DAY 23: Friday, October 12th

3 Cramer's Rule, Volume, and Linear Transformations (Applications of determinants)

Cramer's Rule

Cramer's rule is used to solve linear systems with an invertible matrix coefficient.

Definition 3.1. Let A be an $n \times n$ matrix. Suppose $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$. For any $\vec{b}$ in $\mathbb{R}^n$, let $A_i(\vec{b})$ be the matrix obtained from A by replacing column i by the vector $\vec{b}$. That is

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{i-1} & \vec{b} & \vec{a}_{i+1} & \cdots & \vec{a}_n \end{bmatrix}$$

Theorem 3.2 (Cramer's Rule)**.** Let A be an $n \times n$ invertible matrix. For any $\vec{b}$ in $\mathbb{R}^n$, the unique solution $\vec{x}$ of the equation $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}, \text{ for } i = 1, 2, \ldots n.$$

Example 3.3.

(a) Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:

Since $\det(A) = -1$, the equation $A\vec{x} = \vec{b}$ has a unique solution. We have

$$A_1(\vec{b}) = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \text{ and } A_2(\vec{b}) = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

with

$$\det(A_1(\vec{b})) = 4 \text{ and } \det(A_2(\vec{b})) = -3.$$

By Cramer's Rule, we have

$$\begin{aligned} x_1 &= \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{4}{-1} = -4 \\ x_2 &= \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-3}{-1} = 3 \end{aligned}$$

So $\vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

(b) Consider the linear system for which s is an unspecified parameter.

$$2sx_1 + 2x_2 = 1$$
$$5x_1 + x_2 = -1$$

For which s the system has a unique solution. For such s, describe the solution.

Solution:

The system is equivalent to the matrix equation $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 2s & 2 \\ 5 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence,

$$A_1(\vec{b}) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } A_2(\vec{b}) = \begin{bmatrix} 2s & 1 \\ 5 & -1 \end{bmatrix}$$

We have $\det(A) = 2s - 10 = 2(s-5)$. The system has a unique solution if and only if $\det(A) \neq 0$, that is $s - 5 \neq 0$, so $s \neq 5$. For such an s, the solution is (x_1, x_2) where

$$\begin{aligned} x_1 &= \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{3}{2(s-5)} \\ x_2 &= \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{-2s-5}{2(s-5)}. \end{aligned}$$

A Formula for A^{-1}:

The formula we have seen to compute the inverse of 2×2 matrices extends to higher dimensions.

Definition 3.4. Let A be an $n \times n$ matrix. We define the classical adjoint (or adjugate) of A to be the $n \times n$ matrix given by

$$\mathrm{Adj}A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j} \det(A_{ij})$ the (i,j)-cofactor of A.

Note 3.5.

(a) If C is the matrix formed by the cofactors of A, that is

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

Then $\mathrm{Adj}A = C^T$ (the transpose of C).

(b) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\mathrm{Adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Theorem 3.6. (Adjoint Inverse Formula) Let A be an invertible matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}A$$

Example 3.7. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ by using the adjoint inverse formula.

Determinants as Area or Volume

Determinants have also some geometric interpretations.

Theorem 3.8.

(a) If A is a 2×2 matrix, then the area of the parallelogram determined by its column is $|\det(A)|$.

(b) If A is a 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det(A)|$.

Note 3.9. Let $\vec{v}_1$ and $\vec{v}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\vec{v}_1$ and $\vec{v}_2$ equals the area of the parallelogram determined by $\vec{v}_1$ and $\vec{v}_2 + c\vec{v}_1$.

Example 3.10. Compute the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1)$ and $(6,4)$.

Solution:

By Note 3.9, we can translate the parallelogram to one having the origin $(0,0)$ as a vertex. For example, subtract $(-2,-2)$ from each of the vertices. The new parallelogram has vertices $(0,0), (2,5), (6,1)$ and $(8,6)$, and it has the same area as the same area as the first parallelogram. The new parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

We have $\det(A) = -28$, hence the area od the parallelogram is $|\det(A)| = 28$.

Linear Transformations

If T is a linear transformation with domain $\mathbb{R}^2$ or $\mathbb{R}^3$, and if S is a set in the domain of T, then we denote by $T(S)$ the set of images of points (vectors) in S. We will see how the areas (or volumes) of S and $T(S)$ are related.

Theorem 3.11.

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\text{Area}(T(S)) = |\det(A)|\text{Area}(S).$$

(b) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation determined by a 3×3 standard matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\text{Volume}(T(S)) = |\det(A)|\text{Volume}(S).$$

(c) In general, if S is any region of $\mathbb{R}^2$ or of $\mathbb{R}^3$, then the formulas for $\text{Area}(T(S))$ and $\text{Volume}(T(S))$ in (a) and (b) hold.