# Optimization for Machine Learning

## Lecture 2a: Projected, Proximal and Subgradient Descent

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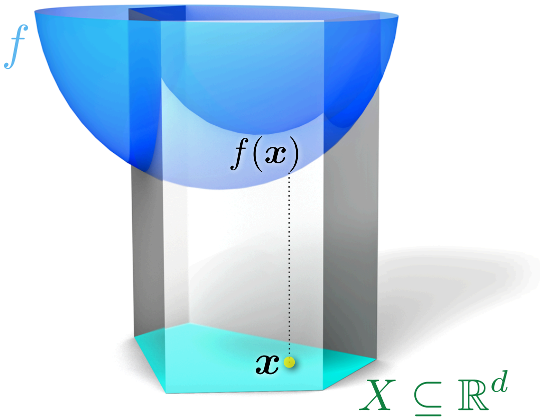
# Chapter 3

# Projected Gradient Descent

# Constrained Optimization

## Constrained Optimization Problem

$$\begin{array}{rl} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in X \end{array}$$




## Solving Constrained Optimization Problems

A Projected Gradient Descent

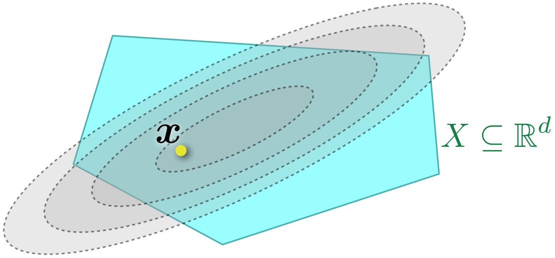
B Transform it into an *unconstrained* problem

# Constrained Optimization

## Solving Constrained Optimization Problems

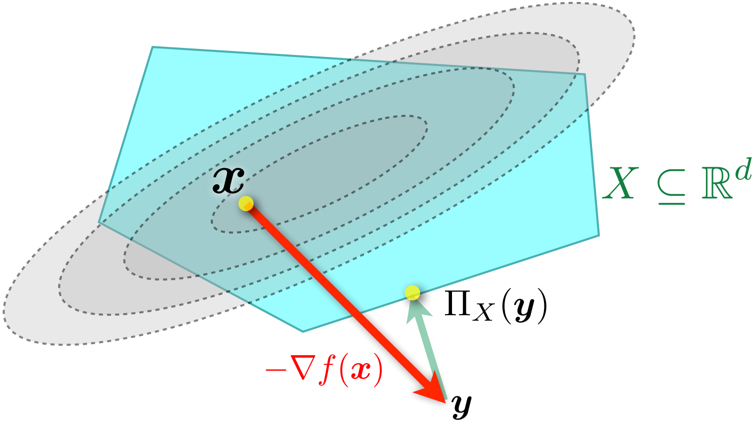
$$\begin{array}{rl} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in X \end{array}$$

- Here: Projected Gradient Descent



# Projected Gradient Descent

Idea: project onto $X$ after every step: $\Pi_X(\mathbf{y}) := \operatorname{argmin}_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{y}\|$




Projected gradient descent: $\mathbf{x}_{t+1} := \Pi_X\left[\mathbf{x}_t - \gamma \nabla f(\mathbf{x}_t)\right]$

# The Algorithm

**Projected gradient descent:**

$$
\begin{aligned}
\mathbf{y}_{t+1} &:= \mathbf{x}_t - \gamma \nabla f(\mathbf{x}_t), \\
\mathbf{x}_{t+1} &:= \Pi_X(\mathbf{y}_{t+1}) := \operatorname*{argmin}_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{y}_{t+1}\|^2.
\end{aligned}
$$

for **timesteps** $t = 0, 1, \dots,$ and **stepsize** $\gamma \geq 0$.

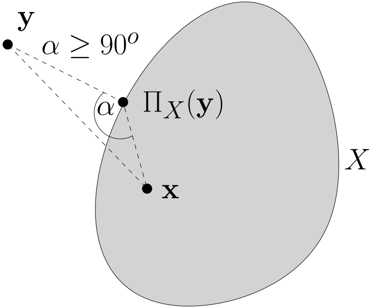
# Properties of Projection

### Fact

*Let $X \subseteq \mathbb{R}^d$ be closed and convex, $\mathbf{x} \in X, \mathbf{y} \in \mathbb{R}^d$. Then*

(i) $(\mathbf{x} - \Pi_X(\mathbf{y}))^\top(\mathbf{y} - \Pi_X(\mathbf{y})) \leq 0$.

(ii) $\|\mathbf{x} - \Pi_X(\mathbf{y})\|^2 + \|\mathbf{y} - \Pi_X(\mathbf{y})\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2$.



# Properties of Projection II

## Fact

*Let* $X \subseteq \mathbb{R}^d$ *be closed and convex,* $\mathbf{x} \in X, \mathbf{y} \in \mathbb{R}^d$. *Then*

(i) $(\mathbf{x} - \Pi_X(\mathbf{y}))^\top (\mathbf{y} - \Pi_X(\mathbf{y})) \leq 0$.

(ii) $\|\mathbf{x} - \Pi_X(\mathbf{y})\|^2 + \|\mathbf{y} - \Pi_X(\mathbf{y})\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2$.

## Proof.

(i) $\Pi_X(\mathbf{y})$ is minimizer of (differentiable) convex function $d_\mathbf{y}(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\|^2$ over $X$.
By first-order characterization of optimality (**Lemma 1.28**),

$$\begin{array}{rcl} 0 & \leq & \nabla d_\mathbf{y}(\Pi_X(\mathbf{y}))^\top (\mathbf{x} - \Pi_X(\mathbf{y})) \\ & = & 2(\Pi_X(\mathbf{y}) - \mathbf{y})^\top (\mathbf{x} - \Pi_X(\mathbf{y})) \\ \Leftrightarrow \quad 0 & \geq & 2(\mathbf{y} - \Pi_X(\mathbf{y}))^\top (\mathbf{x} - \Pi_X(\mathbf{y})) \\ \Leftrightarrow \quad 0 & \geq & (\mathbf{x} - \Pi_X(\mathbf{y}))^\top (\mathbf{y} - \Pi_X(\mathbf{y})) \end{array}$$

$\square$

# Properties of Projection III

## Fact

*Let* $X \subseteq \mathbb{R}^d$ *be closed and convex,* $\mathbf{x} \in X, \mathbf{y} \in \mathbb{R}^d$. *Then*

(i) $(\mathbf{x} - \Pi_X(\mathbf{y}))^\top(\mathbf{y} - \Pi_X(\mathbf{y})) \leq 0$.

(ii) $\|\mathbf{x} - \Pi_X(\mathbf{y})\|^2 + \|\mathbf{y} - \Pi_X(\mathbf{y})\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2$.

## Proof.

(ii)

$$\mathbf{v} := (\mathbf{x} - \Pi_X(\mathbf{y})), \quad \mathbf{w} := (\mathbf{y} - \Pi_X(\mathbf{y})).$$

By (i),

$$\begin{aligned} 0 \geq 2\mathbf{v}^\top\mathbf{w} &= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 \\ &= \|\mathbf{x} - \Pi_X(\mathbf{y})\|^2 + \|\mathbf{y} - \Pi_X(\mathbf{y})\|^2 - \|\mathbf{x} - \mathbf{y}\|^2. \end{aligned}$$

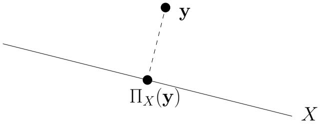
$\square$

# The Projection Step: $\Pi_X(\mathbf{y}) := \mathrm{argmin}_{\mathbf{x}\in X} \|\mathbf{x}-\mathbf{y}\|$

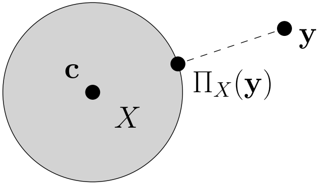
Computing $\Pi_X(\mathbf{y})$ is an optimization problem itself.

It can efficiently be solved in relevant cases:

- Projecting onto an affine subspace (leads to system of linear equations, similar to least squares)

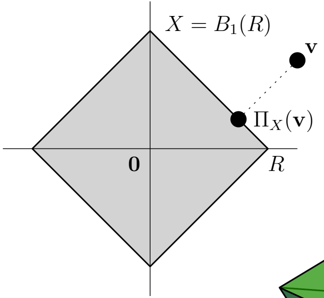


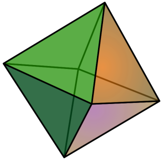

- Projecting onto a Euclidean ball with center $\mathbf{c}$ (simply scale the vector $\mathbf{y}-\mathbf{c}$)



# Projecting onto $\ell_1$-balls (needed in Lasso)

W.l.o.g. restrict to center at $\mathbf{0}$: $B_1(R) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_1 = \sum_{i=1}^d |x_i| \le R\}$.



$B_1(R)$ is the cross polytope ($2d$ vertices, $2^d$ facets). (octahedron, $d = 3$)

Section 3.5: projection can be computed in $\mathcal{O}(d \log d)$ time (can be improved to $\mathcal{O}(d)$)

# Results for projected gradient descent over closed and convex $X$

The same number of steps as gradient over $\mathbb{R}^d$!

- Lipschitz convex functions over $X$: $\mathcal{O}(1/\varepsilon^2)$ steps
- Smooth convex functions over $X$: $\mathcal{O}(1/\varepsilon)$ steps
- Smooth and strongly convex functions over $X$: $\mathcal{O}(\log(1/\varepsilon))$ steps

We will adapt the previous proofs for gradient descent.

BUT:

- Each step involves a projection onto $X$
- may or may not be efficient (in relevant cases, it is)...

# Section 3.6

## Proximal Gradient Descent

# Composite optimization problems

Consider objective functions composed as

$$f(\mathbf{x}) := g(\mathbf{x}) + h(\mathbf{x})$$

where $g$ is a "nice" function, where as $h$ is a "simple" additional term, which however doesn't satisfy the assumptions of niceness which we used in the convergence analysis so far.

In particular, an important case is when $h$ is not differentiable.

# Idea

The classical gradient step for minimizing $g$:

$$\mathbf{x}_{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \; g(\mathbf{x}_t) + \nabla g(\mathbf{x}_t)^\top (\mathbf{y} - \mathbf{x}_t) + \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{x}_t\|^2 \; .$$

For the stepsize $\gamma := \frac{1}{L}$ it exactly minimizes the local quadratic model of $g$ at our current iterate $\mathbf{x}_t$, formed by the smoothness property with parameter $L$.

Now for $f = g + h$, keep the same for $g$, and add $h$ unmodified.

$$\begin{aligned} \mathbf{x}_{t+1} &:= \underset{\mathbf{y}}{\operatorname{argmin}} \; g(\mathbf{x}_t) + \nabla g(\mathbf{x}_t)^\top (\mathbf{y} - \mathbf{x}_t) + \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{x}_t\|^2 + h(\mathbf{y}) \\ &= \underset{\mathbf{y}}{\operatorname{argmin}} \; \frac{1}{2\gamma} \|\mathbf{y} - (\mathbf{x}_t - \gamma \nabla g(\mathbf{x}_t))\|^2 + h(\mathbf{y}) \; , \end{aligned}$$

the proximal gradient descent update.

# The proximal gradient descent algorithm

An iteration of proximal gradient descent is defined as

$$\mathbf{x}_{t+1} := \mathrm{prox}_{h,\gamma}(\mathbf{x}_t - \gamma \nabla g(\mathbf{x}_t)) \ .$$

where the proximal mapping for a given function $h$, and parameter $\gamma > 0$ is defined as

$$\mathrm{prox}_{h,\gamma}(\mathbf{z}) := \operatorname*{argmin}_{\mathbf{y}} \left\{ \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{z}\|^2 + h(\mathbf{y}) \right\} \ .$$

The update step can be equivalently written as

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \gamma G_\gamma(\mathbf{x}_t)$$

for $G_{h,\gamma}(\mathbf{x}) := \frac{1}{\gamma}\Big(\mathbf{x} - \mathrm{prox}_{h,\gamma}(\mathbf{x} - \gamma \nabla g(\mathbf{x}))\Big)$ being the so called generalized gradient of $f$.

# A generalization of gradient descent?

- $h \equiv 0$: recover gradient descent
- $h \equiv \boldsymbol{\iota}_X$: recover projected gradient descent!

  Given a closed convex set $X$, the indicator function of the set $X$ is given as the convex function

$$\boldsymbol{\iota}_X : \mathbb{R}^d \to \mathbb{R} \cup +\infty$$
$$\mathbf{x} \mapsto \boldsymbol{\iota}_X(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

  Proximal mapping becomes

$$\operatorname{prox}_{h,\gamma}(\mathbf{z}) := \underset{\mathbf{y}}{\operatorname{argmin}} \left\{ \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{z}\|^2 + \boldsymbol{\iota}_X(\mathbf{y}) \right\} = \underset{\mathbf{y} \in X}{\operatorname{argmin}} \ \|\mathbf{y} - \mathbf{z}\|^2$$

# Convergence in $\mathcal{O}(1/\varepsilon)$ steps

Same as vanilla case for smooth functions, but now for any $h$ for which we can compute the proximal mapping.

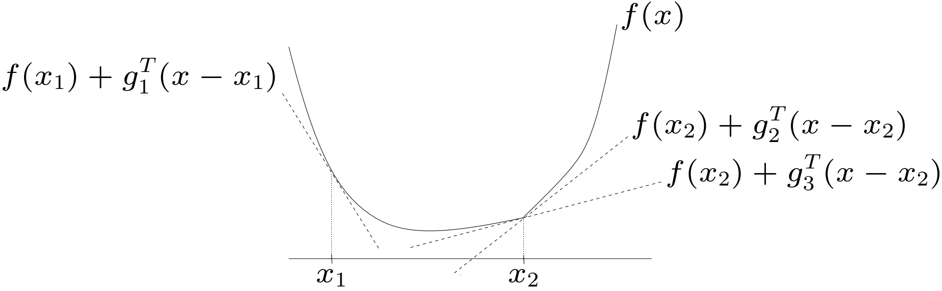
# Subgradients

What if $f$ is not differentiable?

### Definition

$\mathbf{g} \in \mathbb{R}^d$ is a subgradient of $f$ at $\mathbf{x}$ if

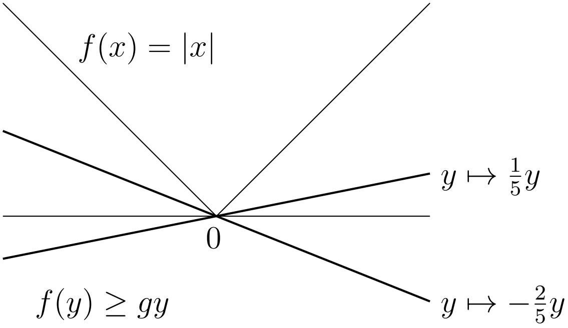
$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{g}^\top(\mathbf{y} - \mathbf{x}) \qquad \text{for all } \mathbf{y} \in \mathbf{dom}(f)$$




$\partial f(\mathbf{x}) \subseteq \mathbb{R}^d$ is the subdifferential, the set of subgradients of $f$ at $\mathbf{x}$.

# Subgradients II

Example:




Subgradient condition at $x = 0$: $f(y) \geq f(0) + g(y - 0) = gy$.

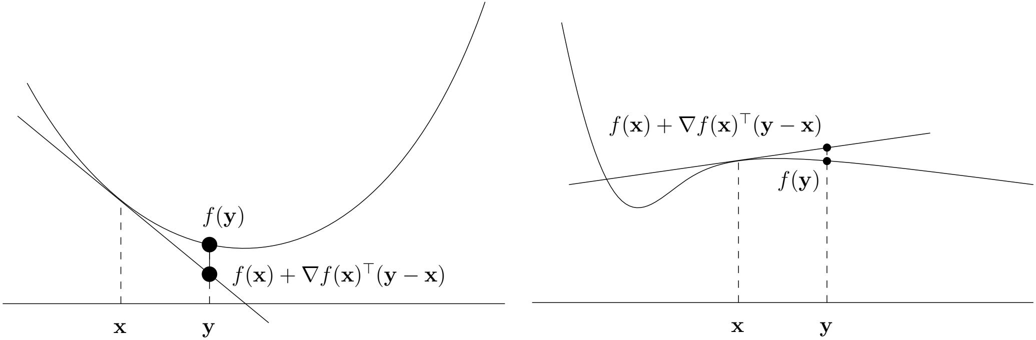
$\partial f(0) = [-1, 1]$

# Subgradients III

## Lemma (Exercise 28)

*If $f : \mathbf{dom}(f) \to \mathbb{R}$ is differentiable at $\mathbf{x} \in \mathbf{dom}(f)$, then $\partial f(\mathbf{x}) \subseteq \{\nabla f(\mathbf{x})\}$.*

Either exactly one subgradient $\nabla f(\mathbf{x})$... ...or no subgradient at all.

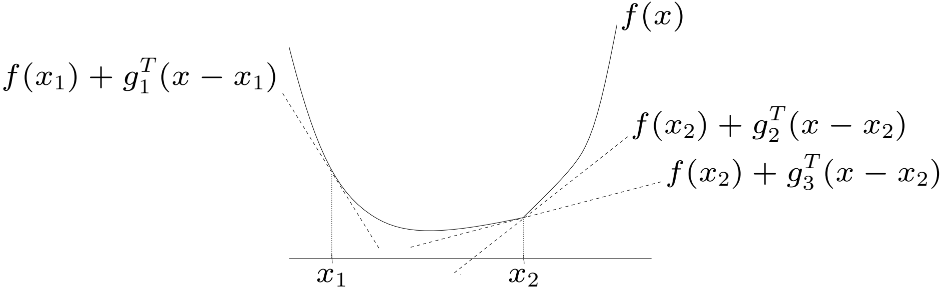


# Subgradient characterization of convexity

"convex = subgradients everywhere"

Lemma (Exercise 29)

*A function $f : \mathbf{dom}(f) \to \mathbb{R}$ is convex if and only if $\mathbf{dom}(f)$ is convex and $\partial f(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in \mathbf{dom}(f)$.*



# Convex and Lipschitz = bounded subgradients

**Lemma (Exercise 30)**

*Let* $f : \mathbf{dom}(f) \to \mathbb{R}$ *be convex,* $\mathbf{dom}(f)$ *open,* $B \in \mathbb{R}_+$*. Then the following two statements are equivalent.*

(i) $\|\mathbf{g}\| \leq B$ *for all* $\mathbf{x} \in \mathbf{dom}(f)$ *and all* $\mathbf{g} \in \partial f(\mathbf{x})$*.*

(ii) $|f(\mathbf{x}) - f(\mathbf{y})| \leq B\|\mathbf{x} - \mathbf{y}\|$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbf{dom}(f)$*.*

# Subgradient optimality condition

**Lemma**

*Suppose that* $f : \mathbf{dom}(f) \to \mathbb{R}$ *and* $\mathbf{x} \in \mathbf{dom}(f)$*. If* $\mathbf{0} \in \partial f(\mathbf{x})$*, then* $\mathbf{x}$ *is a global minimum.*

**Proof.**

By definition of subgradients, $\mathbf{g} = \mathbf{0} \in \partial f(\mathbf{x})$ gives

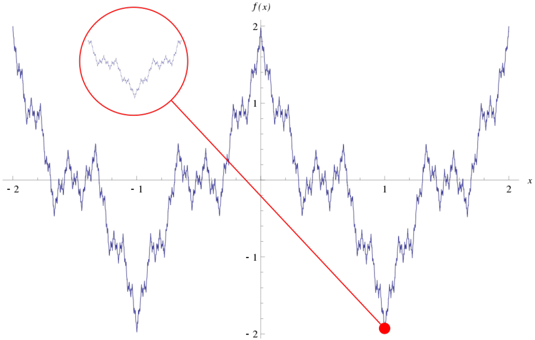
$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{g}^\top(\mathbf{y} - \mathbf{x}) = f(\mathbf{x})$$

for all $\mathbf{y} \in \mathbf{dom}(f)$, so $\mathbf{x}$ is a global minimum. □

# Differentiability of convex functions

How "wild" can a non-differentiable convex function be?

Weierstrass function: a function that is continuous everywhere but differentiable nowhere




https://commons.wikimedia.org/wiki/File:WeierstrassFunction.svg

# Differentiability of convex functions

Theorem ([Roc97, Theorem 25.5])

*A convex function $f : \mathbf{dom}(f) \to \mathbb{R}$ is differentiable almost everywhere.*

In other words:

- Set of points where $f$ is non-differentiable has measure $0$ (no volume).
- For all $\mathbf{x} \in \mathbf{dom}(f)$ and all $\varepsilon > 0$, there is a point $\mathbf{x}'$ such that $\|\mathbf{x} - \mathbf{x}'\| < \varepsilon$ and $f$ is differentiable at $\mathbf{x}'$.

# The subgradient descent algorithm

**Subgradient descent:** choose $\mathbf{x}_0 \in \mathbb{R}^d$.

$$
\begin{aligned}
&\text{Let } \mathbf{g}_t \in \partial f(\mathbf{x}_t) \\
&\mathbf{x}_{t+1} := \mathbf{x}_t - \gamma_t \mathbf{g}_t
\end{aligned}
$$

for **times** $t = 0, 1, \ldots,$ and **stepsizes** $\gamma_t \geq 0$.

Stepsize can vary with time!

This is possible in (projected) gradient descent as well, but so far, we didn't need it.

# Lipschitz convex functions: $\mathcal{O}(1/\varepsilon^2)$ steps

## Theorem

*Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex and $B$-Lipschitz continuous with a global minimum $\mathbf{x}^\star$; furthermore, suppose that $\|\mathbf{x}_0 - \mathbf{x}^\star\| \leq R$. Choosing the constant stepsize*

$$\gamma := \frac{R}{B\sqrt{T}},$$

*subgradient descent yields*

$$\frac{1}{T}\sum_{t=0}^{T-1} f(\mathbf{x}_t) - f(\mathbf{x}^\star) \leq \frac{RB}{\sqrt{T}}.$$

Proof is identical to the one of Theorem 2.1, except...

- In vanilla analyis, now use $\mathbf{g}_t \in \partial f(\mathbf{x}_t)$ instead of $\mathbf{g}_t = \nabla f(\mathbf{x}_t)$.
- Inequality $f(\mathbf{x}_t) - f(\mathbf{x}^\star) \leq \mathbf{g}_t^\top(\mathbf{x}_t - \mathbf{x}^\star)$ now follows from subgradient property instead of first-order charaterization of convexity.

# Optimality of first-order methods

With all the convergence rates we have seen so far, a very natural question to ask is if these rates are best possible or not. Surprisingly, the rate can indeed not be improved in general.

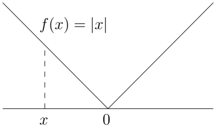
### Theorem (Nesterov)

*For any $T \leq d - 1$ and starting point $\mathbf{x}_0$, there is a function $f$ in the problem class of $B$-Lipschitz functions over $\mathbb{R}^d$, such that any (sub)gradient method has an objective error at least*

$$f(\mathbf{x}_T) - f(\mathbf{x}^\star) \geq \frac{RB}{2(1 + \sqrt{T+1})} \, .$$

# Smooth (non-differentiable) functions?

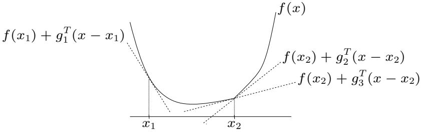
They don't exist (Exercise 31)!




At $0$, graph can't be below a tangent paraboloid.

Can we still improve over $O(1/\varepsilon^2)$ steps for Lipschitz functions?

Yes, if we also require strong convexity (graph is above not too flat tangent paraboloids).



# Strongly convex functions

### "Not too flat"

Straightforward generalization to the non-differentiable case:

**Definition**

Let $f : \mathbf{dom}(f) \to \mathbb{R}$ be convex, $\mu \in \mathbb{R}_+, \mu > 0$. Function $f$ is called strongly convex (with parameter $\mu$) if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{g}^\top(\mathbf{y} - \mathbf{x}) + \frac{\mu}{2}\|\mathbf{x} - \mathbf{y}\|^2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{dom}(f),\ \forall \mathbf{g} \in \partial f(\mathbf{x}).$$

# Strongly convex functions: characterization via "normal" convexity

### Lemma (Exercise 33)

*Let* $f : \mathbf{dom}(f) \to \mathbb{R}$ *be convex,* $\mathbf{dom}(f)$ *open,* $\mu \in \mathbb{R}_+, \mu > 0$*.* $f$ *is strongly convex with parameter* $\mu$ *if and only if* $f_\mu : \mathbf{dom}(f) \to \mathbb{R}$ *defined by*

$$f_\mu(\mathbf{x}) = f(\mathbf{x}) - \frac{\mu}{2}\|\mathbf{x}\|^2, \quad \mathbf{x} \in \mathbf{dom}(f)$$

*is convex.*

# Tame strong convexity

For fast convergence, we consider additional assumptions.

Smoothness? - Not an option in the non-differentiable case (Exercise 31).

Instead: assume that all subgradients $\mathbf{g}_t$ that we encounter during the algorithm are bounded in norm.

May be realistic if...

- we start close to optimality
- we run projected subgradient descent over a compact set $X$

May also fail!

- Over $\mathbb{R}^d$, strong convexity and bounded subgradients contradict each other! (Exercise 35).

# Tame strong convexity: $\mathcal{O}(1/\varepsilon)$ steps

### Theorem

*Let $f : \mathbb{R}^d \to \mathbb{R}$ be strongly convex with parameter $\mu > 0$ and let $\mathbf{x}^\star$ be the unique global minimum of $f$. With decreasing step size*

$$\gamma_t := \frac{2}{\mu(t+1)}, \quad t > 0,$$

*subgradient descent yields*

$$f\left(\frac{2}{T(T+1)}\sum_{t=1}^{T} t \cdot \mathbf{x}_t\right) - f(\mathbf{x}^\star) \leq \frac{2B^2}{\mu(T+1)},$$

*where $B = \max_{t=1}^{T} \|\mathbf{g}_t\|$.* $\uparrow$

convex combination of iterates

# Tame strong convexity: $\mathcal{O}(1/\varepsilon)$ steps II

Proof.

Vanilla analysis ($\mathbf{g}_t \in \partial f(\mathbf{x}_t)$):

$$\mathbf{g}_t^\top(\mathbf{x}_t - \mathbf{x}^\star) = \frac{\gamma_t}{2}\|\mathbf{g}_t\|^2 + \frac{1}{2\gamma_t}\left(\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\right).$$

Lower bound from strong convexity:

$$\mathbf{g}_t^\top(\mathbf{x}_t - \mathbf{x}^\star) \geq f(\mathbf{x}_t) - f(\mathbf{x}^\star) + \frac{\mu}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2.$$

Putting it together (with $\|\mathbf{g}_t\|^2 \leq B^2$):

$$f(\mathbf{x}_t) - f(\mathbf{x}^\star) \leq \frac{B^2\gamma_t}{2} + \frac{(\gamma_t^{-1} - \mu)}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{\gamma_t^{-1}}{2}\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2.$$

Summing over $t = 1, \ldots, T$: we used to have telescoping ($\gamma_t = \gamma, \mu = 0$)...

# Tame strong convexity: $\mathcal{O}(1/\varepsilon)$ steps III

Proof.

So far we have:

$$f(\mathbf{x}_t) - f(\mathbf{x}^\star) \leq \frac{B^2\gamma_t}{2} + \frac{(\gamma_t^{-1} - \mu)}{2} \left\|\mathbf{x}_t - \mathbf{x}^\star\right\|^2 - \frac{\gamma_t^{-1}}{2} \left\|\mathbf{x}_{t+1} - \mathbf{x}^\star\right\|^2.$$

To get telescoping, we would need $\gamma_t^{-1} = \gamma_{t+1}^{-1} - \mu$.

Works with $\gamma_t^{-1} = \mu(1+t)$, but not $\gamma_t^{-1} = \mu(1+t)/2$ (the choice here).

Exercise 36: what happens with $\gamma_t^{-1} = \mu(1+t)$?

Now: what happens with $\gamma_t^{-1} = \mu(1+t)/2$ (the choice here)?

# Tame strong convexity: $\mathcal{O}(1/\varepsilon)$ steps IV

Proof.
So far we have:

$$f(\mathbf{x}_t) - f(\mathbf{x}^\star) \leq \frac{B^2\gamma_t}{2} + \frac{(\gamma_t^{-1} - \mu)}{2} \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{\gamma_t^{-1}}{2} \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 .$$

Plug in $\gamma_t^{-1} = \mu(1+t)/2$ and multiply with $t$ on both sides:

$$\begin{aligned}
t \cdot \big(f(\mathbf{x}_t) - f(\mathbf{x}^\star)\big) &\leq \frac{B^2 t}{\mu(t+1)} + \frac{\mu}{4}\Big(t(t-1)\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - (t+1)t\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\Big) \\
&\leq \frac{B^2}{\mu} + \frac{\mu}{4}\Big(t(t-1)\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - (t+1)t\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\Big).
\end{aligned}$$

# Tame strong convexity: $\mathcal{O}(1/\varepsilon)$ steps V

Proof.
We have

$$t \cdot \big(f(\mathbf{x}_t) - f(\mathbf{x}^\star)\big) \leq \frac{B^2 t}{\mu(t+1)} + \frac{\mu}{4}\Big(t(t-1)\left\|\mathbf{x}_t - \mathbf{x}^\star\right\|^2 - (t+1)t\left\|\mathbf{x}_{t+1} - \mathbf{x}^\star\right\|^2\Big)$$
$$\leq \frac{B^2}{\mu} + \frac{\mu}{4}\Big(t(t-1)\left\|\mathbf{x}_t - \mathbf{x}^\star\right\|^2 - (t+1)t\left\|\mathbf{x}_{t+1} - \mathbf{x}^\star\right\|^2\Big).$$

Now we get telescoping...

$$\sum_{t=1}^{T} t \cdot \big(f(\mathbf{x}_t) - f(\mathbf{x}^\star)\big) \leq \frac{TB^2}{\mu} + \frac{\mu}{4}\Big(0 - T(T+1)\left\|\mathbf{x}_{T+1} - \mathbf{x}^\star\right\|^2\Big) \leq \frac{TB^2}{\mu}.$$

# Tame strong convexity: $\mathcal{O}(1/\varepsilon)$ steps VI

Proof.

Almost done:

$$\underline{\sum_{t=1}^{T} t \cdot \big(f(\mathbf{x}_t) - f(\mathbf{x}^\star)\big)} \leq \frac{TB^2}{\mu} + \frac{\mu}{4}\left(0 - T(T+1)\left\|\mathbf{x}_{T+1} - \mathbf{x}^\star\right\|^2\right) \leq \frac{TB^2}{\mu}.$$

Since

$$\frac{2}{T(T+1)} \sum_{t=1}^{T} t = 1,$$

Jensen's inequality yields

$$f\left(\frac{2}{T(T+1)} \sum_{t=1}^{T} t \cdot \mathbf{x}_t\right) - f(\mathbf{x}^\star) \leq \frac{2}{T(T+1)} \underline{\sum_{t=1}^{T} t \cdot \big(f(\mathbf{x}_t) - f(\mathbf{x}^\star)\big)}.$$

□

# Tame strong convexity: Discussion

$$f\left(\frac{2}{T(T+1)}\sum_{t=1}^{T} t \cdot \mathbf{x}_t\right) - f(\mathbf{x}^\star) \leq \frac{2B^2}{\mu(T+1)},$$

Weighted average of iterates achieves the bound (later iterates have more weight)

Bound is independent of initial distance $\|\mathbf{x}_0 - \mathbf{x}^\star\|$...

...but not really: $B$ typically depends on $\|\mathbf{x}_0 - \mathbf{x}^\star\|$ (for example, $B = \mathcal{O}(\|\mathbf{x}_0 - \mathbf{x}^\star\|)$ for quadratic functions)

Recall: we can only hope that $B$ is small (can be checked while running the algorithm)

What if we don't know the parameter $\mu$ of strong convexity?

$\rightarrow$ **Bad luck!** In practice, try some $\mu$'s, pick best solution obtained

# Bibliography

R. Tyrrell Rockafellar.
*Convex Analysis*.
Princeton Landmarks in Mathematics. Princeton University Press, 1997.