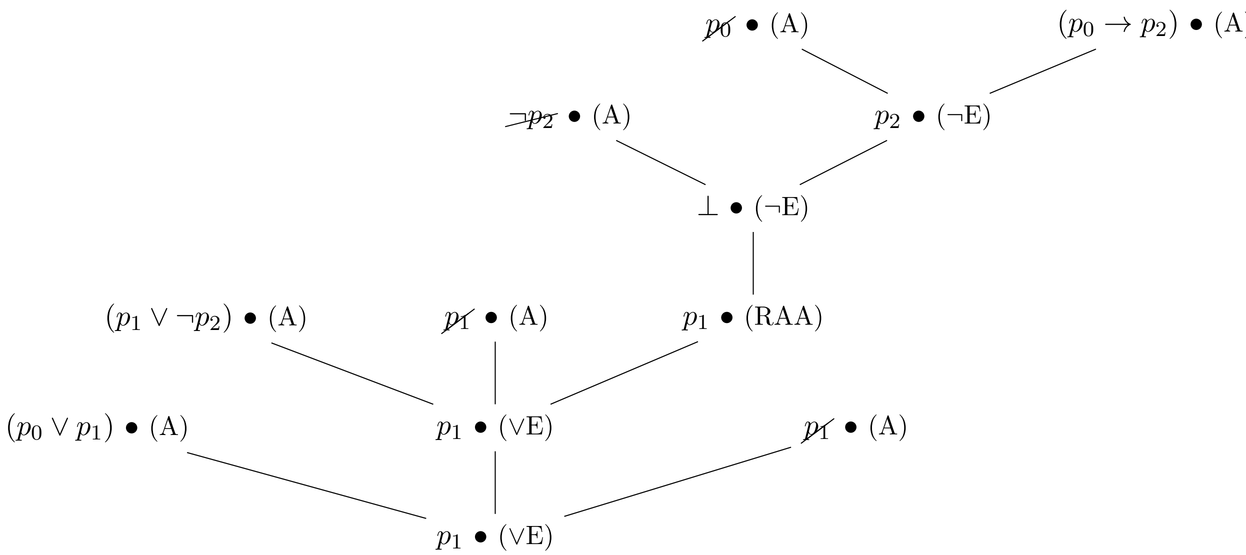
Derivations

1.	$(p_0 \vee p_1)$	Premise
2.	$(p_0 \to p_2)$	Premise
3.	$(p_1 \vee \neg p_2)$	Premise
4.	p_0	Assumption
5.	p_1	Assumption
6.	$\neg p_2$	Assumption
7.	p_2	$\to$E : 2, 4
8.	$\perp$	$\perp$I : 6, 7
9.	p_1	$\perp$E : 8
10.	p_1	$\vee$E : 3, 5, 9
11.	p_1	Assumption
12.	p_1	$\vee$E : 1, 10, 11

1	$(p_0 \vee p_1)$	
2	$(p_0 \to p_2)$	
3	$(p_1 \vee \neg p_2)$	
4	p_0	
5	p_1	
6	p_1	Reit (5)
7	$\neg p_2$	
8	p_2	$\to$-E (4, 2)
9	$\perp$	$\perp$-I (7, 8)
10	p_1	$\perp$-E (9)
11	p_1	$\vee$-E (3, 6, 10)
12	p_1	
13	p_1	Reit (12)
14	p_1	$\vee$-Elim (1, 11, 13)

$$1\;\dfrac{(p_0 \vee p_1) \qquad 2\;\dfrac{(p_1 \vee \neg p_2) \qquad \cancel{p_1}^2 \qquad \dfrac{\dfrac{\cancel{\neg p_2}^2 \qquad \dfrac{\cancel{p_0}^1 \qquad (p_0 \to p_2)}{p_2}\to\mathrm{E}}{\perp}\neg\mathrm{E}}{p_1}\mathrm{RAA}}{p_1}\vee\mathrm{E} \qquad \cancel{p_1}^1}{p_1}\vee\mathrm{E}$$

$$
{}^{1}\dfrac{(p_0 \vee p_1) \qquad {}^{2}\dfrac{(p_1 \vee \neg p_2) \qquad \cancel{p_1}^{2} \qquad \dfrac{\dfrac{\cancel{\neg p_2}^{2} \qquad \dfrac{\cancel{p_0}^{1} \qquad (p_0 \to p_2)}{p_2}\,{\to}\mathrm{E}}{\bot}\,\neg\mathrm{E}}{p_1}\,\mathrm{RAA}}{p_1}\,{\vee}\mathrm{E} \qquad \cancel{p_1}^{1}}{p_1}\,{\vee}\mathrm{E}
$$



Natural Deduction

Example: $\{(\neg p \wedge \neg q)\} \vdash \neg(p \vee q)$

$$1\;\dfrac{2\;\dfrac{\cancel{(p \vee q)}^1 \qquad \dfrac{\cancel{p}^2 \qquad \dfrac{\neg p \wedge \neg q}{\neg p}\,\wedge\mathrm{E}}{\bot}\,\neg\mathrm{E} \qquad \dfrac{\cancel{q}^2 \qquad \dfrac{\neg p \wedge \neg q}{\neg q}\,\wedge\mathrm{E}}{\bot}\,\neg\mathrm{E}}{\bot}\,\vee\mathrm{E}}{\neg(p \vee q)}\,\mathrm{RAA}$$

Gentzen Deduction

A Gentzen deduction of $\Gamma \vdash \varphi$ is a binary tree constructed according to the following rules, with a root $\Gamma \Rightarrow \varphi$ and each leaf is empty.

$$\frac{}{\varphi \Rightarrow \varphi}\,\text{A}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\,\text{WL} \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\,\wedge\text{L} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\,\wedge\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\,\text{WR} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma, \Rightarrow \Delta, \varphi \vee \psi}\,\vee\text{R} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\,\vee\text{L}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi, \varphi \Rightarrow \Delta}\,\text{CL} \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\,\to\text{R} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\,\to\text{R}$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi, \varphi, \Delta}\,\text{CR} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\,\neg\text{R} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\,\text{Cut}$$

$$\frac{\Gamma_1, \psi, \varphi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi, \psi, \Gamma_2 \Rightarrow \Delta}\,\text{PL} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\,\neg\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta_1, \psi, \varphi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi, \psi, \Delta_2}\,\text{PR}$$

Example: $\{(\neg p \wedge \neg q)\} \vdash \neg(p \vee q)$

$$\begin{prooftree}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$p \Rightarrow p$}
\RightLabel{WL}
\UnaryInfC{$p, \neg q \Rightarrow p$}
\RightLabel{PL}
\UnaryInfC{$\neg q, p \Rightarrow p$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg q, p, \neg p \Rightarrow$}
\RightLabel{PL}
\UnaryInfC{$\neg q, \neg p, p \Rightarrow$}
\RightLabel{PL}
\UnaryInfC{$\neg p, \neg q, p \Rightarrow$}
\RightLabel{$\wedge$L}
\UnaryInfC{$(\neg p \wedge \neg q), p \Rightarrow$}
\AxiomC{}
\RightLabel{A}
\UnaryInfC{$q \Rightarrow q$}
\RightLabel{WL}
\UnaryInfC{$q, \neg p \Rightarrow q$}
\RightLabel{PL}
\UnaryInfC{$\neg p, q \Rightarrow q$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg p, q, \neg q \Rightarrow$}
\RightLabel{PL}
\UnaryInfC{$\neg p, \neg q, q \Rightarrow$}
\RightLabel{$\wedge$L}
\UnaryInfC{$(\neg p \wedge \neg q), q \Rightarrow$}
\RightLabel{$\vee$R}
\BinaryInfC{$(\neg p \wedge \neg q), (p \vee q) \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$(\neg p \wedge \neg q) \Rightarrow \neg(p \vee q)$}
\end{prooftree}$$

Hilbert Deduction

A **Hilbert-style proof calculus** consists of a collection of axiom schemes and inference rules. Axioms and Rules:

Axiom 1	$\varphi \to (\psi \to \varphi)$
Axiom 2	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
Axiom 3	$\varphi \to (\varphi \vee \psi)$
Axiom 4	$\psi \to (\varphi \vee \psi)$
Axiom 5	$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
Axiom 6	$(\varphi \wedge \psi) \to \varphi$
Axiom 7	$(\varphi \wedge \psi) \to \psi$
Axiom 8	$\varphi \to (\psi \to (\varphi \wedge \psi))$
Axiom 9	$\neg\varphi \to (\varphi \to \bot)$
Axiom 10	$(\varphi \to \bot) \to \neg\varphi$
Axiom 11	$\neg\neg\varphi \to \varphi$
MP	from φ and $\varphi \to \psi$ infer ψ

A Hilbert-style deduction of φ from from a set of formulas Γ, dented $\Gamma \vdash \varphi$, is a list of formulas $\chi_1, \chi_2, \ldots, \chi_n$ where χ_n is φ and for each $1 \leq i \leq n$, either $\chi_i \in \Gamma$, χ_i is an instance of an axiom, or χ_i follows by MP from χ_j and χ_k with $j, k < i$.

Example: $\{(\neg p \wedge \neg q)\} \vdash \neg(p \vee q)$

1.	$(\neg p \wedge \neg q)$	Assumption
2.	$(\neg p \wedge \neg q) \to \neg p$	Axiom 6
3.	$(\neg p \wedge \neg q) \to \neg q$	Axiom 7
4.	$(p \to \bot) \to ((q \to \bot) \to ((p \vee q) \to \bot))$	Axiom 5
5.	$\neg p$	MP 1, 2
6.	$\neg q$	MP 1, 3
7.	$\neg p \to (p \to \bot)$	Axiom 9
8.	$(p \to \bot)$	MP 5, 7
9.	$\neg q \to (q \to \bot)$	Axiom 9
10.	$(q \to \bot)$	MP 6, 9
11.	$(q \to \bot) \to ((p \vee q) \to \bot)$	MP 8, 4
12.	$(p \vee q) \to \bot$	MP 10, 11
13.	$((p \vee q) \to \bot) \to \neg(p \vee q)$	Axiom 10
14.	$\neg(p \vee q)$	MP 12, 13

Find the following using Natural deductions, Genzten deductions, and Hilbert deductions:

1. $\vdash (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$
2. $\{\neg(p \vee q)\} \vdash \neg p \wedge \neg q$
3. $\vdash \neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$
4. $(p \rightarrow q) \vdash (\neg p \vee q)$
5. $\{(\neg p \vee q)\} \vdash (p \rightarrow q)$
6. $\vdash p \vee \neg p$