Social Choice Theory for Logicians

Lecture 2

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Plan

1. Arrow, Sen, Muller-Satterthwaite
2. Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
3. Voting to get things "right" (Distance-based measures, Condorcet and extensions)
4. Strategizing (Gibbard-Satterthwaite)
5. Generalizations
   5.1 Infinite Populations
   5.2 Judgement aggregation (List & Dietrich)
6. Logics
7. Applications

- X is a (finite or infinite) set of **alternatives** (or **candidates**).

- $N = \{1, \ldots, n\}$ is a set of **voters**

- **Preferences**: $\mathcal{P} = \{R \mid R \subseteq W \times W$ where R is reflexive, transitive and connected$\}$

- Given $R \in \mathcal{P}$, let P be the **strict preference generated by** R: xPy iff xRy and not yRx (we write P_R if necessary)

- A profile is a tuple $(R_1, \ldots, R_n) \in \mathcal{P}^n$

- **Social Welfare Function**: $F : D \to \mathcal{P}$ where $D \subseteq \mathcal{P}^n$ is the domain.

- **Universal Domain** (UD): The domain of F is $\mathcal{P}^n$:

$$D = \mathcal{P}^n$$

- **Independence of Irrelevant Alternatives** (IIA): F satisfies IIA provide for all profiles $\vec{R}, \vec{R'} \in D$,

$$\text{if } [xR_iy \text{ iff } xR'_iy \text{ for all } i \in N], \text{ then } xF(\vec{R})y \text{ iff } xF(\vec{R'})y$$

- **(weak) Pareto** (P): For all profiles $\vec{R} \in D$,

$$\text{if } xP_iy \text{ for all } i \in N \text{ then } xP_{F(\vec{R})}y$$

- Agent i is a **dictator** for F provided for every preference profile and every pair $x, y \in X$,

$$xP_iy \text{ implies } xF(\vec{R})y$$

.

Lemma 1 Suppose that for some x and y, S is decisive for x over y, then S is decisive.

Lemma 2 If S and T are decisive then so is $S \cap T$

Lemma 3 If S is not decisive, then $S^C = N - S$ is decisive.

Arrow's Theorem: There is a singleton decisive set.

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p: Prude reads the book;
$l \to p$: If Lewd reads the book, then so does Prude.

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Prude desires to not read the book, and that Lewd not read it either, but in case Lewd does read the book, Prude wants to read the book to be informed about the dangerous material Lewd has read.

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So, society must be inconsistent!

Muller-Satterthwaite Theorem

E. Muller and M. A. Satterthwaite. *The equivalence of strong positive association and strategy-proofness*. Journal of Economic Theory, 14(2):412-418, 1977.

P. Tang and T. Sandholm. *Coalitional Structure of the Muller-Satterthwaite Theorem*. In *Proceedings of the Workshop on Cooperative Games in Multiagent Systems* (CoopMAS) at AAMAS, 2012.

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A. Sen. *The Possibility of Social Choice*. The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

# voters	30	1	29	10	10	1
2	A	A	B	B	C	C
1	B	C	A	C	A	B
0	C	B	C	A	B	A

$BS(A) = 2 \times 31 + 1 \times 39 + 0 \times 11 = 101$
$BS(B) = 2 \times 39 + 1 \times 31 + 0 \times 11 = 109$
$BS(C) = 2 \times 11 + 1 \times 11 + 0 \times 59 = 33$

$B >_{BC} A >_{BC} C$

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

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	B	C	A	C	A	B
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# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

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Condorcet's Other Paradox: No *scoring rule* will work...
$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$
$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

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$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$

$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$Score(A) > Score(B) \Rightarrow 31s_2 + 39s_1 > 39s_2 + 31s_1 \Rightarrow s_1 > s_2$

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

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Theorem (Fishburn 1974). For all $m \geq 3$, there is some voting situation with a Condorcet winner such that every weighted scoring rule will have at least $m - 2$ candidates with a greater score than the Condorcet winner.

Electing the Condorcet Winner, I

Condorcet Rule: Each voter submits a linear ordering over all the candidates. If there is a Condorcet winner, then that candidate wins the election. Otherwise, all candidates tie for the win.

Copeland's Rule: Each voter submits a linear ordering over all the candidates. A win-loss record for candidate B is calculated as follows:

$$WL(B) = |\{C \mid B >_M C\}| - |\{C \mid C >_M B\}|$$

The Copeland winner is the candidate that maximizes WL.

Electing the Condorcet Winner, II

Dodgson's Method: Each voter submits a linear ordering over all the candidates. For each candidate, determine the fewest number of pairwise swaps needed to make that candidate the Condorcet winner. The candidate(s) with the fewest swaps is(are) declared the winner(s).

Black's Procedure: Each voter submits a linear ordering over all the candidates. If there is a Condorcet winner, then that candidate is the winner. Otherwise, let the winners be the Borda Count winners.

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	C	B	C	A	B	A

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	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

20	28
A	B
B	A
C	C