DAY 1: WEDNESDAY, AUGUST 22nd.

1 System of Linear Equation

Definition 1.1 (Linear Equation)**.** A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $b, a_1, a_2, \ldots, a_n$ are all numbers (constants).

Example 1.2.

(a) $-4x_1 + 5x_2 + 9x_3 = -9$ is a linear equation.

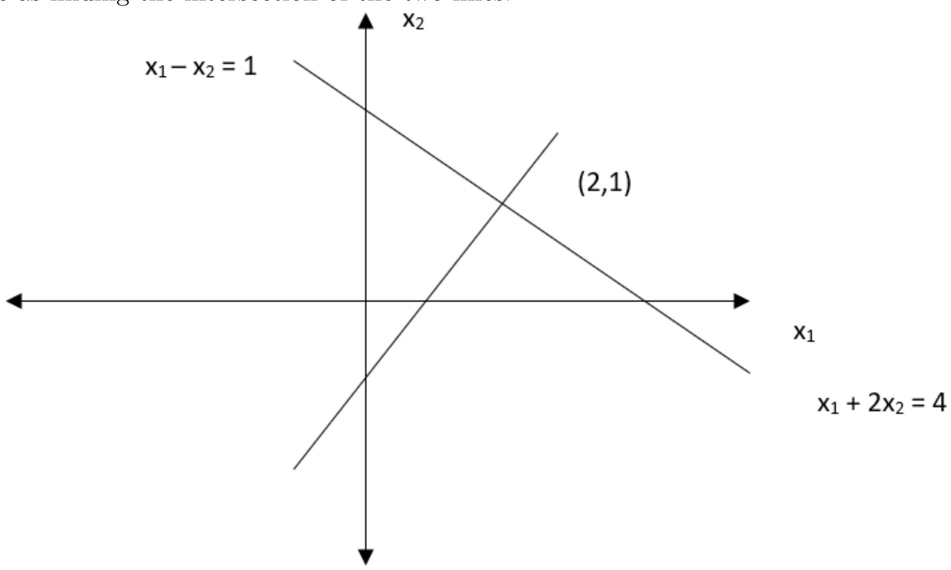
(b) $3x_1(4 + 2x_2) = 5$ is not a linear equation because of the term with x_1x_2.

Definition 1.3 (System of Linear Equations)**.** A **system of linear equations** (or linear system) is a collection of linear equations in the same set of variables. A **solution** of the system in n variables is a set of values $c_1, c_2, \ldots, c_n$ that satisfies all of the equations. The **solution set** of the system is the set of all possible solutions.

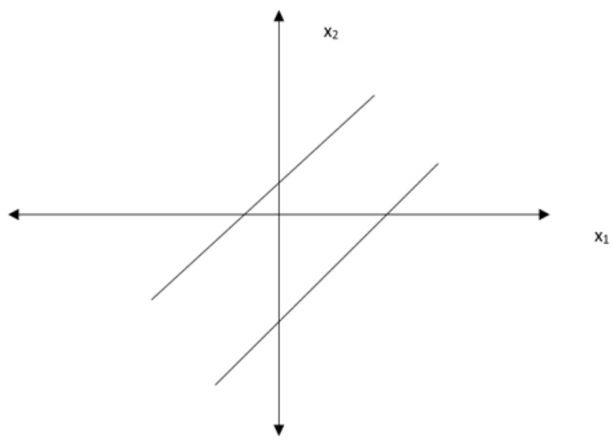
Example 1.4. Consider the following linear system in two variables:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 1 \end{aligned}$$

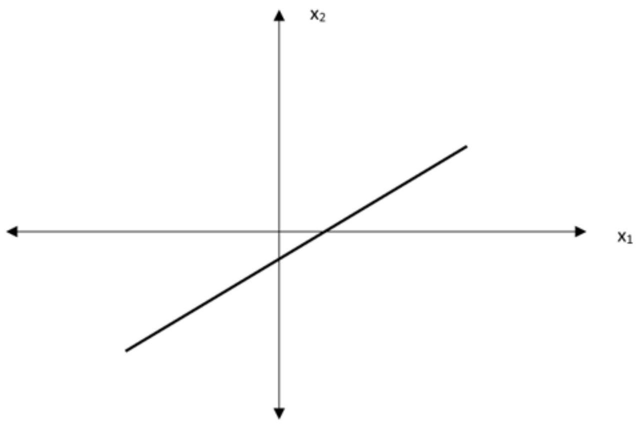
Clearly, each equation in the system is an equation of a line. Finding the solution set of this type of system is the same as finding the intersection of the two lines.



Check that the pair of numbers $(2, 1)$ is a solution of the system.
In the case of a linear system in two variables, there are three cases: the two lines intersect, the two lines are parallel, or the two lines are the same line.



No solution



Infinitely many solutions

A system of linear equations has either

(a) no solution

(b) exactly one solution

(c) infinitely many solutions.

Definition 1.5. A linear system is **consistent** if it has a solution (otherwise inconsistent).

Definition 1.6 (Equivalent Systems)**.** Two linear systems are equivalents if the have the same solution sets.

Example 1.7. The following systems are equivalent (we will see later why).

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \tag{1}$$

and

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \tag{2}$$

Check that the triplet $(29, 16, 3)$ is a solution of both linear systems (1) and (2) (it is actually a unique solution).

When solving a linear system, our goal is to transform the given system into an equivalent one which is much easier to solve, by using operations called elementary row operations.

Definition 1.8. The three basic row operations (for equations) are the following

(a) replacing one equation by the sum of itself and a multiple of another equation,

(b) interchanging two equations

(c) multiplying one equation by a nonzero constant.

DAY 2: FRIDAY, AUGUST 24th

Basic row operations for equations (continued)

Since a solution will make all of the equations in the system true, none of these operations will change the solution set of the system. Suppose that we have a system of linear equations in n variables. The idea is to keep x_1 in the first equation and eliminate it in all of the next equations, keep x_2 is the second equation and eliminate it in all of the next equations, and so on .... The goal is to obtain a new equivalent system that has a *triangular* form.

Example 1.9. Consider the linear system:

$$\text{(A)} \qquad \begin{aligned} 2x_1 + 7x_2 + x_3 &= -2 \\ x_1 + 4x_2 + 4x_3 &= -1 \\ x_2 + 5x_3 &= -4 \end{aligned}$$

Interchange equation (1) and equation (2), we have

$$\begin{aligned} x_1 + 4x_2 + 4x_3 &= -1 \\ 2x_1 + 7x_2 + x_3 &= -2 \\ x_2 + 5x_3 &= -4 \end{aligned}$$

Replace equation (2) by adding to it -2 times equation (1) (i.e. (2) - $2\times$(1)), we have

$$\begin{aligned} x_1 + 4x_2 + 4x_3 &= -1 \\ -x_2 - 7x_3 &= 0 \\ x_2 + 5x_3 &= -4 \end{aligned}$$

Replace equation (3) by adding to it equation (2) (i.e. (3)+(2))

$$\text{(B)} \qquad \begin{aligned} x_1 + 4x_2 + 4x_3 &= -1 \\ -x_2 - 7x_3 &= 0 \\ -2x_3 &= -4 \end{aligned}$$

System (A) and (B) are equivalent since they are obtained from one another by using the basic operations for equations. In particular, they have the same solution set.
We obtain from equation (3) of system (B) that $x_3 = 2$. Use the value of x_3 to get x_2 in equation (2) and then x_1 in equation (1). Check that (49,-14,2) is a solution.

To simplify the operations and notations, we will remove the variables by using matrix notation.

Definition 1.10. A matrix is a rectangular array of numbers. A matrix with m arrows and n is referred to as $m \times n$ matrix.

Example 1.11.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & - & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

is a 3×4 matrix.

The **coefficient matrix** of a linear system consists of all of the coefficients of the variables in the system. The **augmented matrix** for the system is formed by adding the constant terms to the coefficients matrix as the last column.

Example 1.12. The coefficients matrix of the linear system (A) is given by

$$\begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

and its augmented matrix is given by

$$\begin{bmatrix} 2 & 7 & 1 & -2 \\ 1 & 4 & 4 & -1 \\ 0 & 1 & 5 & -4 \end{bmatrix}$$

Now, we translate the elementary operations for equations to elementary row operations in the context of matrices.

Definition 1.13. The elementary row operations for matrices are the following:

(a) (Replacement) Replacing row i by itself plus a multiple of row j: $R_i + cR_j$.

(b) (Interchange) Interchanging rows i and j: $R_i \leftrightarrow R_j$.

(c) (Scaling) Scaling row i by a nonzero constant c: cR_i.

These operations are exactly the same as the operations for equations.

Definition 1.14. Two matrices are row equivalent if one matrix can be obtained from the other by elementary row operations.

FACT: Two linear systems are equivalent if and only if their augmented matrices are row-equivalent.

Example 1.15. We re-solve linear system (A) by using augmented matrix method. Our goal is the same as before, getting more zeros in rows after rows. The linear system is given by

$$\begin{aligned} 2x_1 + 7x_2 + x_3 &= -2 \\ x_1 + 4x_2 + 4x_3 &= -1 \\ x_2 + 5x_3 &= -4 \end{aligned}$$

and its augmented matrix is

$$\begin{bmatrix} 2 & 7 & 1 & -2 \\ 1 & 4 & 4 & -1 \\ 0 & 1 & 5 & -4 \end{bmatrix}$$

We start from analyzing the first row. We can leave the first row unchanged, but for simplification, it is better to have the first entry to be 1, so interchange row (1) and row (2) as we did before in the case of the equations. We have:

$$\begin{bmatrix} 2 & 7 & 1 & -2 \\ 1 & 4 & 4 & -1 \\ 0 & 1 & 5 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 4 & -1 \\ 2 & 7 & 1 & -2 \\ 0 & 1 & 5 & -4 \end{bmatrix}$$

Now we consider this new matrix and look at row (2). We want zero at the first entry. For that, we can replace row (2) by adding row (2) and -2 times row (1).

$$\begin{bmatrix} 1 & 4 & 4 & -1 \\ 2 & 7 & 1 & -2 \\ 0 & 1 & 5 & -4 \end{bmatrix} \xrightarrow{R_2 - 2\times R_1 \to R_2} \begin{bmatrix} 1 & 4 & 4 & -1 \\ 0 & -1 & -7 & 0 \\ 0 & 1 & 5 & -4 \end{bmatrix}$$

With this new matrix, we look at row (3). We want zero at the first and second entry. For that, we can replace row (3) by adding row (3) and row (2). We have:

$$\begin{bmatrix} 1 & 4 & 4 & -1 \\ 0 & -1 & -7 & 0 \\ 0 & 1 & 5 & -4 \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} 1 & 4 & 4 & -1 \\ 0 & -1 & -7 & 0 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

We have looked at each row and we conclude that the following two matrices are row equivalent:

$$\begin{bmatrix} 2 & 7 & 1 & -2 \\ 1 & 4 & 4 & -1 \\ 0 & 1 & 5 & -4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 4 & 4 & -1 \\ 0 & -1 & -7 & 0 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

these matrices are the augmented matrices of the following linear systems respectively:

$$\begin{aligned} 2x_1 + 7x_2 + x_3 &= -2 \\ x_1 + 4x_2 + 4x_3 &= -1 \\ x_2 + 5x_3 &= -4 \end{aligned}$$

and

$$\begin{aligned} x_1 + 4x_2 + 4x_3 &= -1 \\ -x_2 - 7x_3 &= 0 \\ -2x_3 &= -4 \end{aligned}$$

It follows that these two systems are equivalent, i.e. they have the same solution set.

You can show now, by using augmented matrix method, that the two linear systems in Example 1.7 are equivalent.

2 Row Reduction and Echelon Form

When solving a linear sytem, our strategy in the previous section was to replace the system with an equivalent system (i.e. have the same solution sets) that is easier to solve, by using elementary row operations on the augmented matrix of the system. We will refine this method to formalize a way that will enable us to answer the two fundamental questions about linear systems. We are still going to use elementary row operations.

We start with an arbitrary rectangular matrix as the algorithm applies to any matrix whether or not the matrix is viewed as an augmented matrix of a linear system.

Definition 2.1.

- The **leading entry** of a row is the leftmost nonzero entry of the row.

- A **nonzero row** is a row that contains at least one nonzero entry.

Definition 2.2 (Matrix in echelon form)**.** A matrix is in row echelon form (or echelon form) if it satisfies the following conditions:

(a) All nonzero rows are above any rows of zeros.

(b) Each leading entry of a row is in a column to the right of the leading entry of the row above it.

(c) All entries in a column below the leading entry are zeros.

Example 2.3.

(A1) $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{ is in echelon form.}$$

(A2) $$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \text{ is in echelon form}$$

(A3) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 4 & -3 \end{bmatrix} \text{ is not in echelon form, it fails condition (b) and (c).}$$

Definition 2.4.

- A pivot position in a echelon matrix is a location that corresponds to a leading entry. A pivot column is a column that contains a pivot position.

- A pivot is the nonzero number in a pivot position.

Example 2.5. The first and second columns in matrix (A1) are pivot columns.

From a given matrix, we can use elementary row operations to obtain a row equivalent matrix which is in echelon form. If a matrix A is row equivalent to echelon matrix U, we call U an echelon form (or row echelon form) of A.

FACTS:

- Any nonzero matrix may have more than one echelon form using different sequence of row operations.

- The pivot positions in two different echelon forms of a matrix A are always the same (pivots can be different).

We can describe the solution set of linear system by considering an echelon form of its augmented matrix. Let's see an algorithm on how we compute an echelon form of a given matrix.

Algorithm for Row Echelon Form:(Gaussian Elimination)

(1) Start with the leftmost nonzero column (the column has at least one nonzero entry). This column is a pivot column. If the top entry is zero, select a nonzero entry in this column and interchange it with the top row. Now the top entry in this column is nonzero.

(2) Use row replacement operations to eliminate nonzero values below the pivot.

(3) Cover (or ignore) the rows containing pivots and repeat the process for the remaining matrix.

Example 2.6. Consider the following matrix:

$$\text{(A)} \qquad \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix}$$

(1) The leftmost nonzero column is the first column so we start from there. Since the top entry in this column is already nonzero, we do not need to interchange rows to get a nonzero in the top position.

(2) Now we create zeros in all positions below the pivot. Our pivot is 2 and we need to eliminate the last entry, which is 3, of the column. We replace row (3) by adding row (3) to $-\frac{3}{2}$ times row (1). Note that:

$$\frac{3}{2} \times R_1 = [3 \quad 0 \quad -9 \quad -12] \text{ and } R_3 - \frac{3}{2}R_1 \times = [3 \quad 6 \quad -2 \quad -4] - [3 \quad 0 \quad -9 \quad -12] = [0 \quad 6 \quad 7 \quad 8]$$

So we have the following:

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{2} \times R_1 \to R_3} \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix}$$

(3) Consider the new matrix. We complete the first column. Next, we ignore the first row and continue with the remainder of the matrix. The leftmost nonzero column in this case is now column (2), and the top entry, which is 1, is nonzero, so there is no need to interchange rows.

(4) The next is to create zeros in all position below the pivot (which is 1). For that, we need to eliminate 6. We replace row (3) by row (3) plus -6 times row (2). We have:

$$R_3 - 6R_2 = [0 \quad 6 \quad 7 \quad 8] - [0 \quad 6 \quad 12 \quad 18] = [0 \quad 0 \quad -5 \quad -10]$$

So we have:

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{R_3 - 6 \times R_2 \to R_3} \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

(5) Consider now the new matrix. We have completed the second column so we can ignore the two first rows. We are left with the last row which is already in echelon form so are done.

An echelon form of the matrix A is therefore the matrix:

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$