DAY 34: Friday, November 9th

7 Change of Basis

Recall that if V is a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and if $\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$ is a vector in V, then the coordinates of $\vec{u}$ relative to $\mathcal{B}$ is given by

$$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

and the coordinate mapping $T : V \to \mathbb{R}^n$, defined by $T(\vec{u}) = [\vec{u}]_{\mathcal{B}}$ is a one-to-one linear transformation onto $\mathbb{R}^n$.

Example 7.1. Let V be a 2-dimensional vector space and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for V. Suppose

$$\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2 \quad \text{and} \quad \vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$$

Let $\vec{x} = 3\vec{b}_1 + \vec{b}_2$, i.e. $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{C}}$.

Solution

Consider the coordinate mapping $\vec{u} \mapsto T(\vec{u}) = [\vec{u}]_{\mathcal{C}}$. Since T is linear, we have

$$\begin{aligned} [\vec{x}]_{\mathcal{C}} &= [3\vec{b}_1 + \vec{b}_2]_{\mathcal{C}} \\ &= 3[\vec{b}_1] + [\vec{b}_2]_{\mathcal{C}} \end{aligned}$$

This a vector equation (in $\mathbb{R}^2$) which is equivalent to the matrix equation

(vi) $$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Since $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, the equation (vi) becomes

$$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Hence $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, that is $\vec{x} = 6\vec{c}_1 + 4\vec{c}_2$.

Theorem 7.2. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be bases of an n-dimensional vector space V. Then there is a unique matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is

(vii) $$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix}$$

Note 7.3. The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent, hence $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. Multiplying the equality (vi) by $\left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}$ yields

$$\left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$$

Thus the matrix $\left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}$ is the matrix that convert $\mathcal{C}$-coordinates to $\mathcal{B}$- coordinates, that is

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \left(P_{\mathcal{C}\leftarrow\mathcal{B}}\right)^{-1}$$

Change of Basis in $\mathbb{R}^n$

Recall that the set $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, where $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, is called the standard basis for $\mathbb{R}^n$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a nonstandard basis for $\mathbb{R}^n$. We have seen from Section 4 that matrix $P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$, called the change-of-coordinates matrix, converts $\mathcal{B}$-coordinates to $\mathcal{E}$-coordinates. If $\vec{x}$ is in $\mathbb{R}^n$, then

$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

That is $P_{\mathcal{B}} = P_{\mathcal{E}\leftarrow\mathcal{B}}$.

Example 7.4. Let $\vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$, $\vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and $\vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for $\mathbb{R}^2$. Find the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution

By Theorem 7.2, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{bmatrix}$. Hence we need to compute $[\vec{b}_1]_{\mathcal{C}}$ and $[\vec{b}_2]_{\mathcal{C}}$. So we solve the following equations

$$\begin{aligned} \vec{b}_1 &= \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\ \vec{b}_2 &= \beta_1\vec{c}_1 + \beta_2\vec{c}_2 = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \end{aligned}$$

We solve these equations simultaneously by augmenting the coefficient matrix with $\vec{b}_1$ and $\vec{b}_2$ and row reducing it to reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+4R_1\to R_2} \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{bmatrix} \xrightarrow{1/7R_2\to R_2} \begin{bmatrix} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{bmatrix} \xrightarrow{R_1-3R_2\to R_1} \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix}$$

Therefore $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. So $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$

Note 7.5. Note from the previous example that

$$\left[\begin{array}{cc|cc} \vec{c}_1 & \vec{c}_2 & \vec{b}_1 & \vec{b}_2 \end{array}\right] \sim \left[\begin{array}{c|c} I_2 & P_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

Fact 7.6. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ be nonstandard bases for $\mathbb{R}^n$. For each $\vec{x}$ in $\mathbb{R}^n$, we have

$$\begin{aligned} \vec{x} &= P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \\ \vec{x} &= P_{\mathcal{C}}[\vec{x}]_{\mathcal{C}} \end{aligned}$$

with $P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$ and $P_{\mathcal{C}} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix}$. Therefore

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\vec{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

So

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$$

$P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}$