DAY 25: Wednesday, October 17th

Subspaces

Definition 1.8. Let V be a vector space. A subspace of V is a subset H of V (i.e. $H \subseteq V$) that satisfies the following

(a) The zero vector $\vec{0}$ is in H.

(b) For every $\vec{u}$ and $\vec{v}$ in H, $\vec{u} + \vec{v}$ is also in H (closure under addition).

(c) For every $\vec{u}$ in H and for every scalar c, $c\vec{u}$ is in H (closure under scalar multiplication).

Example 1.9. For any vector space V, the subset $\{\vec{0}\}$ is a subspace of V.

Example 1.10. Let m and n be positive integers with $m \leq n$. Then the set $\mathbb{P}_m$ of polynomials of degree at most m is a subspace of the vector space $\mathbb{P}_n$ of polynomials of degree at most n.

Example 1.11. Note that $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ as $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. However the collection of vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, where a, b in $\mathbb{R}$, is subsapce of $\mathbb{R}^3$.

A subspace Spanned by a Set

Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V. Recall that

$$\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p \mid c_1, c_2, \ldots, c_p \in \mathbb{R}\}$$

is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. Clearly, $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ is a subset of V.

Theorem 1.12. Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V. Then $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ is a subspace of V.

We call $H = \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ the subspace spanned (or generated) by $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$, and the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a spanning (or generating) set of H.

Example 1.13. Let H be the set of all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} x_1 - 3x_2 \\ -x_1 + x_2 \\ x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are arbitrary scalars. Show that H is a subspace of $\mathbb{R}^4$.

Solution

For every vector $\vec{u}$ in H, $\vec{u}$ is of the form

$$\begin{aligned}\vec{u} &= \begin{bmatrix} x_1 - 3x_2 \\ -x_1 + x_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ x_2 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2.\end{aligned}$$

It follows that $H = \text{Span}(\vec{v}_1, \vec{v}_2)$. Thus by Theorem 1.12, H is a subspace of $\mathbb{R}^4$.

2 Null Spaces, Column Spaces, and Linear Transformations (Special subspaces)

The Null Space of a Matrix

Definition 2.1. Let A be an $m \times n$ matrix. The null space of A, denoted by $\text{Nul}(A)$, is the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$. That is

$$\text{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}.$$

Proposition 2.2. If A is an $m \times n$ matrix, then the null space $\text{Nul}(A)$ of A is a subspace of $\mathbb{R}^n$.

Proof. We prove that the set $\text{Nul}(A)$ satisfies the three properties for subspace in Definition 1.8. Clearly, $A\vec{0} = \vec{0}$, hence $\vec{0}$ is in $\text{Nul}(A)$. For $\vec{u}$ and $\vec{v}$ in $\text{Nul}(A)$, we have $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$, hence $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0}$. Therefore $\vec{u} + \vec{u}$ is also in $\text{Nul}(A)$. For $\vec{u}$ in $\text{Nul}(A)$ and for a scalar c, since $A\vec{u} = \vec{0}$, we have $A(c\vec{u}) = cA\vec{u} = c \cdot \vec{0} = \vec{0}$. Hence $c\vec{u}$ is in $\text{Nul}(A)$. We conclude that $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$. $\square$

Example 2.3. Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$. Are $\vec{u} = \begin{bmatrix} -10 \\ -6 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\text{Nul}(A)$?

Solution

We have

$$A\vec{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} -10 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } A\vec{v} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Hence $\vec{u}$ is in $\text{Nul}(A)$ and $\vec{v}$ is not in $\text{Nul}(A)$.

Example 2.4. Let H be a subset of $\mathbb{R}^4$ defined by

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a - 2b + 5c = d \text{ and } c - a = b \right\}$$

Show that H is a subspace of $\mathbb{R}^4$ by expressing it as a null space of a matrix.

Solution

We look for a matrix A such that $\text{Nul}(A) = H$. For every vector $\vec{u} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in H, we have

$$\begin{aligned} a - 2b + 5c &= d \\ c - a &= b \end{aligned}$$

By rearranging the equations, we have

$$\begin{aligned} a - 2b + 5c - d &= 0 \\ -a - b + c &= 0 \end{aligned}$$

It follows that every vector $\vec{u}$ in H is a solution of the homogeneous equation $A\vec{x} = \vec{0}$ where $A = \begin{bmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$, i.e. $H = \text{Nul}(A)$. Thus by Proposition 2.2, H is a subspace of $\mathbb{R}^4$.

An explicit description of $\text{Nul}(A)$

To describe $\text{Nul}(A)$ explicitly, we solve the matrix equation $A\vec{x} = \vec{0}$ and write the parametric vector form of the solutions to determine a spanning set for $\text{Nul}(A)$.

Finding a Spanning set for $\text{Nul}(A)$

(1) Solve $A\vec{x} = \vec{0}$ and write the solutions in parametric vector form.

(2) Note that parameters are only the free variables.

(3) If the solutions of $A\vec{x} = \vec{0}$ are of the form

(**) $$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_p\vec{v}_p, \text{ where } t_1, t_2, \ldots, t_p \in \mathbb{R}.$$

then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ spans $\text{Nul}(A)$.

Note 2.5. If $\text{Nul}(A) \neq \{\vec{0}\}$, then the vectors defined in equation (**) are linearly independent. In this case the size of the spanning set is the number of free variables.

Example 2.6. Find a spanning set for the null space of $A = \begin{bmatrix} 1 & -2 & 2 & -3 & -1 \\ 2 & -4 & 5 & -6 & -3 \\ -3 & 6 & -4 & 1 & -7 \end{bmatrix}$.

The Column Space of a Matrix

Definition 2.7. Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix. The column space of A, denoted by $\mathrm{Col}(A)$, is the set of all linear combinations of the columns of A. That is

$$\mathrm{Col}(A) = \mathrm{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$$

Theorem 2.8. The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Note 2.9. Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix.

(a) By definition, $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$ spanned by $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.

(b) The following statements are equivalent:

- $\mathrm{Col}(A) = \mathbb{R}^m$.
- For every $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
- The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is onto.

Example 2.10. Find a matrix A such that $W = \mathrm{Col}(A)$ where

$$W = \left\{ \begin{bmatrix} a+b \\ 2a-b \\ -3a \end{bmatrix} \mid a, b \text{ in } \mathbb{R} \right\}$$

Solution

We write each element of W as a linear combination of some vectors. For each $\vec{u}$ in W, we have

$$\begin{aligned} \vec{u} &= \begin{bmatrix} a+b \\ 2a-b \\ -3a \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ -3a \end{bmatrix} + \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \\ &= a \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2. \end{aligned}$$

It follows that $W = \mathrm{Span}(\vec{v}_1, \vec{v}_2)$. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -3 & 0 \end{bmatrix}$. Then $W = \mathrm{Col}(A)$.

Fact 2.11. Let A be an $m \times n$ matrix. We compare the null space and the column space of A as follows:

Null Space	Column Space
• Nul(A) is a subspace of $\mathbb{R}^n$. • Null space of A is implicitly defined: the vectors in Nul(A) solve $A\vec{x} = \vec{0}$. • It takes time to describe Nul(A) as row reduction of $\begin{bmatrix} A & \vec{0} \end{bmatrix}$ is required. • There is no obvious relation between Nul(A) and the entries in A. • A typical vector $\vec{v}$ in Nul(A) has the property $A\vec{v} = \vec{0}$. • It is easy to check if a given vector $\vec{v}$ is in Nul(A) by computing $A\vec{v}$. • Nul(A) $= \{\vec{0}\}$ if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution. • Nul(A) $= \{\vec{0}\}$ if and only the linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.	• Col(A) is a subspace of $\mathbb{R}^m$. • Col(A) is explicitly defined since it is spanned by described vectors (the columns of A). • It is easy to find vectors in Col(A) since they are linear combinations of the columns of A. • There is an obvious relation between Col(A) and the entries of A since the columns of A generate Col(A). • A typical vector $\vec{v}$ in Col(A) has the property that $A\vec{x} = \vec{v}$ is consistent. • It may take time to check if a given vector $\vec{v}$ is in Col(A) as row reduction of $\begin{bmatrix} A & \vec{v} \end{bmatrix}$ is required. • Col(A) $= \mathbb{R}^m$ if and only if the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^m$. • Col(A) $= \mathbb{R}^m$ if and only if the linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Linear Transformations of Vector Spaces

Definition 2.12. Let V and W be vector spaces. A linear transformation T from V into W, denoted by $T : V \to W$, is a rule that assigns to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W, such that

(a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in V,

(b) $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in V and all scalars c.

Definition 2.13. Let $T : V \to W$ be a linear transformation.

(a) V is called the domain of T, and W is its codomain.

(b) The **kernel** of T, denoted by ker(T), is the set of all vectors $\vec{u}$ in V such that $T(\vec{u}) = \vec{0}$. That is

$$\ker(T) = \{\vec{u} \text{ in } V \mid T(\vec{u}) = \vec{0}\}$$

(c) The **range** or **image** of T, denoted by range(T) or im(T), is the set of all vectors in W which are of the form $T(\vec{v})$ for some $\vec{v}$ in V. That is

$$\text{range}(T) = \text{im}(T) = \{T(\vec{v}) \mid \vec{v} \text{ in } V\}$$

Fact 2.14. Let $T : V \to W$ be a linear transformation of vector spaces. Then,

(a) ker(T) is a subspace of V,

(b) $\text{im}(T)$ is a subspace W.

(c) if $V = \mathbb{R}^n, W = \mathbb{R}^m$, and A is the standard matrix of T, then $\ker(T)$ the null space of A, and $\text{im}(T)$ is the column space of A.