DAY 33: Wednesday, November 7th

6 The Rank of a Matrix

The Row Space

Let A be an $m \times n$ matrix. Then we can view A as a collection of rows instead of a collection of columns. Note that each row of A has n entries, so we can identify each row as a vector in $\mathbb{R}^n$.

Definition 6.1. Let A be an $m \times n$ matrix. The row space of A, denoted by $\mathrm{Row}(A)$, is the set of all linear combinations of the rows of A.

Theorem 6.2. If two matrices A and B are row equivalent then $\mathrm{Row}(A) = \mathrm{Row}(B)$. If B is in row echelon form then the nonzero rows of B form a basis for $\mathrm{Row}(A)$ and $\mathrm{Row}(B)$.

Note 6.3. Theorem 6.2 implies that the dimension of $\mathrm{Row}(A)$ equals the numbers of pivots in A, which is the same as the dimension of $\mathrm{Col}(A)$, i.e.

$$\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col})(A) = \#\text{pivots}$$

Example 6.4. Let $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$. Matrix A is row equivalent to $B = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
Find bases for $\mathrm{Row}(A), \mathrm{Col}(A)$ and $\mathrm{Nul}(A)$.

Solution

By Theorem 6.2, the first three rows of B form a basis for $\mathrm{Row}(A)$ as well as for $\mathrm{Row}(B)$, i.e. the set

$$\mathcal{B}_0 = \{(1,4,0,2,-1),(0,0,1,-1,8),(0,0,0,0,-4)\}$$

is basis for $\mathrm{Row}(A)$.

Since B is a row echelon form of A, we see that columns 1, 3, and 5 are pivot columns hence, the set

$$\mathcal{B}_1 = \{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \}$$

is a basis for $\mathrm{Col}(A)$.

Using the row echelon form B of A, the solutions of the homogeneous equation $A\vec{x} = \vec{0}$ can be written in parametric vector form as

$$\vec{x} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}$$

Hence, the set

$$\mathcal{B}_2 = \{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \}$$

is a basis for $\mathrm{Nul}(A)$.

The Rank Theorem

Definition 6.5. Let A be a matrix. The **rank** of A is the dimension of the column space of A.

Theorem 6.6 (Rank-Nullity Theorem)**.** Let A be an $m \times n$ matrix, then

$$\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n$$

Proof. On one hand, we have

$$\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \#\text{ pivots}$$

On the other hand, we have

$$\dim(\mathrm{Nul}(A)) = \#\text{ free variables} = n - \#\text{ pivots}$$

Therefore,

$$\dim(\mathrm{Nul}(A)) + \mathrm{rank}(A) = (n - \#\text{ pivots}) + \#\text{ pivots} = n.$$

□

Theorem 6.7 (Invertible Matrix Theorem (continued))**.** Let A be an $n \times n$ matrix. Then, the following statements are equivalent:

(a) A is invertible

(b) $\mathrm{Col}(A) = \mathbb{R}^n$

(c) $\dim(\mathrm{Col}(A)) = n$

(d) $\mathrm{rank}(A) = n$

(e) $\mathrm{Nul}(A) = \{\vec{0}\}$

(f) $\dim(\mathrm{Nul}(A)) = 0$.