DAY 4: AUGUST 29th

Augmented Matrix and Row Echelon Form:

Suppose that the augmented matrix of a liner system is reduced to an echelon form matrix.

Definition 2.10.

- A pivot (or basic) variable is a variable corresponding to the pivot (or to the pivot column),
- A free (or independent) variable is a variable which does not correspond to any pivot.

Example 2.11. Suppose that the following matrix is the reduced echelon form of the augmented matrix of a linear system in the variables x_1, x_2, x_3:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Since each variable corresponds to a pivot (or to a pivot column), they are all pivot (basic) variable. The system has no free variable.

Solutions of Linear systems:

FACT: Suppose the augmented matrix of a linear system has been transformed to an echelon form matrix. Then:

(a) if there is a row of the form $[0 \quad 0 \quad \ldots \quad 0 \quad a]$ where $a \neq 0$, then the system is inconsistent (has no solution),

(b) if every variable is a pivot (basic) variable, then the system is consistent and it has a single and unique solution,

(c) if some variables are free (or independent), and if the system is consistent, then it has infinitely many solutions.

Example 2.12. Case 1: the augmented matrix of a linear system is reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

The last row corresponds to $0x_1 + 0x_2 + 0x_3 = 6$ which is impossible, hence, the system has no solution.

Case 2: the augmented matrix of a linear system is reduced to

$$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Every variable is a pivot variable, hence the system has a unique solution. This matrix represents the following equations:

$$
\begin{aligned}
x_1 + 0x_2 + 0x_3 &= -5 \\
0x_1 + x_2 + 0x_3 &= 2 \\
0x_1 + 0x_2 + x_3 &= 6
\end{aligned}
$$

So the solution is $(-5, 2, 6)$.

Case 3: the augmented matrix of a linear system is reduced to

$$
\begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

The variables x_1 and x_3 are pivot variables and x_2 is a free variable. Hence, the system has infinitely many solutions. We have

$$
\begin{aligned}
x_1 + x_3 &= 6 \\
x_2 - x_3 &= 3 \\
x_3 &\text{ is free}
\end{aligned}
$$

Whenever we choose a value of x_3, then we have the value of x_1 and x_2. For example, $x_3 = 0$, then $x_1 = 6$ and $x_2 = 3$. So $(6, 3, 0)$ is a solution.

Parametric Description of Solution Sets:

In the case where the system has free variables, the solution set has parametric description be setting the free variables as parameters (denoted by: $t, r, s, ...$).

Example 2.13. In the following system

$$
\begin{aligned}
x_1 + x_3 &= 6 \\
x_2 - x_3 &= 3 \\
x_3 &\text{ is free}
\end{aligned}
$$

We set x_3 as the parameter t, hence the solution set of the system is described by the following general solution:

$$
\begin{aligned}
x_1 &= 6 - t \\
x_2 &= 3 + t \\
x_3 &= t
\end{aligned}
$$

where t can be any number.

Using Row Reduction to Solve a Linear System.

Given a linear system to solve:

(1) Write the augmented matrix,

(2) Perform rwo reduction to obtain echelon form. If the system is not consistent then there are no solutions and you may stop,

(3) Perform row reduction to obtain reduce echelon form,

(4) Write the system of equations corresponding to the reduced echelon form,

(5) Basic variables correspond to pivots, and free variables correspond to column without pivots,

(6) Write basic variables in terms of free variables, or write the parametric description of the solution.

3 Vectors

Vectors are central in linear algebra. In this section, we explore the relations between vectors and linear system.

Definition 3.1 (Vectors)**.** A vector (column vector) is a matrix with one column. We will use the notation $\vec{u}, \vec{v}, \ldots$.

Example 3.2. .

$\begin{bmatrix} 0 \\ -1 \\ 2 \\ 5 \end{bmatrix}$ is a vector.

Definition 3.3. Two vectors $\vec{u}$ and $\vec{v}$ are equals when their corresponding entries are equals.

Example 3.4.

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Definition 3.5.

- The sum of two vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with the same number of rows is defined by

  $$\vec{w} = \vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

- The product of a scalar α and a vector $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ is given by

$$\alpha\vec{u} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix}$$

Example 3.6. Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ be vectors and $\alpha = 3$ a scalar. We have

$$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix}$$

and

$$\alpha\vec{u} = 3 \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 15 \end{bmatrix}$$

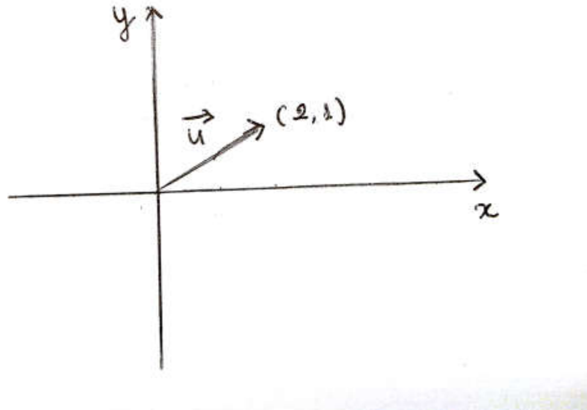
Geometric Interpretation of vectors:

For a positive integer n, a point in $\mathbb{R}^n$ can be viewed as a vector. A point with coordinates $(a_1, a_2, \ldots, a_n)$ corresponds with the vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and the zero vector is given by $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

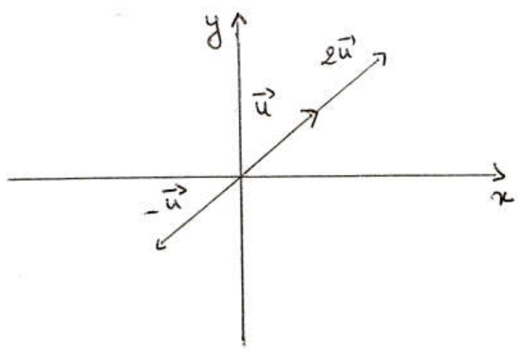
Vector in $\mathbb{R}^2$

A point with coordinate (a, b) in $\mathbb{R}^2$ can be identified as the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

Example 3.7. Consider the vector $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



The scalar multiplication can be viewed as stretching the vector.

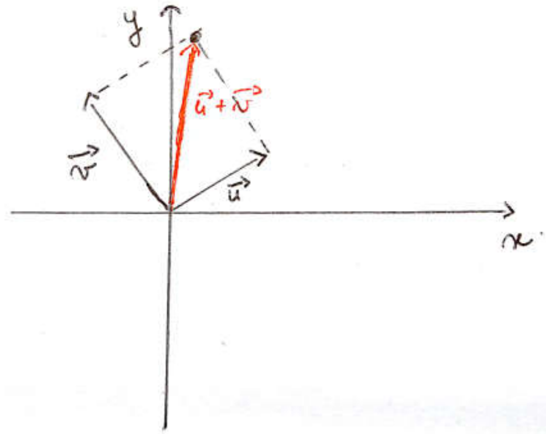


To add two vectors geometrically, we draw the parallelogram determined by the two vectors and the diagonal of this parallelogram is the sum of the the two vectors.

Parallelogram rule for vectors addition:

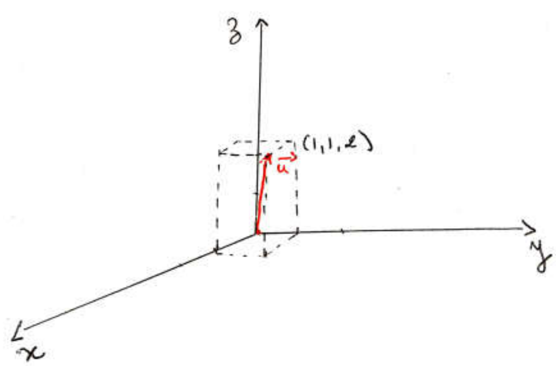
If $\vec{u}$ and $\vec{u}$ are two vectors in $\mathbb{R}^2$, then $\vec{u}+\vec{v}$ corresponds to the fourth vertex of the parallelogram whose opposite vertex is $\vec{0}$ and whose other two vertices correspond to $\vec{u}$ and $\vec{v}$.

Example 3.8. Let $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.



Vectors in $\mathbb{R}^3$

A point with coordinate (a, b, c) in $\mathbb{R}^3$ corresponds to the vector $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$



Theorem 3.9. Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{v}$.

(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

(3) $(\vec{u} + \vec{0}) = \vec{0} = \vec{0} + \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0} = -\vec{u} + \vec{u}$.

(5) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.

(6) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$.

(7) $c(d\vec{u}) = (cd)\vec{u}$.

(8) $1\vec{u} = \vec{u}$.

Linear Combination and spanning

Definition 3.10 (Linear Combination)**.** Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_n$ the vector

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p$$

is called a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Example 3.11. We have

$$2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

The vector $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ is a linear combination of the vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (as $\vec{w} = 2\vec{u} - \vec{v}$).

Suppose that we have the following linear combination:

$$x_1\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + x_2\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + x_3\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

This is the same as

$$\begin{bmatrix} a_1x_1 + b_1x_2 + c_1x_3 \\ a_2x_1 + b_2x_2 + c_2x_3 \\ a_3x_1 + b_3x_2 + c_3x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

By the equality of two vectors, we have the following equalities:

$$a_1x_1 + b_1x_2 + c_1x_3 = d_1$$
$$a_2x_1 + b_2x_2 + c_2x_3 = d_2$$
$$a_3x_1 + b_3x_2 + c_3x_3 = d_3$$

It follows that the vector $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ if the above linear system has solutions.

FACT:

A vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_p\vec{v}_p = \vec{b}$ has the same solutions as the system of linear equations corresponding to the augmented matrix

(A) $$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_p & \vec{b} \end{bmatrix}$$

In particular, $\vec{b}$ can be written (or generated) as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ if and only if the linear system with augmented matrix (A) is consistent.

Example 3.12. Is $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ a linear combination of $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$?

The vector equation is

$$x \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

The augmented matrix of the corresponding linear system is given by

$$\text{(B)} \qquad \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -1 \\ 0 & 2 & 6 \end{bmatrix}$$

EXERCISE: Reduce the matrix (B) to reduced echelon form.

The echelon form of matrix (B) is given by

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the system is consistent, and $x = 2$ and $y = 3$. Therefore $\vec{b} = 2\vec{v}_1 + 3\vec{v}_2$, so $\vec{b}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Definition 3.13 (Spanning)**.** Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$. The subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

$$\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p \colon c_1, c_2, \ldots, c_p \in \mathbb{R}\}.$$

NOTE:

- The span of $\vec{0}$ in $\mathbb{R}^n$ is the single vector $\vec{0}$.
- The span of a single nonzero vector in $\mathbb{R}^2$ and $\mathbb{R}^3$ is a line through $\vec{0}$.
- The span of two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, with $\vec{v} \neq c\vec{u}$, is a plan through $\vec{0}$.

Example 3.14. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -9 \\ -30 \\ 31 \end{bmatrix}$

Span$(\vec{v}_1, \vec{v}_2)$ is a plane in $\mathbb{R}^3$, is $\vec{b}$ in that plane?

This question is the same as: is $\vec{b}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$. The approach to solve this is the same as in Example 3.12.