Appendix A The natural deduction rules

Natural Deduction Rule (Axiom Rule) Let ϕ be a statement. Then

$$\phi$$

is a derivation. Its conclusion is ϕ, and it has one undischarged assumption, namely ϕ.

Natural Deduction Rule ($\wedge$I) If

$$\begin{matrix} D \\ \phi \end{matrix} \quad \text{and} \quad \begin{matrix} D' \\ \psi \end{matrix}$$

are derivations of ϕ and ψ, respectively, then

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix} \qquad\qquad \begin{matrix} D' \\ \psi \end{matrix}}{(\phi \wedge \psi)} \; (\wedge I)$$

is a derivation of $(\phi \wedge \psi)$. Its undischarged assumptions are those of D together with those of D'.

Natural Deduction Rule ($\wedge$E) If

$$\begin{matrix} D \\ (\phi \wedge \psi) \end{matrix}$$

is a derivation of $(\phi \wedge \psi)$, then

$$\dfrac{\begin{matrix} D \\ (\phi \wedge \psi) \end{matrix}}{\phi} \; (\wedge E) \quad \text{and} \quad \dfrac{\begin{matrix} D \\ (\phi \wedge \psi) \end{matrix}}{\psi} \; (\wedge E)$$

are derivations of ϕ and ψ, respectively. Their undischarged assumptions are those of D.

Natural Deduction Rule (→I) Suppose

$$\begin{array}{c} D \\ \psi \end{array}$$

is a derivation of ψ, and ϕ is a formula. Then the following is a derivation of $(\phi \to \psi)$:

$$\dfrac{\begin{array}{c} \cancel{\phi} \\ D \\ \psi \end{array}}{(\phi \to \psi)} \; (\to\mathrm{I})$$

Its assumptions are those of D, except possibly ϕ.

Natural Deduction Rule (→E) If

$$\begin{array}{c} D \\ \phi \end{array} \quad \text{and} \quad \begin{array}{c} D' \\ (\phi \to \psi) \end{array}$$

are derivations of ϕ and $(\phi \to \psi)$, respectively, then

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} D' \\ (\phi \to \psi) \end{array}}{\psi} \; (\to\mathrm{E})$$

is a derivation of ψ. Its assumptions are those of D together with those of D'.

Natural Deduction Rule (↔I) If

$$\begin{array}{c} D \\ (\phi \to \psi) \end{array} \quad \text{and} \quad \begin{array}{c} D' \\ (\psi \to \phi) \end{array}$$

are derivations of $(\phi \to \psi)$ and $(\psi \to \phi)$, respectively, then

$$\dfrac{\begin{array}{c} D \\ (\phi \to \psi) \end{array} \qquad \begin{array}{c} D' \\ (\psi \to \phi) \end{array}}{(\phi \leftrightarrow \psi)} \; (\leftrightarrow\mathrm{I})$$

is a derivation of $(\phi \leftrightarrow \psi)$. Its undischarged assumptions are those of D together with those of D'.

Natural Deduction Rule ($\leftrightarrow$E) If

$$\begin{array}{c} D \\ (\phi \leftrightarrow \psi) \end{array}$$

is a derivation of $(\phi \leftrightarrow \psi)$, then

$$\dfrac{\begin{array}{c} D \\ (\phi \leftrightarrow \psi) \end{array}}{(\phi \to \psi)}\ (\leftrightarrow\text{E}) \qquad \text{and} \qquad \dfrac{\begin{array}{c} D' \\ (\phi \leftrightarrow \psi) \end{array}}{(\psi \to \phi)}\ (\leftrightarrow\text{E})$$

are derivations of $(\phi \to \psi)$ and $(\psi \to \phi)$, respectively. Their undischarged assumptions are those of D.

Natural Deduction Rule ($\neg$E) If

$$\begin{array}{c} D \\ \phi \end{array} \quad \text{and} \quad \begin{array}{c} D' \\ (\neg\phi) \end{array}$$

are derivations of ϕ and $(\neg\phi)$, respectively, then

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} D' \\ (\neg\phi) \end{array}}{\bot}\ (\neg\text{E})$$

is a derivation of $\bot$. Its undischarged assumptions are those of D together with those of D'.

Natural Deduction Rule ($\neg$I) Suppose

$$\begin{array}{c} D \\ \bot \end{array}$$

is a derivation of $\bot$, and ϕ is a statement. Then the following is a derivation of $\neg\phi)$:

$$\dfrac{\begin{array}{c} \cancel{\phi} \\ D \\ \bot \end{array}}{(\neg\phi)}\ (\neg\text{I})$$

Its undischarged assumptions are those of D, except possibly ϕ.

Natural Deduction Rule (RAA) Suppose we have a derivation

$$\begin{array}{c} D \\ \bot \end{array}$$

whose conclusion is $\perp$. Then there is a derivation

$$\begin{array}{c} \cancel{(\neg\phi)} \\ D \\ \dfrac{\perp}{\phi}\ \text{(RAA)} \end{array}$$

Its assumptions are those of D, except possibly $(\neg\phi)$.

Natural Deduction Rule ($\vee$I) If

$$\begin{array}{c} D \\ \phi \end{array}$$

is a derivation with conclusion ϕ, then

$$\dfrac{\begin{array}{c} D \\ \phi \end{array}}{(\phi \vee \psi)}$$

is a derivation of $(\phi\vee\psi)$. Its undischarged assumptions are those of D. Similarly if

$$\begin{array}{c} D \\ \psi \end{array}$$

is a derivation with conclusion ψ, then

$$\dfrac{\begin{array}{c} D \\ \psi \end{array}}{(\phi \vee \psi)}$$

is a derivation with conclusion $(\phi \vee \psi)$. Its undischarged assumptions are those of D.

Natural Deduction Rule ($\vee$E) Given derivations

$$\begin{array}{c} D \\ (\phi \vee \psi) \end{array}, \quad \begin{array}{c} D' \\ \chi \end{array} \quad \text{and} \quad \begin{array}{c} D'' \\ \chi \end{array}$$

we have a derivation

$$\dfrac{\begin{array}{c} \\ D \\ (\phi \vee \psi) \end{array} \quad \begin{array}{c} \cancel{\phi} \\ D' \\ \chi \end{array} \quad \begin{array}{c} \cancel{\psi} \\ D'' \\ \chi \end{array}}{\chi}$$

Its undischarged assumptions are those of D, those of D' except possibly ϕ, and those of D'' except possibly ψ.