DAY 36: Wednesday, November 14th

2 The Characteristic Equation

Finding eigenvalues of a matrix A

Note 2.1.

(a) Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if there exists a nonzero vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$, i.e. $(A - \lambda I_n)\vec{x} = \vec{0}$.

(b) Let A be an $n \times n$ matrix. To find eigenvalues of A, we must find all scalars λ such that the matrix equation $(A - \lambda I_n)\vec{x} = \vec{0}$ has a nontrivial solution. The matrix equation $(A - \lambda I_n)\vec{x} = \vec{0}$ has a nontrivial solution if and only the matrix $A - \lambda I_n$ is not invertible. That is, λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

(c) It follows that the scalar $\lambda = 0$ is an eigenvalue of A if and only if $\det(A) = 0$.

Theorem 2.2. Let A be an $n \times n$ matrix. Then the following statements are equivalent:

(a) A is invertible.

(b) $\det(A) \neq 0$.

(c) The scalar 0 is **not** an eigenvalue of A.

Example 2.3. Find the eigenvalues of $A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$.

Solution

We need to find the λ's such that $\det(A - \lambda I_2) = 0$. We have

$$A - \lambda I_2 = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix}$$

and

$$\det(A - \lambda I_2) = \det \begin{bmatrix} 3-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) = \lambda^2 - 4\lambda + 3$$

Setting $\det(A - \lambda I_2) = 0$, we have $(3-\lambda)(1-\lambda) = 0$. Solving for λ, the eigenvalues of A are 3 and 1.

Definition 2.4. Let A be an $n \times n$ matrix. The polynomial $\det(A - \lambda I_n)$ (in the variable λ) is called the characteristic polynomial of A. The equation $\det(A - \lambda I_n) = 0$ is the characteristic equation of A.

Fact 2.5. A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation $\det(A - \lambda I_n) = 0$.

Note 2.6. Let A be an $n \times n$ matrix. Then

(a) The characteristic polynomial $\det(A - \lambda I_n)$ of A is of degree n. Hence the characteristic equation $\det(A - \lambda I_n) = 0$ has n roots, counted with multiplicity.

(b) The (algebraic) multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example 2.7. Let $A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix}$. Find the characteristic equation of A.

Solution

The characteristic equation of A is given by $\det(A - \lambda I_3) = 0$. We have

$$\det(A - \lambda I_3) = \det \begin{bmatrix} 1-\lambda & 5 & 0 \\ 0 & -6-\lambda & -1 \\ 0 & -2 & -\lambda \end{bmatrix}$$

Using cofactor expansion down the first column we have

$$\det(A - \lambda I_3) = (1-\lambda)\Big((-6-\lambda)(-\lambda) - 2\Big) = -\lambda^3 - 5\lambda^2 + 8\lambda - 2$$

Hence the characteristic equation of A is given by $-\lambda^3 - 5\lambda^2 + 8\lambda - 2 = 0$.

Definition 2.8. Two $n \times n$ matrices A and B are similar if there is an invertible matrix P such that $P^{-1}AP = B$, or equivalently $A = PBP^{-1}$.

Theorem 2.9. If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomials, hence the same eigenvalues.