DAY 42: Monday, December 3rd

Fact 1.16. Let W be a subspace of $\mathbb{R}^n$. Then

(a) $(W^{\perp})^{\perp} = W$

(b) A vector $\vec{z}$ is in $W^{\perp}$ if and only if $\vec{z}$ is orthogonal to every vector in a set that spans W.

(c) $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Theorem 1.17. Let A be an $m \times n$ matrix. Then

$$(\text{Row}(A))^{\perp} = \text{Nul}(A) \quad \text{and} \quad (\text{Col}(A))^{\perp} = \text{Nul}(A^T)$$

Example 1.18. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ be vectors in $\mathbb{R}^3$, and let $W = \text{Span}(\vec{v}_1, \vec{v}_2)$ be a subspace of $\mathbb{R}^3$. Give an explicit description (a parametric form) of the elements of $W^{\perp}$.

Solution

By Fact 1.16 (b), a vector $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is in $W^{\perp}$ if and only if

$$\begin{cases} \vec{z} \cdot \vec{v}_1 = 0 \\ \vec{z} \cdot \vec{v}_2 = 0 \end{cases} \quad \text{which is equivalent to} \quad \begin{cases} z_1 - z_3 = 0 \\ 2z_1 - z_2 + z_3 = 0 \end{cases}$$

So, we solve this linear system to find an explicit description of the vector $\vec{z}$. The augmented matrix of this system is given by

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

This matrix is row reduced to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix}$$

So a solution $\vec{z}$ of the system is of the form

$$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_3 \\ 3z_3 \\ z_3 \end{bmatrix} = z_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad z_3 \in \mathbb{R}.$$

2 Orthogonal Sets and The Gram-Schmidt Process

Definition 2.1. A set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, i.e. $\vec{u}_i \cdot \vec{u}_j = 0$ for every $i \neq j$.

Example 2.2. Show that the set $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \right\}$ is orthogonal.

Solution

We have

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = (1)(0) + (-2)(1) + (1)(2) = 0$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = (1)(-5) + (-2)(-2) + (1)(1) = 0$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = (0)(-5) + (1)(-2) + (2)(1) = 0$$

Theorem 2.3. If the set $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent. Hence, it is a basis for the subspace that it spans.

Definition 2.4. An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W which is an orthogonal set.

Theorem 2.5. Let W be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ be an orthogonal basis for W. Then, for every $\vec{u}$ in W, the weights in the linear combination

$$\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p$$

is given by

$$c_j = \frac{\vec{u} \cdot \vec{b}_j}{\vec{b}_j \cdot \vec{b}_j}, \quad \text{for } j = 1, 2, \cdots, p.$$

Definition 2.6.

(a) A set $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an **orthonormal set** if it is orthogonal and $||\vec{u}_i|| = 1$ for $i = 1, 2, \ldots, p$ (i.e. each vector $\vec{u}_i$ in S is a unit vector).

(b) An **orthonormal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W which is orthonormal (orthogonal set and each element is a unit vector).

(c) If S is an orthonormal set in $\mathbb{R}^n$, then it is an orthonormal basis for the subspace that it spans.

Example 2.7. The standard basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$ is an orthonormal basis for $\mathbb{R}^n$.

Theorem 2.8. An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_m$.

The Gram-Schmidt Process

The Gram-Schmidt algorithm consists to constructing an orthogonal or orthonormal basis for a subspace W of $\mathbb{R}^n$ starting with any basis $\mathcal{B}$ for W.

Theorem 2.9 (Gram-Schmidt Process)**.** Let W be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ be a basis for W. Then, the set $\mathcal{O} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ such that

$$
\begin{aligned}
\vec{v}_1 &= \vec{b}_1 \\
\vec{v}_2 &= \vec{b}_2 - \frac{\vec{b}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\
\vec{v}_3 &= \vec{b}_3 - \frac{\vec{b}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{b}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\
&\vdots \\
\vec{v}_p &= \vec{b}_p - \frac{\vec{b}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{b}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 - \cdots - \frac{\vec{b}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}\vec{v}_{p-1}
\end{aligned}
$$

is an orthogonal basis for W. In addition,

$$\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = \text{Span}(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k) \quad \text{for } 1 \leq k \leq p.$$

Note 2.10. To find an orthonormal basis for W, simply normalize each vector $\vec{v}_i$ constructed in Theorem 2.9.

Example 2.11. Let $\vec{b}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ be vectors in $\mathbb{R}^3$ and let $W = \text{Span}(\vec{b}_1, \vec{b}_2)$. Find an orthogonal basis, and then an orthonormal basis for W.

Solution

We use Gram-Schmidt algorithm to find an orthogonal basis for W. We have

$$
\begin{aligned}
\vec{v}_1 &= \vec{b}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
\vec{v}_2 &= \vec{b}_1 - \frac{\vec{b}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Then the set $\mathcal{O}_1 = \{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for W. Clearly, $\mathcal{O}_1$ is linearly independent and $\vec{v}_1 \cdot \vec{v}_2 = 0$. To find an orthonormal basis for W, we simply normalize each vector in $\mathcal{O}_1$. We have

$$
\begin{aligned}
||\vec{v}_1|| &= \sqrt{\vec{v}_1 \cdot \vec{v}_1} = \sqrt{2} \\
||\vec{v}_2|| &= \sqrt{\vec{v}_2 \cdot \vec{v}_2} = \sqrt{3}
\end{aligned}
$$

Let $\vec{u}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \frac{1}{||\vec{v}_2||}\vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then $\mathcal{O}_2 = \{\vec{u}_1, \vec{u}_2\}$ is orthonormal basis for W.