Social Choice Theory for Logicians

Lecture 5

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Plan

- ✓ Arrow, Sen, Muller-Satterthwaite
- ✓ Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
- ✓ Voting to get things "right" (Distance-based measures, Condorcet and extensions)
- ✓ Strategizing (Gibbard-Satterthwaite)

1. Generalizations
   - 1.1 Infinite Populations
   - ✓ Judgement aggregation (List & Dietrich)
2. Logics
3. Applications

Plan

- The logic of axiomatization results
- Logics for reasoning about aggregation methods
- Preference (modal) logics
- Applications

Setting the Stage: Logic and Games

M. Pauly and W. van der Hoek. *Modal Logic form Games and Information*. Handbook of Modal Logic (2006).

G. Bonanno. *Modal logic and game theory: Two alternative approaches*. Risk Decision and Policy **7** (2002).

J. van Benthem. *Extensive games as process models*. Journal of Logic, Language and Information **11** (2002).

J. Halpern. *A computer scientist looks at game theory*. Games and Economic Behavior **45:1** (2003).

R. Parikh. *Social Software*. Synthese **132: 3** (2002).

What do the (Im)possibility results say?

M. Pauly. *On the Role of Language in Social Choice Theory*. Synthese, 163, 2, pgs. 227 - 243, 2008.

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Δ **relatively axiomatizes** $\mathcal{T}$ iff for all $\varphi \in \mathcal{L}$, $\mathcal{T} \models \varphi$ iff $\Delta \models \varphi$ (i.e., Δ axiomatizes the theory of $\mathcal{T}$)

What do the (Im)possibility results say?

May's Theorem: Δ is the set of aggregation functions w.r.t. 2 candidates, $\mathcal{T}$ is majority rule, $\mathcal{L}$ is the language of set theory, Δ is the properties of May's theorem, then Δ absolutely axiomatizes $\mathcal{T}$.

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Arrow's Theorem: Δ is the set of aggregation functions w.r.t. 3 or more candidates, $\mathcal{T}$ is a dictatorship, $\mathcal{L}$ is the language of set theory, Δ is the properties of May's theorem, then Δ absolutely axiomatizes $\mathcal{T}$.

A Minimal Language

M. Pauly. *Axiomatizing Collective Judgement Sets in a Minimal Logical Language*. 2006.

Let Φ_I be the set of **individual formulas** (standard propositional language)

V_I the set of individual valuations

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Φ_C the set of **collective formulas**: $\Box\alpha \mid \varphi \wedge \psi \mid \neg\varphi$

$\Box\alpha$: *The group collectively accepts* α*.*
V_C the set of collective valuations: $v : \Phi_C \rightarrow \{0, 1\}$

A Minimal Language

Let $\mathcal{CON}_n = \{v \in V_C \mid v(\Box\alpha) = 1 \text{ iff } \forall i \leq n,\ v_i(\alpha) = 1\}$

E. $\Box\varphi \leftrightarrow \Box\psi$ provided $\varphi \leftrightarrow \psi$ is a tautology

M. $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C. $(\Box\varphi \wedge \Box\psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

N. $\Box\top$

D. $\neg\Box\bot$

Theorem [Pauly, 2005] $V_C(\text{KD}) = \mathcal{CON}_n$, provided $n \geq 2^{|\Phi_0|}$.

($\mathcal{D} = V_C$, $\mathcal{T} = \mathcal{CON}_n$, $\Delta = EMCND$, then Δ absolutely axiomatizes $\mathcal{T}$.)

A Minimal Language

Let $\mathcal{MAJ}_n = \{v \in \mathcal{V}_C \mid v([>]\alpha) = 1 \text{ iff } |\{i \mid v_i(\alpha) = 1\}| > \frac{n}{2}\}$

STEM contains all instances of the following schemes

S. $[>]\varphi \to \neg[>]\neg\varphi$

T. $([\geq]\varphi_1 \wedge \cdots \wedge [\geq]\varphi_k \wedge [\leq]\psi_1 \wedge \cdots \wedge [\leq]\psi_k) \to \bigwedge_{1 \leq i \leq k}([=]\varphi_i \wedge [=]\psi_i)$ where $\forall v \in V_I$: $|\{i \mid v(\varphi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

E. $[>]\varphi \leftrightarrow [>]\psi$ provided $\varphi \leftrightarrow \psi$ is a tautology

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Theorem [Pauly, 2005] $V_C(\text{STEM}) = \mathcal{MAJ}$.

($\mathcal{D} = V_C$, $\mathcal{T} = \mathcal{MAJ}_n$, $\Delta = STEM$, then Δ absolutely axiomatizes $\mathcal{T}$.)

- Compare principles in terms of the language used to express them

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T. Daniëls. *Social choice and logic of simple games*. Journal of Logic and Computation, 21, 6, pgs. 883 - 906, 2011.

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- How much "classical logic" is "needed" for the judgement aggregation results?

T. Daniëls and EP. *A general approach to aggregation problems*. Journal of Logic and Computation, 19, 3, pgs. 517 - 536, 2009.

F. Dietrich. *A generalised model of judgment aggregation*. Social Choice and Welfare 28(4): 529 - 565, 2007.

Plan

✓ The logic of axiomatization results

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Judgement Aggregation Logic

T. Agotnes, W. van der Hoek, M. Wooldridge. *On the logic of preference and judgement aggregation*. Autonomous Agent and Multi-Agent Systems, 22, pgs. 4 - 30, 2011.

Some Notation:

- $N = \{1, \ldots, n\}$ a set of agents
- $\mathcal{A}$ is the agenda (set of formulas of some logic $\mathcal{L}$ "on the table" satisfying certain "fullness conditions")
- Let $J(\mathcal{A}, \mathcal{L})$ is the set of *judgements* (eg. maximally consistent subsets of $\mathcal{A}$)
- $\gamma \in J(\mathcal{A}, \mathcal{L})^n$ is a *judgement profile* with γ_i agent i's judgement set

Judgement Aggregation Logic: Semantics

Tables $\langle F, \gamma, p \rangle$

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Example:

	P	$P \to Q$	Q
Individual 1	True	True	True
Individual 2	True	False	False
Individual 3	False	True	False
F_{maj}	True	True	False

$$\mathcal{A} = \{P, Q, P \to Q, \neg P, \neg Q, \neg(P \to Q)\}$$

F is an aggregations function $F : J(\mathcal{A}, \mathcal{L})^n \to J(\mathcal{A}, \mathcal{L})$

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$\gamma \in J(\mathcal{A}, \mathcal{L})^n$ (assuming consistency and completeness)

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$$p \in \mathcal{A}$$

Judgement Aggregation Logic: Language

Atomic Formulas: $\text{At} = \{i, \sigma, \mathbf{h}_p \mid p \in \mathcal{A}, i \in N\}$

Formulas: $\varphi ::= \alpha \mid \Box\varphi \mid \blacksquare\varphi \mid \varphi \wedge \varphi \mid \neg\varphi$

Judgement Aggregation Logic: Language

Judgement Aggregation Logic: Truth

- $F, \gamma, p \models \mathbf{h}_q$ iff $q = p$
- $F, \gamma, p \models i$ iff $p \in \gamma_i$
- $F, \gamma, p \models \sigma$ iff $p \in F(\gamma)$
- $F, \gamma, p \models \Box\varphi$ iff $\forall \gamma' \in J(\mathcal{A}, \mathcal{L})^n$, $F, \gamma', p \models \varphi$
- $F, \gamma, p \models \blacksquare\varphi$ iff $\forall p' \in \mathcal{A}$, $F, \gamma, p' \models \varphi$
- Boolean connectives as usual

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$$F_{maj}, \gamma, P \models \Diamond \underbrace{\blacksquare((1 \leftrightarrow 2) \wedge (2 \leftrightarrow 3) \wedge (1 \leftrightarrow 3))}_{\text{All agents agree on all propositions in the agenda}}$$

$$F_{maj}, \gamma, P \models \Box\blacksquare(\sigma \leftrightarrow \bigvee_{G\subseteq\{1,2,3\},|G|\geq 2} \bigwedge_{i\in G} i)$$

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    Given any judgement profile, any choice of the voters and any $P \in \mathcal{A}$, if society accepts P then for any profile (if the choices are the same w.r.t. P then society should accept P)

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Judgement Aggregation Logic: Results

- Sound and complete axiomatization
- Model checking is decidable, but relatively difficult
- Expressivity:
  - Discursive Dilemma: $\Diamond((\blacksquare MV) \to \bot)$, where $MV := \sigma \leftrightarrow \bigvee_{G \subseteq N, |G| > \frac{n}{2}} \bigwedge_{i \in G} i$,
  - Impossibility results:
    Nondictatorship: $\bigwedge_{i \in N} \Diamond \blacklozenge \neg(\sigma \leftrightarrow i)$,
    Unanimity: $\Box \blacksquare((1 \wedge \cdots \wedge n) \to \sigma)$
    Independence: $\Box \bigwedge_{o \in O} \blacksquare((o \wedge \sigma) \to \Box(o \to \sigma))$
    Given any judgement profile, any choice of the voters and any $P \in \mathcal{A}$, if society accepts P then for any profile (if the choices are the same w.r.t. P then society should accept P)

- Complete axiomatization

U. Endriss. *Logic and Social Choice*. 2011.

Plan

- ✓ The logic of axiomatization results
- ✓ Logics for reasoning about aggregation methods
- Preference (modal) logics
- Applications

Preference (Modal) Logics

x, y objects

$x \succeq y$: x is at least as good as y

Preference (Modal) Logics

x, y objects

$x \succeq y$: x is at least as good as y

1. $x \succeq y$ and $y \not\succeq x$ ($x \succ y$)
2. $x \not\succeq y$ and $y \succeq x$ ($y \succ x$)
3. $x \succeq y$ and $y \succeq x$ ($x \sim y$)
4. $x \not\succeq y$ and $y \not\succeq x$ ($x \perp y$)

Preference (Modal) Logics

x, y objects

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3. $x \succeq y$ and $y \succeq x$ ($x \sim y$)
4. $x \not\succeq y$ and $y \not\succeq x$ ($x \perp y$)

Properties: transitivity, connectedness, etc.

Preference (Modal) Logics

Modal betterness model $\mathcal{M} = \langle W, \succeq, V \rangle$

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Preference Modalities $\langle \succeq \rangle \varphi$: "there is a world at least as good (as the current world) satisfying φ"

$\mathcal{M}, w \models \langle \succeq \rangle \varphi$ iff there is a $v \succeq w$ such that $\mathcal{M}, v \models \varphi$

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$\mathcal{M}, w \models \langle \succ \rangle \varphi$ iff there is $v \succeq w$ and $w \not\succeq v$ such that $\mathcal{M}, v \models \varphi$

Preference (Modal) Logics

1. $\langle \succ \rangle \varphi \rightarrow \langle \succeq \rangle \varphi$
2. $\langle \succeq \rangle \langle \succ \rangle \varphi \rightarrow \langle \succ \rangle \varphi$
3. $\varphi \wedge \langle \succeq \rangle \psi \rightarrow (\langle \succ \rangle \psi \vee \langle \succeq \rangle (\psi \wedge \langle \succeq \rangle \varphi))$
4. $\langle \succ \rangle \langle \succeq \rangle \varphi \rightarrow \langle \succ \rangle \varphi$

Theorem The above logic (with Necessitation and Modus Ponens) is sound and complete with respect to the class of preference models.

J. van Benthem, O. Roy and P. Girard. *Everything else being equal: A modal logic approach to* ceteris paribus *preferences*. JPL, 2008.

Preference Modalities

$\varphi \geq \psi$: the state of affairs φ is at least as good as ψ (ceteris paribus)

G. von Wright. *The logic of preference*. Edinburgh University Press (1963).

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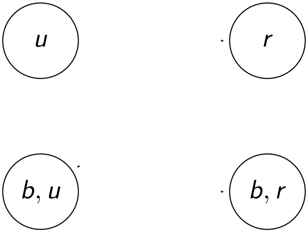
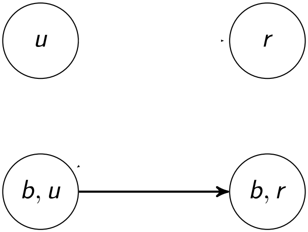
$\langle \Gamma \rangle^{\leq} \varphi$: φ is true in "better" world, *all things being equal*.

J. van Benthem, O. Roy and P. Girard. *Everything else being equal: A modal logic approach to* ceteris paribus *preferences*. JPL, 2008.

All Things Being Equal...





- With boots (b), I prefer my raincoat (r) over my umbrella (u)

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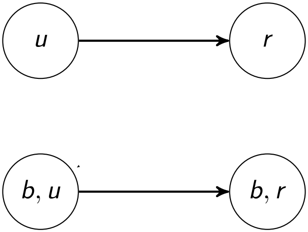

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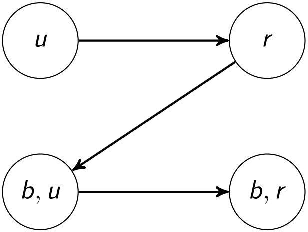
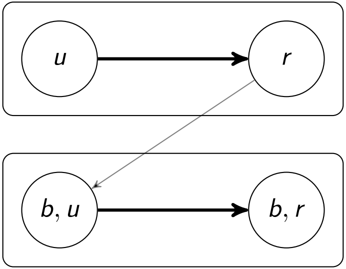


- With boots (b), I prefer my raincoat (r) over my umbrella (u)
- Without boots ($\neg b$), I also prefer my raincoat (r) over my umbrella (u)
- But I do prefer an umbrella and boots over a raincoat and no boots

All Things Being Equal...



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All Things Being Equal...

Let Γ be a set of (preference) formulas. Write $w \equiv_\Gamma v$ if for all $\varphi \in \Gamma$, $w \models \varphi$ iff $v \models \varphi$.

1. $\mathcal{M}, w \models \langle\Gamma\rangle\varphi$ iff there is a $v \in W$ such that $w \equiv_\Gamma v$ and $\mathcal{M}, v \models \varphi$.
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Key Principles:

- $\langle\Gamma'\rangle\varphi \to \langle\Gamma\rangle\varphi$ if $\Gamma \subseteq \Gamma'$
- $\pm\varphi \wedge \langle\Gamma\rangle(\alpha \wedge \pm\varphi) \to \langle\Gamma \cup \{\varphi\}\rangle\alpha$

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Preference Lifting, I

Given a preference ordering $\preceq$ over a set of objects X, we want to **lift** this to an ordering $\hat{\preceq}$ over $\wp(X)$.

Given $\preceq$, what reasonable properties can we infer about $\hat{\preceq}$?

S. Barberá, W. Bossert, and P.K. Pattanaik. *Ranking sets of objects*. In Handbook of Utility Theory, volume 2. Kluwer Academic Publishers, 2004.

Preference Lifting, II

- You know that $x \prec y \prec z$
  Can you infer that $\{x, y\} \mathrel{\hat{\precsim}} \{z\}$?

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  Can you infer that $\{w, x\} \hat{\prec} \{y, z\}$?

Preference Lifting, III

There are different interpretations of $X \mathrel{\hat{\preceq}} Y$:

- You will get one of the elements, but cannot control which.
- You can choose one of the elements.
- You will get the full set.

Preference Lifting, IV

Kelly Principle

(EXT) $\{x\} \hat{\precsim} \{y\}$ provided $x \prec y$

(MAX) $A \hat{\precsim} Max(A)$

(MIN) $Min(A) \hat{\precsim} A$

J.S. Kelly. *Strategy-Proofness and Social Choice Functions without Single-Valuedness*. Econometrica, 45(2), pp. 439 - 446, 1977.

Preference Lifting, IV

Gärdenfors Principle

(G1) $A \hat{\precsim} A \cup \{x\}$ if $a \prec x$ for all $a \in A$

(G2) $A \cup \{x\} \hat{\precsim} A$ if $x \prec a$ for all $a \in A$

P. Gärdenfors. *Manipulation of Social Choice Functions*. Journal of Economic Theory. 13:2, 217 - 228, 1976.

Preference Lifting, IV

Gärdenfors Principle

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P. Gärdenfors. *Manipulation of Social Choice Functions*. Journal of Economic Theory. 13:2, 217 - 228, 1976.

Independence

(IND) $A \cup \{x\} \hat{\preceq} B \cup \{x\}$ if $A \hat{\preceq} B$ and $x \notin A \cup B$

Preference Lifting, V

Theorem (Kannai and Peleg). If $|X| \geq 6$, then no weak order satisfies both the Gärdenfors principle and independence.

Y. Kannai and B. Peleg. *A Note on the Extension of an Order on a Set to the Power Set*. Journal of Economic Theory, 32(1), pp. 172 - 175, 1984.

From Worlds to Sets, I

$\mathcal{M}, w \models \varphi \preceq_{\exists\exists} \psi$ iff there is s, t such that $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ and $s \preceq t$

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$\mathcal{M}, w \models \varphi \preceq_{\forall\exists} \psi$ iff for all s there is a t such that $\mathcal{M}, s \models \varphi$ implies $\mathcal{M}, t \models \psi$, and $s \preceq t$

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$$\varphi \preceq_{\exists\exists} \psi \ := \ \mathsf{E}(\varphi \wedge \Diamond^{\preceq}\psi)$$

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From Worlds to Sets, III

$\mathcal{M}, w \models \varphi \preceq_{\forall\forall} \psi$ iff for all s, for all t, $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, t \models \psi$ implies $s \preceq t$

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From Worlds to Sets,IV

$$\varphi \preceq_{\forall\forall} \psi \ := \ A(\psi \rightarrow \Box^{\preceq}\neg\varphi)$$

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We must assume the ordering $\preceq$ is total

From Sets to Worlds

$P_1 \gg P_2 \gg P_3 \gg \cdots \gg P_n$

$x > y$ iff x and y differ in at least one P_i and the first P_i where this happens is one with $P_i x$ and $\neg P_i y$

F. Liu and D. De Jongh. *Optimality, belief and preference*. 2006.

Logics of Knowledge and Preference

$K(\varphi \succeq \psi)$: "Ann knows that φ is at least as good as ψ"

$K\varphi \succeq K\psi$: "knowing φ is at least as good as knowing ψ

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$\mathcal{M} = \langle W, \sim, \succeq, V \rangle$

J. van Eijck. *Yet more modal logics of preference change and belief revision*. manuscript, 2009.

F. Liu. *Changing for the Better: Preference Dynamics and Agent Diversity*. PhD thesis, ILLC, 2008.

$$A(\psi \rightarrow \langle \succeq \rangle \varphi) \quad \textit{vs.} \quad K(\psi \rightarrow \langle \succeq \rangle \varphi)$$

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Should preferences be restricted to information sets?

$$A(\psi \to \langle\succeq\rangle\varphi) \ \ \textit{vs.} \ \ K(\psi \to \langle\succeq\rangle\varphi)$$

Should preferences be restricted to information sets?

$\mathcal{M}, w \models \langle\succeq \cap \sim\rangle\varphi$ iff there is a v with $w \sim v$ and $w \preceq v$ such that $\mathcal{M}, v \models \varphi$

$K(\psi \to \langle\succeq \cap \sim\rangle\varphi)$

D. Osherson and S. Weinstein. *Preference based on reasons*. Review of Symbolic Logic, 2012.

$\varphi \succeq_X \psi$ "The agent considers φ at least as good as ψ for reason X"

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i envisions a situation in which φ is true and that otherwise differs little from his actual situation. Likewise i envisions a world where ψ is true and otherwise differs little from his actual situation. Finally, there utility according to u_X of the first imagined situation exceeds that of the second.

p: "i purchases a fire alarm"

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$p \succ_1 \neg p$: u_1 measures safety

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$p \prec_2 \neg p$: u_2 measures finances

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$p \succ_1 \neg p$: u_1 measures safety

$p \prec_2 \neg p$: u_2 measures finances

What is the status of $p \succ_{1,2} \neg p$? $p \prec_{1,2} \neg p$?

$(p \succ_1 \top) \succ_2 \top$: it's in your financial interest that your buying a low-power automobile is in you safety interesting — which might well be true inasmuch as low-power vehicles are cheaper.

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$\neg q \succ_1 (p \succ_2 q)$: from the point of view of family pride, you'd rather that your brother not run for mayor than that Miss Smith be the superior candidate.

At a set of atomic proposition, $\mathbb{S}$ a set of **reasons**.

$$\langle W, s, u, V \rangle$$

- W is a set of states
- $s : W \times \wp_{\neq\emptyset}(W) \to W$ is a selection function ($s(w, A) \in A$)
- $u : W \times \mathbb{S} \to \mathfrak{R}$ is a utility function
- $V : \mathsf{At} \to \wp(W)$ is a valuation function

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$$\mathcal{M}, w \models \theta \succeq_X \psi \text{ iff } u_X(s(w, [\![\theta]\!]_{\mathcal{M}})) \geq u_X(s(w, [\![\psi]\!]_{\mathcal{M}}))$$

provided $[\![\theta]\!]_{\mathcal{M}} \neq \emptyset$ *and* $[\![\psi]\!]_{\mathcal{M}} \neq \emptyset$

$$\Diamond\varphi =_{\text{def}} \varphi \succeq_X \varphi$$
$$\Box\varphi =_{\text{def}} \neg(\neg\varphi \succeq_X \neg\varphi)$$

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$$\Box(p \to (p \prec_X \neg p)) \wedge \Box(\neg p \to (\neg p \succ_X p))$$

Regular: if $A \subseteq B$ and $w_1 \in A$ then If $s(w, B) = w_1$ then $s(w, A) = w_1$.

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$\mathcal{M}$ is regular implies $((p \vee q) \succ_X r) \rightarrow ((p \succ_X r) \vee (q \succ_X r))$ is valid.

$\mathcal{M}$ is regular and reflexive then
$((p \prec_1 \top) \succ_2 (q \prec_1 \top)) \rightarrow (\neg p \succ_2 \neg q)$ is valid.

"If it is ecologically better for p than for q to politically backfire the abstaining from p is ecologically better than abstaining from q. "

$\mathcal{M}$ is proximal if for all w and $A \neq \emptyset$, If $s(w, A) = w_1$ then there is no $w_2 \in A$ such that $V^{-1}(w) \Delta V^{-1}(w_2) \subset V^{-1}(w) \Delta V^{-1}(w_1)$, where Δ is the symmetric difference.

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$(((p \wedge r) \succ_X (q \wedge r)) \wedge ((p \wedge \neg r) \succ_X (q \wedge \neg r))) \to (p \succ_X q)$ is invalid in the class of regular and in the class of proximal models, but valid in the class of models that are both proximal and regular.

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$(p \wedge ((p \wedge q) \succ_X r)) \rightarrow (q \succ_X r)$

Plan

- ✓ The logic of axiomatization results
- ✓ Logics for reasoning about aggregation methods
- ✓ Preference (modal) logics
- Applications

Infinite Voting Populations

Given an aggregation method F, let $\mathcal{D} = \{C \mid C \text{ is winning for } F\}$

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Given a set of winning coalitions $\mathcal{D}$, we can define F as follows:

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What is the general relationship between sets of coalitions and aggregators?

Infinite Voting Populations

F. Herzberg and D. Eckert. *Impossibility results for infinite-electorate abstract aggregation rules*. Journal of Philosophical Logic, 41, pgs. 273 - 286, 2012.

F. Herzberg and D. Eckert. *The model-theoretic approach to aggregation: Impossibility results for finite and infinite electorates*. Mathematical Social Sciences, 64, pgs. 41 - 47, 2012.

L. Lauwers and L. van Liedekerke. *Ultraproducts and aggregation*. Journal of Mathematical Economics, 24, pgs. 217 - 237, 1995.

Infinite Voting Populations

F. Herzberg and D. Eckert. *Impossibility results for infinite-electorate abstract aggregation rules*. Journal of Philosophical Logic, 41, pgs. 273 - 286, 2012.

F. Herzberg and D. Eckert. *The model-theoretic approach to aggregation: Impossibility results for finite and infinite electorates*. Mathematical Social Sciences, 64, pgs. 41 - 47, 2012.

L. Lauwers and L. van Liedekerke. *Ultraproducts and aggregation*. Journal of Mathematical Economics, 24, pgs. 217 - 237, 1995.

Theorem. Let $\mathcal{D}$ be a filter and suppose that $F_{\mathcal{D}}$ preserves ψ and assume that there is some $\mathcal{A} \in \Omega^I$ with *finite witness multiplicity* with respect to ψ. Then,

- If $\mathcal{D}$ is an ultrafilter, then it is principal (whence $F_{\mathcal{D}}$ is a dictatorship)
- If φ is free of negation, disjunction and universal quantification then $\mathcal{D}$ contains a finite coalition (whence $F_{\mathcal{D}}$ is an oligarchy)

May's Theorem: Notation

Fix an infinite set W.

Suppose that there are two alternatives, x and y, under consideration.

We assume that each voter has a linear preference over x and y, so for each $w \in W$, either w prefers x to y or y to x, but not both.

Assume that a subset $X \subseteq W$, represents the set of all voters that prefer x to y.

Thus X represents the outcome of a particular vote.

May's Theorem: Notation

There are three possible outcomes to consider: 0 means that alternative y was chosen, $\frac{1}{2}$ means the vote was a tie, and 1 means that alternative x was chosen.

An **aggregation function** is a function $f : 2^W \to \{0, \frac{1}{2}, 1\}$.

A set $X \subseteq W$, $f(X)$ represents the social preference of the group W ($\frac{1}{2}$ is interpreted as a tie).

Properties of f

Consider $f : 2^W \rightarrow \{0, \frac{1}{2}, 1\}$

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Positive Responsiveness if, for all $X, Y \subseteq W$, $X \subsetneq Y$ and $f(X) \neq 0$ implies $f(Y) = 1$.

Anonymity

Anonymity states that it is the number of votes that counts when determining the outcome, not *who* voted for what.

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When W is finite, this condition is straightforward to impose:

Fix an arbitrary order on W, then each subset of W can be represented by a finite sequence of 1s and 0s.

Then f satisfies **anonymity** if f is symmetric in this sequence of 1s and 0s.

Anonymity for an Infinite Population

A **permutation** on a set X is a 1-1 map $\pi : X \to X$.

f is **anonymous** iff for all π and $X \subseteq W$, $f(X) = f(\pi[X])$.

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Too strong! Let X, Y be any (countably) infinite subsets of W, then there is a π such that $\pi[X] = Y$. Hence, for all $X, Y \subseteq W$, $f(X) = f(Y)$.

Anonymity for an Infinite Population

A **finite permutation** on a set X is a 1-1 map $\pi : X \to X$ such that there is a finite set $F \subseteq X$ such that for all $w \in W - F$, $\pi(w) = w$.

f is **finitely anonymous** iff for all finite permutations π and $X \subseteq W$, $f(X) = f(\pi[X])$.

Digression: Bounded Anonymity and Density

Let $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $X(n) = \{m \in X \mid m \leq n\}$

$d(X) = \lim_{n\rightarrow\infty} \frac{X(n)}{n}$

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Unfortunately, $\lim_{n\to\infty} \frac{X(n)}{n}$ does not always exist.
π is a **bounded permutation** iff

$$\lim_{n\to\infty} \frac{|\{k \mid k \leq n < \pi(k)\}|}{n} = 0$$

May's Theorem Generalized

Bounded anonymity: $F(A) = F(\pi[A])$ for all bounded permutations

Density positive responsiveness: f satisfies monotonicity and, if $f(A) = 1/2$ and all sets with density D with $A \cap D \neq \emptyset$ and $d(A) > 1$, we have $f(A \cup D) = 1$.

Theorem (Fey) If an aggregation rule f satisfies neutrality, density positive responsiveness and bounded anonymity, then f agrees with a density majority rule.

M. Fey. *May's Theorem with an Infinite Population*. Social Choice and Welfare (2004).

Broader Applications

- Is it possible to choose rationally among rival scientific theories on the basis of the accuracy, simplicity, scope and other relevant criteria? No

S. Okasha. *Theory choice and social choice: Kuhn versus Arrow*. Mind, 120, 477, pgs. 83 - 115, 2011.

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M. Moureau. *Mr. Accuracy, Mr. Simplicity and Mr. Scope: from social choice to theory choice*. FEW, 2012.

- Is it possible to rationally merge evidence from multiple methods?

J. Stegenga. *An impossibility theorem for amalgamating evidence*. Synthese, 2011.

Broader Applications

- Is it possible to merge classic AGM belief revision with the Ramsey test?

P. Gärdenfors. *Belief revisions and the Ramsey Test for conditionals*. The Philosophical Review, 95, pp. 81 - 93, 1986.

H. Leitgeb and K. Segerberg. *Dynamic doxastic logic: why, how and where to?*. Synthese, 2011.

H. Leitgeb. *A Dictator Theorem on Belief Revision Derived From Arrow's Theorem*. Manuscript, 2011.

Plan

- ✓ Arrow, Sen, Muller-Satterthwaite
- ✓ Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
- ✓ Voting to get things "right" (Distance-based measures, Condorcet and extensions)
- ✓ Strategizing (Gibbard-Satterthwaite)
- ✓ Generalizations
  - ✓ Infinite Populations
  - ✓ Judgement aggregation (List & Dietrich)
- ✓ Logics
- ✓ Applications

Thank you!