Let $FV(\sigma)$ denote the free variables in the formula, σ, and let $\sigma[t/x]$ denote the capture-avoiding substitution of a term t for all free x in σ.

Intuitively, the proof of a universal formula, $\forall x A$, proceeds by proving that, $A[y/x]$, holds for an *arbitrary* element y, given certain assumptions, Γ. We say that y is arbitrary in the sense that no specific information is assumed of y; we formalize this notion by requiring that $\forall \sigma \in \Gamma : (y \notin FV(\sigma))$, abbreviated as $y \notin FV(\Gamma)$. In natural deduction style, we write this rule as:

$$\frac{y \notin FV(\Gamma) \qquad \begin{array}{c}[\Gamma]\\ A[y/x]\end{array}}{\forall x A}\ (\mathrm{I}\forall)$$

In sequent style this is formulated as:

$$\frac{y \notin FV(\Gamma) \qquad \Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A}\ (\mathrm{R}\forall)$$

A problem with this standard rule stems from a discrepancy between provability and truth: it is conceivable that the truth of $\forall x A$ in a specific domain follows from separate proofs of $A[a_1/x]$, $A[a_2/x], ..., A[a_n/x], ...$, where each proof depends on *specific* properties of each a_i. Since the above rule requires the existence of a *uniform* proof of A, that is, a proof that is the same for all instances a_i, then we say that the conditions for provability are *stronger* than those for truth in this logic.

Universal quantifiers are consumed by drawing an arbitrary terms, t, of the quantified formula:

$$\frac{\forall x A}{A(t/x)}\ (\mathrm{E}\forall)$$

In sequent style:

$$\frac{A(t/n), \forall x A, \Gamma \vdash C}{\forall x A, \Gamma \vdash C}\ (\mathrm{L}\forall)$$

Dually, the proof of an existential formula, $\exists x A$ proceeds by proving that a property, $A[t/x]$, holds for some term t from assumptions, Γ. In the style of natural deduction, this is formalized as:

$$\frac{\begin{array}{c}[\Gamma]\\ A[t/x]\end{array}}{\exists x A}\ (\mathrm{I}\exists)$$

In sequent calculus:

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A}\ (\mathrm{R}\exists)$$

To derive a consumption rule for existential quantification, we consider how an arbitrary consequence, C, of $\exists x A$ may be proven. If we know $\exists x A$, we can assume $A[y/x]$ for some variable y. Importantly, we are unable to consume $\exists x A$ if we assume anything else about y. Thus, any assumptions, Γ, must be such that $y \notin FV(\Gamma)$. Finally, if we conclude C, and $y \notin FV(C)$, we can eliminate the assumption $A[y/x]$. From this, we write the following natural deduction rule:

$$\frac{y \notin FV(\Gamma \cup C) \qquad \begin{array}{c}[\Delta]\\ \exists x A\end{array} \qquad \begin{array}{c}[A[y/x]],\Gamma\\ C\end{array}}{C}\ (\mathrm{E}\exists)$$

In sequent calculus:

$$\frac{y \notin FV(\Gamma \cup C) \qquad A(y/x), \Gamma \vdash C}{\exists x A, \Gamma \vdash C}\ (\mathrm{L}\exists)$$

With these rules, we define the sequent logic *L1* as containing: the constant 0 (zero), the unary function S (the successor function), and the following axioms and rules:

$$\begin{aligned}&\mathbf{P1} : \Gamma \vdash \forall n(n = 0 \vee \exists m : n = S(m))\\ &\mathbf{P2} : \Gamma \vdash \forall n \forall m(n = S(m) \to m < n)\\ &\mathbf{P3} : \Gamma \vdash \exists n A \supset \exists n(A \wedge \forall m(m < n \supset \neg A[m/n]))\end{aligned}$$

$$\frac{}{\Gamma, \bot \vdash A}\ (\mathrm{L}\bot)$$

$$\frac{y \notin FV(\Gamma) \qquad \Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A}\ (\mathrm{R}\forall) \qquad \frac{A(t/n), \forall x A, \Gamma \vdash C}{\forall x A, \Gamma \vdash C}\ (\mathrm{L}\forall)$$

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A}\ (\mathrm{R}\exists) \qquad \frac{y \notin FV(\Gamma \cup C) \qquad A(y/x), \Gamma \vdash C}{\exists x A, \Gamma \vdash C}\ (\mathrm{L}\exists)$$

$$\frac{\Gamma, A \vdash C}{\Gamma \vdash A \to C}\ (\mathrm{R}\to) \qquad \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{A \to B, \Gamma \vdash C}\ (\mathrm{L}\to)$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\mathrm{R}\wedge) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ (\mathrm{L}\wedge)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\mathrm{R}\vee) \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ (\mathrm{L}\vee)$$

$$\frac{\Gamma, A(0), \forall n[A(n) \to A(S(n))] \vdash C}{\Gamma, \forall n A \vdash C}\ (\mathrm{Induction})$$

Also, we define *L2* as identical to *L1* without the Induction rule and along with the rule of Excluded Middle:

$$\frac{\Gamma, A \vdash C \qquad \Gamma, \neg A \vdash C}{\Gamma \vdash C}\ \text{(EM)}$$

The derivability relation, $\vdash$, will be written as $\vdash_1$ or $\vdash_2$ to denote derivability in *L1* and *L2*, respectively. Also, by $\sigma(n)$, we refer to a formula σ with 1 free variable that is substituted by n.

Our aim is to prove the following theorems:

Theorem I (**Admissibility of Induction in** *L2*): For any formula, σ, and sequent multiset, Γ:

$$(\Gamma \vdash_2 \sigma(0)) \wedge (\Gamma \vdash_2 \forall n[\sigma(n) \rightarrow \sigma(S(n))]) \Rightarrow (\Gamma \vdash_2 \forall n \sigma(n))$$

Theorem II (**Inadmissibility of Excluded Middle in** *L1*): There exist formulae, σ and γ, and a sequent multiset, Γ, such that:

$$(\Gamma, \sigma \vdash_1 \gamma) \wedge (\Gamma, \neg\sigma \vdash_1 \gamma) \wedge (\Gamma \not\vdash_1 \gamma)$$

Proof of Theorem 1:
Let $\Delta = \Gamma \vdash_2 \sigma(0), \Gamma \vdash_2 \forall n[\sigma(n) \rightarrow \sigma(S(n))], \neg\forall n\sigma$

$1)\Delta \vdash \exists n[\sigma(n) \wedge \forall m[m < n \rightarrow \neg\neg\sigma(m)]]$	Rule: Infimum
$2)\Delta' \vdash N = 0 \vee \exists m : N = S(m)$	PC1
$3)\Delta', N = 0 \vdash \neg sigma(0)$	Rule: = E
$4)\Delta', N = 0 \vdash \bot$	3), Rule: Non-contradiction
$5)\Delta' \vdash \neg N = 0$	4), Rule: Reductio ad absurdum
$6)\Delta', N = S(M) \vdash M < N$	PC2
$7)\Delta', N = S(M) \vdash \neg\neg\sigma(M)$	6), Rule: $\rightarrow$ E
$8)\Delta', N = S(M) \vdash \sigma(M)$	7), Double Negation
$9)\Delta', N = S(M) \vdash \sigma(S(M))$	8), Rule: $\rightarrow$ E
$10)\Delta', N = S(M) \vdash \sigma(N)$	9), Rule: = E
$11)\Delta', N = S(M) \vdash \bot$	10), Rule: Non-contradiction
$12)\Delta' \vdash \neg N = S(M)$	11), Rule: Reductio ad absurdum
$13)\Delta' \vdash \neg\exists m : N = S(m)$	12), Rule: $\exists$ I
$14)\Delta' \vdash \neg N = 0 \wedge \neg\exists m : N = S(m)$	5), 13), Rule: $\wedge$ I
$15)\Delta' \vdash \neg(N = 0 \vee \exists m : N = S(m))$	14), DeMorgan's Theorem
$16)\Delta' \vdash \bot$	2), 15), Rule: Non-contradiction
$17)\Delta \vdash \bot$	16), 1), Rule: $\exists$ E
$18)\Gamma, \sigma(0), \forall n : \sigma(n) \rightarrow \sigma(S(n)) \vdash \neg\exists n : \neg sigma(n)$	17), Rule: non-contradiction
$19)\Gamma, \sigma(0), \forall n : \sigma(n) \rightarrow \sigma(S(n)) \vdash \forall n : sigma(n)$	18), Rule: $\neg\exists$, Rule: Double Negation

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