DAY 30: Monday, October 29th

Proposition 4.7. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for a vector space V. Let $T : V \to \mathbb{R}^n$, with $T(\vec{v}) = \left[\ \vec{v} \ \right]_{\mathcal{B}}$, be the mapping coordinate determined by $\mathcal{B}$. Then, T is one-to-one linear transformation onto $\mathbb{R}^n$.

Fact 4.8. Since the coordinate mapping $T : V \to \mathbb{R}^n$ is a linear transformation, we have

(a) $\left[\ \vec{u} + \vec{v} \ \right]_{\mathcal{B}} = \left[\ \vec{u} \ \right]_{\mathcal{B}} + \left[\ \vec{v} \ \right]_{\mathcal{B}}$ for $\vec{u}, \vec{v}$ in V.

(b) $\left[\ c\vec{v} \ \right]_{\mathcal{B}} = c\left[\ \vec{v} \ \right]_{\mathcal{B}}$ for $\vec{v}$ in V and scalar c.

(c) $\left[\ \vec{v} \ \right]_{\mathcal{B}} = \vec{0}$ if and only if $\vec{v} = \vec{0}$.

Definition 4.9. Let V and W be vector spaces. We say that V and W are isomorphic if there exists a one-to-one linear transformation $T : V \to W$ onto W (we say that T is an isomorphism). In this case we write $V \cong W$.

Note that if V is a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$, then V isomorphic to $\mathbb{R}^n$ as the coordinate mapping $T : V \to \mathbb{R}^n$, $T(\vec{v}) = \left[\ \vec{v} \ \right]_{\mathcal{B}}$, is an isomorphism. This makes V act like $\mathbb{R}^n$.

Note 4.10. For the set $\mathbb{P}_n$ of polynomials of degree at most n, the standard basis is $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$. If $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$, then

$$\left[\ p(t) \ \right]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The mapping coordinate $T : \mathbb{P}_n \to \mathbb{R}^{n+1}$, with $T(p(t)) = \left[\ p(t) \ \right]_{\mathcal{B}}$, is an isomorphism so we can study $\mathbb{P}_n$ as $\mathbb{R}^{n+1}$.

Example 4.11. Show that $p_1(t) = 1 + 2t^2, p_2(t) = 4 + t + 5t^2$ and $p_3(t) = 3 + 2t$ are linearly dependent in $\mathbb{P}_2$.

Solution

Since the coordinate mapping $T : \mathbb{P}_2 \to \mathbb{R}^3$ is an isomorphism, if $T(p_1(t)), T(p_2(t)), T(p_3(t))$ are linearly dependent in $\mathbb{R}^3$, then $p_1(t), p_2(t), p_3(t)$ are also linearly dependent in $\mathbb{P}_2$. We have

$$T(p_1(t)) = \left[\ p_1(t) \ \right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, T(p_2(t)) = \left[\ p_2(t) \ \right]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, T(p_3(t)) = \left[\ p_3(t) \ \right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

The matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix}$ is row equivalent to $B \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, hence $\det(A) = 0$ and the columns of A are linearly dependent in $\mathbb{R}^3$. If follows that $p_1(t), p_2(t), p_3(t)$ are linearly dependent in $\mathbb{P}_2$. In fact we can check that

$$p_3(t) = 2p_2(t) - 5p_1(t).$$

5 The Dimension of a Vector Space

Theorem 5.1. Let V be a vector space that has a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$. Then any set S of vectors of V containing more than n vectors is linearly dependent. That is if $\#S > n$, then S is linearly dependent.

Proof. Let $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ with $p > n$. Then the set $S' = \{\left[\ \vec{u}_1\ \right]_{\mathcal{B}}, \left[\ \vec{u}_2\ \right]_{\mathcal{B}}, \ldots, \left[\ \vec{u}_p\ \right]_{\mathcal{B}}\}$ is a set of vectors of $\mathbb{R}^n$ which is linearly dependent (as $p > n$). Therefore, there exist $c_1, c_2, \ldots, c_p$ scalars not all zero with

$$c_1 \left[\ \vec{u}_1\ \right]_{\mathcal{B}} + c_1 \left[\ \vec{u}_2\ \right]_{\mathcal{B}}, \cdots + c_p \left[\ \vec{u}_p\ \right]_{\mathcal{B}} = \vec{0}$$

Since the coordinate mapping is linear, we have

$$\left[\ c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p\ \right]_{\mathcal{B}} = \vec{0}$$

Now, since the coordinate mapping is one-to-one, we have

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p = \vec{0}$$

Not all of the c_i's are zero, so $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is linearly dependent. $\square$

Theorem 5.2. If V is a vector space with a basis of size n, then every basis for V has exactly n vectors.

Proof. Let $\mathcal{B}_1 = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{B}_2 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ be bases for V. Since $\mathcal{B}_2$ is linearly independent, then by Theorem 4.12, $p \leq n$.

Conversely, as $\mathcal{B}_2$ is a basis for V and $\mathcal{B}_1$ is linearly independent, we have $n \leq p$. We conclude that $p = n$. $\square$

Definition 5.3. If V is vector space spanned by a finite set, then V is said to be finite-dimensional and the dimension of V, denoted by $\dim(V)$ is the number of the vectors in a basis for V. The dimension of the vector space $\{\vec{0}\}$ is defined to be 0. If V is not spanned by a finite set, the V is said to be infinite-dimensional.

Example 5.4. We have $\dim(\mathbb{R}^n) = n$ and $\dim(\mathbb{P}_n) = n + 1$.

Example 5.5. Let $H = \left\{ \begin{bmatrix} -2a + b + 5c \\ b + c \\ a + 3b + c \end{bmatrix} \mid a, b, c \text{ in } \mathbb{R} \right\}$. Find the dimension of H.

Solution

Clearly, H is the set of all linear combinations of the vectors

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

That is $H = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Since $\vec{v}_3 = -2\vec{v}_1 + \vec{v}_2$, by Spanning Set Theorem, we have

$$H = \text{Span}(\vec{v}_1, \vec{v}_2)$$

Furthermore, $\vec{v}_1$ and $\vec{v}_2$ are not multiple of one another, hence $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. Therefore $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for H and $\dim(H) = 2$.

Subspace of a Finite Dimension Space

Theorem 5.6. Let V be a finite-dimensional vector space and let H be a subspace of V. Then any linearly independent set H can be expanded to a basis for H. Furthermore H is also finite-dimensional and $\dim(H) \leq \dim(V)$.

Theorem 5.7 (Basis Theorem)**.** Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

The Dimension of $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Proposition 5.8. The dimension of $\mathrm{Nul}(A)$ is the number of free variables in the equation $A\vec{x} = \vec{0}$, and the dimension of $\mathrm{Col}(A)$ is the number of pivots columns in A.

Example 5.9. Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & 2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$. Determine $\dim(\mathrm{Nul}(A))$ and $\dim(\mathrm{Col}(A))$.

Solution

A is row equivalent to the matrix $B = \begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We see that the variables x_2, x_4 and x_5 are free variables hence $\dim(\mathrm{Nul}(A)) = 3$. Moreover, column 1 and 3 are the pivots columns hence $\dim(\mathrm{Col}(A)) = 2$.