DAY 9: SEPTEMBER 10th

6 Linear Independence

When studying the homogeneous equation $A\vec{x} = \vec{0}$, the important question was whether there exists a nontrivial (nonzero) solution. This can be generalized to some properties of a given set of vectors $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$, for example for the set of the columns of the matrix A.

Definition 6.1. An indexed set of vectors $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent (we may say also that the vectors $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent) if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution (i.e. $(x_1, x_2, \dots, x_p) = (0, 0, \dots, 0)$ or the vector $\vec{x} = \vec{0}$ is the only solution). Otherwise, the set S is said to be linearly dependent if there exist weights $c_1, c_2, \dots, c_p \in \mathbb{R}$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}. \tag{1}$$

Note:

In general, to determine if a collection of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is linearly independent or linearly dependent, we solve the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$. If the trivial solution $(x_1, x_2, \dots, x_p) = (0, 0, \dots, 0)$ is the only solution, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is linearly dependent, otherwise it is linearly dependent. Recall that the vector equation (1) is equivalent to the homogeneous equation $A\vec{x} = \vec{0}$. The matrix equation $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if it has a free variable.

Example 6.2.

(a) Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Determine if the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Solution: Consider the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}.$$

This equation is equivalent to the homogeneous linear system with augmented matrix

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

After some row operations, the augmented matrix is reduced to the following matrix in echelon form:

$$\begin{bmatrix} -1 & 2 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad \begin{array}{r} x_1 - x_2 + 2x_3 = 0 \\ x_2 \;\; \text{free} \\ x_3 \;\; \text{free} \end{array}$$

It is clear that the variables x_1, x_2, x_3 are pivot (basic) variables, so the system has no free variable. Therefore the trivial solution $(x_1, x_2, x_3) = (0, 0, 0)$ is the only solution. Hence, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

(b) Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, and

$\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Is the $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent? If not, give a linear dependence relation among $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

Solution: Consider the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}. \tag{2}$$

If it has a nontrivial solution then $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are linearly independent, otherwise they are linearly independent. This vector equation is equivalent to the homogeneous linear system with augmented matrix

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & 7 & 0 \\ 3 & -6 & -12 & 0 \end{bmatrix}$$

This matrix is row-reduced to the matrix in echelon form

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We conclude that x_1 and x_2 are pivot variables and x_3 is a free variable. Therefore, the system has a nontrivial solution. Hence the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

To give a dependence relation among $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, we compute a solution of the system. Form the row echelon form of the augmented matrix, we have the following equations:

$$x_1 - x_2 - x_3 \quad = 0 \tag{3}$$

$$3x_2 - 9x_3 \quad = 0 \tag{4}$$

$$x_3 \quad \text{free} \tag{5}$$

We choose $x_3 = 1$, the from equation (4) and (3) we have $x_2 = 3$ and $x_1 = 4$. We substitute these values into the vector equation (2) and obtain a linear dependence relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$4\vec{v}_1 + 3\vec{v}_2 + \vec{v}_3 = \vec{0}.$$

Note that even though $0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0}$, because there is another solution with nonzero coefficients, the vectors are linearly independent. Since the system has infinitely many solutions, there are infinitely many linear dependence relations among $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

Linear Independence of Matrix Columns

Let $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_p \end{bmatrix}$ be a matrix. Recall that the matrix equation $A\vec{x} = \vec{0}$ can be written as the vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$. Therefore we have the following

Fact: The columns of a matrix A are linearly independent exactly when the matrix equation $A\vec{x} = \vec{0}$ has only the trivial solution (i.e. when the corresponding linear system has no free variable).

Example 6.3. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 3 & 3 \end{bmatrix}$ be a 3×3 matrix. Are the columns of A linearly independent?

Solution: To determine if the columns of A are linearly independent or not, we study the matrix equation $A\vec{x} = \vec{0}$. This is equivalent to the linear system with augmented matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 5 & 3 & 3 & 0 \end{bmatrix}$$

This matrix is row reduced to

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -6 & -6 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It is clear that every variable is a basic variable. Hence, the equation $A\vec{x} = \vec{0}$ has no nontrivial solution, and so the column of A are linearly independent.

Sets of One of Two Vectors

Facts:

- A collection of one vector $\{\vec{v}\}$ is linearly independent if and only if $\vec{v}$ is not the zero vector $\vec{0}$ ($\vec{v} \neq \vec{0}$).
- A set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent exactly when one vector is a multiple of the other vector. That is, $\vec{v}_1$ and $\vec{v}_2$ are linearly independent if and only if neither of $\vec{v}_1$ and $\vec{v}_2$ is multiple of the other.

Example 6.4.

(a) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The vector $\vec{v}_1$ and $\vec{v}_2$ are not multiple of one another, so the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. We can verify it by solving the homogeneous system with augmented matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{0}. \end{bmatrix}$

(b) Let $\vec{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$. Clearly, $\vec{v}_1$ is a multiple of $\vec{v}_2$ as $\vec{v}_1 = -2\vec{v}_2$. Hence $\vec{v}_1 + 2\vec{v}_2 = \vec{0}$, so $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Note: Geometrically, two vectors are linearly dependent if they lie on the same line.

Sets of Two or more vectors

Theorem 6.5. A set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly independent if $p > n$.

Proof. The vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$ has the same solution set as the linear system with augmented matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p & \vec{0}. \end{bmatrix}$. We know that this homogeneous system is always consistent (as it has the trivial solution). Since $p > n$, there are more variables than equations. it follows that there must be a free variable. Therefore, the system has a nontrivial solution, so $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly independent. $\square$

Example 6.6. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. In this case we have more vectors (there are 3 vectors) than the rows of each vector (each vector has 2 rows), so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. In fact we have:

$$-\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}.$$

We can determine a linear dependence relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by solving the corresponding homogeneous linear system with augmented matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0}. \end{bmatrix}$

Note: Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. We can not conclude anything when $p \leq n$.

Theorem 6.7. A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ of vectors in $\mathbb{R}^n$ that includes the zero vector $\vec{0}$ is linearly dependent.

Proof. By re-arranging the vectors, we may suppose that $\vec{v}_1 = \vec{0}$. Then we have

$$1 \cdot \vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p = \vec{0}$$

Since not all of the coefficients are zero (the coefficient of $\vec{v}_1$ is 1), the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are linearly dependent. $\square$

Example 6.8. Determine by inspection if the given set is linearly dependent

(a) $\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

Solution:

(a) The set contains four vectors which is more than the number of the entries of each vector which is three. Hence by Theorem 6.5, the set is linearly dependent.

(b) Note that this case, the number of the vectors does not exceed the number of the entries of each vector so Theorem 6.5 does not apply here. The set contains the zero vector, hence by Theorem 6.7, the set is linearly dependent.

(c) We have a set of two vectors. It seems that the first vector is 2 times the second vector. This hold for the first and the second entries, but fails for the third entry ($2 \neq -2$). Thus neither of the vector is a multiple of the other and so they are linearly independent.

Theorem 6.9. An indexed set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly independent if and only if one of the vectors in S is a linear combination of the others. In fact, S is linearly dependent if $\vec{v}_1 = \vec{0}$ or there exists j such that $\vec{v}_j$ is linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$.

Proof. Suppose that there exists j such that $\vec{v}_j$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$. That is, there are $c_1, c_2, \ldots, c_{j-1} \in \mathbb{R}$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1} = \vec{v}_j.$$

This implies that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j = \vec{0}.$$

We can complete the sum as

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j + 0\vec{v}_{j+1} + \cdots + 0\vec{v}_p = \vec{0}. \tag{6}$$

Since not all of the coefficients are zero (for example the coefficient of $\vec{v}_j$ is $-1 \neq 0$), the equation (6) is a linear dependence relation among $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. Therefore S is linearly dependent.

Conversely, suppose S is linearly dependent. Then there are coefficients $c_1, c_2, \ldots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}.$$

Let j be the largest index such that c_j is not zero. Then we have

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1} + c_j\vec{v}_j = \vec{0}.$$

It follows that

$$\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_1}{c_j}\vec{v}_2 - \cdots - \frac{c_1}{c_j}\vec{v}_{j-1}$$

That is $\vec{v}_j$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$. $\square$

Note: Let $\vec{w}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \in \mathbb{R}^n$. Then $\vec{w} \in \mathrm{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p)$ if and only if $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{w}$ is linearly dependent.