Kites, Darts, Suns and Stars:
Penrose's Aperiodic Sets and their Applications

by

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I'd like to thank my parents, all of them, for their continuing support in a subject that they respect greatly yet of which they understand little. Many thanks also to all of my friends from my past and present who came to my thesis talk; you all made it the best birthday I'd ever had. I'd also like to thank my other Professor's who have catered their schedules and deadlines to work with my writing of this paper. Lastly I'd like to that Cyn Hernandez for her constant assistance and patience with me during this last semester.

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Chapter 1

Introduction

Tilings are all around us, covering the ceilings we walk under, the bathroom floors we stroll across and the ancient structures we visit. Some of these tilings are organized patterns, like squares, rhombi, regular triangles, and regular hexagons. Others can be seemingly random as in a stone patio placed with all different shaped stones. Some tilings even have rules on how they are tiled and still have no repetitive pattern. In this paper, I will explore how these matching rules affect the creation of particular tilings and the intriguing characteristics of these tilings.

1.1 SOME DEFINITIONS

As defined by Grunbaum and Shephard (G & S) – a reference from which much of the following material used has come – a *plane tiling* $\mathcal{T}$ is a countable family of closed sets $\mathcal{T} = \{T_1, T_2, \ldots\}$ which cover the plane without gaps or overlaps [**4,** pp. 16]. That tiling is *periodic* if its symmetry group contains at least two translations in non-parallel directions [**4,** pp. 29].

G & S define for us that a tile T is called the *prototile* of a tiling $\mathcal{T}$. We also say that the prototile T *admits* the tiling $\mathcal{T}$ if copies of T are solely used to cover the plane. If only one tile T is used, we call them *monohedral* tilings [**4,** pp. 20]. Examples of monohedral tilings (in a finite sense) are bathroom floors that contain repetitions of a square, or a beehive that is covered with only hexagons. The monohedral tilings in Figure 1.1 are also examples of lattices, or space filling structures, in two dimensions. This will be explored more in Chapter 4. Note that these examples are all periodic.

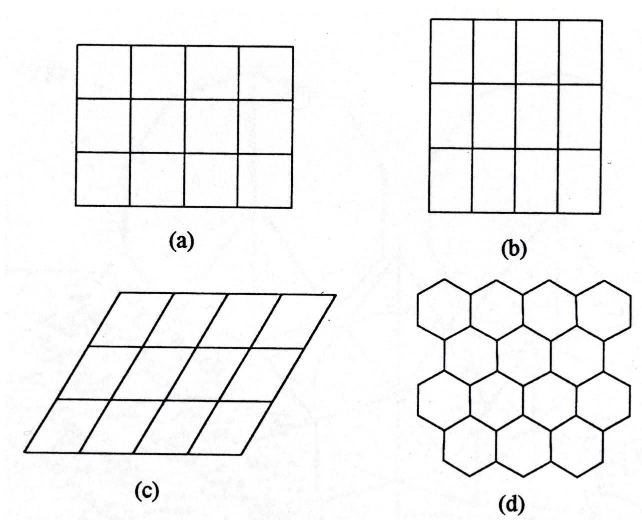


Figure 1.1 – Monohedral tilings of a square (a), a rectangle (b), a parallelogram (c), and a hexagon (d)

In this paper, however, we will only be working with tilings that are admitted by more than one prototile, and are not periodic. Defined nicely by G & S, a set of prototiles is considered to be *aperiodic*, also known as non-periodic, if it will admit infinitely many tilings of the plane, yet no such tiling is periodic [**4,** pp. 520]. This is to say that the tile set will only tile the plane where the symmetry group will contain no translations.

A tiling is also referred to as being aperiodic if it is admitted by an aperiodic set of prototiles.

1.2 HISTORICAL BEGINNING

Johannes Kepler (1571-1630) was one of the first to experiment with tiles and tiling.

Figure 1.2 – Johannes Kepler

Kepler experimented with aperiodic tiling. Tiling with regular pentagons facilitates the tiling process, and Kepler knew this. He was one of the first who considered regular stars to be regular polygons. Kepler designed Figure 1.3, called *Harmonice Mundi* or *The Harmony of the World*, using pentagons, pentacles (the stars), decagons, and double decagons ('monsters' as he referred to them) – all shapes related to the Golden Ratio.

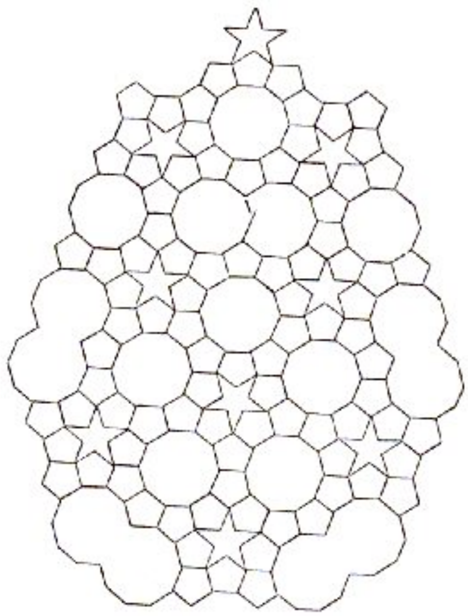


Figure 1.3 – Kepler's *Harmony of the World*

Kepler's tilings may seem to be aperiodic. In fact no single one of his tiles can be repeated to periodically tile the plane. However, given the set of tiles, one can create a section (Figure 1.4) that can tile the plane with translations in two non-parallel directions, making it periodic.

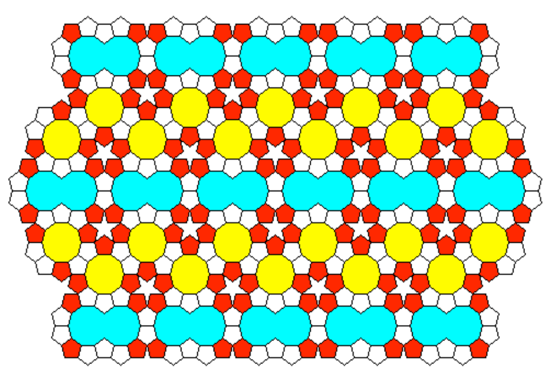


Figure 1.4 – Kepler's tiles organized in a periodic fashion

Tiles that can be copied and put together to create larger copies of the original are known as ‘rep-tiles’. An example of such a set that can tile non-periodically, but not exclusively non-periodically, is known as the “sphinx” tiles.

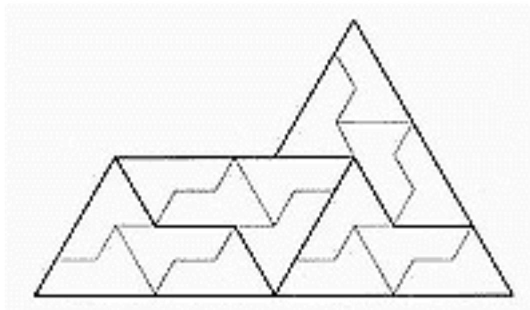


Figure 1.5 – The “sphinx” tile

In Figure 1.5 we see three generations of the sphinx tile. It does not tile only aperiodically; notice that if we copy the sphinx and flip it upside-down, we can place the two tiles together to make a rhombus. This rhombus may be used to tile the plane in a periodic fashion.

Many believed for years that there existed no set of two or more prototiles that *only* tiled non-periodically. Robert Berger proved this wrong using rectangles with colored sides call Wang dominoes. The dominoes, created by Hao Wang, were used to demonstrate that given the existence of a certain decision procedure, the dominoes tiling the plane in a non-periodic fashion could also tile periodically. However, in 1964 Berger proved Wang’s statement to be false. Berger produced a set of over 20,000 dominoes that tiled only non-periodically, disproving a long-believed proposition.

After understanding the blocks more thoroughly, Berger later reduced the set to 104 dominoes that could complete the task. Then, in 1976, Raphel M. Robinson further reduced the set to an astounding 24.

Figure 1.6 – Raphael M. Robinson

However, Robinson's participation in the tiling world did not begin with dominoes; he had earlier in his career worked with polygons. In 1971, Robinson discovered a six-tile set of polygons that forces non-periodicity. The six shapes Robinson discovered, shown in Figure 1.7, were squares composed of either rectangular or triangular notches protruding from them or indentations into which these notches fit.

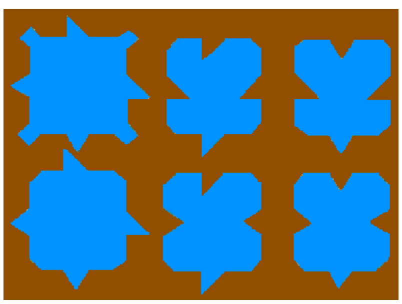


Figure 1.7 - Robinson's six tiles

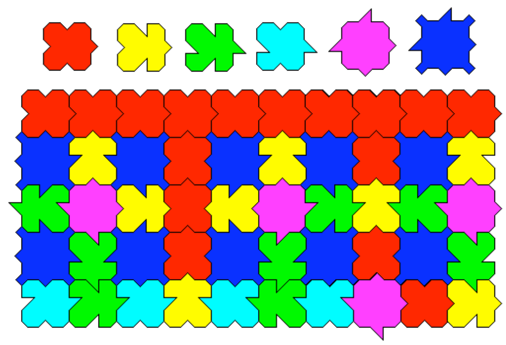


Figure 1.8 – A tiling of Robinson's six tiles

These different formations allow for very particular formations of the shapes making the tiling possible and yet not necessarily 'easy' in the sense that there are no tricks to assist the creation of the tiling.

1.3 PENROSE AND HIS APERIODIC SETS

In 1973 and 1974, in searching for an aperiod tiling to better Robinson, Sir Roger Penrose discovered three aperiodic sets of tiles.

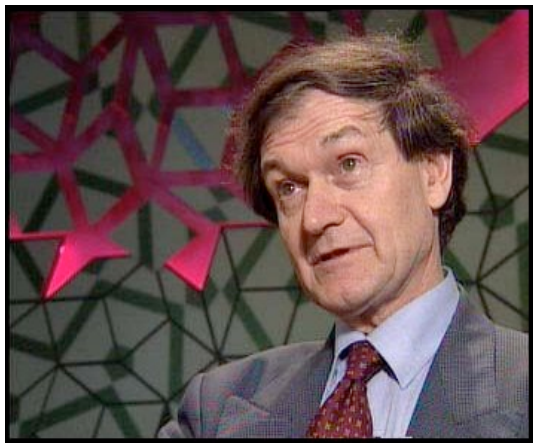


Figure 1.9 - Sir Roger Penrose

These aperiodic sets are intriguing because they are neither orderly nor chaotic. The first was a six-tile set that forces non-periodicity as seen in Figure 1.10. Penrose's tiles were 'easier' to work with. This conclusion is drawn due to the fact that these Penrose tilings contain very interesting properties. The essential shapes are a regular pentagon (of which there are 3), a star, a partial star commonly known as a 'paper boat' or 'jester's cap' and a diamond or 'spike'. The small juts, spikes, and notches in the sides of the six shapes define the matching rules of how the shapes can be fit together.

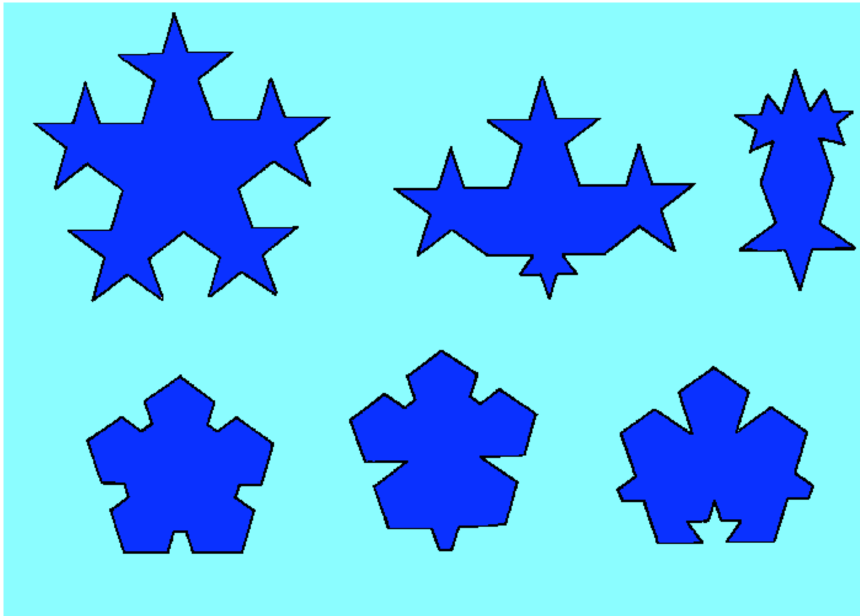


Figure 1.10 – Penrose's aperiodic six-tile set

Soon thereafter, Penrose realized that one of the six tiles could be eliminated. By taking the third pentagon (lower-right in Figure 1.10) and flipping it upside down and lining up the 'crown' notch at the bottom of the paper boat, a new shape is created. And then, we can take two copies of the third pentagon and line them up with the two 'crown' notches on the spike to create yet another shape. Now we have five total shapes and no more of 'crown' notches or indentations. Penrose had done it; he found an aperiodic set of five tiles. That number soon became four and then Penrose, in 1974, very cleverly reduced it to two shapes. John Horton Conway, an English-American mathematician and Princeton professor of mathematics who has discovered many results in his studies of Penrose's tiles, accurately called these two shapes a 'kite' and a 'dart' [**3,** pp. 112-113].

In this paper, much of what we will be working with will be Penrose's kite and dart (Figure 1.11), the second of three mentioned aperiodic sets of tiles (this proof that this set is aperiodic will come in section 3.5):

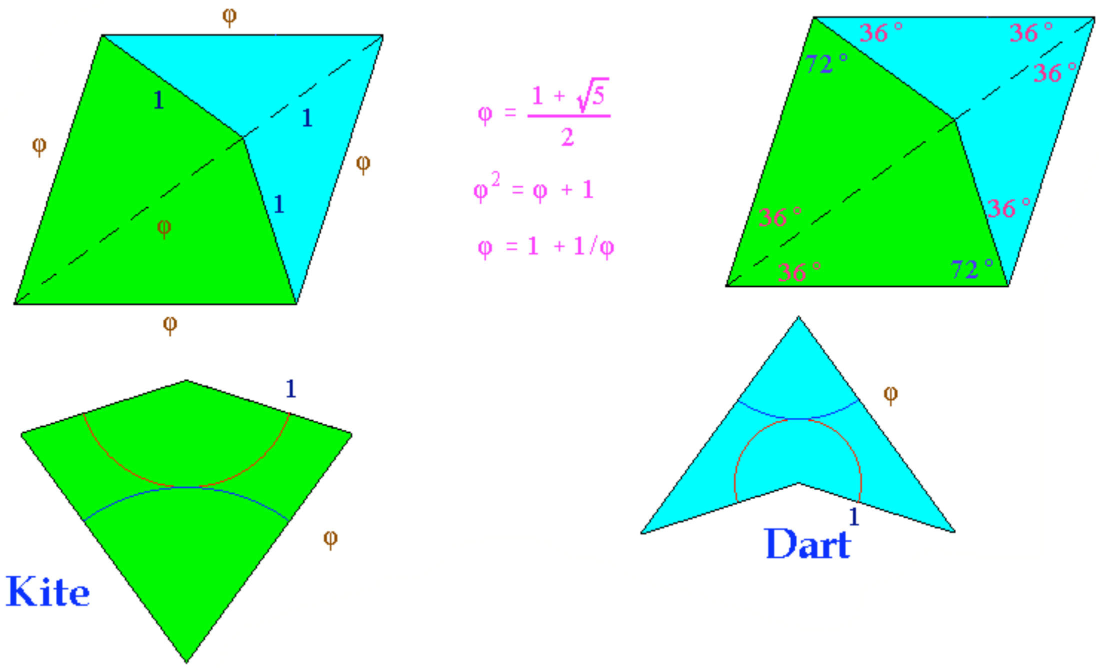


Figure 1.11 – Penrose's kite and dart

It's important to see from where we get these shapes. Well, similar kites and darts can be found in the Pythagorean pentagram.

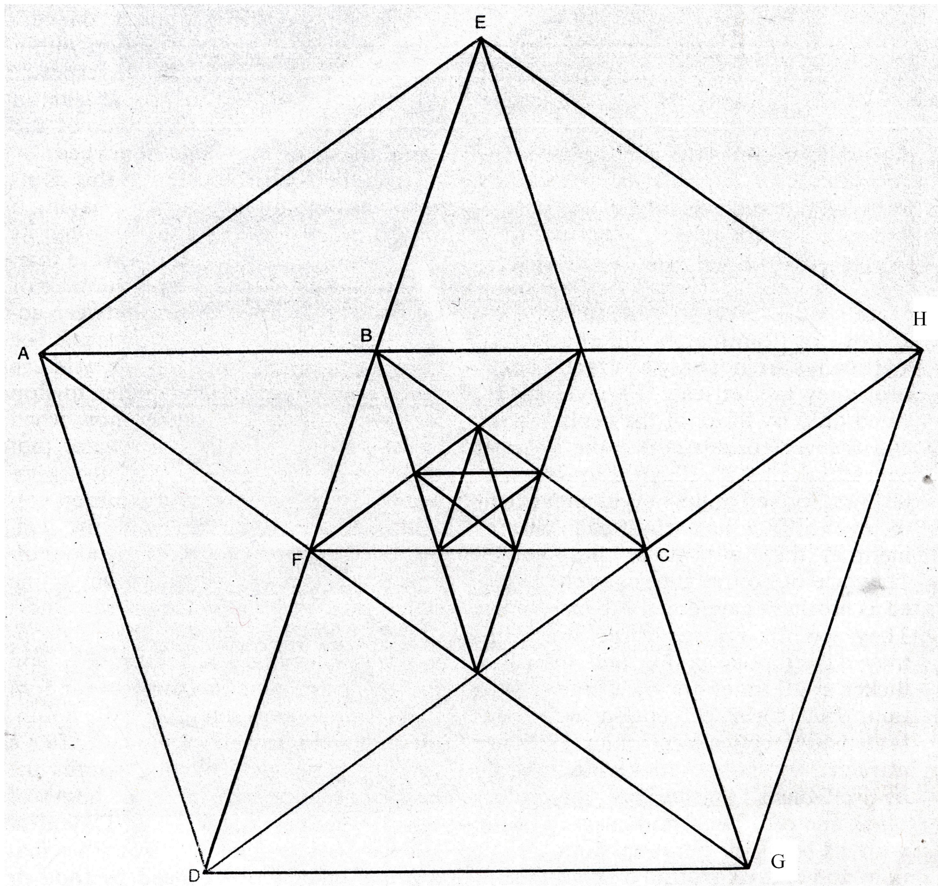


Figure 1.12 – The Pythagorean pentagram (vertices labeled AEHGD in clockwise order)

Our kite is the polygon ABCD; our dart is AECB.

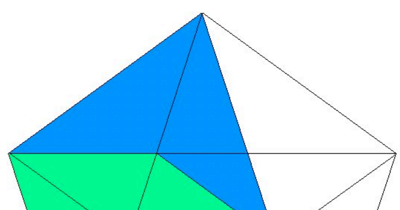


Figure 1.13

The obtuse and acute rhombs about which we will speak of in Chapter 4 are AECD and ABCF respectively (not to scale).

We see that essential to the Pythagorean Pentagram is the Golden Ratio (or Golden Mean) referred to often as phi, φ, or tao, τ. We will be using φ; it is precisely equal to $(1+\sqrt{5}) / 2 = 1.618\ldots$ (or for those interested in the first 50 digits – we are able to compute it due to the precise mathematical formula that we have: 1.61803 39887 49894 84820 45868 34365 63811 77203 09179 80576…). The Golden Ratio holds the property that $\varphi^2 = 1+ \varphi$ and $1/ \varphi = \varphi -1$.

Playing with the intriguing properties that φ exhibits, many have composed fanciful poems and other works about the Golden Ratio. The first verse of the poem "Constantly Mean" published in the journal The Fibonacci Quarterly in 1977 by Paul S. Bruckman of Concord, California [**5,** pp. 81] reads:

The golden mean is quite absurd;
It's not your ordinary surd.
If you invert it (this is fun!),
You'll get itself, reduced by one;
But if increased by unity,
Thus yields its square, take it from me.

In our Pentagram (Figure 1.12), we have a Golden Triangle, DEG, where the ratio of the side to the base is φ. Also there are two isosceles triangles to either side of our Golden Triangle which are called Golden Gnomons (Figure 1.14a), AED and EGH, where the ratio of the side to the base is $1/ \varphi$.

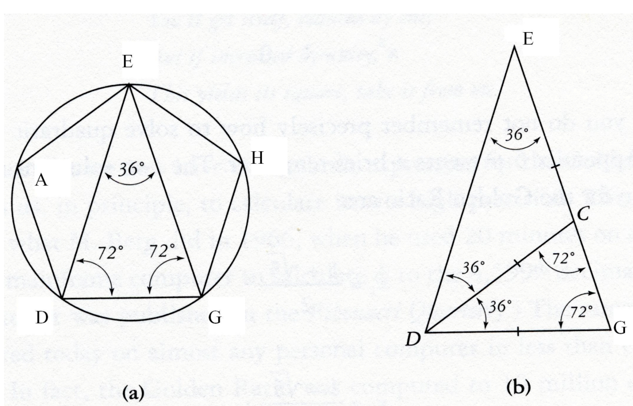


Figure 1.14

Now, as we go to draw line segment DH in Figure 1.14a, we intersect EG at C (Figure 1.14b). In this specific case of a Golden Triangle, we see that interesting results arise. Notice that DC breaks up our Golden Triangle DEG into a Golden Triangle DGC and a Golden Gnomon DEC. Also, note that EG / EC = EC / CG = φ.

It may also be worth noting that the ratio of the diagonal to the side of a regular pentagon (AEHGD) is φ. This comes from the construction of the Pentagram:

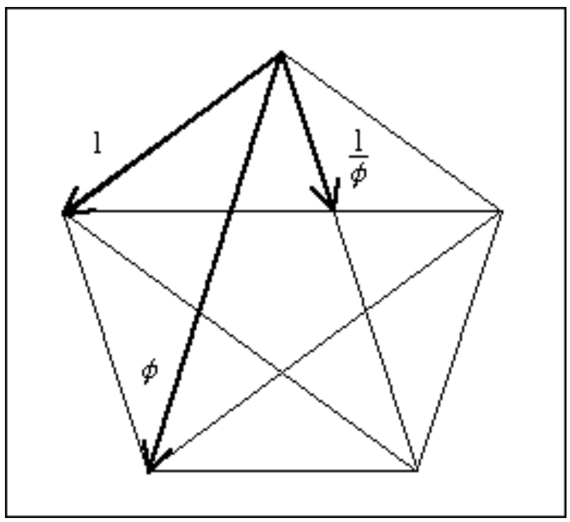


Figure 1.15

Being so closely related to the Pythagorean Pentragram, φ plays a very big role in the construction of our kites and darts:

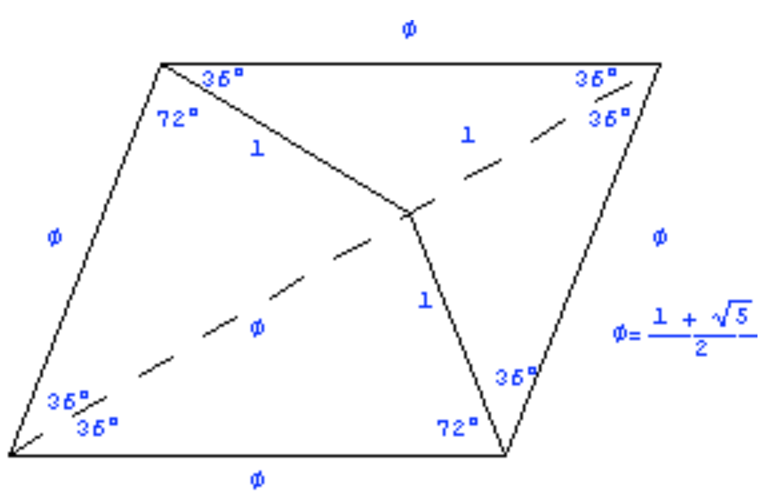


Figure 1.16

For this paper, when referring to a standard or 'original' kite and dart, we assume they are of the dimensions shown in Figure 1.16. Also, the side of length φ in Figure 1.16 will be referred to as the 'long' side and the side of length 1 will be the 'short' side.

Note that the kite is composed of two Golden Triangles and the dart of two Golden Gnomons:

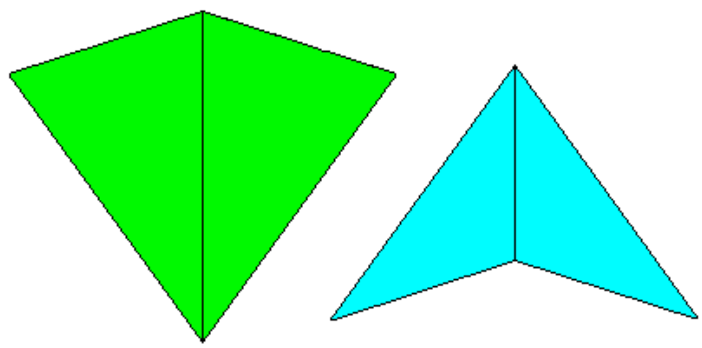


Figure 1.17

It closely follows that the ratio of the area of the kite, A_k, to the area of the dart, A_d, is φ. To show this, let us rearrange the Golden Triangles that compose the kite to make a rhombus.

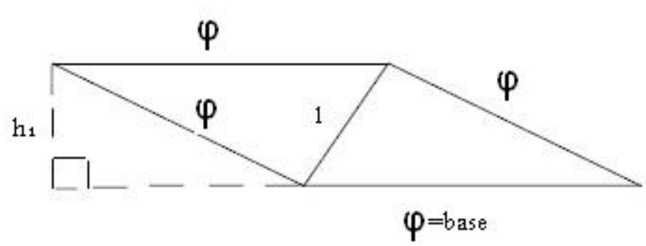


Figure 1.18 (not to scale)

Now to find the area of this rhombus, we must multiply base times height, h_1. The base here is φ. To find h_1, we use some trigonometry; we have sin 36 = h_1 / φ, and it follows that $h_1 = \varphi \sin 36$.

So we see that the area of this rhombus (kite) is $A_k = \varphi^2 \sin 36$.

Also, we see that the two Golden Gnomons of a dart can be rearranged as a rhombus:

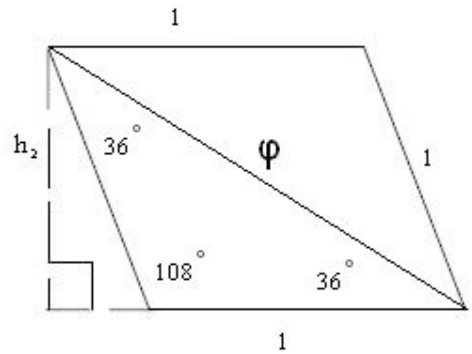


Figure 1.19 (not to scale)

Here the base is 1 and $h_2 = \varphi \sin 36$. So $A_d = 1 (\varphi \sin 36)$. Thus $A_k / A_d = (\varphi^2 \sin 36) / (\varphi \sin 36) = \varphi$.

The acute (thin) rhombus created by the kite's Golden Triangles and the obtuse (fat) rhombus created by the dart's Golden Gnomons compose the third set of Penrose's aperiodic tiles. Unlike the set just created when we rearranged the half kites and half darts to make rhombs, the two rhombs that form Penrose's third aperiodic set have sides all of equal length.

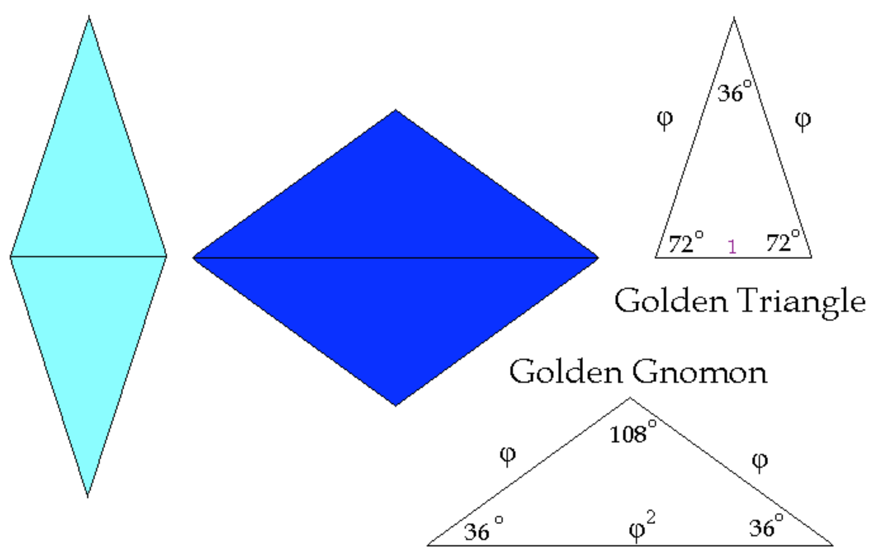


Figure 1.20

For our standard pair to which we will refer in this paper, we modify slightly the scale of the rhombs used above; the length of each of the sides of our rhombs is 1.

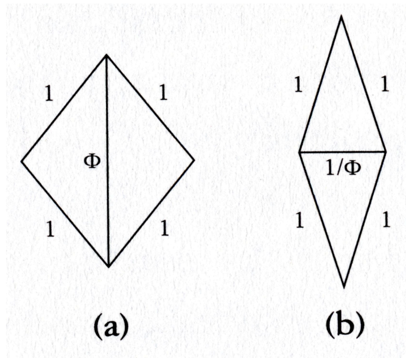


Figure 1.21 – Penrose's Rhombs

Note that similar to our kite and dart the ratio of the area of the obtuse rhombus (Figure 1.21a), A_o, to the acute rhombus (Figure 1.21b), A_a, is φ. This is easily proved in the same manner as the ratio of areas of the kite and dart. Also, the ratio of the length of the long diagonal of the acute rhombus (which would be the vertical diagonal in Figure 1.21b) to the length of the long diagonal of the obtuse rhombus (the horizontal diagonal in Figure 1.21a) is φ. This is can be proved with simple trigonometry.

1.4 FIBONACCI SEQUENCE

When talking about the Golden Ratio, it is practically inevitable that we talk about the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Beginning with the sequence 1, 1…, each successive term is found by adding the previous two terms. Let $f_1 = 1$ and $f_2 = 1$. Then $f_{n+2} = f_{n+1} + f_n$ where f_n is the n^{th} Fibonacci Number.

The Fibonacci Numbers are intrinsically related to the Golden Ratio in that the ratio f_{n+1} / f_n approaches φ as n gets large. To convince ourselves of this, let us compute a few: $f_2/f_1=1/1=1$, $f_3/f_2=2/1=2$, $f_4/f_3=3/2=1.5$, $f_5/f_4=5/3=1.666\ldots$, $f_6/f_5=8/5=1.6$, $f_7/f_6=13/8=1.625$, $f_8/f_7=21/13=1.615\ldots$ and so on. Knowing that φ = 1.618… we see that each successive ratio bounces above and below φ, getting closer with each step. This, however, is only an observation.

To prove that the ratio f_{n+1} / f_n approaches φ as n gets large is straightforward. Let x equal the ratio f_{n+1} / f_n . Looking at the end behavior of x (as n gets large) is equivalent to looking at the behavior of f_{n+2} / f_{n+1} (as n gets large).

Well, we substitute f_{n+2} for $f_{n+1} + f_n$ (which comes from the recursive formula for the Fibonacci numbers $f_{n+2} = f_{n+1} + f_n$) and we have:

$x = f_{n+2} / f_{n+1} = (f_{n+1} + f_n) / f_{n+1} = 1 + f_n / f_{n+1} = 1 + 1/x.$

And now we have $x = 1 + 1/x$. If we multiply through by x we have

$x^2 = x + 1.$

Then it follows that

$x^2 - x - 1 = 0.$

By using the quadratic formula we get that

$x = (1 + \sqrt{5}) / 2 = \varphi$ and $x = (1 - \sqrt{5}) / 2 = 1 - \varphi.$

In our case, it does not make sense to consider a negative ratio, so we conclude that $x = f_{n+1} / f_n$ approaches φ as n gets larger. This property, as Livio tells us, is credited to the German astronomer Johannes Kepler, even though an anonymous Italian may have previously discovered it [**5,** pp. 101].

We notice that an even Fibonacci Number in the numerator produces a ratio less than φ, and an odd one produces a ratio greater than φ. We can prove this by induction.

The statement that we are trying to prove is:

f_{n+1} / f_n {< φ when n is odd; > φ when n is even}.

Let us test the first few initial values. For n=1, we have $f_2 / f_1 = 1$, which confirms our statement. For n=2, we have $f_3 / f_2 = 2$, confirming our statement, and for n=3, we have $f_4 / f_3 = 1.5$, which again confirms our statement.

Next we assume the following statement (comprised of two cases) true for all $n<k$.

Let Case 1 be $k \geq 4$ is even. Case 2 is $k \geq 5$ is odd.

For Case 1, let $a = f_{k-3}$, $b = f_{k-2}$, $c = f_{k-1}$, and $d = f_k$. By induction hypothesis, $b/a < \varphi$ and $c/b > \varphi$. Remember we want to show that $d/c < \varphi$. Due to the construction process of the Fibonacci Numbers, we know that $d = b+c$. So we have:

$d/c = (c+b)/c = 1 + b/c$.

Since $c/b > \varphi$, we know that $1/\varphi > b/c$, and it follows that:

$1 + b/c < 1 + 1/\varphi$.

However, $1 + 1/\varphi = \varphi$. So we conclude that $d/c < \varphi$. Case 1 is thus proved. Case 2 is similarly proved (a, b, c, d are the same; assume $b/a > \varphi$ and $c/b < \varphi$ and we want $d/c > \varphi$).

Thus $f_{n+1} / f_n < \varphi$ when n is odd and $f_{n+1} / f_n > \varphi$ when n is even. Or we can say that $f_{2n+1} / f_{2n} > \varphi$ and $f_{2n} / f_{2n-1} < \varphi$ for $n \geq 1$. To rewrite, it will always be the case that $f_{2n} / f_{2n-1} < \varphi < f_{2n+1} / f_{2n}$ for $n \geq 1$.

Chapter 2

Penrose's Set of Six

2.1 PENTAGONS, SPIKE, STAR AND 'PAPER BOAT'

The first discovery of a set of aperiodic tiles by Roger Penrose came from a realization that in taking a pentagon, one can split it up in such a way that smaller pentagons are created.

Figure 2.1 – A pentagon split into five smaller ones

In between the smaller pentagons exist little triangular gaps shaped like half spikes. Now, imagine that we repeat this process with each of the five little pentagons. Thus we are left with 25 smaller pentagons. However, it is the gaps in between these pentagons that interest us. Between the central pentagon in Figure 2.1 and any of the pentagons surrounding it the two little half spikes come together to make a complete spike:

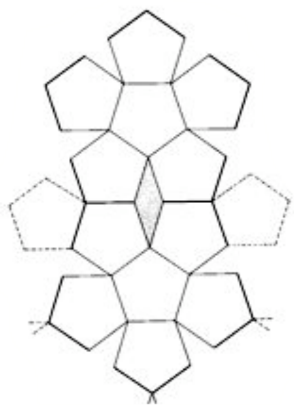


Figure 2.2 – The spike shaped gap formed by splitting up two adjacent pentagons

When this process of breaking each pentagon into five smaller ones is continued, the shapes that compose the gaps in between the pentagons are three: a spike, a star and a 'paper boat' (or 'jester's cap') (Figure 2.3). No new shapes are introduced as the process continues. Penrose thus concludes that we can foresee this subdivision process continuing indefinitely [**6,** pp. 32].

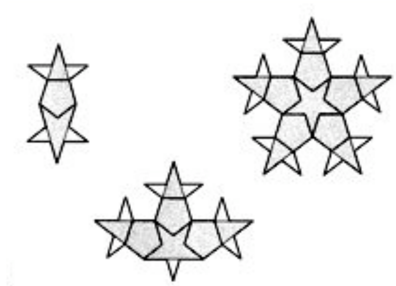


Figure 2.3

So Penrose concluded that using these four shapes (the three just mentioned plus the pentagon) one can tile a plane aperiodically. However, given copies of those four shapes and attempting to create a tiling one could easily construct a plane 'incorrectly'. Penrose tells us that 'correctness' can be *forced* if matching rules are applied. Without matching rules for the edges, the set of tiles admits periodic tilings [**4,** pp. 531]. One of many ways that these matching rules are expressed is by putting jigsaw-like notches on the sides of the shapes as shown in Figure 2.4. Note that there are three different pentagon shapes – thus making this Penrose set of *six* tiles.

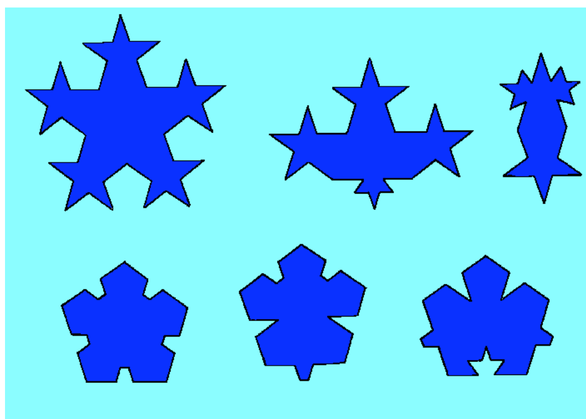


Figure 2.4 – Penrose's set of six tiles with matching conditions

The set of these six prototiles is an aperiodic set. The proof follows directly from Statement 3.5.1 that will come in Chapter 3.

Following these matching rules, these shapes can tile the plane:

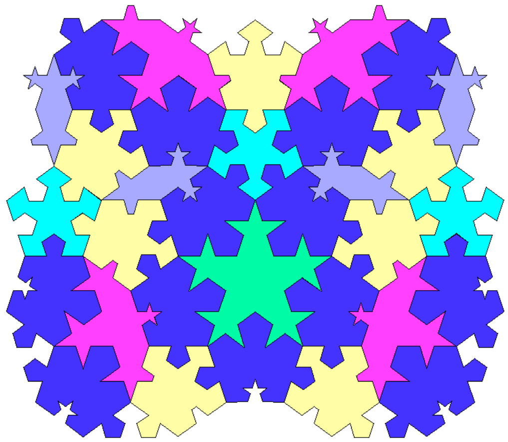


Figure 2.5 – A section of a tiling of Penrose's six with matching conditions

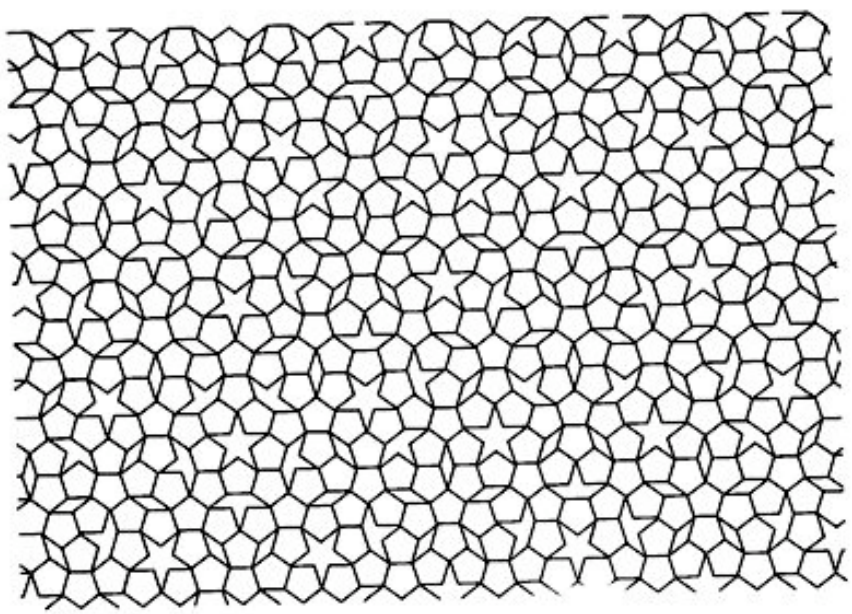


Figure 2.6 – A section of a tiling of Penrose's six not showing matching conditions

Many unexpected and pleasing visual characteristics appear. For example, notice that tiling of Penrose's six contains decapods throughout (polygons with ten sides) and that ten pentagons surround each decapod.

2.2 COMPOSITION AND INFLATION

The set of Penrose's six possesses a very interesting property that comes from the fact that each shape has nested within it other shapes of the set. Figure 2.7 is an example of how each of the six are grouped together to form larger tiles – this process is called composition (a more formal definition to come in Chapter 3). The numbers 0, 1 and 2 represent an alternate way of matching the tiles. In this method, a 0 must match a 0, a 1 must match a 1, and so on.

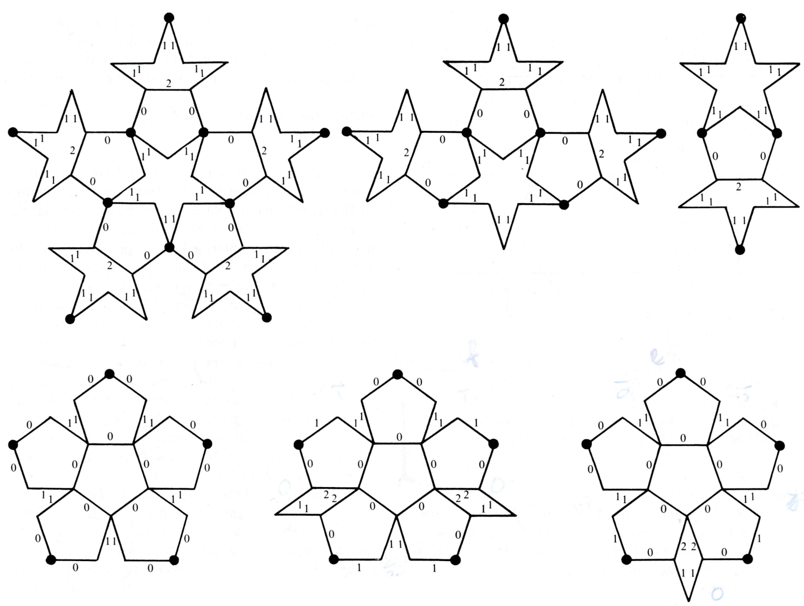


Figure 2.7 – How each of Penrose's six are grouped together when composed

Since we can compose the individual tiles, it makes sense that we can take a tiling and compose it as well. This process of composing a tiling will be more fully explained when talking about kites and darts in the next chapter.

While composition consists of grouping together tiles to make larger shapes, decomposition is the process of splitting up tiles in a special manner to make smaller shapes. One can imagine by looking at Figure 2.7 precisely how the tiles are split up in decomposition.

Figure 2.8 is an example of how to compose a tiling where the light lines make the original tiling and the heavy lines show how the tiles are grouped together to make larger tiles. The heavy lines represent another Penrose tiling where the tiles are larger.

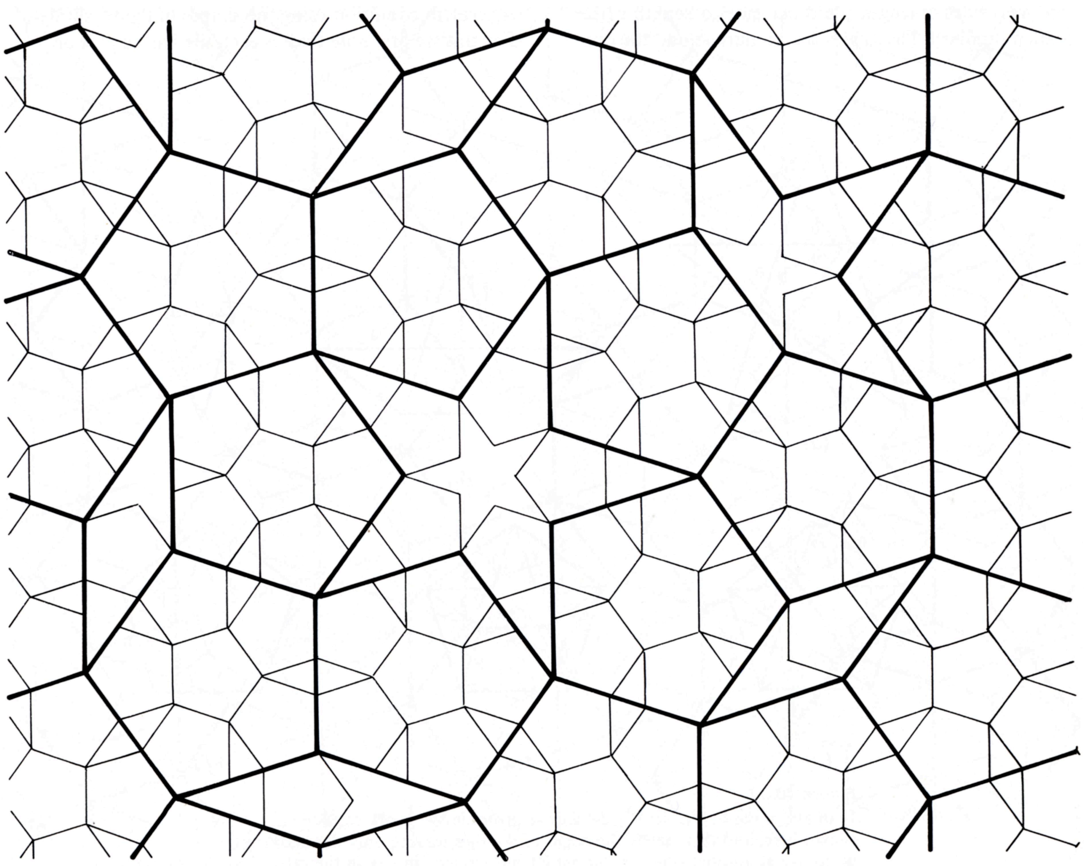


Figure 2.8 – Composition of a tiling of Penrose's six

Now that we see through decomposition that we can break up the tiles into smaller tiles belonging to the same set, we enlarge the smaller ones so that they are the size of the originals. The process of increasing the size of the tiles (so that when split up the small tiles that result are the size of the original tiles) then decomposing them is referred to as inflation.

We conclude that there are two ways to make a tiling. One can take a single tile and inflate it over and over. By repeating this process indefinitely, we can imagine that an infinite tiling is formed. Another way would be to start with a single tile and place others around it following the matching rules. If one piece fits it does not imply that it is allowed in that spot; a few steps later there may be a spot where *no* piece can fit. One must then remove a few tiles and start again. This process can be slow and frustrating, but very rewarding after completing a section successfully.

Chapter 3

Kites and Darts

3.1 MATCHING CONDITIONS

Our second aperiodic set of tiles is composed of the kite and dart. A Penrose tiling of kites and darts is created by starting with a single vertex and, following specific matching conditions, adding on tiles. Penrose's kites and darts fit together in accordance to the following matching conditions:

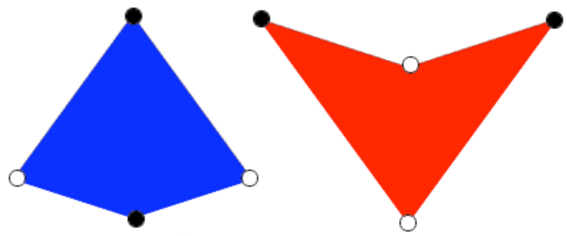


Figure 3.1

Let us start with a kite (darker shape on left). Of the four sides, let us call the length of the longer of two x_k which is φ times as great as the length of the shorter side, y_k. In other words $\varphi y_k = x_k$. The same is true of the dart (lighter shape on right). Let x_d be the long side and y_d be the short one, thus $\varphi y_d = x_d$.

Notice that in Figure 3.1, the vertices are represented as either open or filled circles. The matching conditions require that when two shapes placed next to one other, the vertex type must align (open circles to open and filled circles to filled). Having this constraint restricts the formation of a rhombus (Figure 3.2). This prohibited alignment would allow for periodic tiling.

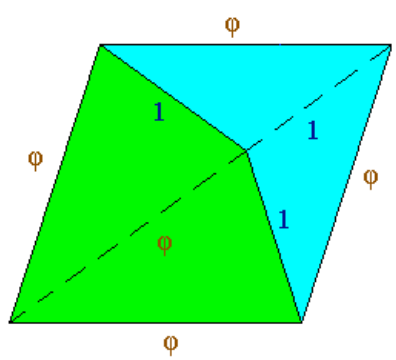


Figure 3.2

In addition, the rules permit us only to align sides with equal length. In other words, we may only align x_k with x_k or x_d; y_k may only align with y_k or y_d.

When the matching conditions are followed, one can tile the entire plane; Figure 3.3 shows a finite section of that tiling with matching rules illustrated.

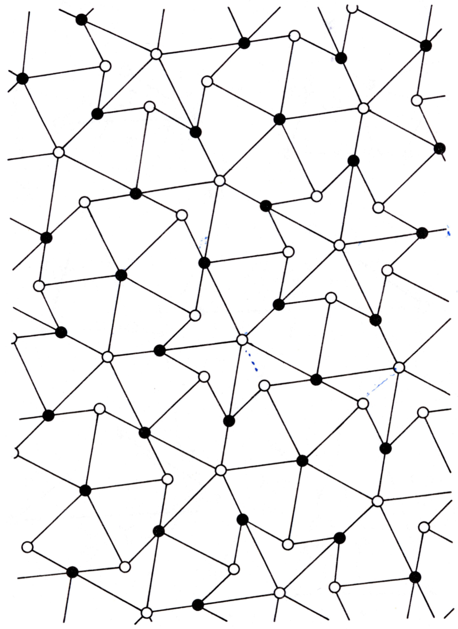


Figure 3.3 – Tiling of Penrose's kites and darts with matching rules shown

We see from the two shapes in Figure 3.4 another way to describe the matching conditions. The darker arches (lower arch on left image of kite, higher arch on right image of dart) and lighter arches (higher on left, lower on right) must align.

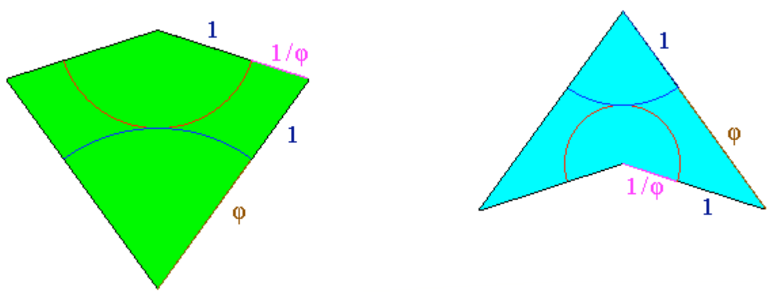


Figure 3.4 – Alternate matching conditions

In a Penrose tiling made using the curves as matching rules (Figure 3.5), we notice that many of the curved lines rejoin and close. In fact, at most only two curves in a tiling will remain unclosed [**3,** pp. 116].

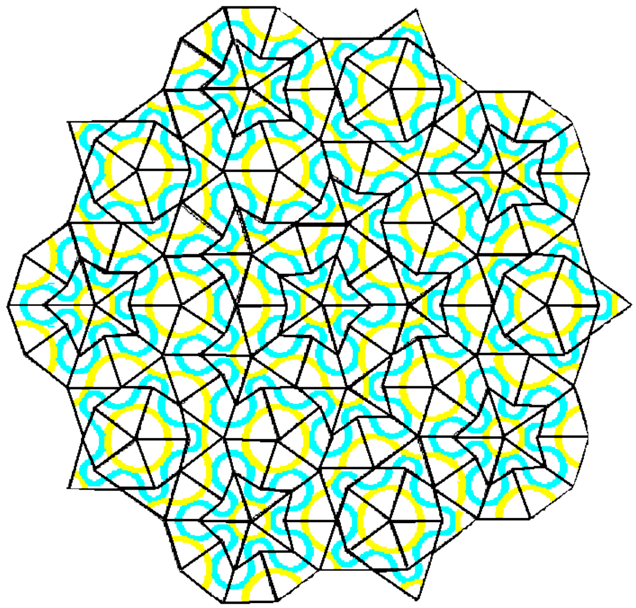


Figure 3.5

Conway and Penrose each independently proved that every curve that closes has fivefold symmetry and the tiles contained by the curves also exhibit fivefold symmetry [**3**, pp. 116].

3.2 VERTEX NEIGHBORHOOD

G&S defines a *vertex neighborhood* N(V) of a vertex V in a tiling $\mathscr{T}$ as a minimal patch of tiles that completely surrounds V. Conway gave the vertex neighborhoods their playful and fanciful names.

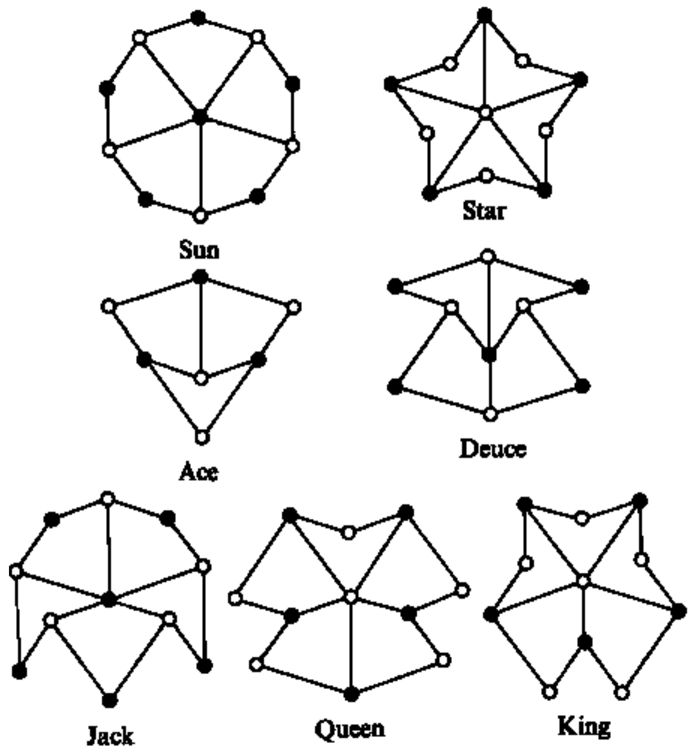


Figure 3.6 – The seven possible vertex neighborhoods

Statement **3.2.1**: *Every vertex in any Penrose tiling will have one of the vertex neighborhoods as seen in Figure 3.4.*

The statement can be shown simply by rearranging the kites and darts (obeying the matching conditions) creating different arrangements until all the possible combinations of tiles are exhausted.

Statement **3.2.2:** *Every point P in a Penrose tiling belongs to an ace.*

Let P be a random point in a Penrose tiling $\mathscr{T}$. Assume for our first case that P falls inside a dart. After carefully reviewing our matching conditions and trying different configurations, we see that in the concavity of the dart two kites are forced. This creates an ace. To convince ourselves of this, we look at the seven different vertex neighborhoods and see that the vertex in a dart's concavity only appears once as a neighborhood, and a dart is formed around it. Now assume that P falls inside a kite. Along the 'short' sides of a kite, there are three possible matchings. The kite will either have two darts on the two short sides, a kite and a dart, or a dart and a kite (refer to a deuce and jack to confirm this). In every situation, the kite has a dart on one of its short sides. Thus the kite is fit into the dart's concavity and this will force another kite in the concavity creating an ace.

If one begins a tiling with a sun, it is always possible to add tiles (following matching rules) so that fivefold symmetry is preserved. However, in preserving fivefold symmetry, one is forced to build the pattern shown in Figure 3.7. The tiling is uniquely determined to infinity [**3,** pp. 117-118].

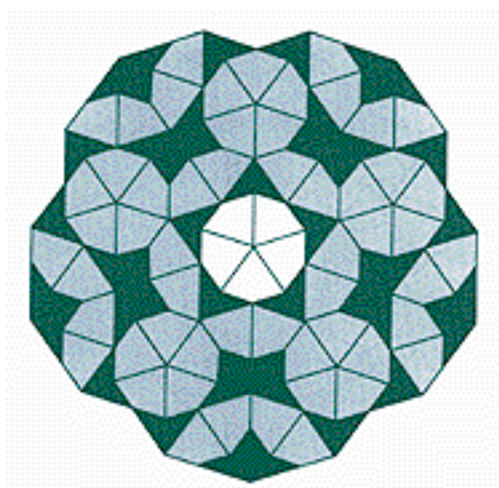


Figure 3.7

Similarly, the star (which forces the 10 kites around it) contains the same property.

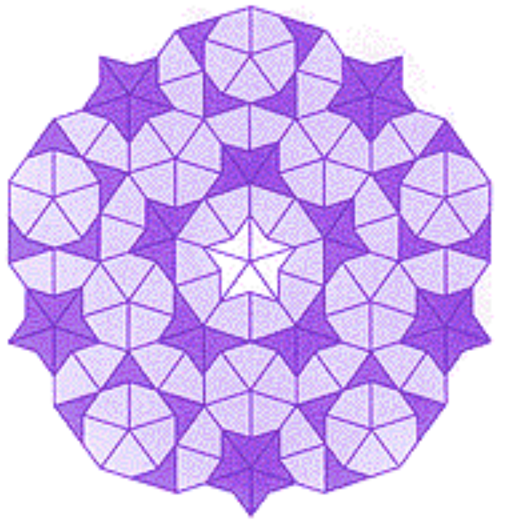


Figure 3.8

According to Gardner, the star and sun patterns are the only infinite Penrose tilings with perfect fivefold symmetry [**3,** pp. 117-118].

A surprising connection between these two universes emerges when we inflate one of them (a process mentioned in section 2.2 and to be defined in section 3.4). Inflation or deflation (the reverse process of inflation) of one results in the other.

3.3 COMPOSITION

Composition, or an 'inflation' property as Penrose speaks of it, is a process of taking the individual shapes of an infinite Penrose tiling and grouping them together following our matching rules [**6,** pp. 35]. Let us consider an infinite tiling (Figure 3.9) consisting of Penrose's kites and darts.

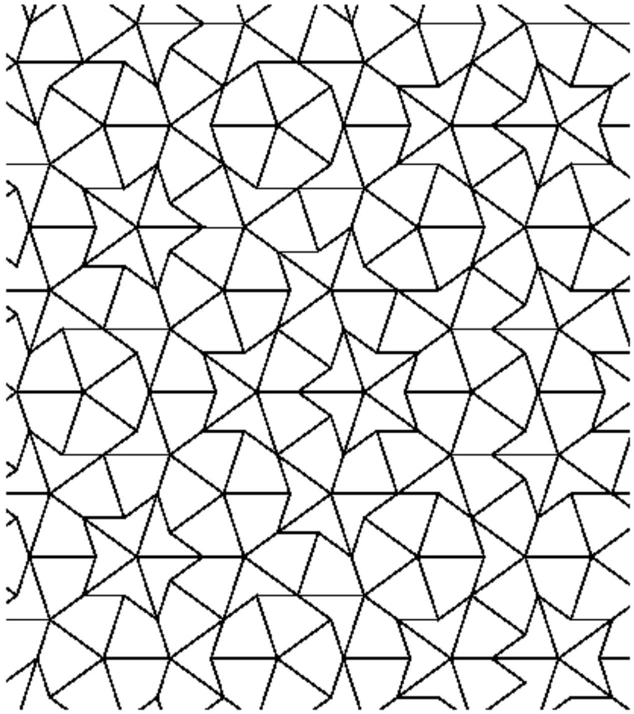
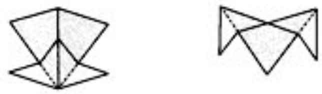


Figure 3.9 – A Penrose tiling of kites and darts

The process of composition of kites and darts involves bisecting the darts symmetrically. The infinite plane at this point is filled with kites and half-darts (which are Golden Gnomons). The Golden Gnomon is an isosceles triangle with sides of length 1, 1 and φ. It is then possible to join the half-darts and kites to create larger kites and darts. In the process of joining the tiles, not that after bisecting a dart, those two half-darts do not rejoin together to form the dart that was originally bisected.

The process of *composition*, as thoroughly defined by G&S, is “the process of taking unions of tiles so as to build up larger tiles which are basically the same shapes as those from which we started and whose edge modifications and labels specify a matching condition equivalent to the original one” [**4,** pp. 532].



(a) (b)

Figure 3.10 - Composition of Penrose's kites and darts

For Penrose's tiling of kites and darts, there is a unique method of composing following Figure 3.10.

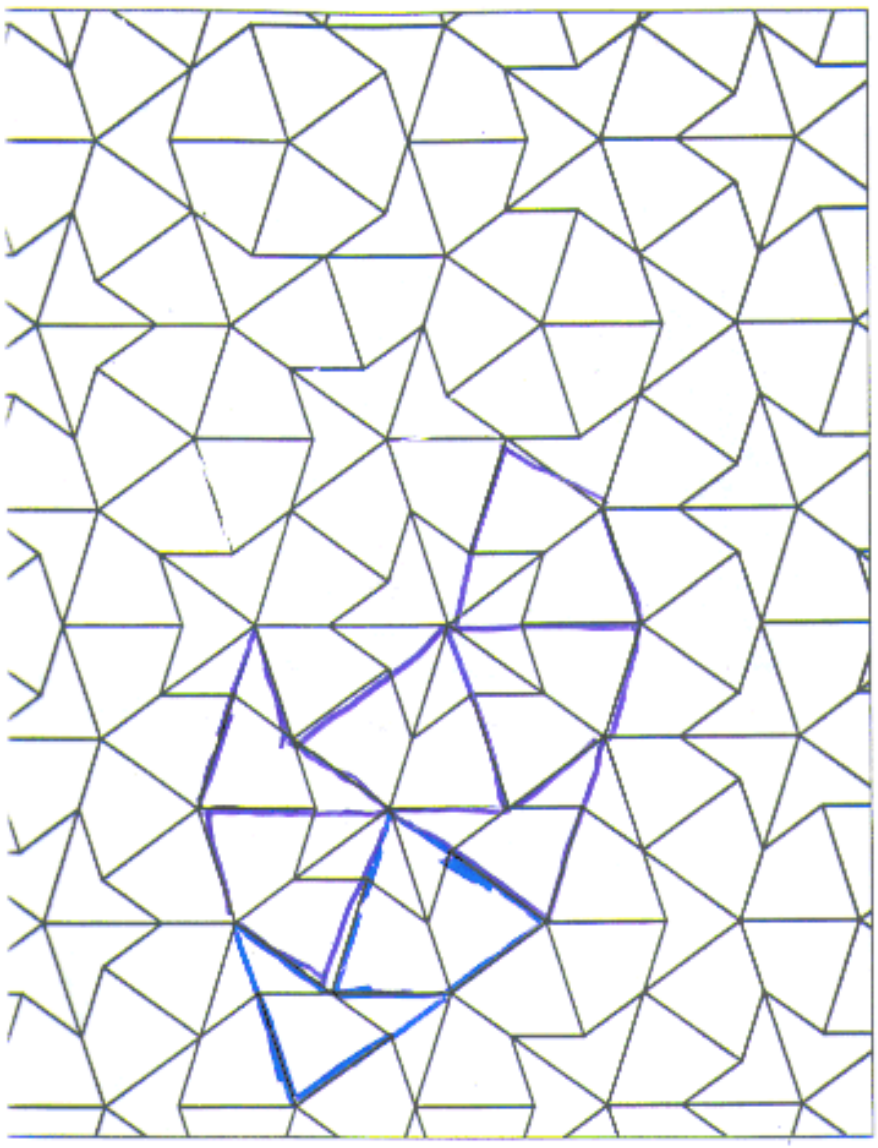


Figure 1. A Penrose "kite and dart" tiling.

Figure 3.11 – Tiling showing composition of Penrose's kites and darts

In the tiling in Figure 3.11, we see that this process has been applied to a few tiles. These 'new', or composed kites and darts (hand drawn with slightly darker edges) are slightly larger than the original kites and darts.

Statement **3.3.1**: *The length of the sides of the composed kites and darts is φ times as large as that of the original tiles.*

The composed kite is formed by two kites and two half darts. From Figure 3.11, we can see that the length of the long side of a composed kite is composed of one long side of an original kite and one short side of an original dart, thus it is $\varphi + 1$ which is φ^2. The short side of the composed kite is equal to a long side of an original dart, which is φ. Similarly, the long side of a composed dart is φ^2 and the short side is φ. We have thus shown that in the process of composition, we have increased the length of the sides of the kites and darts by a factor of φ.

We are working with an infinite tiling, so we remember that the size of the overall tiling has not changed in size (it remains infinite in size), only the individual tiles change size.

Statement **3.3.2**: *It is always possible to compose the tiles of a Penrose tiling and do so uniquely.*

This is to say that it is always possible in any tiling to bisect the darts symmetrically and collect two half-darts and one kite to make a new dart and then two half-darts and two kites make a new kite.

The example in Figure 3.11 is enough to convince us that this process is possible. To prove that it is always possible, let us address the following cases. After splitting the darts in half, we are left with a tiling with half-darts and kites. We must first prove that each of these half darts belongs to a ‘new’ tile after composing. Next, we must prove that each of the kites also belongs to ‘new’ tile after composing. This would prove that there are no gaps between our ‘new’ tiles after composing the tiling. Then, not only must we prove that the half darts and the original kites *belong* to a ‘new’ tile, we will prove that they *only* belong to *one* ‘new’ tile and we can always tell what that tile will be depending on the nearby tiles in our original tiling. This would exhaust the possibility that the ‘new’ tiles overlap and that our process is unique. If we succeed in proving both of these cases, we will have sufficiently showed that it is always possible to uniquely compose a Penrose tiling.

First let us examine some random half-dart in a tiling. Remember that this half dart is an isosceles triangle with two short sides and one long side. Well one of the short sides is the connection point to the other half dart and we know that when we regroup our shapes, the two half darts cannot rejoin. Thus we know that one of these short sides will be part of the outside of a new dart.

The point between the two short sides is the concavity vertex and thus the other short side will always have a kite adjacent to it. We must now examine the two possibilities of shapes that could be adjacent to the long side of the half dart. There will either be a kite or a dart. Consider that the shape is a dart.

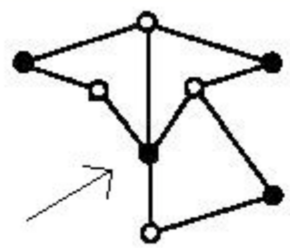
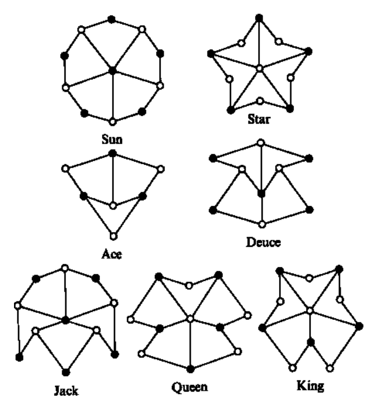


Figure 3.12

We see that we have a vertex where the short and long sides of the half dart meet. If we observe our seven different vertex neighborhoods (Figure 3.6 reproduced here for convenience), we see that the deuce is the only one where the vertex is shared by two darts in this fashion. This thus forces a kite as the remaining shape and produces precisely Figure 3.10a.



Now let us consider the placement of a kite along the long side of the half dart.

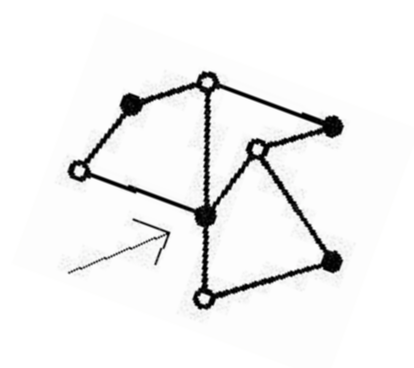


Figure 3.13

Again, we see that the only vertex neighborhood that has two kites and a dart in this manner is a jack. Thus we see that another dart is forced. The two darts and the kite that shares vertices with the darts' concavity create precisely Figure 3.10b.

Thus we have shown that every half dart exclusively belongs to a larger or composed kite or dart.

Now let us examine a kite. When considering the possible shapes that lie adjacent to a kite in a Penrose tiling, let us only concern ourselves with the short sides of the kite. From Figure 3.10 we see that adjacent the long side of a kite nothing is ever joined during composition. Thus we consider all vertex neighborhoods where the vertex is the intersection point between the two short sides of the kite. More specifically these are the deuce and the jack. In the case of the deuce, we see that, like before, the deuce is identical to Figure 3.10a, thus it makes a 'new' kite. In the case of the jack, we have already said that that kite will join with two half darts (recall that the darts in the picture of the deuce in Figure 3.6 are bisected during composition) and make a 'new' dart.

To recapitulate, we have shown that in any Penrose tiling, if we bisect the darts symmetrically every tile belongs to a larger kite or dart. Moreover, every tile belongs specifically to *only one* particular composed kite or dart. Thus it is always possible to *uniquely* compose the tiling making another Penrose tiling.

3.4 INFLATION

It is possible to talk about inflation (defined rigorously by G & S) in two ways. The first involves a single tile, be it either a kite or a dart, and the second involves an infinite Penrose tiling. Let us begin by examining the process of inflating a single tile.

Inflation is the process of enlarging the length of the tiles' edges by φ and then decomposing the tiles according to specific rules, resulting in tiles of the original size [**4,** pp. 534]. The process of decomposition of kites and darts is done according to Figure 3.14.

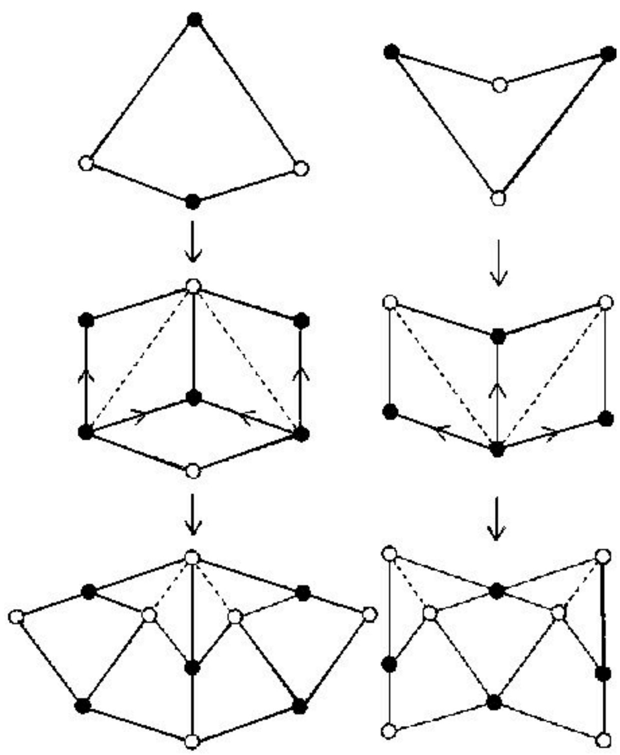


Figure 3.14 – Decomposition of kite and dart

The procedure of decomposing is simply the reverse of composition, a process that requires taking the tile and breaking it up into smaller tiles (which are either kites or darts on a smaller scale). Note that when decomposing following the steps shown in

Figure 3.14, the decomposed sections (shown in the bottom step in Figure 3.14 composed of the smaller kites and darts) are slightly wider than the originals in the sense that they have had sections of tiles added to the sides. Along the 'long' side of the original kite and dart, the shapes are expanded slightly. More specifically, along this 'long' side we add on a kite and half dart (see the last step in Figure 3.14). We note that after the decomposition of the kite or dart, the shading of any vertex (indicating the matching conditions) belonging to any of the original shapes inverts (switch from filled in to empty or vice versa) to be consistent with the new smaller kites and darts.

Note that similarly to composition, the resulting kites and darts have been altered by a factor of φ – in this case reduced. That is to say that after decomposing a single time the 'long' side of the kites and darts is $\varphi / \varphi = 1$ and the 'short' side is $1 / \varphi$. The order in which one enlarges and decomposes when inflating a tile is irrelevant. After (or before) decomposing the tile(s), one must enlarge the tile(s) by a factor of φ. Thus the smaller tiles resulting from the decomposition enlarge to become the size of the original tiles.

When speaking of decomposing a second time, that is say decomposing a finite section of tiles, we see that this decomposition may become a problem due to adjacent shapes both expanding and having the overlapped area possibly not match up. However, when decomposing any adjacent tiles, the additional kite and half dart always overlap perfectly and hence avoid any problems. Note that there are only three possible matching along the 'long' side of any kite and dart. It is possible (a) to have two kites, both 'facing' the same direction, (b) to have two darts both 'facing' the same direction, and (c) to have a kite and dart match up following matching conditions along their 'long' sides. Note in any of these three cases, when decomposing, the small addition along the 'long' sides always overlaps. Thus we have shown that given any correctly tiled Penrose tiling (matching conditions followed) we may decompose the tiles.

We have thus shown that composition and decomposition are unique inverses of each other.

We have considered taking a single tile and inflating it. Then, with a larger section (still finite), we have concluded that we can inflate it again. Since we proved that it is always possible to inflate a finite section, we conclude that inflating any finite section (including even a single tile) an infinite number of times we arrive at a Penrose tiling. This is sufficient to prove that the entire plane can be tiled.

3.5 APERIODIC SET

Statement **3.5.1**: *The set of Penrose's kite and dart is an aperiodic set* [**4**, pp. 534, 542].

Assume that we have a tiling of Penrose's kites and darts. Thus we know that the process of composition is unique; in other words there is are very specific rules one follows in order to compose a tiling, and it can only be done in one way. Suppose, for contradiction, that the symmetry group of the tiling contains one translation t. This is similar to saying that a tiling is periodic (except that the symmetry group of a periodic tiling contains *two* non-parallel translations). One translation will suffice if we succeed in demonstrating this contradiction. Thus for some random point P in a tile, we have t(P) in a separate identical tile with an identical neighborhood surrounding it. The distance between the two we will call d. Since composition is unique, after composing, P and t(P) will still be in identical and separate tiles. We know this because "uniqueness of composition implies that t must also be a symmetry of the k-composition tiling" [**4**, pp. 524], where the k-composition tiling can be simply understood in this context as the composition tiling. As already covered, we know that with every composition, the lengths of the sides of the tiles enlarge by a factor of φ. Well, if we compose the tiling enough times, we will be able to make the shorter side of one of the tiles greater than d. Thus we compose a sufficient number of times so that the translation vector would lie entirely in a single tile. This is a contradiction, and thus no translation is a symmetry of our tiling of Penrose's kites and darts and so the tiling is non-periodic.

Statement **3.5.2**: *Penrose's set of six prototiles and Penrose's rhombs are both aperiodic sets* [**4**, pp. 534, 542].

Since the set of Penrose's six and the set of Penrose's rhombs (to be discussed in Chapter 4) both have unique compositions, the proof of this statement follows directly from the proof of Statement 3.5.1.

3.6 CARTWHEEL

In order to create a Cartwheel, as described by G&S, one starts with an ace, consisting of two kites and a dart. We will call this $\mathcal{C}_0$.

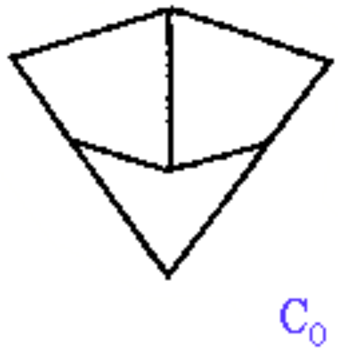
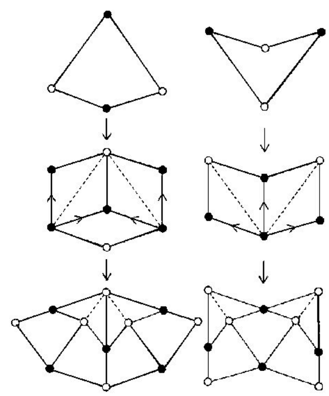


Figure 3.15 - The Ace (C_0)

Then, following the steps delineated in Figure 3.14 (reproduced here for convenience), we inflate the ace (as defined by G&S). Recall that this is a process of decomposition and then enlargement of the tiles. So, when we inflate our ace, we get $\mathcal{C}_1$ as shown in Figure 3.16.



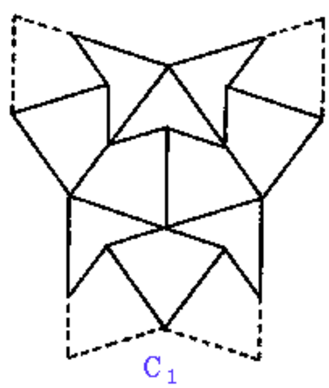


Figure 3.16 – C_1

In general, we can say that an inflation of $\mathcal{C}_n$ gives us $\mathcal{C}_{n+1}$. Figures 3.17 - 3.19 show the next three inflations, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$.

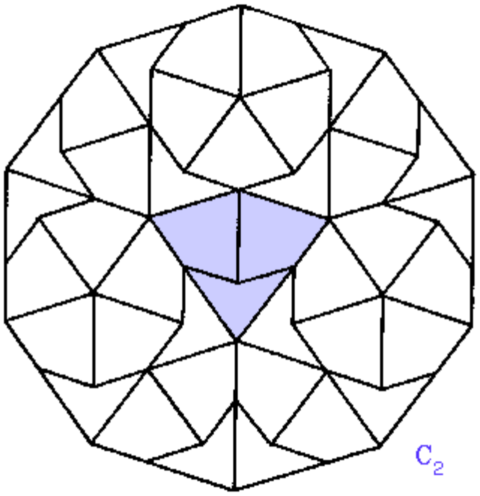


Figure 3.17 – The first order cartwheel (C_2)

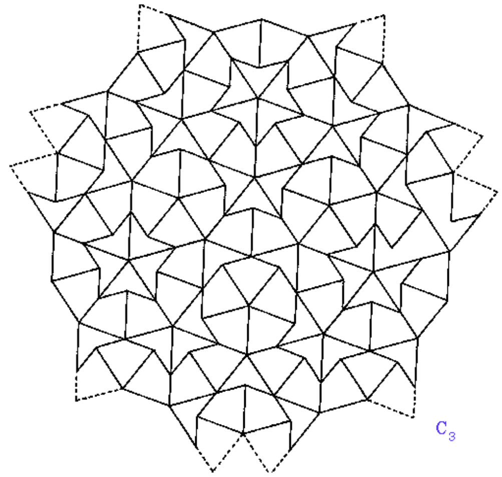


Figure 3.18 – C_3

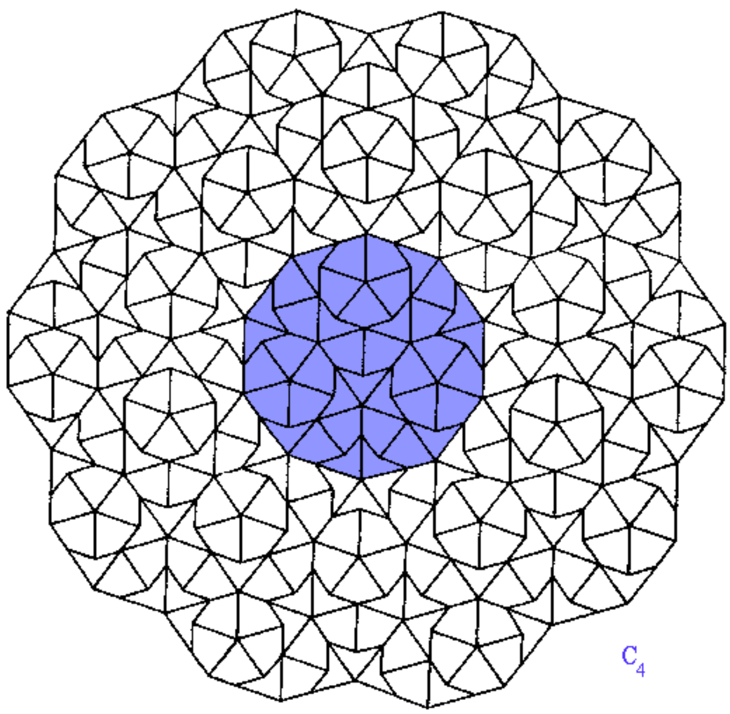


Figure 3.19 – The second order cartwheel (C_4)

The shaded areas of $\mathcal{C}_n$, where n is odd, we have chosen to trim because the resulting shapes, when inflated, will make even-numbered patches that have a special property. $\mathcal{C}_n$, where n is even, has the property that the union of all the tiles is a polygon with the symmetry group of a regular pentagon [**4**, pp. 561]. We define all of these even-numbered patches as *cartwheels*. $\mathcal{C}_{2n}$ is called a *cartwheel of order n.*

Statement **3.6.1**: *In every tiling by Penrose kites and darts, every tile* $\mathcal{T}$ *lies in a cartwheel* $\mathcal{C}_{2n}$ *of every order* $n \geq 1$ [**4,** pp. 562].

To show this, let T be a tile in any tiling $\mathcal{T}$. As already shown, we know that T belongs in an ace. Thus we know that T belongs to $\mathcal{C}_0$. Well, as G&S states, it is apparent from Figures 3.16-3.19 that for each n>1, the cartwheel $\mathcal{C}_{2n}$ contains its predecessor $\mathcal{C}_{2(n-1)}$. Also, we can see that it does so concentrically. Thus if T belongs in $\mathcal{C}_0$, then T lies in every $\mathcal{C}_{2n}$ for n a positive integer.

Statement **3.6.2 (part a)**: *In every tiling by Penrose kites and darts, every one of the seven kinds of vertex neighborhoods necessarily occurs…*

We observe from Figure 3.17 that six different kinds of vertex neighborhoods (the seventh being the star, which is missing) appear in the first-order cartwheel $\mathcal{C}_2$. We then see that the star emerges in the second-order cartwheel $\mathcal{C}_4$. Knowing that every tile belongs to a second-order cartwheel and thus a second-order cartwheel exists in any Penrose Tiling, we can conclude that statement (above) is proved.

Statement **3.6.2 (part b)**:…*and does so infinitely often* [**4,** pp. 562].

Since there are infinite tiles in a Penrose Tiling, it clearly follows that there are an infinite number of aces as well (an ace is composed of three tiles). Thus we also know that there are infinite second-order cartwheels (each containing an ace at its center). Thus we have proved the above statement.

Statement **3.6.3**: *Every patch $\mathscr{A}$ of tiles in a tiling $\mathscr{T}$ by Penrose kites and darts is congruent to infinitely many patches in every tiling by the same prototiles* [**4,** pp. 562].

Let $\mathscr{T}$ be a Penrose tiling and let $\mathscr{A}$ be any finite section inside that tiling. Let the diameter $\mathscr{A}$ be d($\mathscr{A}$). Now compose $\mathscr{T}$ n times. Thus the length between vertices is at smallest φ^n (remember that the shortest length between vertices in our tiling $\mathscr{T}$ is 1 and composing increases the distance between vertices by a factor of φ). After composing $\mathscr{T}$ n times, $\mathscr{A}$ contains at most one vertex V. It is possible that $\mathscr{A}$ does not contain a vertex, and in that case we can extend $\mathscr{A}$ slightly to include one without loss of generality. So now $\mathscr{A}$ contains exactly one vertex V. That vertex has one of our seven vertex neighborhoods, N(V), surrounding it. We know that the union of tiles surrounding N(V) contains $\mathscr{A}$.

Next, take some other tiling $\mathscr{T}'$ and compose it n times. In that composed tiling, there are infinite examples of every vertex neighborhood (Statement 3.6.2b). More specifically, there are infinite examples of vertex neighborhoods congruent to our N(V). We may choose one of those and call it N(V') so that the vertex belonging to that neighborhood is V'. Now decompose N(V') n times. Remember that in section 3.4 we proved that decomposition is unique and the inverse process to composition; and so decomposing N(V') n times will give us a section congruent to the patch we would obtain by decomposing N(V) n times. Given that N(V) contains $\mathscr{A}$, we conclude that N(V') also contains $\mathscr{A}$. Thus the statement is proved.

Since any finite patch has an equal (infinitely many equals) in any other tiling and since one cannot physically produce an entire tiling due to it being infinite, it is impossible to say that two tilings are equal by comparing sections of them. In section 3.8 we will discuss how to differentiate between tilings.

3.7 RATIO OF KITES TO DARTS

Statement **3.7.1**: *There are φ times as many kites as darts in any finite region of the plane that is exactly covered by copies of the tiles ϕ^n K and ϕ^n D where pK and pD are the results of enlarging the kites and darts (K and D respectively) by a factor of p* [**4**, pp. 563 – 564].

Let K denote a kite and D denote a dart and pK and pD are the results of enlarging the kites and darts by a factor of p. As explained in section 3.4, we know that when we inflate a kite (enlarge it by φ) we get two kites and a dart. And when we inflate a dart, we get a kite and a dart. We denote this by:

$\varphi K = 2K + D$

$\varphi D = K + D$

Now let us continue and inflate again (enlarge by φ):

$\varphi^2 K = \varphi\,(\varphi K) = \varphi\,(2K + D) = \varphi\,(2K) + \varphi\,(D) = 2(\varphi K) + (K+D) = 5K + 3D$,

$\varphi^2 D = \varphi\,(\varphi D) = \varphi\,(K + D) = \varphi\,(K) + \varphi\,(D) = 3K + 2D$.

And once more it follows:

$\varphi^3 K = \varphi(\varphi^2 K) = \varphi(5K + 3D) = 5(\varphi K) + 3(\varphi D) = 13K + 8D$,

$\varphi^3 D = \varphi(\varphi^2 D) = \varphi(3K + 2D) = 3(\varphi K) + 2(\varphi D) = 8K + 5D$.

Now we can conclude that the general case will follow:

$\varphi^n K = f_{2n+1}K + f_{2n}D$, (*)

$\varphi^n D = f_{2n}K + f_{2n-1}D$,

where f_n is the *n*th term in the Fibonacci Sequence.

Let S be any region of the plane that is *exactly* covered by copies of the tiles ϕ^n K and ϕ^n D. Let r(S) be the ratio of kites to darts in our Penrose tiling. Well, if we were to take from S all the copies of the tiles ϕ^n K and collect them together as S', it follows from * that we would have precisely f_{2n+1} kites and f_{2n} darts. So r(S')= f_{2n+1} / f_{2n} . Also, if S" were a collection of all copies of the tiles ϕ^n D, we would have r(S")= f_{2n} / f_{2n-1} from *. Clearly r(S') < r(S"); this follows from an conclusion made in section 1.4. Now, as we look at the entire collection S, we can conclude that r(S) lies between the two ratios, that is that $f_{2n} / f_{2n-1} \leq r(S) \leq f_{2n+1} / f_{2n}$. So when n is large, r(S) is closely approximated by φ.

Statement **3.7.2**: *There are φ times as many kites as darts in every infinite Penrose tiling* [**4**, pp. 564].

Let R be any convex region of the plane. Let A be the area of R and let P be the perimeter.

Let us assume that this plane is covered by a Penrose tiling inflated n times. That is, it is tiled by copies of the tiles ϕ^n K and ϕ^n D (kites and darts).

We conclude that there exists S a subregion of R that is covered exactly by copies of the tiles φ^n K and φ^n D. Let Q be the region lying inside of R that is left by subtracting S; that is, R = Q ∪ S and Q ∩ S = ∅. It helps to think of Q as a 'band' that wraps around S. Let X be the area of S and Y be the area of Q. Note that the sum of X and Y is equal to A. We can assume that is it possible to construct an S in such a way that there are no complete tiles φ^n K or φ^n D in Q.

The length of the long side of φ^n K and φ^n D tiles is φ^{n+1}. Recall from section 3.4 that with every inflation, the sides of the kites and darts increase by a factor of φ. After inflating n times, the sides will be increased by φ^n. The length of the long side of both the kite and the dart is φ and after inflating n times, the length is φ^{n+1}.

Since there are no copies of φ^n K or φ^n D tiles in Q, we can conclude that the distance from the perimeter of R to the perimeter of S is always less than φ^{n+1}. Thus we can now conclude that the 'band' Q, if it were unraveled and stretched out, would fit in a box P long and φ^{n+1} high. Thus we have

$Y < (\varphi^{n+1})\ P.$

Negating each side, it follows that

$-\ Y > -\ (\varphi^{n+1})\ P.$

and then by adding A to each side we have

$A - Y > A - (\varphi^{n+1})\ P.$

Remember that X + Y = A, and so

$X > A - (\varphi^{n+1})\ P.$

Dividing everything by A we have

$X/A > (A - (\varphi^{n+1})\ P) / A = 1 - (\varphi^{n+1})\ P/A.$

It is good here to note that X/A is always less than 1 because S lies completely inside of R, so $X < A$. We'd like to now examine what happens as P/A gets smaller.

As P/A gets smaller, the convex region R gets larger in all directions, that is to say R tends toward the entire plane. So, as P/A approaches 0, then $(\varphi^{n+1})P/A$ approaches 0, and $(1 - (\varphi^{n+1})P/A)$ approaches 1 from the left. But $(1 - (\varphi^{n+1})P/A)$ is less that Y/A which is less than 1. So as P/A gets smaller, it pushes X/A closer and closer to 1. Which means X approaches A. That is to say that the region S tends to R.

So, let us consider a sequence of regions R where each is increasing in size. Thus P/A gets smaller, and so S gets closer to approximating our regions. Therefore the ratio of kites to darts in these regions tends to φ (as it does in our S – Statement 3.7.1).

An alternate proof to the Penrose's set of kites and darts being an aperiodic set (Statement 3.5.1) closely follows. As previously discussed and proved, the ratio r(s) of kites to darts in an infinite Penrose tiling is φ, the golden ratio. Note that $\varphi = ½ (1 + \sqrt{5})$ which is irrational. Penrose calls this relative density. It is impossible for a periodic tiling to have an irrational relative density thus proving that Penrose tilings are aperiodic [**6,** pp. 36].

3.8 UNCOUNTABLY MANY TILINGS

An equivalent version of composition (used in Robinson's analysis of Penrose's kites and darts) is given to us by G & S in Figure 3.20.

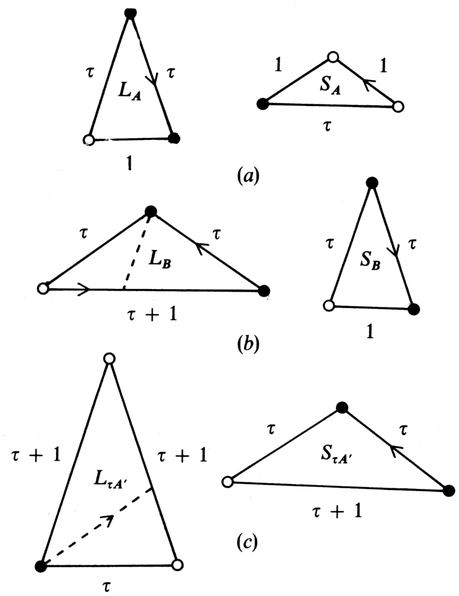


Figure 3.20 – Alternate version of composition using triangles

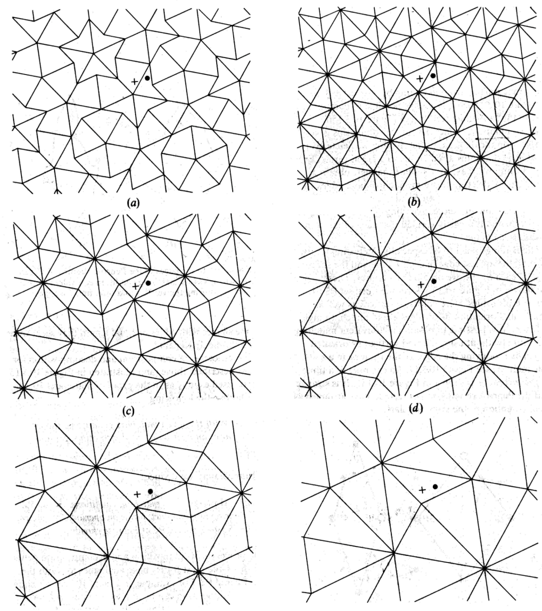


Figure 3.21 – Tilings corresponding to the different compositions

To apply this method of composition, one would first take a Penrose tiling of kites and darts as in Figure 3.21a. Then, by bisecting all the shapes symmetrically down the center, our Penrose tiling becomes a tiling of triangles (Figure 3.21b) – this is referred to as an A-tiling. In this tiling, we have the Golden Triangles (the larger triangle) left from bisecting the kites, called L_A in Figure 3.20a; we also have Golden Gnomons (the smaller triangle) called S_A in Figure 3.20a. One would then combine one L_A and one S_A to form an L_B (B-tile), and some L_A triangles remain unchanged and become a B-tile S_B. As seen in Figure 3.21c, after composing the A-tiling we get a B-tilings. For clarity, Figure 3.22 shows composition of triangles.

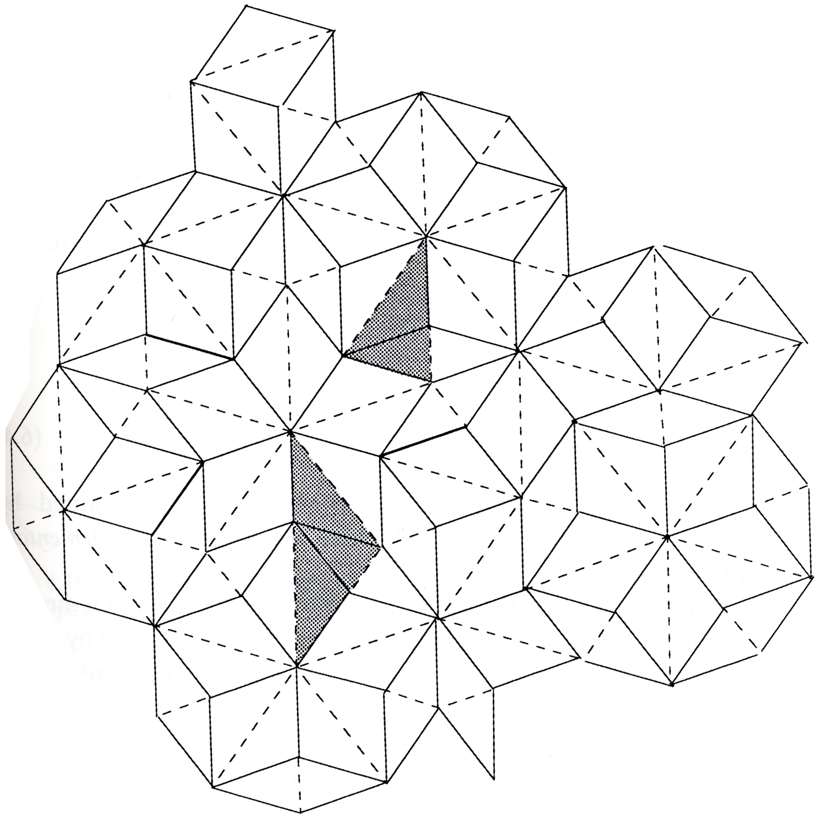


Figure 3.22 – Composition of triangles

Notice that in this inflation step, as we saw before with our kites and darts, the length of the sides increase by a factor of φ. Also notice in Figure 3.20 that the matching conditions represented by the filled in or open circles are kept. Finally, by inflating our B-tiles L_B and S_B we arrive at $L_{\varphi A'}$ (composed of one L_B and one S_B) and $S_{\varphi A'}$ (composed of one L_B). G&S explains that the prime (') indicates that the circles at the vertices representing the matching conditions have been interchanged.

G&S uses this system to describe to us a method of labeling each and every tiling. First let us begin with a random point P that lies in a tiling $\mathscr{T}_i$ (made of triangles - or half kites and darts) where i= A, B, φA', φB', φ^2A, … and where $\mathscr{T}_A$ corresponds to the A-tiling, $\mathscr{T}_B$ corresponds to the B-tiling and so on. Let us assign to every tiling an infinite sequence of integers (x_0, x_1, x_2, …) where $x_0 = 0$ if P is in the large A-tile of $\mathscr{T}_A$

(represented on the left hand side of Figure 3.17) and $x_0 = 1$ if P is in the small tile of $\mathscr{T}_A$. Then let $x_1 = 0$ or 1 according as P lies in a large or small B-tile of $\mathscr{T}_B$ and so on. We will call this an *index sequence*. Each composition of the tiling gives us a new term in the sequence. Since a tiling can be composed an infinite number of times, it makes sense that the index sequence is infinitely long.

Remember that each large tile is composed of a large and small tile of the previous tiling (Figure 3.20), and each small tile is simply a large tile of the previous tiling. Thus if P lies in a large tile (and the corresponding value in the sequence is a 0), after inflating once P may either be in a large tile or a small tile. Thus we conclude that either a 0 or a 1 may follow a 0 in the sequence. However, if P lies in a small tile (corresponding to a 1) and the tiling is inflated, certainly P will lie in a large tile in that inflation. Thus, in a sequence representing a tiling there are no two consecutive 1's and thus every 1 is followed by a 0.

Until now, we have seen that given a tiling, it is possible to create its index sequence. We must now ask if given any index sequence, there is a tiling that can be created.

Statement **3.8.1**: *Given any infinite sequence of 0's and 1's (with no two consecutive 1's), we can construct a properly matched tiling* [**4**, pp. 568].

It will not be too hard to convince ourselves of this. Given a sequence, we start with either a small or large tile depending on the first number of the sequence. We know that our point P to which the index sequence belongs exists inside that first tile. Next we inflate; a large tile would be split and become a large and a small, and a small would become a large tile. For this statement to be true, our point P must lie in the appropriate inflated tile that refers to the second number in the sequence. If any number is a 0 (large tile), then the next number may legally be a 0 or a 1 (either large or small). However both options are available after we inflate. If the number is a 1, then the following will necessarily be a 0 and again we are all set because the shape we have after inflation is necessarily large. This argument certainly works for the first tile. Thus we conclude by induction that it is always possible to have the point P lie in the tile corresponding to the next number in the sequence. And so the statement is proved.

Statement **3.8.2**: *Given two points P_1 and P_2 in the same tiling, the sequences that represent their tiling differ in a finite number of places* [**4**, pp. 568].

We've already established that since the inflation process is unique, that is to say that it follows a set of rules, a tiling will always inflate in the same way. Now assume that we have two points P_1 and P_2 in the same tiling. Let the distance between the two points be d. Now let us inflate our tiling y times so that both P_1 and P_2 lie in the same tile (either large or small). This is always possible because the tiles increase in size (by a factor of φ) with every inflation and one can inflate an infinite number of times. Now, after y (finite) terms P_1 and P_2 will lie in the same tile. After that point, the sequences of P_1 and P_2 the sequences will be identical.

Statement **3.8.3**: *The index sequences of two points P_1 and P_2 chosen from two different tilings differ in an infinite number of terms* [**4**, pp. 568].

This can be easily proven by contradiction following our previous conclusion.

Statement **3.8.4**: *There exists an uncountable infinity of distinct tilings by Penrose kites and darts* [**4**, pp. 569].

Let us choose a random point P from any Penrose tiling and assign the index sequence of our point P to that tiling. And we know that the sequence for every tiling will be a different infinite list of 0's and 1's (with no two consecutive 1's). G&S reminds us that there exists an uncountable infinity of sequences using the digits 0 and 1 with no consecutive terms equal to 1. Thus we can conclude that there exists uncountably many Penrose tilings. To find out more on this topic, the reader is invited to read G & S section 10.5.

Due to the specific composition process that also exists for Penrose's set of six and his rhombs, they also hold the property that there are uncountably infinite different tilings. This proof follows very closely from the proof of Statement 3.8.4.

3.9 RECOMPOSITION

Up until this point, we have seen so many similarities between Penrose's aperiodic set of six tiles and his kite and dart set. It is natural to wonder if the tilings of the two sets are related in any way. To answer this, we first look at a definition given to us by G&S: Given two sets of prototiles, let us say that $\mathscr{T}_1$ is *any* tiling of the first set. If it is possible to mark the first set in such a way that in $\mathscr{T}_1$ the marks align with the edges and vertices of *some* tiling $\mathscr{T}_2$ (admitted by the second set), then we say that $\mathscr{T}_2$ is obtained from $\mathscr{T}_1$ (or that second set of prototiles is obtained by the first) by *recomposition* [**4,** pp. 545].

Now, we return to our interest to the relationship between Penrose's six and the kites and darts. It turns out that the two sets of prototiles can be obtained by recomposition. We look at the following figure for the appropriate markings.

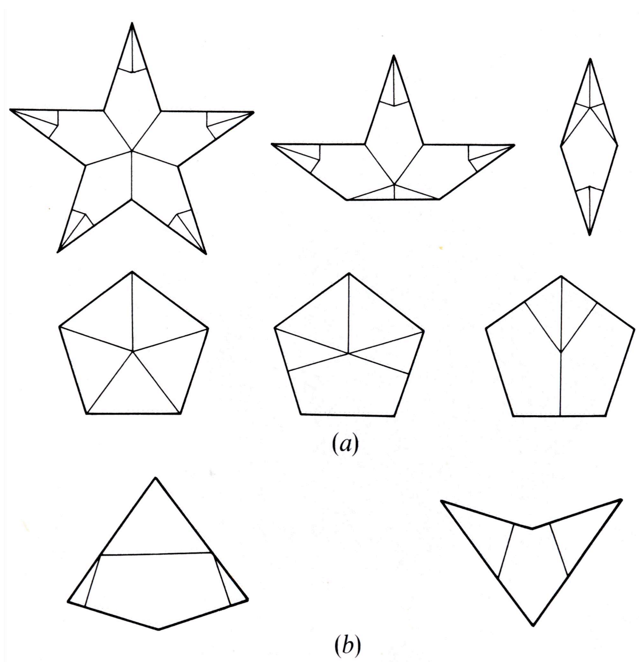


Figure 3.23 – Markings that show that Penrose's six and kite and dart six are recompositions of each other

Penrose's six can be obtained from kites and darts (Figure 3.23a) or vise versa (Figure 3.23b). In order to show how to recompose a tiling, let us shade in between the markings on our kite and dart in Figure 3.23b.

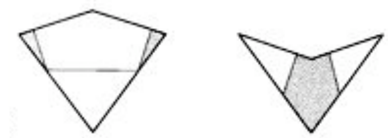


Figure 3.24 – Shaded markings showing instructions for recomposition

On the left side of Figure 3.25 we have a tiling of kites and darts. Moving from left to right of the Figure we notice that the tiles near the center have been given the markings and shading from Figure 3.24. We see that after applying the markings, a tiling of one set can become a tiling of the other.

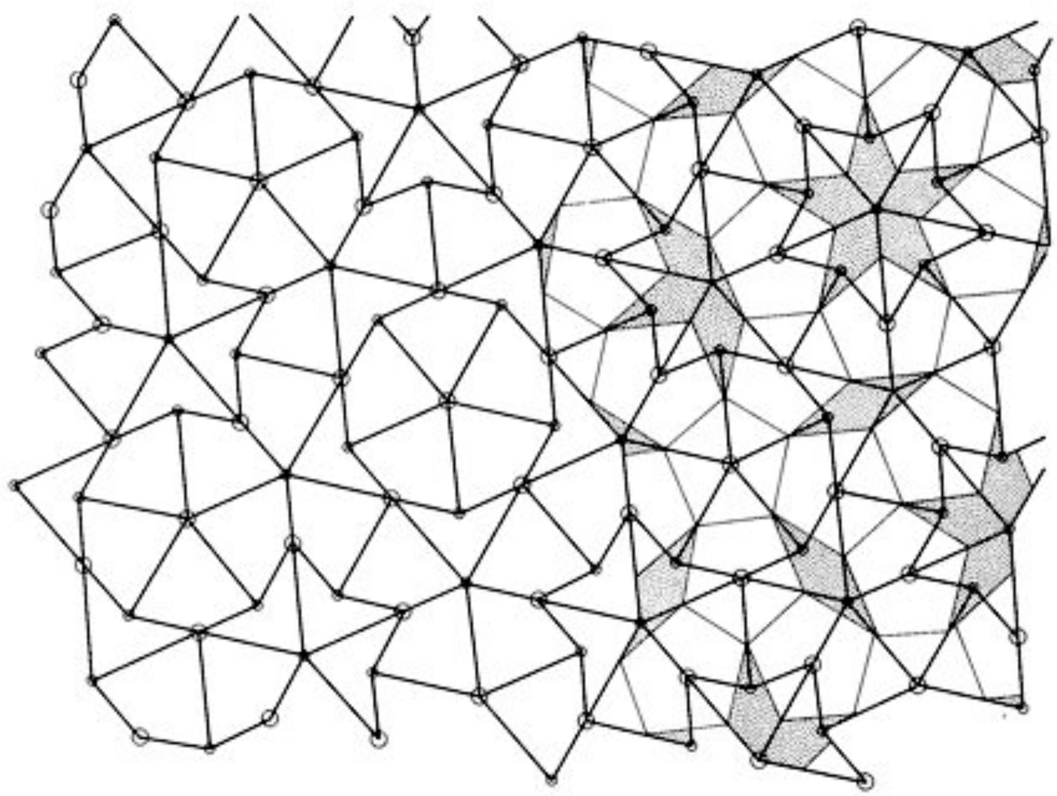


Figure 3.25 – A tiling illustrating recomposition

We will see in Section 4.1 that the kite and dart set and Penrose's rhombs are also recompositions of each other.

3.10 PENROSE'S CHICKENS

There are many methods of portraying the matching conditions needed to force a set of prototiles to tile aperiodically. Another is to curve the sides of the kite and dart as shown:

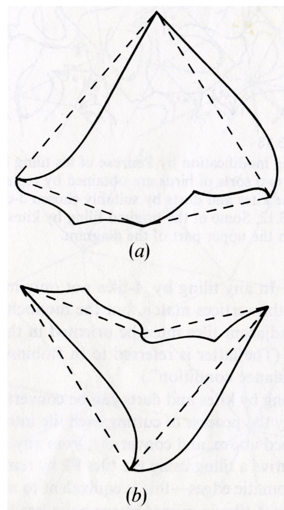


Figure 3.26

Many shapes have been drawn into these modified prototiles as to create a 'jigsaw' quality. One example is to turn the kite and dart into a fat and thin bird as shown [**6,** pp. 36].

Figure 3.27

These birds, strangely enough, can also be decomposed as shown below.

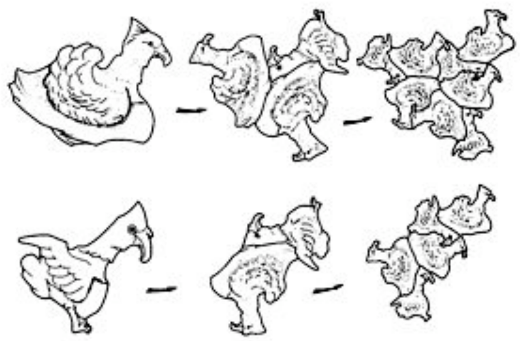


Figure 3.24

Penrose's alternative offered is modeled after the work of M.C. Escher [**4,** pp. 537]. His bird of choice was the chicken and the tiling that followed is nothing short of entertaining.

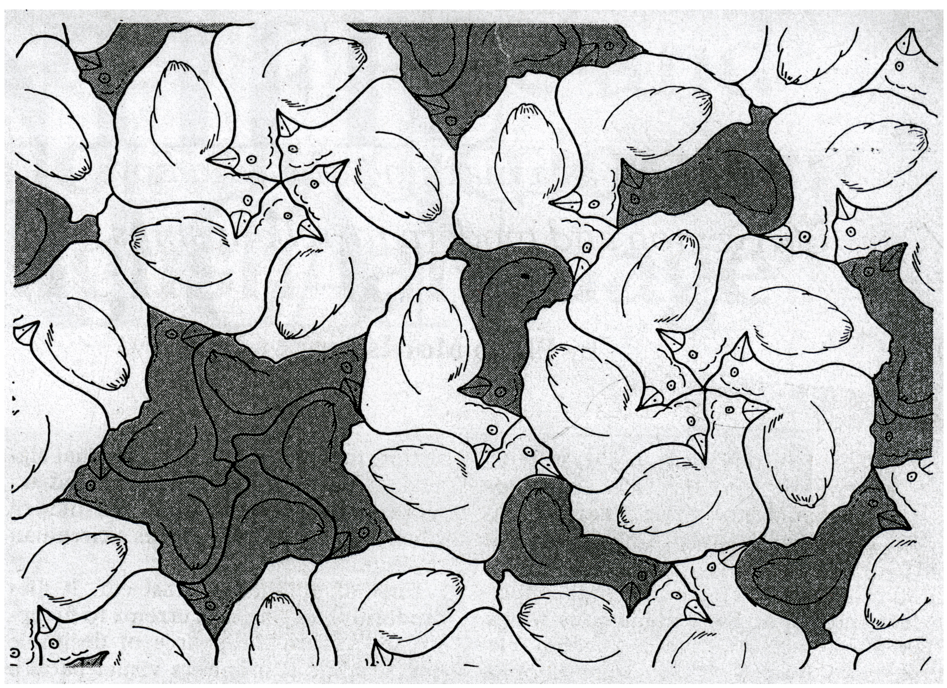


Figure 3.29

In Figure 3.29 we see that the kites are represented with light chickens and the darts are dark chickens. We can see more specifically the process of converting kites and darts to chickens in the following image.

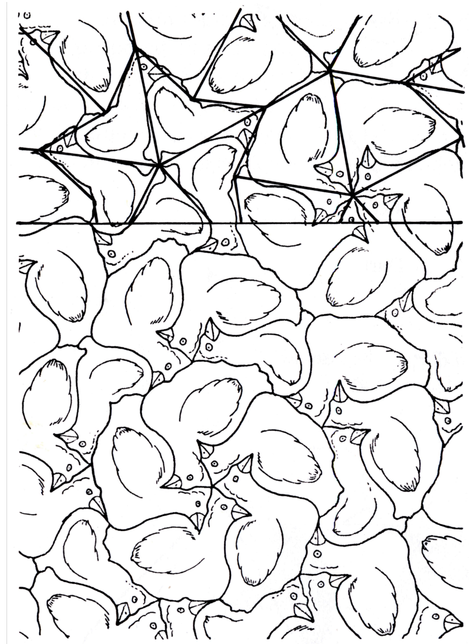


Figure 3.30

Chapter 4

Rhombs and Quasicrystals

4.1 PENROSE'S RHOMBS

The third of three sets of aperiodic tiles is Penrose's rhombs. One is an obtuse, or fat, rhombus and the other is acute, or thin. They too can be found in the Pythagonean pentagram (Figure 4.1), or derived directly from the kite and dart (as shown in section 1.3).

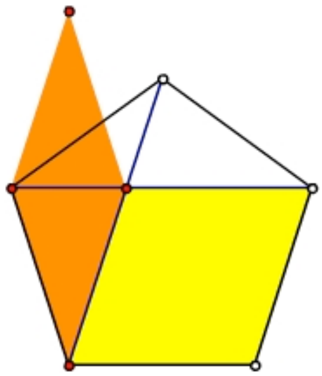


Figure 4.1 – Penrose’s rhombs come from pentagram

The rhombs, just like the kite and dart set, have matching conditions.

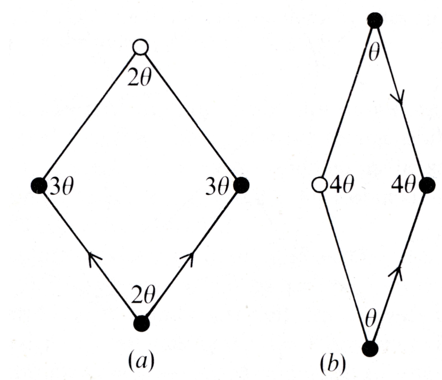


Figure 4.2 – Matching rules for Penrose’s rhombs

These conditions are a little more specific since all of the edges are equal in length. According to the Figure 4.2, edges containing a single arrow may only match up with other edges that have a single arrow; the arrows must also be pointing in the same direction. The edges with no arrow must also match up and like before the filled and open circles (at the vertices) must align.

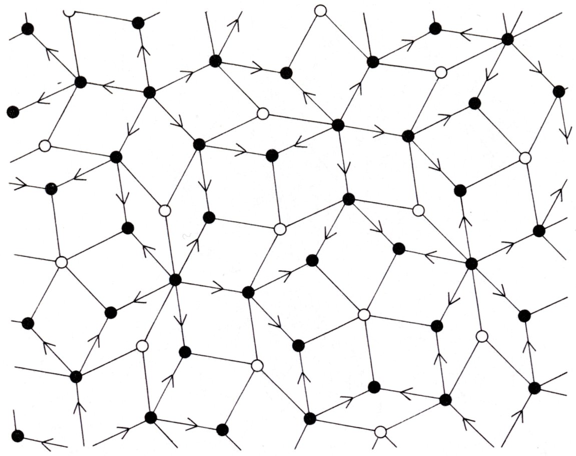


Figure 4.3 – Tiling of Penrose's rhombs showing matching conditions

Another way in which the rhombs parallel the kite and dart set is that in an infinite tiling, the number of obtuse rhombs will be approximately φ times as many as the acute rhombs. This proof follows directly from that done for the kite and dart set (Statement 3.7.2).

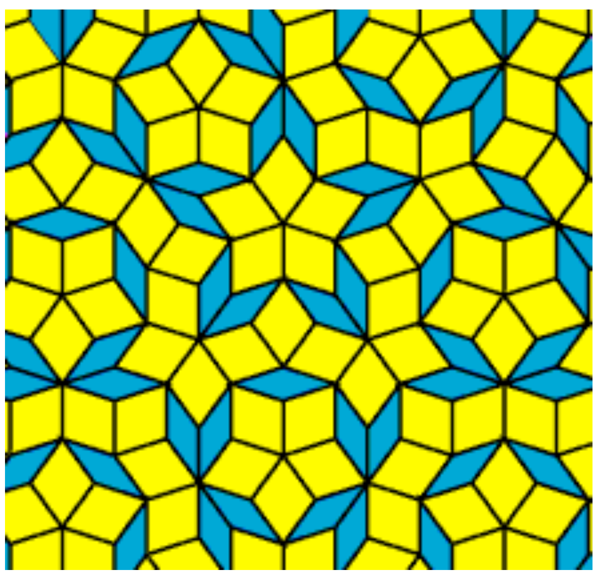
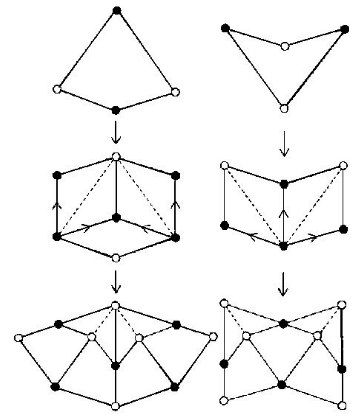


Figure 4.4 – Tiling of Penrose's rhombs without matching conditions shown

These rhombs tile the plane only aperiodically (Statement 3.5.2). To conclude this we must first show that the set of rhombs compose uniquely. This is easily shown in an analogous way to our argument of kites and darts.

Figure 3.14 (reproduced here for convenience) shows us the specific rules on how to decompose a kite and dart. Our rhombs also have specific matching rules; note that the middle step in Figure 3.14 contains Penrose's rhombs and maintains their matching rules. If we were to continue this diagram down one step we would have the rhombs' unique decomposition.



As we have already discussed, composition is the inverse operation of composition and it also applies to Penrose's rhombs.

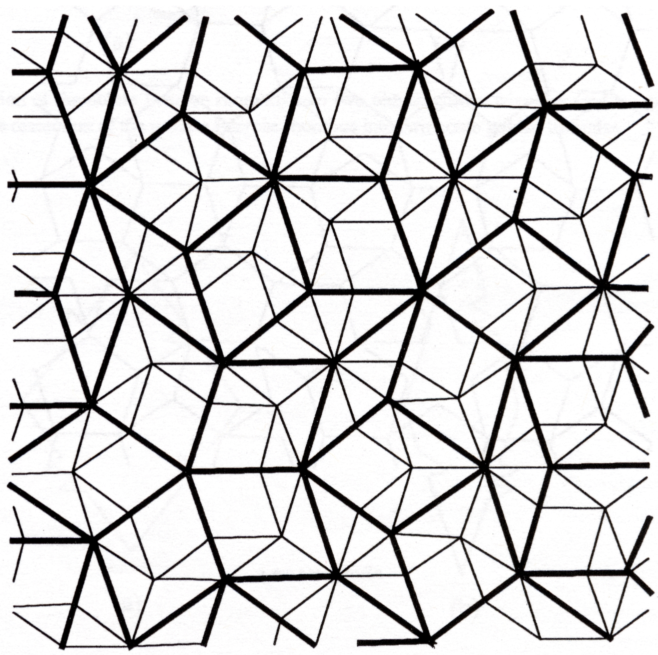


Figure 4.5 – Composition of Penrose's rhombs

Penrose's rhombs and kite and dart set are recompositions of each other, as shown in Figure 4.6. In 4.6a the kites and darts are bisected as can be seen by the dotted lines. Then in 4.6b the remaining triangles are merged together (thicker lines) to form rhombs.

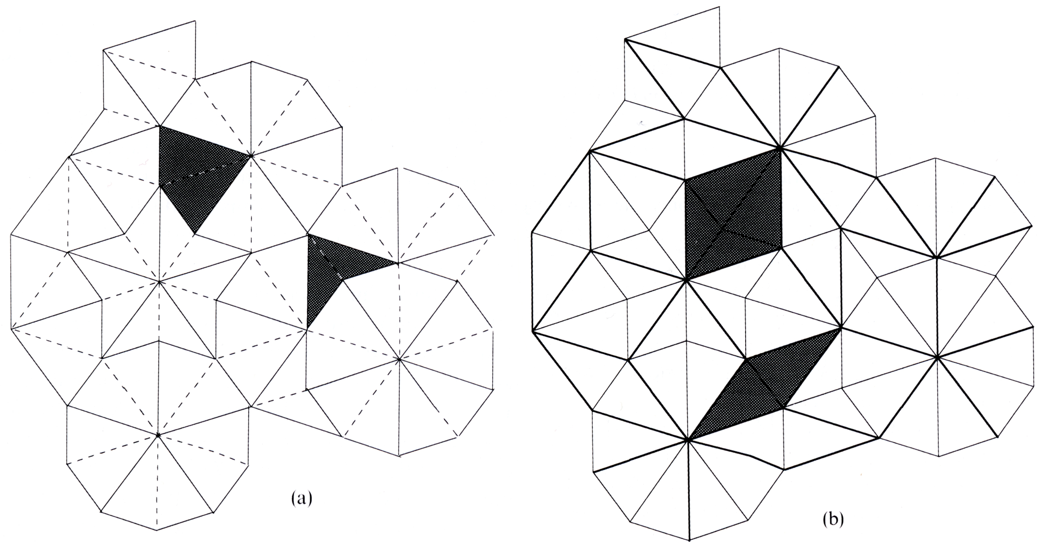


Figure 4.6 – Process of recomposition between Penrose's rhombs and kite and dart set

4.2 RHOMBOHEDRA

In 1976, Robert Ammann discovered a pair of three-dimensional blocks called rhombohedra that can fill up space without gaps in the same way that two-dimensional rhombs can tile a plane [**5,** pp. 206]. These three-dimensional tilings were aperiodic as well.

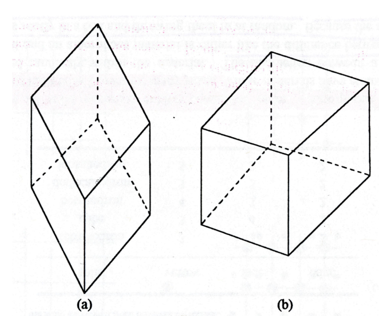


Figure 4.7 – Rhombohedral cells

These rhombohedra are based on Penrose's work. Senechal tells us that of the two rhombohedra, one is thick and one is thin; all the faces are congruent [**7,** pp. 220]. Dunlap clarifies for us that each face is the golden rhombus.

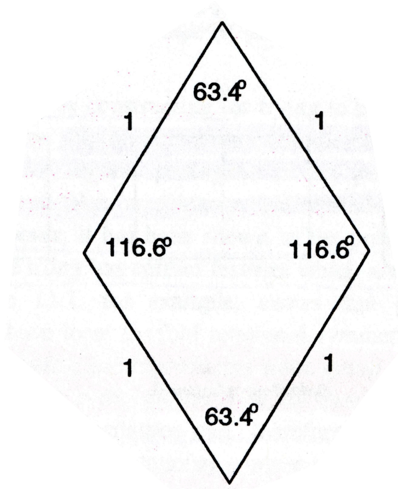


Figure 4.8 – The golden rhombus

This golden rhombus is slightly different from Penrose's rhombs introduced in Section 1.3 (Figure 1.21) because the ratio of the long diagonal to the short one in the golden rhombus is φ.

4.3 QUASICRYSTALS

The common belief has been that every pure solid in nature fits into one of two classes of solid materials, glassy or crystalline. An amorphous, or glassy, material has atoms randomly positioned within the material. Thus there is no simply defined unit cell and thus no rotational or translational symmetry [**2,** pp. 115].

A crystalline material is formed when the atoms are placed on a lattice. A lattice, as defined by G & S, is formed by the set of images of a fixed point *O* under the translations *na* + *mb*. We are also told that the most common lattice is the unit square lattice, which is the set of points in the Euclidean place with integer coordinates.

The most common crystals have a three dimensional structure. Thus the lattice is also three-dimensional; the simplest one is an arrangement of cubes. In salt, the unit cell is a cube.

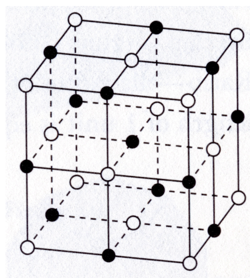


Figure 4.9 – Cubic unit cell

The way that the atoms are placed on the lattice is called the basis [**2,** pp. 111]. To summarize, a crystal has the combination of a lattice and a basis. Figure 4.10 gives us an easily understood depiction of these characteristics in a two-dimensional context.

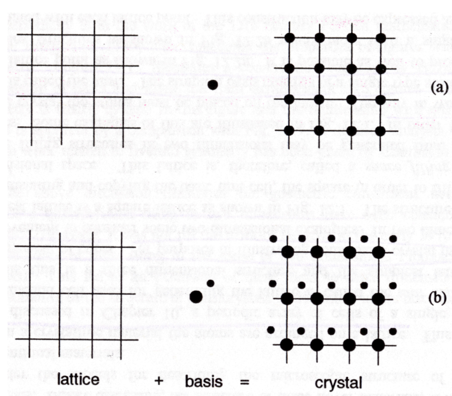


Figure 4.10 – In a two-dimensional context, a crystal is made of a lattice and a basis

The lattices made of squares, rectangles, parallelograms, and hexagons (Figure 1.1) all portray twofold symmetry; the hexagonal lattice also has threefold and sixfold symmetry and the square lattice has fourfold symmetry.

The construction of a tiling comprised of only pentagons is impossible. Thus the common belief was that crystallographic fivefold symmetry (as well as sevenfold, ninefold, elevenfold and others) was forbidden due to this demonstration. However, after Penrose's work in a two-dimensional plane containing evidence of fivefold symmetry (Figure 4.10) emerged, the quasilattice became the focus of many studies on crystals. The quasilattice was said to have fivefold rotational symmetry without having translational symmetry.

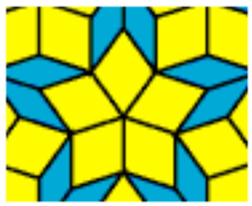


Figure 4.11 – Section of a Penrose tiling showing characteristics of fivefold symmetry

Experiments in fivefold symmetry that were conducted in the three-dimensional setting previous to Penrose's work had used dodecahedrons and icosahedrons (both contain fivefold symmetry) to prove that a three-dimensional tiling without gaps was impossible. However, extending the concept a crystal that is based on a quasilattice instead of the traditional lattice is possible in three dimensions.

The breakthrough came through using two-dimensional mappings called diffraction patterns. Baeyer best described it as follows: "Diffraction patterns are the windows physicists use to peer inside materials. When beams of electrons or X-rays pass through a solid material, they are diffracted, or scattered, by the atoms inside" [**1**, pp. 71]. The diffraction pattern of a crystal, as seen in the example given in Figure 4.12, has sharp, well-defined dots that are arranged in directions that represented the arrangement of the atoms (usually showing two, three, four, and sixfold symmetry). The diffraction pattern of a glassy material, however, has dots that are either fuzzy or missing.

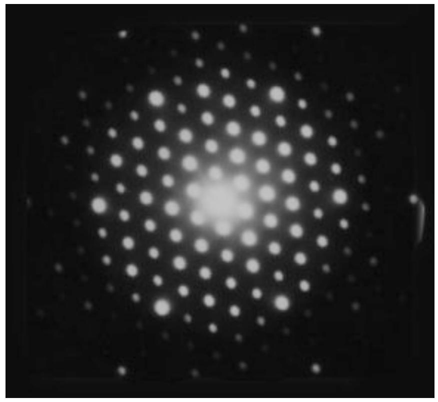


Figure 4.12 – A diffraction pattern of a beam of electrons passing through a crystal lattice

In 1984, Paul Steinhardt, a physicist at the University of Pennsylvania, and one of his graduate students Dov Levine simulated using a computer program an X-ray being shot through a computer generated structure (Figure 4.13). This structure was composed of the Golden Rhombohedra that were inspired by Penrose's work [**1,** pp. 76].

Figure 4.13 – Calculated diffraction pattern based on Golden Rhombohedra

The diffraction pattern that they discovered portrayed un-mistakable sharp dots that had fivefold symmetry. However, up until that point it was believed that fivefold symmetry was forbidden in the atomic makeup of solids. Levine and Steinhardt believed that since the structure that they had entered in the computer had a nonperiodic arrangement, it should have produced a blurry diffraction pattern typical to a glassy substance.

Levine and Steinhardt quickly contacted Robert Ammann, a recreational mathematician. They concluded at that time that the tiles were neither periodically spaced nor randomly spaced. The order was then called quasiperiodic. The predicted substance from which the new kind of diffraction pattern came was named a quasicrystal.

Having discovered the quasicrystal in the environment of the simulated computer program, they were left with the job of actually locating this substance in nature.

In the fall of 1984, soon after Levine and Steinhardt's discovery, a colleague of Steinhardt Harvard physicist David Nelson had produced a diffraction pattern of a real alloy of aluminum and manganese that had an obvious fivefold symmetry [**1**, pp. 76]. The wait was over.

Soon after Nelson's discovery more than a hundred alloys with fivefold symmetry were discovered. Then even more astonishing discoveries came as scientists found quasicrystals containing sevenfold, ninefold, elevenfold, and other previously forbidden symmetries. It had been one of the few beautiful moments in history when the mathematics of something physical had preceded the actual discovery of it in nature.

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