MATH 3110 - Fall 2018

Practice 3

Exercise I

Consider the following linear system:

$$(E) \qquad \begin{aligned} 2x_1 - x_2 + x_3 &= 2 \\ 2x_1 - x_3 &= 3 \\ -x_2 + hx_3 &= k \end{aligned}$$

where h and k are parameters.

(a) Write the matrix equation corresponding to (E).

(b) Determine the value(s) of h and k such that (E):

- (E) has a unique solution.
- (E) has infinitely many solutions.
- (E) has no solution.

(c) Find the general solution of (E) when $h = 2$ and $k = -1$.

Exercise II

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -5 & -1 \\ 0 & 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}$.

(a) Compute $\det(A)$ by using cofactor expansion, and compute the inverse of A by using the Inverse Formula.

(b) Compute $\det(B)$ by using row reduction, and compute the inverse of B by using Gauss-Jordan elimination.

Exercise III

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & -1 \\ 1 & -4 & 1 \end{bmatrix}$$

(a) Compute $T(\vec{u})$ where $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(b) Show that $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is in $\mathrm{Im}(T)$ (the image of T). Give a vector $\vec{x}$ with $T(\vec{x}) = \vec{v}$.

(c) Is T onto $\mathbb{R}^3$? Justify your answer.

(d) Is T one to one? Justify your answer.

Exercise III (6 points)

Let $W = \left\{ \begin{bmatrix} 2a-b+c \\ -2a+2b \\ 3a+b+4c \end{bmatrix} \mid a,b,c \in \mathbb{R} \right\}$ be a subset of $\mathbb{R}^3$.

(a) Show that W is a subspace of $\mathbb{R}^3$.

(b) Show that the set $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, with $\vec{b}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, is a basis for W.

(c) Show that $\vec{x} = \begin{bmatrix} -5 \\ 4 \\ -10 \end{bmatrix}$ is in W.

(d) What is the $\mathcal{B}$-coordinate $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$?

(e) Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be another basis of W, such that $\vec{b}_1 = 2\vec{c}_1 - \vec{c}_2$ and $\vec{b}_2 = -3\vec{c}_1 + \vec{c}_2$. What is the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$? Compute the $\mathcal{C}$-coordinate $[\vec{x}]_{\mathcal{C}}$ of the vector $\vec{x}$ in part c).

(f) Find an orthogonal basis and an orthonormal basis for W.

Exercise IV

Let $A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & 1 & -3 & 3 \\ -2 & 1 & 3 & -3 \end{bmatrix}$ be a 3×4 matrix.

(a) What is the largest possible rank of A? Justify your answer.

(b) Is $\vec{z} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ in Nul(A)?

(c) what is the value of m such that Col(A) is a subspace of $\mathbb{R}^m$?

(d) The matrix A is reduced to the matrix $B = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

((d).1) Determine the rank of A and the dimension of Nul(A).

((d).2) Find a basis $\mathcal{B}_C$ for Col(A).

((d).3) Find a basis $\mathcal{B}_N$ for Nul(A).

((d).4) Find a basis $\mathcal{B}_R$ of Row(A).

Exercise V

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -5 & -1 & 5 \\ 1 & 0 & -2 \end{bmatrix}$ be a 3×3 matrix.

(a) Compute the characteristic polynomial of A.

(b) Determine the eigenvalues of A? Give their respective multiplicity.

(c) Determine a basis for the eigenspace corresponding to each eigenvalue of A.

(d) Diagonalize A.

(e) Find an LU factorization of A.

Exercise VI

Let A be a 3×3 matrix which is factored in the form PDP^{-1} as follows

$$A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -1 \end{bmatrix}$$

Using the Diagonalization Theorem, find the eigenvalues of A and a basis for each corresponding eigenspace.