Quantum teleportation protocol

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Abstract

Equations typed in LaTeX to explain quantum teleportation protocol implemented with Qiskit in python. Complete description available here.

Equations

We define three states $|\psi\rangle$, $|\phi\rangle$, and $|\omega\rangle$. $|\psi\rangle$ will be the state we want to send, $|\phi\rangle$ will be an ancillary qubit, and $|\omega\rangle$ will be the qubit to which we want to send the state originally in $|\psi\rangle$. To summarize:

$$|\psi\rangle : \text{ sender qubit, } |\phi\rangle : \text{ ancillary qubit, } |\omega\rangle : \text{ receiver qubit.} \tag{1}$$

This is the circuit we are going to describe with equations in this paper:

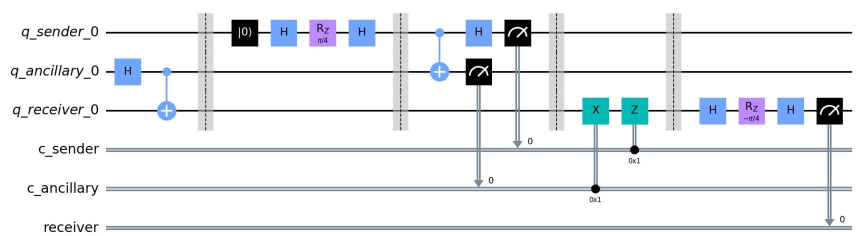

Figure 1: Quantum teleportation protocol circuit

As you can see, the first thing we do is entangle $|\phi\rangle$ with $|\omega\rangle$. This gives us the state:

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2}$$

Note that in this equation and onward, we will describe our state as $|\psi\phi\omega\rangle$. For example, the state $|010\rangle$ means that $|\psi\rangle = |0\rangle$, $|\phi\rangle = |1\rangle$, and $|\omega\rangle = |0\rangle$. After entangling $|\phi\rangle$ and $|\omega\rangle$, we prepare $|\psi\rangle$ to the state we want to send. Let's now focus on this state and how it will result after applying the corresponding gates.

$$|\psi\rangle = |0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{3}$$

$$\xrightarrow{R_z} \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{8}}|0\rangle + e^{i\frac{\pi}{8}}|1\rangle\right) \tag{4}$$

$$\xrightarrow{H} \frac{1}{2}\left(e^{-i\frac{\pi}{8}}(|0\rangle + |1\rangle) + e^{i\frac{\pi}{8}}(|0\rangle - |1\rangle)\right) = \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle \tag{5}$$

The simplified state at (5) is the one we will be sending to $|\omega\rangle$. Now, we are going to entangle $|\psi\rangle$ with $|\phi\rangle$ using a CNOT gate and then we are going to send $|\psi\rangle$ through a Hadamard gate. Before applying any gates, let's see our state up to here. We can found this state by replacing $|\psi\rangle$ in (2) with the state for $|\psi\rangle$ we got in (5).

$$\left(\cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}(|000\rangle + |011\rangle) - \sin\frac{\pi}{8}(|100\rangle + |111\rangle)\right) \tag{6}$$

First we are going to send our state through the CNOT gate, where $|\psi\rangle$ acts as the control qubit and $|\phi\rangle$ as the target qubit, and then we are going to send $|\psi\rangle$ through a Hadamard gate. The following equations describe this operations. For simplicity, we will denote the state we are starting with as $|\psi\phi\omega\rangle$, this state is the one we ended up with in (6).

$$|\psi\phi\omega\rangle \xrightarrow{CNOT|\psi\phi\rangle} \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}(|000\rangle + |011\rangle) - \sin\frac{\pi}{8}(|110\rangle + |101\rangle)\right) \tag{7}$$

$$\xrightarrow{H|\psi\rangle} \frac{1}{2}\left(\cos\frac{\pi}{8}(|000\rangle + |100\rangle + |011\rangle + |111\rangle) - \sin\frac{\pi}{8}(|010\rangle - |110\rangle + |001\rangle - |101\rangle)\right) \tag{8}$$

Here it gets a little bit tricky, since we are going to measure $|\psi\rangle$ and $|\phi\rangle$. As you may know, there are 4 possible states we might end up with, these are $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. And for each of this four states, you can have either $|0\rangle$ or $|1\rangle$ for our last qubit: $|\omega\rangle$. This gives us the 8 possible states we can see in (8). Since we are only going to measure the first two qubits, $|\omega\rangle$ will remain in a superposition of states, we will examine the 4 possibilities in the following equations:

$$\text{When } |\psi\phi\rangle = |00\rangle\,,\ |\omega\rangle = \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle \tag{9}$$

$$\text{When } |\psi\phi\rangle = |01\rangle\,,\ |\omega\rangle = -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle \tag{10}$$

$$\text{When } |\psi\phi\rangle = |10\rangle\,,\ |\omega\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle \tag{11}$$

$$\text{When } |\psi\phi\rangle = |11\rangle\,,\ |\omega\rangle = \sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle \tag{12}$$

Now, as you can see in the circuit, we are going to send $|\omega\rangle$ through some gates depending on the measurements we made. We are going to apply a Pauli-X gate if $|\phi\rangle = |1\rangle$, followed by a Pauli-Z gate if $|\psi\rangle = |1\rangle$. After applying this conditional gates, $|\omega\rangle$ should be in the state of $|\psi\rangle$ showed in (5), meaning that the teleportation was completed successfully. Let's see what happens exactly in each case, we will start with the case in which $|\psi\phi\rangle = |00\rangle$. Well, in this case no further gate is applied, we just have the following state.

$$|\omega\rangle = \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle \tag{13}$$

Now, let's look at what happens when $|\psi\phi\rangle = |01\rangle$. We will only apply a Pauli-X gate.

$$|\omega\rangle = -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle \xrightarrow{X} \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle \tag{14}$$

Now, we look at the case where $|\psi\phi\rangle = |10\rangle$. In this case, we only apply a Pauli-Z gate.

$$|\omega\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle \xrightarrow{Z} \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle \tag{15}$$

Finally, we look at what happens when $|\psi\phi\rangle = |11\rangle$. This time, we apply a Pauli-X gate followed by a Pauli-Z gate.

$$|\omega\rangle = \sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle \xrightarrow{X} \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle \tag{16}$$

$$\xrightarrow{Z} \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle \tag{17}$$

At this point, the quantum teleportation protocol is over. $|\omega\rangle$ is in the state we showed on (5), meaning that we successfully sent the state from $|\psi\rangle$ into $|\omega\rangle$. To achieve this, we needed to send two bits of information through classical channels, which allowed us to perform the controlled operations showed in (13), (14), (15), and (17). It is common to think that this use of classical channels throws away the whole purpose of teleporting a quantum state, but it really doesn't. This classical communication is really the only way of ensuring that we teleport the state successfully; if we didn't apply this step, we would be stuck with one of the four possibilities shown above without the person that has $|\omega\rangle$ having a clue about what $|\psi\phi\rangle$ is, therefore this person would only have the intended state $1/4$ of the time.

In the circuit shown in figure 1, we apply some additional gates to $|\omega\rangle$. This is just a way to ensure that the teleportation protocol was successful when running in IBM's quantum computers. As you can see, this gates are the same that were applied to the state $|\psi\rangle$ when preparing it, but backwards (which turns out to be the same in this case). As you may know, quantum computation is reversible, so performing this operations will turn $|\omega\rangle$ back to the state $|0\rangle$ (we are applying the operations shown in (5) but in reverse). This makes it easier to measure if the protocol was successful since the computer will measure $|\omega\rangle$ to be $|0\rangle$ with certainty (without taking into account noise of actual quantum hardware).