

Problem Set II

Econometrics I - FGV EPGE
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Problem 1 – (points: 1)

In this problem, we will play with different concepts.

a) Consider the following process:

$$y_t = \cos(t + \theta), \forall t \in \mathbb{R}$$

where θ is a discrete random variable, distributed with uniform probability over the set $[0, \pi/2, \pi, 3\pi/2]$.

Show that this process is covariance stationary (or weakly stationary).

b) Show that this process is **not** strictly stationary.

c) Now, we consider a different process. Let z_t be such that:

$$z_t = \begin{cases} 1, & \forall t \text{ with probability } 0.5 \\ 0, & \forall t \text{ with probability } 0.5 \end{cases}$$

Argue that this process is weakly stationary by computing its first two moments and its autocovariance function. Now, argue that the time-series mean of *any* realized path will *never* converge towards the mean of z_t .

Problem 2 – (points: 2)

Consider the following AR(2) process:

$$(I - 1.1L + 0.18L^2)y_t = \varepsilon_t$$

where ε_t is a white noise process. Is this process stationary? If so, compute its mean, its variance, and its whole autocovariance function.

Problem 3 – (points: 1)

This question is a "warm-up" exercise that will be useful for the following problem.

Consider the polynomial $f(x) = a_p x^p + a_{p-1} x^{p-1} + \dots + a_1 x + a_0$, where $a_i \in \mathbb{R}$ for all $i = 0, 1, \dots, p$ and $a_p \neq 0$. Further assume that $a_0 \neq 0$. Let λ be any root for this polynomial.

- Argue that $\lambda \neq 0$;
- Consider another polynomial $g(x) = a_p + a_{p-1} x + \dots + a_1 x^{p-1} + a_0 x^p$. Show that $1/\lambda$ is a root of $g(\cdot)$;
- Argue that if $f(\cdot)$ has all its roots outside the unit circle, then $g(\cdot)$ has all its roots inside the unit circle.

Problem 4 – (points: 3)

In this question, you will prove a claim we made in the slides. Consider a general $AR(p)$ model:

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t$$

where ε_t is a white noise process. Define the operator $\Phi(L) = I - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$. We assume that all roots of its associated polynomial $f(x) = 1 - a_1 x - a_2 x^2 - \dots - a_p x^p$ lie outside the unit circle. As we saw in the slides, this implies that there exists an operator $\Psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \dots$ such that $\Psi(L)\Phi(L) = I$.

In that case, we can write $y_t = \Psi(L)\mu + \Psi(L)\varepsilon_t$.

- Compute $\Psi(L)\mu$ explicitly as a function of μ , and (ϕ_1, \dots, ϕ_p) .

Hint: what happens when you apply $\Phi(L)$ to a constant?

- We argued in class that the invertibility condition for $\Phi(L)$ was enough to ensure that this process is stationary because its $MA(\infty)$ representation is stationary. We will now work in steps to prove this claim.

Recall that the MA representation will be stationary if $\sum_{i=0}^{\infty} \psi_i^2 < \infty$. We need to prove this result. Our approach will be directly characterizing ψ_i as a function of (ϕ_1, \dots, ϕ_p) .

By definition, we have that:

$$(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p)(\psi_0 + \psi_1 L + \psi_2 L^2 + \dots) = 1$$

After performing these infinite multiplications, constant terms, terms that depend on L , terms that depend on L^2 , terms that depend on L^3 , and so on. Any terms that depend on powers of L should be identically zero, by definition.

Show that

$$\psi_k = \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2} + \dots + \phi_p \psi_{k-p}, \quad \forall k > p. \quad (1)$$

3. Recognize that (1) is a linear, homogenous, p -order difference equation in the ψ 's. Find its characteristic polynomial and explain why all its roots lie inside the unit circle.
4. It is a classic mathematical fact that all solutions to (1) are linear combinations of at most p terms of form $m(k) \cdot r^k$ where $m(\cdot)$ is a polynomial in k of order at most p , and r is a root of the characteristic polynomial. If such a fact is too obscure, [click here](#).

Using this fact, explain why $\sum_{i=0}^{\infty} \psi_i^2 < \infty$.

Hint: think about the different tests for the convergence of infinite series.

Problem 5 – (points: 3)

In this question, you will analyze two common price indexes for the Brazilian economy: IPCA and IGP-M. The data for this question is contained on the `ipca_igpm.csv` file. You have access to the monthly time series of the month-over-month changes for these price indexes.

- a) What are the main conceptual differences between IPCA and IGP-M? What are they measuring?
- b) Create a time-series plot with the two series.
- c) Assume both series are stationary. Compute their mean, variance, and autocovariances up to lag 24. Which one displays the most volatility?
- d) Plot the autocorrelation function of both series. How would you describe the differences in persistence between them?
- e) Given your answer in item a), are the results from items b) and c) in line with your expectations? Discuss your intuition.