

Investigation of the Higher Order Zeeman Effect

By

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DEDICATION

I would like to dedicate this thesis to ...

ACKNOWLEDGEMENTS

...

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LIST OF ABBREVIATIONS

CHAPTER 1

Introduction

1.1 Background and Motivation

1.2 Overview of Research Projects

1.3 Methodology Overview

1.4 Research Significance

1.5 Dissertation Structure

CHAPTER 2

Theoretical Methods

2.1 Overview

This chapter provides an overview of all theoretical methods used throughout this dissertation. It gives the theoretical building blocks upon which the dissertation is constructed, ranging from the set of units used in the problem, to perturbation equation solutions and methods for verifying calculations. As stated in the introduction, this thesis is based upon calculating matrix elements between two sets of wavefunctions connected by an operator. These matrix elements arise from perturbation theory and are necessary for computing energy shifts due to external fields such as those present in the higher-order Zeeman effect. The foundation of theory is presented here, but the more specific calculations related to solving for the Zeeman effect are presented in the relevant chapter below.

The chapter begins first by discussing atomic units in Sec. 2.2, then introduces the Schrödinger equation for the one-electron problem as well as its solution for hydrogenic wavefunctions in Sec. 2.3. Next, Perturbation Theory is introduced in Sec. 2.5, and how it is to approximate the new eigenvalues and eigenstates of the Zeeman-perturbed Hamiltonian is discussed. Sec. 2.4 reviews various integration techniques that are implemented both analytically and programmatically in order to simplify the calculation process, followed by an introduction to solving recursion relations in Sec. 2.6. This is the core step in determining the perturbed wavefunctions used throughout the rest of the study. Finally, this chapter concludes with the discussion of the Dalgarno Interchange Theorem and how it serves as a critical check on the

correctness of the obtained perturbed wavefunctions.

2.2 Atomic Units

First proposed by Hartree in November of 1928, atomic units have since become the standard across all of atomic physics. Atomic units were designed with the purpose of *”eliminating various universal constants from equations and also to avoid high powers of 10 in numerical work”* [20]. Hartree proposed that physical quantities such as the mass are measured in terms of the mass of the electron m_e . The charge measured in terms of the electron charge, e , the vacuum permittivity in terms of $4\pi\epsilon_0$, and the angular momentum in terms of \hbar . Their standard definitions in both S.I. units and atomic units are stated in Table. 2.2.1 [2, 3, 4, 5, 6, 7]. Written mathematically;

$$\hbar = m_e = e = 4\pi\epsilon_0 = 1 . \quad (1)$$

Scaling these units out of the problem greatly simplifies the calculation. Once a result is found, multiplying the unscaled units back into the answer retrieves the desired result in standard SI units. While the use of atomic units can make any work done by hand much simpler to manage, it also benefits numerical work that is done computationally. Machine error is an unavoidable reality of programmed solutions

Symbol	Name	Value in S.I	Value in a.u.
\hbar	Reduced Planck’s constant	$1.054\,571\,817\dots \times 10^{-34} \text{ J s}$	1
m_e	Electron mass	$9.109\,383\,713\,9(28) \times 10^{-31} \text{ kg}$	1
e	Electron charge	$1.602\,176\,634 \times 10^{-19} \text{ C}$	1
ϵ_0	Electric permittivity	$8.854\,187\,818\,8(14) \times 10^{-12} \text{ F/m}$	$\frac{1}{4\pi}$
c	Speed of light	$299\,792\,458 \text{ m/s}$	137.035 999 177(21)
α	Fine structure constant	$7.297\,352\,564\,3(11) \times 10^{-3}$	$\frac{1}{137.035\,999\,177(21)}$

Fig. 2.2.1: Table of fundamental constants expressed in S.I. units and atomic units.

and working in units which avoids extreme values both large and small helps to mitigate the uncertainty present in high precision calculations. In addition, exact analytical results need not be limited by the values of the fundamental constants. This thesis utilizes atomic units both analytically and numerically to simplify the calculation process throughout.

A consequence of utilizing atomic units is the need to redefine key fundamental physical quantities in terms of these scaled values. An example of key importance in the investigation of the higher-order Zeeman effect is the atomic unit of magnetic field strength, otherwise referred to as the atomic unit of magnetic flux density [1].

$$B_{\text{a.u.}} = \frac{\hbar}{ea_0^2} . \quad (2)$$

This value is composed of the Bohr radius a_0 , electron charge e , and \hbar , the unit of action. The Bohr radius is defined as the most probable distance between an electron in the ground state of hydrogen and its nucleus. It can be described by the following equation [19]

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2m_e} = \frac{\hbar}{m_e c \alpha} . \quad (3)$$

The Bohr radius itself is defined using a similar combination of fundamental constants which after applying the rules for atomic units, should equal 1. It can now be seen that all units in the atomic unit of field strength also become 1 as intended. At the end of the problem, the desired result in S.I units is retrieved by simply multiplying the determined result by the atomic unit of magnetic field strength when its constants are not scaled to 1. This is equivalent to multiplying the result in atomic units by [1].

$$B_{\text{a.u.}} = 2.35051757077(73) \times 10^5 \text{ T} \quad (4)$$

The numbers in parenthesis here denote the uncertainty of the quantity. Another key benefit in atomic units is that it allows us to define almost all fundamental quantities in terms of two constants; the speed of light, c and the fine structure constant, α . This allows one to express key physical laws in a way that highlights fundamental dimensionless relationships rather than specific unit-dependant values.

The fine structure constant was introduced in 1916 by Arnold Sommerfeld when he was investigating the gap in the spectral lines of the hydrogen atom [24]. He compared his work with the Michelson Morley experiment from 1887 where the spectral lines were shown and it was deduced that the universe is not permeated by the aether [21]. It is defined as the following dimensionless quantity

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.035\,999\,177(21)} \quad (5)$$

which in atomic units is just the following relationship with the speed of light

$$\alpha = \frac{1}{c} . \quad (6)$$

The fine structure constant is often used as an essential expansion parameter, treating relativistic effects as a perturbation in α .¹ Applying the atomic unit scaling to the fine structure constant shown in equation (6) confirms that the second equivalence in equation (3) remains equal to 1.

Applying this technique to problems in atomic physics gives a powerful way of simplifying calculations while providing an easy conversion factor to SI units for comparison of theory and experiment. This dissertation considers all equations and quantities in atomic units unless further specified otherwise.

¹More on this is discussed in Sec. 3.6.3

2.3 One-electron Schrödinger equation

Discovered by Schrödinger in 1926, the Schrödinger equation is a nonrelativistic solution to any quantum-mechanical system [23]. Consider the two-body problem with a nucleus and a single orbiting electron². The electron interacts with the nucleus via the Coulomb interaction described by the potential

$$V(\vec{r}) = -\frac{Ze^2}{4\pi\epsilon_0\vec{r}}. \quad (7)$$

The energy of the system is written using the Hamiltonian, where the potential is the Coulomb potential. This can be written as

$$H = \frac{p^2}{2m} - \frac{Ze^2}{4\pi\epsilon_0\vec{r}}. \quad (8)$$

All observable quantities correspond to hermitian operators that act on the wavefunction ψ . The wavefunction ψ is a normalized vector in Hilbert space, \mathcal{H} containing all information pertaining to the system. The Hamiltonian is thus a hermitian operator that satisfies the eigenvalue problem

$$\hat{H}|\psi\rangle = E_n|\psi\rangle \quad (9)$$

This is the Schrödinger equation. It can be applied to the two-body problem by substituting for the momentum its quantum operator analog. This results in the expression

$$\hat{H}\psi = \frac{-\hbar^2}{2m_e}\nabla^2\psi - \frac{Ze^2}{4\pi\epsilon_0\vec{r}}\psi = E_n\psi. \quad (10)$$

²It is assumed here that the nucleus is a point particle with infinite mass.

This is a second-order partial differential equation. Assuming spherical symmetry allows one to split the solution to any system into two separable parts; The radial part and the angular part.

$$\psi_{nlm} = R_{nl}(r)Y_m^l(\cos \theta) \quad (11)$$

For a two body hydrogenic system the solutions for each are written in terms of Laguerre polynomials and spherical harmonics. The radial equation has the following solutions [8]

$$R_{nl}(r) = \frac{2Z}{n^2} \sqrt{\frac{Z(n-l-1)!}{(n+l)!}} \left(\frac{2Zr}{n}\right)^l e^{\frac{-Zr}{n}} L_{n-l-1}^{(2l+1)}(2Zr/n) \quad (12)$$

Where Z is the nuclear charge atomic number, l is the angular momentum quantum number, n is the principle quantum number, and $L_{n-l-1}^{(2l+1)}(2Zr/n)$ is the generalized Laguerre polynomial (which is defined in Appendix D.3). The radial part of the solution can also be written in terms of the confluent hypergeometric function [?]]

$$R_{nl}(r) = \frac{1}{(2l+1)} \sqrt{\frac{(n+l)!}{(n-l-1)!2n}} \left(\frac{2Z}{n}\right)^{\frac{3}{2}} e^{\frac{-Zr}{n}} \times \left(\frac{2Zr}{n}\right) F\left(-(n-l-1), 2l+2, \frac{2Zr}{n}\right) \quad (13)$$

Additionally, the spherical harmonics are

$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi} \quad (14)$$

where l is the angular momentum quantum number, m is the magnetic quantum number, and $P_l^m(\cos \theta)$ is the associated Legendre polynomial (which is discussed

further in Appendix D.4). A key feature of the spherical harmonics which is utilized in Sec. 2.4 is that the spherical harmonics form a normalized complete set of orthogonal basis functions. The ability to separate the solution into two independent parts plays a crucial role in simplifying the task at hand and is exploited when evaluating integrals involving these wavefunctions in Chapter 3.

2.4 Integration Techniques

In this section a standard approach to solving the integrals pertaining to the various matrix elements required throughout is discussed. A matrix element of an operator V , such as $\langle \psi^n | V | \psi^m \rangle$, where V is a perturbed Hamiltonian, can be used to approximate the behaviour of quantum systems under small disturbances. Calculating their matrix elements gives insight to the system's behaviour, revealing properties such as transition rates or behaviour in an external field ???. The calculation of these matrix elements requires integration over the states as well as the operator acting on a state, which can involve radial and angular pieces. As discussed in the previous section, the ability to separate the solution into a radial part as well as an angular part significantly simplifies the solution process relative to a direct calculation. Instead of having to integrate a single function over all three dimensions in spherical coordinates, the solution is split into two steps which are calculated independently. Assuming that $V(\vec{r})$ can also be partitioned into $V(\vec{r}) = V_r V_\Omega$,

$$\langle \psi^n | V | \psi^m \rangle = \int_0^\infty r^2 \psi^n(r) V_r \psi^m(r) dr \int_\Omega \sin \theta \psi^n(\theta, \phi) V_\Omega \psi^m(\theta, \phi) d\Omega \quad (15)$$

where Ω represents the solid angles for θ and ϕ . The total problem can then be solved with the simple combination of calculated integrals.

$$\langle \psi^n | V | \psi^m \rangle = I_{\text{radial}} \cdot I_{\text{angular}} . \quad (16)$$

This section serves as a general introduction to the process of calculating matrix elements, but Sec. 2.4.1 and Sec. 2.4.2 discuss further the specifics and techniques used to calculate each piece of the solution.

2.4.1 The angular part

As mentioned previously, the angular part of the solution to any matrix element is

$$I_{\text{angular}} = \int_{\Omega} \sin \theta \psi^n(\theta, \phi) V_{\Omega} \psi^m(\theta, \phi) d\Omega . \quad (17)$$

Since hydrogenic wavefunctions are used to calculate matrix elements for the higher order Zeeman effect, integrals of this form with one or more spherical harmonics within the integrand appear. The calculation of the angular integrals is simplified by exploiting the nature of the spherical harmonic function. Since it forms a complete set of orthogonal basis functions, for any integral involving θ and ϕ , the integrand is written in terms of spherical harmonic functions. Using the orthogonality of the basis functions the integral of any two spherical harmonics is

$$\int_0^{\pi} \int_0^{2\pi} Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) d\theta d\phi = \delta_{ll'} \delta_{mm'} . \quad (18)$$

For all angular integrals where the spherical harmonics contain different angular momentum or magnetic quantum numbers, the result is zero. If given a scenario where three spherical harmonics are multiplied together (i.e, there are three spherical harmonic functions to integrate), the result can be expressed in terms of Wigner $3j$ -symbols according to [?]]

$$\int_0^\pi \int_0^{2\pi} \sin \theta \, d\theta d\phi \, Y_{l_1}^{m_1}(\theta, \phi) Y_{l_2}^{m_2}(\theta, \phi) Y_{l_3}^{m_3}(\theta, \phi) = \quad (19)$$

$$\sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

where the set of 6 parameters inside the brackets at the end of the expression is called a 3j symbol, which has the following definition [15]:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3} (2j_2+1)^{\frac{1}{2}} \langle j_1, m_1, j_2, m_2 | j_1, j_2, j_3 - m_3 \rangle \quad (20)$$

where $\langle j_1, m_1, j_2, m_2 | j_1, j_2, j_3 - m_3 \rangle$ is known as a vector coupling coefficient and is defined as [15]

$$\begin{aligned} \langle j_1, m_1, j_2, m_2 | j_1, j_2, j, m \rangle &= \delta(m_1 + m_2 + m) \times \quad (21) \\ &\sqrt{\frac{(2j+1)(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!}{(j_1+j_2+j+1)!}} \times \\ &\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!} \times \\ &\sum_z \frac{(-1)^z}{z!(j_1+j_2-j-z)!(j_1-m_1-z)!(j_2+m_2-z)!(j-j_2+m_1+z)!(j-j_1-m_2+z)!} \end{aligned}$$

This provides a closed form solution to any angular integrals involving three angular momenta. Since the spherical harmonics form a complete set of orthonormal functions, any expression in terms of 3 or more angular momenta is written by substituting linear combinations of spherical harmonics for the angular parts of the integral. The 3-j symbols are applied to such a system allowing one to avoid the direct integration and replace it with the closed form solution in equations (20) and (21).

The ability to substitute the angular part of the integral with a closed form solution in terms of Wigner-3j symbols allows for easy computation providing exact analytic solutions to part of the problem being discussed.

2.4.2 The radial part

The radial part of the hydrogenic wavefunction is $R_{nl}(r)$, stated in equation (12). The perturbed wavefunctions will thus also resemble a similar form since they serve as small corrections to the original solution³. Removing the terms not dependant on r outside of the integrand (which are dependant on the perturbed wavefunction being investigated as well as the angular momentum and principle quantum numbers), a series of integrals that resemble the following form emerge⁴

$$I_{\text{radial}} = \int_0^\infty r^j e^{-\alpha r} dr \quad (22)$$

where $j \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}^+$. This integral is a special one, being in the form of the gamma function integral⁵, $\Gamma(z)$. The Gamma function integral has the following solution

$$\int_0^\infty r^j e^{-\alpha r} dr = \frac{\Gamma(j+1)}{\alpha^{j+1}} \quad (23)$$

where as long as $j+1 \in \mathbb{Z}^{0,+}$, the gamma function simplifies to a factorial, and the following relation emerges;

$$\int_0^\infty r^j e^{-\alpha r} dr = \frac{j!}{\alpha^{j+1}}. \quad (24)$$

³See Sec. 2.5 for more details.

⁴This is shown explicitly in Sec. D.3.

⁵This function is technically Gamma function-esque, but can be manipulated into the form of the Gamma function. This is shown in greater detail in Sec. D.2.

Fitting the integrals to the gamma function not only makes the task of computing the integrals significantly easier, but also makes the integration computationally stable. The solution provides a simple formula to implement when calculating radial integrals, and allows one to skip the implementation of well known numerical integration methods such as Euler's method or any Runge-Kutta methods [22]. While these methods are powerful, they do have significant drawbacks compared to the derived analytical solutions due to error propagation and machine-precision and computation time.

The ability to compute these radial hydrogenic wavefunction integrals exactly is not only a satisfying result mathematically, but opens the door for more difficult problems in atomic physics where machine error is the determining step in the accuracy of a high-precision calculation⁶.

2.5 Perturbation Theory

To quantify how a magnetic field affects atomic energy levels, the shifts induced by the Zeeman effect must be determined. Since this interaction modifies the Hamiltonian slightly compared to the field-free case, perturbation theory is utilized to provide a systematic way to approximate the new eigenvalues and eigenstates.

To start we write the perturbed Hamiltonian as a sum of two terms;

$$\hat{H} = H^0 + \lambda H' \tag{25}$$

where H^0 is the unperturbed Hamiltonian and H' is the perturbation. λ is a perturbation expansion parameter used to group together terms involving some power of λ . expanding the wavefunction and the energy of the system as a power series in λ gives

⁶Such problems include the calculation of Hylleraas wavefunctions for three body atomic systems such as H^- , He, or Li^+ , where the radial integrals follow a similar but more complex closed form solution.

[19]

$$\psi_n = \psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2 + \dots \quad (26)$$

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots \quad (27)$$

where each superscript is a higher order correction to the original wavefunction. Each higher order term is less impactful to the solution (by a factor of λ) but still contributes a small amount. The first order perturbation equation is

$$(H^0 - E^0) |\psi^1\rangle = (V - E^1) |\psi^0\rangle . \quad (28)$$

To find the first order correction to the energy, E^1 , simply multiply through by $\langle\psi^0|$. This gives

$$\langle\psi^0|H^0 - E^0|\psi^1\rangle = \langle\psi^0|V|\psi^0\rangle - E^1\langle\psi^0|\psi^0\rangle . \quad (29)$$

The first term is zero because $(H^0 - E^0) |\psi^0\rangle = 0$. $\langle\psi^0|\psi^0\rangle = 1$ because of the normalization condition for the original unperturbed wavefunction. The normalization condition states that for each order of the perturbation

$$\langle\psi_n|\psi_n\rangle = 1 . \quad (30)$$

But since $|\psi^0\rangle$ is already normalized, the following condition must be imposed;

$$\langle\psi^0|\psi^1\rangle = 0 . \quad (31)$$

Which ensures that the two states are orthogonal. The end result is

$$E^1 = \langle \psi^0 | V | \psi^0 \rangle . \quad (32)$$

In general, any higher order energy can be found with the following expression, which stems from the same process performed here for the first order energy.

$$E^n = \langle \psi_{n-1} | V | \psi_0 \rangle . \quad (33)$$

Similarly to the standard quantum mechanical Hamiltonian, the perturbation equation is a second order partial differential equation. However, the perturbing term in the equation as well as the higher order corrections to the energy serve as inhomogeneous terms. To solve this it is assumed that the higher order wavefunction is of the form of a power series in r , and the method of Frobenius is used check if a solution of this form exists. This is discussed in the following section.

To calculate a higher order correction to the wavefunction, a similar process is repeated. This time starting with the second order equation

$$(H^0 - E^0) |\psi^2\rangle = - (V - E^1) |\psi^1\rangle + E^2 |\psi^0\rangle . \quad (34)$$

As it can be seen above, the second order solution requires the solution from the first order equation. The requirement of solving the previous correction in order to get the current one makes the processing of calculating higher order corrections quite laborious, and developing a method to solve for higher order corrections computationally removes the burden of solving lengthy equations by hand. The adaptation of solving the higher order perturbation equations computationally is discussed in Sec. A.

2.6 Recursion relations

When solving the perturbation equation, one ends up with an inhomogeneous second order partial differential equation. To solve for the radial piece of the corrected wavefunction, one can use the method of Frobenius. The method of Frobenius was developed by Ferdinand Georg Frobenius in 1869 and is a method of solving equations of the following form [25]:

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + k^2u = 0 . \quad (35)$$

The method of Frobenius involves assuming the solution is of the form of a power series, and then solving for terminating recursion relations to check if a finite solution exists. The method of solving these recurrence relations is what will be highlighted in this section.

A recurrence relation is an equation where the n^{th} term of a sequence is dependant on some combination of its previous terms in the sequence. A simple example of a recurrence relation is the Fibonacci numbers [17].

$$F_n = F_{n-1} + F_{n-2} . \quad (36)$$

But these equations can only be solved if there is an indicial equation. An indicial equation is an equation which gives the condition that somewhere, the sequence stops, or is no longer defined by its previous term. In this case, it is the equation for the lowest starting power of r in the power series solution. A good example of this is the factorial function. The n^{th} factorial number is determined by

$$n! = n(n-1)! \quad (37)$$

where $0!$ is not dependant on $-1!$, but is equivalent to 1. The recurrence relation stops here. If the recurrence relation did not have an indicial equation, and kept referring to the previous iteration forever, there would be no solution to $n!$. Recurrence relations can be written going downwards, as shown by the Fibonacci numbers and the factorial examples, or they can be written going upwards. This is best shown by the following recurrence relation

$$\sum_{j=0}^{\infty} \left[Z(j-1)a_{j-1} - \frac{j(j+1)}{2}a_j \right] r^{j-2} = - \left(\frac{1}{r} + Z \right) . \quad (38)$$

The recurrence relation can be identified by writing out values for different j

$$\begin{array}{ll} j = 0 & -Za_{-1} = 0 \\ j = 1 & a_1 = -1 \\ j = 2 & Za_1 - 3a_2 = -Z \\ & -Z + 3a_2 = -Z \\ & a_2 = 0 \\ j = 3 & 2Za_2 + 6a_3 = 0 \\ & a_3 = 0 \\ \vdots & \vdots \end{array} \quad (40)$$

It can be seen here that for every term after a_1 , the series gives zero. The recurrence relation was used for each iteration including the indicial equation for $j = 0$ and it has been shown that the sequence terminates, providing a non-divergent answer. If this was the solution to the PDE given by the Frobenius method, it would mean that the solution is in the form of a power series and the PDE is solved using the closed form solution.

2.7 The Dalgarno Interchange Theorem

Now that the perturbation equation, the matrix elements associated with it, and the ground work for the Hydrogenic wavefunctions has been laid out, a way to confirm the calculations is desirable. This presents itself as the Dalgarno Interchange Theorem. The theorem states that given any two perturbations V and W , thier first order equations can be written as

$$(H^0 - E^0) \psi^1 + V\psi^0 = E^1\psi^0 \quad (41)$$

$$(H^0 - E^0) \varphi^1 + W\psi^0 = F^1\psi^0 . \quad (42)$$

Multiplying equation (41) by φ^1 and equation (42) by ψ^1 and then integrating and subtracting (41) from (42) gives

$$\langle \psi^0 | V | \psi^1 \rangle - \langle \psi^0 | W | \varphi^1 \rangle = 0 . \quad (43)$$

Therefore

$$\langle \psi^0 | V | \psi_1 \rangle = \langle \psi^0 | W | \varphi^1 \rangle . \quad (44)$$

So given any two perturbations, the respective perturbation equations can be solved to retrieve φ^1 and ψ^1 . Using these solutions the two matrix elements can be computed with the perturbed wavefunction and its opposing perturbation from the opposing equation, and the result should be the same! This method serves as a check that the solutions are correct. Once the perturbing terms for the magnetic dipole operator and the relativistic correction have been calculated, the Dalgarno Interchange theorem can be used to verify that the calculations were correct.

The synthesis of all these methods is reached by perturbing the nuclear charge of the hydrogen atom to model a ${}^3\text{He}^+$ atom, and then solving the perturbation equation to find the higher order Zeeman effects. This will require the calculation of matrix elements for $\frac{1}{r}$ for relativistic corrections, and for the cubic Zeeman effect, the r^2 matrix element for magnetic effects. these matrix elements can be computed by utilizing the discussed integration techniques above, as well as verify the calculated results for each using the Dalgarno interchange theorem.

CHAPTER 3

The Higher Order Zeeman Effect

3.1 Overview

In this chapter the Zeeman Effect is introduced, and the motivation, direct applications, and the higher order Zeeman Effect is discussed. The main focus of this chapter is to show the effect of the quadratic Zeeman Effect, and show how using the magnetic dipole operator in conjunction with the relativistic corrections to ${}^3\text{He}^+$ yields a cubic Zeeman Effect. The effects of both the quadratic and cubic corrections are discussed in great detail, and the impact of the effect on high precision measurements is displayed for various magnetic field strengths.

Sec. 3.2 starts with the history of the Zeeman effect, its origins and discovery. Afterwards the motivation for the project in Sec. 3.3 is discussed. Here, some current experiments in the field such as the $g - 2$ experiment conducted at the Max Planck Institute as well as applications to high-precision magnetometry are highlighted. Some additional applications in the field of atomic physics such as ... are introduced as well. In Sec. 3.4, the ordinary Zeeman effect is discussed, introducing its theory and application to atomic systems such as ${}^3\text{He}^+$. After introducing the ordinary Zeeman effect the quadratic Zeeman effect is introduced, where it is derived using the canonical momentum and a description of its impact on an atom subjected to a magnetic field is given. Moving towards higher order systems, the cubic Zeeman effect is introduced. Starting with the effects that contribute to the cubic Zeeman effect such as the magnetic dipole operator in Sec. 3.6.2 and relativistic corrections

to the $^3\text{He}^+$ ion in Sec. 3.6.3, these effects are combined to yield a B^3 contribution to the energy splitting within the presence of an external magnetic field. Afterwards, Sec. 3.7 discusses the results of the calculation and its applications.

3.2 History

The Zeeman effect was first introduced by Pieter Zeeman, who discovered in 1896 that in the presence of a static magnetic field, spectral lines could be split into many components. After the discovery of quantum mechanics, the behaviour was found to be described as a perturbation of the Hamiltonian using the magnetic moment of the atom and the magnetic field.

Since its discovery, the Zeeman effect has played a large role in the field of atomic physics and magnetometry, which is the study of the intensity of magnetic field across space and time. There have been several calculations to include the relativistic corrections [26, 27], field inhomogeneities, and quadratic effects in hydrogenic systems [18]. However, little is known about its behavior in helium atoms such as $^3\text{He}^+$ and ^3He , which is of key interest in magnetometry and the muon magnetic moment anomaly ($\mu_g - 2$), for which there is a 5.0σ discrepancy [9] with the standard model prediction.

3.3 Motivation

3.3.1 The $g - 2$ experiment

The Dirac equation is a very successful and well studied equation in quantum mechanics. Its success comes from its ability to predict 2 important phenomena; the existence of antimatter and the magnetic dipole moment of the electron. The Dirac equation predicts that the magnetic dipole of the electron should be twice that of the classical prediction. This result is expressed in terms of the g-factor which the Dirac equation predicts is equal to 2. While the Dirac prediction is much closer to experimental findings, there is still a difference between the experimentally measured value

of g and the equations prediction. This is called the $g - 2$ anomaly. The anomaly is represented by

$$a = \frac{g - 2}{2} . \quad (1)$$

The discrepancy of g is caused by higher-order contributions from quantum field theory and to this day is yet to be properly explained.

$$a_{\mu}^{\text{SM}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{EW}} + a_{\mu}^{\text{hadron}} \quad (2)$$

The first two terms can be derived from first principles, but the hadronic term cannot be calculated precisely on its own and is estimated from experimental results. The effort to measure the muon magnetic moment precisely is an active area of research. The work presented in this thesis aids in the investigation of the $g - 2$ anomaly by providing corrections to the Zeeman splitting in $^3\text{He}+$, the element used in the magnetometry experiment to measure the anomaly.

3.3.2 High-precision magnetometry

3.3.3 Connection to Atomic Physics

3.4 The Zeeman effect in Hydrogen

3.5 The Zeeman effect in Helium-3

3.6 The higher-order Zeeman effect

3.6.1 The quadratic Zeeman effect

The non-relativistic Hamiltonian for an n electron system in atomic units can be expressed as follows

$$\hat{H} = \sum_{i=1}^n \frac{(\vec{p}_i - e\vec{A}_i)^2}{2m} + V \quad (3)$$

using the canonical momentum instead of the classical momentum is essential to account for electromagnetic interactions. The vector potential \vec{A} can be described in terms of the magnetic field

$$\vec{A} = \frac{B}{2} (y\hat{x} - x\hat{y}) \quad (4)$$

which when expanded gives us the operator corresponding to the quadratic Zeeman effect

$$\hat{H}_Z^{(2)} = \frac{B^2 e^2}{8m} \sum_{i=1}^n (x_i^2 + y_i^2) \quad (5)$$

This operator can be expressed in spherical coordinates, written in terms of Legendre

Polynomials

$$\hat{H}_Z^{(2)} = \frac{B^2 e^2}{12m} \sum_{i=1}^n r_i^2 (P_0(\cos \theta) - P_2(\cos \theta)) \quad (6)$$

${}^3\text{He}^+$ is a system which contains only one electron, so we can drop the summation which accounts for all electrons to get our final quadratic Zeeman operator

$$\hat{H}_Z^{(2)}({}^3\text{He}^+) = \frac{B^2 e^2}{12m} r^2 (P_0(\cos \theta) - P_2(\cos \theta)) \quad (7)$$

3.6.2 The magnetic dipole moment operator

The magnetic dipole moment operator represents the interaction of a magnetic dipole moment with an external magnetic field. It is described via the following relation

$$Q_{M1} = \mu_B \left(1 - \frac{2p^2}{3m^2 c^2} + \frac{Ze^2}{3mc^2 r} \right) \vec{\sigma} \cdot \vec{B} \quad (8)$$

Where μ_B is the Bohr magneton

$$\mu_B = \frac{e\hbar}{2mc} \quad (9)$$

The second term in the brackets of the magnetic dipole moment operator accounts for the relativistic correction to the kinetic energy of the electron, and the third term is the potential energy due to the Coulomb interaction between the electron and the nucleus. The first term corresponds to the ordinary Zeeman Effect, which does not contribute to the sum over states due to orthogonality.

The ordinary Zeeman effect contributes to Q_{M1} in ${}^3\text{He}^+$ because it has non-zero spin due to the missing electron. For systems such as ${}^3\text{He}$, the ordinary Zeeman effect

will not contribute.

3.6.3 The relativistic correction to ${}^3\text{He}^+$

Combining the magnetic dipole moment with the quadratic Zeeman operator, we can write down the relativistic corrections for ${}^3\text{He}^+$. Written in terms of pseudostates the relativistic correction is

$$C_{\text{rel}}^{(2)} = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\langle \psi_0 | H_Z^{(2)} | \psi_n \rangle \langle \psi_n | Q_{M1} | \psi_0 \rangle}{E_0 - E_n} \quad (10)$$

3.7 Results

CHAPTER 4

Conclusion and Future Work

4.1 Overview

4.2 Consolidation of Results

4.3 Synthesis of Conclusions

4.4 Future Work

4.5 Theoretical and Practical Impact

APPENDIX A

Methods of solving for higher-order perturbations programmatically

APPENDIX B

Derivation of the Hydrogen Atom

APPENDIX C

Derivation of the Recursion Relations

APPENDIX D

Special Functions

D.1 Overview

D.2 Gamma Function

D.3 Laguerre Polynomials

D.4 Legendre Polynomials

D.5 Spherical Harmonics

D.6 Confluent Hypergeometric Function

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- [2] Codata value: electron mass.
- [3] Codata value: elementary charge.
- [4] Codata value: fine-structure constant.
- [5] Codata value: reduced planck constant.
- [6] Codata value: speed of light in vacuum.
- [7] Codata value: vacuum electric permittivity.
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