DAY 35: Monday, November 12th

Chap V: EIGENVECTORS and EIGENVALUES

1 Eigenvectors and Eigenvalues

Definition 1.1. Let A be an $n \times n$ matrix. A nonzero vector $\vec{x}$ in $\mathbb{R}^n$ is called an **eigenvector** of A if there exists some scalar λ such that $A\vec{x} = \lambda\vec{x}$.

A scalar λ is called an **eigenvalue** of A if there exists a nonzero vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$.

If $A\vec{x} = \lambda\vec{x}$ then we say that $\vec{x}$ is an eigenvector corresponding to λ.

Note 1.2. An eigenvector $\vec{x}$ must be s nonzero vector, i.e. we exclude the trivial case $A\vec{0} = \vec{0}$. However, it is possible that the scalar 0 is an eigenvalue of A.

Example 1.3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(a) Is $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenvector of A?

(b) Show that 5 is an eigenvalue of A.

Solution

(a) To check if $\vec{x}$ is an eigenvector of A, we compute $A\vec{x}$ and see if we have the form $A\vec{x} = \lambda\vec{x}$ for some scalar λ. We have

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

It follows that $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue $\lambda = 0$.

(b) The scalar 5 is an eigenvalue of A if and only if the equation $A\vec{x} = 5\vec{x}$ has a nontrivial solution. This equation is equivalent to

$$A\vec{x} - 5\vec{x} = \vec{0}$$

That is

$$(A - 5I_2)\vec{x} = \vec{0}$$

This a homogeneous matrix equation of the form $B\vec{x} = \vec{0}$ where

$$B = A - 5I_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

Reducing the augmented matrix of $B\vec{x} = \vec{0}$, we have

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that a nonzero solutions of $B\vec{x} = \vec{0}$ has the form

$$\vec{x} = \vec{x}_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad x_2 \neq 0 \text{ in } \mathbb{R}.$$

That is, the equation $A\vec{x} = 5\vec{x}$ has nonzero solutions, namely $\vec{x} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$, $x_2 \neq 0$ in $\mathbb{R}$, hence 5 is an eigenvalue of A. For example taking $x_2 = 1$, the vector $\vec{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 5 of A as

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 5 \end{bmatrix} = 5 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Note 1.4. The equation $A\vec{x} = \lambda\vec{x}$ is equivalent to the homogeneous equation $(A - \lambda I_n)\vec{x} = \vec{0}$. Hence, λ is an eigenvalue of A if and only if $(A - \lambda I_n)\vec{x} = \vec{0}$ has a nontrivial solution. In other words, if λ is an eigenvalue of A, then the eigenvectors of A corresponding to λ is the null space of $A - \lambda I_n$

Definition 1.5. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The null space $\text{Nul}(A - \lambda I_n)$ is called the **eigenspace** of A corresponding to the eigenvalue λ.

Example 1.6. The scalar 2 is an eigenvalue of the matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find a basis for the corresponding eigenspace.

Solution

The eigenspace of A corresponding to 2 is the subspace $\text{Nul}(A - 2I_3)$. We have

$$A - 2I_3 = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

The augmented matrix of the equation $(A - 2I_3)\vec{x} = \vec{0}$ is row reduced to

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The parametric vector form of the solutions is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

Hence a basis for the eigenspace of A corresponding to 2 is $\mathcal{B} = \{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \}$. So it is a 2-dimensional subspace of $\mathbb{R}^3$.

Theorem 1.7. The eigenvalues of a triangular matrix are its diagonal entries.

Example 1.8. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. The eigenvalues of A are 4 and 1.

Fact 1.9. The equation $A\vec{x} = 0\vec{x}$ which is equivalent to $A\vec{x} = \vec{0}$ has a solution if and only if A is not invertible. Therefore, 0 is an eigenvalue of A if and only if A is not invertible.

Theorem 1.10. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are eigenvectors corresponding to disctinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly independent.