# Derivations with Equality

(**Symmetry of** $=$) Suppose that $s$ and $t$ are terms. Then,

$$\{s = t\} \vdash t = s$$

$$\dfrac{s = t \qquad \dfrac{}{(x = s)[s/x]}\,(=I)}{(x = s)[t/x]}\,(=E)$$

Suppose that $f$ is a unary function symbol and $s$ and $t$ are any terms. Then,

$$\{s = t\} \vdash (f(s) = f(t))$$

$$\dfrac{s = t \qquad \dfrac{}{(f(x) = f(s))[s/x]}\,(=I)}{(f(x) = f(s))[t/x]}\,(=E)$$

- For all terms $r$ and $s$, $r[s/x][s/x]$ is $r[s/x]$
- For all terms $r$ and $s$, $(r = r[s/x])[s/x]$ is $(r[s/x] = r[s/x])$
- For all terms $r$, $s$ and $t$, $r[s/x][t/x]$ is $r[s/x]$
- For all terms $r$, $s$ and $t$, $(r = r[s/x])[t/x]$ is $(r[t/x] = r[s/x])$

(**Leibniz's Rule**) Suppose that $r$, $s$ and $t$ are terms. Then,

$$\{s = t\} \vdash (r[s/x] = r[t/x])$$

$$\dfrac{s = t \qquad \dfrac{}{(r = r[s/x])[s/x]}\,(=I)}{(r = r[s/x])[t/x]}\,(=E)$$

(**Transitivity of** $=$) Suppose that $r$, $s$ and $t$ are terms and $x$ is not a variable in $r$, $s$ or $t$. Then,

$$\{(s = r), (r = t)\} \vdash (s = t)$$

$$\dfrac{s = r \qquad (x = t)[r/x]}{(x = t)[s/x]}\,(=E)$$

Axioms for addition:

- A1 For each $n \in \mathbb{N}$ let $\bar{n} \bar{+} \bar{0} = \bar{n}$ be an axiom.
- A2 For each $m \in \mathbb{N}$ and $n \in \mathbb{N}$, $\bar{m} \bar{+} \overline{n+1} = \bar{S}(\bar{m} \bar{+} \bar{n})$ is an axiom.

Instances of A1 include

- $\bar{0} \bar{+} \bar{0} = \bar{0}$
- $\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})$
- $\bar{S}\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}\bar{S}(\bar{0})$
- $\bar{S}\bar{S}\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}\bar{S}\bar{S}(\bar{0})$

Instances of A2 include

- $\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0})$
- $\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}(\bar{0}))$
- $\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}))$
- $\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}))$

Let $\Gamma_A = \{\varphi \mid \varphi \text{ is an instance of A1 or and instance of A2}\}$

1. $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})$

   $\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0}) \in \Gamma_A$, so $\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})$ is a derivation with all undischarged assumptions in $\Gamma_A$.

2. $\Gamma_A \vdash \bar{S}(\bar{0}) = \bar{S}(\bar{0}) \bar{+} \bar{0}$

   The following derivation verifies the above:

$$\dfrac{\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0})}{\bar{S}(\bar{0}) = \bar{S}(\bar{0}) \bar{+} \bar{0}}\ (=\text{sym})$$

3. $\Gamma_A \vdash \bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0})$

$$\dfrac{\bar{0} \bar{+} \bar{0} = \bar{0} \qquad (\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0}))}{(\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0}))}\ (=E)$$

$$\dfrac{\bar{0} \bar{+} \bar{0} = \bar{0} \qquad (\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(x))[\bar{0} \bar{+} \bar{0}/x]}{(\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(x))[\bar{0}/x]}\ (=E)$$

4. $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})$

$$\dfrac{\dfrac{\bar{S}(\bar{0}) \bar{+} \bar{0} = \bar{S}(\bar{0}) \qquad \bar{S}(\bar{0}) \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{S}(\bar{0}) \bar{+} \bar{0})}{\bar{S}(\bar{0}) \bar{+} \bar{S}(\bar{0}) = \bar{S}\bar{S}(\bar{0})}\ (=E) \qquad \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{S}(\bar{0}) \bar{+} \bar{S}(\bar{0}))}{\bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})}\ (=E)$$

5. $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0})$

   It is enough to show that $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})$ (see item 4) and $\Gamma_A \vdash \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})$:

$$\dfrac{\dfrac{\dfrac{\bar{0} \bar{+} \bar{0} = \bar{0} \qquad \bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0})}{\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0})}{\scriptstyle (=E)} \qquad \bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}(\bar{0}))}{\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}(\bar{0})}{\scriptstyle (=E)} \qquad \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{S}\bar{S}(\bar{0}))}{\bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0}) = \bar{S}\bar{S}\bar{S}(\bar{0})}{\scriptstyle (=E)}$$

   Then by symmetry and transitivity of $=$, we have that $\Gamma_A \vdash \bar{S}(\bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0}) = \bar{0} \bar{+} \bar{S}\bar{S}\bar{S}(\bar{0})$.

Axioms for multiplication:

P1 For each $n \in \mathbb{N}$ let $\bar{n} \bar{\times} \bar{0} = \bar{0}$ be an axiom.

P2 For each $m \in \mathbb{N}$ and $n \in \mathbb{N}$, $\bar{m} \bar{\times} \overline{n+1} = (\bar{m} \bar{\times} \bar{n}) \bar{+} \bar{m}$ is an axiom.

Instances of P1 include

- $\bar{0} \bar{\times} \bar{0} = \bar{0}$
- $\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0}$
- $\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0}$
- $\bar{S}\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0}$

Instances of P2 include

- $\bar{0} \bar{\times} \bar{S}(\bar{0}) = (\bar{0} \bar{\times} \bar{0}) \bar{+} \bar{0}$
- $\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = (\bar{S}(\bar{0}) \bar{\times} \bar{0}) \bar{+} \bar{S}(\bar{0})$
- $\bar{S}(\bar{0}) \bar{\times} \bar{S}\bar{S}(\bar{0}) = (\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0})) \bar{+} \bar{S}(\bar{0})$
- $\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = (\bar{S}\bar{S}(\bar{0}) \bar{\times} \bar{0}) \bar{+} \bar{S}\bar{S}(\bar{0})$

$\Gamma_P = \Gamma_A \cup \{\varphi \mid \varphi \text{ is an instance of P1 or is an instance of P2}\}$

$$\Gamma_P \vdash \bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = \bar{S}(\bar{0})$$

$$\dfrac{\dfrac{\bar{0} \bar{+} \bar{0} = \bar{0} \qquad \bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0} \bar{+} \bar{0})}{\bar{0} \bar{+} \bar{S}(\bar{0}) = \bar{S}(\bar{0})}(=E) \qquad \dfrac{\bar{S}(\bar{0}) \bar{\times} \bar{0} = \bar{0} \qquad \bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = \bar{S}(\bar{0}) \bar{\times} \bar{0} \bar{+} \bar{S}(\bar{0})}{\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = \bar{0} \bar{+} \bar{S}(\bar{0})}(=E)}{\bar{S}(\bar{0}) \bar{\times} \bar{S}(\bar{0}) = \bar{S}(\bar{0})}$$