DAY 28: Wednesday, October 24th

4 Coordinate Systems

Let V be a vector space and let $\mathcal{B}$ be a basis for V. If $\mathcal{B}$ contains n vectors, then the coordinate system makes V act like $\mathbb{R}^n$.

Theorem 4.1 (Unique Representation Theorem)**.** Let V be a vector space and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for V. Then for every $\vec{v}$ in V, there exists a unique set of scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n.$$

Proof. Since $\mathcal{B}$ is a basis for V then $V = \text{Span}(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$. So every element $\vec{v}$ of V is a linear combination of the elements of $\mathcal{B}$. That is, there exist $c_1, c_2, \ldots, c_n$ such that

(i) $$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

Suppose that we have also

(ii) $$\vec{v} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n$$

Subtracting (i) and (ii), we obtain

(iii) $$\vec{0} = (c_1 - d_1)\vec{b}_1 + (c_2 - d_2)\vec{b}_2 + \cdots + (c_n - d_n)\vec{b}_n$$

Since $\mathcal{B}$ is linearly independent (as $\mathcal{B}$ is a basis), the weights in (iii) are all zero. That is

$$c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots, c_n - d_n = 0,$$

hence $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$ and the representation in (i) is unique. $\square$

Definition 4.2. Let V be a vector space and let $\mathcal{B}$ be a basis for V. Let $\vec{v}$ be a vector in V with

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

The coordinates of $\vec{v}$ relative the $\mathcal{B}$ (or $\mathcal{B}$-coordinates of $\vec{v}$) are the weights $c_1, c_2, \ldots, c_n$. We write

$$\begin{bmatrix} \vec{v} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The map $T : V \to \mathbb{R}^n$ defined by $T(\vec{v}) = \begin{bmatrix} \vec{v} \end{bmatrix}_{\mathcal{B}}$ is called Coordinate mapping (determined by $\mathcal{B}$).

Example 4.3. Let $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$. Clearly $\mathcal{B}$ is basis for $\mathbb{R}^2$. Compute the $\mathcal{B}$-coordinate of $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution

If

$$\vec{u} = x_1\vec{b}_1 + x_2\vec{b}_2 \tag{iv}$$

then $\begin{bmatrix} \vec{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The vector equation (iv) is equivalent to the linear system with augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The matrix is already in row echelon form and we have $x_2 = 2$ and $x_1 = 3$. That is

$$\vec{u} = 3\vec{b}_1 + 2\vec{b}_2$$

Therefore the coordinate of $\vec{u}$ relative to $\mathcal{B}$ is given by $\begin{bmatrix} \vec{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Coordinates in $\mathbb{R}^n$

Note 4.4. The entries in vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ are the coordinates of $\vec{x}$ relative to the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$. We call it standard coordinates.

Fact 4.5. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. Let $\vec{u}$ be in $\mathbb{R}^n$ with

$$\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \tag{v}$$

that is $\begin{bmatrix} \vec{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. The equality in (v) is the same as

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{u}$$

that is

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{u} \end{bmatrix}_{\mathcal{B}} = \vec{u}$$

Since $\mathcal{B}$ is basis, the matrix $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$ is invertible. So we also have

$$\begin{bmatrix} \vec{u} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}^{-1} \vec{u}$$

Definition 4.6. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. The matrix $P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$ is called the change-of-coordinates matrix.

Proposition 4.7. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis for a vector space V. Let $T : V \to \mathbb{R}^n$, with $T(\vec{v}) = \left[\ \vec{v}\ \right]_{\mathcal{B}}$, be the mapping coordinate determined by $\mathcal{B}$. Then, T is one-to-one linear transformation onto $\mathbb{R}^n$.

Fact 4.8. Since the coordinate mapping $T : V \to \mathbb{R}^n$ is a linear transformation, we have

(a) $\left[\ \vec{u} + \vec{v}\ \right]_{\mathcal{B}} = \left[\ \vec{u}\ \right]_{\mathcal{B}} + \left[\ \vec{v}\ \right]_{\mathcal{B}}$ for $\vec{u}, \vec{v}$ in V.

(b) $\left[\ c\vec{v}\ \right]_{\mathcal{B}} = c\left[\ \vec{v}\ \right]_{\mathcal{B}}$ for $\vec{v}$ in V and scalar c.

(c) $\left[\ \vec{v}\ \right]_{\mathcal{B}} = \vec{0}$ if and only if $\vec{v} = \vec{0}$.

Definition 4.9. Let V and W be vector spaces. We say that V and W are isomorphic if there exists a one-to-one linear transformation $T : V \to W$ onto W (we say that T is an isomorphism). In this case we write $V \cong W$.

Note that if V is a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$, then V isomorphic to $\mathbb{R}^n$ as the coordinate mapping $T : V \to \mathbb{R}^n$, $T(\vec{v}) = \left[\ \vec{v}\ \right]_{\mathcal{B}}$, is an isomorphism. This makes V act like $\mathbb{R}^n$.

Note 4.10. For the set $\mathbb{P}_n$ of polynomials of degree at most n, the standard basis is $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$. If $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$, then

$$\left[\ p(t)\ \right]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The mapping coordinate $T : \mathbb{P}_n \to \mathbb{R}^{n+1}$, with $T(p(t)) = \left[\ p(t)\ \right]_{\mathcal{B}}$, is an isomorphism so we can study $\mathbb{P}_n$ as $\mathbb{R}^{n+1}$.

Example 4.11. Show that $p_1(t) = 1 + 2t^2, p_2(t) = 4 + t + 5t^2$ and $p_3(t) = 3 + 2t$ are linearly dependent in $\mathbb{P}_2$.

Solution

Since the coordinate mapping $T : \mathbb{P}_2 \to \mathbb{R}^3$ is an isomorphism, if $T(p_1(t)), T(p_2(t)), T(p_3(t))$ are linearly dependent in $\mathbb{R}^3$, then $p_1(t), p_2(t), p_3(t)$ are also linearly dependent in $\mathbb{P}_2$. We have

$$T(p_1(t)) = \left[\ p_1(t)\ \right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, T(p_2(t)) = \left[\ p_2(t)\ \right]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, T(p_3(t)) = \left[\ p_3(t)\ \right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

The matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix}$ is row equivalent to $B \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, hence $\det(A) = 0$ and the columns of A are linearly dependent in $\mathbb{R}^3$. If follows that $p_1(t), p_2(t), p_3(t)$ are linearly dependent in $\mathbb{P}_2$. In fact we can check that

$$p_3(t) = 2p_2(t) - 5p_1(t).$$