Logic, Interaction and Collective Agency

Lecture 2

ESSLLI'10, Copenhagen

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D. Lewis. *Convention*. 1969.

M. Chwe. *Rational Ritual*. 2001.

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What should they do?

R. Sugden. *The Logic of Team Reasoning*. Philosophical Explorations (6)3, pgs. 165 - 181 (2003).

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		B	
		l	r
A	l		
	r		

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		B: l	B: r
A	l	10,10	0,0
	r	0,0	11,11

A: What should I do?

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(symmetric reasoning for B)

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A: What should *we* do? **Team Reasoning**: an escape from the infinite regress? why should this "mode of reasoning" be endorsed?

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Plan for Today

- Group Informational Attitudes

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It is not Common Knowledge who "defined" Common Knowledge!

The first formal definition of common knowledge?

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Shared situation: There is a *shared situation s* such that (1) *s* entails φ, (2) *s* entails everyone knows φ, plus other conditions

H. Clark and C. Marshall. *Definite Reference and Mutual Knowledge*. 1981.

M. Gilbert. *On Social Facts*. Princeton University Press (1989).

P. Vanderschraaf and G. Sillari. *"Common Knowledge", The Stanford Encyclopedia of Philosophy (2009)*.
http://plato.stanford.edu/entries/common-knowledge/.

The "Standard" Account

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R. Fagin, J. Halpern, Y. Moses and M. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.

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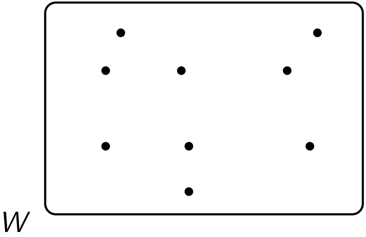
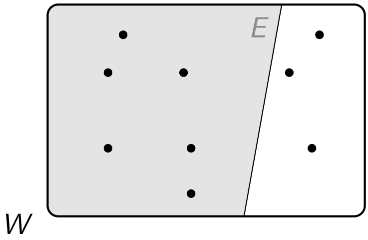


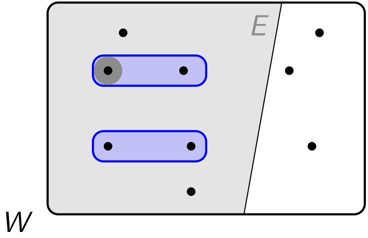
W is a set of **states** or **worlds**.

The "Standard" Account



An **event**/**proposition** is any (definable) subset $E \subseteq W$

The "Standard" Account



At each state, agents are assigned a set of states they *consider possible* (according to their information).
The information may be (in)correct, partitional, ...

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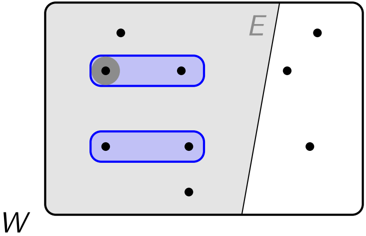

Knowledge Function: $K_i : \wp(W) \rightarrow \wp(W)$ where $K_i(E) = \{w \mid R_i(w) \subseteq E\}$

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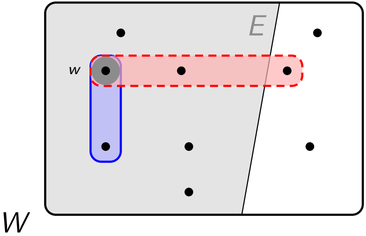
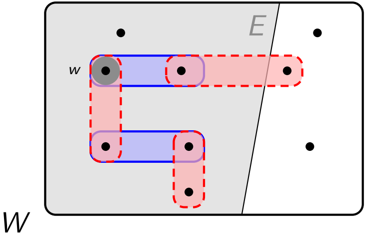


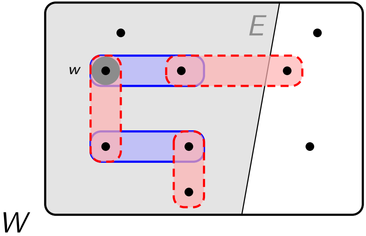
$w \in K_A(E)$ and $w \notin K_B(E)$

The "Standard" Account



The model also describes the agents' **higher-order knowledge/beliefs**

The "Standard" Account



Everyone Knows: $K(E) = \bigcap_{i \in \mathcal{A}} K_i(E)$, $K^0(E) = E$, $K^m(E) = K(K^{m-1}(E))$

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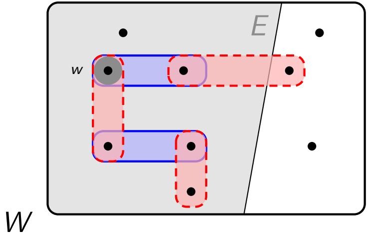

Common Knowledge: $C : \wp(W) \to \wp(W)$ with

$$C(E) = \bigcap_{m \geq 0} K^m(E)$$

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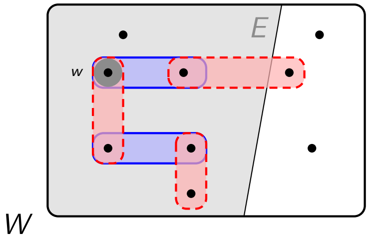

$w \in K(E)$ $\qquad$ $w \notin C(E)$

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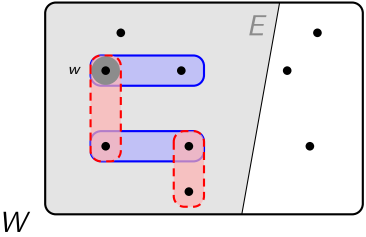

$w \in C(E)$

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> Suppose you are told "Ann and Bob are going together,"' and respond "sure, that's common knowledge." What you mean is not only that everyone knows this, but also that the announcement is pointless, occasions no surprise, reveals nothing new; in effect, that the situation after the announcement does not differ from that before. ...the event "Ann and Bob are going together" — call it E — is common knowledge if and only if some event — call it F — happened that entails E and also entails all players' knowing F (like all players met Ann and Bob at an intimate party). *(Aumann, pg. 271, footnote 8)*

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An event F is **self-evident** if $K_i(F) = F$ for all $i \in \mathcal{A}$.

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Fact. $w \in C(E)$ if every finite path starting at w ends in a state in E

The following axiomatize common knowledge:

- $C(\varphi \to \psi) \to (C\varphi \to C\psi)$
- $C\varphi \to (\varphi \wedge EC\varphi)$ (Fixed-Point)
- $C(\varphi \to E\varphi) \to (\varphi \to C\varphi)$ (Induction)

An Example

Two players Ann and Bob are told that the following will happen. Some positive integer n will be chosen and *one* of n, $n + 1$ will be written on Ann's forehead, the other on Bob's. Each will be able to see the other's forehead, but not his/her own.

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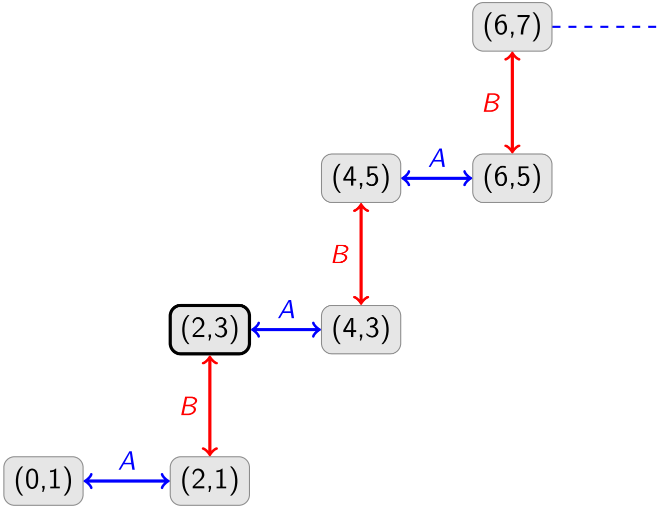
Is it common knowledge that their numbers are less than 1000?


(6,7)
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B
(0,1)
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- (Tarski) Every monotone operator has a greatest (and least) fixed point
- Let $K^*(E)$ be the greatest fixed point of f_E.
- **Fact**. $K^*(E) = C(E)$.

The Fixed-Point Definition

Separating the fixed-point/iteration definition of common knowledge/belief:

J. Barwise. *Three views of Common Knowledge*. TARK (1987).

J. van Benthem and D. Saraenac. *The Geometry of Knowledge*. Aspects of Universal Logic (2004).

A. Heifetz. *Iterative and Fixed Point Common Belief*. Journal of Philosophical Logic (1999).

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- $D_G(\varphi) \to \bigwedge_{i \in G} K_i \varphi$

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$w \in K_G(E)$ iff $R_G(w) \subseteq E$ (without necessarily $R_G(w) = \bigcap_{i \in G} R_i(w)$)

A. Baltag and S. Smets. *Correlated Knowledge: an Epistemic-Logic view on Quantum Entanglement*. Int. Journal of Theoretical Physics (2010).

Common p-belief

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The typical example of an event that creates common knowledge is a **public announcement**.

Shouldn't one always allow for some small probability that a participant was absentminded, not listening, sending a text, checking facebook, proving a theorem, asleep, ...

"We show that the weaker concept of "common belief" can function successfully as a substitute for common knowledge in the theory of equilibrium of Bayesian games."

D. Monderer and D. Samet. *Approximating Common Knowledge with Common Beliefs*. Games and Economic Behavior (1989).

Common p-belief: definition

Let $(W, \{R_i\}_{i \in \mathcal{A}}, p, V)$ be a Bayesian model.

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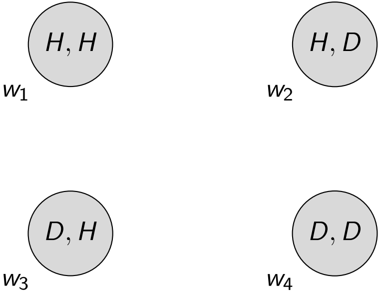
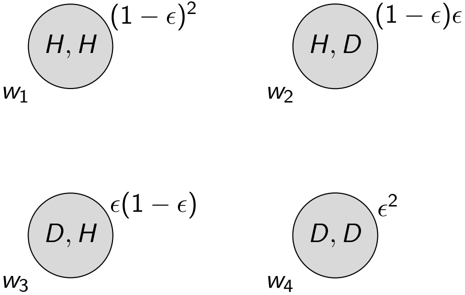
An event F is **common p-belief** at w if there exists and evident p-belief event E such that $w \in E$ and for all $i \in \mathcal{A}$, $E \subseteq B_i^p(F)$

Common p-belief: example



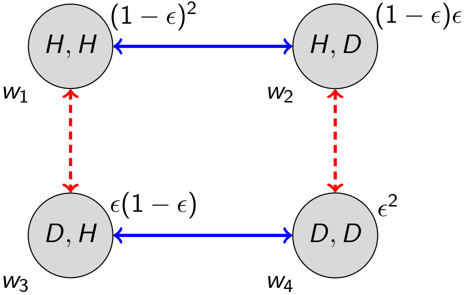
Two agents either hear (H) or don't hear (D) the announcement.

Common p-belief: example



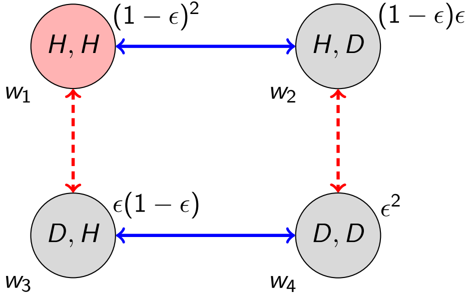
The probability that an agent hears is $1 - \epsilon$.

Common p-belief: example



The agents *know* their "type".

Common p-belief: example



The event "everyone hears" ($E = \{w_1\}$)

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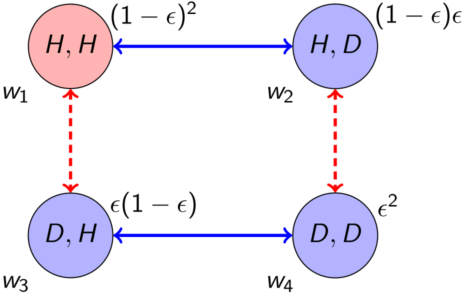

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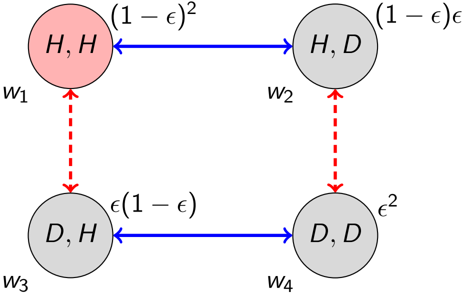

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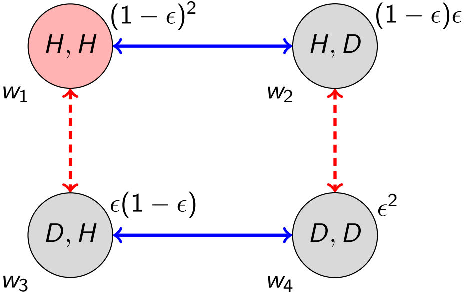

The event "everyone hears" ($E = \{w_1\}$) is **not** common knowledge, but it is common $(1-\epsilon)$-belief:
$B_i^{(1-\epsilon)}(E) = \{w \mid p(E \mid \Pi_i(w)) \geq 1-\epsilon\} = \{w_1\} = E$, for $i = 1, 2$

Some Issues

✓ What *does* a group know/believe/accept? vs. what *can* a group (come to) know/believe/accept?

C. List. *Group knowledge and group rationality: a judgment aggregation perspective*. Episteme (2008).

✓ Other "group informational attitudes": distributed knowledge, common belief, ...

- Levels of knowledge
- Common knowledge/belief of *rationality*
- Where does common knowledge come from?

Levels of Knowledge

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What are the *states of knowledge* created in a group when communication takes place? What happens when communication is not the the whole group, but pairwise?

R. Parikh and P. Krasucki. *Communication, Consensus and Knowledge*. Journal of Economic Theory (1990).

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States of Knowledge in Games

R. Parikh. *Levels of knowledge, games and group action*. Research in Economics 57, pp. 267 - 281 (2003).

States of Knowledge in Games

	G	N
g		
n		

States of Knowledge in Games

	G	N
g		$(1,0)$
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	G	N
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$$C_{p,m}c$$

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What about other levels of knowledge?

R. Parikh and P. Krasucki. *Levels of knowledge in distributed computing*. Sadhana-Proceedings of the Indian Academy of Science 17 (1992).

What is a *State*?

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What about in *game situations*?
Answer: a *description* of the first-order and higher-order information of the players

R. Fagin, J. Halpern and M. Vardi. *Model theoretic analysis of knowledge*. Journal of the ACM 91 (1991).

Is an Epistemic Model "Common Knowledge"?

"The implicit assumption that the information partitions...are themselves common knowledge...constitutes no loss of generality... the assertion that each individual 'knows' the knowledge operators of all individual has no real substance; it is part of the framework."

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"it is an informal but *meaningful* meta-assumption....It is not trivial at all to assume it is "common knowledge" which partition every player has."

A. Heifetz. *How canonical is the canonical model? A comment on Aumann's interactive epistemology*. International Journal of Game Theory (1999).

A General Question

How many levels/states of knowledge (beliefs) are there?

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It depends on how you count:

- Parikh and Krasucki: Countably many *levels* of knowledge

  ▸ Why?

- Parikh and EP: Uncountably many levels of belief

  ▸ Why?

- Hart, Heiftetz and Samet: Uncountably many *states* of knowledge

  ▸ Why?

Returning to the Motivating Questions

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  It depends...

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  It depends...

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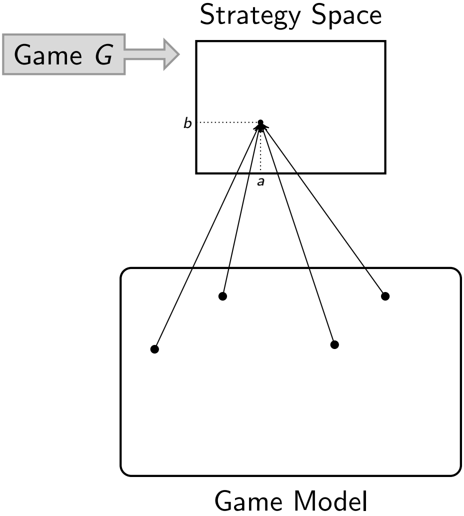
“Rational” Strategies



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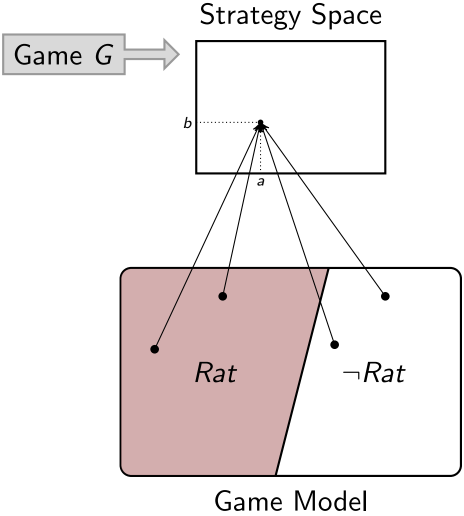

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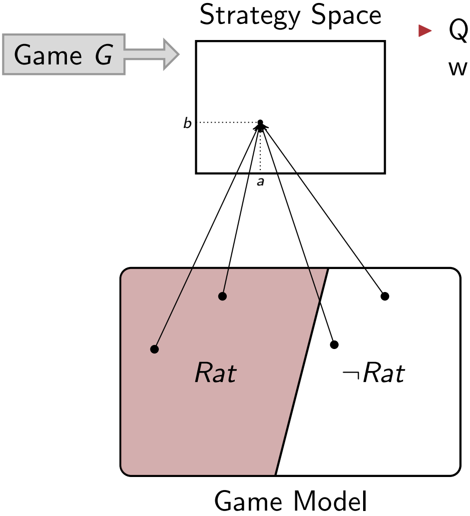

- Q1: Can we always find a model where $Rat \neq \emptyset$?

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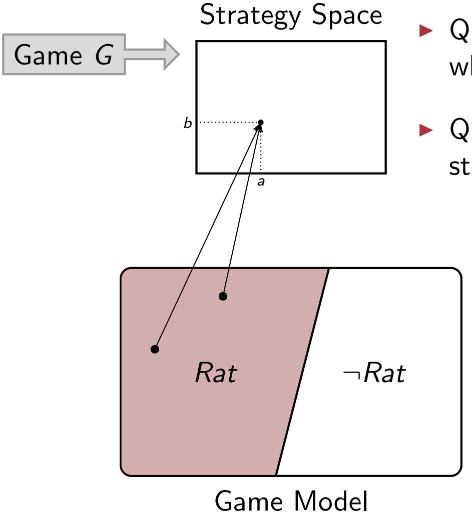

- Q1: Can we always find a model where $Rat \neq \emptyset$?
- Q2: Can we *characterize* the strategies that are *always* in Rat?

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  the agents choices are uniform in their information sets

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- *Common Knowledge* of "rational choice"
  there is no "Ann-Bob path" that leads outside of Rat

Returning to the quesitons

Simple Characterization Theorem In any *Bayesian model* of a finite strategic game, (the projection of) any state where the players *are rational* and there is *common knowledge of rationality* is exactly the set of strategies that survive *iterated removal of strictly dominated strategies*. (*Question 2*)

A. Brandenburger and E. Dekel. *Rationalizability and correlated equilibria*. *Econometrica*, 55:1391-1402, 1987.

Common Knowledge of Rationality

Bob

Ann		L	R
	U	1,2	0,1
	D	0,1	1,0

Common Knowledge of Rationality

		Bob L	R
Ann	U	1,2	0,1
	D	0,1	1,0

There is no prior such that R is rational for Bob.

Common Knowledge of Rationality

		Bob: L	Bob: R
Ann	U	1,2	0,1
Ann	D	0,1	1,0

If Ann knows this, then she does not consider R a option for Bob

Common Knowledge of Rationality

Ann \ Bob	L	R
U	1,2	0,1
D	0,1	1,0

So, U is the only rational choice.

Other natural properties...

- Do not *initially* rule out any strategies of the other players (admissibility)
- If two strategies are rational for an opponent, then neither can be "ruled out" (picking vs. choosing: *i knows* which options *j* will choose from, but *i* cannot know which optioned *j picked*)
- Do not *initially* rule out any *types* of the other players

...lead to puzzles

R. Cubitt and R. Sugden. *Rationally Justiable Play and the Theory of Non-cooperative games*. Economic Journal, 104, pgs. 798 - 803, 1994.

R. Cubitt and R. Sugden. *Common reasoning in games*. Manuscpript, 2008.

A. Brandenburger and H. J. Keisler. *An Impossibility Theorem on Beliefs in Games*. *Studia Logica* (2006).

Proving too Much Puzzle

		Bob L	Bob R
Ann	F	0,0	0,0
Ann	S	-1,3	2,2
Ann	T	-1,3	1,5

Proposition In every* Bayesian model satisfying *privacy of tie-breaking* of the above game, $Rat_A = \{F\}$ & $Rat_B = \{L, R\}$

Then player 1 must assign probability greater that $\frac{2}{3}$ *to player 2 playing L. But why is this justified?*

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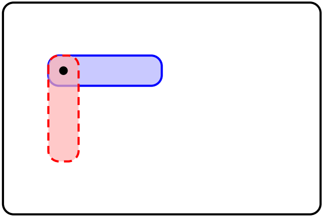

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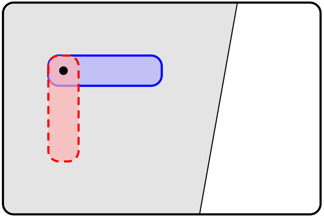
1. T *cannot* be rational for Ann



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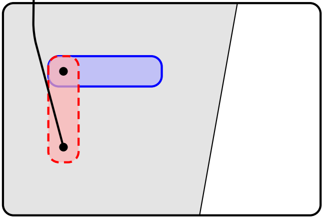

1. T *cannot* be rational for Ann
2. if Bob assigns a nonzero probability to S, then R is not rational

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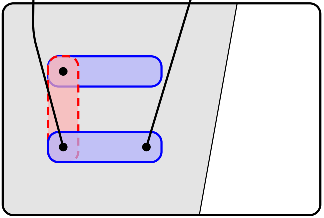

1. T *cannot* be rational for Ann
2. if Bob assigns a nonzero probability to S, then R is not rational
3. S is rational for Ann provided she considers it possible that Bob plays R

Ann \ Bob	L	R
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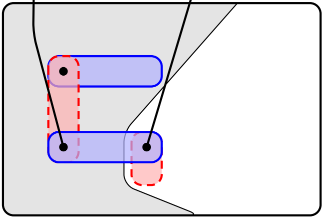

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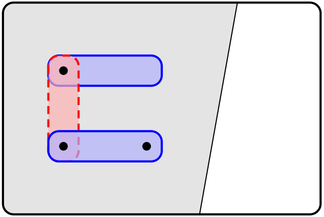

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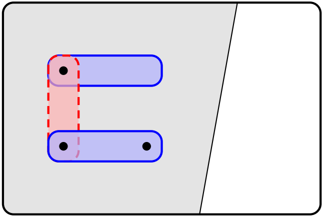

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2. if Bob assigns a nonzero probability to S, then R is not rational
3. S is rational for Ann provided she considers it possible that Bob plays R
4. F is the only rational choice for Ann
5. both L and R are rational responses if it is commonly known that Ann will play F

Paradox

Paradox

	in_2	out_2
in_1	$1, 1, 1$	$1, 1, 1$
out_1	$1, 1, 1$	$0, 1, 1$

in_3

	in_2	out_2
in_1	$1, 1, 1$	$1, 0, 1$
out_1	$1, 1, 0$	$0, 0, 0$

out_3

There is no Bayesian model of the above game satisfying privacy of tie-breaking.

Paradox

	in_2	out_2
in_1	$\color{blue}{1}, 1, 1$	$\color{blue}{1}, 1, 1$
$\color{red}{out_1}$	$\color{red}{1}, 1, 1$	$\color{red}{0}, 1, 1$

in_3

	in_2	out_2
in_1	$\color{blue}{1}, 1, 1$	$\color{blue}{1}, 0, 1$
$\color{red}{out_1}$	$\color{red}{1}, 1, 0$	$\color{red}{0}, 0, 0$

out_3

1. If 1 considers out_2 possible, then it is common knowledge that out_1 is not possible

Paradox

in_3:

	in_2	out_2
in_1	1, 1, 1	1, 1, 1
out_1	1, 1, 1	0, 1, 1

out_3:

	in_2	out_2
in_1	1, 1, 1	1, 0, 1
out_1	1, 1, 0	0, 0, 0

1. If 1 considers out_2 possible, then it is common knowledge that out_1 is not possible
2. If 2 considers out_3 possible, then it is common knowledge that out_2 is not possible

Paradox

	in_2	out_2
in_1	$1, 1, 1$	$1, 1, 1$
out_1	$1, 1, 1$	$0, 1, 1$

in_3

	in_2	out_2
in_1	$1, 1, 1$	$1, 0, 1$
out_1	$1, 1, 0$	$0, 0, 0$

out_3

1. If 1 considers out_2 possible, then it is common knowledge that out_1 is not possible
2. If 2 considers out_3 possible, then it is common knowledge that out_2 is not possible
3. If 3 considers out_1 possible, then it is common knowledge that out_3 is not possible

Paradox

	in_2	out_2
in_1	1, 1, 1	1, 1, 1
out_1	1, 1, 1	0, 1, 1

in_3

	in_2	out_2
in_1	1, 1, 1	1, 0, 1
out_1	1, 1, 0	0, 0, 0

out_3

4. If 1 does not consider out_2 possible, then 2 & 3 must consider in_1 & out_1 possible

Paradox

	in_2	out_2
in_1	1, 1, 1	1, 1, 1
out_1	1, 1, 1	0, 1, 1

in_3

	in_2	out_2
in_1	1, 1, 1	1, 0, 1
out_1	1, 1, 0	0, 0, 0

out_3

4. If 1 does not consider out_2 possible, then 2 & 3 must consider in_1 & out_1 possible
5. If 2 does not consider out_3 possible, then 1 & 3 must consider in_2 & out_2 possible

Paradox

$$\begin{array}{c|c|c|} & in_2 & out_2 \\ \hline in_1 & 1,1,1 & 1,1,1 \\ \hline out_1 & 1,1,1 & 0,1,1 \\ \hline \multicolumn{3}{c}{in_3} \end{array} \qquad \begin{array}{c|c|c|} & in_2 & out_2 \\ \hline in_1 & 1,1,1 & 1,0,1 \\ \hline out_1 & 1,1,0 & 0,0,0 \\ \hline \multicolumn{3}{c}{out_3} \end{array}$$

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5. If 2 does not consider out_3 possible, then 1 & 3 must consider in_2 & out_2 possible
6. If 3 does not consider out_1 possible, then 1 & 2 must consider in_3 & out_3 possible

Paradox

	in_2	out_2
in_1	$1, 1, 1$	$1, 1, 1$
out_1	$1, 1, 1$	$0, 1, 1$

in_3

	in_2	out_2
in_1	$1, 1, 1$	$1, 0, 1$
out_1	$1, 1, 0$	$0, 0, 0$

out_3

- If i considers out_{i+1} possible, then it is common knowledge that out_i is not possible
- If i does not consider out_{i+1} possible, then $i + 1$ & $i + 2$ must consider in_i & out_i possible

Paradox

in_3:

	in_2	out_2
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out_1	$1, 1, 1$	$0, 1, 1$

out_3:

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- 1 does consider out_2 possible $\implies$ 3 does not consider out_1 possible $\implies$ 2 considers out_3 possible $\implies$ 1 does not consider out_2 possible

Digression: Admissibility

Related to an old puzzle in game theory about admissibility (best response to some belief of player i with positive probability on all the strategies of the opponents):

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A strategy is admissibility iff it is not *weakly dominated*.

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Does assuming that it is commonly known that players play only admissible strategies lead to a process of iterated removal of weakly dominated strategies? no!

L. Samuelson. *Dominated Strategies and Common Knowledge*. Games and Economic Behavior (1992).

Explanation

Some Issues

- ✓ What *does* a group know/believe/accept? vs. what *can* a group (come to) know/believe/accept?

C. List. *Group knowledge and group rationality: a judgment aggregation perspective*. Episteme (2008).

- ✓ Other "group informational attitudes": distributed knowledge, common belief, ...
- ✓ Levels of knowledge
- ✓ Common knowledge/belief of *rationality*
- ▶ Where does common knowledge come from?

R. Cubitt and R. Sugden. *Common Knowledge, Salience and Convention: A Reconstruction of David Lewis' Game Theory*. Economics and Philosophy, 19, pgs. 175-210 , 2003..

Reason to Believe

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- Anyone who accept the rules of arithmetic has a reason to believe $618 \times 377 = 232,986$, but most of us do not hold have firm beliefs about this.
- Definition: $R_i(\varphi)$ means φ is true within some logic of reasoning that is *endorsed* by (that is, accepted as a normative standard by) person i...φ must be either regarded as *self-evident* or derivable by rules of inference (deductive or inductive)

A indicates to i that φ

A is a "state of affairs"

$A\ ind_i\ \varphi$: i's reason to believe that A holds *provides* i's reason for believing that φ is true.

(A1)For all i, for all A, for all φ: $[R_i(A\ holds) \wedge (A\ ind_i\ \varphi)] \Rightarrow R_i(\varphi)$

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- $[(A \textit{ ind}_i \varphi) \wedge (\varphi \textit{ entails } \psi)] \Rightarrow A \textit{ ind}_i \psi$
- $[(A \textit{ ind}_i\ R_j[A' \textit{ holds}]) \wedge R_i(A' \textit{ ind}_j \varphi)] \Rightarrow A \textit{ ind}_i R_j(\varphi)$

Reflexive Common Indicator

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- $A\ ind_i\ \varphi$
- $(A\ ind_i\ \psi) \Rightarrow R_i[A\ ind_j\ \psi]$

Let $R^G(\varphi)$: $R_i\varphi, R_j\varphi, \ldots$, $R_i(R_j\varphi)$, $R_j(R_i(\varphi))$, $\ldots$
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iterated reason to believe φ.

Theorem. (Lewis) For all states of affairs A, for all propositions φ, and for all groups G: if A holds, and if A is a reflexive common indicator in G that φ, then $R^G(\varphi)$ is true.

Lewis and Aumann

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Example. Suppose there is an agent $i \notin G$ that is *authoritative* for each member of G. So, for $j \in G$, "i states to j that φ is true" *indicates to j that φ*. Suppose that *separately and privately* to each member of G, i states that φ and $R^G(\varphi)$ are true.Then, we have $R^i\varphi$ and $R_i(R^G(\varphi))$ for each $i \in G$. *But there is no common indicator that φ is true.* The agents $j \in G$ may have no reason to believe that everyone heard the statement from i or that all agents in G treat i as authoritative.

How does this help?

		B	
		l	r
A	l	10,10	0,0
	r	0,0	11,11

A: What should *we* do? **Team Reasoning**: why should this "mode of reasoning" be endorsed?

Reason to Believe Logic

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Assume each person's logic at least contains propositional logic:
$inf(R) : \varphi_1, \dots \varphi_n, \neg(\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \neg\psi) \to \psi$

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Agent i is the **subject of the proposition** φ_i if φ_i makes an assertion about a current or future act of is will

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- assert that i endorses some inference rule; or
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$R_i(\varphi_i)$ vs. $R_j(\varphi_i)$: Suppose i reliable takes a bus every Monday. The other commuters may all make the inductive inference that i will take the bus next Monday (M_i). In fact, we may assume that this is a *common mode of reasoning*, so everyone reliably makes the inference that i will catch the bus next Monday. So, $R_j(M_i)$, $R_iR_j(M_i)$, but i should still be *free* to choose whether he wants to take the bus on Monday, so $\neg R_i(M_i)$ and $\neg R_j(R_i(M_i))$, etc.

Common Reason to Believe

Awareness of Common Reason: for all $i \in G$ and all propositions φ,

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Common Attribution of Common Reason: for all $i \in G$, for all propositions φ for which i is not the subject

$$inf(R^G) : \varphi \rightarrow R_i(\varphi)$$

Common Reason to Believe to Common Belief

Theorem The three previous properties can generate any hierarchy of belief (i has reason to believe that j has reason to believe that... that φ) for any φ with $R^G(\varphi)$.

Team Maximising

$inf(R_i) : R^N[opt(v, N, s^N)]$,
$R^N[$ each $i \in N$ endorses team maximising with respect to N and v $]$,
$R^N[$ each member of N acts on reasons $] \rightarrow ought(i, s_i)$

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$R_i[ought(i, s_i)]$: i has reason to choose s_i

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i acts on reasons if for all s_i, $R_i[ought(i, s_i)] \Rightarrow choice(i, s_i)$

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$opt(v, N, s^N)$: s^N is maximal for the group N w.r.t. v

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Recursive definition: i's endorsement of the rule depends on i having a reason to believe everyone else endorses the rule...

Next

Team modes of reasoning, group identification,frames and team preferences

Levels of Knowledge

Fix a set of agents $\mathcal{A} = \{1, \ldots, n\}$.

$\Sigma_K = \{K_1, \ldots, K_n\}$ and $\Sigma_C = \{C_U\}_{U \subseteq \mathcal{A}}$

Level of Knowledge: $Lev_{\mathcal{M}}(p, s) = \{x \in \Sigma^* \mid \mathcal{M}, s \models xp\}$ (where $\Sigma = \Sigma_K$ or $\Sigma = \Sigma_C$).

[If Σ is a finite set, then Σ^* is the set of finite strings over Σ]
[Recall the definition of truth in a Kripke structure]

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R. Parikh and P. Krasucki. *Levels of knowledge in distributed computing*. Sadhana-Proceedings of the Indian Academy of Science 17 (1992).

R. Parikh. *Levels of knowledge, games and group action*. Research in Economics 57, pp. 267 - 281 (2003).

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A level of knowledge is simply a set of finite strings over Σ_C.
Why isn't it obvious that there are *uncountably* many levels of knowledge?

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Consider the sets:

- $L_1 = \{K_1, K_2\}$ and $L_2 = \{K_1, K_2, K_1K_2\}$
- $L_1 = \{K_1, K_3, K_1K_2K_3\}$ and $L_2 = \{K_1, K_2, K_3, K_1K_2K_3\}$

Levels of Knowledge

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Consider the sets:

- $L_1 = \{K_1, K_2\}$ and $L_2 = \{K_1, K_2, K_1K_2\}$ (*different level of knowledge*)
- $L_1 = \{K_1, K_3, K_1K_2K_3\}$ and $L_2 = \{K_1, K_2, K_3, K_1K_2K_3\}$ (*same level of knowledge*)

Levels of Knowledge: Preliminaries

Given any two strings $x, y \in \Sigma^*$, we say x is **embeddable** in y, written $x \leq y$, if all symbols of x occur in y in the same order but not necessarily consecutively.

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1. $x \leq x$ and $\epsilon \leq x$ for all $x \in \Sigma^*$
2. $x \leq y$ if there exists $x', x'', y', y'', (y, y'' \neq \epsilon)$ such that $x = x'x''$, $y = y'y''$ and $x' \leq y'$, $x'' \leq y''$.

$\leq$ is the smallest relation satisfying (1) and (2).

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Example:
$aba \leq aaba$
$aba \leq abca$
$aba \not\leq aabb$

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well founded if every infinite subset of X has a $(\preceq -)$minimal element.

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a **well-partial order** (WPO) if $(X, \preceq)$ is a partial order and every linear order that extends $(X, \preceq)$ (i.e., a linear order $(X, \preceq')$ with $\preceq \subseteq \preceq'$) is well-founded.

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A set $\{a_1, a_2, \ldots\}$ of incomparable elements is a well-founded partial order but not a WPO.

Well-Partial Orders

Fact. $(X, \preceq)$ is a WPO iff $\preceq$ is well-founded and every subset of mutually incomparable elements is finite

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Theorem (Higman). If Σ is finite, then $(\Sigma^*, \leq)$ is a WPO

G. Higman. *Ordering by divisibility in abstract algebras*. Proc. London Math. Soc. 3 (1952).

D. de Jongh and R. Parikh. *Well-Partial Orderings and Hierarchies*. Proc. of the Koninklijke Nederlandse Akademie van Wetenschappen 80 (1977).

WPO and Downward Closed Sets

Given $(X, \preceq)$ a set $A \subseteq X$ is **downward closed** iff $x \in A$ implies for all $y \preceq x$, $y \in A$.

Theorem. (Parikh & Krasucki) If Σ is finite, then there are only countably many $\leq$-downward closed subsets of Σ^* and all of them are *regular*.

Levels of Knowledge

Theorem. Consider the alphabet $\Sigma_C = \{C_U\}_{U \subseteq \mathcal{A}}$. For all strings $x, y \in \Sigma_C^*$, if $x \preceq y$ then for all pointed models $\mathcal{M}, s$, if $\mathcal{M}, s \models yP$ then $\mathcal{M}, s \models xP$.

Levels of Knowledge

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Corollary 1. Every level of knowledge is a downward closed set.
Corollary 2. There are only countably many levels of knowledge.

Realizing Levels of Knowledge

Theorem. (R. Parikh and EP) Suppose that L is a downward closed subset of Σ_K^*, then there is a finite Kripke model $\mathcal{M}$ and state s such that $\mathcal{M}, s \models xP$ iff $x \in L$. (i.e., $L = Lev_{\mathcal{M}}(p, s)$).

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States of Knowledge

S. Hart, A. Heifetz and D. Samet. *"Knowing Whether," "Knowing That," and The Cardinality of State Spaces*. Journal of Economic Theory 70 (1996).

States of Knowledge

Let W be a set of states and fix an event $X \subseteq W$.

Consider a sequence of finite boolean algebras $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ defined as follows:
$\mathcal{B}_0 = \{\emptyset, X, \neg X, \Omega\}$
$\mathcal{B}_n = \mathcal{B}_{n-1} \cup \{K_i E \mid E \in \mathcal{B}_{n-1}, i \in \mathcal{A}\}$

The events $\mathcal{B} = \cup_{i=1,2,\ldots} \mathcal{B}_i$ are said to be **generated by** X.

States of Knowledge

Definition. Two states w, w' are **separated** by X if there exists an event E which is generated by X such that $w \in E$ and $w' \in \neg E$.

Question: How many states can be in an information structure (W, Π_1, Π_2) such that an event X separates any two of them?

States of Knowledge

Consider a K-list $(E_1, E_2, E_3, \ldots)$ of events generated by X.

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$(X, K_1X, \neg K_2K_1X, \neg K_1\neg K_2K_1X, K_2\neg K_1\neg K_2K_1X)$ is inconsistent.

Knowing Whether

Let $J_i E \;:=\; K_i E \vee K_i \neg E$.

Lemma. Every J-list is consistent.

Theorem. (Hart, Heifetz and Samet) There are uncountably many states of knowledge.

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What about beliefs?

In Aumann/Kripke structures belief operators are just like knowledge operators except we replace the truth axiom/property ($K\varphi \to \varphi$) with a consistency property ($\neg B\bot$).

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Theorem. (R. Parikh and EP) There are uncountably many levels of belief.

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The Issue

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The condition that the players incorporate admissibility *into their rationality calculations seems to conflict with the condition that the players think the other players are rational*

Common Knowledge of Admissibility

Theorem Iterated admissibility is not equivalent to common knowledge of admissibility.

L. Samuelson. *Dominated Strategies and Common Knowledge*. Games and Economic Behavior (1992).

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X_1	2,4	5,4	-1,0
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Other Results

1. We have seen that IA and common knowledge of admissibility diverge.
2. There exist games in which assuming that admissibility is common knowledge does not provide players with sufficient information to determine which strategies should be eliminated on admissibility grounds.
3. There exists games in which assuming that admissibility is common knowledge yields a contradiction

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A. Brandenburger, A. Friedenberg, H. J. Keisler. *Admissibility in Games*. Econometrica (2008).

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