Double Perturbation Theory

Suppose we have two perturbations V and W. Start from the Schrödinger equations

$$(H_0 + \lambda V + \nu W)\Psi = E\Psi$$

and expand

$$\Psi = \psi_{0,0} + \lambda\psi_{1,0} + \nu\psi_{0,1} + \lambda^2\psi_{2,0} + \nu^2\psi_{0,2} + \lambda\nu\psi_{1,1}$$

$$E = E_{0,0} + \lambda E_{1,0} + \nu E_{0,1} + \lambda^2 E_{2,0} + \nu^2 E_{0,2} + \lambda\nu E_{1,1}$$

λ and ν are perturbation parameters.
Collect equal powers of λ and ν.

$$\lambda^0\nu^0: \quad (H_0 - E_{0,0})\psi_{0,0} = 0 \qquad (1)$$

$$\lambda^1\nu^0: \quad (H_0 - E_{0,0})\psi_{1,0} + V\psi_{0,0} = E_{1,0}\psi_{0,0} \qquad (2)$$

$$\lambda^0\nu^1: \quad (H_0 - E_{0,0})\psi_{0,1} + W\psi_{0,0} = E_{1,0}\psi_{0,0} \qquad (3)$$

$$\lambda^1\nu^1: \quad (H_0 - E_{0,0})\psi_{1,1} + V\psi_{0,1} + W\psi_{1,0} = E_{1,0}\psi_{0,1} + E_{0,1}\psi_{1,0} + E_{1,1}\psi_{0,0} \qquad (4)$$

Multiply (4) by $\psi_{0,0}$ and integrate to obtain

$$E_{1,1} = \langle\psi_{0,0}|V|\psi_{0,1}\rangle + \langle\psi_{0,0}|W|\psi_{1,0}\rangle$$

In terms of spectral representations,

$$\langle\psi_{0,0}|V|\psi_{0,1}\rangle = \sum_n \frac{\langle\psi_0|V|\psi_n\rangle\langle\psi_n|W|\psi_0\rangle}{E_0 - E_n} \qquad (5)$$

$$\langle\psi_{0,0}|W|\psi_{1,0}\rangle = \sum_n \frac{\langle\psi_0|W|\psi_n\rangle\langle\psi_n|V|\psi_0\rangle}{E_0 - E_n} \qquad (6)$$

$\therefore$ (5) = (6).