

Learning Augmented Joint-Space Task-Oriented Dynamical Systems: A Linear Parameter Varying and Synergetic Control Approach

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Abstract—In this paper, we propose a stable joint-space dynamical system that captures desired behaviors in joint-space while converging towards a reachable target in task-space. Our method is fast to compute and smoothly moves through classic kinematic singularities by avoiding the use of the pseudo-inverse Jacobian; moreover, it Lyapunov-stably converges towards its task-space attractor. To encode complex joint-space behaviors while meeting these stability criteria, the dynamical system is constructed as a Linear Parameter Varying (LPV) system combining different motor synergies, and we provide a method for learning these synergy matrices from demonstrations. Specifically, we use dimensionality reduction to find low-dimensional synergy regions, and then estimate the parameters of the corresponding synergies by solving a convex semi-definite optimization problem that minimizes the joint velocity prediction error from the demonstrations. Our proposed approach is validated on a variety of motions for a 7-DOF KUKA LWR4+ robot arm.

I. INTRODUCTION

We'll remove the subsection headings later.

Some comments:

- Change PbD to LfD everywhere

A. Motivation for learning joint+task-space behaviors

See comments in tex

Robot motion planning in joint-space has long been a major field of study [1]. For manipulation problems with an objective defined in task space (i.e. target or desired trajectory), we can often find a myriad of joint-space trajectories to achieve the same task-space goal. In many cases, however, certain joint-space trajectories are favored over others; for example, when we expect the robot to follow a desired joint-space behavior or “style”, as illustrated in Fig. 1.

Learning robot motion generators that follow desired joint and/or task-space behaviors is, in fact, the central focus of Programming by Demonstration (PbD) [2] [3]. Historically, much of the original work in PbD focused on learning motions solely in joint-space by parameterizing motions using GMMs [4], HMMs [5], GPs [6], or other probabilistic models [7], [8]. Concurrently, a significant body of work from the graphics community was targeted at learning joint-space motion “styles” from human motion capture data, through a variety of latent-space and stylistic probabilistic motion models [9], [10]. Yet, over time, research shifted



Fig. 1: Two robot motions in **joint-space** accomplishing *similar* behavior in **task-space** (pouring chips in a bowl). The left example avoids a known obstacle in its workspace, while the right one does not.

to pure task-space learning [11], [12], [13], which has two main advantages over pure joint-space: *generalization* and *re-usability*. By learning motions in task-space, one can concisely encode the task-relevant features and generate new motions which meet these task-space criteria. Furthermore, a task-space trajectory is robot-independent, and thus can be reused across robots with differing morphologies.

A major drawback of these initial task-space methods was that despite learning motions similar to the demonstrations, there was no guarantee when applying the learned motion to a new problem (e.g. a different initial/target position) that the end-effector would stably converge to its ultimate task-space target. This led to the introduction and subsequent popularity of task-space learning techniques such as SEDS [14], τ -SEDS [15] and LMDS [16] which encode task-space behaviors as asymptotically stable dynamical systems (DS) guaranteed to converge towards a desired task-space target.

For all the previously mentioned task-space techniques, the robot is ultimately controlled by projecting the desired task-space velocity into joint-space via Jacobian Pseudo-Inverse based Inverse-Kinematic (IK) approximations and variants thereof [1]. When the main focus is on executing a complex task-space behavior, regardless of a specific joint-space constraint, this approach has been deemed sufficient [17], [18]. However, for other applications, such an approach yields significant problems [19]. First of all, finding the pseudo-inverse is computationally taxing. Moreover, when the Jacobian matrix cannot be inverted (i.e. when the robot is near a singularity) its behavior becomes erratic, requiring layers of additional engineering to generate smooth trajectories and ensure the desired task-space behavior. These problems encapsulate the main source of inaccuracies in tasks that require generating fast dynamical motions, such as catching or reaching for moving objects [20], [21].

Although DS-based task-space learning techniques ensure target convergence and stability, they still suffer from this reliance on IK methods, possibly generating erratic or discontinuous motions during execution. Further, for certain tasks, rather than generating arbitrary joint-space trajectory returned by the IKs solver, it might be desirable to learn

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behaviors and “styles” in joint-space directly. For example, throwing a ball is a motion less characterized by task-space behaviors (the ball moving backwards and then forwards) and more characterized by the joint behaviors: the elbow folds, the shoulder moves back and then swings forward, followed by the elbow straightening and the wrist snapping down. For a visual example of the need for such combined task/joint-space behavior learning, see Fig. 1. This is precisely the shortcoming associated with LfD in task-space: it fails to learn the joint-space behavior of the system. The current formulation of the DS-based learning techniques introduced thus far (e.g. SEDS, τ -SEDS LMDS, etc.) do not allow for integrating combined task/joint-space behavior learning.

B. Approaches that tackle the joint+task problem

See comments in tex

[22] notably attempted to address this problem by learning separate motion policies in task space and the null space of the Jacobian (which would not affect task-space position), and driving the robot with a weighted sum of the two. However, this approach no longer guarantees convergence to the desired task-space target and is still reliant on computing the pseudo-inverse Jacobian.

[23] learns from demonstration in both joint and task space within the “thin-plate spline” trajectory warping framework, but does not propose a dynamical system for generating motions.

C. Our proposed Approach

Dynamical System, Why?

Linear Parameter Varying, Why?

Lower-Dimensional Manifold, Why?

In this work, we seek to devise an augmented Joint-space Task-oriented Dynamical System (JT-DS) that not only incorporates task-space attractors, but also avoids the problems generated by pseudo-inverse approximations. [24] proposed an approach with similar properties to this desiderata, where two concurrent DS, one in task-space and one in joint-space, are modulated by enforcing kinematic coherence constraints to avoid singularities. The resulting DS avoids singularities through generalization of the pseudo-inverse approximations. However, because the two DS have their own unique attractors, the non-linear interaction between them imposed by the kinematic constraints does not ensure that the combined DS has a unique attractor. This gives rise to spurious attractors or cycles, and thus must be carefully tuned in order to avoid them. In this work, we depart from any modulation or combination of DS and instead propose a novel dynamical system which:

- 1) computes a **complex motion in joint space** that provably and stably converges to a **fixed task-space target**.
- 2) **avoids** the need for computing **pseudo-inverses**, and cleanly moves through kinematic “singularities”.
- 3) is formulated such that complex **joint-space behaviors** can be **learned** from demonstrations in **synergy space**.

The most similar approach to our proposed DS is the Jacobian transpose (JT) control [25] method. JT control is

an inverse kinematics method that yields a dynamical system in joint space $\dot{q} = f(q)$ which converges stably (in the sense of Lyapunov) over time to a desired end-effector position x^* , without the need for pseudo-inverse computations. It shares some of our approach’s advantages: fast computation and provable task-space stability. However, despite some previous work designing velocity adjustments by hand [26], this is the first known work to employ a JT system to learn behaviors from demonstrations.

This paper is organized as follows. Section II formalizes the problem. The proposed dynamical system is introduced in Section III. In Section IV, a probabilistic model is introduced to approximate the parameters of the dynamical system. In addition, a convex optimization problem is formalized to estimate these parameters. The effectiveness of the proposed method is shown through simulations and experiments on a 7-DOF robot-arm in Section V. The paper is finalized with a discussion over our method and results in Section VI.

II. PROBLEM STATEMENT

Consider a robotic system with d task-space dimensions and m degrees of freedom. We direct the system using a joint position or joint velocity controller, which can have joint position limits and a maximum joint velocity. We are further provided with a set of N demonstrated joint-space trajectories $D = \{\{q_{t,n}, \dot{q}_{t,n}\}_{t=1,\dots,T_n}\}_{n=1,\dots,N}$, where T_n is the number of the sample points of the n^{th} demonstration. We refer to the system’s joint-space position as $q = [q^1 \dots q^m]^T \in \mathbb{R}^m$, and to its task-space position as $x \in \mathbb{R}^{d,1}$. The kinematics of the robot are assumed to be known, hence, the robot’s forward kinematics is indicated by $x = H(q)$ and its Jacobian is $J(q) = \frac{dx}{dq} \in \mathbb{R}^{m \times d}$.

We wish to formulate a dynamical system $\dot{q} = f(q)$ which satisfies the following two criteria:

- (I) The dynamical system must be Lyapunov stable² with respect to a fixed task-space target x^* . This can be expressed by ensuring that the following Lyapunov function

$$V(q) = (H(q) - x^*)^T P (H(q) - x^*) \quad (1)$$

is stable; i.e. $\dot{V}(q) \leq 0 \forall q \in Q$ and $V(q) = 0 \forall q \in Q^*$. Where $Q = \{q | q_{min}^i < q^i < q_{max}^i, \forall i \in \{1, \dots, d\}\}$ and $Q^* = \{q | H(q) = x^* \wedge q \in Q\}$. $P \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix. $V(\cdot)$ can be thought of as a metric for the task-space distance-to-go.

- (II) The dynamical system should encapsulate the desired joint-space behaviors such that the following metric is minimized

$$e_{total} = \frac{1}{NT_n} \sum_{n=1}^N \sum_{t=0}^{T_n} \|\dot{q}_{d;t,n} - f(q_{t,n})\| \quad (2)$$

¹For sake of brevity and simplicity, the time index, t , is dropped throughout the paper.

²Unless otherwise specified, “stability” in this paper always refers to global Lyapunov stability. We make no claim to proving global asymptotic stability, which is in fact impossible to achieve in a joint-constrained kinematic system. For example, any finite-size manipulator will always have regions beyond its reach that it cannot converge to.

where \dot{q}_d is the desired “true” velocity, and $f(\cdot)$ is the motion generation policy.

We make two decisions by the choice of behavior error metric (2). First, the metric implicitly suggests that behaviors are defined by expert motions of the behavior, and that fulfilling a behavior means moving similarly to the experts’ motion. Second, the type of feature of the trajectory that is being minimized is important. If instead the error feature were joint position distance $\sum_i \|q_{d,i} - q_i\|$ where q_d is a vector of demonstrated positions, then executing a “behavior” would imply mimicking position, but not velocity. Alternatively, if the error feature were joint velocity direction $\sum_i \left| \frac{\dot{q}_{d,i}}{\|\dot{q}_{d,i}\|} - \frac{\dot{q}_i}{\|\dot{q}_i\|} \right|$ mimicking a “behavior” would involve following the motion profile, but not the motion’s speed (so for example a slap and a push would exhibit the same behavior). By choosing the combined direction and magnitude of the joint velocity as the error, we are choosing to mimic the magnitude and direction of the motion, which [9] suggests is visually most similar to the human definition of “joint motion style”.

III. AUGMENTED JOINT-SPACE TASK-ORIENTED DYNAMICAL SYSTEM

Change semi-definite to definite everywhere

Change overall $A(q)$ to $\mathcal{A}(q)$

Introduce ‘behavior synergies’ here

We propose the following augmented Joint-space Task-oriented Dynamical System (JTDS) to achieve the two criteria presented in 1 and 2.

$$\dot{q} = f(q) = -A(q)J^T(q)P(H(q) - x^*) \quad (3)$$

where $P \in \mathbb{R}^{d \times d}$, and $A(q) \in \mathbb{R}^{m \times m}$ is constructed using the Linear Parameter Varying system paradigm [27], [20], where the overall $A(q)$ is a changing linear combination of constant matrices, each of which encodes a local joint-space synergy.

$$A(q) = \sum_{k=1}^K \theta_k(q) A_k \quad (4)$$

where $A_k \in \mathbb{R}^{m \times m}$ are the different synergies and $\theta_k(q) \in \mathbb{R}^1 \forall k \in \{1, \dots, K\}$ are the scheduling parameters³ modulating each local synergy through time as well as space. Based on (3), one can utilize different synergies in different regions, and compose them to create a more complex multi-behavior motion.

Before proving that the proposed dynamical system satisfies all of the criteria, let us establish an intuitive understanding of the components of the system. While it may seem daunting at first, each of the elements has a straightforward explanation. One can intuitively understand the control law as follows: $(H(q) - x^*)$ denotes the position error w.r.t. the task target (warped according to P), and by multiplying that error

³The scheduling parameters can be a function of time t , the states of the system q or external signals $d(t)$, i.e. $\theta_k(t, q(t), d(t))$. In this paper, we only consider it as a function of the states of the system. It is noteworthy that the presented stability proof can easily be extended for $\theta_k(t, q(t), d(t))$.

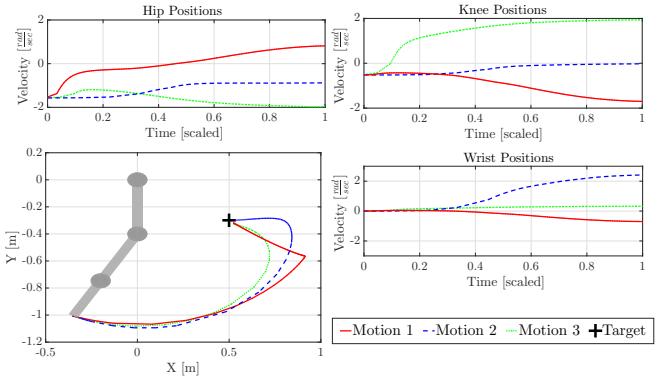


Fig. 2: Three example 3-DOF motions (A, B, C), each with a different constant joint augmentation matrix A (emphasizing the hip, knee, and ankle respectively), A: $A(q) = \text{diag}(5, 1, 1)$, B: $A(q) = \text{diag}(1, 5, 1)$, and C: $A(q) = \text{diag}(1, 1, 5)$. On the left, the task-space traces of each motion. On the right, the time-scaled joint positions of each joint. Each motion tends to use its “primary” joint most and uses the other available joints to compensate for what the primary joint cannot do.

by the transposed Jacobian $J^T(q)$, the error is projected into joint space (similar to Jacobian transpose control [25], [28]). The positive semi-definite matrix $A(q)$ warps the resulting joint-space velocity; Fig. 2 illustrates the effects of $A(q)$ on the generated motion. Thus the controller can be thought of as a proportional controller in joint space. Lastly, we refer to the P matrix as the **task augmentation matrix** (because it augments the task error, and thus the direction of motion) and $A(q)$ as the **joint augmentation matrix** (because it augments the outputted joint velocities).

Proposition 1: The flow of motion generated by the dynamical system (3) accomplishes criteria (I) if (3) meets the following constraints.

$$\begin{cases} 0 \prec P \\ 0 \preceq A_k & \forall k \in \{1, \dots, K\} \\ 0 \leq \theta_k(\cdot) \end{cases} \quad (5a)$$

Proof: See Appendix I. ■

It is worth mentioning that the constraints in (5a) do not ensure that JT-DS (3) asymptotically converges to the desired target position x^* , as $\exists q \in Q - Q^*$ such that both $\dot{V}(q) = 0$ and $H(q) \neq x^*$. This happens if the target position is kinematically unreachable; i.e. for each task-space axis i , $(J^T(q)P(H(q) - x^*))_i = 0$ (meaning that moving joints increases the value of (1)), or the joint has reached a kinematic limit. Instead, the proof in Appendix I shows that the system converges in the sense of Lyapunov, meaning that its distance to the attractor is always monotonically decreasing.

Criterion (II) (i.e. encoding specific joint-space behaviors) is achieved by embedding the desired dynamics in the matrices $A_k(q) \forall k \in \{1, \dots, K\}$. We describe in the next section an approach to automatically learn these matrices from demonstrated data.

IV. LEARNING JOINT-SPACE TASK-ORIENTED DYNAMICAL SYSTEM

The behavior of the JT-DS algorithm can be best understood through the lens of synergy control [?]. In robotic synergy control, a robot's movements can be decomposed into a small number of synergies: principal components of the actuation-space that are sufficient to accurately recreate the desired robotic behaviors. In our case, the synergies are represented by A_1, A_2, \dots, A_k , and $A(q)$ represents the resulting motion constructed from a superposition of different synergies (through (3)).

A central question becomes how to modulate the synergies in different regions to yield our desired behavior. First, we assume that our desired behavior can be efficiently described⁴ using some sub-manifold of the joint-space, called the *synergy space*. We would like to define the robot's policy in synergy space such that in different regions of the space, we will utilize different synergies. We are thus left with three problems: finding an underlying synergy-space Z in which the behavior can be accurately controlled (defined by a mapping $\phi : Q \rightarrow Z$), finding a policy for modulating the synergies in different regions of the synergy space (defined by $\theta_k(q) \forall k \in \{1, \dots, K\}$), and finding parameters for the synergies themselves (defined by $A_k \forall k \in \{1, \dots, K\}$). Our choice of parameters must also obey the constraints laid out in (5a). We propose the following 3-step procedure.

- (I) We first construct our "synergy-space", which provides us a lower-dimensional manifold through which to control the robot, by projecting the demonstration data (i.e. collections of joint positions q) into a lower-dimensional embedding $\phi(q) \in \mathbb{R}^{\delta \leq d}$. In each experiment, we used either Principal Component Analysis (PCA) [29] or Kernel PCA (KPCA) [30].
- (II) We then jointly estimate the optimal number K of synergy "regions" and the parameters of the scalar functions that determine the scheduling parameters $\theta_k(q)$ for weighting these synergies, by fitting a Gaussian Mixture Model (GMM) on the projected joint positions $\phi(q)$ seen in the demonstrations.
- (III) Finally, once the local synergy regions have been found (described by each of the Gaussian distributions $\theta_k(q)$), we compute the corresponding joint synergy matrices $A_k \forall k \in \{1, \dots, K\}$ for each region by formulating a convex optimization problem that finds the optimal set of A_k 's that minimize the overall velocity error with respect to the demonstrations (2).

A. Embedding Joint Configurations in Lower-Dimensional Space

The search for a lower-dimensional representation of the joint positions stems from the desire to identify regions of *similar* "joint behaviors" (defined in Section I), in which it would be appropriate to define synergies. It is important to

⁴Given some original space A and some behavior policy $\pi_A(a)$ in A , we say that a latent mapping $\phi(\cdot)$ and latent space $B : \{b = \phi(a) | a \in A\}$ "efficiently describe" π_A if there exists some policy $\pi_B(b)$ in B such that we can deterministically reconstruct $\pi_A(a)$ given $\pi_B(\phi(a))$

understand that distances in joint space (defined as $\|\Delta q\|$) might not numerically reflect what a human intuitively perceives as the distance between joint configurations. Motor control studies have postulated that most human arm motions, be they reaching motions or following trajectories as straight/curved lines, are the result of compromising between planning a straight line in the task space and a straight line in the joint space [31], [32]. This suggests that human arm motion in general tends to move on a plane, and thus can be represented in such lower-dimensional space.

To this end, any type of topology-preserving latent-space embedding or dimensionality reduction approach can theoretically be used for $\phi(q)$. In this work, we choose to construct the projection $\phi(q)$ of q through a linear map $W : q \in \mathbb{R}^d \rightarrow \phi(q) \in \mathbb{R}^{\delta \leq d}$ given by $\phi(q) = Wq$, where the projection matrix $W \in \mathbb{R}^{\delta \times d}$ is computed through Principal Component Analysis (PCA). Although the projected dimensions extracted from PCA are linear combinations of the original variables, this has been proven to be sufficient to encapsulate the correlations in joint-space data [33].

B. Discovering Local Behavior Synergies

Given the set of projected joint position trajectories $D = \{\{\phi(q_{t,n})\}_{t=1, \dots, T}\}_{n=1, \dots, N}$ where $\phi(q_{t,n})$ is the lower-dimensional embedding of $q_{t,n}$, t is the time-step and N is the number of demonstrations, we seek to learn a set of regions of distinct local synergies, each defined by their corresponding scheduling parameter $\theta_k(q) = \theta'_k(\phi(q))$. Moreover, we would like for our scheduling parameters $\theta'_k(\phi(q))$ to have the following properties: (i) $0 \prec \theta'_k(\phi(q))$ and (ii) $\sum_{k=1}^K \theta'_k(\phi(q)) = 1$.

Scheduling parameters for LPV systems with such properties have been modeled in previous work as probability distributions [20], [21]. In this work, we adopt this approach and use a GMM to estimate the joint distribution over the projected joint positions⁵, $p(\phi(q)) = \sum_{k=1}^K \pi_k \mathcal{N}(\phi(q); \mu_k, \Sigma_k)$, where π_k are the prior probabilities and $\{\mu_k, \Sigma_k\}$ are the mean and covariance matrices that parametrize the k -th multivariate Gaussian distribution. Each such distribution represents a local region of projected joint positions $\phi(q)$, and will be used to construct the scheduling parameter θ'_k of our LPV system. We will define $\theta'_k(\phi(q))$ as $p(k|\phi(q))$:

$$\theta'_k(\phi(q)) = \frac{\pi_k \mathcal{N}(\phi(q); \mu_k, \Sigma_k)}{\sum_{k=1}^K \pi_k \mathcal{N}(\phi(q); \mu_k, \Sigma_k)} \quad (6)$$

which is the probability of the projected joint-position $\phi(q)$ belonging to the k -th synergy. Therefore, each synergy region is associated with a Gaussian component of the GMM, cumulatively describing all the synergy regions of the dynamical system. We use the standard Expectation Maximization (EM) training algorithm to estimate the parameters of the GMM.

C. Estimating the Synergy Matrices

Given the parameters of $\theta'_k(\phi(q)) \forall k \in \{k = 1, \dots, K\}$, from (4) one can construct $A(q)$ as a linear combination of

⁵It must be noted that, although we present GMM as the approach to estimate the scheduling parameters, alternative algorithms can be used.

local A_k matrices weighted by their scheduling parameters $\theta'_k(\phi(q))$ as follows:

$$A(q) = \frac{\sum_{k=1}^K A_k \pi_k \mathcal{N}(\phi(q); \mu_k, \Sigma_k)}{\sum_{k=1}^K \pi_k \mathcal{N}(\phi(q); \mu_k, \Sigma_k)}. \quad (7)$$

Notice the resemblance of (7) to the Nadaraya-Watson kernel estimator [47], [48]⁶ with a Gaussian pdf as its kernel function. Hence (7) can be considered a type of kernel estimator, with the key distinction that the weighting functions $\theta'_k(\phi(q))$ are not determined by individual points (as in the original Nadaraya-Watson kernel estimator) but by the components of a GMM, similar to the weighting functions derived in Gaussian Mixture Regression (GMR) [49].

Intuitively, each “synergy region” in synergy space is defined by a Gaussian distribution, and the closer the robot is to a region, the more that region’s synergy (A_k) influences the robot’s current joint-space motions. Finding the appropriate synergy matrices A_k for the set of local behaviors can be reduced to a convex semidefinite optimization, with the goal of minimizing (2). To achieve this, the following optimization is proposed which uses mean square error as a means to minimize the joint velocity error from (2) as follows:

$$\min_{A_1, \dots, A_K} \sum_{n=1}^N \sum_{t=0}^{T_n} \|\dot{q}_{t,n} - f(q_{t,n})\| \quad (8)$$

subject to

$$0 \preceq A_k, \forall k \in \{1, \dots, K\}.$$

where $f(q_{t,n})$ is calculated by combining our dynamical system formulation (3) with (7), and x_n^* is defined as the endpoint of the n^{th} demonstrated trajectory.

⁶The Nadaraya-Watson kernel estimator is used to estimate an unknown regressive function $m(x) = \mathbb{E}\{Y|X\}$, which takes the general form of $\hat{m}(x) = \frac{\sum_{i=1}^n y_i \mathcal{K}(x, x_i)}{\sum_{i=1}^n \mathcal{K}(x, x_i)}$ where $\mathcal{K}(x, x_i)$ is a kernel function denoting the distance or similarity of x_i to the given location x . [47], [48]

V. EXPERIMENTAL VALIDATION

A. JT-DS Learning Performance Evaluation

To evaluate the proposed JT-DS learning algorithm we collected kinesthetic demonstrations on a 7-DOF KUKA LWR 4+ robot arm, for 7 different tasks (see Table I and Fig. 4) which require mimicking the demonstrated joint-space behavior while reaching for a single target in task space:

- (1-2) *Tennis strokes*: For these particular behaviors, we guide the robot to perform a forehand and backhand stroke.
- (3-5) *Pouring motions*: We guide the robot to pour chips in a bowl with three different joint-space behaviors. The first is without an obstacle in the workspace. The second is with an obstacle in the upper hemisphere of the workspace, where the human is seen folding the elbow and lowering the shoulder of the robot arm to avoid the obstacle. The third instance is with an obstacle in the lower hemisphere, and now the human raises the shoulder to avoid it.
- (6) *Footstep-like motion*: We demonstrate a footstep motion which begins with a straight leg, moving through a singularity, and finally bringing the knee up.
- (7) *Singularity motions*: Movement was constrained to the boundary of the workspace by fixing $q^i = 0 \forall i \in \{3, \dots, 7\}$, the second joint was fixed to $q^2 \in \{10^\circ, 20^\circ, \dots, 100^\circ\}$, and only the first joint was moved by a human. This restricted the motion to a series of arcs of different radii along the robot’s motion boundary.

For each behavior, we collected N demonstrations consisting of (14-D) time-series $D = \{\{q_{t,n}, \dot{q}_{t,n}\}_{t=1, \dots, T_n}\}_{n=1, \dots, N}$ where each demonstration has different durations T_n . We also record the final task-space configurations $x^{*(n)}$, which we assume is the attractor in task-space. The recordings are collected at a rate of 500 Hz. However, our learning approach does not require such a dense representation of the trajectories, hence we down-sample them to 25 Hz. The total number of demonstrations N and samples per dataset $M = \sum_{n=1}^N T_n$ are provided in Table I. By performing our evaluation on all these datasets, we seek to:

- (I) Verify that the learned *behavior synergies* are mimicking the demonstrated joint-space behaviors and generalize them to new initial joint configurations.
- (II) Analyze the role of dimensionality reduction in our proposed approach and further find the method which yields the best trade-off between accuracy and model complexity; i.e. the least number of dimensions δ to represent our activation function $\theta(q)$ and the least number of local behavior synergies K .

We thus evaluate three learning approaches with different dimensionality reduction algorithms: (1) None, (2) PCA and (3) K-PCA with RBF kernel. We perform 10-fold cross-validation on all datasets, for each learning approach, with a training/testing ratio of 60%. To evaluate performance, we take the square root of the MSE defined in (2), which we refer to as the joint-velocity RMSE.

Performing such cross-validation in our case, is however not trivial, as our $A(q)$ formulation has several hyperparameters: the dimensionality δ of our lower-dimensional



(a) Tennis Strokes (1-2).



(b) Pouring Motions (3-4)



(c) Pouring Motion (5) and Foot-step (6).

Fig. 4: Demonstrated Joint-Space Behaviors with Task-Space Targets. [videos]

Behavior Dataset	Dim. Red. Approach	Optimal Parameters — Joint Velocity RMSE [rad/s]			
		Optimal δ	Optimal K	RMSE Train	RMSE Test
(1) Forehand Stroke ($N = 10, M = 1424$)	None	7	5.5 ± (2.99)	0.271 ± (0.038)	0.827 ± (0.697)
	K-PCA ($\sigma_{opt} = 5.488$)	PCA 3.9 ± (0.316)	4.1 ± (0.316) 3.2 ± (0.421)	4.8 ± (2.347) 0.305 ± (0.021)	0.276 ± (0.033) 0.433 ± (0.104)
(2) Backhand Stroke ($N = 11, M = 1223$)	None	7	5.1 ± (2.643)	0.192 ± (0.033)	0.728 ± (0.709)
	K-PCA ($\sigma_{opt} = 6.238$)	PCA 3.1 ± (0.316)	3.1 ± (0.316) 3.1 ± (0.316)	3.5 ± (0.971) 0.219 ± (0.013)	0.216 ± (0.021) 0.469 ± (0.164)
(3) Pouring - Free ($N = 9, M = 1032$)	None	7	4.1 ± (1.286)	0.163 ± (0.026)	7.609 ± (18.99)
	K-PCA ($\sigma_{opt} = 3.92$)	PCA 2.3 ± (0.483)	4.8 ± (2.044) 4.3 ± (1.767)	0.161 ± (0.034) 0.171 ± (0.106)	0.623 ± (0.301) 0.511 ± (0.305)
(4) Pouring - Obstacle 1 ($N = 11, M = 1232$)	None	7	6.2 ± (2.573)	0.247 ± (0.039)	3.049 ± (3.113)
	K-PCA ($\sigma_{opt} = 7.695$)	PCA 3.1 ± (0.421)	3.8 ± (0.421) 3.1 ± (0.421)	3.8 ± (1.549) 0.288 ± (0.020)	0.292 ± (0.018) 0.486 ± (0.125)
(5) Pouring - Obstacle 2 ($N = 7, M = 1406$)	None	7	4.3 ± (1.418)	0.122 ± (0.014)	3.095 ± (3.406)
	K-PCA ($\sigma_{opt} = 2.86$)	PCA 2.9 ± 0.316	3.7 ± (2.213) 3.9 ± (0.316)	0.126 ± (0.02) 0.129 ± (0.013)	0.825 ± (0.764) 0.738 ± (0.507)
(6) Foot Step ($N = 8, M = 1058$)	None	7	4.4 ± (2.366)	0.080 ± (0.008)	0.368 ± (0.265)
	K-PCA ($\sigma_{opt} = 1.513$)	PCA 1	3.2 ± (0.421) 3	3.2 ± (0.421) 0.091 ± (0.007)	0.084 ± (0.004) 0.123 ± (0.011)
(7) Singularity Motions ($N = 10, M = 1467$)	None	7	6.2 ± (2.898)	0.054 ± (0.009)	0.137 ± (0.098)
	K-PCA ($\sigma_{opt} = 1.769$)	PCA 2	6 ± (2.357) 5.3 ± (2.452)	0.053 ± (0.010)	0.055 ± (0.018) 0.105 ± (0.046)

TABLE I: Performance Comparison of Learning Approach with different Dim. Red. Schemes. We present the mean (std) for the optimal δ , K and joint-velocity RMSE on training and testing set, found for every learning scheme over 10 runs.

embedding $\phi(q)$ and the number of local behavior synergies K . Moreover, when using K-PCA (with RBF kernel $k(q, q') = \exp(-\frac{\|q-q'\|^2}{2\sigma^2})$) we must also find the optimal width σ . Thus, for each fold and each learning approach we find the optimal hyper-parameters δ , K and σ (when applicable), as follows:

- (I) **δ for PCA:** We choose δ such that the projection is capable of explaining 95% of the variance in the data.
- (II) **δ and σ for K-PCA:** The interaction of these two parameters plays a major roll in the resulting projection obtained from K-PCA. Hence, we do a grid search on a log-spaced range of $\sigma_{range} = [\sigma_{min} : \sigma_{max}]$ values, where $\sigma_{min} = \frac{1}{\kappa\sqrt{2}} \max_{i \in M, j \in M} \{ \|q_i - q_j\|^2 \}$ and $\sigma_{max} = \frac{2\kappa}{\sqrt{2}} \max_{i \in M, j \in M} \{ \|q_i - q_j\|^2 \}$ for $\kappa = cte$. This yields a feasible range for σ that is guided by the pairwise Euclidean distances between all points in the dataset. Moreover, K-PCA is not limited to providing a $\delta \ll d$, in fact it can generate $\delta \leq M$. This is a nuisance as in our datasets $M \approx 1000$. To alleviate this we truncate σ_{range} by compute the explained variance of the eigenvectors in feature space for different values of $\sigma \in \sigma_{range}$. We then remove σ values from σ_{range} whose number of *optimal* eigenvectors $\delta > d$ and resample it. By doing this procedure, we can ensure that for all values in truncated $\bar{\sigma}_{range}$ we will obtain $\delta \leq d$. We then choose σ_{opt} by running 10-fold cross-validation for all $\sigma \in \bar{\sigma}_{range}$.
- (III) **K for GMM:** We choose the optimal number of components K , by evaluating and selecting the best resulting model using the Bayesian Information Criterion (BIC) [42]. Typically, one chooses the optimal K manually, by visually identifying the point at which the BIC curve produced from K_{range} stops changing or plateaus. To automate this process, we devised an approach which selects the optimal K as the one which yields the highest inflection point on the second order derivative of the BIC curve.

Once these optimal hyper-parameters are estimated for each fold, we solve for the definite convex optimization problem in order to find our synergy matrices A 's which best minimize the objective function (8). We then select the initial joint configuration q_0 and target in task-space x^* from each training/testing dataset, simulate the joint-space trajectories $\{q_1, \dots, q_T\}$, with our learned $\mathcal{A}(q)$, and compute the joint-velocity RMSE between these simulations and the training/testing trajectories, as reported in Table I

As can be seen, for all datasets there is a significant increase in performance on the testing sets when using either dimensionality reduction (DR) approaches. This suggests that using DR to represent our activation functions $\theta(q)$ yields better generalization capabilities than the contrary. This is most notable for the three pouring motions, where the joint-velocity RMSE testing error for a JT-DS model learnt without DR is an order of magnitude higher than with DR. Such an error indicates that the demonstrated joint-behavior was over-fitted on the training set, which is also exhibited in the number of K needed to represent the motion. For all datasets, the DR methods provided $\delta < d/2$, either comparable or less number of local behaviors synergies K and better RMSE errors on testing sets as opposed to no DR. By finding a lower-dimensional manifold to represent the joint trajectories, we are getting rid of outliers, noise and redundancies that might arise from the raw joint demonstrations. Hence, through DR we are capable of robustly extracting the local behavior synergies from raw demonstrations.

Both PCA and K-PCA yield comparable results, with K-PCA providing a slight improvement on some datasets. This suggests that perhaps a linear DR method might be sufficient for such tasks. However, if we seek to maximize accuracy a non-linear DR method should be employed. One of the drawbacks of K-PCA is its computational complexity for out-of-sample evaluations, which involves computing the kernel function between the new data-point and all the samples M . This is, however, not so taxing for our method as our datasets range in $M \approx 1000$ and in previous work [38] we have experimentally found that evaluating < 3000 RBF



(a) Execution of Pour Obstacle 2 learned through JT-DS.

(b) Execution of Pour Obstacle 2 learned through SEDS.

Fig. 5: Snapshots of the robot experiments. A corresponding video is available on-line [video link].

kernel computations (on a 3.4-GHz i7 PC with 8GB RAM) in a control-loop with 2ms rate is feasible. If one requires extremely fast computation; i.e. a control-loop rate of < 2ms then PCA should be used as opposed to K-PCA.

Implementation Details: The learning pipeline is implemented in MATLAB. We used the Matlab Toolbox for Dimensionality Reduction [41] for implementations of PCA, K-PCA and its out-of-sample extension. The YALMIP framework [40] was used to solve the definite convex optimization problem. Source code for the entire learning pipeline and cross-validation with simulation of multi-DOF robot arms can be found in:

<https://github.com/epfl-lasa/JT-DS-Learning>

For execution of the learnt JT-DS models on a real 7-DOF the KUKA LWR 4+ robot arm, we provide the following code in C++: <https://github.com/epfl-lasa/JT-DS-lib>. The robot is controlled on the joint position level (linearly interpolating from joint velocities computed from JT-DS) at a rate of 500 Hz. The resultant joint angles are filtered by a critically damped filter to avoid high torques.

B. JT-DS Execution Performance Evaluation

We now seek to elucidate the distinctive properties of our JT-DS model by comparing its performance to a DS-based Cartesian motion generator + IK solver approach for behaviors (5-7). For the Cartesian motion generator we use the SEDS approach [14] which learns an asymptotically stable DS in Cartesian space from demonstrations. We then generate joint trajectories through a damped least square IK solver. From herein we refer to this approach as SEDS+IK.

1) Following Desired Joint and Task-Based Behaviors:

We compare tracking capabilities of our JT-DS method with those of SEDS+IK for behaviors (5) and (6). In Fig. 5a and 5b we can qualitatively see the difference between these two approaches for behavior (5). The JT-DS algorithm mimicked the joint-space behavior of the demonstration (e.g. folding the elbow, raising the shoulder), successfully avoiding the obstacle while still converging to the desired Cartesian position. Meanwhile, SEDS+IK only learned the demonstrated behavior in task-space, it does not have the capabilities of constraining motion in joint-space. This ultimately led to one of its joints colliding with the obstacle. It should be noted that the JT-DS motion did not follow the demonstrations in task-space very closely (as we would expect), but did ultimately converge to its target position. For the foot-step behavior (6) JT-DS followed the demonstrations closely, while SEDS became unstable in the singularity (Fig. 6 and 7). This demonstrates the JT-DS algorithm's ability to move cleanly in and out of singularities, overcoming the Achilles heel of



Fig. 6: Execution of problematic joint-space behaviors learned through JT-DS: (left) Foot step-like motion and (right) Singularity motions.

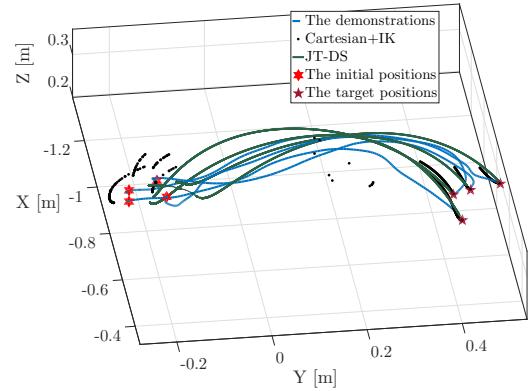


Fig. 7: End-effector trajectories for the footstep motion in Cartesian space. The JT-DS motion moves smoothly closely resembles the demonstrated trajectories. On the other hand, the SEDS-based Cartesian motion generator (whose values here are simulated because they can not physically executable) quickly becomes unstable, as evidenced by the dotted paths starting on the left side and abruptly disappearing.

equivalent Cartesian-based motion generators.

2) *Systematic Assessment:* To systematically determine the performance properties (tracking error, computation time, and convergence time) of JT-DS (3) and compare it to a SEDS+IK, we simulated 400 simple motions on each system. The systems were tasked with moving to a fixed target randomly chosen from the region $[-0.0081 \pm 0.3 \quad -0.0188 \pm 0.3 \quad 0.4974 \pm 0.21]^T$. The Cartesian motion generator was SEDS-based, and mapping from Cartesian motions to joint-space motions was done using a damped least-squares IK solver. To save on computation time, both the JT-DS and SEDS algorithms were taught behaviors defined by a single synergy, i.e. uniform behaviors. The results, summarized in Table II, indicate that the computation time of the proposed approach is significantly faster than the Cartesian-based DS, because the algorithm does not require calculating the Jacobian pseudo-inverse. As the generated joint motion by (3) is directly transmitted to the robot, the tracking error is zero. Nevertheless, the convergence time of (3) is slightly higher than SEDS+IK.

3) *Transiting through singular configurations:* One of the main advantages of the proposed DS is its ability to

	Norm. converg. [s/m]	Comp. time [ms]	Track. error [m]
SEDS + IK	10.6566 ± 2.1364	71.7752 ± 4.9189	0.0163 ± 0.0063
JT-DS	14.1926 ± 5.26453	12.1736 ± 0.8573	0.0 ± 0.0

TABLE II: Each simulated trajectory is initialized at $q = [0 \dots 0]^T$. The convergence duration is the time required to move within 0.001m of the target. The normalized convergence duration is the convergence duration divided by the distance between the initial and target positions.

generate accurate paths in classical singular configurations. To evaluate this we generated behavior (7), which were demonstrated motions are entirely within a classic kinematic singularity. Fig. 8a shows the *demonstrated* motions and the motion *generated* by JT-DS (3). The algorithm never requires the pseudo-inverse of the Jacobian matrix, so the generated motion perfectly follows the demonstrations throughout the workspace boundary. In Fig. 6 we show the learnt singular motion successfully executed on the real robot.

VI. DISCUSSION

In this paper, we have presented a dynamical system in joint space that is provably Lyapunov-stable in task space and which replicates demonstrated joint-space behaviors. The desired motions are fast to compute, and smoothly handle singularities by avoiding the pseudo-inverse Jacobian. We showed the system’s ability to learn different joint-space behaviors on a redundant robotic platform.

One of the most important points when validating a learning from demonstration method is to evaluate the system’s behavior away from demonstrations. When the current joint configuration is far from any of the local synergy regions, computing the scheduling parameters $\theta_k(\cdot)$ becomes numerically infeasible (all the Gaussians in (7) $\rightarrow 0$), and so $\sum_{k=1}^K \frac{A_k}{K}$ is used to move the robot, which is still guaranteed to move towards the target. Moreover, in overlapping local regions where multiple A_k ’s might be in conflict, the presented system compromises between them while still stably moving to the target. The reason for this is that rather than “determining” the velocity of the system, our A matrix only warps it. This means that adding multiple “conflicting” synergies together amounts to nothing more than repeatedly warping the velocity, and still maintains the original stability property.

One of the major advantages of the proposed dynamical system is that it avoids the undesirable effects of traditional Inverse Kinematic (IK) solvers. Fast dynamical motions require not only fast and accurate motion planning, but also precise IK mapping, which is not practically feasible. By utilizing the JT-DS method, even a fast system could reactively generate stable motions to its target.

A drawback of the approach is that the controller is not guaranteed to converge to the task-space target x^* even if a path to a valid configuration q^* exists. As mentioned in Sec. III, the controller will always minimize (or keep even) the warped distance to goal $(H(q) - x^*)^T P(H(q) - x^*)$, and in the process may get stuck at its kinematic joint limits. If the only valid trajectory to reach the goal requires that we temporarily move away from the target (and increase

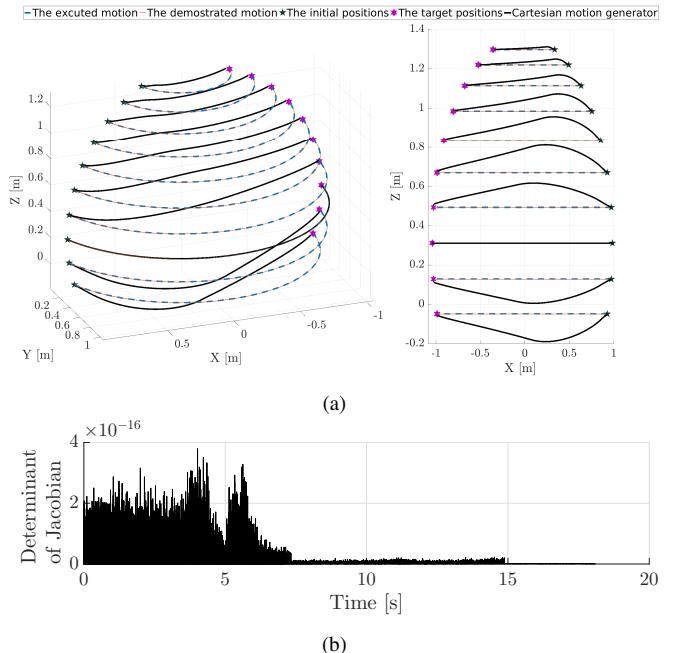


Fig. 8: The experiment generated $K = 3$ Gaussian components. As the first joint is the only joint which was not fixed during the demonstration, the learned augmentation matrices had only one nonzero entry $A_k(1, 1) \neq 0 \forall k \in \{1, 2, 3\}$. In (a), the end-effector positions for the demonstrations and executed motions are plotted in Cartesian space. The JT-DS trajectory was generated closed-loop, while the SEDS trajectory was generated open-loop (otherwise it would be unstable). In (b), the determinants of the Jacobian ($|J(q)J^T(q)|$) for all 10 JT-DS motions are superimposed. Due to sensory noise, the determinants are not exactly zero, but are very close.

the warped distance-to-goal), this controller will not find that trajectory. There is a simple fix, however: rather than providing a single task-space objective x^* , provide a series of sequential task-space objectives $\mathbf{x}^* = \{x_0^*, x_1^*, \dots, x_n^*\}$ and have the controller move to each in succession. The smaller sub-trajectories give the robot less of an opportunity to deviate from the learned (valid) joint behavior and thus help it avoid being caught by the robot’s joint limits. Further, specifying a sequence of task-space targets provides greater control over the robot’s task-space trajectory while still exhibiting the learned joint-space behaviors.

One application that we have discussed for this algorithm is obstacle avoidance. In an environment with static obstacles, the system would, through demonstration, learn joint-space behaviors that avoid those obstacles. One can compare this form of joint-space obstacle avoidance with other methods from the literature [28], [50]. The main difference is that these previous methods require explicit knowledge of the obstacles’ position, whereas our algorithm learns the obstacle positions only implicitly by learning motions that avoid those positions. This lack of explicit obstacle location can be an advantage: encoding obstacles’ geometry can be expensive and cumbersome. However, it is also a drawback: without knowing the exact location of obstacles, the algorithm cannot be guaranteed to avoid collisions, especially away from demonstrations.

APPENDIX I

PROVING STABILITY OF THE DYNAMICAL SYSTEM

We wish to prove Proposition 1, that is, that JT-DS (3) (reproduced below)

$$\dot{q} = f(q) = -A(q)J^T(q)P(H(q) - x^*)$$

accomplishes criterion (I), where A is positive semi-definite, and P is positive definite.

Theorem 1.1 (Proof of Lyapunov Stability): JT-DS (3) is stable in the sense of Lyapunov with respect to the Lyapunov candidate

$$V(q) = \frac{1}{2}(H(q) - x^*)^T P(H(q) - x^*)$$

That is,

$$0 \prec V(q) \quad V(q^*) = 0 \quad \forall q \neq q^* \quad (9)$$

where q^* is any joint configuration such that $H(q^*) = x^*$.

The first two statements are trivially true because P is positive definite and the rest of V is a square that is only 0 when $H(q) = x^*$. To prove the last statement, we find $\frac{dV}{dt}$:

$$\begin{aligned} \frac{dV(q)}{dt} &= (H(q) - x^*)^T P J(q) \dot{q} \\ &= -(H(q) - x^*)^T P J(q) A(q) J^T(q) P (H(q) - x^*) \\ &= -(H(q) - x^*)^T P J(q) \sum_{k=1}^K \underbrace{\theta_k(q)}_{\geq 0} \underbrace{A_k}_{\geq 0} \\ &\quad \cdot J^T(q) P (H(q) - x^*) \leq 0 \end{aligned} \quad (10)$$

By observation, each and every term in the final expression is multiplied by its transpose (creating a square) except for A , which is positive semi-definite. This means that the expression is guaranteed to be negative semi-definite. Therefore, JT-DS (3) is globally Lyapunov stable; i.e. $\|H(q) - x^*\|$ and \dot{q} are bounded.

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