DAY 18: October 1st

Elementary Matrices

Recall the three elementary row operations¿ Let R_i and R_j denote the ith row and jth row of a matrix A. The row operations are

(a) $R_i \leftrightarrow R_j$: interchange rows R_i and R_j.

(b) cR_i, with $c \in \mathbb{R}$: replace R_i by cR_i.

(c) $R_i + cR_j$: replace R_i by $R_i + cR_j$.

Definition 2.9. An elementary matrix is any $n \times n$ matrix that can be obtained by performing a single elementary row operation to I_n.

Example 2.10. The following matrices are elementary matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+2R_3 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_2$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_1 \to R_1} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then we have

$$\begin{aligned} E_1A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+2a_{31} & a_{22}+2a_{32} & a_{23}+2a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ E_2A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ E_3A &= \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned}$$

We observe that following reduction: $A \xrightarrow{R_2+2R_3 \to R_2} E_1A$, $A \xrightarrow{R_2 \leftrightarrow R_3} E_2A$, and $A \xrightarrow{3R_1 \to R_1} E_3A$.

Fact 2.11. Let $\mathcal{R}$ denote an elementary row operation. Then

(a) If $I_n \xrightarrow{\mathcal{R}} E$, the for any matrix A with n rows, $A \xrightarrow{\mathcal{R}} EA$.

(b) So, if A can be row reduced to B by a sequence of row operations $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ and $I_n \xrightarrow{\mathcal{R}_i} E_i$ then $B = E_k E_{k-1} \cdots E_2 E_1 A$. In particular, we have

$$A \xrightarrow{\mathcal{R}_1} E_2(E_1 A) \xrightarrow{\mathcal{R}_3} \cdots \xrightarrow{\mathcal{R}_k} E_k E_{k-1} \cdots E_2 E_1 A = B$$

(c) Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transform E back into I. Indeed, let $\mathcal{R}$ be the row operation that reduces I_n to E, i.e. $I_n \xrightarrow{\mathcal{R}} EI_n = E$ and let $\bar{\mathcal{R}}$ be the operation that transforms E back to I and let $\bar{E}$ be the elementary matrix that does the operation, that is $E \xrightarrow{\bar{\mathcal{R}}} \bar{E}E = I$. Then $\bar{E}$ is the inverse of E.

Example 2.12. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$. So E is the elementary matrix given by $I_3 \xrightarrow{R_3 - 4R_1 \to R_3} E$. To transform E back to I_3, we add 4 times row 1 to row 3. The elementary matrix that does it is

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Hence $E^{-1} = \bar{E}$ is the inverse of E.

Theorem 2.13. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, in which case the sequence of elementary row operations which transform A to I_n also transform I_n into A^{-1}.

Proof. Recall that A is invertible if and only if every equation $A\vec{x} = \vec{b}$ has a unique solution. This is true if and only if the row reduced echelon form of A has a pivot in every row)(existence of solution) and column (uniqueness). Thus A is invertible if and only if the row reduced echelon form of A is I_n.

Now suppose that A is invertible and that $A \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} I_n$. Suppose also that $I_n \xrightarrow{\mathcal{R}_i} E_i$. Then

$$A \xrightarrow{\mathcal{R}_1} E_2(E_1 A) \xrightarrow{\mathcal{R}_3} \cdots \xrightarrow{\mathcal{R}_k} E_k E_{k-1} \cdots E_2 E_1 A = I_n$$

Thus

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1$$

□

Note:

If $A \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} I_n$, then $\begin{bmatrix} A & : & I_n \end{bmatrix} \xrightarrow{\mathcal{R}_1} \xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} \begin{bmatrix} I_n & : & A^{-1} \end{bmatrix}$

Algorithm to Find A^{-1}

Given a matrix A, to find A^{-1}

(a) Start with an augmented matrix $\begin{bmatrix} A & : & I_n \end{bmatrix}$.

(b) Row reduce the matrix to reduced row echelon form.

(c) If the reduced echelon form is of the form $\begin{bmatrix} I_n & : & B \end{bmatrix}$ then $A^{-1} = B$. If the matrix is of any other form, then A is not invertible.

Example 2.14. Compute the inverse of $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

3 Characterizations of Invertible Matrices

Recall that an $n \times n$ square matrix A is invertible if and only if it can be row reduced to the identity matrix I_n. The identity matrix has pivot in every row and column, so A has a pivot in every row and column. We combine all equivalent statements concerning pivots in every row and column in the following theorem.

Theorem 3.1 (The Invertible Matrix Theorem)**.** Let A be a square $n \times n$ matrix. Then, the following statements are equivalent (all are true or all are false)

(a) A is an invertible matrix.

(b) A is row equivalent to the identity matrix I_n.

(c) A has n pivots.

(d) The equation $A\vec{x} = \vec{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.

(g) The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I_n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

(l) A^T is an invertible matrix.

Fact 3.2. Let A and B be square matrices. If $AB = I$, then A and B are both invertible with $B = A^{-1}$ and $A = B^{-1}$.

Invertible Linear Transformation

We have seen in section 2 that matrix multiplication corresponds to composition of linear transformations. When a matrix A is invertible, the equation $A^{-1}A\vec{x} = \vec{x}$ can be viewed as a statement about linear transformation.

Definition 3.3 (Invertible Linear Transformation)**.** A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S\big(T(\vec{x})\big) = \vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n \tag{1}$$

$$T\big(S(\vec{x})\big) = \vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n \tag{2}$$

The next theorem states that if the transformation S exists then it is unique and must be a linear transformation. In this case, we call S the **inverse** of T and write it as T^{-1}.

Theorem 3.4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S is given by $S(\vec{x}) = A^{-1}\vec{x}$ is the unique transformation satisfying conditions (1) and (2).