Complex continued fractions, automatically

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Multidimensional continued fractions and Euclidean dynamics

Lorentz Center

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In this talk *continued fractions* are expressions of the form

$$[a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

with a_i elements of some (possibly infinite) discrete set S.

I am particularly interested in:

(i) algorithms that generate continued fractions that converge and that represent elements (from some field F) essentially uniquely;

(ii) characterizing the sequences generated by such algorithms in a finite way.

In this talk: $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Regular (real) continued fractions:

$$[0; a_1, a_2, a_3, \ldots] = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

with a_i elements of $\mathbb{Z}_{\geq 1}$.

Infinite continued fractions represent real numbers in $[0, 1)$ in an essentially unique way. No further restrictions on a_i.

For irrational x the infinite sequence of a_i is obtained by putting $x_0 = x$ and repeating

$$a_i = \lfloor x_i \rfloor, \quad x_{i+1} = \frac{1}{x_i - a_i}.$$

Nearest integer continued fractions for real numbers are of the form

$$[0; b_1, b_2, b_3, \ldots] = 0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}}$$

with $b_i \in \mathbb{Z}_{\geq 2}$, and moreover,

$$b_j = 2 \quad \Rightarrow \quad b_{j+1} \geq 2 \quad \text{and} \quad b_j = -2 \quad \Rightarrow \quad b_{j+1} \leq -2.$$

Infinite nearest integer continued fractions represent real numbers in $[-\frac{1}{2}, \frac{1}{2})$ in an essentially unique way.

For irrational x the infinite sequence of b_i is obtained by putting $x_0 = x$ and repeating

$$b_i = \lfloor x_i \rceil, \quad x_{i+1} = \frac{1}{x_i - b_i}.$$

A *deterministic finite automaton* starts in a distinguished initial state, reads letters from a finite input word, which determine transitions to finite number of states, to end in one of the final states, which is either accepting or rejecting for the input word.

$A = (Q, \Sigma, \delta, q_0, F)$:

Q finite set of states,

Σ finite alphabet,

$\delta : Q \times \Sigma \to Q$ transitions,

$q_0 \in Q$ initial state,

$F \subset Q$ accepting states.

general example

4.1 Finite Automata

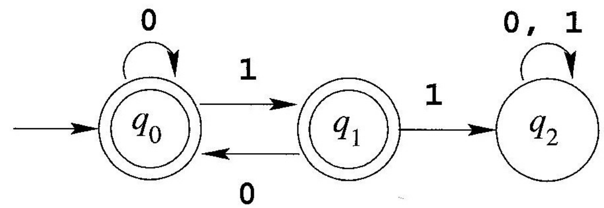

Figure 4.1: DFA Accepting Strings with No Two Consecutive 1's.

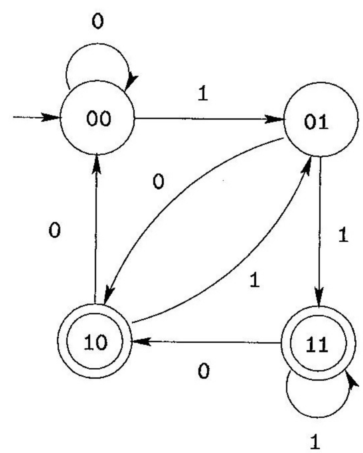

Figure 4.5: Minimal DFA for $L = \{0, 1\}^*1\{0, 1\}$.

source: Allouche, Shallit, Automatic sequences

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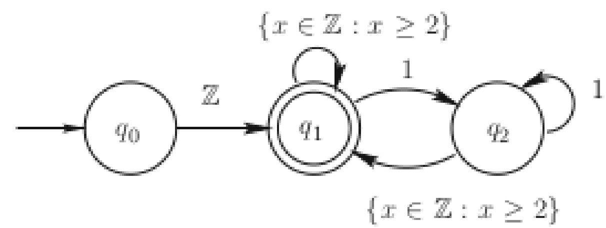


Fig. 1 Automaton for the simple continued fraction algorithm SCF

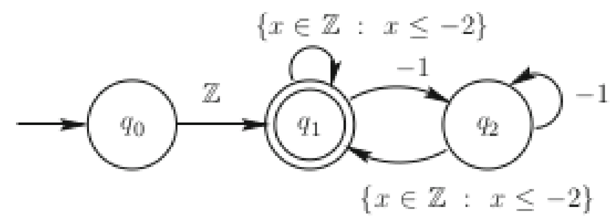


Fig. 2 Automaton for the ceiling algorithm CCF

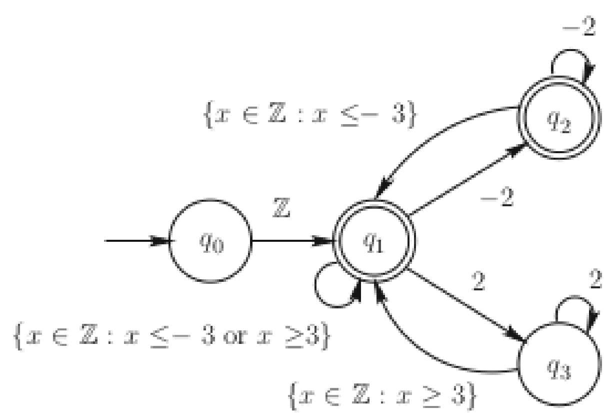


Fig. 3 Automaton for the nearest integer continued fraction algorithm NICF

source: Shallit, Description of finite generalized continued fractions by automata

Adolf Hurwitz's complex continued fractions

ÜBER DIE ENTWICKLUNG COMPLEXER GRÖSSEN IN KETTENBRÜCHE

VON

A. HURWITZ
in KÖNIGSBERG i/Pr.

Es möge (S) ein System von Zahlen bezeichnen, welches die Eigenschaft besitzt, dass die Summe, die Differenz und das Produkt irgend zweier Zahlen des Systems wieder Zahlen des Systems sind.[1] Wenn die complexen Grössen in der üblichen Weise durch die Punkte einer Ebene dargestellt werden, so wird den Zahlen von (S) ein gewisses System von Punkten entsprechen. Ich nehme an, das System (S) sei so beschaffen, dass von diesen Punkten in jedem endlichen Gebiete der Ebene nur eine endliche Anzahl liegt. Daraus folgt, dass ausser der Null keine andere Zahl von (S) existirt, deren absoluter Betrag kleiner als 1 ist. Denn die Potenzen dieser Zahl würden sämmtlich Zahlen von (S) sein und im Innern des um den Nullpunkt mit dem Radius 1 beschriebenen Kreises liegen. Eine letzte Voraussetzung, die ich in Betreff des Systems (S) mache, ist die, dass die Zahl 1 dem Systeme angehört.

Von einer Grösse x_0 ausgehend bilde ich nun die Gleichungskette:

$$(1)\qquad x_0 = a_0 + \frac{1}{x_1}, \qquad x_1 = a_1 + \frac{1}{x_2}, \qquad x_2 = a_2 + \frac{1}{x}, \quad \ldots$$

$$x_n = a_n + \frac{1}{x_{n+1}}, \quad \ldots$$

[1] Eine Theorie solcher Zahlsysteme ist in den bekannten Arbeiten von KRONECKER und DEDEKIND, vorzugsweise für den Fall algebraischer Zahlen, entwickelt. Vgl. insbesondere das XI. Supplement zu DIRICHLET's Vorlesungen über Zahlentheorie. Dritte Auflage.

Acta mathematica. 11. Imprimé le 6 Mars 1888.

In this talk *complex continued fractions* are expressions of the form

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots}}}}$$

with c_j Gaussian integers satisfying $|c_j| > 1$, (except $c_0 \in \mathbb{Z}[i]$), denoted $[c_0; c_1, c_2, c_3, \ldots]$.

For $z = x + yi \notin \mathbb{Q}(i)$ the infinite sequence of c_j for Adolf Hurwitz's complex continued fraction is obtained by putting $z_0 = z$ and repeating

$$c_j = \lfloor x_j \rceil + \lfloor y_j \rceil i, \quad z_{j+1} = \frac{1}{z_j - c_j}.$$

There are rather complicated restrictions on c_i to characterize the output.

A. Hurwitz's complex continued fractions

Fig. 1.

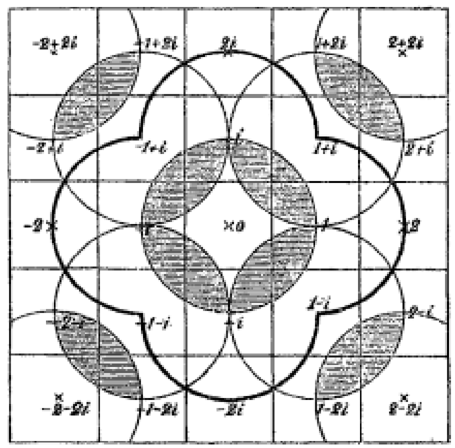
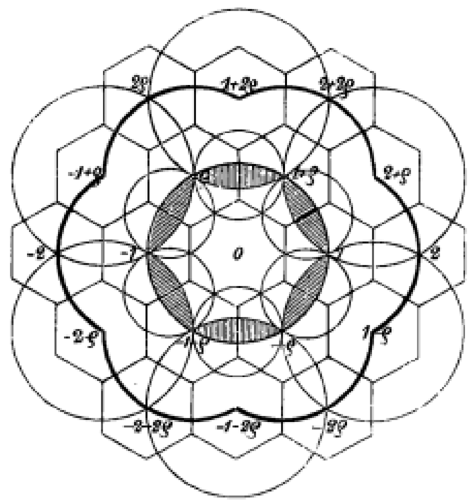


Tabelle unmöglicher Zahlfolgen.

	a_n	a_{n+1}	a_{n+2}	a_{n+3}
I.	$-2, 2i, -1+i, -2+i, -1+2i$	$1+i$		
II.	$2, 2i, 1+i$	$-2+2i$		
III.	$2+i, 1+2i$	$-2+2i$	$1+i$	
IV.	$-2, 2i, -1+i$	$2+2i$		
V.	$-2+i, -1+2i$	$2+2i$	$-2+2i$	$1+i$

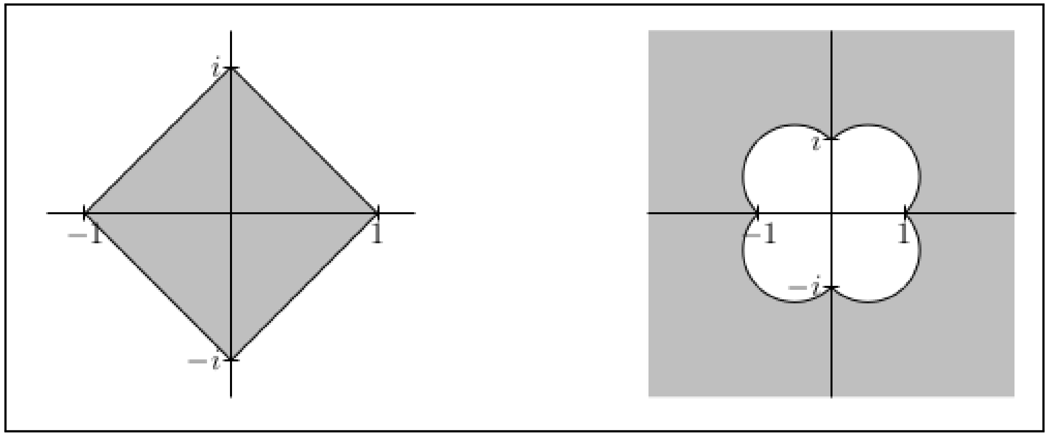
A. Hurwitz's other complex continued fractions

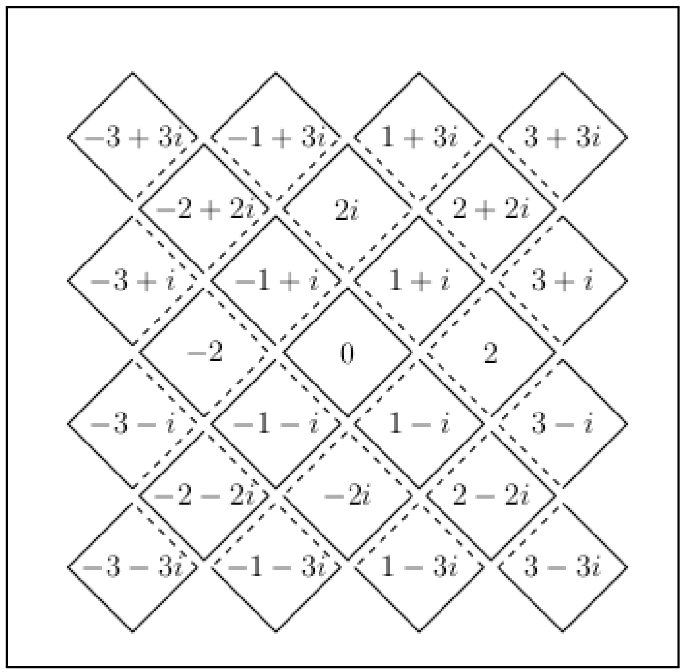
Fig. 2.



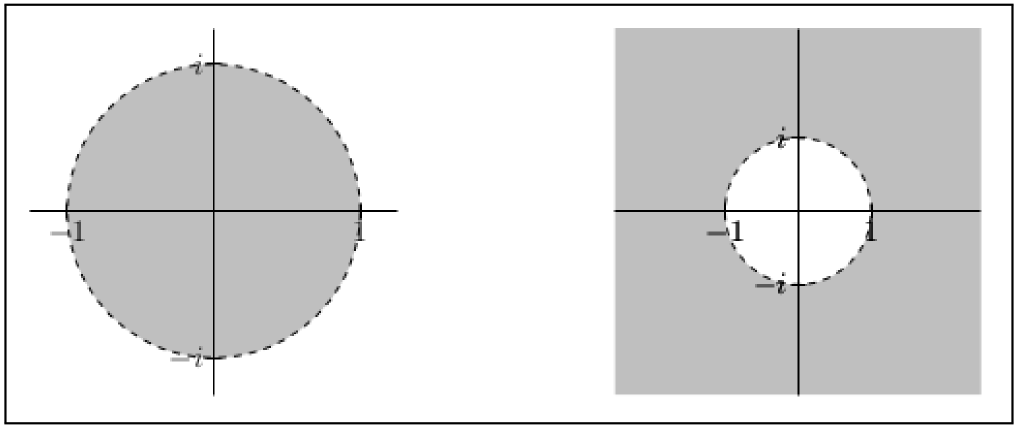
Here partial quotients are from the Eisenstein ring $\mathbb{Z}[\zeta_3]$; in principle possible for S any ring of integers in a Euclidean imaginary quadratic field. See also: Dani, *Continued fraction expansions for complex numbers — a general approach*, Acta Arithmetica 2015.

J. Hurwitz's complex continued fractions

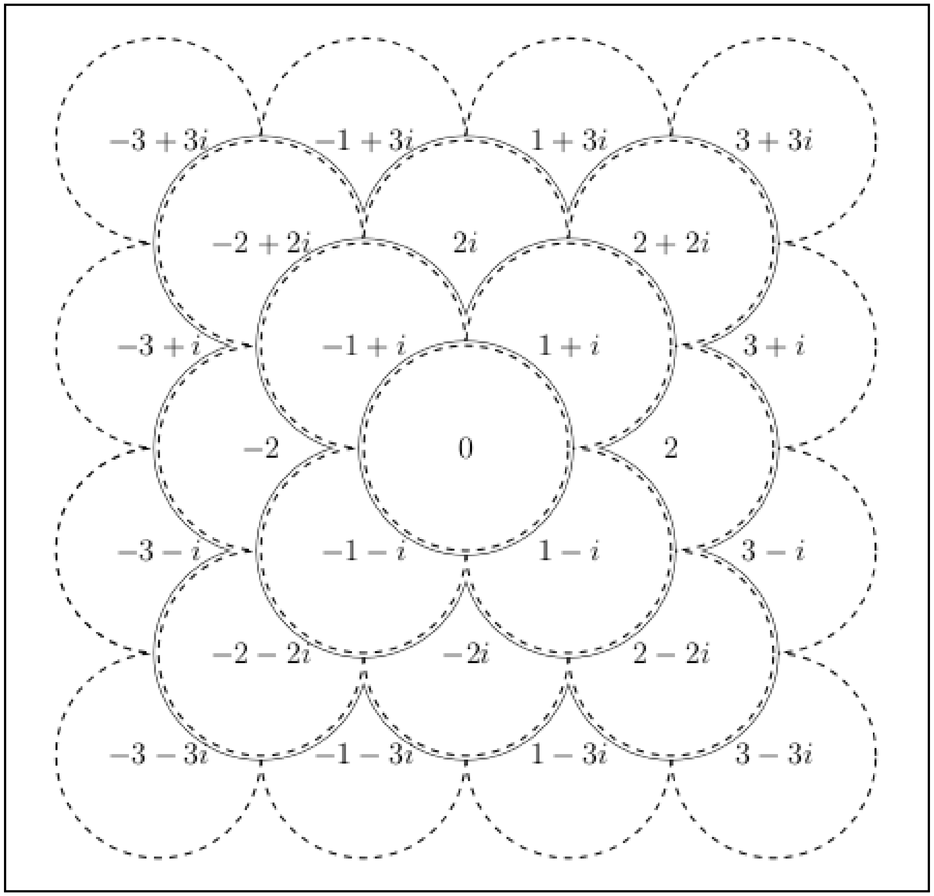




and its dual



i
-1
1
$-i$
i
-1
1
$-i$



$-3+3i$
$-1+3i$
$1+3i$
$3+3i$
$-2+2i$
$2i$
$2+2i$
$-3+i$
$-1+i$
$1+i$
$3+i$
-2
0
2
$-3-i$
$-1-i$
$1-i$
$3-i$
$-2-2i$
$-2i$
$2-2i$
$-3-3i$
$-1-3i$
$1-3i$
$3-3i$

My restrictions

I will be considering only complex continued fration algorithms for which

- the set of partial quotients is $S = \mathbb{Z}[i]$;
- the tiling (tiles being the subsets of $\mathbb{C}$ with given partial quotient) is translation invariant;
- the fundamental tile is finite union of 'nice' connected subsets of $\mathbb{C}$, contained in the unit circle and containing 0.

Think of regions bounded by circle arcs and line segments.

desirable properties

- convergence
- periodicity for quadratics
- recognizable output
- dynamical properties.

Hurwitz's Complex Continued Fractions

A Historical Approach and Modern Perspectives.

Dissertationsschrift zur Erlangung

des naturwissenschaftlichen Doktorgrades

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aus Kronach

Würzburg 2014

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SOME ERGODIC PROPERTIES OF A COMPLEX CONTINUED FRACTION ALGORITHM

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(*Received May 7, 1976*)

ABSTRACT

Some ergodic properties of a continued fraction algorithm for complex numbers are given.

on Adolf Hurwitz's algorithm

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CrossMark

On the construction of the natural extension of the Hurwitz complex continued fraction map

Hiromi Ei[1] · Shunji Ito[2] · Hitoshi Nakada[3] · Rie Natsui[4]

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Abstract
We consider the Hurwitz complex continued fraction map associated to the Gaussian field $\mathbb{Q}(i)$. We characterize the density function of the absolutely continuous invariant measure for the map associated to the Hurwitz continued fractions. For this reason, we construct a representation of its natural extension map (in the sense of an ergodic measure preserving map) on a subset of $\mathbb{C} \times \mathbb{C}$. This subset is constructed by the closure of pairs of the n-th iteration of a complex number by the Hurwitz complex continued fraction map and $-\frac{Q_n}{Q_{n-1}}$, where Q_n is the denominator of the n-th convergent of the Hurwitz continued fractions. The absolutely continuous invariant measure for the natural extension map is induced from the invariant measure for Möbius transformations on the set of geodesics over three dimension upper-half space. Then the absolutely continues invariant measure for the Hurwitz continued fraction map is given by its marginal measure.

$$\mathrm{RCF}(k) = \{x : x \in \mathbb{R} \mid a_i \leq k \text{ for } i \geq 1\}.$$

Conjecture. *There do not exist real algebraic numbers of degree greater than 2 over $\mathbb{Q}$ that have bounded partial quotients.*

$$\mathrm{HCF}(r) = \{z : z \in \mathbb{C} \mid |c_j| \leq r \text{ for } j \geq 1\}.$$

Theorem (Doug Hensley, WB)**.** *There exist complex numbers z of any even degree over $\mathbb{Q}(i)$ that have bounded HCF-partial quotients.*

ANNALS OF MATHEMATICS
Vol. 48, No. 4, October, 1947

ON THE SUM AND PRODUCT OF CONTINUED FRACTIONS

BY MARSHALL HALL, JR.

(Received November 6, 1946)

1. Introduction

For almost all x the development

$$x = [u_0, u_1, u_2, \cdots, u_i, \cdots] \tag{1.1}$$

as a regular continued fraction contains arbitrarily large numbers among the partial quotients $u_1, u_2, \cdots, u_i, \cdots$. Those real numbers whose fractional parts are continued fractions with partial quotients not exceeding n will be designated as the set $F(n)$. The sets $F(n)$ are all of measure zero. Two principal results of this paper are (Theorem 3.1) that every real number is representable as a sum of two numbers of the set $F(4)$, and (Theorem 3.2) that every real number greater than unity is representable as a product of two numbers of the set $F(4)$.

Theorem (Marshall Hall, 1947)**.**

$$\mathrm{RCF}(4) + \mathrm{RCF}(4) = \mathbb{R}$$

Theorem (Marshall Hall, 1947). $\mathrm{RCF}(4) + \mathrm{RCF}(4) = \mathbb{R}$

Theorem (Cusick 1973; Divis 1973). $\mathrm{RCF}(3) + \mathrm{RCF}(3) \neq \mathbb{R}$

Theorem (Hlavka 1975). $\mathrm{RCF}(4) + \mathrm{RCF}(3) = \mathbb{R}$
$\mathrm{RCF}(4) + \mathrm{RCF}(2) \neq \mathbb{R}$, $\mathrm{RCF}(7) + \mathrm{RCF}(2) = \mathbb{R}$

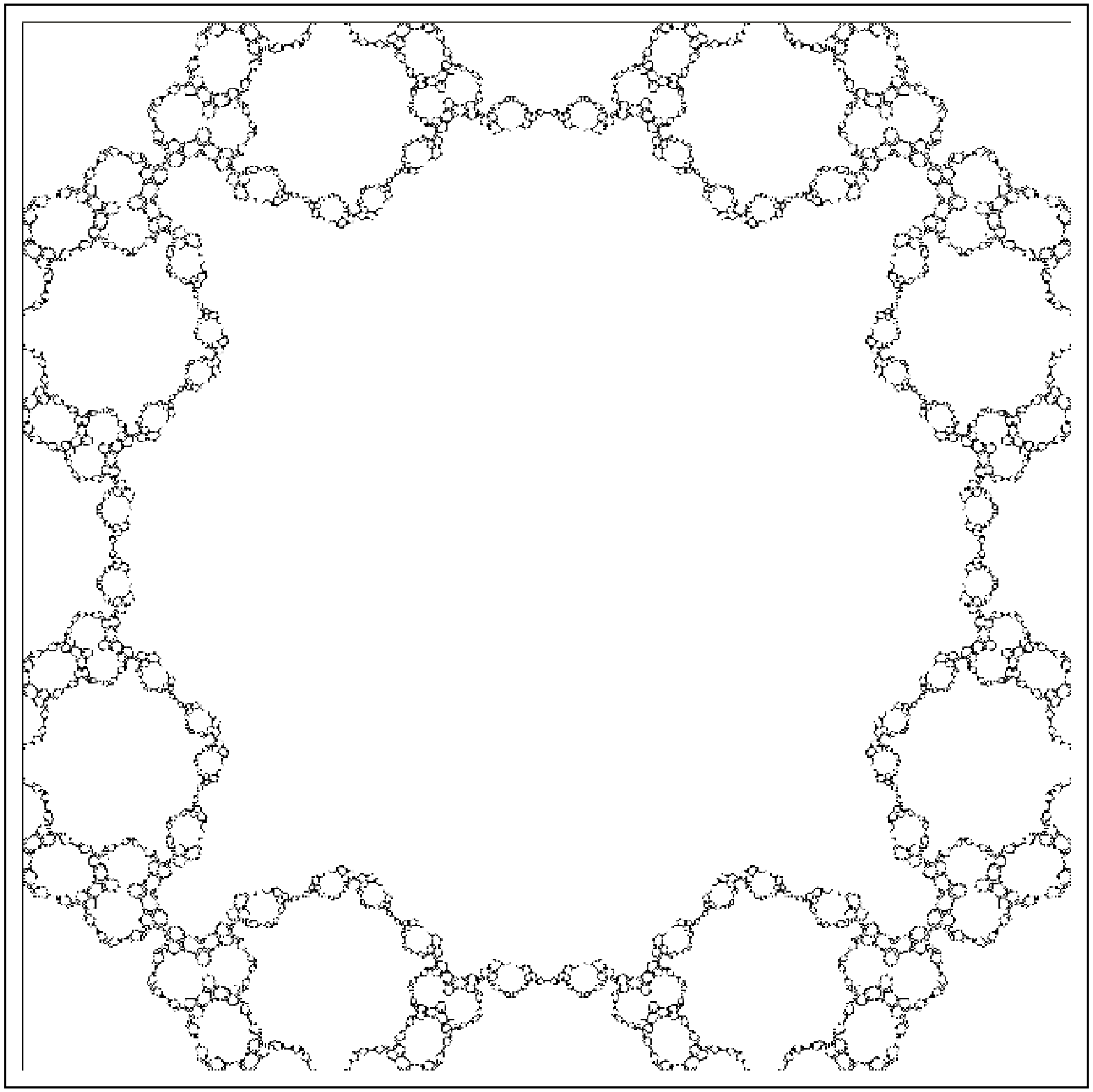
Theorem (Astels 2001). $\mathrm{RCF}(5) + \mathrm{RCF}(2) \neq \mathbb{R}$

$$\mathrm{NICF}(k) = \{x : x \in \mathbb{R} \mid |b_i| \leq k \text{ for } i \geq 1\}.$$

Theorem (Noud Aldenhoven, 2012). $\mathrm{NICF}(6) + \mathrm{NICF}(6) = \mathbb{R}$.

Theorem (Alex Brouwers, 2018). $\mathrm{NICF}(5) + \mathrm{NICF}(5) = \mathbb{R}$,
$\mathrm{NICF}(4) + \mathrm{NICF}(4) \neq \mathbb{R}$.

Theorem (Alex Brouwers, 2018). $\mathrm{HCF}(\sqrt{5}) + \mathrm{HCF}(\sqrt{5}) = \mathbb{C}$.



HCF($\sqrt{5}$)

Shallit's case

Integer Functions and Continued Fractions

Jeffrey O. Shallit
Department of Mathematics
Princeton University
Princeton, New Jersey
April 15, 1979

Shallit's case

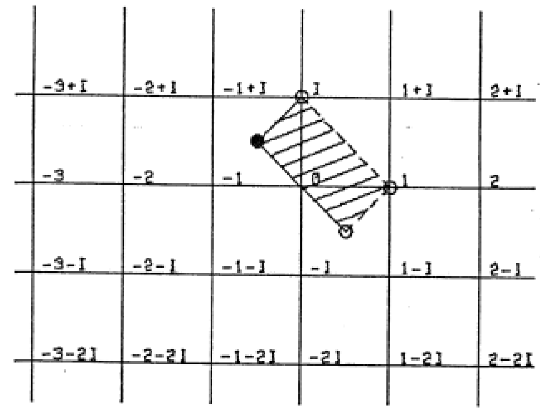


Figure 8

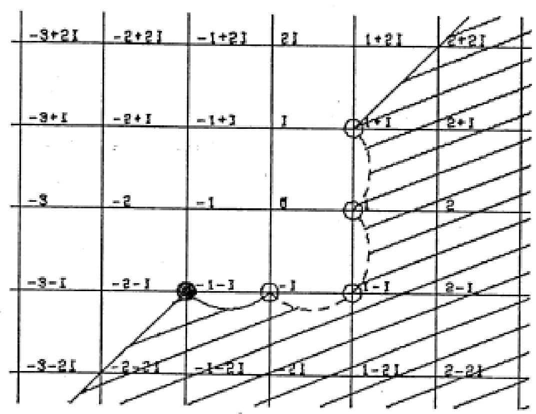


Figure 9

Shallit's case

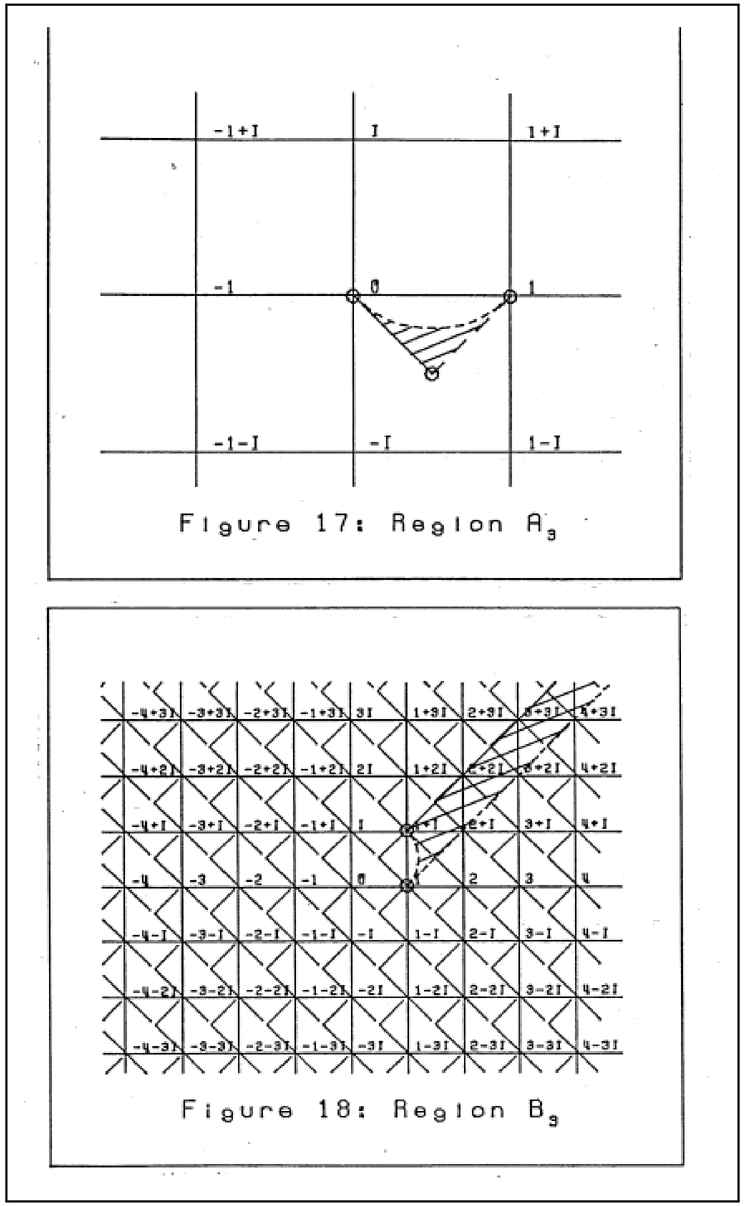


Figure 17: Region A_3

Figure 18: Region B_3

Theorem (Shallit, 1979). *There exists a finite automaton recognizing the admissible sequences for the continued fraction based on McDonnell's complex floor function.*

Theorem (Cijsouw, Bosma, 2015). *There exists a finite automaton recognizing the admissible sequences for the Hurwitz continued fraction based on the complex nearest integer function.*

Current project: automate this.

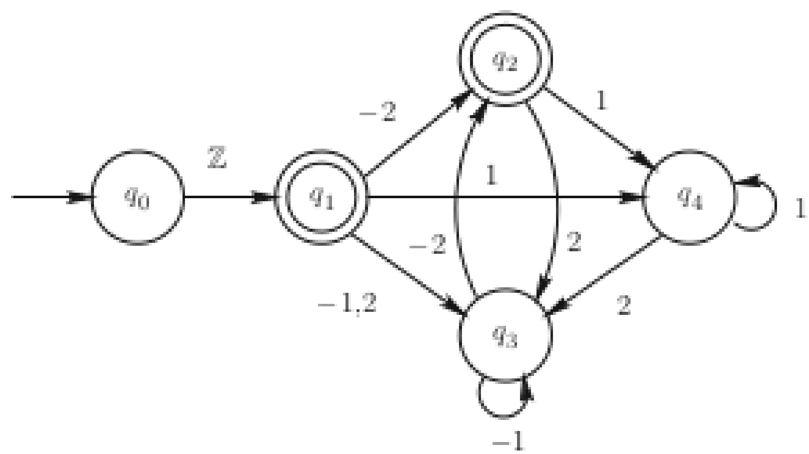
Goal: do nice examples and generalize this theorem.

Theorem (Shallit, 2013). *If the fundamental tile of a real continued fraction is the finite union of subintervals of $[-1, 1]$ then there exists an automaton recognizing the admissible sequences if and only if all intervals have endpoints that are rational or quadratic irrational.*

Example. *A real example: take for the fundamental tile*

$$(-1,-\frac{1}{2}] \cup \{0\} \cup (\frac{1}{2},1).$$

Fig. 5 Automaton generating bounded partial quotients



These expansions were introduced by Lehner [11] and further studied by Dajani and Kraaikamp [5]. An interesting feature of this expansion is that the partial quotients all lie in the set $\{-2,-1,1,2\}$. For example, the expansion of $\frac{52}{43}$ is

Various scribblings on the whiteboard: lift this to the complex case, and various other constructions.

Devise your own favourite complex continued fraction (but I claim priority)!