Opinion Dynamics

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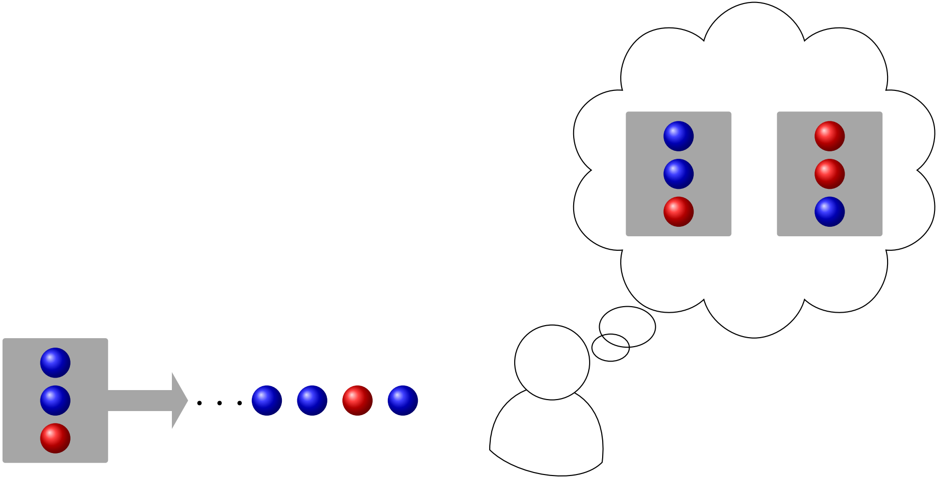

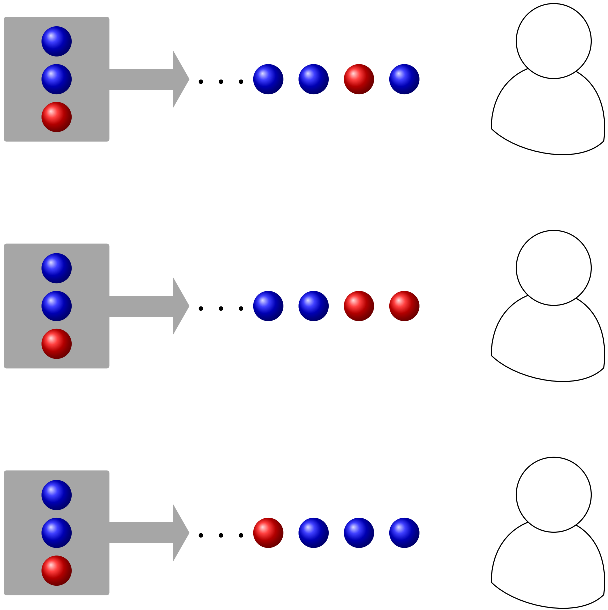

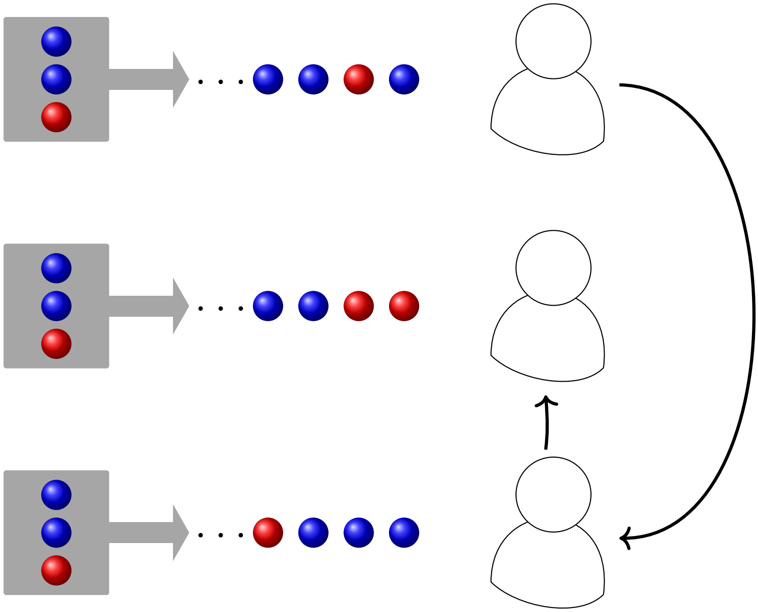

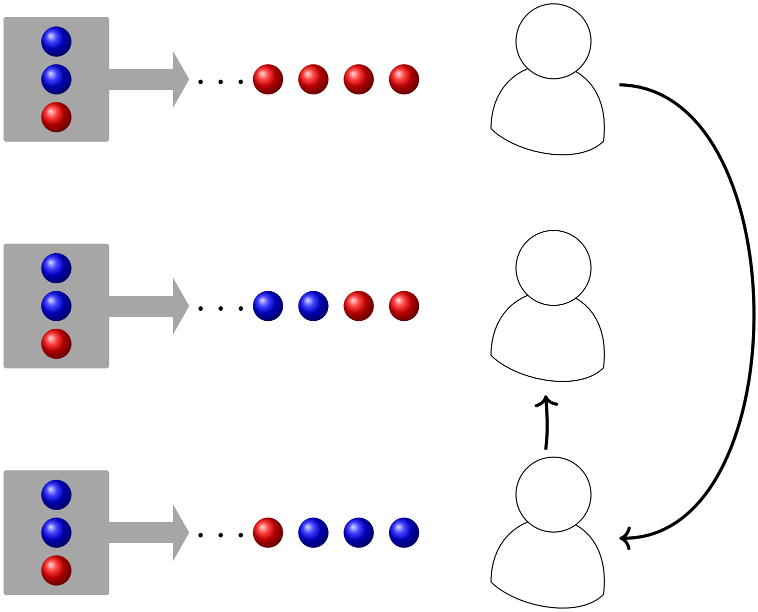

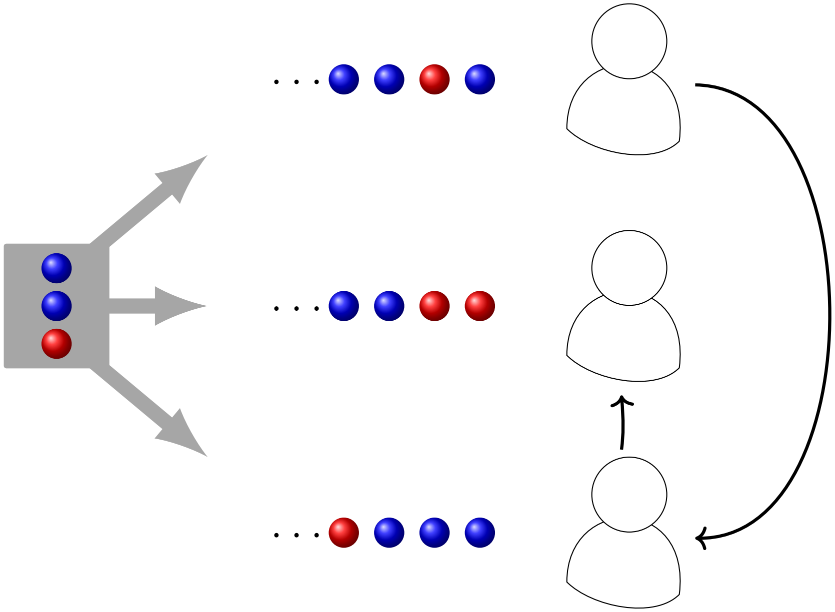

Suppose that W is a set of states (the *set of outcomes*).

A σ-algebra is a set $\Sigma \subseteq \wp(W)$ such that

- $W \in \Sigma$
- If $A \in \Sigma$, then $\overline{A} \in \Sigma$
- If $\{A_i\}$ is a countable collection of sets from Σ, then $\bigcup_i A_i \in \Sigma$

A **probability function** is a function $Pr : \Sigma \to [0, 1]$ satisfying:

- $Pr(W) = 1$
- $Pr(A \cup B) = Pr(A) + Pr(B)$ whenever $A \cap B = \varnothing$

(W, Σ, Pr) is called a probability space.

Probability

Kolmogorov Axioms:

1. For each E, $0 \leqslant Pr(E) \leqslant 1$
2. $Pr(W) = 1$, $Pr(\varnothing) = 0$
3. If $E_1, \ldots, E_n, \ldots$ are pairwise disjoint ($E_i \cap E_j = \varnothing$ for $i \neq j$), then $Pr(\bigcup_i E_i) = \sum_i Pr(E_i)$

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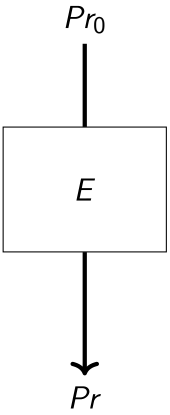
- $Pr(\overline{E}) = 1 - Pr(E)$ ($\overline{E}$ is the complement of E)
- If $E \subseteq F$ then $Pr(E) \leqslant Pr(F)$
- $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$


Pr_0
E
Pr

Conditional Probability

The probability of E *given* F, dented $Pr(E \mid F)$, is defined to be

$$Pr(E \mid F) = \frac{Pr(E \cap F)}{Pr(F)}.$$

provided $Pr(F) > 0$.

Deliberation - Basic Model

Each individual has an opinion represented by a real number in some interval (e.g, $[0, 1]$).

Each individual is influenced by another to a certain degree, again represented by some non-negative number.

Individuals then update their beliefs by averaging their own belief with all those who have influence over them.

Learning from others

If Q reports that the probability of E is q, then $P_{new}(E) = P(E \mid Q(E) = q)$

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Shortcut: $P_{new}(E) = \frac{P(E)+Q(E)}{2}$

C. Genest and M. J. Schervish (1985). *Modeling expert judgments for bayesian updating*. The Annals of Statistics, 13(3), pgs. 1198-1212.

R. Bradley (2017). *Learning from others: conditioning versus averaging*. Theory and Decision.

J. W. Romeijn and O. Roy (2017). *All agreed: Aumann meets DeGroot*. Theory and Decision.

Aggregating Probabilities

C. Genest and J. V. Zidek. *Combining probability distributions: A critique and an annotated bibliography*. Statistical Science,1(1), pp. 114 - 135, 1986.

F. Dietrich and C. List. *Probabilistic opinion pooling*. in Oxford Handbook of Probability and Philosophy, 2016.

Aggregating Probabilities

Let $(W, \mathcal{E})$ be an algebra of events

Let $\mathcal{P}$ be the set of probability functions on $(W, \mathcal{E})$

Probabilistic aggregation function: $F : \mathcal{P}^n \rightarrow \mathcal{P}$

Aggregation Functions

Linear pooling: for all $A \in \mathcal{E}$, $f(\mathrm{P})(A) = w_1 P_1(A) + \cdots w_n P_n(A)$, with $\sum_i w_i = 1$

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Geometric pooling: for all $w \in W$, $f(\mathrm{P})(w) = c \cdot [P_1(w)]^{w_1} \cdots [P_n(w)]^{w_n}$ with $\sum_i w_i = 1$ and $c = \frac{1}{\sum_{w' \in W} [P_i(w')]^{w_1} \cdots [P_i(w')]^{w_n}}$

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Multiplicative pooling: for all $w \in W$, $f(\mathrm{P})(w) = c \cdot [P_1(w)] \cdots [P_n(w)]$ with $c = \frac{1}{\sum_{w' \in W} [P_i(w')] \cdots [P_i(w')]}$

Note that multiplicative pooling $=$ geometric pooling with weights all equal to 1.

Example, I

$W = \{w_1, w_2\}$

$\mathsf{P} = (P_1, P_2, P_3)$ with $P_1(w_1) = 0.9$, $P_2(w_1) = 0.1$, $P_3(w_1) = 0.6$

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$f_{lin}(\mathrm{P})(w_1) = \frac{1}{3} * 0.9 + \frac{1}{3} * 0.1 + \frac{1}{3} * 0.6 = 0.5333$

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$f_{geom}(\mathrm{P})(w_1) = \frac{\sqrt{0.9*0.1*0.6}}{\sqrt{0.9*0.1*0.6}+\sqrt{0.1*0.9*0.4}} = 0.5337$

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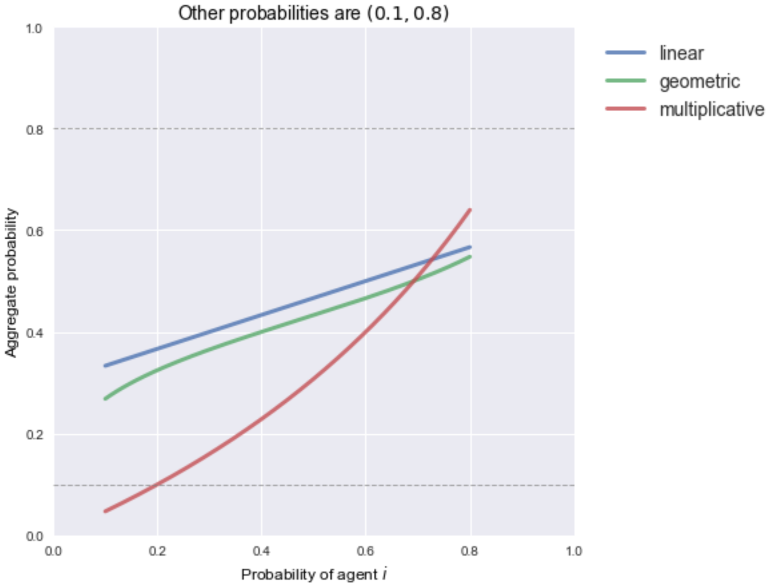
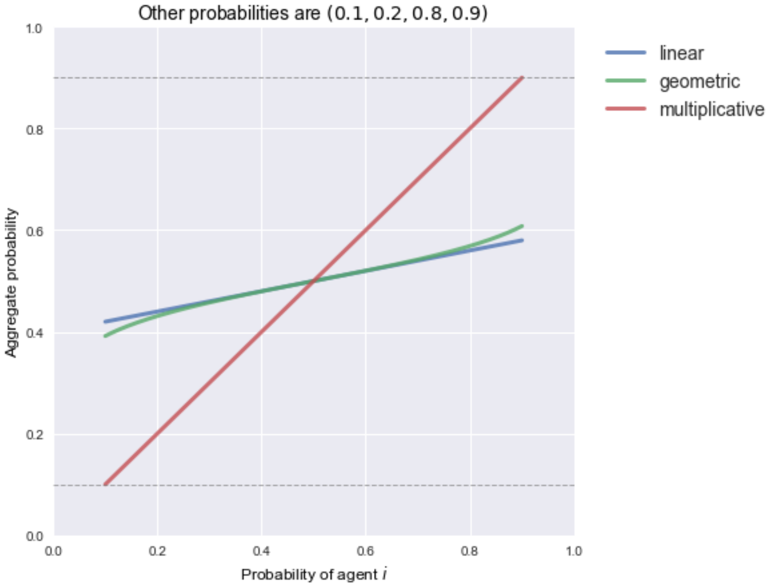
$$f_{mult}(\mathrm{P})(w_1) = \frac{0.9*0.1*0.6}{0.9*0.1*0.6+0.1*0.9*0.4} = 0.6$$

Example, II



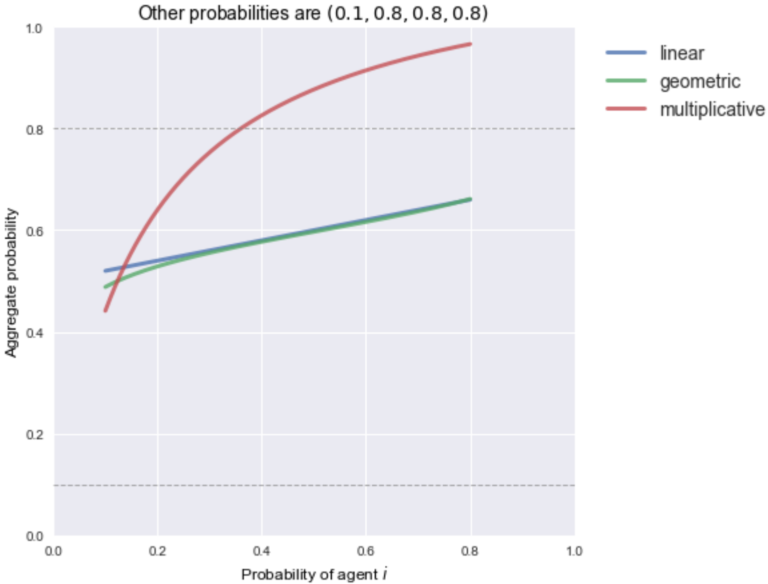
Other probabilities are (0.1, 0.8)
1.0
0.8
0.6
0.4
0.2
0.0
Aggregate probability
0.0
0.2
0.4
0.6
0.8
1.0
Probability of agent i
linear
geometric
multiplicative

Example, III



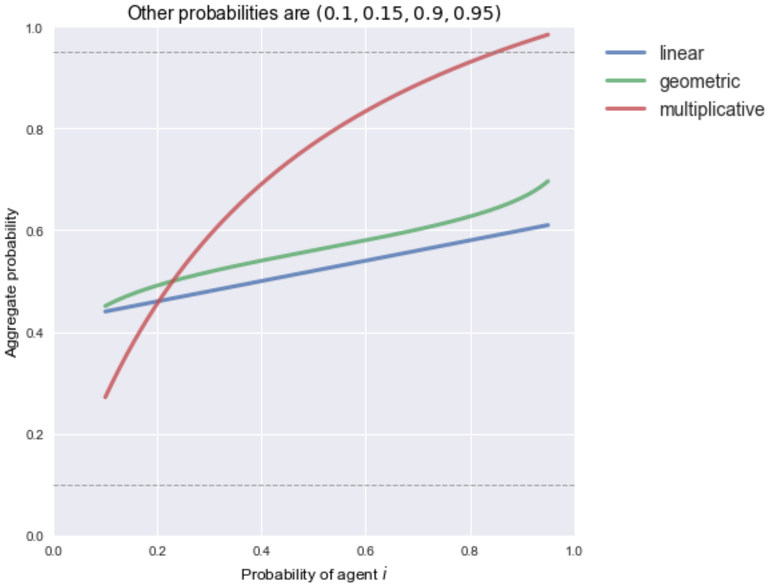
Other probabilities are $(0.1, 0.2, 0.8, 0.9)$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Aggregate probability
Probability of agent i
linear
geometric
multiplicative

Example, IV



Other probabilities are (0.1, 0.8, 0.8, 0.8)
1.0
0.8
0.6
0.4
0.2
0.0
Aggregate probability
0.0
0.2
0.4
0.6
0.8
1.0
Probability of agent i
linear
geometric
multiplicative

Example, V



Other probabilities are (0.1, 0.15, 0.9, 0.95)
linear
geometric
multiplicative
Aggregate probability
Probability of agent i
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0

Linear Pooling

J. Aczel and C. Wagner. *A characterization of weighted arithmetic means*. SIAM Journal on Algebraic and Discrete Methods 1(3), pp. 259 - 260, 1980.

K. J. McConway. *Marginalization and Linear Opinion Pools*. Journal of the American Statistical Association, 76(374), pp. 410 - 414, 1981.

Eventwise Independence For each event $A \in \mathcal{E}$, there exists a function $D_A : [0,1]^n \to [0,1]$ such that for each $\mathrm{P} = (P_1, \ldots, P_n)$,

$$f(\mathrm{P})(A) = D_A(P_1(A), \ldots, P_n(A))$$

Unanimity preservation For every profile $\mathrm{P} = (P_1, \ldots, P_n)$ in the domain of the aggregation function f, if all P_i are identical, then $f(\mathrm{P})$ is identical to them.

Theorem (Aczel and Wagner 1980; McConway 1981) Suppose $|W| > 2$. The linear pooling functions are the only eventwise-independent and unanimity-preserving aggregation functions (with domain $\mathcal{P}^n$).

The DeGroot/Lehrer Model

- There is a proposition about which several individuals disagree.
- Each individual i initially assigns some probability p_i to the proposition.

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- Each individual i assigns every individual j (including himself!) a non-zero weight $w_{i,j}$. The weights represent how reliable i believes j is relative to others in the group.
  - $0 \leqslant w_{i,j} \leqslant 1$
  - For any individual, the weights sum to 1, i.e., $\sum_{i,j} w_{i,j} = 1$

The DeGroot/Lehrer Model

- Time is divided into discrete stages.
- Let i's degree of belief on stage t be represented by $p_{i,t}$ At stage $t+1$, individual i updates his belief to be a weighted-average of everyone's beliefs from stage t.

$$p_i^{(t+1)} = \sum_j w_{i,j} p_j^{(t)}$$

Initial opinion: $\mathrm{P}^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots, p_n^{(0)})^T$

$0 \leqslant w_{ij} \leqslant 1$: the weight that i assigns j, where $\sum_j w_{ij} = 1$

$p_i^{(1)} = \sum_j w_{ij} p_j^{(0)}$

Let W be a matrix with $\mathrm{W}_{ij} = w_{ij}$

$\mathrm{P}^{(1)} = \mathrm{W}\mathrm{P}^{(0)}$

$\mathrm{P}^{(k)} = \mathrm{W}\mathrm{P}^{(k-1)} = \mathrm{W}^k\mathrm{P}^{(0)}$

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$W^t = \begin{cases} W & t \text{ odd} \\ I & t \text{ even} \end{cases}$$

If $p_{1,0} \neq p_{2,0}$, then the agents beliefs oscillate forever.

$$W = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$W^t \to \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix}$$

All initial probabilities will converge.

A **walk** in W is a sequence of nodes $i_1, i_2, \dots i_K$, not necessarily distinct, such that $W_{i_k i_{k+1}} > 0$ for each $1 \leqslant k \leqslant K-1$. The **length of the walk** is defined to be $K-1$

A **path** in W is a walk consisting of distinct nodes.

A **cycle** is a walk $i_1, i_2, \dots i_K$ such that $i_1 = i_K$. The length of a cycle with K (not necessarily distinct) entries is defined to be $K-1$. A cycle is **simple** if the only node appearing twice in the sequence is the starting (and ending) node.

The matrix W is **strongly connected** if there is path in W from any node to any other node.

Proposition. If W is a strongly connected matrix, the following are equivalent:

1. W is convergent: $\lim_{t\to\infty} W^t P$ exists for all $P \in [0,1]^n$
2. W is aperiodic: the greatest common divisor of the lengths of all the simple cycles in W is 1.
3. There is a unique $s \in [0,1]^n$ such that $\sum_i s_i = 1$ such that for every $P \in [0,1]^n$

$$\left(\lim_{t\to\infty} W^t P^T\right)_i = sP$$

where $s_i = \sum_j W_{ji} s_j$.

The DeGroot/Lehrer Model

- Why should individuals assign non-zero weight to others?
- Why should individuals repeat the averaging process?
- Why should the weights remain constant?

Repeated Averaging

> [R]efusing to shift from state 1 to state 2 is equivalent to assigning a weight of 0 to other members of the group at this stage. This amounts to the assumption that there is no chance that one is mistaken and no chance that others in the group with whom one disagrees are correct. In short, the only alternative to the iterated aggregation converging toward a consensual probability assignment is individual dogmatism at some stage.

(Lehrer, 1976, pg. 331)

Constant Weights

> The constancy condition is sustained by the assumption that members of the community...acquire no new information... The constancy assumption amounts to the requirement that a person who forms an estimate of the reliability of others as indicators of truth apply that estimate consistently until he obtains new information.
>
> (Lehrer, 1976, pg. 330)

Hegselmann and Krause

An individual's peers change over time.

In the model, an individual considers only those whose opinions are sufficiently similar to his or her own.

Since our opinions change, our peer groups can change.

Hegselmann and Krause wanted to explain why groups might become polarized and not reach a consensus.

The Hegselmann-Krause Model

Agent i's belief is represented by a real number r_i

- A subjective probability of a proposition
- A numerical estimate of some quantity

The truth is represented by a real-number T.

- 0 or 1 might represent the truth-value of some proposition.
- The *real* value of some quantity

The Hegselmann-Krause Model

There is some number ρ (for all agents) that represents how "close" other's opinions must be to one's own in order for one to take them seriously. If ρ is close to zero, then one only considers the opinions of those who are similar to oneself. Agent i is assigned some number τ_i between 0 and 1 that represents how strongly she is "attracted" to the truth.

- $\tau_i = 0$ will represent an agent who only listens to her peers.
- $\tau_i = 1$ will represent an agent who has immediate access to the truth.

The Hegselmann-Krause Model

Time is divided into discrete stages $1, 2, 3, \ldots$ At stage $t+1$, agent i averages The beliefs of her peers whose opinions are within distance ρ of her own and the truth T. The truth is given weight τ_i and the remaining weight $1-\tau_i$ is divided evenly among peers.

Let $b_{i,t}$ be i's belief at time t

Let $N_\rho(i, t)$ be those peers whose opinions differ from i's by no more than ρ at stage t

$$N_\rho(i, t) = \{j \mid |b_{j,t} - b_{i,t}| < \rho\}$$

Let $N = |N_\rho(i, t)|$. Then,

$$b_{i,t+1} = \tau_i \times T + (1 - \tau_i) \times \sum_{j \in N_\rho(i,t)} \frac{1}{N} \cdot b_{j,t}$$