DAY 41: Friday, November 30th

Chap VI: ORTHOGONALITY and LEAST SQUARES

1 Inner Product, Length, and Orthogonality

Definition 1.1. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. The **inner product** (or **dot product**) $\vec{u} \cdot \vec{v}$ of $\vec{u}$ and $\vec{v}$ is defined by $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. That is

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example 1.2. Compute $\vec{u} \cdot \vec{v}$ and $\vec{v} \cdot \vec{u}$ for $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.

Solution

We have

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = (-1)(-2) + (2)(0) + (1)(3) = 5$$

and

$$\vec{v} \cdot \vec{u} = \begin{bmatrix} -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = (-2)(-1) + (0)(2) + (3)(1) = 5$$

Theorem 1.3. Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

(d) $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

(e) $(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p) \cdot \vec{w} = c_1(\vec{u}_1 \cdot \vec{w}) + c_2(\vec{u}_2 \cdot \vec{w}) + \cdots + c_p(\vec{u}_p \cdot \vec{w})$

Definition 1.4. The **length** (or **norm**) $||\vec{v}||$ of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is defined by

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \qquad \text{(i.e. } ||\vec{v}||^2 = \vec{v} \cdot \vec{v}\text{)}$$

Note 1.5. In $\mathbb{R}^2$, the definition of the length $||\vec{v}|| = \sqrt{v_1^2 + v_2^2}$ of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ coincides with the standard definition of length of the line segment from the origin to the point $\vec{v}$. With this definition, the Pythagorean Theorem holds (i.e. the length l of the vector which is the hypotenuse of the right triangle with horizontal length v_1 and vertical height v_2 satisfies $v_1^2 + v_2^2 = l^2$).

Fact 1.6. For any vector $\vec{v}$ and for any scalar c we have

$$||c\vec{v}|| = |c|||\vec{v}||$$

Definition 1.7.

(a) A vector of length 1 is called a **unit vector**.

(b) If $\vec{v} \neq \vec{0}$ then $\frac{1}{||\vec{v}||}\vec{v}$ is a unit vector and is in **the same direction** as $\vec{v}$.

(c) The process in part b) is called **normalizing** $\vec{v}$.

Example 1.8. Find a unit vector $\vec{u}$ in $\mathbb{R}^4$ in the same direction as $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

Solution

First, compute the length of $\vec{v}$:

$$|||\vec{v}||^2 = \vec{v} \cdot \vec{v} = (-2)^2 + (0)^2 + (1)^2 + (2)^2 = 9$$

So

$$|||\vec{v}|| = \sqrt{9} = 3$$

Now, multiply $\vec{v}$ by $\frac{1}{||\vec{v}||}$ to obtain the vector $\vec{u}$ (the process is normalizing $\vec{v}$):

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v} = \frac{1}{3}\vec{v} = \frac{1}{3}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 0 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Check that $\vec{u}$ is indeed a unit vector (i.e. $||\vec{u}|| = 1$). It suffices to only check that $||\vec{u}||^2 = 1$. We have

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u} = \left(\frac{-2}{3}\right)^2 + (0)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{4}{9} + 0 + \frac{1}{9} + \frac{4}{9} = 1$$

Distance in $\mathbb{R}^n$

Definition 1.9. For $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$, denoted by $\text{dist}(\vec{u}, \vec{v})$, is the length of the vector $\vec{u} - \vec{v}$. That is

$$\text{dist}(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||.$$

Example 1.10. Compute the distance of $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

Solution

By definition $\text{dist}(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||$. We have

$$\begin{aligned} \vec{u} - \vec{v} &= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \\ ||\vec{u} - \vec{v}|| &= \sqrt{(-3)^2 + (-1)^2 + (1)^2} = \sqrt{11}. \end{aligned}$$

Hence $\text{dist}(\vec{u}, \vec{v}) = \sqrt{11}$.

Definition 1.11. Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\vec{u} \cdot \vec{v} = 0$.

Note 1.12. We have

$$\begin{aligned} ||\vec{u} + \vec{v}||^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot (\vec{u} + \vec{v}) + \vec{v} \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= ||\vec{u}||^2 + ||\vec{v}||^2 + 2\vec{u} \cdot \vec{v} \end{aligned}$$

Theorem 1.13 (Pythagorean Theorem)**.** .
Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$.

Example 1.14. The vector $\vec{0}$ is orthogonal to every vector $\vec{v}$ in $\mathbb{R}^n$ since $\vec{0} \cdot \vec{v} = 0$.

Definition 1.15. Let W be a subspace of $\mathbb{R}^n$.

(a) A vector $\vec{z}$ in $\mathbb{R}^n$ is said to be orthogonal to W if $\vec{z}$ is orthogonal to every vector in W.

(b) The **orthogonal complement** of W, denoted by $W^\perp$, is the collection of all vectors orthogonal to W.

$$W^\perp = \{\vec{z} \in \mathbb{R}^n \mid \ \vec{z} \cdot \vec{w} = 0 \text{ for every } \vec{w} \in W\}$$