3.10pt

# Social Choice Theory for Logicians

## Lecture 4

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# Plan

- ✓ Arrow, Sen, Muller-Satterthwaite
- ✓ Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
- ✓ Voting to get things "right" (Distance-based measures, Condorcet and extensions)
- ✓ Strategizing (Gibbard-Satterthwaite)

1. Generalizations
   1.1 Infinite Populations
   1.2 Judgement aggregation (List & Dietrich)
2. Logics
3. Applications

Consider 3 votes, each with a confidence level $p = 2/3$.

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$$\binom{3}{2} * (2/3)^2 * 1/3^1 + \binom{3}{3} 2/3^3 * 1/3^0$$

$$= 3 * 4/27 + 1 * 8/27$$

$$= 20/27$$

# Condorcet Jury Theorem

State of the world **x** takes values 0 and 1

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**Condorcet Jury Theorem**. Suppose Independence and Competence. As the group size increases, the probability $Pr(M_n)$ that a majority votes correctly (i) increases and (ii) converges to one.

D. Austen-Smith and J. Banks. *Aggregation, Rationality and the Condorcet Jury Theorem*. The American Political Science Review, 90, 1, pgs. 34 - 45, 1996.

D. Estlund. *Opinion Leaders, Independence and Condorcet's Jury Theorem*. Theory and Decision, 36, pgs. 131 - 162, 1994.

F. Dietrich. *The premises of Condorcet's Jury Theorem are not simultaneously justified*. Episteme, 2008.

# Judgement Aggregation

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Preference aggregations vs. *judgement aggregation*

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- Judgements of preference, value judgements, beliefs
- What should be done? What is the best alternative?
- The Pareto conditions (see forthcoming work by W. Rabinowicz, S. Hartmann and S. Rafiee Rad)

# Doctrinal Paradox

Suppose that three experts *independently* formed opinions about three propositions. For example,

1. $p$: "Carbon dioxide emissions are above the threshold $x$"
2. $p \rightarrow q$: "If carbon dioxide emissions are above the threshold $x$, then there will be global warming"
3. $q$: "There will be global warming"

# Doctrinal Paradox

| | $p$ | $p \to q$ | $q$ |
|---|---|---|---|
| Expert 1 | | | |
| Expert 2 | | | |
| Expert 3 | | | |

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| | $p$ | $p \to q$ | $q$ |
|---|---|---|---|
| Expert 1 | True | True | |
| Expert 2 | | | |
| Expert 3 | | | |

# Doctrinal Paradox

| | $p$ | $p \to q$ | $q$ |
|---|---|---|---|
| Expert 1 | True | True | True |
| Expert 2 | | | |
| Expert 3 | | | |

# Doctrinal Paradox

| | $p$ | $p \to q$ | $q$ |
|---|---|---|---|
| Expert 1 | True | True | True |
| Expert 2 | True | | False |
| Expert 3 | | | |

# Doctrinal Paradox

| | $p$ | $p \to q$ | $q$ |
|---|---|---|---|
| Expert 1 | True | True | True |
| Expert 2 | True | False | False |
| Expert 3 | | | |

# Doctrinal Paradox

| | $p$ | $p \to q$ | $q$ |
|---|---|---|---|
| Expert 1 | True | True | True |
| Expert 2 | True | False | False |
| Expert 3 | False | True | False |

# Doctrinal Paradox

| | $p$ | $p \rightarrow q$ | $q$ |
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| Expert 1 | True | True | True |
| Expert 2 | True | False | False |
| Expert 3 | False | True | False |
| Group | | | |

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# The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

$p$: a valid contract was in place

$q$: there was a breach of contract

$r$: the court is required to find the defendant liable.

| | $p$ | $q$ | $(p \wedge q) \leftrightarrow r$ | $r$ |
|---|---|---|---|---|
| 1 | yes | yes | yes | yes |
| 2 | yes | no | yes | no |
| 3 | no | yes | yes | no |

# The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

Should we accept $r$?

| | $p$ | $q$ | $(p \wedge q) \leftrightarrow r$ | $r$ |
|---|---|---|---|---|
| 1 | yes | yes | yes | yes |
| 2 | yes | no | yes | no |
| 3 | no | yes | yes | no |

# The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

Should we accept $r$? No, a simple majority votes no.

| | $p$ | $q$ | $(p \wedge q) \leftrightarrow r$ | $r$ |
|---|---|---|---|---|
| 1 | yes | yes | yes | yes |
| 2 | yes | no | yes | no |
| 3 | no | yes | yes | no |

# The Logic of Group Decisions, II

(Kornhauser and Sager 1993)

Should we accept $r$? Yes, a majority votes yes for $p$ and $q$ and $(p \wedge q) \leftrightarrow r$ is a legal doctrine.

| | $p$ | $q$ | $(p \wedge q) \leftrightarrow r$ | $r$ |
|---|---|---|---|---|
| 1 | yes | yes | yes | yes |
| 2 | yes | no | yes | no |
| 3 | no | yes | yes | no |

# Many Variants!

See
`http://personal.lse.ac.uk/LIST/doctrinalparadox.htm`
for many generalizations!

Kornhauser and Sager. *Unpacking the court*. Yale Law Journal, 1986.

C. List and P. Pettit. *Aggregating Sets of Judgments: An Impossibility Result*. Economics and Philosophy 18: 89-110, 2002.

## The Judgement Aggregation Model: The Propositions

**Propositions**: Let $\mathcal{L}$ be a logical language (called **propositions** in the literature) with the usual boolean connectives.

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**Consistency**: The standard notion of logical consistency.

*Aside*: We actually need

1. $\{p, \neg p\}$ are inconsistent
2. all subsets of a consistent set are consistent
3. $\emptyset$ is consistent and each $S \subseteq \mathcal{L}$ has a consistent maximal extension (not needed in all cases)

# The Judgement Aggregation Model: The Agenda

**Definition** The **agenda** is a non-empty set $X \subseteq \mathcal{L}$, interpreted as the set of propositions on which judgments are made (note: $X$ is a union of proposition-negation pairs $\{p, \neg p\}$).

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**Example**: In the discursive dilemma:
$X = \{a, \neg a, b, \neg b, a \rightarrow b, \neg(a \rightarrow b)\}$.

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**Definition**: Given an agenda $X$, each individual $i$'s judgement set is a subset $A_i \subseteq X$.

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**Definition**: Given an agenda $X$, each individual $i$'s judgement set is a subset $A_i \subseteq X$.

**Rationality Assumptions**:

1. $A_i$ is **consistent**
2. $A_i$ is **complete**, if for each $p \in X$, either $p \in A_i$ or $\neg p \in A_i$

## The Judgement Aggregation Model: Aggregation Rules

Let $X$ be an agenda, $N = \{1, \ldots, n\}$ a set of voters, a **profile** is a tuple $(A_i, \ldots, A_n)$ where each $A_i$ is a judgement set. An **aggregation function** is a map from profiles to judgment sets. I.e., $F(A_1, \ldots, A_n)$ is a judgement set.

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**Examples**:

- **Propositionwise majority voting**: for each $(A_1, \ldots, A_n)$,

$$F(A_1, \ldots, A_n) = \{p \in X \mid |\{i \mid p \in A_i\}| \geq |\{i \mid p \notin A_i\}|\}$$

- **Dictator of** $i$: $F(A_1, \ldots, A_n) = A_i$
- **Reverse Dictator of** $i$: $F(A_1, \ldots, A_n) = \{\neg p \mid p \in A_i\}$

# The Judgement Aggregation Model: Input Condition

**Universal Domain**: The domain of $F$ is the set of all possible profiles of consistent and complete judgement sets.

# The Judgement Aggregation Model: Output Condition

**Collective Rationality**: $F$ generates consistent and complete collective judgment sets.

# The Judgement Aggregation Model: Responsiveness Conditions

**Systematicity**: For any $p, q \in X$ and all $(A_1, \ldots, A_n)$ and $(A_1^*, \ldots, A_n^*)$ in the domain of $F$,

$$\text{if [for all } i \in N,\ p \in A_i \text{ iff } q \in A_i^*]$$
$$\text{then } [p \in F(A_1, \ldots, A_n) \text{ iff } q \in F(A_1^*, \ldots A_n^*)\ ].$$

# The Judgement Aggregation Model: Responsiveness Conditions

**Systematicity**: For any $p, q \in X$ and all $(A_1, \ldots, A_n)$ and $(A^*_1, \ldots, A^*_n)$ in the domain of $F$,

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$$\text{then } [p \in F(A_1, \ldots, A_n) \text{ iff } q \in F(A^*_1, \ldots A^*_n)\ ].$$

- independence
- neutrality

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$$\text{then } [p \in F(A_1, \dots, A_n) \text{ iff } q \in F(A_1^*, \dots A_n^*) \text{ ]}.$$

- independence
- neutrality

**Independence**: For any $p \in X$ and all $(A_1, \dots, A_n)$ and $(A_1^*, \dots, A_n^*)$ in the domain of $F$,

$$\text{if [for all } i \in N,\ p \in A_i \text{ iff } p \in A_i^*]$$
$$\text{then } [p \in F(A_1, \dots, A_n) \text{ iff } p \in F(A_1^*, \dots A_n^*) \text{ ]}.$$

# The Judgement Aggregation Model: Responsiveness Conditions

**Anonymity**: For all profiles $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = F(A_{\pi(1)}, \ldots, A_{\pi(n)})$ where $\pi$ is a permutation of the voters.

**Unanimity**: For all profiles $(A_1, \ldots, A_n)$ if $p \in A_i$ for each $i$ then $p \in F(A_1, \ldots, A_n)$

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**Unanimity**: For all profiles $(A_1, \ldots, A_n)$ if $p \in A_i$ for each $i$ then $p \in F(A_1, \ldots, A_n)$

**Monotonicity**: For any $p \in X$ and all $(A_1, \ldots A_i, \ldots, A_n)$ and $(A_1, \ldots, A_i^*, \ldots, A_n)$ in the domain of $F$,

$$\text{if } [p \notin A_i,\ p \in A_i^* \text{ and } p \in F(A_1, \ldots, A_i, \ldots A_n)]$$
$$\text{then } [p \in F(A_1, \ldots, A_i^*, \ldots A_n)].$$

# The Judgement Aggregation Model: Responsiveness Conditions

**Non-dictatorship**: There exists no $i \in N$ such that, for any profile $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = A_i$

## Baseline Result

**Theorem (List and Pettit, 2001)** If $X \subseteq \{a, b, a \wedge b\}$, there exists no aggregation rule satisfying universal domain, collective rationality, systematicity and anonymity.

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See `personal.lse.ac.uk/LIST/doctrinalparadox.htm` for many generalizations!

# Agenda Richness

Whether or not judgment aggregation gives rise to serious impossibility results depends on how the propositions in the agenda are *interconnected*.

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**Definition** A set $Y \subseteq \mathcal{L}$ is **minimally inconsistent** if it is inconsistent and every proper subset $X \subsetneq Y$ is consistent.

# Agenda Richness

**Definition** An agenda $X$ is **minimally connected** if

1. (non-simple) it has a minimal inconsistent subset $Y \subseteq X$ with $|Y| \geq 3$
2. (*even-number-negatable*) it has a minimal inconsistent subset $Y \subseteq X$ such that

$$Y - Z \cup \{\neg z \mid z \in Z\} \text{ is consistent}$$

for some subset $Z \subseteq Y$ of even size.

## Impossibility Theorems

**Theorem (Dietrich and List, 2007)** If (and only if) an agenda is non-simple and even-number negatable, every aggregation rule satisfying universal domain, collective rationality, systematicity and unanimity is a dictatorship (or inverse dictatorship).

# Impossibility Theorems

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**Theorem (Nehring and Puppe, 2002)** If (and only if) an agenda is non-simple, every aggregation rule satisfying universal domain, collective rationality, systematicity unanimity, and monotonicity is a dictatorship.

# Characterization Result

$p \in X$ conditionally entails $q \in X$, written $p \vdash^* q$ provided there is a subset $Y \subseteq X$ consistent with each of $p$ and $\neg q$ such that $\{p\} \cup Y \vdash q$.

**Totally Blocked**: $X$ is totally blocked if for any $p, q \in X$ there exists $p_1, \ldots, p_k \in X$ such that

$$p = p_1 \vdash^* p_2 \vdash^* \cdots \vdash^* p_k = q$$

## Characterization Result

**Theorem (Dietrich and List, 2007, Dokow Holzman 2010)** If (and only if) an agenda is totally blocked and even-number negatable, every aggregation rule satisfying universal domain, collective rationality, independence and unanimity is a dictatorship.

**Theorem (Nehring and Puppe, 2002, 2010)** If (and only if) an agenda is totally blocked, every aggregation rule satisfying universal domain, collective rationality, independence unanimity, and monotonicity is a dictatorship.

# Many Variants!

Christian List. *The Theory of Judgement Aggregation: A Survey*. Synthese, forthcoming.

How should we aggregate judgements *without* independence?

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- Premiss-based aggregation
- Distance-based

## What is a premiss?

An employee-owned bakery must decide whether to buy a pizza oven ($P$) or a fridge to freeze their outstanding Tiramisu ($F$). The pizza oven and the fridge cannot be in the same room. So they also need to decide whether to rent an extra room in the back ($R$). They all agree that they will rent the room if they decide to buy both the pizza oven and the fridge: $((P \wedge F) \rightarrow R)$, but they are contemplating renting the room regardless of the outcome of the vote on the appliances.

F. Cariani. *Judgement Aggregation*. Philosophy Compass, 6, 1, pgs. 22 - 32, 2011.

# Distance-Based Aggregation


G. Pigozzi. *Belief merging and the discursive dilemma: an argument-based account of paradoxes in judgement aggregation*. Synthese 152, pgs. 285 - 298, 2006.

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

C. Duddy and A. Piggins. *A measure of distance between judgement sets*. Manuscript, 2011.

Given $(A_1, \ldots, A_n)$, select the set consistent and complete $A$ that minimizes the total distance from the individual judgement sets: find $A$ such that $\sum_{i \in N} d(A, A_i)$ is minimized.

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**Hamming Metric**: $d(A, A') =$ the number of propositions for which $A$ and $A'$ disagree

$d_H(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\}) = 2$

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$d_H(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\}) = 2$

Duddy and Piggins: shouldn't
$d(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\} = 1$?

# Duddy and Piggins Measure

Judgement set $C$ is between judgement sets $A$ and $B$ if $A, B$ and $C$ are distinct and, on each proposition $C$ agrees with $A$ or with $B$ (or both). (*$C$ is a compromise between $A$ and $B$*)

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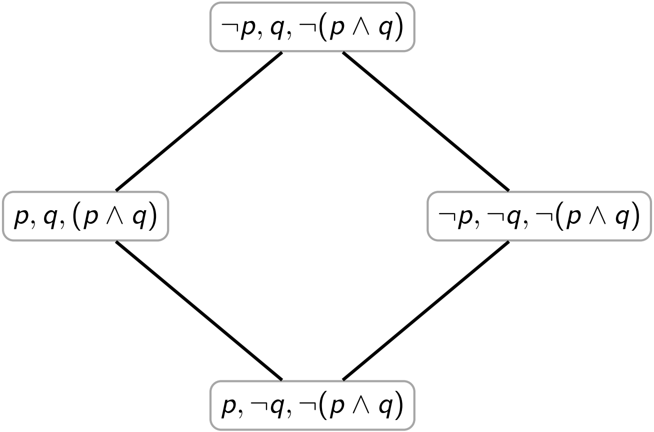
Judgement set $C$ is between judgement sets $A$ and $B$ if $A, B$ and $C$ are distinct and, on each proposition $C$ agrees with $A$ or with $B$ (or both). (*$C$ is a compromise between $A$ and $B$*)

Draw a graph where the nodes are possible judgement sets and there is an edge between $A$ and $B$ provided there is no judgement set between them.

The distance between $A$ and $B$ is the length of the shortest path from $A$ to $B$.


$\neg p, q, \neg(p \wedge q)$
$p, q, (p \wedge q)$
$\neg p, \neg q, \neg(p \wedge q)$
$p, \neg q, \neg(p \wedge q)$

# Axioms

**Axiom 1** $d(A, B) = 0$ iff $A = B$
**Axiom 2** $d(A, B) = d(B, A)$
**Axiom 3** $d(A, B) \leq d(A, C) + d(C, B)$

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For all $A, B, C$, $C$ is between $A$ and $B$ provided $A \neq B \neq C$ and $(A \cap B) \subset C$.

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**Axiom 4** If there is a judgement set between $A$ and $B$ then there exists $C$ different from $A$ and $B$ such that
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**Axiom 5** If there is no judgement set between $A$ and $B$ with $A \neq B$ then $d(A, B) = 1$

**Theorem** (Duddy & Piggins) The previously defined metric is the unique metric satisfying Axioms 1 - 5.

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |
| DP-metric | $T$ | $T$ | $T$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |
| DP-metric | $T$ | $T$ | $T$ |
| Hamming | $F$ | $T$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |
| DP-metric | $T$ | $T$ | $T$ |
| Hamming | $F$ | $T$ | $F$ |
| Premise | $T$ | $T$ | $T$ |

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

Differing on $\{a, b \wedge c\}$ may be considered more consequential than differing on $\{a, a \wedge b\}$.

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Let $\mathcal{F}$ be the set of *all* judgement sets and $\mathcal{F}^\circ$ the set of all consistent judgement sets.

$d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$

**Axiom 1** $d(A, B) = 0$ iff $A = B$
**Axiom 2** $d(A, B) = d(B, A)$
**Axiom 3** $d(A, B) \leq d(A, C) + d(C, B)$

$d(J, J') = \sum_{i \leq n} d(J_i, J'_i)$

For a profile $P$, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. $P$ is majority consistent provided $M(P) \in \mathcal{F}^\circ$

Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

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Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

- $Full_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority consistent profiles $J'$ in $\mathcal{F}^\circ$

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- $\mathit{Full}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority consistent profiles $J'$ in $\mathcal{F}^\circ$
- $\mathit{Output}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority profiles $J'$ in $\mathcal{F}$ (*allowing inconsistencies*)

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Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

- $\mathit{Full}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority consistent profiles $J'$ in $\mathcal{F}^\circ$
- $\mathit{Output}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority profiles $J'$ in $\mathcal{F}$ (*allowing inconsistencies*)
- $\mathit{Endpoint}_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles $J'$

For a profile $P$, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. $P$ is majority consistent provided $M(P) \in \mathcal{F}^\circ$

Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

- $Full_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority consistent profiles $J'$ in $\mathcal{F}^\circ$
- $Output_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority profiles $J'$ in $\mathcal{F}$ (*allowing inconsistencies*)
- $Endpoint_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles $J'$
- $Prototype_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $\sum_{i \leq n} d(J_i, K)$ over all $K \in \mathcal{F}^\circ$

For $J, K$ let $Ham(J, K)$ denote the Hamming distance (the number of items on which $J$ and $K$ disagree)

$$d(J, K) = \begin{cases} 0.9 & \text{if } J \text{ and } K \text{ disagree only on } a \wedge b \\ \sqrt{Ham(p, q)} & \text{otherwise} \end{cases}$$

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$

|  | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$
- $Output_d(J) = TTT$ $(d(TFF, TFT) = 0.9)$

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$
- $Output_d(J) = TTT$ $(d(TFF, TFT) = 0.9)$
- $Endpoint_d(J) = TTT$ $(d(TTF, TTT) = 0.9)$

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

- $Full_d(J) = TFF \; (d(FTF, FFF) = 1)$
- $Output_d(J) = TTT \; (d(TFF, TFT) = 0.9)$
- $Endpoint_d(J) = TTT \; (d(TTF, TTT) = 0.9)$
- $Prototype_d(J) = \{TTT, TFF\} \; (\sum_i d(J_i, TTT) = 3\sqrt{2}$, $\sum_i d(J_i, TFF) = 3\sqrt{2}$, $\sum_i d(J_i, FTF) = 4\sqrt{2}$, $\sum_i d(J_i, FFF) = 2\sqrt{3} + 3)$

Tomorrow: Logic!