DAY 22: Wednesday, October 10th

Theorem 1.6. Let A be an $n \times n$ square matrix. Then, the determinant $\det(A)$ of A can be computed by a cofactor expansion across any row or down any column.

(a) The expansion across the ith row is (i is fixed)

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(b) The expansion down the jth column is (j is fixed)

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Note 1.7. The plus or minus sign in the (i,j)-cofactor depends on the position of a_{ij} in the matrix. The sign of $(-1)^{i+j}$ is determined in the following pattern.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Example 1.8. Use a cofactor expansion down the third column to compute the determinant of $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix}$.

We have

$$\begin{aligned} \det(A) &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \\ &= 0 \cdot (-1)^{1+3}\det(A_{13}) + 0 \cdot (-1)^{2+3}\det(A_{23}) + 1 \cdot (-1)^{3+3}\det(A_{33}). \\ &= 0 - 0 + \det \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \\ &= 0 - 0 + (2 - (-2)) = 4 \end{aligned}$$

Definition 1.9. (triangular matrix) An $n \times n$ matrix A is triangular if all of its entries above (or below) the main diagonal are all zero.

Theorem 1.10. If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A, i.e. $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Note 1.11. If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then $\det(A)$ is also denoted by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

2 Properties of determinants

The determinant has many properties and most of them lie in how it changes when row operations are performed.

Theorem 2.1. Let A and B be $n \times n$ matrices. Then

(a) If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det(A) = -\det(B)$.

(b) If $A \xrightarrow{R_i + cR_j \to R_i} B$, then $\det(A) = \det(B)$.

(c) if $A \xrightarrow{cR_i \to R_i} B$, then $\det(A) = \frac{1}{c}\det(B)$.

Example 2.2.

(a) Compute the determinant of $A = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -2 & 1 \\ -1 & -4 & 3 \end{bmatrix}$.

The strategy is to reduce A to echelon form and then use the fact that the determinant of a triangular matrix is the product of the diagonal entries. We have

$$\begin{aligned} \det(A) &= \begin{vmatrix} 0 & 3 & -1 \\ 1 & -2 & 1 \\ -1 & -4 & 3 \end{vmatrix} \text{ Interchange rows 1 and 2. This will reverse the sign of the determinant} \\ &= -\begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ -1 & -4 & 3 \end{vmatrix} \text{ We replace } R_2 + R_3 \to R_3 \\ &= -\begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & -6 & 4 \end{vmatrix} \text{ The row replacement does not change the determinant. Now we replace } R_3 + 2R_2 \to R_3 \\ &= -\begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{vmatrix} \text{ The row replacement does not change the determinant.} \\ &= -1 \times 3 \times 2 = -6 \end{aligned}$$

Since the matrix is triangular, the determinant is just the product of the entries of the main diagonal.

(b) **Exercise**: Compute the determinant of $A = \begin{bmatrix} 3 & -6 & 3 \\ 0 & 1 & -1 \\ 1 & -2 & -1 \end{bmatrix}$

Note 2.3. Let E be an elementary matrix. Then

$$\det(E) = \begin{cases} 1 & \text{if } E \text{ is a row replacement} \\ -1 & \text{if } E \text{ is an interchange} \\ r & \text{if } E \text{ is a scale by } r \end{cases}$$

Fact 2.4. Suppose that A is a square matrix and that we put A into row echelon form U by using row replacements and row interchanges only (no row scaling). If there are r interchanges then by properties (a) and

(b) of Theorem 2.1, we have

$$\det(A) = (-1)^r \det(U).$$

Since U is in row echelon form, then U is triangular. Hence $\det(U)$ is the product of the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$.

If A is invertible, then A can be reduced to I_n and so the entries u_{ii} of U are all pivots (nonzero). Thus $\det(A) = (-1)^r \det(U) \neq 0$. If A is not invertible, then at least one of the diagonal entries u_{ii} of U is zero, so $\det(A) = (-1)^r \det(U) = 0$.

Proposition 2.5. Suppose A is an $n \times n$ matrix and U is an echelon form of A obtained through the use of row replacements and row interchanges only. Let r denote the number of row interchanges used to obtain U. Then

$$\det(A) = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

Theorem 2.6. A square matrix A is invertible if and only if $\det(A) \neq 0$.

Corollary 2.7. If the matrix A has two rows or two columns the same, then $\det(A) = 0$ and A is nonivertible (singular).

Proposition 2.8. Let A be an $n \times n$ matrix. Then

$$\det(A) = \det(A^T)$$

Proposition 2.9. If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Note 2.10. In general $\det(A + B) \neq \det(A) + \det(B)$.