

Derivations with Equality

(Symmetry of =) Suppose that s and t are terms. Then,

$$\{s = t\} \vdash t = s$$

$$\frac{s = t \quad \overline{(x = s)[s/x]} \text{ (=I)}}{(x = s)[t/x]} \text{ (=E)}$$

Suppose that f is a unary function symbol and s and t are any terms. Then,

$$\{s = t\} \vdash (f(s) = f(t))$$

$$\frac{s = t \quad \overline{(f(x) = f(s))[s/x]} \text{ (=I)}}{(f(x) = f(s))[t/x]} \text{ (=E)}$$

- For all terms r and s , $r[s/x][s/x]$ is $r[s/x]$
- For all terms r and s , $(r = r[s/x])[s/x]$ is $(r[s/x] = r[s/x])$
- For all terms r , s and t , $r[s/x][t/x]$ is $r[s/x]$
- For all terms r , s and t , $(r = r[s/x])[t/x]$ is $(r[t/x] = r[s/x])$

(Leibniz's Rule) Suppose that r , s and t are terms. Then,

$$\{s = t\} \vdash (r[s/x] = r[t/x])$$

$$\frac{s = t \quad \overline{(r = r[s/x])[s/x]} \text{ (=I)}}{(r = r[s/x])[t/x]} \text{ (=E)}$$

(Transitivity of =) Suppose that r , s and t are terms and x is not a variable in r , s or t . Then,

$$\{(s = r), (r = t)\} \vdash (s = t)$$

$$\frac{s = r \quad (x = t)[r/x]}{(x = t)[s/x]} \text{ (=E)}$$

Axioms for addition:

A1 For each $n \in \mathbb{N}$ let $\bar{n} \bar{+} \bar{0} = \bar{n}$ be an axiom.

A2 For each $m \in \mathbb{N}$ and $n \in \mathbb{N}$, $\bar{m} \bar{+} \overline{n+1} = \bar{S}(\bar{m} \bar{+} \bar{n})$ is an axiom.

Instances of A1 include

- $\bar{0} \bar{+} \bar{0} = \bar{0}$
- $\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})$
- $\bar{S}\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}\bar{S}(\bar{0})$
- $\bar{S}\bar{S}\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}\bar{S}\bar{S}(\bar{0})$

Instances of A2 include

- $\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0})$
- $\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}(\bar{0}))$
- $\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}))$
- $\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}))$

Let $\Gamma_A = \{\varphi \mid \varphi \text{ is an instance of A1 or and instance of A2}\}$

$$1. \Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})$$

$\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0}) \in \Gamma_A$, so $\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})$ is a derivation with all undischarged assumptions in Γ_A .

$$2. \Gamma_A \vdash \bar{S}(\bar{0}) = \bar{S}(\bar{0}) \bar{+} \bar{0}$$

The following derivation verifies the above:

$$\frac{\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})}{\bar{S}(\bar{0}) = \bar{S}(\bar{0}) \bar{+} \bar{0}} (=sym)$$

$$3. \Gamma_A \vdash \bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0})$$

$$\frac{\bar{0} \bar{+} \bar{0} = \bar{0} \quad (\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0}))}{(\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0}))} (=E)$$

$$\frac{\bar{0} \bar{+} \bar{0} = \bar{0} \quad (\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(x))[\bar{0} \bar{+} \bar{0}/x]}{(\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(x))[\bar{0}/x]} (=E)$$

$$4. \Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})$$

$$\frac{\frac{\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0}) \quad \bar{S}(\bar{0}) \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{S}(\bar{0}) \bar{+} \bar{0})}{\bar{S}(\bar{0}) \bar{+} \bar{S}(\bar{0}) = \bar{S}\bar{S}(\bar{0})} (=E) \quad \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{S}(\bar{0}) \bar{+} \bar{S}(\bar{0}))}{\bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})} (=E)$$

5. $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0})$

It is enough to show that $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})$ (see item 4) and $\Gamma_A \vdash \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})$:

$$\frac{\frac{\bar{0} \bar{+} \bar{0} = \bar{0} \quad \bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0})}{\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0})} (=E) \quad \bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}(\bar{0}))}{\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}(\bar{0})} (=E) \quad \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}))}{\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})} (=E)$$

Then by symmetry and transitivity of $=$, we have that $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0})$.

Axioms for multiplication:

P1 For each $n \in \mathbb{N}$ let $\bar{n} \bar{\times} \bar{0} = \bar{0}$ be an axiom.

P2 For each $m \in \mathbb{N}$ and $n \in \mathbb{N}$, $\bar{m} \bar{\times} \overline{n+1} = (\bar{m} \bar{\times} \bar{n}) \bar{+} \bar{m}$ is an axiom.

Instances of P1 include

- $\bar{0} \bar{\times} \bar{0} = \bar{0}$
- $\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0}$
- $\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0}$
- $\bar{S}\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0}$

Instances of P2 include

- $\bar{0} \bar{\times} \bar{S}(\bar{0}) = (\bar{0} \bar{\times} \bar{0}) \bar{+} \bar{0}$
- $\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = (\bar{S}(\bar{0}) \bar{\times} \bar{0}) \bar{+} \bar{S}(\bar{0})$
- $\bar{S}(\bar{0}) \bar{\times} \bar{S}\bar{S}(\bar{0}) = (\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0})) \bar{+} \bar{S}(\bar{0})$
- $\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = (\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0})$

$\Gamma_P = \Gamma_A \cup \{\varphi \mid \varphi \text{ is an instance of P1 or is an instance of P2}\}$

$$\Gamma_P \vdash \bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = \bar{S}(\bar{0})$$

$$\frac{\frac{\bar{0} \bar{+} \bar{0} = \bar{0} \quad \bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0})}{\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0})} (=E) \quad \frac{\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0} \quad \bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = (\bar{S}(\bar{0}) \bar{\times} \bar{0}) \bar{+} \bar{S}(\bar{0})}{\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = \bar{0} \bar{+} \bar{S}(\bar{0})} (=E)}{\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = \bar{S}(\bar{0})} (=E)$$