f_j$ follows the PDE we derived earlier (1).

The reason is pretty clear; we can find a one-to-one mapping relation between the probability density function and the characteristic function. The Feynman-Kac theorem states that if $f_j(x,\nu,t;\phi)$ satisfies the Heston PDE, then the solution of $f_j(x,\nu,t;\phi)$ is the conditional expectation $f_j(x,\nu,t;\phi) = \mathbb{E}[f_j(x,\nu,T;\phi)|\mathcal{F}_t]$. Therefore, the PDE for the characteristic functions is:

$$-\frac{\partial f_j}{\partial \tau} + \rho\sigma\nu\frac{\partial^2 f_j}{\partial x \partial \nu} + \frac{\nu}{2}\frac{\partial^2 f_j}{\partial x^2} + \frac{\sigma^2 \nu}{2}\frac{\partial^2 f_j}{\partial \nu^2} + (r + u_j \nu)\frac{\partial f_j}{\partial x} + (a - b_j \nu)\frac{\partial f_j}{\partial \nu} = 0,$$

the transformation from $t$ to $\tau$ giving the negative sign of the first term.

(1) Given the expression for $f_j$ proposed in Heston (1993), derive $\frac{\partial f_j}{\partial t}$, $\frac{\partial f_j}{\partial x}$, $\frac{\partial f_j}{\partial \nu}$, $\frac{\partial^2 f_j}{\partial x^2}$, $\frac{\partial^2 f_j}{\partial \nu^2}$ and $\frac{\partial^2 f_j}{\partial x \partial \nu}$.

(2) Show that the above PDE can produce two differential equations:

$$\begin{cases} \dfrac{\partial D_j}{\partial \tau} = \rho\sigma i\phi D_j - \dfrac{\phi^2}{2} + \dfrac{\sigma^2}{2}D_j^2 + u_j i\phi - b_j D_j, \\ \dfrac{\partial C_j}{\partial \tau} = ri\phi + aD_j. \end{cases}$$

**QUESTION 4**

(a) Why do we observe a 'volatility smile' in the market? And more specifically, what volatility smile is observed for equities?

(b) Option traders sometimes refer to deep out-of-the-money options as being options on volatility. Why do you think they do this?

(c) The Heston model seems to produce a better fit to options with long maturity but not a good fit for short-term maturity options. Discuss why this might be the case.