Problem Set III

Econometrics I - FGV EPGE
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Problem 1 – (points: 1)

Consider an AR(p) process:

$$y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \varepsilon_t, \tag{1}$$

where $\varepsilon_t \sim$ i.i.d. $(0, \sigma^2)$. This questions will explore the dynamics of conditional moments of y_{t+h} given $\mathcal{I}_t$, where $\mathcal{I}_t = \{y_t, y_{t-1}, \ldots\}$ represents the information set available at time t.

a) Consider $p \times 1$ vector $Y_t = (y_t - \mu, y_{t-1} - \mu, \ldots, y_{t-p+1} - \mu)'$. Show that there exists a $p \times p$ matrix A and a $p \times 1$ vector U_t such that:

$$Y_t = AY_{t-1} + U_t, \quad \forall t$$

Additionally, show that $\Omega \equiv \mathbb{E}[U_t U_t']$ is a $p \times p$ matrix with all elements equal to zero, except for the first element of the main diagonal, which is equal to σ^2.

Hint: Matrix A will only have 0's and 1's, except for the first row.

b) Show that $\mathbb{E}[Y_{t+h}|\mathcal{I}_t] = A^h Y_t$.

c) Find an expression for $\text{Var}[Y_{t+h}|\mathcal{I}_t]$ that depends only on A, Ω and h;

d) Show that the eigenvalues of A are the roots of the polynomial

$$\Phi(z) = (-1)^p \left(z^p - \phi_1 z^{p-1} - \phi_2 z^{p-2} - \cdots - \phi_p \right),$$

i.e., this is its characteristic polynomial;

Hint 1: Recall that the determinant of a triangular matrix is equal to the product of its main diagonal elements.

Hint 2: Recall that if we multiply a column of a matrix by a constant and add the result to another column, the determinant does not change. Try applying operations on $(A - \lambda I)$ to make it triangular – this is a good refresher in Linear Algebra, isn't it?

e) Even if you have not completed the previous item, argue that the eigenvalues of A are all smaller than one in absolute value if the AR(p) process is stationary;

f) Find the limits of $\mathbb{E}[Y_{t+h}|\mathcal{I}_t]$ and $\text{Var}[Y_{t+h}|\mathcal{I}_t]$ as $h \to \infty$ if the process is stationary. What is the intuition for this result?

Problem 2 – (points: 1)

In this question, we will explore one example of a stationary process that is not an ARMA process. Let $\psi_j = \frac{1}{j^2}$ for $j \neq 0$ and $\psi_0 = 1$. Consider the process y_t defined in the following way:

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \varepsilon_t \sim \text{i.i.d. } (0, \sigma^2).$$

1. Compute the mean and variance of y_t;
2. Compute the autocovariance function $\gamma(h) = Cov(y_t, y_{t-h})$ for $h = 1, 2, 3, \ldots$; Is this process covariance-stationary? Why?
3. Show that there are positive constants c_1 and c_2 such that $c_1 \leq h^2 \cdot \gamma(h) \leq c_2$. Conclude that $\gamma(h) = O(1/h^2)$;
4. Now, assume by contradiction that y_t is an ARMA(p, q) process for some finite p and q. Let $\tilde{\gamma}(h)$ be the h-th autocovariance implied by the coefficients of this ARMA process. Show that

   $$\lim_{h \to \infty} \frac{\gamma(h)}{|\tilde{\gamma}(h)|} = +\infty$$

   Conclue that y_t can never be an ARMA(p, q) process.

Problem 3 – (points: 1)

Empirical question with ARMA estimation and lag selection. Also add an item so they can do forecasts.

Problem 4 – (points: 1)

Empirical question