DAY 3: MONDAY, AUGUST 27th

2 Row Reduction and Echelon Form

When solving a linear sytem, our strategy in the previous section was to replace the system with an equivalent system (i.e. have the same solution sets) that is easier to solve, by using elementary row operations on the augmented matrix of the system. We will refine this method to formalize a way that will enable us to answer the two fundamental questions about linear systems. We are still going to use elementary row operations.

We start with an arbitrary rectangular matrix as the algorithm applies to any matrix whether or not the matrix is viewed as an augmented matrix of a linear system.

Definition 2.1.

- The **leading entry** of a row is the leftmost nonzero entry of the row.
- A **nonzero row** is a row that contains at least one nonzero entry.

Definition 2.2 (Matrix in echelon form)**.** A matrix is in row echelon form (or echelon form) if it satisfies the following conditions:

(a) All nonzero rows are above any rows of zeros.

(b) Each leading entry of a row is in a column to the right of the leading entry of the row above it.

(c) All entries in a column below the leading entry are zeros.

Example 2.3.

(A1) $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{ is in echelon form.}$$

(A2) $$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \text{ is in echelon form}$$

(A3) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 4 & -3 \end{bmatrix} \text{ is not in echelon form, it fails condition (b) and (c).}$$

Definition 2.4.

- A pivot position in a echelon matrix is a location that corresponds to a leading entry. A pivot column is a column that contains a pivot position.
- A pivot is the nonzero number in a pivot position.

Example 2.5. The first and second columns in matrix (A1) are pivot columns.

From a given matrix, we can use elementary row operations to obtain a row equivalent matrix which is in echelon form. If a matrix A is row equivalent to echelon matrix U, we call U an echelon form (or row echelon form) of A.

FACTS:

- Any nonzero matrix may have more than one echelon form using different sequence of row operations.
- The pivot positions in two different echelon forms of a matrix A are always the same (pivots can be different).

We can describe the solution set of linear system by considering an echelon form of its augmented matrix. Let's see an algorithm on how we compute an echelon form of a given matrix.

Algorithm for Row Echelon Form:(Gaussian Elimination)

(1) Start with the leftmost nonzero column (the column has at least one nonzero entry). This column is a pivot column. If the top entry is zero, select a nonzero entry in this column and interchange it with the top row. Now the top entry in this column is nonzero.

(2) Use row replacement operations to eliminate nonzero values below the pivot.

(3) Cover (or ignore) the rows containing pivots and repeat the process for the remaining matrix.

Example 2.6. Consider the following matrix:

(A)
$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix}$$

(1) The leftmost nonzero column is the first column so we start from there. Since the top entry in this column is already nonzero, we do not need to interchange rows to get a nonzero in the top position.

(2) Now we create zeros in all positions below the pivot. Our pivot is 2 and we need to eliminate the last entry, which is 3, of the column. We replace row (3) by adding row (3) to $-\frac{3}{2}$ times row (1). Note that:

$\dfrac{3}{2} \times R_1 = [3 \quad 0 \quad -9 \quad -12]$ and $R_3 - \dfrac{3}{2} R_1 \times = [3 \quad 6 \quad -2 \quad -4] - [3 \quad 0 \quad -9 \quad -12] = [0 \quad 6 \quad 7 \quad 8]$

So we have the following:

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{2} \times R_1 \to R_3} \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix}$$

(3) Consider the new matrix. We complete the first column. Next, we ignore the first row and continue with the remainder of the matrix. The leftmost nonzero column in this case is now column (2), and the top entry, which is 1, is nonzero, so there is no need to interchange rows.

(4) The next is to create zeros in all position below the pivot (which is 1). For that, we need to eliminate 6. We replace row (3) by row (3) plus -6 times row (2). We have:

$$R_3 - 6R_2 = [0 \quad 6 \quad 7 \quad 8] - [0 \quad 6 \quad 12 \quad 18] = [0 \quad 0 \quad -5 \quad -10]$$

So we have:

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{R_3 - 6\times R_2 \to R_3} \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

(5) Consider now the new matrix. We have completed the second column so we can ignore the two first rows. We are left with the last row which is already in echelon form so are done.

An echelon form of the matrix A is therefore the matrix:

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

Reduced Echelon Form:

We can actually push it further to reduced echelon form where the the leading entries are all 1. The reduced echelon form of the augmented matrix of a linear system provides large amount of information about the system.

Definition 2.7 (Reduced Echelon Form)**.** A matrix is in reduced echelon form (or reduced row echelon form) if it has the following properties:

(a) it is in echelon form,

(b) all of the pivots (leading entries) are 1's,

(c) every entry above the pivot is also zero.

Example 2.8.

(B1) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form.

(B2) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ is in reduced echelon form

(B3) $\begin{bmatrix} 0 & -1 & -4 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is not in reduce echelon form, it fails condition (b) and (c).

FACT:

Each matrix is row equivalent to one and only one reduced echelon matrix.

Reduced Row Echelon Form Algorithm:

(1) Complete the row echelon form.

(2) Scale pivots to be 1 (start with the rightmost pivot).

(3) Use row replacement operation to change all the values above the pivot to be zero.

Example 2.9. We continue to reduce matrix (A) in Example 2.6 to a reduced echelon form. We have seen that an echelon form of (A) is given by the matrix

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

(1) This matrix is in echelon form but not reduced echelon form as there are pivots different from 1 and there are entries above pivots different from 0.

(2) The rightmost pivot is -5. To change this pivot to be 1, we divide row (3) by -5:

$$-\frac{1}{5} \times R_3 = -\frac{1}{5}[0 \quad 0 \quad -5 \quad -10] = [0 \quad 0 \quad 1 \quad 2].$$

We have:

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_3 \to R_3} \begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(3) Now we eliminate the nonzero entries above the pivot which are 2 and -6. For that, we replace row (2) by row (2) minus 2 times row (3), and we replace row (1) by row (1) plus 6 times row (3):

$$R_2 - 2R_3 = [0 \quad 1 \quad 2 \quad 3] - [0 \quad 0 \quad 2 \quad 4] = [0 \quad 1 \quad 0 \quad -1]$$

$$R_1 + 6R_3 = [2 \quad 0 \quad -6 \quad -8] - [0 \quad 0 \quad 6 \quad 12] = [2 \quad 0 \quad 0 \quad 4]$$

We have

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 - 2R_3 \to R_2]{R_1 + 6R_1 \to R_1} \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(4) The next pivot is already one, and every entry above it is zero so there is nothing more to do.

(5) The last pivot is 2 which can be scaled to 1 by multiplying row (1) by $\frac{1}{2}$.

$$\frac{1}{2}R_1 = \frac{1}{2}[2 \quad 0 \quad 0 \quad 4] = [1 \quad 0 \quad 0 \quad 2]$$

We have

$$\begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The last matrix is now in reduce echelon form.

Augmented Matrix and Row Echelon Form:

Suppose that the augmented matrix of a liner system is reduced to an echelon form matrix.

Definition 2.10.

- A pivot (or basic) variable is a variable corresponding to the pivot (or to the pivot column),
- A free (or independent) variable is a variable which does not correspond to any pivot.

Example 2.11. Suppose that the following matrix is the reduced echelon form of the augmented matrix of a linear system in the variables x_1, x_2, x_3:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Since each variable corresponds to a pivot (or to a pivot column), they are all pivot (basic) variable. The system has no free variable.

Solutions of Linear systems:

FACT: Suppose the augmented matrix of a linear system has been transformed to an echelon form matrix. Then:

(a) if there is a row of the form $[0 \quad 0 \quad \dots \quad 0 \quad a]$ where $a \neq 0$, then the system is inconsistent (has no solution),

(b) if every variable is a pivot (basic) variable, then the system is consistent and it has a single and unique solution,

(c) if some variables are free (or independent), then the system is consistent and it has infinitely many solutions.

Example 2.12. Case 1: the augmented matrix of a linear system is reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

The last row corresponds to $0x_1 + 0x_2 + 0x_3 = 6$ which is impossible, hence, the system has no solution.

Case 2: the augmented matrix of a linear system is reduced to

$$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Every variable is a pivot variable, hence the system has a unique solution. This matrix represents the following equations:

$$x_1 + 0x_2 + 0x_3 = -5$$

$$0x_1 + x_2 + 0x_3 = 2$$

$$0x_1 + 0x_2 + x_3 = 6$$

So the solution is $(-5, 2, 6)$.

Case 3: the augmented matrix of a linear system is reduced to

$$\begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables x_1 and x_3 are pivot variables and x_2 is a free variable. Hence, the system has infinitely many solutions. We have

$$\begin{aligned} x_1 + x_3 &= 6 \\ x_2 - x_3 &= 3 \\ x_3 &\text{ is free} \end{aligned}$$

Whenever we choose a value of x_3, then we have the value of x_1 and x_2. For example, $x_3 = 0$, then $x_1 = 6$ and $x_2 = 3$. So $(6, 3, 0)$ is a solution.

Parametric Description of Solution Sets:

In the case where the system has free variables, the solution set has parametric description be setting the free variables as parameters (denoted by: $t, r, s, ...$).

Example 2.13. In the following system

$$\begin{aligned} x_1 + x_3 &= 6 \\ x_2 - x_3 &= 3 \\ x_3 &\text{ is free} \end{aligned}$$

We set x_3 as the parameter t, hence the solution set of the system is described by the following general solution:

$$\begin{aligned} x_1 &= 6 - t \\ x_2 &= 3 + t \\ x_3 &= t \end{aligned}$$

where t can be any number.

Using Row Reduction to Solve a Linear System.

Given a linear system to solve:

(1) Write the augmented matrix,

(2) Perform rwo reduction to obtain echelon form. If the system is not consistent then there are no solutions and you may stop,

(3) Perform row reduction to obtain reduce echelon form,

(4) Write the system of equations corresponding to the reduced echelon form,

(5) Basic variables correspond to pivots, and free variables correspond to column without pivots,

(6) Write basic variables in terms of free variables, or write the parametric description of the solution.