DAY 37: Friday, October 16th

3 Diagonalization

Definition 3.1. An $n \times n$ matrix D is diagonal if every non-diagonal entry of D is zero. That is, if $D = (d_{ij})_{i,j}$ then $d_{ij} = 0$ for $i \neq j$.

Example 3.2. Let $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (D is a diagonal matrix). Let $P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}$

(a) Compute D^k for $k = 1, 2, \ldots$.

(b) Compute A^k where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

Solution

(a) We have

$$D^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} (-1)^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 2^2 \end{bmatrix}$$

and

$$D^3 = DD^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} (-1)^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 2^2 \end{bmatrix} = \begin{bmatrix} (-1)^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & 2^3 \end{bmatrix}$$

So, in general

$$D^k = \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 2^k \end{bmatrix}$$

(b) We have

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDDP^{-1} = PD^2P^{-1}$$

and

$$A^3 = AA^2 = (PDP^{-1})(PD^2P^{-1}) = PD(P^{-1}P)D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

So in general for $k \geq 1$, we have

$$A^k = PD^kP^{-1}$$

By using for example the Adjoint Inverse Formula, we have

$$P^{-1} = \begin{bmatrix} -3 & 0 & -2 \\ 2 & 1 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

Hence

$$\begin{aligned} A^k &= PD^kP^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} -3 & 0 & -2 \\ 2 & 1 & 1 \\ -2 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -3\cdot(-1)^k + 2^{k+2} & 0 & -2\cdot(-1)^k + 2^{k+1} \\ 2\cdot 3^k - 2^{k+1} & 3^k & 3^k - 2^k \\ 6\cdot(-1)^k - 3\cdot 2^{k+1} & 0 & 4\cdot(-1)^k - 3\cdot 2^k \end{bmatrix} \end{aligned}$$

Definition 3.3. An $n \times n$ matrix A is diagonalizable if there exist an $n \times n$ invertible matrix P and an $n \times n$ diagonal matrix D such that $A = PDP^{-1}$. That is, A is diagonalizable if it is similar to a diagonal matrix.