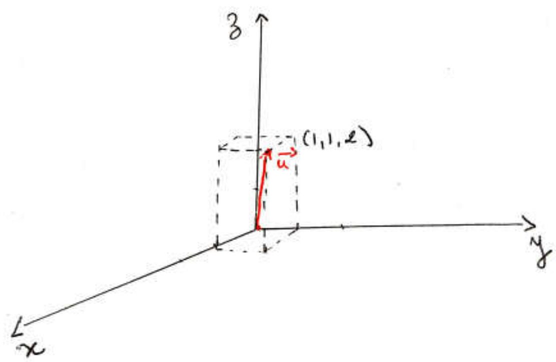
DAY 5: AUGUST 31st

(VECTOR EQUATIONS: continued)

Vectors in $\mathbb{R}^3$

A point with coordinate (a, b, c) in $\mathbb{R}^3$ corresponds to the vector $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$



Theorem 3.9. Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{v}$.

(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

(3) $(\vec{u} + \vec{0}) = \vec{0} = \vec{0} + \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0} = -\vec{u} + \vec{u}$.

(5) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.

(6) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$.

(7) $c(d\vec{u}) = (cd)\vec{u}$.

(8) $1\vec{u} = \vec{u}$.

Linear Combination and spanning

Definition 3.10 (Linear Combination)**.** Given vectors $\vec{v}_1, \vec{v_2}, \ldots, \vec{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_n$ the vector

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p$$

is called a linear combination of $\vec{v}_1, \vec{v_2}, \ldots, \vec{v}_p$.

Example 3.11. We have

$$2\begin{bmatrix}1\\0\\-1\end{bmatrix}-\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}1\\-2\\-1\end{bmatrix}$$

The vector $\vec{w}=\begin{bmatrix}1\\-2\\-1\end{bmatrix}$ is a linear combination of the vectors $\vec{u}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ (as $\vec{w}=2\vec{u}-\vec{v}$).

Suppose that we have the following linear combination:

$$x_1\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}+x_2\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+x_3\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}$$

This is the same as

$$\begin{bmatrix}a_1x_1+b_1x_2+c_1x_3\\a_2x_1+b_2x_2+c_2x_3\\a_3x_1+b_3x_2+c_3x_3\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}$$

By the equality of two vectors, we have the following equalities:

$$a_1x_1+b_1x_2+c_1x_3=d_1$$
$$a_2x_1+b_2x_2+c_2x_3=d_2$$
$$a_3x_1+b_3x_2+c_3x_3=d_3$$

It follows that the vector $\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}$ is a linear combination of $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix},\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix},\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$ if the above linear system has solutions.

FACT:

A vector equation $x_1\vec{v}_1+x_2\vec{v}_2+\ldots+x_p\vec{v}_p=\vec{b}$ has the same solutions as the system of linear equations corresponding to the augmented matrix

(A) $$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_p & \vec{b}\end{bmatrix}$$

In particular, $\vec{b}$ can be written (or generated) as a linear combination of $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_p$ if and only if the linear system with augmented matrix (A) is consistent.

Example 3.12. Is $\vec{b}=\begin{bmatrix}2\\-1\\6\end{bmatrix}$ a linear combination of $\vec{v}_1=\begin{bmatrix}1\\-2\\0\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}0\\1\\2\end{bmatrix}$?

The vector equation is

$$x\begin{bmatrix}1\\-2\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\2\end{bmatrix}=\begin{bmatrix}2\\-1\\6\end{bmatrix}$$

The augmented matrix of the corresponding linear system is given by

(B)
$$\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -1 \\ 0 & 2 & 6 \end{bmatrix}$$

EXERCISE: Reduce the matrix (B) to reduced echelon form.

The echelon form of matrix (B) is given by

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the system is consistent, and $x = 2$ and $y = 3$. Therefore $\vec{b} = 2\vec{v}_1 + 3\vec{v}_2$, so $\vec{b}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Definition 3.13 (Spanning)**.** Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$. The subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

$$\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p \colon c_1, c_2, \ldots, c_p \in \mathbb{R}\}.$$

NOTE:

- The span of $\vec{0}$ in $\mathbb{R}^n$ is the single vector $\vec{0}$.
- The span of a single nonzero vector in $\mathbb{R}^2$ and $\mathbb{R}^3$ is a line through $\vec{0}$.
- The span of two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, with $\vec{v} \neq c\vec{u}$, is a plan through $\vec{0}$.

Example 3.14. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -9 \\ -30 \\ 31 \end{bmatrix}$

$\mathrm{Span}(\vec{v}_1, \vec{v}_2)$ is a plane in $\mathbb{R}^3$, is $\vec{b}$ in that plane?

This question is the same as: is $\vec{b}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$. The approach to solve this is the same as in Example 3.12.