DAY 16: September 26th

Definition 1.11 (Powers of matrix)**.** Let A be a $n \times n$ matrix and let k be a positive integer. We define A^k to be the kth power of A by

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$$

We put $A^0 = I_n$ and $A^1 = A$

Example 1.12. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Then

$$A^2 = AA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

The Transpose of a Matrix

Definition 1.13 (Transpose of a matrix)**.** Let A be an $m \times n$ matrix. The transpose A^T of A is the $n \times m$ matrix whose columns are formed from the corresponding rows of A. That is

$$(A^T)_{ij} = a_{ji}.$$

Example 1.14. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$. Then

$$A^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}$$

Proposition 1.15. Let A and B be matrices whose sizes are appropriate so the following operations are defined. Let r be a scalar. Then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$. The transpose of a product of matrices equals the product of the transposes in the reverse order.

Note

Note that generally $(AB)^T \neq A^T B^T$. Most of the time the product $A^T B^T$ is not even defined.

2 The Inverse of a Matrix

Recall that for a positive integer n, I_n denotes the $n \times n$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

and if A is an $m \times n$ matrix, then

$$I_m A = I_n A$$

In this section, we focus on square matrices A ($n \times n$ matrices) and look for their multiplicative inverses.

Definition 2.1 (Invertible matrix)**.** An $n \times n$ square matrix A is said to be invertible if there exists an $n \times n$ matrix B such that

$$AB = I_n = BA.$$

- the matrix B is called the inverse of A and it is denoted by A^{-1}.
- If A is not invertible then A is said to be singular.

Example 2.2. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ Then

$$AB = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $B = A^{-1}$

The case of 2×2 Matrices

Definition 2.3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. We define the determinant of A by the quantity

$$\det(A) = ad - bc$$

Theorem 2.4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then is invertible if and only if $\det(A)$ is nonzero. In this case, we have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.5. Let $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$. So $\det(A) = 8 \cdot 4 - 6 \cdot 5 = 2 \neq 0$. Therefore A is invertible with

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}.$$

Check that $AA^{-1} = A^{-1}A = I_2$.

Theorem 2.6. Let A be an invertible $n \times n$ matrix. Then, for every $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$.

Proof. Let $\vec{b}$ be $\mathbb{R}^n$.

Existence of a solution of $A\vec{x} = \vec{b}$: We have

$$\begin{aligned} A(A^{-1}\vec{b}) &= (AA^{-1})\vec{b} \\ &= I_n\vec{b} = \vec{b} \end{aligned}$$

Hence $\vec{x} = A^{-1}\vec{b}$ is a solution.

Uniqueness of the solution: If $\vec{u}$ is any solution of $A\vec{x} = \vec{b}$, then we have

$$A\vec{u} = \vec{b}$$

Multiplying both sides by A^{-1}, we have

$$\begin{aligned} A^{-1}A\vec{u} &= A^{-1}\vec{b} \\ I_n\vec{u} &= A^{1}\vec{b} \\ \vec{u} &= A^{-1}\vec{b}. \end{aligned}$$

□

Proposition 2.7. Let A and B be $n \times n$ matrices.

(a) If A is ivertible, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

(b) If A and B are invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(c) If A is invertible, then so is A^T, and

$$(A^T)^{-1} = (A^{-1})^T$$

Remark 2.8. We check property (2) of the above proposition. We have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA = AA^{-1} = I_n$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$$

Therefore $(AB)^{-1} = B^{-1}A^{-1}$

Elementary Matrices

Recall the three elementary row operations¿ Let R_i and R_j denote the ith row and jth row of a matrix A. The row operations are

(a) $R_i \leftrightarrow R_j$: interchange rows R_i and R_j.

(b) cR_i, with $c \in \mathbb{R}$: replace R_i by cR_i.

(c) $R_i + cR_j$: replace R_i by $R_i + cR_j$.

Definition 2.9. An elementary matrix is any $n \times n$ matrix that can be obtained by performing a single elementary row operation to I_n.

Example 2.10. The following matrices are elementary matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+2R_3 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_2$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_1 \to R_1} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then we have

$$\begin{aligned} E_1A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+2a_{31} & a_{22}+2a_{32} & a_{23}+2a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ E_2A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ E_3A &= \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned}$$

We observe that following reduction: $A \xrightarrow{R_2+2R_3 \to R_2} E_1A$, $A \xrightarrow{R_2 \leftrightarrow R_3} E_2A$, and $A \xrightarrow{3R_1 \to R_1} E_3A$.

Fact 2.11. Let $\mathcal{R}$ denotes an elementary operation. Then

(a) If $I_n \xrightarrow{\mathcal{R}} E$, the for any matrix A with n rows, $A \xrightarrow{\mathcal{R}} AE$.

(b) So, if A can be row reduced to B by a sequence of row operations $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ and $I_n \xrightarrow{\mathcal{R}_i} E_i$ then $B = E_kE_{k-1}\cdots E_2E_1A$. In particular, we have

$$A \xrightarrow{\mathcal{R}_1} E_2(E_1A) \xrightarrow{\mathcal{R}_3} \cdots \xrightarrow{\mathcal{R}_k} E_kE_{k-1}\cdots E_2E_1A = B$$

(c) Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transform E back into I. Indeed, let $\mathcal{R}$ be the row operation that reduces I_n to E, i.e. $I_n \xrightarrow{\mathcal{R}} EI_n = E$ and let $\bar{\mathcal{R}}$ be the operation that transforms E back to I and let $\bar{E}$ be the elementary matrix that does the operation, that is $E \xrightarrow{\bar{\mathcal{R}}} \bar{E}E = I$. Then $\bar{E}$ is the inverse of E.

Example 2.12. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$. So E is the elementary matrix given by $I_3 \xrightarrow{R_3 - 4R_1 \to R_3} E$. To transform E back to I_3, we add 4 times row 1 to row 3. The elementary matrix that does it is

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Hence $E^{-1} = \bar{E}$ is the inverse of E.

Theorem 2.13. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, in which case the sequence of elementary row operations which transform A to I_n also transform I_n into A^{-1}.

Proof. Recall that A is invertible if and only if every equation $A\vec{x} = \vec{b}$ has a unique solution. This is true if and only if the row reduced echelon form of A has a pivot in every row)(existence of solution) and column (uniqueness). Thus A is invertible if and only if the row reduced echelon form of A is I_n.

Now suppose that A is invertible and that $A \xrightarrow{\mathcal{R}_1}\xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} I_n$. Suppose also that $I_n \xrightarrow{\mathcal{R}_i} E_i$. Then

$$A \xrightarrow{\mathcal{R}_1} E_2(E_1A) \xrightarrow{\mathcal{R}_3} \cdots \xrightarrow{\mathcal{R}_k} E_kE_{k-1}\cdots E_2E_1A = I_n$$

Thus

$$A^{-1} = E_kE_{k-1}\cdots E_2E_1$$

□

Note:

If $A \xrightarrow{\mathcal{R}_1}\xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} I_n$, then $\left[\begin{array}{ccc} A & : & I_n \end{array}\right] \xrightarrow{\mathcal{R}_1}\xrightarrow{\mathcal{R}_2} \cdots \xrightarrow{\mathcal{R}_k} \left[\begin{array}{ccc} I_n & : & A^{-1} \end{array}\right]$

Algorithm to Find A^{-1}

Given a matrix A, to find A^{-1}

(a) Start with an augmented matrix $\left[\begin{array}{ccc} A & : & I_n \end{array}\right]$.

(b) Row reduce the matrix to reduced row echelon form.

(c) If the reduced echelon form is of the form $\left[\begin{array}{ccc} I_n & : & B \end{array}\right]$ then $A^{-1} = B$. If the matrix is of any other form, then A is not invertible.

Example 2.14. Compute the inverse of $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$