Lecture 5: ARMA Models and the Wold Decomposition

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Intro

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- So far, we learned what MA and AR models are;
- Very different ways of modelling dependence over time;
- An ARMA model combines both ways of modelling dependence $\implies$ very flexible model;
- Wold's Decomposition: ARMA models are *the* class of stationary processes you should worry about!
- We will also talk about how to *estimate* an ARMA model;
- We will talk about forecasting with ARMA models as well - really useful in the real world!

Questions?

ARMA(p, q) Models

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- Let ε_t be a white noise with variance σ^2;
- An ARMA(p, q) process y_t satisfies the following dynamics:

$$y_t = \mu + \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$$

- Equivalently:

$$\phi(L) y_t = \mu + \theta(L) \varepsilon_t$$

  where $\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$ and $\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$;

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  where $\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$ and $\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$;
- What conditions will ensure that y_t is stationary?

When is an ARMA(p, q) stationary?

- If the roots of $\phi(L)$ are outside of the unit circle, then $\phi^{-1}(L)$ is well-defined;
- We can invert $\phi(L)$ to get:

$$y_t = \frac{\mu}{1 - \phi_1 - ... - \phi_p} + \theta(L)\varepsilon_t = \frac{\mu}{1 - \phi_1 L - ... - \phi_p L^p} + \psi(L)\varepsilon_t$$

where $\psi(L) = \theta(L)\phi^{-1}(L)$;

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where $\psi(L) = \theta(L)\phi^{-1}(L)$;

- The stationarity of y_t depends only on the roots of $\phi(L)$, not on the roots of $\theta(L)$;
- From here, it is clear that: $\mathbb{E}[y_t] = \frac{\mu}{1-\phi_1-...-\phi_p}$

What kind of autocovariance structure do we get?

- Notice that if $j > q$, then:

$$\gamma_j = \phi_1 \gamma_{j-1} + ... + \phi_p \gamma_{j-p}$$

- Roots of $\phi(L)$ are outside the unit circle $\implies \gamma_j \to 0$ as $j \to \infty$ (exponentially fast!);

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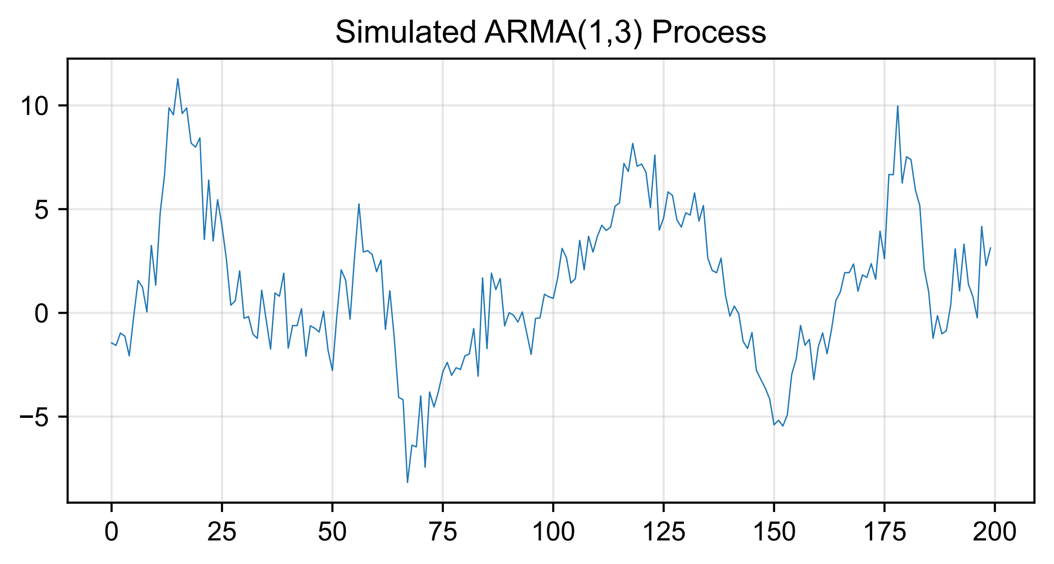
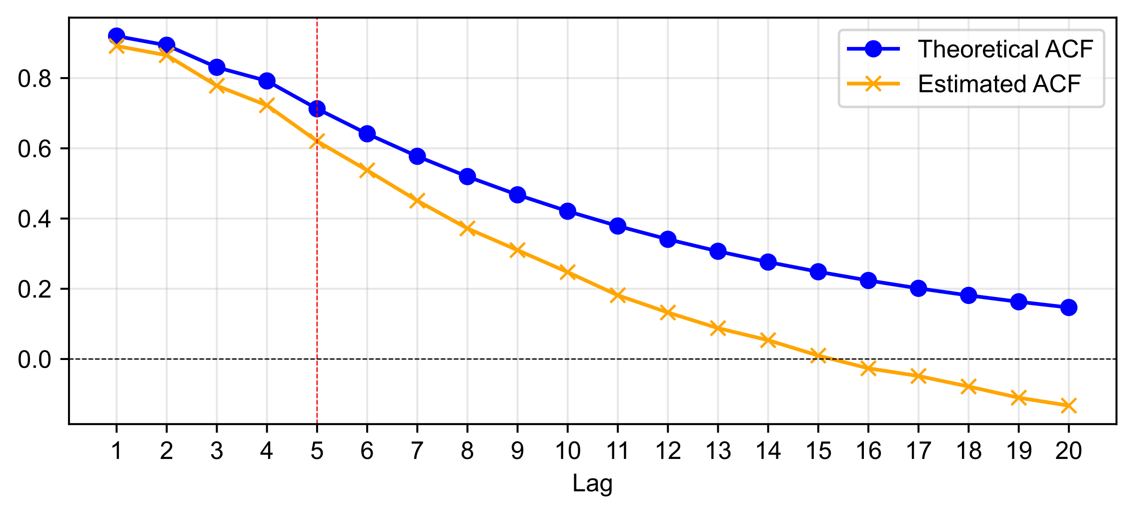
What is important here:

After lag q the decay should be fast and exponential. The MA part will never create trouble for stationarity.

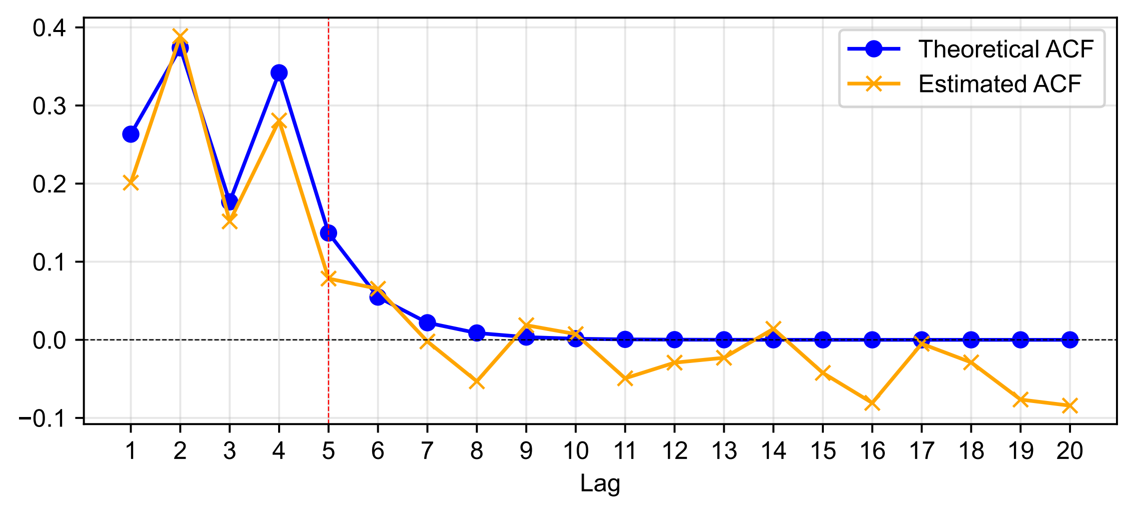
Simulated Example: $y_t = 0.9y_{t-1} + \varepsilon_t - 0.8\varepsilon_{t-1} + 0.6\varepsilon_{t-2} + 0.4\varepsilon_{t-3} + 0.8\varepsilon_{t-4}$



Theoretical vs Estimated ACF



Let's repeat it with $\phi_1 = 0.4$



Example for ARMA(1,1)

- Consider an ARMA(1,1) process:

$$y_t = \mu + \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}, \qquad |\phi_1| < 1$$

- We already know that $\gamma_2 = \phi_1 \gamma_1$;
- More generally, $\gamma_j = \phi_1 \gamma_{j-1}$ for $j \geq 2$. We just need to find γ_0 and γ_1;

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$$\begin{aligned}
\gamma_0 &= \phi_1 \gamma_1 + Cov(\varepsilon_t, y_t) + \theta_1 Cov(\varepsilon_{t-1}, y_t) = \phi_1 \gamma_1 + \sigma^2 + \theta_1 \phi_1 \sigma^2 + \theta_1^2 \sigma^2 \\
\gamma_1 &= \phi_1 \gamma_0 + Cov(\varepsilon_t, y_{t-1}) + \theta_1 Cov(\varepsilon_{t-1}, y_{t-1}) = \phi_1 \gamma_0 + \theta_1 \sigma^2
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Now we can solve this linear system!

Example of ARMA(1,1)

- Just solve the linear system:

$$
\begin{aligned}
\gamma_0 &= \frac{(1+\theta_1^2+2\theta_1\phi_1)\sigma^2}{1-\phi_1^2} \\
\gamma_1 &= \left[\theta+\phi+\frac{(\theta+\phi)^2\phi}{1-\phi^2}\right]\sigma^2 \\
\gamma_j &= \phi_1^{j-1}\gamma_1, \quad j \geq 2
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- Wait a minute... what would happen if $\theta_1 = -\phi_1$?

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- In fact, ARMA models can be *overparameterized*;
- Remember that we can always factor the lag polynomials:

$$\phi(L) = (1 - \phi_1 L)(1 - \phi_2 L) \dots (1 - \phi_p L)$$

$$\theta(L) = (1 + \theta_1 L)(1 + \theta_2 L) \dots (1 + \theta_q L)$$

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- If there are common roots that cancel out, we can have the exact same correlation structure with $p - 1$ lags of y_t and $q - 1$ lags of ε_t;
- This is a theoretical justification to prefer small values of p and q when estimating an ARMA model!
- When we write "ARMA(p,q)" we implicitly mean the *minimal representation*!

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- Our data: $\{y_1, y_2, \dots, y_{T-1}, y_T\} \implies$ a *strip* of *one* realized path with T observations;
- Exact forecasts will depend on the *infinite* past of the process;
- We will use the *available* information to compute the approximate forecasts;

Forecasting with Infinite Data

- Let's say you have a stationary ARMA model and has already computed the MA(∞) representation:

$$y_t - \mu = \psi(L)\varepsilon_t = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} = \epsilon_t + \sum_{i=1}^{\infty} \psi_i \epsilon_{t-i}$$

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- Let's say you want to compute $\mathbb{E}[y_{t+s} | \varepsilon_t, \varepsilon_{t-1}, ...]$;
- Just using the definition:

$$y_{t+s} - \mu = \varepsilon_{t+s} + \psi_1 \varepsilon_{t+s-1} + ... + \psi_{s-1}\varepsilon_{t+1} + \psi_s \varepsilon_t + \sum_{i=s+1}^{\infty} \psi_i \varepsilon_{t+s-i}$$

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- This implies: $\mathbb{E}[y_{t+s}|\varepsilon_t, \varepsilon_{t-1}, ...] = \mu + \sum_{i=s}^{\infty} \psi_i \varepsilon_{t+s-i}$

Forecasting with Infinite Data

- There is a shorthand notation that is useful here. Notice that:

$$\frac{\psi(L)}{L^s} = L^{-s} + \psi_1 L^{1-s)} + \psi_2 L^{2-s} + ... + \psi_{s-1} L^{-1} + \psi_s + \psi_{s+1} L + \psi_{s+2} L^2 ...$$

- The *annihilation operator* denoted by $[.]_+$ only considers the positive powers of L:

$$\left[\frac{\psi(L)}{L^s}\right]_+ \equiv \psi_s + \psi_{s+1} L + \psi_{s+2} L^2 + ...$$

- Now we can write:

$$\mathbb{E}[y_{t+s}|\varepsilon_t, \varepsilon_{t-1}, ...] = \mu + \left[\frac{\psi(L)}{L^s}\right]_+ \varepsilon_t$$

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- We never really observe the shocks ε_t. How can this be useful?
- Sometimes we can actually back them out from infinite data!
- Let's go back to the ARMA(p, q) representation: $\phi(L)(y_t - \mu) = \theta(L)\varepsilon_t$
- $\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$ and $\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$;
- If all roots of $\phi(L)$ are outside the unit circle, then $\phi^{-1}(L)$ is well-defined;
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- In this case we can write:

$$\theta(L)^{-1}\phi(L)(y_t - \mu) = \varepsilon_t \implies \varepsilon_t = \eta(L)(y_t - \mu)$$

where $\eta(L) = \theta^{-1}(L)\phi(L) = \eta_0 + \eta_1 L + \eta_2 L^2 + ...$

The General Formula

- So, for an ARMA(p, q) process with all roots outside the unit circle:

$$\mathbb{E}[y_{t+s}|y_t, y_{t-1}, ...] = \mu + \left[\frac{\psi(L)}{L^s}\right]_+ \eta(L)(y_t - \mu)$$

- In practice, a computer will do the necessary algebra to get the coefficients;
- But it is really important to understand *why* a computer can do that!
- This is also called the **Wiener-Kolmogorov** prediction formula;
- The crucial assumptions are stationarity and invertibility, i.e., roots of $\phi(L)$ and $\theta(L)$ must be outside the unit circle!

Example: AR(1)

- Assume that $(1-\phi L)y_t = \varepsilon_t$ with $|\phi| < 1$;
- Then we know that $\psi(L) = \phi(L)^{-1} = 1 + \phi L + \phi^2 L^2 + \ldots$ and $\eta(L) = \phi(L) = 1 - \phi L$;

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- The annihilation operator gives:

$$\left[\frac{\psi(L)}{L^s}\right]_+ = \phi^s + \phi^{s+1}L + \phi^{s+2}L^2 + ... = \phi^s(1 + \phi L + \phi^2 L^2 + ...) = \phi^s \left(\frac{1}{1-\phi L}\right)$$

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- So we get:

$$\mathbb{E}[y_{t+s} | y_t, y_{t-1}, ...] = \mu + \phi^s (y_t - \mu)$$

- What happens when $s \to \infty$? What's the intuition for this result?
- You will do the AR(p) case in the problem set!

Example: MA(q)

- Now let's assume that $y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$;
- Then $\psi(L) = \theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$, and $\eta(L) = \theta(L)^{-1}$;
- If $s > q$, then the annihilation operator gives 0;
- In that case: $\mathbb{E}[y_{t+s} | y_t, y_{t-1}, ...] = \mu$. What's the intuition for this result?

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- In that case: $\mathbb{E}[y_{t+s} | y_t, y_{t-1}, ...] = \mu$. What's the intuition for this result?
- In case $s \leq q$, then:

$$\left[\frac{\psi(L)}{L^s}\right]_+ = \theta_s + \theta_{s+1} L + ... + \theta_q L^{q-s}$$

- The final forecast is:

$$\mathbb{E}[y_{t+s} | y_t, y_{t-1}, ...] = \mu + \frac{(\theta_s + \theta_{s+1} L + ... + \theta_q L^{q-s})}{\theta(L)^{-1}} (y_t - \mu)$$

Questions?

Forecasting with *Finite* Data

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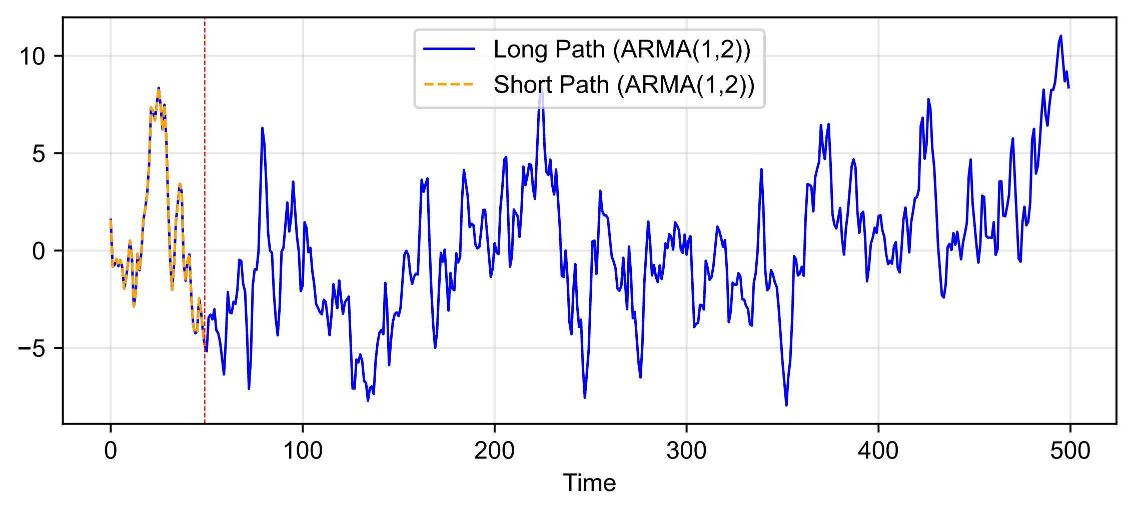
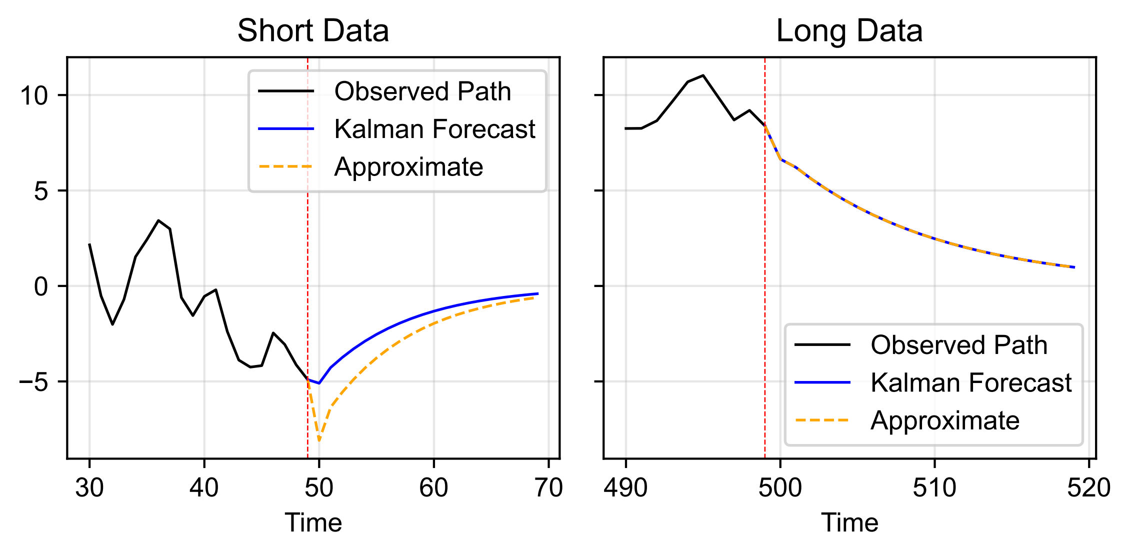
- Bad news about what we did: we assumed we had infinite data. That is never the case...
- There are two main ways of dealing with this problem:
  1. Create an "approximate" forecast;
  2. Use some method that explictly accounts for "missing data";
- The most common method for (2) is something called the "Kalman Filter";
- We don't have time to cover it, but it's super useful in Macro/Finance/estimation of DSGE models, etc!
- Most statistical software use (2) as the method to construct forecasts;
- But learning (1) is instructive and yields the same results is T is large!

Example: ARMA(1,2)



Example: ARMA(1,2) - Small vs Large Sample Forecasts



The End

References

- Chapter 3 from Hamilton's book for and the basics of ARMA models;
- Chapter 4 from Hamilton's book for Forecasting and Wold's Decomposition;
- Chapter 5 from Hamilton's book for Estimation of ARMA models;