DAY 26: Friday, October 19th

The Column Space of a Matrix

Definition 2.7. Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix. The column space of A, denoted by $\mathrm{Col}(A)$, is the set of all linear combinations of the columns of A. That is

$$\mathrm{Col}(A) = \mathrm{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$$

Theorem 2.8. The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Note 2.9. Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix.

(a) By definition, $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$ spanned by $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.

(b) The following statements are equivalent:

- $\mathrm{Col}(A) = \mathbb{R}^m$.
- For every $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
- The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is onto.

Example 2.10. Find a matrix A such that $W = \mathrm{Col}(A)$ where

$$W = \left\{ \begin{bmatrix} a+b \\ 2a-b \\ -3a \end{bmatrix} \mid a, b \text{ in } \mathbb{R} \right\}$$

Solution

We write each element of W as a linear combination of some vectors. For each $\vec{u}$ in W, we have

$$\begin{aligned} \vec{u} &= \begin{bmatrix} a+b \\ 2a-b \\ -3a \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ -3a \end{bmatrix} + \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \\ &= a \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2. \end{aligned}$$

It follows that $W = \mathrm{Span}(\vec{v}_1, \vec{v}_2)$. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -3 & 0 \end{bmatrix}$. Then $W = \mathrm{Col}(A)$.

Fact 2.11. Let A be an $m \times n$ matrix. We compare the null space and the column space of A as follows:

Null Space	Column Space
• $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.	• $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.
• Null space of A is implicitly defined: the vectors in $\text{Nul}(A)$ solve $A\vec{x} = \vec{0}$.	• $\text{Col}(A)$ is explicitly defined since it is spanned by described vectors (the columns of A).
• It takes time to describe $\text{Nul}(A)$ as row reduction of $\begin{bmatrix} A & \vec{0} \end{bmatrix}$ is required.	• It is easy to find vectors in $\text{Col}(A)$ since they are linear combinations of the columns of A.
• There is no obvious relation between $\text{Nul}(A)$ and the entries in A.	• There is an obvious relation between $\text{Col}(A)$ and the entries of A since the columns of A generate $\text{Col}(A)$.
• A typical vector $\vec{v}$ in $\text{Nul}(A)$ has the property $A\vec{v} = \vec{0}$.	• A typical vector $\vec{v}$ in $\text{Col}(A)$ has the property that $A\vec{x} = \vec{v}$ is consistent.
• It is easy to check if a given vector $\vec{v}$ is in $\text{Nul}(A)$ by computing $A\vec{v}$.	• It may take time to check if a given vector $\vec{v}$ is in $\text{Col}(A)$ as row reduction of $\begin{bmatrix} A & \vec{v} \end{bmatrix}$ is required.
• $\text{Nul}(A) = \{\vec{0}\}$ if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.	• $\text{Col}(A) = \mathbb{R}^m$ if and only if the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^m$.
• $\text{Nul}(A) = \{\vec{0}\}$ if and only the linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.	• $\text{Col}(A) = \mathbb{R}^m$ if and only if the linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Linear Transformations of Vector Spaces

Definition 2.12. Let V and W be vector spaces. A linear transformation T from V into W, denoted by $T : V \to W$, is a rule that assigns to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W, such that

(a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in V,

(b) $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in V and all scalars c.

Definition 2.13. Let $T : V \to W$ be a linear transformation.

(a) V is called the domain of T, and W is its codomain.

(b) The **kernel** of T, denoted by $\ker(T)$, is the set of all vectors $\vec{u}$ in V such that $T(\vec{u}) = \vec{0}$. That is

$$\ker(T) = \{\vec{u} \text{ in } V \mid T(\vec{u}) = \vec{0}\}$$

(c) The **range** or **image** of T, denoted by $\text{range}(T)$ or $\text{im}(T)$, is the set of all vectors in W which are of the form $T(\vec{v})$ for some $\vec{v}$ in V. That is

$$\text{range}(T) = \text{im}(T) = \{T(\vec{v}) \mid \vec{v} \text{ in } V\}$$

Fact 2.14. Let $T : V \to W$ be a linear transformation of vector spaces. Then,

(a) $\ker(T)$ is a subspace of V,

(b) $\operatorname{im}(T)$ is a subspace W.

(c) if $V=\mathbb{R}^n, W=\mathbb{R}^m$, and A is the standard matrix of T, then $\ker(T)$ the null space of A, and $\operatorname{im}(T)$ is the column space of A.

3 Linearly Independent Sets, Bases

Definition 3.1. Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V.

(a) The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly independent if the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

(b) The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly dependent if there exist scalars $c_1, c_2, \ldots, c_p$ not all zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0} \tag{II}$$

Equation (II) is called a linear dependence relation among $\vec{v}_1, \vec{v_2}, \ldots, \vec{x}_p$.

Fact 3.2. As in the case of $\mathbb{R}^n$, if V is a vector space, then

(a) A single vector $\vec{v}$ in V is linearly independent if and only if $\vec{v} \neq \vec{0}$.

(b) A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other.

(c) Any set of vectors containing the zero vector is linearly dependent.

Theorem 3.3. Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in V. Suppose $p \geq 2$ and $\vec{v}_1 \neq 0$. The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly dependent if and only if there is some $\vec{v}_j$, $j > 1$, such that $\vec{v}_j$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$.

Example 3.4. Consider the vector space $\mathbb{P}_2$ (the set of all polynomials of degree at most 2). Let $p_1(t) = t-1, p_2(t) = t^2+2t-3$ and $p_3(t) = t^2-1$. We have

$$\begin{aligned} p_2(t) - 2p_1(t) &= (t^2+2t-3) - 2(t-1) \\ &= t^2 - 1 = p_3(t) \end{aligned}$$

That is

$$p_3(t) = p_2(t) - 2p_1(t),$$

so

$$p_3(t) - p_2(t) + 2p_1(t) = 0$$

Hence the set $\{p_1(t), p_2(t), p_3(t)\}$ is linearly dependent.

Definition 3.5. Let V be a vector space and let H be a subspace of V.

(a) A basis for V is an indexed set of vectors $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ in V such that $\mathcal{B}$ is a linearly independent set and V is spanned by $\mathcal{B}$, that is

$$V = \operatorname{Span}(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p)$$

(b) A basis of H is a collection of vectors $\mathcal{B}_H = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ in H (hence in V) such that $\mathcal{B}_H$ is a linearly independent set and $H = \text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r)$.

Fact 3.6 (Invertible Matrix Theorem)**.** Recall that if A is an $n \times n$ matrix, the following statements are equivalent:

(a) A is invertible

(b) $\det(A) \neq 0$

(c) the columns of A are linearly independent

(d) the columns of A span $\mathbb{R}^n$.

Example 3.7. Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix}$ be an invertible matrix. By Fact 3.6, the set $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is linearly independent and $\mathbb{R}^n = \text{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$. Therefore the columns of A form a basis for $\mathbb{R}^n$.

Example 3.8. Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be the columns of the $n \times n$ identity matrix I_n. That is

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since $\det(I_n) \neq 0$, the set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is basis, called standard basis, for $\mathbb{R}^n$.

Example 3.9. Let $\mathcal{B}_1 = \left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$ and $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$. Determine if $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for $\mathbb{R}^3$.

Solution

Since there are exactly three vectors in $\mathcal{B}_1$ and $\mathcal{B}_2$, we can use any of the conditions of the invertible matrix theorem.

For $\mathcal{B}_1$, let $A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$. After some row replacements, A has a three pivots, hence A is invertible.

Therefore the columns of A are linearly independent and $\mathbb{R}^3 = \text{Span}\left(\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right)$. Hence $\mathcal{B}_1$ is a basis for $\mathcal{R}^3$.

For $\mathcal{B}_2$, let $B = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{bmatrix}$. After some row replacements, B is row equivalent to $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence B is not invertible, so $\mathcal{B}_2$ is linearly dependent and it is not a basis for $\mathbb{R}^3$.

Fact 3.10. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ be a set of vectors in $\mathbb{R}^n$.

(a) If $p > n$, then there are more vectors than entries, so S is linearly dependent. Thus S is not a basis for $\mathbb{R}^n$.

(b) If $p < n$, then $\mathbb{R}^n \neq \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$, so S is not a basis for $\mathbb{R}^n$.

Example 3.11. Let $\mathbb{P}_n$ be the set of all polynomials of degree at most n. Let $S = \{1, t, t^2, \ldots, t^n\}$. First, every polynomial in $\mathbb{P}_n$ can be written as a linear combination of the elements of S. Hence $\mathbb{P}_n = \text{Span}(1, t, t^2, \ldots, t^n)$. Second, if

$$c_0.1 + c_1 t + \cdots + c_n t^n = 0$$

then $c_0 = c_1 = \cdots = c_n = 0$. Hence S is linearly independent. Therefore, S is a basis for $\mathbb{P}_n$.

The Spannning Set Theorem

Theorem 3.12 (The Spanning Set Theorem)**.** Let V be a vector space, let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in V, and let $H = \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$

(a) If there exists i such that $\vec{v}_i$ is a linear combination of the other vectors in S, then the set $S' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots \vec{v}_n\}$ (we remove $\vec{v}_i$ from S) still spans H. That is

$$H = \text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots \vec{v}_n).$$

(b) If $H \neq \{\vec{0}\}$, then some subset of S is a basis for H.

Example 3.13. Let $v_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$, and let $H = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Find a basis for H.

Solution

By solving the homogeneous equation $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \vec{x} = \vec{0}$, we find that $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are linearly dependent and $\vec{v}_2 = 2\vec{v}_1 - \vec{v}_3$. Hence by the spanning set theorem $H = \text{Span}(\vec{v}_1, \vec{v}_3)$. Since $\vec{v}_1$ and $\vec{v}_3$ are not multiple of each other, they are linearly independent. It follows that $\{\vec{v}_1, \vec{v}_3\}$ is a basis for H.

Bases for $\text{Nul}(A)$ and $\text{Col}(A)$

Note 3.14. We have already seen how to find a basis for $\text{Nul}(A)$. For that, row reduce A to obtain a matrix in row echelon form and express the null space in parametric vector form. The vectors appearing will be a basis for $\text{Nul}(A)$.

Example 3.15. Let $A = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for $\text{Nul}(A)$.

Solution

The matrix A is already in row echelon form, and we see that x_2 and x_4 are free variables. An element of the $\text{Nul}(A)$ is then of the form (it is a solution of $A\vec{x} = \vec{0}$):

$$\vec{x} = \begin{bmatrix} 2x_2 + x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}$$

Therefore, $\mathrm{Nul}(A) = \mathrm{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$ and $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Nul}(A)$.

Theorem 3.16. The pivot columns of a matrix A form a basis for $\mathrm{Col}(A)$.

Example 3.17. Let $A = \begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \\ 1 & 4 & 5 & -9 \end{bmatrix}$. Find a basis for $\mathrm{Col}(A)$.

Solution

The matrix A is row reduced to the matrix $\begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. It follows that column 1, 2, and 4 are pivot columns. Therefore $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 3 \\ -9 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}(A)$.

Note 3.18. A basis is

- a spanning set which is as small as possible,
- a linearly independent set which is as bis as possible.