DAY 8: SEPTEMBER 7th

5 Solution Sets of Linear Systems

In this section, we use vector notation to give explicit and geometric description of the solution set of a given linear system.

Definition 5.1. A system of linear equation is called homogeneous if it can be written as $A\vec{x} = \vec{0}$.

Example 5.2. The following linear system is homogeneous:

$$x_1 - 3x_2 + 3x_3 = 0$$
$$2x_1 + x_2 - 4x_3 = 0$$

Note: Clearly, the zero vector $\vec{x} = \vec{0}$ is always a solution of $A\vec{x} = \vec{0}$. It is called the **trivial solution.**. The most important question is whether there exists a nontrivial solution, that is a vector $\vec{x} \neq \vec{0}$ that satisfies $A\vec{x} = \vec{0}$.

Fact: The homogeneous system $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if it has a free variable.

Example 5.3. Consider the following homogeneous linear system:

$$x_1 - 2x_2 + 3x_3 = 0$$
$$-2x_1 - 3x_2 - 4x_3 = 0$$
$$2x_1 - 11x_2 + 8x_3 = 0$$

The corresponding matrix equation is:

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & -3 & -4 \\ 2 & 11 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To determine if the system has a nontrivial solution, we do row reduction on the augmented matrix to see if there is a free variable. The augmented matrix is given by:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & 11 & 8 & 0 \end{bmatrix}.$$

Applying $R_2 + 2R_1 \to R_2$ and $R_3 - 2R_1 \to R_3$, we have:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & -7 & 2 & 0 \end{bmatrix}.$$

Now applying $R_3 + R_2 \to R_3$, we have

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is now in row echelon form and we can see that x_3 does not correspond to any pivot so it is a free variable. It follows that the homogeneous system has a nontrivial solution.1

Parametric Vector Form

Suppose that after row reduction, the augmented matrix of an homogeneous system is given by:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that x_2 and x_3 are free variables. The corresponding equations are given by:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ x_2 &\ \text{ free} \\ x_3 &\ \text{ free} \end{aligned}$$

A a vector, our general solution $\vec{x}$ is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

In particular, since x_2 and x_3 can be chosen to any number, a solution is a linear combination of the vectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ Therefore, the solution set of the linear system in this particular example is $\text{Span}(\vec{v}_1, \vec{v}_2)$. Since neither of $\vec{v}_1$ nor $\vec{v}_2$ is a scalar multiple of of the other, $\text{Span}(\vec{v}_1, \vec{v}_2)$ is a plan through the origin. We can set x_2 and x_3 as parameters r and t respectively, and we obtain the parametric vector form of the general solution.

Note: The solution set of $A\vec{x} = \vec{0}$ can always be written as $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ for some vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. To give an explicit description of the solutions, we use parametric vector form.

Definition 5.4. An equation of the form

$$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_p\vec{v}_p$$

is called **parametric vector equation**.

Example 5.5. For the above example, we set $x_2 = r$ and $x_3 = t$ and then we have the parametric vector equation of the solution as

$$\vec{x} = r \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = r\vec{v}_1 + t\vec{v}_2$$

Solutions of Nonhomogeneous Systems

Suppose now that the system in the above example is not homogeneous (it has some nonzero constants, i.e. we have the form $A\vec{x} = \vec{b}$). In addition, suppose that after row reduction to an echelon form, the augmented matrix is given by:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding equations are given by

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ x_2 &\quad \text{free} \\ x_3 &\quad \text{free} \end{aligned}$$

In particular, the system has infinitely many solutions. Hence, as a vector, the general solution of $A\vec{x} = \vec{b}$ in this case is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

We have seen that

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = r\vec{v}_1 + t\vec{v}_2$$

is the parametric expression of the homogeneous equation. Therefore, we may write $\vec{x} = \vec{p} + r\vec{v}_1 + t\vec{v}_2$ where $\vec{p}$ is a specific solution to the system and $r\vec{v}_1 + t\vec{v}_2$ is the general form of a solution of the homogeneous system.

Theorem 5.6. Suppose that the equation $A\vec{x} = \vec{b}$ has a solution $\vec{p}$. Then all solutions to the equation have the form:

$$\vec{w} = \vec{p} + \vec{v}_h$$

where $\vec{v}_h$ is a solution of the corresponding homogeneous equation $A\vec{x} = \vec{0}$.