DAY 24: Monday, October 15th

Chap IV: VECTOR SPACES

1 Vector Spaces and Subspaces

Definition 1.1. A vector space is a nonempty set V of objects, called vectors, with two operations: addition and scalar multiplication, such that for every $\vec{u}, \vec{v}$, and $\vec{w}$ in V and for all scalars c and d:

(1) $\vec{u}+\vec{v}$ is in V (closure under addition).

(2) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (commutative).

(3) $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ (associative).

(4) There is a zero vector $\vec{0}$ in V such that $\vec{0}+\vec{u}=\vec{u}$.

(5) For each $\vec{u}$ in V, there is vector $-\vec{u}$ in V such that $\vec{u}+(-\vec{u})=\vec{0}$ (negatives).

(6) The scalar multiple $c\vec{u}$ of $\vec{u}$ by c is in V (closure under scalar multiplication).

(7) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ (distributive).

(8) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ (distributive).

(9) $c(d\vec{u})=(cd)\vec{u}$ (associative).

(10) $1\vec{u}=\vec{u}$ (identity).

Fact 1.2. For each vector $\vec{u}$ in a vector space V and for each scalar c:

(a) $0\vec{u}=\vec{0}$.

(b) $c\vec{0}=\vec{0}$.

(c) $-\vec{u}=(-1)\vec{u}$ (negative $\vec{u}$ or the additive inverse of $\vec{u}$).

Example 1.3. The set $\mathbb{R}^n$, where the addition and scalar multiplication are component-wise, is a vector space.

Example 1.4. For a positive integer n, let $\mathbb{P}_n$ denote the set of polynomials of degree at most n. Let $p(t)$ be a polynomial

$$p(t)=a_0+a_1t+a_2t^2+\cdots+a_nt^n, \quad a_i\in\mathbb{R}.$$

The degree of $p(t)$ is the highest power of t whose coefficient is not zero. If $p(t)=a_0$, where $a_0\neq 0$ in $\mathbb{R}$, then the degree of $p(t)$ is zero. We call $p(t)=0$ the zero polynomial and we include it in $\mathbb{P}_n$ even if its degree is not defined. If $p(t)=a_0+a_1t+a_2t^2+\cdots+a_nt^n, q(t)=b_0+b_1t+b_2t^2+\cdots+b_nt^n$ be polynomials in $\mathbb{P}_n$ and if c is a scalar then

$$p(t)+q(t)=(a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2+\cdots+(a_n+b_n)t^n$$

and

$$cp(t)=ca_0+ca_1t+ca_2t^2+\cdots+ca_nt^n$$

With these two operations, the set $\mathbb{P}_n$ is a vector space. The zero polynomial is the zero vector.

Example 1.5. Let $\mathcal{C}((0,1))$ be the set of all continuous real-valued functions.

$$\mathcal{C}((o,1)) = \{f : (0,1) \to \mathbb{R} \mid f \text{ is continuous}\}.$$

If f and g are in $\mathcal{C}((0,1))$ and if c is a scalar, we define

$$(f+g)(x) = f(x) + g(x) \text{ and } (cf)(x) = cf(x) \text{ for each } x \in (0,1).$$

Then $\mathcal{C}((o,1))$ is a vector space. The zero function defined by $f(x) = 0$, for each x in $\mathbb{R}$, is the zero vector.

Example 1.6. Let $M_{m\times n}(\mathbb{R})$ denote the collection of $m \times n$ matrices (with real number entries). Together with component-wise addition and component-wise scalar multiplication, $M_{m\times n}(\mathbb{R})$ is vector space.

Example 1.7. The collection of all polynomials (any degree is possible) is a vector space.

Subspaces

Definition 1.8. Let V be a vector space. A subspace of V is a subset H of V (i.e. $H \subseteq V$) that satisfies the following

(a) The zero vector $\vec{0}$ is in H.

(b) For every $\vec{u}$ and $\vec{v}$ in H, $\vec{u} + \vec{v}$ is also in H (closure under addition).

(c) For every $\vec{u}$ in H and for every scalar c, $c\vec{u}$ is in H (closure under scalar multiplication).

Example 1.9. For any vector space V, the subset $\{\vec{0}\}$ is a subspace of V.

Example 1.10. Let m and n be positive integers with $m \leq n$. Then the set $\mathbb{P}_m$ of polynomials of degree at most m is a subspace of the vector space $\mathbb{P}_n$ of polynomials of degree at most n.

Example 1.11. Note that $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ as $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. However the collection of vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, where a, b in $\mathbb{R}$, is subsapce of $\mathbb{R}^3$.

A subspace Spanned by a Set

Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V. Recall that

$$\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p \mid c_1, c_2, \ldots, c_p \in \mathbb{R}\}$$

is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. Clearly, $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ is a subset of V.

Theorem 1.12. Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V. Then $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ is a subspace of V.

We call $H = \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ the subspace spanned (or generated) by $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$, and the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a spanning (or generating) set of H.

Example 1.13. Let H be the set of all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} x_1 - 3x_2 \\ -x_1 + x_2 \\ x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are arbitrary scalars. Show that H is a subspace of $\mathbb{R}^4$.

Solution

For every vector $\vec{u}$ in H, $\vec{u}$ is of the form

$$\begin{aligned}
\vec{u} &= \begin{bmatrix} x_1 - 3x_2 \\ -x_1 + x_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ x_2 \end{bmatrix} \\
&= x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2.
\end{aligned}$$

It follows that $H = \mathrm{Span}(\vec{v}_1, \vec{v}_2)$. Thus by Theorem 1.12, H is a subspace of $\mathbb{R}^4$.