DAY 13: September 19th

8 The Matrix of a Linear Transformation

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we show in this section that it is actually a matrix transformation $\vec{x} \mapsto A\vec{x}$.

Definition 8.1. Let $\vec{e}_i$ denote the **standard basis vector** having 1 in the ith row, and zero elsewhere.

$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \longleftarrow ith \quad \text{row} \\ \\ \\ \\ \end{array}$$

In particular $\vec{e}_i$ is the ith column of the $n \times n$ identity matrix.

Example 8.2. Consider the 3×3 identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}$$

Theorem 8.3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a unique matrix A such that for every $\vec{x}$ in $\mathbb{R}^n$ we have

$$T(\vec{x}) = A\vec{x}.$$

In fact, A is the $m \times n$ matrix whose ith column is the vector $T(\vec{e}_i)$. In particular

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

The matrix A is called the **standard matrix** for T.

Proof. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be in $\mathbb{R}^n$. Then

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n.$$

Hence $T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$. Since T is a linear transformation, we have

$$\begin{aligned} T(\vec{x}) &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n) \\ &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= A\vec{x}. \end{aligned}$$

$\square$

Example 8.4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Compute $T(\vec{u})$ where $\vec{u} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.
- Give the matrix A such that $T(\vec{x}) = A\vec{x}$ for every $\vec{x}$ in $\mathbb{R}^2$ (the matrix A is the standard matrix of T).

Solution:

- We have

$$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  Hence

$$\begin{aligned} T(\vec{u}) &= T\left(4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 4T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 3T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= 4\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix} \end{aligned}$$

- The standard matrix A of T is given by

$$\begin{aligned} A &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \end{aligned}$$

  Check that

$$A\vec{u} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}.$$

Properties of Linear Transformation

Definition 8.5. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if every $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one vector $\vec{u}$ in $\mathbb{R}^n$. That is, T is onto if for all $\vec{b}$ in $\mathbb{R}^m$, there exists $\vec{u}$ in $\mathbb{R}^n$ such that $T(\vec{u}) = \vec{b}$.

Facts:

- T is onto when the range of T is all of the codomain $\mathbb{R}^m$.
- T is not onto when there is some $\vec{b}$ in $\mathbb{R}^m$ for which the equation $T(\vec{x}) = \vec{b}$ has no solution.

Definition 8.6. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if every $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one vector $\vec{u}$ in $\mathbb{R}^n$. That is T is said to be one-to-one if for every $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, $T(\vec{u}) = T(\vec{v})$ implies $\vec{u} = \vec{v}$.

Facts:

A transformation T is one-to-one if for every $\vec{b}$ in $\mathbb{R}^m$ the equation $T(\vec{x}) = \vec{b}$ never has multiple solutions (unique or none).

Example 8.7. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation whose standard matrix is given by

$$A = \begin{bmatrix} -1 & 2 & 5 & 1 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

- Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? (the same as: is T onto?)
- Is T a one-to-one transformation?

Solution:

- To check if T is onto, we need to verify if for every $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$, the equation $T(\vec{x}) = \vec{b}$ has a solution. This equation is the same as $A\vec{x} = \vec{b}$. The later equation is equivalent to the linear system with augmented matrix
$$\begin{bmatrix} -1 & 2 & 5 & 1 & b_1 \\ 0 & -2 & 3 & 2 & b_2 \\ 0 & 0 & 0 & -4 & b_3 \end{bmatrix}.$$
The matrix is already in row echelon form and we see that the system is always consistent for every $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. It follows that for every $\vec{b} \in \mathbb{R}^3$, there exists $\vec{x} \in \mathbb{R}^4$ such that $T(\vec{x}) = A\vec{x} = \vec{b}$, so T is onto.

- To check if T is one-to-one, we need to verify if for every $\vec{b}$ in $\mathbb{R}^3$, there exist at most one vector $\vec{x}$ in $\mathbb{R}^4$ such that $T(\vec{x}) = \vec{b}$, that is $A\vec{x} = \vec{b}$. This again equivalent to the linear system with augmented matrix
$$\begin{bmatrix} -1 & 2 & 5 & 1 & b_1 \\ 0 & -2 & 3 & 2 & b_2 \\ 0 & 0 & 0 & -4 & b_3 \end{bmatrix}$$

This shows that the system is consistent. Since there are more variables than equations, we have a free variable (x_4 is free), then the system has infinitely many solutions, so T is not one-to-one.

Theorem 8.8. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Proof. Suppose that the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution. Let $\vec{b}$ be in $\mathbb{R}^m$ and $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ such that $T(\vec{u}) = T(\vec{v}) = \vec{b}$. Then $T(\vec{u}) - T(\vec{v}) = \vec{0}$. Since T is linear, we have $T(\vec{u} - \vec{v}) = \vec{0}$. As $\vec{0}$ is the only solution of $T(\vec{x}) = \vec{0}$, we have $\vec{u} - \vec{v} = \vec{0}$, so $\vec{u} = \vec{v}$. It follows that T is one-to-one.

Suppose T is one-to-one. Then every equation including $T(\vec{x}) = \vec{0}$ has at most one solution. Since $T(\vec{0}) = \vec{0}$ is always true, $T(\vec{x}) = \vec{0}$ has only the trivial solution. $\square$

Facts:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A spans $\mathbb{R}^m$.
- T is one-to-one if and only if the columns of A are linearly independent.

Note:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. As consequences the above facts, we have the following:

- If there is a pivot in every row of the row echelon form of A, then T is onto.
- If there is a pivot in every column of the row echelon form of A, then T is one-to-one.