DAY 21: Monday, October 8th

Chap III: DETERMINANTS

1 Introduction to Determinants

Recall that an $n \times n$ square matrix A is invertible if and only if it can be row reduced to the identity matrix I_n. That is A has pivot in every column. In the case of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A is invertible if and only if the determinant $\det(A) = ad - bc$ is different from 0. In this section, we recursively define the determinant of larger matrices by using invertibility properties.

We start from a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We assume that A is inveritble and that $a_{11} \neq 0$. We reduce A to row echelon form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[a_{11}R_3 \to R_3]{a_{11}R_2 \to R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \xrightarrow[R_3 - a_{31}R_1 \to R_3]{R_2 - a_{21}R_1 \to R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Since A is invertible, either the $(2,2)$-entry or the $(3,2)$-entry of the last matrix is nonzero. We assume that the $(2,2)$-entry is nonzero, otherwise, we make a row interchange. We continue the row echelon form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix} \xrightarrow{(a_{11}a_{22} - a_{12}a_{21})R_3 \to R_3}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & (a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{22} - a_{12}a_{21}) & (a_{11}a_{33} - a_{13}a_{31})(a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix} \xrightarrow{R_3 - (a_{11}a_{32} - a_{12}a_{31})R_2 \to R_3}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & \alpha \end{bmatrix}$$

where, after cancellation and factoring,

$$\alpha = a_{11}\big(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\big) = a_{11}\Delta.$$

Since A is invertible and $a_{11} \neq 0$, the number

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

must be nonzero (A must have a pivot at the 3rd column). The quantity Δ is the determinant of the 3×3 matrix A. To recursively define the determinant for larger matrices, we rewrite the determinant of the 3×3

matrix in terms of the determinant of 2×2 matrices. We have

$$
\begin{aligned}
\Delta = & \quad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\
= & \quad (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22}) \\
= & \quad a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
\end{aligned} \tag{1}
$$

Definition 1.1. Let A be an $n \times n$ square matrix. We define A_{ij} to be the submatrix of A formed by deleting row i and column j from A (A_{ij} is also called ijth minor of A).

Be careful with the notation a_{ij} (the (i, j)-entry of A) and A_{ij}.

Example 1.2.

$$
A = \begin{bmatrix} -1 & 3 & 4 & -3 \\ 0 & 0 & -2 & 1 \\ 3 & 5 & 6 & -4 \\ -4 & 0 & -3 & -3 \end{bmatrix}
$$

Then

$$
A_{34} = \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & -2 \\ -4 & 0 & -3 \end{bmatrix}
$$

We have from equality (1) that

$$
\Delta = a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + a_{13} \cdot \det(A_{13}).
$$

In general, the determinant of an $n \times n$ matrix is defined by the determinant of $(n-1) \times (n-1)$ submatrices.

Definition 1.3 (Determinant of an $n \times n$ matrix)**.** Let A be an $n \times n$ matrix. The determinant of A is

$$
\begin{aligned}
\det(A) &= a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + \cdots + (-1)^{n+1} \cdot \det(A_{1n}) \\
&= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \cdot \det(A_{1j})
\end{aligned}
$$

Example 1.4. Compute the determinant of $A = \begin{bmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{bmatrix}$ By definition, we have

$$
\begin{aligned}
\det(A) &= 2 \cdot \det(A_{11}) + (-3)\det(A_{12}) + (-4) \cdot \det(A_{13}) \\
&= 2\det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} - 3\det \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - 4\det \begin{bmatrix} 4 & 0 \\ 5 & 1 \end{bmatrix} \\
&= 2(-5) - 3(-1) - 4(4) = -23
\end{aligned}
$$

Definition 1.5 (Cofactors of a matrix)**.** The (i, j)-cofactor of an $n \times n$ matrix A is the number C_{ij} given by

$$
C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})
$$

Rewriting the definition of the determinant using cofactors gives

$$
\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.
$$

This formula is the cofactor expansion across the first row of A.

Theorem 1.6. Let A be an $n \times n$ square matrix. Then, the determinant $\det(A)$ of A can be computed by a cofactor expansion across any row or down any column.

(a) The expansion across the ith row is (i is fixed)

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(b) The expansion down the jth column is (j is fixed)

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Note 1.7. The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix. The sign of $(-1)^{i+j}$ is determined in the following pattern.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Example 1.8. Use a cofactor expansion down the third column to compute the determinant of $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix}$.

We have

$$\begin{aligned} \det(A) &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \\ &= 0 \cdot (-1)^{1+3} \det(A_{13}) + 0 \cdot (-1)^{2+3} \det(A_{23}) + 1 \cdot (-1)^{3+3} \det(A_{33}). \\ &= 0 - 0 + \det \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \\ &= 0 - 0 + (2 - (-2)) = 4 \end{aligned}$$

Definition 1.9. (triangular matrix) An $n \times n$ matrix A is triangular if all of its entries above (or below) the main diagonal are all zero.

Theorem 1.10. If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A, i.e. $\det(A) = a_{11}a_{22} \cdots a_{nn}$.

Note 1.11. If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then $\det(A)$ is also denoted by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$