Investigation of the Higher Order Zeeman Effect

By

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A Thesis
Submitted to the Faculty of Graduate Studies
through the Department of Physics
in Partial Fulfillment of the Requirements for
the Degree of Bachelors of Science (With Thesis)
at the University of Windsor

Windsor, Ontario, Canada

2025

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DEDICATION

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AKNOWLEDGEMENTS

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CHAPTER 1

The Higher Order Zeeman Effect

1.1 Overview

In this chapter the Zeeman Effect is introduced, and the motivation, direct applications, and the higher order Zeeman Effect is discussed. The main focus of this chapter is to show the effect of the quadratic Zeeman Effect, and show how using the magnetic dipole operator in conjunction with the relativistic corrections to $^3He^+$ yields a cubic Zeeman Effect. The effects of both the quadratic and cubic corrections are discussed in great detail, and the impact of the effect on high precision measurements is displayed for various magnetic field strengths.

Sec. 1.2 starts with the history of the Zeeman effect, its origins and discovery. Afterwards the motivation for the project in Sec. 1.3 is discussed. Here, some current experiments in the field such as the $g - 2$ experiment conducted at the Max Planck Institute as well as applications to high-precision magnetometry are highlighted. Some additional applications in the field of atomic physics such as ... are introduced as well. In Sec. 1.4, the ordinary Zeeman effect is discussed, introducing its theory and application to atomic systems such as $^3He^+$. After introducing the ordinary Zeeman effect the quadratic Zeeman effect is introduced, where it is derived using the canonical momentum and a description of its impact on an atom subjected to a magnetic field is given. Moving towards higher order systems, the cubic Zeeman effect is introduced. Starting with the effects that contribute to the cubic Zeeman effect such as the magnetic dipole operator in Sec. 1.6 and relativistic corrections to the $^3He^+$ ion in Sec. 1.6.1, these effects are combined to yield a B^3 contribution to the energy splitting within the presence of an external magnetic field. Afterwards, Sec. 1.7 discusses the results of the calculation and its applications.

1.2 History

The Zeeman effect was first introduced by Pieter Zeeman, who discovered in 1896 that in the presence of a static magnetic field, spectral lines could be split into many components. After the discovery of quantum mechanics, the behaviour was found to be described as a perturbation of the Hamiltonian using the magnetic moment of the atom and the magnetic field.

Since it's discovery, the Zeeman effect has played a large role in the field of atomic physics and magnetometry, which is the study of the intensity of magnetic field across space and time. There have been several calculations to include the relativistic corrections [26, 27], field inhomogeneities, and quadratic effects in hydrogenic systems [18]. However, little is known about its behavior in helium atoms such as $^3He^+$ and ^3He, which is of key interest in magnetometry and the muon magnetic moment anomaly ($\mu_g - 2$), for which there is a 5.0 σ discrepancy [9] with the standard model prediction.

1.3 Motivation

1.3.1 The $g - 2$ experiment

The Dirac equation is a very successful and well studied equation in quantum mechanics. Its success comes from its ability to predict 2 important phenomena; the existence of antimatter and the magnetic dipole moment of the electron. The Dirac equation predicts that the magnetic dipole of the electron should be twice that of the classical prediction. This result is expressed in terms of the g-factor which the Dirac equation predicts is equal to 2. While the Dirac prediction is much closer to experimental findings, there is still a difference between the experimentally measured value of g and the equations prediction. This is called the $g-2$ anomaly. The anomaly is represented by

$$a = \frac{g-2}{2}\ . \tag{1}$$

The discrepancy of g is caused by higher-order contributions from quantum field theory and to this day is yet to be properly explained.

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{EW}} + a_\mu^{\mathrm{hadron}} \tag{2}$$

The first two terms can be derived from first principles, but the hadronic term cannot be calculated precisely on its own and is estimated from experimental results. The effort to measure the muon magnetic moment precisely is an active area of research. The work presented in this thesis aids in the investigation of the $g-2$ anomaly by providing corrections to the Zeeman splitting in ^{3}He+, the element used in the magnetometry experiment to measure the anomaly. Accounting for higher order corrections to the Zeeman effect may help consolidate the discrepancy between theory and experiment and help researchers further understand the muon magnetic moment and its impact on muonic systems.

1.3.2 High-precision magnetometry

1.3.3 Connection to Atomic Physics

1.4 The Zeeman effect in Hydrogen

When an atom is placed in an external magnetic field, its energy levels are shifted. The shifting of energy levels is known as the Zeeman effect. The effect can be written as a perturbation to the Hamiltonian [19]

$$\hat{H}_Z = -\left(\vec{\mu_l} + \vec{\mu_s}\right) \cdot \vec{B} \ . \tag{3}$$

μ_l is the orbital dipole moment, and μ_s is the spin magnetic dipole moment, which have the definitions

$$\vec{\mu_l} = -\frac{e}{2m}\vec{L} \qquad\qquad \vec{\mu_s} = -\frac{e}{m}\vec{S} \tag{4}$$

So the first order Zeeman effect Hamiltonian is

$$\hat{H}_Z = \frac{e}{2m}\left(\vec{L} + 2\vec{S}\right) \cdot \vec{B} \tag{5}$$

and has the following eigen energy solutions

$$E_{n,m_s,m_l} = -\frac{E_0}{n^2} + \mu_B B(m_l + 2m_s) \ . \tag{6}$$

So it is seen that depending on the magnetic quantum number, the energy levels split apart. Their corresponding new energies depend on this magnetic quantum number as well as the principle quantum number n, and scale linearly with magnetic field strength B.

1.5 The quadratic Zeeman effect

The quadratic Zeeman effect is derived using the Schrodinger equation and the canonical momentum. The canonical momentum is a conserved quantity that describes a moving charged particle. It can be written as

$$\vec{p} = m\vec{v} + e\vec{A}\ . \tag{7}$$

Where $m\vec{v}$ is the classical definition of the momentum, and $e\vec{A}$ is the extension from electrodynamics that accounts for the impact of an external magnetic field on a charged particle. This term is required in order to ensure that the conservation of momentum holds true, since charged particles subject to an external magnetic field travel in a circular path dependant on the direction of the field.

The canonical momentum then is also written in replacement to the typical momentum operator in quantum mechanics, giving the canonical momentum operator

$$\hat{p}_{\text{canonical}} = i\hbar\vec{\nabla} + e\hat{A}\ . \tag{8}$$

Where $\hat{A}$ is the vector potential operator. For an external magnetic field of strength B pointing in the $\hat{k}$ direction the operator becomes

$$\hat{A} = \frac{B}{2}\left(y\hat{i} - x\hat{j}\right)\ . \tag{9}$$

Substituting this in for the vector potential operator in the canonical momentum and placing it into the Hamiltonian equation one gets

$$\hat{H} = \frac{\left(i\hbar\vec{\nabla} + \frac{Be}{2}\left(y\hat{i} - x\hat{j}\right)\right)^2}{2m_e} - \frac{Ze^2}{4\pi\epsilon_0\vec{r}}\ . \tag{10}$$

Which when expanded gives

$$\hat{H} = \frac{-\hbar^2\nabla^2}{2m} - \frac{i\hbar eB}{4mc}\vec{\nabla}\cdot\left[y\hat{i} - x\hat{j}\right] - \frac{i\hbar eB}{4mc}\left[y\hat{i} - x\hat{j}\right]\cdot\vec{\nabla} + \frac{e^2B^2}{8mc}\left(x^2 + y^2\right) - \frac{Ze^2}{4\pi\epsilon_0 r}\ . \tag{11}$$

The B^2 term is the quadratic Zeeman effect and is written on its own as

$$\hat{H}_Z = \frac{B^2 e^2}{8m_e}(x^2 + y^2) \ . \tag{12}$$

Using $x^2 + y^2 = r^2 - z^2 = \frac{2}{3}r^2 \left[P_0(\cos\theta) - P_2(\cos\theta)\right]$ where $P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right)$ and $P_0(\cos\theta) = 1$ are a Legendre polynomials,

$$\hat{H}_Z = \frac{B^2 e^2}{8m_e}(P_0(\cos\theta) - P_2(\cos\theta)) \ . \tag{13}$$

1.6 The magnetic dipole moment operator

The magnetic dipole moment operator represents the interaction of a magnetic dipole moment with an external magnetic field. It is described via the following relation

$$Q_{M1} = \mu_B \left(1 - \frac{2p^2}{3m^2c^2} + \frac{Ze^2}{3mc^2 r}\right) \vec{\sigma} \cdot \vec{B} \tag{14}$$

Where μ_B is the Bohr magneton

$$\mu_B = \frac{e\hbar}{2mc} \tag{15}$$

The second term in the brackets of the magnetic dipole moment operator accounts for the relativistic correction to the kinetic energy of the electron, and the third term is the potential energy due to the Coulomb interaction between the electron and the nucleus. The first term corresponds to the ordinary Zeeman Effect, which does not contribute to the sum over states due to orthogonality.

The ordinary Zeeman effect contributes to Q_{M1} in $^3He^+$ because it has non-zero spin due to the missing electron. For systems such as ^3He, the ordinary Zeeman effect will not contribute.

1.6.1 The relativistic correction to ^{3}He$^+$

Combining the magnetic dipole moment with the quadratic Zeeman operator, we can write down the relativistic corrections for ^{3}He$^+$. Written in terms of pseudostates the relativistic correction is

$$C_{\mathrm{rel}}^{(2)} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\langle\psi_0|H_Z^{(2)}|\psi_n\rangle\langle\psi_n|Q_{M1}|\psi_0\rangle}{E_0 - E_n} \tag{16}$$

1.7 Results

REFERENCES

[1] Codata value: atomic unit of magnetic flux density.

[2] Codata value: electron mass.

[3] Codata value: elementary charge.

[4] Codata value: fine-structure constant.

[5] Codata value: reduced planck constant.

[6] Codata value: speed of light in vacuum.

[7] Codata value: vacuum electric permittivity.

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