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II. "INVESTIGATIONS INTO LOGICAL DEDUCTION"*

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Introduction

GERHARD GENTZEN's inaugural dissertation for the University of Göttingen, "*Untersuchungen über das logische Schliessen*," of which Mr. Manfred Szabo here presents an English translation, is an important step in the deeper analysis of the logical calculus as it was begun in the thesis of Jacques Herbrand. In it Gentzen develops a new form of logical calculus, the "calculus of sequents," which he introduces by starting first from a "natural calculus" and by then transforming it for his technical purposes. Its central theorem, an "elimination theorem," is proved in detail in the first part of the paper. The second part deals with applications of this elimination theorem as well as containing the proof of the equivalence of the calculus of sequents with the usual logical calculus.

The *Untersuchungen* have had a profound influence on the development of mathematical logic and proof theory. The treatment of the calculus of sequents has been extended by Oiva Ketonen (in Finland) and by Haskell B. Curry (in the U.S.A.), and the elimination theorem of Gentzen has been transferred by Kurt Schütte to the usual logical calculus. The application of the elimination theorem to proof theory has been developed by Paul Lorenzen and Kurt Schütte through the use of "infinite induction." Lately Gaisi Takeuti (in Japan) has generalized the Gentzen calculus to a system of type theory in which the theory of real numbers can be formalized. Here the elimination theorem is not yet proved, but Takeuti shows that upon the assumption of the generalized elimination theorem the consistency of his system follows.

Thus Gentzen's *Untersuchungen* are of great current interest, and an English edition of this treatise is to be very much welcomed.

Paul Bernays

(Part I)

Synopsis

The investigations that follow concern the domain of *predicate logic*, called by H–A[1] the "lower functional calculus." It comprises the types of inference that are continually used in all parts of mathematics. What remains to be added to these are axioms and forms of inference that may be considered as being proper to the particular branches of mathematics, e.g., in elementary number theory the axioms of the natural numbers, of addition, multiplication, and exponentiation, as well as the inference of complete induction; in geometry the geometric axioms.

In addition to *classical logic* I shall also deal with *intuitionist logic* as formalized, for example, by Heyting[2].

The present investigations into classical and intuitionist predicate logic fall essentially into two only loosely connected parts.

1. My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return. I intended, first of all, to set up a formal system which came as close as possible to actual reasoning. The result was a "*calculus of natural deduction*." ("*NJ*" for intuitionist, "*NK*" for classical predicate logic.) This calculus then turned out to have specific properties; in particular, the "law of the excluded middle," which intuitionists reject, occupies a special position.

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[1] Hilbert-Ackermann, *Grundzüge der theoretischen Logik*, in this paper referred to as H–A.

[2] A. Heyting, *Die formalen Regeln der intuitionistischen Logik und Mathematik*, Sitzungsber. d. Preuß. Akad. d. Wiss., phys.-math. Kl. 1930, pp. 42–65.

I shall develop the calculus of natural deduction in the second Section of this paper together with some remarks concerning it.

2. A closer investigation of the specific properties of the natural calculus have finally led me to a very general theorem which will be referred to below as the "*Hauptsatz*."

The *Hauptsatz*[3] says that every purely logical proof can be reduced to a determinate, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying "it is not roundabout." No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result.

The *Hauptsatz* is valid both for classical and for intuitionist predicate logic.

In order to be able to enunciate and prove the *Hauptsatz* in a convenient form, I had to provide a logical calculus especially suited to the purpose. For this the natural calculus proved unsuitable. For, although it already contains the properties essential to the validity of the *Hauptsatz*, it does so only with respect to its intuitionist form, in view of the fact that the law of excluded middle, as pointed out earlier, occupies a special position in relation to these properties.

In Section III of this paper, therefore, I shall develop a new calculus of logical deduction containing all the desired properties in both their intuitionist and their classical form. ("*LJ*" for intuitionist, "*LK*" for classical predicate logic.) The *Hauptsatz* will then be enunciated and proved by means of that calculus.

The *Hauptsatz* permits of a variety of *applications*. To illustrate this I shall develop a decision procedure (IV, §1) for intuitionist propositional logic in Section IV, and shall in addition give a new proof of the consistency of classical arithmetic without complete induction (IV, §3).

Sections III and IV may be read independently of Section II.

3. Section I contains the stipulation of the terms and notations used in this paper.

In Section V, I prove the *equivalence* of the logical calculi *NJ*, *NK*, and *LJ*, *LK*, developed in this paper, by means of a calculus modeled on the formalisms of Russell, Hilbert, and Heyting (and which may easily be compared with them). ("*LHJ*" for intuitionist, "*LHK*" for classical predicate logic.)

4. Only the first part of my paper containing Sections I to III is presented here. Sections IV and V follow in the second part.

Section I.

Stipulation of Terms

To the concepts "object," "function," "predicate," "proposition," "theorem," "axiom," "proof," "inference," etc., in logic and mathematics there correspond, in the formalization of these disciplines, certain symbols or combinations of symbols. We divide these into:

1. *Symbols*.
2. *Expressions*, i.e., finite series of symbols.
3. *Figures*, i.e., finite sets of symbols, with some ordering.

Symbols count as special cases of expressions and figures, expressions as special cases of figures.

In this paper we shall consider symbols, expressions, and figures of the following kind:

1. *Symbols*.

These divide into constant symbols and variables.

1.1 Constant symbols:

Symbols for particular objects: 1, 2, 3, . . .

Symbols for particular functions: $+$, $-$, $\cdot$.

Symbols for particular propositions: **T** ("the true proposition"), **F** ("the false proposition").

Symbols for particular predicates: $=$, $<$.

Logical symbols[4]*:* & "and," $\vee$ "or," $\supset$ "if . . . then," $\supset\subset$ "is equivalent to," $\neg$ "not", $\forall$ "for all," $\exists$ "there is a."

We shall also use the terms: conjunction symbol, disjunction symbol, implication symbol, equivalence symbol, negation symbol, universal quantifier, existential quantifier.

Auxiliary symbols:), (, $\rightarrow$.

1.2 Variables:

Object variables. These we divide into *free* object variables: $a, b, c, \ldots m$ and *bound* object variables: $n, \ldots x, y, z$.

Propositional variables: $A, B, C, \ldots$

[3] An important special case of the *Hauptsatz* had already been proved in a completely different way by Herbrand, cf. Section iv, §2.

[4] We take the symbols $\vee$, $\supset$, $\exists$ from Russell. Russell's symbols for "and," "equivalent," "not," "all," viz: $\cdot$, $\equiv$, $\sim$, ( ), are already being used with a different meaning in mathematics. We shall therefore take Hilbert's &, whereas Hilbert's symbols for equivalence, all, and not, viz.: $\sim$, ( ), $^{-}$, again have already different meanings. Besides, the negation symbol represents a departure from the linear arrangement of symbols and is inconvenient for some purposes. We shall therefore use Heyting's symbols for equivalence and negation, and for "all" we shall use a symbol [namely $\forall$] corresponding to $\exists$.

We assume that an indefinite number of variables is available; if the alphabet does not suffice, we add numerical subscripts such as a_7, C_3.

1.3. Boldface and Greek letters serve as "syntactic variables," i.e., not as symbols of the logic formalized, but as variables of our considerations *concerning* that logic. Their meaning is explained as they are used.

2. *Expressions.*

2.1. The concept of a propositional expression, called a "formula" for short (defined inductively):

(The concept of a formula is ordinarily used in a more general sense; the special case defined below might thus perhaps be described as a "purely logical formula.")

2.11. A symbol for a particular proposition (i.e., the symbols $\mathbf{T}$ and $\mathbf{F}$) is a formula.

A propositional variable followed by a number (possibly zero) of free object variables is a formula, e.g., *Abab*.

The object variables are called the *arguments* of the propositional variables.

Formulae of the two kinds mentioned are also called *elementary formulae.*

2.12. If $\mathbf{A}$ is a formula, then $\neg\mathbf{A}$ is also a formula.

If $\mathbf{A}$ and $\mathbf{B}$ are formulae, then $\mathbf{A}\&\mathbf{B}$, $\mathbf{A}\vee\mathbf{B}$, $\mathbf{A}\supset\mathbf{B}$ are formulae.

(We shall not introduce the symbol $\supset\subset$ into our presentation; it is in fact superfluous, since $\mathbf{A}\supset\subset\mathbf{B}$ may be regarded as an abbreviation of $(\mathbf{A}\supset\mathbf{B})$ & $(\mathbf{B}\supset\mathbf{A})$.

2.13. A formula not containing the bound object variable $\mathbf{x}$ yields another formula, if we prefix either $\forall\mathbf{x}$ or $\exists\mathbf{x}$. At the same time we may substitute $\mathbf{x}$ in a number of places for a free object variable occurring in the formula.

2.14. Brackets (or parentheses) are to be used to show the structure of a formula unequivocally. Example of a formula:

$$\exists x\ (\ (\ (\neg Abxa) \vee Bx) \supset (\ \forall z(A\&B)\ )\ )$$

By special convention the number of brackets may be reduced, but (with one exception, *vide* 2.4) no use will be made of this, since we do not have to write down many formulae.

2.2. The number of logical symbols occurring in a formula is called the *grade of a formula*. (Thus an elementary formula is of grade 0.)

The logical symbol of a non-elementary formula that has been added last, in the construction of the formula according to 2.12 and 2.13, is called the *terminal symbol of a formula.*

Formulae that may have arisen in the course of the construction of a formula according to 2.12 and 2.13, including the formula itself, are called *subformulae.*

Example: the subformulae of $A\&\forall xBxa$ are A, $\forall xBxa$, $A\&\forall xBxa$ as well as all formulae of the form $B\mathbf{a}a$, where $\mathbf{a}$ represents any free object variable (this variable may also be a, for example). The grade of $A\&\forall xBxa$ is 2, the terminal symbol is &.

2.3. The concept of a *sequent:*

(This concept will not be used until Section III, and it is only then that the purpose of its introduction becomes apparent.)

A sequent is an expression of the form

$$\mathbf{A}_1, \ldots, \mathbf{A}_\mu \longrightarrow \mathbf{B}_1, \ldots, \mathbf{B}_\nu,$$

where $\mathbf{A}_1, \ldots, \mathbf{A}_\mu, \mathbf{B}_1, \ldots, \mathbf{B}_\nu$ may represent any formula whatever. (The $\longrightarrow$, like commas, is an auxiliary symbol and not a logical symbol.)

The formulae $\mathbf{A}_1, \ldots, \mathbf{A}_\mu$ forms the *antecedent*, and the formulae $\mathbf{B}_1, \ldots, \mathbf{B}_\nu$, the *succedent* of the sequent. Both expressions may be empty.

2.4. The sequent $\mathbf{A}_1, \ldots, \mathbf{A}_\mu \longrightarrow \mathbf{B}_1, \ldots, \mathbf{B}_\nu$ has exactly the same intuitive meaning as the formula

$$(\mathbf{A}_1\& \ldots \&\mathbf{A}_\mu) \supset (\mathbf{B}_1 \vee \ldots \vee \mathbf{B}_\nu).$$

(By $\mathbf{A}_1\&\mathbf{A}_2\&\mathbf{A}_3$ we mean $(\mathbf{A}_1\&\mathbf{A}_2)\&\mathbf{A}_3$, likewise for $\vee$.)

If the antecedent is empty, the sequent reduces to the formula $\mathbf{B}_1 \vee \ldots \vee \mathbf{B}_\nu$.

If the succedent is empty, the sequent means the same as the formula $\neg(\mathbf{A}_1\& \ldots \&\mathbf{A}_\mu)$ or $(\mathbf{A}_1\& \ldots \&\mathbf{A}_\mu)\supset\mathbf{F}$.

If both parts of the formula are empty, the sequent means the same as $\mathbf{F}$, i.e., a false proposition.

Conversely, to every formula there corresponds an equivalent sequent, e.g., the sequent whose antecedent is empty and whose succedent consists precisely of that formula.

The formulae making up a sequent are called *S–formulae* (i.e., sequent formulae). By this we intend to indicate that we are not considering the formula by itself, but as it appears in the sequent. Thus we say, for example:

"A formula occurs in several places in a sequent as an *S*–formula"; which may also be expressed as follows:

"Several distinct *S*–formulae (which shall simply mean: having distinct occurrences in the sequent) are formally identical."

3. *Figures.*

We require inference figures and proof figures.

Such figures consist of formulae or sequents, as the case may be. In what follows (3.1 to 3.3, 3.5) we shall be speaking only of formulae, but whatever is said applies analogously to sequents; all we need to do is to replace the word "formula," wherever it occurs, by the word "sequent."

3.1. An *inference figure* may be written in the following way:

$$\frac{\mathbf{A}_1, \ldots, \mathbf{A}_\nu}{\mathbf{B}} \quad (\nu \geqslant 1),$$

where $\mathbf{A}_1, \ldots, \mathbf{A}_\nu$, $\mathbf{B}$ are formulae. $\mathbf{A}_1, \ldots, \mathbf{A}_\nu$ are then called the *upper formulae* and $\mathbf{B}$ the *lower formula* of the inference figure. (The concepts of an upper sequent and of a lower sequent of an inference figure consisting of sequents are to be understood correspondingly.)

We shall have to consider only particular inference figures and they will be stated for each calculus as they arise.

3.2. A proof figure, called a *derivation* for short, consists of a number of formulae (at least one), which combine to form inference figures in the following way: Each formula is a lower formula of at most one inference figure; each formula (with the exception of exactly one: the *endformula*) is an upper formula of at least one inference figure; and the system of inference figures is non-circular, i.e., there is in the derivation no cycle (no series whose last member is again succeeded by its first member) of formulae of which each upper formula of an inference figure has the lower formula as the next one in the series.

3.3. The formulae of a derivation that are not lower formulae of an inference figure are called *initial formulae* of the derivation.

A derivation is in "tree form" if every one of its formulae is an upper formula of *at most one* inference figure.

Thus all formulae except the endformula are upper formulae of *exactly one* inference figure.

We shall have to treat only of derivations in tree form.

The formulae which compose a derivation so defined are called *D-formulae* (i.e., derivation formulae). By this we wish to indicate that we are not considering merely the formula as such, but also its position in the derivation. In this sense we shall be using, for example, expressions such as:

"A formula occurs in a derivation as a *D-formula*."

"Two distinct *D*–formulae (i.e., formulae occurring merely in distinct places in the derivation) are formally identical, viz., identical to the same formula."

Thus by "**A** is the *same D*–formula as **B**" we mean that **A** and **B** are not only formally identical, but occur also in the same place in the derivation. We shall use the words "formally identical" to indicate identity of form regardless of place.

For object variables, however, we shall not introduce a special term that would associate the variable with a specific place of occurrence in the formula. Thus we say, e.g.: "*The same* object variable occurs in two distinct *D*–formulae."

3.4. The inference figures of the derivation are called *D–inference figures* (i.e., derivation inference figures).

In a derivation consisting of sequents the *S*–formulae of the *D*–sequents are called *D–S–formulae* (i.e., derivation sequent formulae).

3.5. A *branch* in a derivation is (following Hilbert) a series of *D*–formulae whose first formula is an initial formula and whose last formula is the endformula, and of which each formula except the last is an upper formula of a *D*–inference figure whose lower formula is the next formula in the branch.

We say that "a *D*–formula stands *above* (*below*) another *D*–formula" if there exists a branch in which the former occurs before (after) the latter.

We are here thinking of the fact that a derivation is written in tree form with the initial formula above and the endformula below. (Examples may be found in II, §4).

Furthermore, we say that "a *D*–inference figure occurs above (below) a *D*–formula," if all formulae of the inference figure occur above (below) that *D*–formula.

A derivation with the endformula **A** is also called a "derivation of **A**."

The initial formulae of a derivation may be *basic formulae* or *assumption formulae*; more about their nature will have to be said as we reach the different calculi.

Section II.

The Calculus of Natural Deduction

§ 1

Examples of Natural Deduction

We wish to set up a formalism that reflects as accurately as possible the actual logical reasoning involved in mathematical proofs.

By means of a number of examples we shall first of all show what form actual reasoning tends to take and shall examine, for this purpose, three "valid formulae" and try to see their validity in the most natural way possible.

1.1. *First example*:

$(X \vee (Y \& Z)) \supset ((X \vee Y) \& (X \vee Z))$ can be recognized as a valid formula (H–A, p. 28, formula 19).

The argument runs as follows: Suppose that either X or $Y \& Z$ holds. We distinguish the two cases: 1. X holds, 2. $Y \& Z$ holds. In the first case it follows that $X \vee Y$ holds, and also $X \vee Z$; hence $(X \vee Y)$ & $(X \vee Z)$ also holds. In the second case $Y \& Z$ holds, which means that both Y and Z hold. From Y follows $X \vee Y$; from Z follows $X \vee Z$. Thus $(X \vee Y) \& (X \vee Z)$ again holds. The latter formula has thus been derived, in general, from $X \vee (Y \& Z)$, i.e.,

$$(X \vee (Y \& Z)) \supset ((X \vee Y) \& (X \vee Z)) \text{ holds.}$$

1.2. *Second example*:

$$(\exists x \forall y Fxy) \supset (\forall y \exists x Fxy).$$

(H–A, formula 36, p. 60). The argument runs as follows: Suppose there is an x such that for all y Fxy holds. Let a be such an x. Then for all y: Fay holds. Now let b be an arbitrary object. Then Fab holds. Thus there is an x, viz., a, such that Fxb holds. Since b was arbitrary, our result therefore holds for all objects, i.e., for all y there is an x, such that Fxy holds. This yields our assertion.

1.3. *Third example*:

$(\neg \exists x Fx) \supset (\forall y \neg Fy)$ is to be recognized as intuitionistically valid. We reason as follows: Assume there were no x for which Fx held. From this we wish to infer: For all y, $\neg Fy$ holds. Now suppose a were some object for which Fa held. It would then follow that there was an x for which Fx held, viz., a would be such an object. This contradicts our hypothesis that $\neg \exists x Fx$. We have therefore a contradiction, i.e., Fa cannot hold. But since a was completely arbitrary, it follows that for all y, $\neg Fy$ holds. Q.E.D.

We intend now to integrate proofs of the kind carried out in these three examples in an exactly defined calculus (in §4, we shall show how these examples are presented in that calculus).

§2.

Construction of the Calculus NJ

2.1. We intend now to present a calculus for "natural" intuitionist derivations of valid formulae. The restriction to intuitionist reasoning is only provisional; we shall explain below (cf. §5) our reasons for doing so and shall show in what way the calculus has to be extended for classical reasoning (by including the law of the excluded middle).

Externally, the essential difference between "*NJ*–derivations" and derivations in the systems of Russell, Hilbert, and Heyting is the following: In the latter systems true formulae are derived from a series of "logical basic formulae" by means of a few forms of inference. Natural deduction, however, does not, in general, start from logical basic propositions, but rather from *assumptions* (cf. examples in §1) to which logical deductions are applied. By means of a later inference the result is then again made independent of the assumption.

Calculi of the former kind will be referred to as *logistic* calculi.

2.2. After this preliminary remark we define the concept of an *NJ–derivation* as follows:

(Examples in §4).

An *NJ*–derivation consists of formulae ordered in tree form (I, 3.3).

(By demanding that the formulae are in tree form we are deviating somewhat from the analogy with actual reasoning. This is so, since in actual reasoning we necessarily have (1) a linear sequence of propositions due to the linear ordering of our utterances, and (2) we are accustomed to applying repeatedly a result once it has been obtained, whereas the tree form permits only of a single use of a derived formula. These two deviations permit us to define the concept of a derivation in a more convenient form and are not essential.)

The initial formulae of the derivation are *assumption formulae*. Each of these is correlated with precisely one *D*–inference figure (and in fact occurs "above" [I. 3.5] the lower formula of that figure, as will be explained more fully below).

All formulae that occur below an assumption formula, but still above the lower formula of the *D*–inference figure with which that assumption formula is correlated, that assumption formula itself included, are said to be *dependent* on that assumption formula.

(Thus the inference makes all succeeding propositions independent of the assumption which is correlated with it.)

According to what we have said the endformula of the derivation depends on no assumption formula.

2.21. We shall now state the permissible *inference figures*.

The inference figure schemata below are to be understood in the following way:

We obtain an $\mathcal{NJ}$–inference figure from one of the schemata by replacing **A**, **B**, **C**, **D** by arbitrary formulae; and $\forall \mathbf{xFx}$ ($\exists \mathbf{xFx}$) by an arbitrary formula containing $\forall$ or $\exists$ for its terminal symbol, where **x** indicates the bound object variable belonging to that terminal symbol; and **Fa** by the formula obtained from **Fx** by replacing the bound variable **x**, wherever it occurs, by the free object variable **a**.

(For **a** we may, for instance, take a variable already occurring in **Fx**. For the inference figures $\forall$–*I* and $\exists$–*E*, this possibility will, however, be excluded by the restriction on variables which follows below, but it remains for $\forall$–*E* and $\exists$–*I*. Nor need **x** occur at all in **Fx**, in which case **Fa** is, of course, identical with **Fx**. — **Fa** is obviously always a subformula of $\forall \mathbf{xFx}$ ($\exists \mathbf{xFx}$), according to the definition of a subformula in I, 2.2.)

Symbols written in square brackets have the following meaning: An arbitrary number (possibly zero) of formulae of this form, all formally identical, may be correlated with the inference figure as assumption formulae. They must then be initial formulae of the derivation and, moreover, occur in branches of proofs to which the particular upper formula of the inference figure belongs. (I.e., that upper formula above which the square bracket occurs in the scheme. This formula may already itself be an assumption formula.)

The fact that there is a correlation in a derivation between a *D*–inference figure and the related assumption formulae must somehow be made explicit, for example, by jointly numbering them (cf. examples in §4).

The designations of the various inference figure schemata: &–*I*, &–*E*, etc., stand for the following: An inference figure formed according to a particular schema is an "introduction" (*I*) or an "elimination" (*E*) of the conjunction (&), the disjunction ($\vee$), the universal quantifier ($\forall$), the existential quantifier ($\exists$), the implication ($\supset$), or of the negation ($\neg$). More about this in §5.

The Inference Figure Schemata:

&–*I*	&–*E*		$\vee$–*I*		$\vee$–*E*
$\dfrac{\mathbf{A} \quad \mathbf{B}}{\mathbf{A \& B}}$	$\dfrac{\mathbf{A \& B}}{\mathbf{A}}$	$\dfrac{\mathbf{A \& B}}{\mathbf{B}}$	$\dfrac{\mathbf{A}}{\mathbf{A \vee B}}$	$\dfrac{\mathbf{B}}{\mathbf{A \vee B}}$	$\dfrac{\mathbf{A \vee B} \quad \overset{[\mathbf{A}]}{\mathbf{C}} \quad \overset{[\mathbf{B}]}{\mathbf{C}}}{\mathbf{C}}$

$\forall$–*I*	$\forall$–*E*	$\exists$–*I*	$\exists$–*E*
$\dfrac{\mathbf{Fa}}{\forall \mathbf{xFx}}$	$\dfrac{\forall \mathbf{xFx}}{\mathbf{Fa}}$	$\dfrac{\mathbf{Fa}}{\exists \mathbf{xFx}}$	$\dfrac{\exists \mathbf{xFx} \quad \overset{[\mathbf{Fa}]}{\mathbf{C}}}{\mathbf{C}}$

$\supset$–*I*	$\supset$–*E*	$\neg$–*I*	$\neg$–*E*	
$\dfrac{\overset{[\mathbf{A}]}{\mathbf{B}}}{\mathbf{A \supset B}}$	$\dfrac{\mathbf{A} \quad \mathbf{A \supset B}}{\mathbf{B}}$	$\dfrac{\overset{[\mathbf{A}]}{\mathbf{F}}}{\neg \mathbf{A}}$	$\dfrac{\mathbf{A} \quad \neg \mathbf{A}}{\mathbf{F}}$	$\dfrac{\mathbf{F}}{\mathbf{D}}$

The free object variable of a $\forall$–*I* or $\exists$–*E*, represented by **a** in the respective schema, is called a *proper variable*. (This, of course, presupposes that there is such a variable, i.e., that the bound object variable represented by **x** occurs in the formula represented by **Fx**.)

Restrictions on Variables:

An $\mathcal{NJ}$–derivation is subject to the following restriction (for the significance of this restriction cf. §3):

The proper variable of an $\forall$–*I* must not occur in the formula represented in the schema by $\forall \mathbf{xFx}$, nor in any assumption formula upon which that formula depends.

The proper variable of an $\exists$–*E* must not occur in the formula represented in the schema by $\exists \mathbf{xFx}$; nor in an upper formula represented by **C**; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae represented by **Fa** correlated with the $\exists$–*E*.

This concludes the definition of the "$\mathcal{NJ}$–derivation."

§3.

Intuitive Sense of $\mathcal{NJ}$–Inference Figures

We shall explain the intuitive sense of a number of inference figure schemata and thus try to show how the calculus in fact reflects "actual reasoning."

$\supset$–*I*: Expressed in words, this schema corresponds to the following inference: If **B** has been proved by means of assumption **A**, we have (this time without the assumption): from **A** follows **B**. (Further assumptions may, of course, have been made and the result continues to depend on them.)

$\vee$–*E* ("Distinction of Cases"): If $\mathbf{A} \vee \mathbf{B}$ has been proved, one can distinguish two cases: What one first assumes is that **A** holds and derives, let us say, **C** from it. If it is then possible to derive **C** also by assuming that **B** holds, then **C** holds generally, i.e., it is now independent of both assumptions (cf. 1.1).

$\forall$–*I*: If **Fa** has been proved for an "arbitrary **a**," then $\forall$**xFx** holds. The presupposition that **a** is "completely arbitrary" can be expressed more precisely as: **Fa** must not depend on any assumption in which the object variable **a** occurs. And this, together with the obvious requirement that *every occurrence* of **a** in **Fa** must be replaced by an **x** in **Fx**, constitutes precisely the part of the "restrictions on variables" relative to the schema of the $\forall$–*I*.

$\exists$–*E*: We have $\exists$**xFx**. We then say: Suppose **a** is an object for which **F** holds, i.e., assume that **Fa** holds. (It is, of course, obvious that for **a** we must take an object variable which does not yet occur in $\exists$**xFx**.) If, on this assumption, we then prove a proposition **C** which no longer contains **a** and does not depend on any other assumption containing **a**, we have proved **C** independently of the assumption **Fa**. We have here stated the part of the "restrictions on variables" that concerns the $\exists$–*E*. (A certain analogy exists between the $\exists$–*E* and the $\vee$–*E* since the existential quantifier is indeed the generalization of $\vee$, and the universal quantifier the generalization of &.)

$\neg$–*E*: **A** and $\neg$ **A** signify a contradiction, and this cannot obtain (law of contradiction). This is formally expressed by the inference figure $\neg$–*E*, where **F** designates "the contradiction," "the false."

$\neg$–*I*: (*Reductio ad absurdum.*) If we can derive any false proposition (**F**) on an assumption **A**, then **A** is not true, i.e., $\neg$ **A** holds.

The schema $\dfrac{\mathbf{F}}{\mathbf{D}}$

If a false proposition holds, any arbitrary proposition also holds.

The interpretation of the remaining inference figure schemata is straightforward.

§4.

Representation of the three examples of §1 *as NJ–Derivations*

First example (1.1):

$$\dfrac{\dfrac{\overset{2}{X\vee(Y\&Z)} \qquad \dfrac{\dfrac{\overset{1}{X}}{X\vee Y}\vee\text{–}I \quad \dfrac{\overset{1}{X}}{X\vee Z}\vee\text{–}I}{(X\vee Y)\&(X\vee Z)}\&\text{–}I \qquad \dfrac{\dfrac{\dfrac{\overset{1}{Y\&Z}}{Y}\&\text{–}E}{X\vee Y}\vee\text{–}I \quad \dfrac{\dfrac{\overset{1}{Y\&Z}}{Z}\&\text{–}E}{X\vee Z}\vee\text{–}I}{(X\vee Y)\&(X\vee Z)}\&\text{–}I}{(X\vee Y)\&(X\vee Z)}\vee\text{–}E\,1}{(X\vee(Y\&Z))\supset((X\vee Y)\&(X\vee Z))}\supset\text{–}I\,2$$

In this example the tree form must appear somewhat artificial since it does not bring out the fact that it is *after* the enunciation of $X\vee(Y\&Z)$ that we distinguish the cases X, $Y\&Z$.

Second example (1.2):

$$\dfrac{\dfrac{\overset{2}{\exists x\ \forall y\,Fxy} \qquad \dfrac{\dfrac{\dfrac{\overset{1}{\forall y\,Fay}}{Fab}\forall\text{–}E}{\exists x\,Fxb}\exists\text{–}I}{\forall y\,\exists x\,Fxy}\forall\text{–}I}{\forall y\,\exists x\,Fxy}\exists\text{–}E\,1}{(\exists x\ \forall y\,Fxy)\supset(\forall y\,\exists x\,Fxy)}\supset\text{–}I\,2$$

If we were to use *linear* ordering, then here too the assumption of the $\exists$–*E* would quite naturally *follow* the upper formula on the left, as was the case in our treatment of that example in §1.

Third example (1.3):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{Fa}}{\exists xFx}\exists\text{–}I \qquad \overset{1}{\neg\ \exists xFx}}{\mathbf{F}}\neg\text{–}E}{\neg\ Fa}\neg\text{–}I2}{\forall y\ \neg\ Fy}\forall\text{–}I}{(\neg\ \exists xFx)\supset(\forall y\ \neg\ Fy)}\supset\text{–}I1$$

§5.

Some Remarks Concerning the Calculus NJ. The Calculus NK

5.1. The calculus *NJ* lacks a certain formal elegance. This has to be put against the following advantages:

5.11. A close affinity to actual reasoning, which had been our fundamental aim in setting up the calculus. The calculus lends itself in particular to the formalization of mathematical proofs.

5.12. In most cases the derivations for valid formulae in our calculus are *shorter* than their counterparts in logistic calculi. This is so primarily because in logistic derivations one and the same formula usually occurs a number of times (as part of other formulae), whereas this happens only very rarely in the case of *NJ*–derivations.

5.13. The designations given to the various inference figures (2.21) make it plain that our calculus is remarkably *systematic*. To every logical symbol &, $\vee$, $\forall$, $\exists$, $\supset$, $\neg$, belongs precisely one inference figure which "introduces" the symbol—as the terminal symbol of a formula—and one which "eliminates" it. The fact that the inference figures &-*E* and $\vee$-*I* each have two forms constitutes a trivial, purely external deviation and is of no interest. The introductions represent, as it were, the "definitions" of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, the formula, whose terminal symbol we are dealing with, may be used only "in the sense afforded it by the introduction of that symbol." An example may clarify what is meant: We were able to introduce the formula $\mathbf{A} \supset \mathbf{B}$ when there existed a derivation of **B** from the assumption formula **A**. If we then wished to use that formula by eliminating the $\supset$-symbol [we could, of course, also use it to form longer formulae, e.g., $(\mathbf{A} \supset \mathbf{B}) \vee \mathbf{C}$, $\vee$-*I*], we could do this precisely by inferring **B** directly, once **A** has been proved, for what $\mathbf{A} \supset \mathbf{B}$ attests is just the existence of a derivation of **B** from **A**. Note that in saying this we need not go into the "intuitive sense" of the $\supset$-symbol.

By making these ideas more precise it should be possible to display the *E*-inferences as single-valued functions of their corresponding *I*-inferences, on the basis of certain requirements.

5.2. It is possible to eliminate the *negation* from our calculus by regarding $\neg \mathbf{A}$ as an abbreviation of $\mathbf{A} \supset \mathbf{F}$. This is permissible, since by replacing every $\neg \mathbf{A}$ by $\mathbf{A} \supset \mathbf{F}$, and thus removing all $\neg$-symbols from an $\mathcal{NJ}$-derivation, we obtain another $\mathcal{NJ}$-derivation (the inference figures $\neg$-*I* and $\neg$-*E* then become special cases of the $\supset$-*I* and the $\supset$-*E*) and vice versa: If we replace every occurrence of $\mathbf{A} \supset \mathbf{F}$ by $\neg \mathbf{A}$ in an $\mathcal{NJ}$-derivation, we obtain another $\mathcal{NJ}$-derivation.

The inference figure schema $\dfrac{\mathbf{F}}{\mathbf{D}}$ occupies a special place among the schemata: It does not belong to a logical symbol, but to the propositional symbol **F**.

5.3. The "*law of the excluded middle*" and the *calculus NK*.

From the calculus $\mathcal{NJ}$ we obtain a complete classical calculus *NK* by adding the "law of the excluded middle" (*tertium non datur*), i.e.: As initial formulae of a derivation we now also allow in addition to the assumption formulae, "basic formulae" of the form $\mathbf{A} \vee \neg \mathbf{A}$, where **A** is to be replaced by an arbitrary formula.

We have thus allotted to the law of the excluded middle, in a purely exterior way, a special position, and we have done this because we considered that formulation the "most natural." It would be perfectly feasible to introduce a new inference figure schema, say $\dfrac{\neg\neg \mathbf{A}}{\mathbf{A}}$ (a schema analogous to the one formed by Hilbert and Heyting) in place of the basic formula schema $\mathbf{A} \vee \neg \mathbf{A}$. However, such a schema still falls outside the framework of the $\mathcal{NJ}$-inference figures, because it represents a new elimination of the negation whose admissibility does not follow at all from our method of introducing the $\neg$-symbol by the $\neg$-*I*.

Section III.

Deductive Calculi LJ, LK and the Hauptsatz

§1.

The Calculi LJ and LK (*Logistic Intuitionist and Classical Calculus*)

1.1. Preliminary remarks concerning the construction of the calculi *LJ* and *LK*.

What we want to do is to formulate a deductive calculus (for predicate logic) which is "logistic" on the one hand, i.e., in which the derivations do not, as in the calculus $\mathcal{NJ}$, contain assumption formulae, but which, on the other hand, takes over from the calculus $\mathcal{NJ}$ the division of the forms of inference into introductions and eliminations of the various logical symbols.

The most obvious method of converting an $\mathcal{NJ}$-derivation into a logistic one is this: We replace a *D*-formula **B**, which depends on the assumption formulae $\mathbf{A}_1, \ldots, \mathbf{A}_\mu$, by the new formula $(\mathbf{A}_1 \& \ldots \& \mathbf{A}_\mu) \supset \mathbf{B}$. This we do in all *D*-formulae.

We thus obtain formulae which are already valid *in themselves*, i.e., whose truth is no longer *conditional* on the truth of certain assumption formulae. This procedure, however, introduces the new logical symbols & and $\supset$, necessitating additional inference figures for & and $\supset$, and thus upsets the systematic character of our method of introducing and eliminating symbols. For this reason we have introduced the concept of a *sequent* (I, 2.3). Instead of the formula $(\mathbf{A}_1 \& \ldots \& \mathbf{A}_\mu) \supset \mathbf{B}$, e.g., we therefore write the sequent

$$\mathbf{A}_1, \ldots, \mathbf{A}_\mu \longrightarrow \mathbf{B}$$

The sequent does not distinguish itself from the above formula in its intuitive meaning, but only in its formal structure (cf. I, 2.4).

Even now new inference figures are required that cannot be integrated into our system of introductions and eliminations; but we have the advantage of being able to reserve them special places within our system, since they no longer refer to logical symbols, but merely to the structure of the sequents. We therefore call these "structural inference figures," and the others "operational inference figures."

In the *classical* calculus *NK* the law of the excluded middle occupied a special place among the forms of inference (II, 5.3), because it could not be integrated into our system of introductions and eliminations. In the classical logistic calculus *LK* to be presented below, that peculiarity is removed. What makes this possible is that we admit into our system sequents with *several* formulae in the succedent, whereas the indicated transition from the calculus *NJ* has resulted only in sequents with *one* formula in the succedent. (For the intuitive meaning of the general sequents cf. I, 2.4.) The symmetry thus obtained is more suited to classical logic. On the other hand, the restriction to at most one formula in the succedent will be retained for the intuitionist calculus *LJ*. (Cf. below —An empty succedent means the same as if **F** stood in the succedent.)

We have thus outlined a number of points that underlie the construction of the calculi that follow. Their form is largely determined, however, by considerations connected with the "*Hauptsatz*" (§2) whose proof follows later. That form cannot therefore be justified more fully at this stage.

1.2. We now define the concepts of a "*LK*–derivation" and a "*LJ*–derivation" as follows:

An *LJ*– or *LK*–derivation consists of sequents arranged in tree form (I, 3.3).

The *initial sequents* of the derivation are basic sequents of the form

$$\mathbf{D} \longrightarrow \mathbf{D},$$

where **D** may be an arbitrary formula.

Each *inference figure* of the derivation results from one of the schemata below by a substitution of the following kind (cf. II, 2.21):

Replace **A**, **B**, **D**, **E** by an arbitrary formula; for $\forall\mathbf{xFx}$ ($\exists\mathbf{xFx}$) put an arbitrary formula having $\forall$ ($\exists$) for its terminal symbol, where **x** designates the associated bound object variable; for **Fa** put that formula which is obtained from **Fx** by replacing every occurrence of the bound object variable **x** by the free object variable **a**.

For Γ, Δ, Θ, Λ put arbitrary (possibly empty) sequences of formulae separated by commas.

The following restriction is furthermore placed on *LJ–inference figures* (this is the *only* respect in which the concepts of a *LJ*– and a *LK*–derivation differ):

"In the succedent of each *D*–sequent no more than one *S*–formula is permissible."

The designations of the various schemata for operational inference figures &–*IS*, &–*IA*, etc., are intended to mean: An inference figure formed according to the schema is an introduction (*I*) in the succedent (*S*) or antecedent (*A*) of the conjunction (&), the disjunction ($\vee$), the universal quantifier ($\forall$), the existential quantifier ($\exists$), the negation ($\neg$), or the implication($\supset$).

The Inference Figure Schemata

1.21. Schemata for structural inference figures:

Thinning:

in the antecedent $\qquad$ in the succedent

$$\frac{\Gamma \longrightarrow \Theta}{\mathbf{D}, \Gamma \longrightarrow \Theta}, \qquad \frac{\Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \mathbf{D}};$$

Contraction:

in the antecedent $\qquad$ in the succedent

$$\frac{\mathbf{D}, \mathbf{D}, \Gamma \longrightarrow \Theta}{\mathbf{D}, \Gamma \longrightarrow \Theta}, \qquad \frac{\Gamma \longrightarrow \Theta, \mathbf{D}, \mathbf{D}}{\Gamma \longrightarrow \Theta, \mathbf{D}};$$

Interchange:

in the antecedent $\qquad$ in the succedent

$$\frac{\Delta, \mathbf{D}, \mathbf{E}, \Gamma \longrightarrow \Theta}{\Delta, \mathbf{E}, \mathbf{D}, \Gamma \longrightarrow \Theta}, \qquad \frac{\Gamma \longrightarrow \Theta, \mathbf{E}, \mathbf{D}, \Lambda}{\Gamma \longrightarrow \Theta, \mathbf{D}, \mathbf{E}, \Lambda};$$

Cut:

$$\frac{\Gamma \longrightarrow \Theta, \mathbf{D} \quad \mathbf{D}, \Delta \longrightarrow \Lambda}{\Gamma, \Delta \longrightarrow \Theta, \Lambda}$$

1.22. Schemata for operational inference figures:

$$\&\text{–}IS\text{:} \quad \frac{\Gamma \longrightarrow \Theta, \mathbf{A} \quad \Gamma \longrightarrow \Theta, \mathbf{B}}{\Gamma \longrightarrow \Theta, \mathbf{A} \,\&\, \mathbf{B}},$$

$$\&\text{–}IA\text{:} \quad \frac{\mathbf{A}, \Gamma \longrightarrow \Theta}{\mathbf{A} \,\&\, \mathbf{B}, \Gamma \longrightarrow \Theta} \quad \frac{\mathbf{B}, \Gamma \longrightarrow \Theta}{\mathbf{A} \,\&\, \mathbf{B}, \Gamma \longrightarrow \Theta},$$

$$\vee\text{–}IS\text{:} \quad \frac{\mathbf{A}, \Gamma \longrightarrow \Theta \quad \mathbf{B}, \Gamma \longrightarrow \Theta}{\mathbf{A} \vee \mathbf{B}, \Gamma \longrightarrow \Theta},$$

$$\vee\text{–}IA\text{:} \quad \frac{\Gamma \longrightarrow \Theta, \mathbf{A}}{\Gamma \longrightarrow \Theta, \mathbf{A} \vee \mathbf{B}} \quad \frac{\Gamma \longrightarrow \Theta, \mathbf{B}}{\Gamma \longrightarrow \Theta, \mathbf{A} \vee \mathbf{B}},$$

$\forall$–*IS*: $\dfrac{\Gamma \longrightarrow \Theta, \mathbf{Fa}}{\Gamma \longrightarrow \Theta, \forall\mathbf{xFx}}$,

$\exists$–*IA*: $\dfrac{\mathbf{Fa}, \Gamma \longrightarrow \Theta}{\exists\mathbf{xFx}, \Gamma \longrightarrow \Theta}$.

Restrictions on Variables: The object variable in the last two schemata, which is designated by **a** and is called the *proper variable* of the $\forall$–*IS* ($\exists$–*IA*), must not occur in the lower sequent of the inference figure (i.e., not in Γ, Θ, and **Fx**).

$\forall$–*IA*: $\dfrac{\mathbf{Fa}, \Gamma \longrightarrow \Theta}{\forall\mathbf{xFx}, \Gamma \longrightarrow \Theta}$,

$\neg$–*IS*: $\dfrac{\mathbf{A}, \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \neg\mathbf{A}}$,

$\exists$–*IS*: $\dfrac{\Gamma \longrightarrow \Theta, \mathbf{Fa}}{\Gamma \longrightarrow \Theta, \exists\mathbf{xFx}}$,

$\neg$–*IA*: $\dfrac{\Gamma \longrightarrow \Theta, \mathbf{A}}{\neg\mathbf{A}, \Gamma \longrightarrow \Theta}$,

$\supset$–*IS*: $\dfrac{\mathbf{A}, \Gamma \longrightarrow \Theta, \mathbf{B}}{\Gamma \longrightarrow \Theta, \mathbf{A} \supset \mathbf{B}}$,

$\supset$–*IA*: $\dfrac{\Gamma \longrightarrow \Theta, \mathbf{A} \quad \mathbf{B}, \Delta \longrightarrow \Lambda}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Delta \longrightarrow \Theta, \Lambda}$.

1.3. Example of an *LJ*–derivation (following II, 4.3):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fa \longrightarrow Fa}{Fa \longrightarrow \exists xFx}\,\exists\text{–}IS \qquad \dfrac{\dfrac{\exists xFx \longrightarrow \exists xFx}{\neg\exists xFx, \exists xFx \longrightarrow}\,\neg\text{–}IA}{\exists xFx, \neg\exists xFx \longrightarrow}\,\text{Interchange}}{Fa, \neg\exists xFx \longrightarrow}\,\text{Cut}}{\neg\exists xFx \longrightarrow \neg Fa}\,\neg\text{–}IS}{\neg\exists xFx \longrightarrow \forall y\neg Fy}\,\forall\text{–}IS}{\longrightarrow (\neg\exists xFx) \supset (\forall\neg Fy)}\,\supset\text{–}IS$$

1.4. Example of an *LK*–derivation (derivation of the "law of the excluded middle"):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \longrightarrow A}{\longrightarrow A, \neg A}\,\neg\text{–}IS}{\longrightarrow A, A \vee \neg A}\,\vee\text{–}IS}{\longrightarrow A \vee \neg A, A}\,\text{Interchange}}{\longrightarrow A \vee \neg A, A \vee \neg A}\,\vee\text{–}IS}{\longrightarrow A \vee \neg A}\,\text{Contraction}$$

§2.

Some Remarks Concerning the Calculi LJ and LK

The Hauptsatz

(We shall make no further use, in this paper, of remarks 2.1 to 2.3.)

2.1. The schemata are not all mutually independent, i.e., certain schemata could be eliminated with the help of the remaining ones. Yet if they were left out, the "*Hauptsatz*" would no longer be valid.

2.2. In general, we could *simplify* the calculi in various respects if we attached no importance to the *Hauptsatz*. To indicate this briefly: the inference figures &–*IS*, $\vee$–*IA*, &–*IA*, $\vee$–*IS*, $\forall$–*IA*, $\exists$–*IS*, $\neg$–*IS*, $\neg$–*IA*, and $\supset$–*IA* in the calculus *LK* could be replaced by basic sequents according to the following schemata:

A, B$\longrightarrow$A&B A $\vee$ B$\longrightarrow$A, B A&B$\longrightarrow$A A&B$\longrightarrow$B A$\longrightarrow$A $\vee$ B B$\longrightarrow$A $\vee$ B $\forall$xFx$\longrightarrow$Fa Fa$\longrightarrow$$\exists$xFx $\longrightarrow$A, $\neg$A (law of the excluded middle)

$\neg$A, A$\longrightarrow$ (law of contradiction), **A$\supset$B, A$\longrightarrow$B**.

These basic sequents and our inference figures may easily be shown to be equivalent.

The same possibility exists for the calculus *LJ*, with the exception of the inference figures $\vee$–*IA* and $\neg$–*IS*, since *LJ*–*D*–sequents may not in fact contain two *S*–formulae in the succedent (cf. V, §5).

2.3. The distinction between *intuitionist* and *classical* logic is, externally, of a quite different nature in the calculi *LJ* and *LK* from that in the calculi *NJ* and *NK*. In the case of the latter, the distinction rests on the inclusion or exclusion of the law of the excluded middle, whereas for the calculi *LJ* and *LK* the difference is characterized by the restrictions on the succedent. (The fact that both distinctions are equivalent will become evident as a result of the equivalence proofs in Section V for all calculi discussed in this paper.)

2.4. If $\supset$–*IS* and the $\supset$–*IA* are excluded, the calculus *LK* is *dual* in the following sense: If we reverse all sequents of an *LK*–derivation (in which the $\supset$–symbol does not occur), i.e., if for $\mathbf{A}_1, \ldots, \mathbf{A}_\mu \longrightarrow \mathbf{B}_1, \ldots, \mathbf{B}_\nu$ we put $\mathbf{B}_\nu, \ldots, \mathbf{B}_1 \longrightarrow \mathbf{A}_\mu, \ldots, \mathbf{A}_1$; and if we exchange, in inference figures with two upper sequents, the right and left-hand upper sequents, including their derivations, and also replace every occurence of & by $\vee$, $\forall$ by $\exists$, $\vee$ by &, and $\exists$ by $\forall$ (in the case of & and $\vee$ we also have to change the respective scope of the symbols, e.g., for $\mathbf{B} \vee \mathbf{A}$ we have to put $\mathbf{A}\&\mathbf{B}$), then another *LK*–derivation results.

This can be seen at once from the schemata. (Special care was taken to order them in such a way as to bring out their symmetry.)

(Cf. H–A's duality principle, p. 62).

2.41. In any case, the $\supset$–symbol may, in a well-known manner, be eliminated from the calculus *NK*, by regarding $\mathbf{A} \supset \mathbf{B}$ as an abbreviation for $(\neg \mathbf{A}) \vee \mathbf{B}$. It may easily be shown that the schemata for the $\supset$–*IS* and the $\supset$–*IA* may then be replaced by the schemata for $\vee$ and $\neg$.

The calculus *NJ* has no corresponding property.

2.5. The most important fact for us with regard to the calculi *LJ* and *LK* is the following:

HAUPTSATZ: Every *LJ*– or *LK*–derivation can be transformed into an *LJ*– or *LK*–derivation that has the same endsequent and in which the inference figure termed a "cut" does not occur.

2.51. The proof follows in §3.

In order to give greater clarity to the meaning of the *Hauptsatz*, we prove a simple *corollary* (2.513).

For this purpose we introduce a number of expressions (which will be needed frequently later on) relating to operational inference figures.

2.511. That *S*–formula which contains the logical symbol in its schema will be called the *principal formula* of an inference figure.

For the &–*IS* and the &–*IA* this is simply the *S*–formula of the form $\mathbf{A}\&\mathbf{B}$; for the $\vee$–*IS* and the $\vee$–*IA* it is $\mathbf{A} \vee \mathbf{B}$; for the $\forall$–*IS* and the $\forall$–*IA* it is $\forall\mathbf{x}\mathbf{F}\mathbf{x}$; for the $\exists$–*IS* and the $\exists$–*IA* it is $\exists\mathbf{x}\mathbf{F}\mathbf{x}$; for the $\neg$–*IS* and the $\neg$–*IA* it is $\neg\mathbf{A}$; and for the $\supset$–*IS* and the $\supset$–*IA* it is $\mathbf{A} \supset \mathbf{B}$.

The *S*–formulae designated by **A**, **B**, **Fa** in the schemata we call *side formulae* of their corresponding inference figures.

They are always subformulae of the principal formula (according to the definition of a subformula in I, 2.2).

2.512. We can now easily read off the following facts from the inference figure schemata: A principal formula occurs always in the lower sequent, a side formula always in the upper sequents of an operational inference figure.

In the case of a formula occurring as an *S*–formula in an upper sequent of an arbitrary inference figure without being a side formula or the **D** of a cut, it occurs also in the lower sequent as an *S*–formula.

These two facts entail the following:

If anywhere in an *LJ*– or *LK*–derivation a formula occurs as an *S*–formula, and if we trace the branch of the derivation from the formula concerned up to the endsequent, the formula can only then vanish from that branch if it is the **D** of a cut or the side formula of an operational inference figure. In the latter case, however, there appears, in the next sequent, the *principal formula* of the inference figure of which our side formula is a subformula. To that principal formula we can then, continuing downwards, apply the same consideration, and so on. Thus we obtain the following corollary:

2.513. *Corollary of the Hauptsatz (subformula property)*: In an *LJ*– or *LK*–derivation without cuts, all occurring *D–S*–formulae are *subformulae* of the *S*–formula that occurs in the endsequent.

2.514. Intuitively speaking, these properties of derivations without cuts may be expressed as follows: The *S*–formulae become larger as we descend lower down in the derivation, never shorter. The final result is, as it were, gradually built up from its constituent elements. The proof represented by the derivation is not roundabout in that it contains only concepts which recur in the final result (cf. the synopsis at the beginning of this paper).

Example: The derivation given above (1.3) for $\longrightarrow (\neg\exists\mathbf{x}\mathbf{F}\mathbf{x}) \supset (\forall\mathbf{y}\neg\mathbf{F}\mathbf{y})$ may be written without a cut as follows:

$$\dfrac{\dfrac{\dfrac{Fa \longrightarrow Fa}{Fa \longrightarrow \exists xFx}\ \exists\text{–}IS}{\neg\exists xFx,\ Fa \longrightarrow}\ \neg\text{–}IA}{Fa,\ \neg\exists xFx \longrightarrow}\ \text{Interchange}$$

etc., as above.

§3.

Proof of the Hauptsatz

The *Hauptsatz* runs as follows:

Every *LJ*– or *LK*–derivation can be transformed into another *LJ*– or *LK*—derivation with the same endsequent, in which no cuts occur.

3.1. *Proof of the Hauptsatz for LK–derivations.*

We introduce a new inference figure (in order to facilitate the proof) that constitutes a modified form of the cut, and which we call a *mix*.

The schema of that figure runs as follows:

$$\frac{\Gamma \longrightarrow \Theta \quad \Delta \longrightarrow \Lambda}{\Gamma,\ \Delta^* \longrightarrow \Theta^*,\ \Lambda}$$

In order to obtain an inference figure from this schema, Θ and Δ must be replaced by sequences of formulae, separated by commas, in each of which occurs at least once (as a member of the sequence) a formula of the form **M**, called the "mix formula"; and Θ^* and Δ^* must be replaced by the same sequences of formulae, save that all formulae of the form **M** occurring as members of the sequence are omitted. (**M** is an arbitrary formula.) Γ and Λ must be replaced, as in the other schemata, by arbitrary (possibly empty) sequences of formulae, separated by commas.

Example of a mix:

$$\frac{A \longrightarrow B, \neg A \quad B \vee C, B, B, D, B \longrightarrow}{A, B \vee C, D \longrightarrow \neg A}$$

B is the mix formula.

We notice at once that every cut may be transformed into a mix by means of a number of thinnings and interchanges. (Conversely, every mix may be transformed into a cut by means of a certain number of preceding interchanges and contractions, though we do not use this fact.)

In the following we shall consider only derivations in which no cuts occur, but which may contain mixes instead.

Since derivations in the old sense may be transformed into derivations of the new kind, it suffices, for the proof of the *Hauptsatz*, to show that a derivation of the new type may be transformed into a derivation with no mix.

Furthermore, the following lemma is already sufficient:

Lemma: Any derivation with a mix for its lowest inference figure, and not containing any other mix, may be transformed into a derivation (with the same endsequent) in which no mix occurs.

From this the complete theorem easily follows:

In an arbitrary derivation consider a mix above whose lower sequent no further mix occurs. The derivation for this lower sequent is then of the kind mentioned in the lemma, i.e., it may be transformed in such a way that it no longer contains a mix. In doing so, the rest of the derivation remains unchanged. This operation is then repeated until every mix has systematically been eliminated.

It now remains for us to establish the *proof of the lemma*. (This proof extends into 3.2 incl.)

We have to consider a derivation whose lowest inference figure is a mix and which contains no other mix besides.

The grade of the mix formula will be called the "grade of the derivation" (defined in I, 2.2).

We shall call "rank of the derivation" the sum of its rank on the left and its rank on the right. These two terms are defined as follows:

The left rank is the greatest number of sequents connected on a branch such that the last sequent is the *left-hand* upper sequent of the mix, where each formula of the branch contains the mix formula in the *succedent*.

The right rank is (correspondingly) the greatest number of sequents connected on a branch whose lowest sequent is the *right-hand* upper sequent of the mix, and each formula of the branch contains the mix formula in the *antecedent*.

The lowest possible rank is evidently 2.

To prove the lemma we perform two complete inductions, one according to the grade γ, the other according to the rank ρ of the derivation, i.e., we prove the theorem for a derivation of the grade γ, assuming it to hold for derivations of a lower grade (in so far as there are such derivations, i.e., as long as γ is not equal to zero), supposing, therefore, that derivations of a lower grade may already be transformed into derivations not containing a mix.

Furthermore, we shall begin by considering the case where the rank ρ of the derivation equals 2 (3.11), and after that the case of $\rho > 2$ (3.12), where we assume that the theorem already holds for derivations of the same grade, but of a lower rank.

In the following bold-face capital letters will generally serve as syntactic variables for *formulae*, and Greek capital letters as syntactic variables for (possibly empty) *sequences of formulae*.

In transforming derivations, we shall occasionally meet "identical inference figures," i.e., inference figures with the same upper and lower sequent. Since we have not admitted such figures in our calculus, they must be eliminated as soon as they occur; this is trivially possible by omitting one of the two sequents.

The mix formula of the mix that occurs at the end of the derivation is designated by **M.** It is of grade γ.

3.10. *Re-designating of free object variables* in preparation for the transformation of derivations.

We wish to obtain a derivation that has the following properties:

3.101. For every $\forall$–*IS* ($\exists$–*IA*) it holds that: Its proper variable occurs in the derivation only in sequents *above* the lower sequent of the $\forall$–*IS* ($\exists$–*IA*) and does not occur as a proper variable in any other $\forall$–*IS* ($\exists$–*IA*).

3.102. This is achieved by re-designating the free object variables in the following way:

We take a $\forall$–*IS* ($\exists$–*IA*) above whose lower sequent either no further inference figures of this kind occur, or if they do, they have already been dealt with in a way to be outlined.

In all sequents above the lower sequent of this inference figure we replace the proper variables by one and the same free object variable which, so far, has not yet occurred in the derivation. This obviously leaves the validity of the $\forall$–*IS* ($\exists$–*IA*) as such unchanged. (The proper variables did in fact not occur in its lower sequent.) Furthermore, the remainder of the derivation remains correct, as is shown by the immediately following lemma:

By applying this method systematically to every single $\forall$–*IS* and $\exists$–*IA*, the derivation thus remains correct throughout and at the conclusion obviously has the desired property (3.101). Furthermore, as was essential, the grade and rank of the derivation as well as its endsequent have remained unaltered.

3.103. Now we give the still outstanding proof of the following *lemma*. (It is enunciated in a somewhat more general form than is immediately necessary, since we shall have to apply it again later on (3.113.33).)

> "An *LK*-basic sequent or inference figure turns into a basic sequent or inference figure of the same kind, if we replace a free object variable, which is *not the proper variable* of the inference figure, in all its occurrences in the basic sequent or inference figure, by one and the same free object variable, provided again that that is *not the proper variable* of the inference figure."

This holds trivially except for the $\forall$–*IS*, the $\forall$–*IA*, the $\exists$–*IS* and the $\exists$–*IA*. Yet even here there is no cause for concern: the restrictions on variables are not violated, since we may neither substitute nor replace the proper variable. (This is the reason why both restrictions on variables are necessary.) Furthermore, the formula resulting from **Fa** is still obtained by substituting **a** for **x** in the formula resulting from **Fx**.

Having prepared the way (3.10), we now proceed to the actual transformation of the derivation which serves to eliminate the mix occurring in it.

As already mentioned, we distinguish two cases: $\rho = 2$ (3.11) and $\rho > 2$ (3.12).

3.11. Suppose $\rho = 2$.

We distinguish between a number of particular cases, of which cases 3.111, 3.112, 3.113.1, 3.113.2 are especially simple in that they allow the mix to be immediately eliminated. The other cases (3.113.3) are the most important since their consideration brings out the basic idea behind the whole transformation. Here we use the induction hypothesis with respect to γ, i.e., we reduce each one of the cases to transformed derivations of a lower grade.

3.111. Suppose the left-hand upper sequent of the mix at the end of the derivation is a *basic sequent*. The mix then reads:

$$\frac{\mathbf{M} \longrightarrow \mathbf{M} \quad \Delta \longrightarrow \Lambda}{\mathbf{M}, \Delta^* \longrightarrow \Lambda}$$

which is transformed into:

$$\frac{\Delta \longrightarrow \Lambda}{\mathbf{M}, \Delta^* \longrightarrow \Lambda} \quad \text{possibly several interchanges and contractions.}$$

That part of the derivation which is above $\Delta \longrightarrow \Lambda$ remains the same, and we thus already have a derivation without a mix.

3.112. Suppose the right-hand upper sequent of the mix is a *basic sequent*. The treatment of this case is symmetric to that of the previous one. We have only to regard the two schemata as duals (cf. 2.4).

3.113. Suppose that neither the left- nor the right-hand upper sequent of the mix is a basic sequent. Then both are *lower sequents of inference figures* since $\rho = 2$, and the right and left rank both equal 1, i.e.: In the sequents directly above the *left-hand* upper sequent of the mix, the mix formula **M** does not occur in the *succedent*; in the sequents directly above the *right-hand* upper sequent **M** does not occur in the *antecedent*.

Now the following holds generally: If a formula occurs in the antecedent (succedent) of the lower sequent of an inference figure, it is either a principal formula or the **D** of a thinning, or else it also occurs in the antecedent (succedent) in at least one upper sequent of the inference figure.

This can be seen immediately by looking at the inference figure schemata (1.21, 1.22).

If we now consider the assumptions of the following three cases, we see at once that they exhaust all the possibilities that exist within case 3.113.

3.113.1. Suppose the left-hand upper sequent of the mix is the lower sequent of a *thinning*. Then the conclusion of the derivation runs:

$$\frac{\dfrac{\Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \mathbf{M}} \quad \Delta \longrightarrow \Lambda}{\Gamma, \Delta^* \longrightarrow \Theta, \Lambda}$$

This is transformed into:

$$\frac{\Gamma \rightarrow \Theta}{\overline{\Gamma, \Delta^* \rightarrow \Theta, \Lambda}} \text{ possibly several thinnings and interchanges.}$$

That part of the derivation which occurs above $\Delta \rightarrow \Lambda$ disappears.

3.113.2. Suppose the right-hand upper sequent of the mix is the lower sequent of a thinning. This case can be dealt with symmetrically to the previous one.

3.113.3. The mix formula **M** occurs both in the succedent of the left-hand upper sequent and in the antecedent of the right-hand upper sequent solely as the *principal formula* of one of the operational inference figures.

Depending on whether the terminal symbol of **M** is &, $\vee$, $\forall$, $\exists$, $\neg$, $\supset$, we distinguish the cases 3.113.31 to 3.113.36 (a formula without logical symbols cannot be a principal formula).

3.113.31. Suppose the terminal symbol of **M** is &. In that case the end of the derivation runs:

$$\dfrac{\dfrac{\Gamma_1 \rightarrow \Theta_1, \mathbf{A} \qquad \Gamma_1 \rightarrow \Theta_1, \mathbf{B}}{\Gamma_1 \rightarrow \Theta_1, \mathbf{A} \,\&\, \mathbf{B}}\,\&\text{-}IS \qquad \dfrac{\mathbf{A}, \Gamma_2 \rightarrow \Theta_2}{\mathbf{A} \,\&\, \mathbf{B}, \Gamma_2 \rightarrow \Theta_2}\,\&\text{-}IA}{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}\text{mix}$$

(and correspondingly for the other form of the &-*IA*, treated analogously.)

We transform it into:

$$\dfrac{\dfrac{\Gamma_1 \rightarrow \Theta_1, \mathbf{A} \qquad \mathbf{A}, \Gamma_2 \rightarrow \Theta_2}{\Gamma_1, \Gamma_2^* \rightarrow \Theta_1^*, \Theta_2}\text{mix}}{\overline{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}} \text{ possibly several thinnings and interchanges.}$$

We can now apply the induction hypothesis with respect to γ to that part of the derivation whose lowest sequent is $\Gamma_1, \Gamma_2^* \rightarrow \Theta_1^*, \Theta_2$, because it has a lower grade than γ. (**A** obviously contains fewer logical symbols than **A**&**B**.) This means that the whole derivation may be transformed into one with no mix.

3.113.32. Suppose the terminal symbol of **M** is $\vee$. This case is to be dealt with symmetrically to the previous one.

3.113.33. Suppose the terminal symbol of **M** is $\forall$. Then the end of the derivation runs:

$$\dfrac{\dfrac{\Gamma_1 \rightarrow \Theta_1, \mathbf{Fa}}{\Gamma_1 \rightarrow \Theta_1, \forall \mathbf{xFx}}\,\forall\text{-}IS \qquad \dfrac{\mathbf{Fb}, \Gamma_2 \rightarrow \Theta_2}{\forall \mathbf{xFx}, \Gamma_2 \rightarrow \Theta_2}\,\forall\text{-}IA}{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}\text{mix.}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma_1 \rightarrow \Theta_1, \mathbf{Fb} \qquad \mathbf{Fb}, \Gamma_2 \rightarrow \Theta_2}{\Gamma_1, \Gamma_2^* \rightarrow \Theta_1^*, \Theta_2}\text{mix}}{\overline{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}} \text{ possibly several thinnings and interchanges.}$$

Above the left-hand upper sequent of the mix, $\Gamma_1 \rightarrow \Theta_1$, **Fb**, we write the same part of the derivation which previously occurred above $\Gamma_1 \rightarrow \Theta_1$, **Fa**, yet having replaced every occurrence of the free object variable **a** by **b**. It now follows from the lemma 3.103, together with 3.101, that in performing this operation the part of the derivation *above* $\Gamma_1 \rightarrow \Theta_1$, **Fb** has again become a correct part of the derivation. (By virtue of 3.101 neither **a** nor **b** can be the proper variable of an inference figure occurring in that part of the derivation.) The same consideration may be applied to that part of the derivation which *includes* the sequent $\Gamma_1 \rightarrow \Theta_1$, **Fb**, since it too results from $\Gamma_1 \rightarrow \Theta_1$, **Fa** by substitution of **b** for **a**. It is now in fact clear that by virtue of the restriction on variables for $\forall$-*IS*, **a** could have occurred neither in Γ_1 and Θ_1, nor in **Fx**. Furthermore, **Fa** results from **Fx** by substituting **a** for **x**, and **Fb** from **Fx** by substituting **b** for **x**. This is why **Fb** results from **Fa** by substituting **b** for **a**.

The mix formula **Fb** in the new derivation has a lower grade than γ. Therefore, according to the induction hypothesis, the mix may be eliminated.

3.113.34. Suppose the terminal symbol of **M** is $\exists$. This case is resolved symmetrically to the previous one.

3.113.35. Suppose the terminal symbol of **M** is $\neg$. Then the end of the derivation runs:

$$\dfrac{\dfrac{\mathbf{A}, \Gamma_1 \rightarrow \Theta_1}{\Gamma_1 \rightarrow \Theta_1, \neg \mathbf{A}}\,\neg\text{-}IS \qquad \dfrac{\Gamma_2 \rightarrow \Theta_2, \mathbf{A}}{\neg \mathbf{A}, \Gamma_2 \rightarrow \Theta_2}\,\neg\text{-}IA}{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}\text{mix.}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma_2 \rightarrow \Theta_2, \mathbf{A} \qquad \mathbf{A}, \Gamma_1 \rightarrow \Theta_1}{\Gamma_2, \Gamma_1^* \rightarrow \Theta_2^*, \Theta_1}\text{mix}}{\overline{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}} \text{ possibly several interchanges and thinnings.}$$

The new mix may be eliminated by virtue of the induction hypothesis.

3.113.36. Suppose the terminal symbol of **M** is $\supset$. Then the end of the derivation runs:

$$\dfrac{\dfrac{\mathbf{A}, \Gamma_1 \to \Theta_1, \mathbf{B}}{\Gamma_1 \to \Theta_1, \mathbf{A} \supset \mathbf{B}}\supset\text{-}IS \quad \dfrac{\Gamma \to \Theta, \mathbf{A} \quad \mathbf{B}, \Delta \to \Lambda}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Delta \to \Theta, \Lambda}\supset\text{-}IA}{\Gamma_1, \Gamma, \Delta \to \Theta_1, \Theta, \Lambda}\text{mix}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \to \Theta, \mathbf{A} \quad \dfrac{\mathbf{A}, \Gamma_1 \to \Theta_1, \mathbf{B} \quad \mathbf{B}, \Delta \to \Lambda}{\mathbf{A}, \Gamma_1, \Delta^* \to \Theta_1^*, \Lambda}\text{mix}}{\Gamma, \Gamma_1^*, \Delta^{**} \to \Theta^*, \Theta_1^*, \Lambda}\text{mix}}{\Gamma_1, \Gamma, \Delta \to \Theta_1, \Theta, \Lambda}\text{possibly several interchanges and thinnings}$$

(The asterisks are, of course, intended to be understood as follows: Δ^* and Θ^* result from Δ and Θ_1 by omitting all *S*–formulae of the form **B**; Γ_1^*, Δ^{**} and Θ^* result from Γ_1, Δ^* and Θ by omitting all *S*–formulae of the form **A**.)

Now we have two mixes, but both mix formulae are of a lower grade than γ. We first apply the induction hypothesis to the upper mix (i.e., to that part of the derivation whose lowest figure it is). Thus the upper mix may be eliminated. We can then also eliminate the lower mix.

3.12. Suppose $\rho > 2$.

To begin with, we distinguish two main cases: First case: The right rank is greater than 1 (3.121). Second case: The right rank is equal to 1 and the left rank is therefore greater than 1 (3.122). The second case may essentially be dealt with symmetrically to the first.

3.121. *Suppose the right rank is greater than* 1.

I.e.: The right-hand upper sequent of the mix is the lower sequent of an inference figure, let us call it **If**, and **M** occurs in the antecedent of at least one upper sequent of **If**.

The basic idea behind the transformation procedure is the following:

In the case of $\rho = 2$, we generally reduced the derivation to one of a lower *grade*. Now, however, we shall proceed to reduce the derivation to one of the same grade, but of a lower *rank*, in order to be able to use the induction hypothesis with respect to ρ.

The only exception is the first case, 3.121.1, where the mix may at once be altogether eliminated.

In the remaining cases the reduction to derivations of a lower rank is achieved in the following way: The mix is, as it were, moved up one level within the derivation, beyond the inference figure **If**. (Case 3.121.231, for example, illustrates this point particularly well.) To speak more precisely, the left-hand upper sequent of the mix (which from now on will be designated by $\Pi \to \Sigma$), at present occurring beside the *lower sequent* of **If**, is instead written next to the *upper sequents* of **If**. These now become upper sequents of new mixes. The lower sequents of these mixes are now used as upper sequents of a new inference figure that takes the place of **If**. This new inference figure takes us back either directly, or after having added further inference figures, to the original endsequent. Each new mix obviously has a rank smaller than ρ, since the left rank remains unchanged and the right rank is diminished by at least 1.

In the strict application of this basic idea special circumstances still arise which make it necessary to distinguish the corresponding cases and to deal with them separately.

3.121.1. Suppose **M** occurs in the antecedent of the left-hand upper sequent of the mix. The end of the derivation runs:

$$\dfrac{\Pi \to \Sigma \quad \Delta \to \Lambda}{\Pi, \Delta^* \to \Sigma^*, \Lambda}, \text{ thus } \mathbf{M} \text{ occurs in } \Pi.$$

This is transformed into:

$$\dfrac{\Delta \to \Lambda}{\Pi, \Delta^* \to \Sigma^*, \Lambda}\text{possibly several thinnings, contractions and interchanges.}$$

3.121.2. Suppose **M** does not occur in the antecedent of the left-hand upper sequent of the mix. (This assumption is used for the first time in 3.121.222.)

3.121.21. Suppose **If** is a thinning, contraction, or interchange in the *antecedent*. Then the end of the derivation runs:

$$\dfrac{\Pi \to \Sigma \quad \dfrac{\Psi \to \Theta}{\Xi \to \Theta}\mathbf{If}}{\Pi, \Xi^* \to \Sigma^*, \Theta}\text{mix.}$$

This is transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi \to \Sigma \quad \Psi \to \Theta}{\Pi, \Psi^* \to \Sigma^*, \Theta}\text{mix}}{\Psi^*, \Pi \to \Sigma^*, \Theta}\text{possibly several interchanges}}{\Xi^*, \Pi \to \Sigma^*, \Theta}\S}{\Pi, \Xi^* \to \Sigma^*, \Theta}\text{possibly several interchanges}$$

The inference figure marked § is of the same kind as **If**, in so far as the *S*–formulae designated in the schema of **If** (in 1.21) by **D** and **E**, were not equal to **M**. If **D** or **E** is equal to **M**, we have an identical inference figure (Ψ^* equals Ξ^*).

The derivation for the lower sequent of the new mix has the same left rank as the old derivation, whereas its right rank is lower by 1. Thus the mix may be completely eliminated by virtue of the induction hypothesis.

3.121.22. Suppose **If** is an inference figure with *one* upper sequent, but not containing a thinning, contraction, or interchange in the antecedent. Then the end of the derivation runs:

$$\dfrac{\Pi \longrightarrow \Sigma \qquad \dfrac{\Psi, \Gamma \longrightarrow \Omega_1}{\Xi, \Gamma \longrightarrow \Omega_2}\,\mathbf{If}}{\Pi, \Xi^*, \Gamma^* \longrightarrow \Sigma^*, \Omega_2}\,\text{mix.}$$

Here we have comprised in Γ the same S–formulae that are designated by Γ in the schema of the inference figure (1.21, 1.22). Hence Ψ may be empty or consist of a side formula of the inference figure, and Ξ may be empty or consist of the principal formula of the inference figure.

First of all, the end of the derivation is transformed into:

$$\dfrac{\dfrac{\dfrac{\Pi \longrightarrow \Sigma \qquad \Psi, \Gamma \longrightarrow \Omega_1}{\Pi, \Psi^*, \Gamma^* \longrightarrow \Sigma^*, \Omega_1}\,\text{mix}}{\Psi, \Gamma^*, \Pi \longrightarrow \Sigma^*, \Omega_1}\,\text{possibly several interchanges and thinnings}}{\Xi, \Gamma^*, \Pi \longrightarrow \Sigma^*, \Omega_2}$$

The lowest inference is obviously an inference figure of the same kind as **If** (taking Γ^*, Π as the Γ of the inference figure and including Σ^* in the Θ of the inference figure).

We must only be careful not to violate the restrictions on variables (if **If** is a $\forall$–*IS* or $\exists$–*IA*): Any such violation is precluded by 3.101, which entails that a proper variable that may have occurred in **If** cannot have occurred in Π and Σ.

The mix may be eliminated from the new derivation by virtue of the induction hypothesis.

We therefore obtain a derivation with no mix and which is terminated by the following inference figure:

$$\dfrac{\Psi, \Gamma^*, \Pi \longrightarrow \Sigma^*, \Omega_1}{\Xi, \Gamma^*, \Pi \longrightarrow \Sigma^*, \Omega_2}$$

In general, the endsequent is not yet of the form aimed at.

Hence we proceed as follows:

3.121.221. Suppose Ξ does not contain **M**.

In that case we perform a number of interchanges, if necessary, and obtain the endsequent of the original derivation.

3.121.222. Suppose Ξ contains **M**. Then Ξ is the principal formula of **If** and is equal to **M**. We then add:

$$\dfrac{\dfrac{\Pi \longrightarrow \Sigma \qquad \mathbf{M}, \Gamma^*, \Pi \longrightarrow \Sigma^*, \Omega_2}{\Pi, \Gamma^*, \Pi^* \longrightarrow \Sigma^*, \Sigma^*, \Omega_2}\,\text{mix}}{\Pi, \Gamma^* \longrightarrow \Sigma^*, \Omega_2}\,\text{possibly several contractions and interchanges}$$

Once again, this is the endsequent of the original derivation.

(Above $\Pi \longrightarrow \Sigma$ we once again write the derivation associated with it.)

Thus we have another mix in the derivation. The left rank of our derivation is the same as that of the original derivation. The right rank is now equal to 1. This is so because directly above the right-hand upper sequent occurs the sequent.

$$\Psi, \Gamma^*, \Pi \longrightarrow \Sigma^*, \Omega_1$$

M no longer occurs in its antecedent, for Γ^* does not contain **M**, nor does Π, because of 3.121.2; and Ψ contains at most one *side formula* of **If**, which cannot be equal to **M**, since the *principal formula* of **If** is equal to **M**.

Hence this mix, too, may be eliminated by virtue of the induction hypothesis.

3.121.23. Suppose **If** is an inference figure with *two* upper sequents, i.e., an &–*IS*, $\vee$–*IA*, or a $\supset$–*IA*.

(In view of the application to intuitionist logic (3.2) we shall deal with each possibility in greater detail than would be necessary for the classical case.)

3.121.231. Suppose **If** is an &–*IS*.

Then the end of the derivation runs:

$$\dfrac{\Pi \longrightarrow \Sigma \qquad \dfrac{\Gamma \longrightarrow \Theta, \mathbf{A} \qquad \Gamma \longrightarrow \Theta, \mathbf{B}}{\Gamma \longrightarrow \Theta, \mathbf{A} \,\&\, \mathbf{B}}\,\&\text{–}IS}{\Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta, \mathbf{A} \,\&\, \mathbf{B}}\,\text{mix}$$

(**M** occurs in Γ.) This is transformed into:

$$\dfrac{\dfrac{\Pi \longrightarrow \Sigma \quad \Gamma \longrightarrow \Theta, \mathbf{A}}{\Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta, \mathbf{A}}\,\text{mix} \qquad \dfrac{\Pi \longrightarrow \Sigma \quad \Gamma \longrightarrow \Theta, \mathbf{B}}{\Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta, \mathbf{B}}\,\text{mix}}{\Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta, \mathbf{A} \,\&\, \mathbf{B}}\,\&\text{–}IS$$

Both mixes may be eliminated by virtue of the induction hypothesis.

3.121.232. Suppose **If** is a $\vee$–*IA*.

Then the end of the derivation runs:

$$\dfrac{\Pi \longrightarrow \Sigma \qquad \dfrac{\mathbf{A}, \Gamma \longrightarrow \Theta \qquad \mathbf{B}, \Gamma \longrightarrow \Theta}{\mathbf{A} \vee \mathbf{B}, \Gamma \longrightarrow \Theta}\,\vee\text{–}IA}{\Pi, (\mathbf{A} \vee \mathbf{B})^*, \Gamma^* \longrightarrow \Sigma^*, \Theta}\,\text{mix}$$

($(\mathbf{A} \vee \mathbf{B})^*$ stands either for $\mathbf{A} \vee \mathbf{B}$ or for nothing according as $\mathbf{A} \vee \mathbf{B}$ is unequal or equal to **M**.)

M certainly occurs in Γ. (For otherwise **M** would be equal to $\mathbf{A} \vee \mathbf{B}$, and the right rank would be equal to 1 contrary to 3.121.)

To begin with, we transform the end of the derivation into:

$$\dfrac{\dfrac{\dfrac{\Pi \longrightarrow \Sigma \quad \mathbf{A}, \Gamma \longrightarrow \Theta}{\Pi, \mathbf{A}^*, \Gamma^* \longrightarrow \Sigma^*, \Theta}\text{ mix}}{\mathbf{A}, \Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta}\text{ possibly several interchanges and thinnings} \qquad \dfrac{\dfrac{\Pi \longrightarrow \Sigma \quad \mathbf{B}, \Gamma \longrightarrow \Theta}{\Pi, \mathbf{B}^*, \Gamma^* \longrightarrow \Sigma^*, \Theta}\text{ mix}}{\mathbf{B}, \Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta}\text{ possibly several interchanges and thinnings}}{\mathbf{A} \vee \mathbf{B}, \Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta}\ \vee\text{-}IA.$$

Both mixes may be eliminated by virtue of the induction hypothesis.

From here on the procedure is the same as that in 3.121.221 and 3.121.222, i.e., we distinguish two cases according as $\mathbf{A} \vee \mathbf{B}$ is unequal or equal to **M**. In the first case we may have to add several interchanges to obtain the endsequent of the original derivation; in the second case we add a mix with $\Pi \longrightarrow \Sigma$ for its left-hand upper sequent, and thus once again obtain the endsequent of the original derivation by going on to perform a number of contractions and interchanges, if necessary. The mix concerned may be eliminated, since the associated right rank equals 1. (All this as in 3.121.222.)

3.121.233. Suppose **If** is a $\supset$-*IA*.

Then the end of the derivation runs:

$$\dfrac{\Pi \longrightarrow \Sigma \qquad \dfrac{\Gamma \longrightarrow \Theta, \mathbf{A} \qquad \mathbf{B}, \Delta \longrightarrow \Lambda}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Delta \longrightarrow \Theta, \Lambda}\supset\text{-}IA}{\Pi, (\mathbf{A} \supset \mathbf{B})^*, \Gamma^*, \Delta^* \longrightarrow \Sigma^*, \Theta, \Lambda}\text{ mix}$$

3.121.233.1. Suppose **M** occurs in Γ and Δ.

In that case we begin by transforming the derivation into:

$$\dfrac{\dfrac{\Pi \longrightarrow \Sigma \quad \Gamma \longrightarrow \Theta, \mathbf{A}}{\Pi, \Gamma^* \longrightarrow \Sigma^*, \Theta, \mathbf{A}}\text{ mix} \qquad \dfrac{\dfrac{\Pi \longrightarrow \Sigma \quad \mathbf{B}, \Delta \longrightarrow \Lambda}{\Pi, \mathbf{B}^*, \Delta^* \longrightarrow \Sigma^*, \Lambda}\text{ mix}}{\mathbf{B}, \Pi, \Delta^* \longrightarrow \Sigma^*, \Lambda}\text{ possibly several interchanges and thinnings}}{\mathbf{A} \supset \mathbf{B}, \Pi, \Gamma^*, \Pi, \Delta^* \longrightarrow \Sigma^*, \Theta, \Sigma^*\, \Lambda}\supset\text{-}IA$$

Both mixes may be eliminated by virtue of the induction hypothesis. Then we proceed as in 3.121.221 and 3.121.222. (All that may happen in the first case is that beside interchanges a number of contractions become necessary.)

3.121.233.2. Suppose **M** does not occur in both Γ and Δ simultaneously. **M** must occur in either Γ or Δ because of 3.121. Consider the case of **M** occurring in Δ but not in Γ. The second case is treated analogously.

The end of the derivation is transformed into:

$$\dfrac{\Gamma \longrightarrow \Theta, \mathbf{A} \qquad \dfrac{\dfrac{\Pi \longrightarrow \Sigma \quad \mathbf{B}, \Delta \longrightarrow \Lambda}{\Pi, \mathbf{B}^*, \Delta^* \longrightarrow \Sigma^*, \Lambda}\text{ mix}}{\mathbf{B}, \Pi, \Delta^* \longrightarrow \Sigma^*, \Lambda}\text{ possibly several interchanges and thinnings}}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Pi, \Delta^* \longrightarrow \Theta, \Sigma^*, \Lambda}\supset\text{-}IA$$

The mix may be eliminated by virtue of the induction hypothesis. We then proceed as in 3.121.221 and 3.121.222. (In the second case, where $\mathbf{A} \supset \mathbf{B}$ is equal to **M**, the right rank belonging to the new mix equals 1 as always, since **M** does not occur in $\mathbf{B}, \Pi, \Delta^*$ for the usual reason, nor does it occur in Γ according to the assumption of the case under consideration.)

3.122. *Suppose the right rank is equal to* 1. In that case the *left rank is greater than* 1.

This case is, in essence, treated symmetrically to 3.121. Special attention is required only for those inference figures with no symmetric counterpart, viz., the $\supset$-*IS* and the $\supset$-*IA*.

The inference figures **If** with *one* upper sequent were incorporated, in 3.121.22, in the general schema:

$$\dfrac{\Psi, \Gamma \longrightarrow \Omega_1}{\Xi, \Gamma \longrightarrow \Omega_2}$$

The symmetric schema runs:

$$\dfrac{\Omega_1 \longrightarrow \Gamma, \Psi}{\Omega_2 \longrightarrow \Gamma, \Xi}$$

which also includes a $\supset$-*IS* without any further change. (Γ here represents the formulae designated by Θ in the schemata 1.21, 1.22.)

3.112.1. On the other hand, the case, where the inference figure **If** is a $\supset$-*IA*, must be treated separately. Although this treatment will seem very similar to that in 3.121.233, it is not entirely symmetric.

Thus the end of the derivation runs:

$$\dfrac{\dfrac{\Gamma \longrightarrow \Theta, \mathbf{A} \quad \mathbf{B}, \Delta \longrightarrow \Lambda}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Delta \longrightarrow \Theta, \Lambda}\supset\text{-}IA \qquad \Sigma \longrightarrow \Pi}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Delta, \Sigma^* \longrightarrow \Theta^*, \Lambda^*, \Pi}\text{ mix}$$

3.122.11. Suppose **M** occurs both in Θ and Λ. In that case we transform the end of the derivation into:

$$\cfrac{\cfrac{\cfrac{\Gamma \rightarrow \Theta, \mathbf{A} \quad \Sigma \rightarrow \Pi}{\Gamma, \Sigma^* \rightarrow \Theta^*, \mathbf{A}^*, \Pi}\text{ mix}}{\text{possibly several interchanges and thinnings}} \qquad \cfrac{\mathbf{B}, \Delta \rightarrow \Lambda \quad \Sigma \rightarrow \Pi}{\mathbf{B}, \Delta, \Sigma^* \rightarrow \Lambda^*, \Pi}\text{ mix}}{\cfrac{\mathbf{A} \supset \mathbf{B}, \Gamma, \Sigma^*, \Delta, \Sigma^* \rightarrow \Theta^*, \Pi, \Lambda^*, \Pi}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Delta, \Sigma^* \rightarrow \Theta^*, \Lambda^*, \Pi}\text{ possibly several contractions and interchanges.}}\supset\text{-}IA$$

Both mixes may be eliminated by virtue of the induction hypothesis.

3.122.12. Suppose **M** does not occur in both Θ and Λ simultaneously. It must occur in one of them. We consider the case of **M** occurring in Λ but not in Θ; the alternative case is completely analogous.

We transform the end of the derivation into:

$$\cfrac{\Gamma \rightarrow \Theta, \mathbf{A} \qquad \cfrac{\mathbf{B}, \Delta \rightarrow \Lambda \quad \Sigma \rightarrow \Pi}{\mathbf{B}, \Delta, \Sigma^* \rightarrow \Lambda^*, \Pi}\text{ mix}}{\mathbf{A} \supset \mathbf{B}, \Gamma, \Delta, \Sigma^* \rightarrow \Theta, \Lambda^*, \Pi}\supset\text{-}IA$$

The mix may be eliminated by virtue of the induction hypothesis.

3.2. *Proof of the Hauptsatz for LJ–derivations.*

In order to transform an *LJ*–derivation into an *LJ*–derivation *without cuts*, we apply *exactly the same procedure* as for *LK*–derivations.

Since an *LJ*–derivation is a special case of an *LK*–derivation, it is clear that the transformation can be carried out. We have only to convince ourselves that with every transformation step an *LJ*–derivation becomes another *LJ*–derivation, i.e., that the *D*–sequents of the transformed derivation do not contain more than one *S*–formula in the succedent, given that this was the case before.

We therefore examine each step of the transformation from that point of view.

3.21. Replacement of cuts by mixes. An *LJ*–cut runs:

$$\frac{\Gamma \rightarrow \mathbf{D} \quad \mathbf{D}, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Lambda},$$

where Λ contains at most one *S*–formula. We transform this cut into:

$$\cfrac{\cfrac{\Gamma \rightarrow \mathbf{D} \qquad \mathbf{D}, \Delta \rightarrow \Lambda}{\Gamma, \Delta^* \rightarrow \Lambda}\text{ mix}}{\Gamma, \Delta \rightarrow \Lambda}\text{ possibly several interchanges and thinnings in the antecedent.}$$

This replacement gives us a new *LJ*–derivation.

3.22. By replacing the free object variable (3.10) we trivially get another *LJ*–derivation from a previous one.

3.23. The transformation proper (3.11 and 3.12).

We have to show for each of the cases 3.111 to 3.122.12 that the given transformations do not introduce any sequents with more than one *S*–formula in the succedent.

3.231. Let us begin with the cases 3.11:

In the cases 3.111, 3.113.1, 3.113.31, 3.113.35 and 3.113.36, only such formulae occur in each succedent of the sequent of a new derivation as already occurred in the succedent of the sequent of the original derivation.

Essentially the same applies in 1.113.33. The only difference is an additional substitution of free object variables, which does not, of course, alter the number of succedent formulae of a sequent.

Cases 3.112, 3.113.32, and 3.113.34 were dealt with symmetrically to cases 3.111, 3.113.1, 3.113.31, and 3.113.33, i.e., in order to get one case from another, we read the schemata from right to left instead of from left to right (as well as changing logical symbols, a process which is here of no consequence). Hence in the antecedent of one case we get precisely the same as in the succedent of another. For the antecedents of cases 3.111, 3.113.1, 3.113.31 and 3.113,33, the same applies as for the succedents, viz., in every antecedent of a sequent of the new derivation only such formulae occur as already occurred in an antecedent of a sequent of the original derivation.

This disposes of all symmetric cases: 3.112, 3.113.2, 3.113.32 and 3.113.34.

3.23. Now let us look at the cases, 3.12:

3.232.1. For the cases 3.121 it holds generally that Σ^* is empty, since in $\Pi \rightarrow \Sigma$, Σ must contain only *one* formula, and that formula must be equal to **M**.

It is now obvious that in every succedent of a sequent only such formulae occur as already occurred in the succedent of a sequent of the original derivation.

3.232.2. In the cases 3.122 it is somewhat more difficult to see that from an *LJ*–derivation we always get another *LJ*–derivation. We must direct our attention, as we have already done in considering symmetric cases, to the *antecedents* in the schemata 3.121.

At this point we distinguish two further subcases:

3.232.21. The case which is symmetric to 3.121.1 is trivial, since in every antecedent of a sequent of a new derivation (in case 3.121.1) only such formulae occur as already occurred in an antecedent of a sequent of the original derivation.

3.232.22. In the cases that are symmetric to 3.121.2, the mix in the end of the derivation runs:

$$\frac{\Omega \dashrightarrow \mathbf{M} \quad \Sigma \dashrightarrow \Pi}{\Omega, \Sigma^* \dashrightarrow \Pi}$$

where Π contains at most *one* S–formula, and where $\Omega \dashrightarrow \mathbf{M}$ is the lower sequent of an *LJ*–inference figure in which at least one upper sequent contains **M** as a succedent formula.

If we now look at the inference figure schemata 1.21, 1.22, it becomes easily apparent that such an inference figure can only be a thinning, contraction, or interchange in the antecedent, or a $\vee$–*IA*, a &–*IA*, a $\exists$–*IA*, a $\forall$–*IA*, and a $\supset$–*IA*. Let us disregard for the moment the $\vee$–*IA* and the $\supset$–*IA*. Then all the possibilities enumerated above fall within the case symmetric to 3.121.22, where both Ψ and Ξ always remain empty. (Γ corresponds to the Θ of the inference figure.) Thus we have the case which is symmetric to 3.121.221. Furthermore, Γ is equal to **M**, i.e., Γ^* is empty, and Π contains at most one formula. Hence in the new derivation there never in fact occurs more than one formula in the succedent of a sequent.

The case of a $\vee$–*IA* is symmetric to 3.121.231. Again, Γ is equal to **M**, Γ^* is empty, and Π contains at most one formula; all is thus in order.

There now remains the case of a $\supset$–*IA*, i.e., 3.122.1. In an *LJ*– $\supset$–*IA*, the Θ of the schema (1.22) is empty. Thus we have the case set out under 3.122.12. Λ^* is also empty, and Π contains at most one formula, which means that here, too, we again obtain an *LJ*–derivation from an *LJ*–derivation.

GLOSSARY

All-Zeichen—universal quantifier
Annahmeformel—assumption formula
Antezedenz—antecedent
Äußerstes Zeichen—terminal symbol
Eigenvariable—proper variable
Es-gibt-Zeichen—existential quantifier
Faden—branch
Folgt-Zeichen—implication symbol
Grad—grade
Grundformel—basic formula
Hauptformel—principal formula
Hauptsatz—*Hauptsatz*
Herleitung—derivation
Hilfssatz—lemma
Inhaltlicher Sinn—intuitive sense
Logische-Zeichen-Schlußfigur— operational inference figure
Mischformel— mix formula
Mischung—mix
Mittelungszeichen—syntactic variable
Nebenformel—side formula
Nicht-Zeichen—negation symbol
Oberformel—upper formula
Obersequenz—upper sequent
Oder-Zeichen—disjunction symbol
Rang—rank
Satz—theorem
Schließen—deduction
Schluß—inference
Schnitt—cut
Sequenz—sequent
Spiegelbidlich—dual
Stammbaumform—tree form
Struktur-Schlußfigur—structural inference figure
Sukzedenz—succedent
Teilformel—subformula
Und-Zeichen—conjunction symbol
Untersequenz—lower sequent
Verdünnung—thinning
Vertauschung—interchange
Zeichen für Bestimmtes—constant symbol
Zusammenzeihung—contraction