UNLocBoX

A matlab convex optimization toolbox using proximal splitting methods

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Abstract

Nowadays the trend to solve optimization problems is to use specific algorithms rather than very general ones. The UNLocBoX provides a general framework allowing the user to design his own algorithms. To do so, the framework try to stay as close from the mathematical problem as possible. More precisely, the UNLocBoX is a Matlab toolbox designed to solve convex optimization problem of the form

$$\min_{x \in \mathcal{C}} \sum_{n=1}^{K} f_n(x),$$

using proximal splitting techniques. It is mainly composed of solvers, proximal operators and demonstration files allowing the user to quickly implement a problem.

1 Introduction

The UNLocBoX is designed to solve convex optimization problems of the form

$$\min_{x \in \mathbb{R}^N} f_1(x) + f_2(x), \tag{1}$$

or more generally

$$\min_{x \in \mathbb{R}^N} \sum_{n=1}^{K} f_n(x), \tag{2}$$

where the f_i are lower semi-continuous convex functions from $\mathbb{R}^N$ to $(-\infty, +\infty]$. We assume $\lim_{\|x\|_2 \to \infty} \left\{ \sum_{n=1}^{K} f_n(x) \right\} = \infty$ and the f_i have non-empty domains, where the domain of a function f is given by

$$\mathrm{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$

In problem (2), and when both f_1 and f_2 are smooth functions, gradient descent methods can be used to solve (1); however, gradient descent methods cannot be used to

solve (1) when f_1 and/or f_2 are not smooth. In order to solve such problems more generally, we implement several algorithms including the forward-backward algorithm [1]-[3] and the Douglas-Rachford algorithm [4, 5]-[8].[1]

Both the forward-backward and Douglas-Rachford algorithms fall into the class of *proximal splitting algorithms*. The term *proximal* refers to their use of proximity operators, which are generalizations of convex projection operators. The *proximity operator* of a lower semi-continuous convex function $f : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathrm{prox}_f(x) := \underset{y \in \mathbb{R}^N}{\arg\min} \left\{ \frac{1}{2}\|x - y\|_2^2 + f(y) \right\}. \tag{3}$$

Note that the minimization problem in (3) has a unique solution for every $x \in \mathbb{R}^N$, so $\mathrm{prox}_f : \mathbb{R}^N \to \mathbb{R}^N$ is well-defined. The proximity operator is a useful tool because (see, e.g., [9, 10]) x^* is a minimizer in (1) if and only if for any $\gamma > 0$,

$$x^* = \mathrm{prox}_{\gamma(f_1+f_2)}(x^*). \tag{4}$$

The term *splitting* refers to the fact that the proximal splitting algorithms do not directly evaluate the proximity operator $\mathrm{prox}_{\gamma(f_1+f_2)}(x)$, but rather try to find a solution to (4) through sequences of computations involving the proximity operators $\mathrm{prox}_{\gamma f_1}(x)$ and $\mathrm{prox}_{\gamma f_2}(x)$ separately. The recent survey [11] provides an excellent review of proximal splitting algorithms used to solve (1) and related convex optimization problems.

The toolbox is essentially made of three kind of functions:

- Solvers: the core of the toolbox
- Proximity operators: they solve small minimization problems and allow a quick implementation of common problems.
- Demonstration files: examples to help you to use the toolbox

The design of the UNLocBoX was largely inspired by the LTFAT toolbox [12]. The authors are jointly developing mat2doc a documentation system that allows to generate documentation from source files.

2 Solvers / algorithm

The forward-backward algorithm can be used to solve

$$\min_{x \in \mathbb{R}^N} f_1(x) + f_2(x),$$

when either f_1 or f_2 is a continuously differentiable convex function with a Lipschitz continuous gradient. A function f has a β-Lipschitz-continuous gradient ∇f if

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq \beta \|x - y\|_2 \qquad \forall x, y \in \mathbb{R}^N, \tag{5}$$

[1] In fact, the toolbox implements generalized versions of these algorithms that can solve problems with sums of a general number of such functions, but for simplicity, we discuss here the simplest case of two functions.

where $\beta > 0$.

If, without loss of generality, f_2 is the function with a β-Lipschitz-continuous gradient ∇f_2, then x^* is a solution to (1) if and only if for any $\gamma > 0$ (see, e.g., [3, Proposition 3.1]),

$$x^* = \mathrm{prox}_{\gamma f_1}\left(x^* - \gamma \nabla f_2(x^*)\right). \tag{6}$$

The forward-backward algorithm finds a point satisfying (6) by computing a sequence $\left\{x^{(k)}\right\}_{k=0,1,\ldots}$ via

$$x^{(k+1)} = \mathrm{prox}_{\gamma f_1}\left(x^{(k)} - \gamma \nabla f_2(x^{(k)})\right). \tag{7}$$

For any $x^{(0)}$, the sequence $\left\{x^{(k)}\right\}_{k=0,1,\ldots}$ converges to a point satisfying (6), which is therefore a minimizer of (1). For a detailed convergence analysis that includes generalizations of (7) which may result in improved convergence rates, see [3, Theorem 3.4].

The associated Matlab function `forward_backward` takes four parameters

```
function sol = forward_backward (x0, f1, f2, param).
```

`x0` $\in \mathbb{R}^N$ is the starting point for the algorithm. `f1` and `f2` are two Matlab structures that represent the functions f_1 and f_2. Each of these structures needs at least two fields. The Matlab structure `f1` contains the fields `f1.eval` and `f1.prox`. The former is a Matlab function that takes as input a vector $x \in \mathbb{R}^N$ and returns the value $f_1(x)$, the latter is a Matlab function that takes as input a vector $x \in \mathbb{R}^N$, a strictly positive real number τ and returns the vector $\mathrm{prox}_{\tau f_1}(x)$ (In Matlab, write: `f1.prox=@(x, T) prox_f1(x, T)`, where `prox_f1(x, T)` solves the problem $\mathrm{prox}_{Tf_1}(x)$ given in equation 3). In the same way, the Matlab structure `f2` contains the fields `f2.eval` and `f2.grad`. The former is a Matlab function that takes as input a vector $x \in \mathbb{R}^N$ and returns the value $f_2(x)$, the latter is also a Matlab function that takes as input a vector $x \in \mathbb{R}^N$ and returns the vector $\nabla f_2(x)$ (In Matlab, write: `f2.grad=@(x) grad_f2(x)`, where `grad_f2(x)` return the value of $\nabla f_2(x)$). Finally, `param` is a Matlab structure that containing a set of optional parameters. The list of parameters is described in the help of the function itself. The following two fields are specific to the `forward_backward` function:

- `param.method`: "ISTA" or "FISTA". Specify the method used to solve problem (1). ISTA stands for the original forward-backward algorithm while FISTA stands for its accelerated version (for details, see [13]).
- `param.gamma`: step-size parameter γ. This constant should satisfy: $\gamma \in [\epsilon, 2/\beta - \epsilon]$, for $\epsilon \in]0, \min\{1, 1/\beta\}[$.
- `param.lambda`:λ weight of the update term used in ISTA method $\in [\epsilon, 1]$. By default, it's equal to one.

2.1 Douglas-Rachford

The Douglas-Rachford algorithm [4][5] is a more general algorithm to solve

$$\min_{x \in \mathbb{R}^N} f_1(x) + f_2(x)$$

that does not require any assumptions on the smoothness of f_1 or f_2. For any constant $\gamma > 0$, a point $x^* \in \mathbb{R}^N$ is a solution to (1) if and only if there exists a $y^* \in \mathbb{R}^N$ such that [8, Proposition 18]

$$x^* = \text{prox}_{\gamma f_2}(y^*), \text{ and} \tag{8}$$

$$\text{prox}_{\gamma f_2}(y^*) = \text{prox}_{\gamma f_1}\left(2\text{prox}_{\gamma f_2}(y^*) - y^*\right). \tag{9}$$

To find x^* and y^* that satisfy (8) and (9), the Douglas-Rachford algorithm computes the following sequence of points, for any fixed $\gamma > 0$ and stepsize $\lambda \in (0, 2)$:

$$x^{(k)} = \text{prox}_{\gamma f_2}(y^{(k)}), \text{ and} \tag{10}$$

$$y^{(k+1)} = y^{(k)} + \lambda\left(\text{prox}_{\gamma f_1}\left(2x^{(k)} - y^{(k)}\right) - x^{(k)}\right). \tag{11}$$

The convergence of the Douglas-Rachford algorithm is analyzed in [7, Corollary 5.2] and [8, Theorem 20], which also show that the algorithm can be accelerated by allowing the step size λ to change at every iteration of the algorithm. Note that while the Douglas-Rachford algorithm does not require any smoothness assumptions on f_1 or f_2, it requires two proximal steps at each iteration, as compared to one proximal step per iteration for the forward-backward algorithm.

The associated Matlab function `douglas_rachford` takes four parameters

```
function sol = douglas_rachford (x0, f1, f2, param).
```

As in Section 2.1, `x0` $\in \mathbb{R}^N$ is the starting point for the algorithm. `f1` and `f2` are two Matlab structures that represent the functions f_1 and f_2. The Matlab structure `f1` contains the fields `f1.eval` and `f1.prox`. The former is a Matlab function that takes as input a vector $x \in \mathbb{R}^N$ and returns the value $f_1(x)$, the latter is a Matlab function that takes as input a vector $x \in \mathbb{R}^N$, a strictly positive real number τ and returns the vector $\text{prox}_{\tau f_1}(x)$. The Matlab structure `f2` contains the exact same fields but for the function f_2. Finally, `param` contains a list of parameters. The list of parameters is described in the help of the function itself. The following two fields are specific to the `douglas_rachford` function:

- `param.lambda`: λ acts as a stepsize parameter. Let $\epsilon \in]0, 1[$, λ should be in the interval $[\epsilon, 2 - \epsilon]$. Its default value is 1.
- `param.gamma`: $\gamma > 0$ controls the speed of convergence. Its default value is 1.

2.2 Alternating-direction method of multipliers (ADMM)

Augmented Lagrangian techniques are classical approaches for solving problem like:

$$\min_{x \in \mathbb{R}^N} f_1(x) + f_2(Lx). \tag{12}$$

First we reformulate (12) to

$$\min_{\substack{x \in \mathbb{R}^N, y \in \mathbb{R}^M \\ Lx = y}} f_1(x) + f_2(y).$$

We then solve this problem using the augmented Lagrangian technique.

Warning: the proximal operator of function f_1 is defined to be:

$$\mathrm{prox}^L_{f_1 \tau}(z) = \arg\min_{x \in \mathbb{R}^N} \tau f_1(x) + \frac{1}{2}\|Lx - z\|_2^2$$

The ADMM algorithm can be used when f_1 and f_2 are in $\Gamma_0(\mathbb{R}^N)$ and in $\Gamma_0(\mathbb{R}^M)$ with $L^T L$ invertible and $(\mathrm{ridom} f_1) \cap L(\mathrm{ridom} f_2) \neq \emptyset$. The associated Matlab function `ADMM` takes five parameters:

```
function sol = admm(x_0, f1, f2, param).
```

As in Section 2.1, `x0` $\in \mathbb{R}^N$ is the starting point for the algorithm. `f1` and `f2` are two Matlab structures that represent the functions f_1 and f_2. The Matlab structure `f1` contains the fields `f1.eval` and `f1.prox`. The former is a Matlab function that takes as input a vector $x \in \mathbb{R}^N$ and returns the value $f_1(x)$, the latter is a Matlab function that takes as input a vector $x \in \mathbb{R}^N$, a strictly positive real number τ and returns the vector $\mathrm{prox}_{\tau f_1}(x)$. The Matlab structure `f2` contains the exact same fields but for the function f_2. Finally, `param` contains a list of parameters. The list of parameters is described in the help of the function itself. The following field is specific to the `admm` function:

- `param.gamma`: $\gamma > 0$ controls the speed of convergence. Its default value is 1.
- `param.L`: L is an operator or a matrix. Its default value is the identity.

2.3 Other solvers

The UNLocBoX contains some other solvers that should not be forgotten. Very important are the generalization of forward backard (generalized forward backward), Douglas Rachford (PPXA) and ADMM (SDMM). Those allows to solve problems of the type (2). A list of the solver can be found at `http://unlocbox.sourceforge.net/doc/solver/index.php`.

3 Proximal operator

For every function that is minimized by the UNLocBoX, the gradient or the proximal operator has to be determined. The toolbox provide some of the most common ones (see table 3). A complete list can be found at `http://unlocbox.sourceforge.net/doc/prox/index.php`.

All proximal operator functions take as input three parameters: `x, lambda, param`. First, `x` is the initial signal. Then `lambda` is the weight of the objective function. Finally, `param` is a Matlab structure that containing a set of optional parameters.

Function	**Explanation**
prox_l1	ℓ_1 norm proximal operator. Solve: $$\min_x \frac{1}{2}\|x-z\|_2^2 + \lambda\|\Psi(x)\|_1$$
Prox_linf	ℓ_∞ norm proximal operator. Solve: $$\min_x \frac{1}{2}\|x-z\|_2^2 + \lambda\|\Psi(x)\|_\infty$$
prox_tv	TV norm proximal operator. Solve: $$\min_x \frac{1}{2}\|x-z\|_2^2 + \lambda\|x\|_{TV}$$
prox_l12	ℓ_{12} norm[2] proximal operator. Solve: $$\min_x \frac{1}{2}\|x-z\|_2^2 + \lambda\|x\|_{12}$$
prox_l1inf	$\ell_{1\infty}$ norm[3] proximal operator. Solve: $$\min_x \frac{1}{2}\|x-z\|_2^2 + \lambda\|x\|_{1\infty}$$
prox_nuclear_norm	ℓ_* norm proximal operator. Solve: $$\min_x \frac{1}{2}\|x-z\|_2^2 + \lambda\|x\|_*$$

Table 1: List of proximal operators

If we would like to restrict the set of admissible functions to a subset $\mathcal{C}$ of $\mathbb{R}^L$, i.e.

find the optimal solution to (2) considering only solutions in $\mathcal{C}$, we can use projection operator instead of proximal operators. Indeed proximal operators are generalization of projections. For any nonempty, closed and convex set $\mathcal{C} \subset \mathbb{R}^L$, the *indicator function* [11] of $\mathcal{C}$ is defined as

$$i_{\mathcal{C}} : \mathbb{R}^L \to \{0, +\infty\} : x \mapsto \begin{cases} 0, & \text{if } \ x \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}, \tag{13}$$

The corresponding proximity operator is given by the projection onto the set $\mathcal{C}$:

$$\begin{aligned} P_{\mathcal{C}}(y) &= \arg\min_{x \in \mathbb{R}^L} \left\{ \frac{1}{2}\|y - x\|_2^2 + i_{\mathcal{C}}(x) \right\} \\ &= \arg\min_{x \in \mathcal{C}} \left\{ \|y - x\|_2^2 \right\} \end{aligned}$$

Such restrictions are called *constraints* and can be given, e.g. by a set of linear equations that the solution is required to satisfy. Table 3 summaries some of the projections function present in the toolbox.

proj_b1	Projection on B1-ball. lambda is an unused parameter for compatibility reasons. Solve: $\min_x \|x - z\|_2^2 \quad s.t. \quad \|x\|_1 < \epsilon$
proj_b2	Projection on B2-ball. lambda is an unused parameter for compatibility reasons. Solve: $\min_x \|x - z\|_2^2 \quad s.t. \quad \|x\|_2 < \epsilon$

Table 2: List of projection operators

4 Example

The problem Let's suppose we have a noisy image with missing pixels. Our goal would be to find the closest image to the original one. We begin first by setting up some assumptions about the problem.

Assumptions In this particular example, we firstly assume that we know the position of the missing pixels. This can be the result of a previous process on the image or a simple assumption. Secondly, we assume that the image is not special. Thus, it is composed of well delimited patches of colors. Thirdly, we suppose that known pixels are subject to some Gaussian noise with a variance of ϵ.

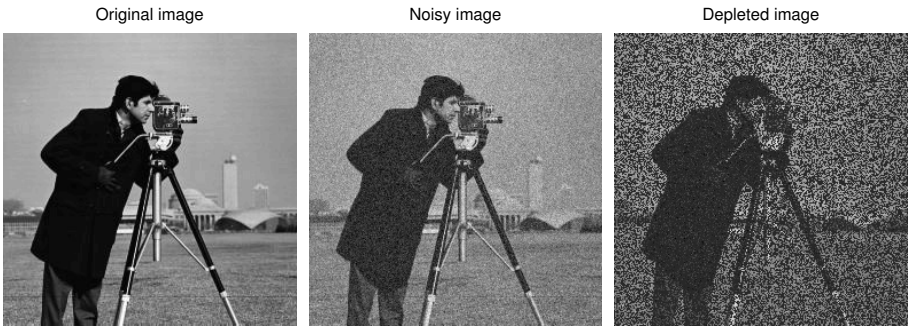

Figure 1: This figure shows the image chosen for this example: the cameraman.

Formulation of the problem At this point, the problem can be expressed in a mathematical form. We will simulate the masking operation by a mask A. This first assumption leads to a constraint.

$$Ax = y$$

where x is the vectorized image we want to recover, y are the observe noisy pixels and A a linear operator representing the mask. However due to the addition of noise this constraint is a little bit relaxed. We rewrite it in the following form

$$\|Ax - y\|_2 \leq \epsilon$$

Note that ϵ can be chosen equal to 0 to satisfy exactly the constraint. In our case, as the measurements are noisy, we set ϵ to the standard deviation of the noise.

Using the prior assumption that the image has a small TV-norm (image composed of patch of color and few degradee), we will express the problem as

$$\arg\min_x \|x\|_{TV} \qquad \text{subject to} \qquad \|b - Ax\|_2 \leq \epsilon \qquad \text{(Problem I)}$$

where b is the degraded image and A an linear operator representing the mask. ϵ is a free parameter that tunes the confidence to the measurements. This is not the only way to define the problem. We could also write:

$$arg\min_x \|b - Ax\|_2 + \lambda\|x\|_{TV} \qquad \text{(Problem II)}$$

with the first function playing the role of a data fidelity term and the second a prior assumption on the signal. λ adjusts the tradeoff between measurement fidelity and prior assumption. We call it the *regularization parameter*. The smaller it is, the more we trust the measurements and vice-versa. ϵ play a similar role as λ. Note that there exist a bijection between the parameters λ and ϵ leading to the same solution. The bijection function is not trivial to determine. Choosing between one or the other problem will affect the solvers and the convergence rate.

Solving problem I The UNLocBoX solvers take as input functions with their proximity operator or with their gradient. In the toolbox, functions are modelized with structure object with at least two fields. One field contains an operator to evaluate the function and the other allows to compute either the gradient (in case of differentiable function) or the proxitity operator (in case of non differentiable functions). In this example, we need to provide two functions:

- $f_1(x) = ||x||_{TV}$
  The proximal operator of f_1 is defined as:

$$prox_{f1,\gamma}(z) = arg\min_x \frac{1}{2}\|x-z\|_2^2 + \gamma\|z\|_{TV}$$

  This function is defined in Matlab using:

```
paramtv.verbose=1;
paramtv.maxit=50;
f1.prox=@(x, T) prox_tv(x, T, paramtv);
f1.eval=@(x) tv_norm(x);
```

  This function is a structure with two fields. First, *f1.prox* is an operator taking as input x and T and evaluating the proximity operator of the function (T plays the role of γ is the equation above). Second, and sometime optional, *f1.eval* is also an operator evaluating the function at x.
  The proximal operator of the TV norm is already implemented in the UNLocBoX by the function `prox_tv`. We tune it by setting the maximum number of iterations and a verbosity level. Other parameters are also available (see documentation `http://unlocbox.sourceforge.net/doc.php`).

  - *paramtv.verbose* selects the display level (0 no log, 1 summary at convergence and 2 display all steps).
  - *paramtv.maxit* defines the maximum number of iteration.

- f_2 is the indicator function of the set S defines by $||Ax-b||_2 < \epsilon$. We define the proximity operator of f_2 as

$$prox_{f2,\gamma}(z) = arg\min_x \frac{1}{2}\|x-z\|_2^2 + i_S(x),$$

  with $i_S(x)$ is zero if x is in the set S and infinite otherwise. This previous problem has an identical solution as:

$$arg\min_z \|x-z\|_2^2 \qquad \text{subject to} \qquad \|Az-by\|_2 \leq \epsilon$$

  It is simply a projection on the B2-ball. In matlab, we write:

```
param_proj.epsilon=epsilon;
param_proj.A=A;
param_proj.At=A;
param_proj.y=y;
f2.prox=@(x,T) proj_b2(x,T,param_proj);
f2.eval=@(x) eps;
```

The *prox* field of *f2* is in that case the operator computing the projection. Since we suppose that the constraint is satisfied, the value of the indicator function is 0. For implementation reasons, it is better to set the value of the operator *f2.eval* to *eps* than to 0. Note that this hypothesis could lead to strange evolution of the objective function. Here the parameter *A* and *At* are mandatory. Note that *A* = *At*, since the masking operator can be performed by a diagonal matrix containing 1's for observed pixels and 0's for hidden pixels.

At this point, a solver needs to be selected. The UNLocBoX contains many different solvers. You can try them and observe the convergence speed. Just remember that some solvers are optimized for specific problems. In this example, we present two of them `forward_backward` and `douglas_rachford`. Both of them take as input two functions (they have generalization taking more functions), a starting point and some optional parameters.

In our problem, both functions are not smooth on all points of the domain leading to the impossibility to compute the gradient. In that case, solvers (such as forward backward) using gradient descent cannot be used. As a consequence, we will use Douglas Rachford instead. In matlab, we write:

```
param.verbose=1;
param.maxit=100;
param.tol=10e-5;
param.gamma=1;
sol = douglas_rachford(y,f1,f2,param);
```

- *param.verbose* selects the display level (0 no log, 1 summary at convergence and 2 display all steps).
- *param.maxit* defines the maximum number of iteration.
- *param.tol* is stopping criterion for the loop. The algorithm stops if

$$\frac{n(t) - n(t-1)}{n(t)} < tol,$$

  where $n(t)$ is the objective function at iteration *t*
- *param.gamma* defines the stepsize. It is a compromise between convergence speed and precision. Note that if *gamma* is too big, the algorithm might not converge.

The solution is displayed in figure 2

Solving problem II Solving problem II instead of problem I can be done with a small modification of the previous code. First we define another function as follow:

```
param_l2.A=A;
param_l2.At=A;
```

Figure 2: This figure shows the reconstructed image by solving problem I using Douglas Rachford algorithm.

```
param_l2.y=y;
param_l2.verbose=1;
f3.prox=@(x,T) prox_l2(x,lambda*T,param_l2);
f3.grad=@(x) 2*lambda*A(A(x)-y);
f3.eval=@(x) lambda*norm(A(x)-y,'fro');
```

The structure of *f3* contains a field *f3.grad*. In fact, the l2 norm is a smooth function. As a consequence the gradient is well defined on the entire domain. This allows using the forward backward solver. However, we can in this case also use the Douglas Rachford solver. For this we have defined the field *f3.prox*.

We remind that forward backward will not use the field *f3.prox* and Douglas Rachford will not use the field *f3.grad*. The solvers can be called by:

```
sol21 = forward_backward(y,f1,f3,param);
```

Or:

```
sol22 = douglas_rachford(y,f3,f1,param);
```

These two solvers will converge (up to numerical errors) to the same solution. However, convergence speed might be different. As we perform only 100 iterations with both of them, we do not obtain exactly the same result. Results is shown in figure 3.

Remark: The parameter *lambda* (the regularization parameter) and *epsilon* (The radius of the l2 ball) can be chosen empirically. Some methods allow to compute those parameters. However, this is far beyond the scope of this tutorial.

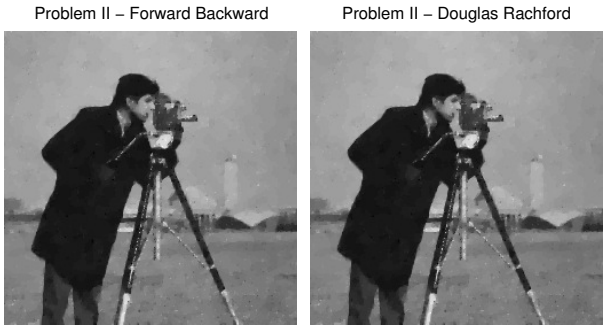

Figure 3: This figure shows the reconstructed image by solving problem II.

Appendix: example's code

```
%% Initialisation
clear all;
close all;

% Loading toolbox
global GLOBAL_useGPU;
init_unlocbox();

verbose=2; % verbosity level

%% Load an image

% Original image
im_original=cameraman;

% Displaying original image
imagescgray(im_original,1,'Original image');

%% Creation of the problem

sigma_noise = 20/255;
im_noisy=im_original+sigma_noise*randn(size(im_original));

% Create a matrix with randomly 50 % of zeros entry
p=0.5;
matA=rand(size(im_original));
matA=(matA>(1-p));
% Define the operator
A=@(x) matA.*x;

% Depleted image
y=matA.*im_noisy;

% Displaying noisy image
imagescgray(im_noisy,2,'Noisy image');

% Displaying depleted image
imagescgray(y,3,'Depleted image');

%% Setting the proximity operator

% setting the function f1 (norm TV)
paramtv.useGPU = GLOBAL_useGPU;    % Use GPU
paramtv.verbose = verbose-1;
paramtv.maxit = 50;
f1.prox=@(x, T) prox_tv(x, T, paramtv);
f1.eval=@(x) tv_norm(x);

% setting the function f2
param_proj.epsilon = sqrt(sigma_noise^2*length(im_original(:))*p);
param_proj.A = A;
param_proj.At = A;
param_proj.y = y;
```

```
param_proj.verbose = verbose-1;
f2.prox=@(x,T) proj_b2(x,T,param_proj);
f2.eval=@(x) eps;

% setting the function f3
lambda = 10;
param_l2.A = A;
param_l2.At = A;
param_l2.y = y;
param_l2.verbose = verbose-1;
param_l2.tight = 0;
param_l2.nu = 1;
f3.prox=@(x,T) prox_l2(x,lambda*T,param_l2);
f3.grad=@(x) 2*lambda*A(A(x)-y);
f3.eval=@(x) lambda*norm(A(x)-y,'fro')^2;

%% Solving problem I

% setting different parameters for the simulation
param.verbose = verbose;    % display parameter
param.maxit = 100;    % maximum number of iterations
param.tol = 1e-5;    % tolerance to stop iterating
param.gamma = 1 ;     % Convergence parameter
% solving the problem with Douglas Rachord
sol = douglas_rachford(y,f1,f2,param);

%% Displaying the result
imagescgray(sol,4,'Problem I - Douglas Rachford');

%% Solving problem II (forward backward)
param.gamma=0.5/lambda;     % Convergence parameter
param.tol=1e-5;
% solving the problem with Douglas Rachord
sol21 = forward_backward(y,f1,f3,param);

%% Displaying the result
imagescgray(sol21,5,'Problem II - Forward Backward');

%% Solving problem II (Douglas Rachford)
param.gamma=0.5/lambda;     % Convergence parameter
sol22 = douglas_rachford(y,f3,f1,param);

 %% Displaying the result
imagescgray(sol22,6,'Problem II - Douglas Rachford');

%% Close the UNLcoBoX
close_unlocbox();
```

References

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