DAY 11: September 14th

7 Linear Transformation

Let A be a $m \times n$ matrix and $\vec{x}$ a vector in $\mathbb{R}^n$. The product $A\vec{x}$ can be viewed as the "action" of the matrix A on the vector $\vec{x}$: we can think of it as transforming the vector $\vec{x}$ into another vector $\vec{b} = A\vec{x}$.

Example 7.1. Consider the product:

$$\begin{bmatrix} -1 & -3 \\ -1 & 0 \\ 3 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

It shows that the matrix $\begin{bmatrix} -1 & -3 \\ -1 & 0 \\ 3 & -1 \\ 3 & 2 \end{bmatrix}$ transform the vector $\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$ into the vector $\vec{b} = \begin{bmatrix} -5 \\ 1 \\ -5 \\ 1 \end{bmatrix}$ in $\mathbb{R}^4$.

Note: Recall that if A is $m \times n$ matrix, the we can only multiply A on the right by a vector in $\mathbb{R}^n$. The resulting vector will always be in $\mathbb{R}^m$.

Definition 7.2. A function $f : D \to C$ is a rule that assigns for each element $d \in D$ a unique element $c \in C$ so that $c = f(d)$. We refer to D as the domain of f, and C as the codomain of f. The set $\{f(d) : d \in D\}$ of values taken by the function is called image of range of f.

We will be interested in functions that have $\mathbb{R}^n$ as a domain and R^m as a codomain.

Definition 7.3. A transformation (or function, or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted by $T : \mathbb{R}^n \to \mathbb{R}^m$, takes, as input, a vector $\vec{x}$ in $\mathbb{R}^n$ and outputs vector $T(\vec{x})$ in $\mathbb{R}^m$. We say that:

- $\mathbb{R}^n$ is the domain of T,
- $\mathbb{R}^m$ is the codomain of T,
- for $\vec{x} \in \mathbb{R}^n$, $T(\vec{x})$ is the image of $\vec{x}$,
- the set of all images $T(\vec{x})$ is called the ranges or image of T.

Matrix Transformation

We are more interested on transformation associated with matrix multiplication.

Definition 7.4. A matrix transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that operates by multiplication by a $m \times n$ matrix A, that is if $\vec{x} \in \mathbb{R}^n$, then $T(\vec{x}) = A\vec{x}$. This transformation takes a vector $\vec{x} \in \mathbb{R}^n$ to the vector $A\vec{x} \in \mathbb{R}^m$. For simplicity, we sometimes denote such a matrix transformation by $\vec{x} \to A\vec{x}$.

Notes:

- The domain of the transformation T, associated with the matrix A, is $\mathbb{R}^n$ when A has n columns, and the codomain of T is $\mathbb{R}^m$ when A has n rows.
- Since the image $T(\vec{x})$ is of the form $A\vec{x}$, the range of T the set of all linear combinations if the columns of A.

Example 7.5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. This transformation can be described in general by

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_2 \\ 2x_1 + 3x_2 \end{bmatrix}.$$

The types of questions that we may ask for the transformation T include:

(1) Find the image $T(\vec{u})$ of $\vec{u}$ under the transformation T.

(2) Find a vector $\vec{x}$ in $\mathbb{R}^2$ such that $T(\vec{x}) = \vec{b}$.

(3) Is there more that one vector whose image under T is $\vec{b}$?

(4) Determine if the vector $\vec{c}$ is the range of the transformation T.

These questions correspond to some questions addressed in the previous sections about matrix equation and their corresponding linear systems.

(1) To find the image of $\vec{u}$ under T, we just need to compute the product $A\vec{u}$.

(2) To find a vector $\vec{x} \in \mathbb{R}^2$ such that $T(\vec{x}) = \vec{b}$, one needs to solve the matrix equation $A\vec{x} = \vec{b}$ and give a solution. If we call $\vec{v}_1, \vec{v}_2$ the columns of A, this matrix equation is equivalent to linear system with augmented matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{b} \end{bmatrix}$.

(3) To determine if there are more that one vector whose image under T is the given $\vec{b}$, one needs to know if the solution of the matrix equation $A\vec{x} = \vec{b}$ is unique or infinitely many.

(4) To determine if the vector $\vec{c}$ is in the range of the transformation T, it is the same as to check if there exists $\vec{x} \in \mathbb{R}^2$ such that $T(\vec{x}) = \vec{c}$, so one need to check if the matrix equation $A\vec{x} = \vec{c}$ has a solution or not.

Linear Transformation

Definition 7.6. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if, for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ (scalar):

(a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$,

(b) $T(c\vec{u}) = cT(\vec{u})$.

DAY 12: September 17th

(Linear Transformation continued)

Note:

Let A be a $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by $T(\vec{x}) = A\vec{x}$. Since for every $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we have $A(\vec{u}+\vec{v}) = A\vec{u}+A\vec{v}$ and $A(c\vec{u}) = cA\vec{u}$, the transformation T is linear. In particular, every matrix transformation is a linear transformation.

Proposition 7.7. Let T be a linear transformation. Then for all vectors $\vec{u}$ and $\vec{v}$ in the domain of T and for all scalars c and d in $\mathbb{R}$, we have

(a) $T(\vec{0}) = \vec{0}$, and

(b) $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.

Proof.

(a) Since $0\vec{u} = \vec{0}$ and since T is a linear transformation, we have

$$T(\vec{0}) = T(0\vec{u}) = 0T(\vec{u}) = \vec{0}.$$

(b) Since T is a linear transformation and $c\vec{u}$ and $c\vec{v}$ are vectors in the domain of T, we have

$$T(c\vec{u} + d\vec{v}) = T(c\vec{u}) + T(c\vec{v}) = cT(\vec{u}) + cT(\vec{v}).$$

□

Note: A repeated application of property (2) in Proposition 7.7 gives a generalization:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \cdots + T(c_p\vec{v}_p = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_pT(\vec{v}_p)$$

Example 7.8.

(a) **The identity transformation:** The square matrix (same number of rows as columns) with 1's on the diagonal and 0's elsewhere is called an identity matrix. For $n = 3$, the identity matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for every $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$. Then we have

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

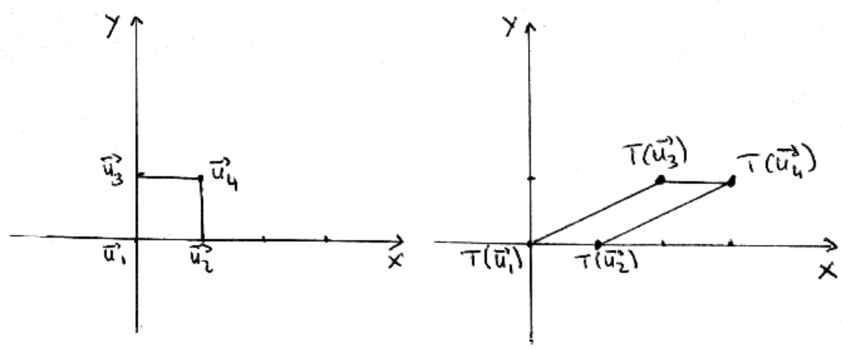
That is $T(\vec{x}) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^3$.

(b) A **shear transformation** in the x-direction is a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ for some $k \in \mathbb{R}$.

Suppose $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Let $\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{u}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$T(\vec{u}_1) = A\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, T(\vec{u}_2) = A\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, T(\vec{u}_3) = A\vec{u}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, T(\vec{u}_4) = A\vec{u}_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Geometrically, this map translates horizontal segments of the unit square proportionally in their x-value.



(c) Define the map $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\vec{x}) = r\vec{x}$ for some real number r. If $0 \leq r \leq 1$ the map is called **contraction**. If $r > 1$, it is called **dilation**. Show that T is a linear transformation.

Solution: Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^n$ and c, d be scalars. It suffices to show that $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$. For that, we have

$$\begin{aligned} T(c\vec{u} + d\vec{v}) &= r(c\vec{u} + d\vec{v}) \text{ by the definition of } T \\ &= rc\vec{u} + rc\vec{v} \\ &= c(r\vec{u}) + c(r\vec{v}) \\ &= cT(\vec{u}) + dT(\vec{v}). \end{aligned}$$

It follows that T is a linear transformation.

8 The Matrix of a Linear Transformation

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we show in this section that it is actually a matrix transformation $\vec{x} \mapsto A\vec{x}$.

Definition 8.1. Let $\vec{e}_i$ denote the **standard basis vector** having 1 in the ith row, and zero elsewhere.

$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longleftarrow ith \quad \text{row}$$

In particular $\vec{e}_i$ is the ith column of the $n \times n$ identity matrix.

Example 8.2. Consider the 3×3 identity matrix.

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \uparrow \; \uparrow \; \uparrow \\ \vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3 \end{array}$$

Theorem 8.3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a unique matrix A such that for every $\vec{x}$ in $\mathbb{R}^n$ we have

$$T(\vec{x}) = A\vec{x}.$$

In fact, A is the $m \times n$ matrix whose ith column is the vector $T(\vec{e}_i)$. In particular

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

The matrix A is called the **standard matrix** for T.

Proof. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be in $\mathbb{R}^n$. Then

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n.$$

Hence $T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$. Since T is a linear transformation, we have

$$\begin{aligned} T(\vec{x}) &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n) \\ &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= A\vec{x}. \end{aligned}$$

$\square$

Example 8.4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\Big(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\Big) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \text{ and } T\Big(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Compute $T(\vec{u})$ where $\vec{u} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.
- Give the matrix A such that $T(\vec{x}) = A\vec{x}$ for every $\vec{x}$ in $\mathbb{R}^2$ (the matrix A is the standard matrix of T).

Solution:

- We have

$$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  Hence

$$\begin{aligned} T(\vec{u}) &= T\Big(4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) = 4T\Big(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\Big) + 3T\Big(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) \\ &= 4\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix} \end{aligned}$$

- The standard matrix A of T is given by

$$\begin{aligned} A &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\Big(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\Big) & T\Big(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \end{aligned}$$

  Check that

$$A\vec{u} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}.$$

Properties of Linear Transformation

Definition 8.5. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if every $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\vec{u}$ in $\mathbb{R}^n$. That is, T is onto if for all $\vec{b}$ in $\mathbb{R}^m$, there exists $\vec{u}$ in $\mathbb{R}^n$ such that $T(\vec{u}) = \vec{b}$.

Facts:

- T is onto when the range of T is all of the codomain $\mathbb{R}^m$.
- T is not onto when there is some $\vec{b}$ in $\mathbb{R}^m$ for which the equation $T(\vec{x}) = \vec{b}$ has no solution.

Definition 8.6. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if every $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\vec{u}$ in $\mathbb{R}^n$. That is T is said to be one-to-one if for every $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, $T(\vec{u}) = T(\vec{v})$ implies $\vec{u} = \vec{v}$.

Facts:

A transformation T is one-to-one if for every $\vec{b}$ in $\mathbb{R}^m$ the equation $T(\vec{x}) = \vec{b}$ never has multiple solutions (unique or none).

Example 8.7. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation whose standard matrix is given by

$$A = \begin{bmatrix} -1 & 2 & 5 & 1 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

- Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? (the same as: is T onto?)
- Is T a one-to-one transformation?

Solution:

- To check if T is onto, we need to verify if for every $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$, the equation $T(\vec{x}) = \vec{b}$ has a solution. This equation is the same as $A\vec{x} = \vec{b}$. The later equation is equivalent to the linear system with augmented matrix
$$\begin{bmatrix} -1 & 2 & 5 & 1 & b_1 \\ 0 & -2 & 3 & 2 & b_2 \\ 0 & 0 & 0 & -4 & b_3 \end{bmatrix}.$$
The matrix is already in row echelon form and we see that the system is always consistent for every $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. It follows that for every $\vec{b} \in \mathbb{R}^3$, there exists $\vec{x} \in \mathbb{R}^4$ such that $T(\vec{x}) = A\vec{x} = \vec{b}$, so T is onto.

- To check if T is one-to-one, we need to verify if for every $\vec{b}$ in $\mathbb{R}^3$, there exist at most one vector $\vec{x}$ in $\mathbb{R}^4$ such that $T(\vec{x}) = \vec{b}$, that is $A\vec{x} = \vec{b}$. This again equivalent to the linear system with augmented matrix
$$\begin{bmatrix} -1 & 2 & 5 & 1 & b_1 \\ 0 & -2 & 3 & 2 & b_2 \\ 0 & 0 & 0 & -4 & b_3 \end{bmatrix}$$

This shows that the system is consistent. Since there are more variables than equations, we have a free variable (x_4 is free), then the system has infinitely many solutions, so T is not one-to-one.

Theorem 8.8. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Proof. Suppose that the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution. Let $\vec{b}$ be in $\mathbb{R}^m$ and $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ such that $T(\vec{u}) = T(\vec{v}) = \vec{b}$. Then $T(\vec{u}) - T(\vec{v}) = \vec{0}$. Since T is linear, we have $T(\vec{u} - \vec{v}) = \vec{0}$. As $\vec{0}$ is the only solution of $T(\vec{x}) = \vec{0}$, we have $\vec{u} - \vec{v} = \vec{0}$, so $\vec{u} = \vec{v}$. It follows that T is one-to-one.

Suppose T is one-to-one. Then every equation including $T(\vec{x}) = \vec{0}$ has at most one solution. Since $T(\vec{0}) = \vec{0}$ is always true, $T(\vec{x}) = \vec{0}$ has only the trivial solution. $\square$

Facts:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A spans $\mathbb{R}^m$.
- T is one-to-one if and only if the columns of A are linearly independent.

Note:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. As consequences the above facts, we have the following:

- If there is a pivot in every row of the row echelon form of A, then T is onto.
- If there is a pivot in every column of the row echelon form of A, then T is one-to-one.