DAY 38: Monday, November 19th

Definition 3.3. An $n \times n$ matrix A is diagonalizable if there exist an $n \times n$ invertible matrix P and an $n \times n$ diagonal matrix D such that $A = PDP^{-1}$. That is, A is diagonalizable if it is similar to a diagonal matrix.

Example 3.4. The matrix $A = \begin{bmatrix} 16 & -35 \\ 6 & -13 \end{bmatrix}$ is diagonalizable as

$$A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

with $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

Theorem 3.5 (The Diagonalization Theorem)**.** An $n \times n$ matrix A is diagonalizable if and only if A has an n linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns if P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Note 3.6. An $n \times n$ matrix A is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ made up of eigenvectors of A. Such basis is called an eigenvector basis

Diagonalizing Matrices

Let A be an $n \times n$ matrix. To diagonalize A, there are four steps to implement the description in Theorem 3.4:
Step 1: Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (it's possible that $\lambda_i = \lambda_j$ for some $i \neq j$) of A by solving the equation characteristic $\det(A - \lambda I_n) = 0$.
Step 2: Find eigenvectors $\vec{v}_i$ that correspond respectively to λ_i, by solving the homogeneous equation $(A - \lambda_i I_n)\vec{x} = \vec{0}$, such that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent. If it fails then, by Theorem 3.4, A is not diagonalizable.
Step 3: Construct the matrix P from the vectors in Step 2 (assuming $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent).

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

Step 4: Construct D from the corresponding eigenvalues. The order of the eigenvalues must match the order chosen for the columns of P.

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

Check that $A = PDP^{-1}$.

Example 3.7. Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible.

Solution

Step 1: Eigenvalues. We have

$$\det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+2)^2$$

The equation characteristic is given by $-(\lambda-1)(\lambda+2)^2 = 0$. Hence, the eigenvalues are $\lambda_1 = 1, \lambda_2 = -2$, and $\lambda_3 = -2$.

Step 2: Linearly independent eigenvectors.

For $\lambda = 1$, the parametric vector form of the homogeneous equation $(A - I_3)\vec{x} = \vec{0}$ is given by (by reducing the augmented matrix):

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, x_3 \in \mathbb{R}.$$

Choosing $x_3 = 1$, the vector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A that corresponds to the eigenvalue $\lambda = 1$.

For $\lambda = -2$, the parametric vector form of the solutions of the equation $(A + 2I_3)\vec{x} = \vec{0}$ is given by

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x_2, x_3 \in \mathbb{R}$$

Choosing $(x_2 = 1, x_3 = 0)$, and $(x_2 = 0, x_3 = 1)$, vectors $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvetcors of A corresponding to eigenvalue $\lambda = -2$.

We can check (by computing for example the determinant) that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. This implies that A is diagonalizable.

Step 4: The matrix P. We have

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4: The matrix D. We have

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

To verify that the diagonalization is true, check that $AP = PD$ instead of $A = PDP^{-1}$ to avoid computing P^{-1}.

Theorem 3.8. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem 3.9. Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$.

(a) For $k = 1, 2, \ldots, p$, the dimension of the eigenspace $\mathrm{Nul}(A - \lambda_k I_n)$ for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

(b) The matrix A is diagonalizable if and only if the sum of the dimension of the distinct eigenspaces equals n. This happens if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

(c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Example 3.10. Let $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$.

(a) Show that A is diagonalizable.

(b) Find matrices P and D, such that D is diagonal and $A = PDP^{-1}$ (or diagonalize A).

Solution

(a) To show that A is diagonalizable, we compute the eigenvalues of A and compare their multiplicities with the dimension of the corresponding eigenspaces:

- if A has four distinct eigenvalues, by Theorem 3.8, it is diagonalizable.
- if A has less than four distinct eigenvalues, then, by Theorem 3.9 A is diagonalizable if and only if the dimension of the eigenspace corresponding to each eigenvalue is equal to the multiplicity of the eigenvalue.

Since A is triangular, the eigenvalues of A are 5 and -3, each with multiplicity 2.
The row echelon form of the augmented matrix of the equation $(A - 5I_4)\vec{x} = \vec{0}$ is given by

$$\begin{bmatrix} 1 & 4 & -8 & 0 & 0 \\ 0 & 2 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence there are two free variables. It follows that the dimension of the eigenspace $\mathrm{Nul}(A - 5I_4)$ of A corresponding to the eigenvalue $\lambda = 5$ is 2 which is equal to the multiplicity of $\lambda = 5$.
The row echelon form of the augmented matrix of $(A + 3I_4)\vec{x} = \vec{0}$ is given by

$$\begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence there are two free variables. It follows that the dimension of the eigenspace $\mathrm{Nul}(A + 3I_4)$ of A corresponding to the eigenvalue $\lambda = -3$ is 2 which is equal to the multiplicity of $\lambda = -3$.
Both eigenspaces $\mathrm{Nul}(A - 5I_4)$ and $\mathrm{Nul}(A + 3I_4)$ have the same dimension as the multiplicity of $\lambda = 5$ and $\lambda = -3$ respectively. It follows from Theorem 3.9 that A is diagonalizable.

(b) To find the matrix P, we need to find bases for the eigenspaces.

The parametric vector form of the solutions of $(A-5I_4)\vec{x}=\vec{0}$ is given by

$$\vec{x}=x_3\begin{bmatrix}-8\\4\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-16\\4\\0\\1\end{bmatrix}, x_3,x_4\in\mathbb{R}$$

Hence $\mathcal{B}_{\lambda_1}=\{\begin{bmatrix}-8\\4\\1\\0\end{bmatrix},\begin{bmatrix}-16\\4\\0\\1\end{bmatrix}\}$ is a basis for the eigenspace $\mathrm{Nul}(A-5I_4)$.

The parametric vector form of the the solutions of $(A+3I_4)\vec{x}=\vec{0}$ is given by

$$\vec{x}=x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, x_3,x_4\in\mathbb{R}$$

Hence $\mathcal{B}_{\lambda_2}=\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\}$ is basis for the eigensapce $\mathrm{Nul}(A+3I_4)$.

By Theorem 3.9, the set $\{\begin{bmatrix}-8\\4\\1\\0\end{bmatrix},\begin{bmatrix}-16\\4\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\}$ is linearly independent, and the matrix P is given by

$$P=\begin{bmatrix}-8&-16&0&0\\4&4&0&0\\1&0&1&0\\0&1&0&1\end{bmatrix}$$

The diagonal matrix D is given by

$$D=\begin{bmatrix}5&0&0&0\\0&5&0&0\\0&0&-3&0\\0&0&0&-3\end{bmatrix}$$

So $A=PDP^{-1}$.