Lecture 1: Non-Parametric Estimation

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Introduction

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So far, you have concentrated on *parametric models*:

$$Y_i = g(X_i, \theta) + u_i$$

where g is a **known** function and θ needs to be estimated;

- Example: $g(X, \theta) = X'\theta$, good and old OLS;

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- Example: $g(X, \theta) = X'\theta$, good and old OLS;
- But what happens if you don't know g?
- If you have a structural model, theory alone might give you g;
- Just assume some g that looks cute and call it a day?
- We can do better $\implies$ non-parametric estimation;

Motivation

Our goal will be estimating:

$$m(x) = \mathbb{E}[Y|X = x]$$

- Examples: expected wage given education, expected returns given exchange rates...
- For now we assume that both Y and X are scalars;
- We assume the researcher has access to a random sample $\{(Y_i, X_i)\}_{i=1}^n$;

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- For now we assume that both Y and X are scalars;
- We assume the researcher has access to a random sample $\{(Y_i, X_i)\}_{i=1}^{n}$;
- With OLS, you always assume that $m(.)$ is an affine function;
- Importantly: in OLS, $\frac{\partial m(x)}{\partial x} =$ constant;

What if X is discrete?

- Assume that $X \in \{x_1, x_2, ..., x_l\}$ for some l. What's the natural estimator?

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$$\hat{m}(x) = \frac{\sum_{i=1}^{n} I_{X_i = x} \cdot Y_i}{\sum_{i=1}^{n} I_{X_i = x}}$$

- Example: binary X, a treatment;
- You will prove in the problem set that this estimator is consistent under mild conditions;
- How would you describe this estimator in words?

What about the continuous case?

- Now assume $X \in \mathbb{R} \implies$ the event $\{X_i = x\}$ has zero probability;
- Well... if $m(.)$ is continuous... maybe observing points in a neighborhood of x is good enough!
- Problem: how large is a neighborhood?

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- Well... if $m(.)$ is continuous... maybe observing points in a neighborhood of x is good enough!
- Problem: how large is a neighborhood?

The **Binned Estimator**, given $h > 0$ (called *bandwidth*), is defined as:

$$\hat{m}(x) = \frac{\sum_{i=1}^{n} I_{\{|X_i - x| \leq h\}} \cdot Y_i}{\sum_{i=1}^{n} I_{\{|X_i - x| \leq h\}}}$$

- How do you think h will play a role here? Small vs large h?

What about the continuous case?

Notice that we can also write

$$\hat{m}(x) = \sum_{i=1}^{n} w_i(x) \cdot Y_i, \qquad w_i(x) \equiv \frac{I_{\{|X_i - x| \leq h\}}}{\sum_{j=1}^{n} I_{\{|X_j - x| \leq h\}}}$$

in which these w_i's are weakly positive and $\sum_{i=1}^{n} w_i(x) = 1$.

- We can view this estimator and a weighted average of Y, with x-dependent weights;
- True or false: can we interpret these weights as a *probability distribution*?

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- How do these weights change with h? Smoothly? Sharply?

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- True or false: can we interpret these weights as a *probability distribution*?
- How do these weights change with h? Smoothly? Sharply?
- One important drawback is that weights change discontinuously with h!
- $\hat{m}(x)$ is always a step function even if $m(.)$ is continuous!

Kernels

Kernels

- Ideally: points that are close to x should matter more for estimation, in a smooth way;
- It seems that we would like to have a "continuous way" of measuring distances;
- (Continuous) kernels are exactly what we need: they are fancy weights;

Definition (Second-Order Kernel)

A second-order kernel function $K(u)$ satisfies:

1. $0 \leq K(u) \leq \bar{K} < \infty$; $\implies$ the kernel is positive and bounded;
2. $K(u) = K(-u)$; $\implies$ the kernel is symmetric around zero;
3. $\int_{-\infty}^{\infty} K(u)du = 1$; $\implies$ this is like asking weights to sum up to 1;
4. $\int_{-\infty}^{\infty} |u|^r K(u)du < \infty$ for positive integers r; $\implies$ "not too fat tails";

Examples?

Kernel	Formula	R_K
Rectangular	$K(u) = \begin{cases} \frac{1}{2\sqrt{3}} & \text{if } \lvert u\rvert < \sqrt{3} \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{2\sqrt{3}}$
Gaussian	$K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$	$\frac{1}{2\sqrt{\pi}}$
Epanechnikov	$K(u) = \begin{cases} \frac{3}{4\sqrt{5}}\left(1 - \frac{u^2}{5}\right) & \text{if } \lvert u\rvert < \sqrt{5} \\ 0 & \text{otherwise} \end{cases}$	$\frac{3\sqrt{5}}{25}$
Triangular	$K(u) = \begin{cases} \frac{1}{\sqrt{6}}\left(1 - \frac{\lvert u\rvert}{\sqrt{6}}\right) & \text{if } \lvert u\rvert < \sqrt{6} \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sqrt{6}}{9}$

Figure 1: Examples of Kernels

A More General Estimator

For a given bandwidth $h > 0$, the **Nadaraya-Watson (NW)** estimator is given by

$$\hat{m}(x) = \frac{\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) \cdot Y_i}{\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)}$$

- Common choices of kernel: Gaussian and Epanechnikov;
- Choosing the bandwidth is more important than the kernel;

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- Common choices of kernel: Gaussian and Epanechnikov;
- Choosing the bandwidth is more important than the kernel;
- What happens when $h \to 0$? And when $h \to \infty$?
- You will prove that this estimator nests the binned estimator;

Questions?

Aymptotic Properties

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- Do we have any guarantees that this methodology works?
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Asymptotic Setup:

- We will assume that the sample size $n \to \infty$;
- But we will also need $h = h(n)$ to be converging towards zero $\implies h \to 0$;
- But it cannot go to zero too fast $\implies n \cdot h \to \infty$
- Is this intuitive? Why?

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- Is this intuitive? Why?
- The asymptotic theory will also be *pointwise*, i.e., for a fixed value of x;
- The asymptotic distribution of our estimator will change as x changes;
- Very important: we will work through the case of an **interior point** x;

Asymptotic Theory

We start writing $Y_i = m(X_i) + U_i$, where $\mathbb{E}[U_i|X_i] = 0$ and $\sigma^2(x) \equiv \text{Var}(U_i|X_i = x)$.

Let $x \in \mathbb{R}$ and write $Y_i = m(x) + (m(X_i) - m(x)) + U_i$

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Let $x \in \mathbb{R}$ and write $Y_i = m(x) + (m(X_i) - m(x)) + U_i$

Then we can write:

$$\begin{aligned}
\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) Y_i = & \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) m(x) \\
& + \underbrace{\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) (m(X_i) - m(x))}_{\equiv \hat{\Delta}_1(x)} \\
& + \underbrace{\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) U_i}_{\equiv \hat{\Delta}_2(x)}
\end{aligned}$$

Asymptotic Theory

- Assume x has some distribution with density $f()$. Obviously, we assume $f(x) > 0$.
- Let $f_n(x) \equiv \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)$;
- You will prove in your problem set that $f_n(x) \xrightarrow{p} f(x)$ as $n \to \infty$;
- This is what we call the non-parametric kernel density estimator;
- This is just a fancy way of writing a histogram;
- For now, we will take this result for granted;

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- This is just a fancy way of writing a histogram;
- For now, we will take this result for granted;
- The previous expression simplifies to

$$\hat{m}(x) - m(x) = \frac{1}{f_n(x)} \left[\hat{\Delta}_1(x) + \hat{\Delta}_2(x)\right]$$

Asymptotic Theory

We will show that $\sqrt{nh} \cdot \hat{\Delta}_1(x)$ converges in probability and $\sqrt{nh} \cdot \hat{\Delta}_2(x)$ has a limiting distribution.

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- First we analyze $\hat{\Delta}_2(x)$. Notice that

$$\mathbb{E}\left[\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) U_i\right] = \frac{1}{h}\mathbb{E}\left[K\left(\frac{X_i - x}{h}\right) U_i\right] = 0$$

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- We also see that:

$$\begin{aligned}
\text{Var}\left[\hat{\Delta}_2(x)\right] &= \frac{1}{nh^2}\mathbb{E}\left[\left(K\left(\frac{X_i - x}{h}\right) U_i\right)^2\right] \\
&= \frac{1}{nh^2}\mathbb{E}\left[K\left(\frac{X_i - x}{h}\right)^2 \sigma^2(X_i)\right] \\
&= \frac{1}{nh^2}\int_{-\infty}^{\infty} K\left(\frac{z - x}{h}\right)^2 \sigma^2(z) f(z)\, dz
\end{aligned}$$

Asymptotic Theory

- We perform a change of variable: $u = \frac{z-x}{h}$
- We can only do this because we are assuming that x is in the interior of its support;
- The last integral becomes:

$$\frac{1}{nh^2} \int_{-\infty}^{\infty} K\left(\frac{z-x}{h}\right)^2 \sigma^2(z) f(z)\, dz = \frac{1}{nh} \int_{-\infty}^{\infty} K(u)^2\, \sigma^2(x+hu)\, f(x+hu)\, du$$

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- For any continuous function $g(.)$ and fixed u, we have $g(x+hu) = g(x) + o(1)$ as $h \to 0$;

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$$\frac{1}{nh}\int_{-\infty}^{\infty} K(u)^2\,\sigma^2(x+hu)\,f(x+hu)\,du = \frac{\sigma^2(x)f(x)}{nh}\int_{-\infty}^{\infty} K(u)^2 du + o\left(\frac{1}{nh}\right)$$

Asymptotic Theory

- We define $R(K) \equiv \int_{-\infty}^{\infty} K(u)^2 du$ as the *roughness* of the kernel;
- So, we have shown that

$$\text{Var}\left[\hat{\Delta}_2(x)\right] = \frac{\sigma^2(x) f(x) R(K)}{nh} + o\left(\frac{1}{nh}\right)$$

- Important to notice: the variance of this term only vanishes if $nh \to \infty$;
- Fast $h \to 0 \implies$ slow convergence of this variance towards zero;
- By the Lindberg-Feller CLT:

$$\sqrt{nh}\hat{\Delta}_2(x) \xrightarrow{d} N(0, \sigma^2(x) f(x) R(K))$$

- Notice how the asymptotic variance depends on x!

Asymptotic Theory

Now we analyze our other guy: $\hat{\Delta}_1(x)$ using similar tricks:

$$\begin{aligned}
\mathbb{E}[\hat{\Delta}_1(x)] &= \frac{1}{h}\,\mathbb{E}\left[K\left(\frac{X_i - x}{h}\right)(m(X_i) - m(x))\right] \\
&= \frac{1}{h}\int_{-\infty}^{\infty} K\left(\frac{z - x}{h}\right)(m(z) - m(x))\, f(z)\, dz \\
&= \int_{-\infty}^{\infty} K(u)\,(m(x + hu) - m(x))\, f(x + hu)\, du
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\end{aligned}
$$

We add the assumptions that $f \in \mathbb{C}^1$ and that $m \in \mathbb{C}^2$ and expand them:

$$
\begin{aligned}
m(x + hu) - m(x) &= m'(x)hu + \frac{1}{2}m''(x)h^2u^2 + o(h^2) \\
f(x + hu) &= f(x) + f'(x)hu + o(h)
\end{aligned}
$$

Asymptotic Theory

Distribute the terms:

$$(m(x+hu)-m(x))\cdot f(x+hu) = m'(x)f(x)hu+m'(x)f'(x)h^2u^2+\frac{1}{2}m''(x)f(x)h^2u^2+o(h^2)$$

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This will lead to:

$$\begin{aligned}\mathbb{E}[\hat{\Delta}_1(x)] &= h^2\left(\int_{-\infty}^{\infty} u^2K(u)^2du\right)\left(m'(x)f'(x)+\frac{1}{2}m''(x)f(x)\right)+o(h^2)\\ &= h^2\kappa_2 f(x)B(x)+o(h^2)\end{aligned}$$

where $B(x) = \left(m'(x)\frac{f'(x)}{f(x)}+\frac{1}{2}m''(x)\right)$ and $\kappa_2 \equiv \left(\int_{-\infty}^{\infty} u^2K(u)^2du\right)$

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Questions: is $B(x)$ random? How does h impact $\mathbb{E}[\hat{\Delta}_1(x)]$?

Asymptotic Theory

- Computing $\text{Var}(\hat{\Delta}_1(x))$ is a boring computation;
- See section 19.26 from Hansen's book. He can show that:

$$\text{Var}(\hat{\Delta}_1(x)) = o\left(\frac{1}{nh}\right)$$

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$$\text{Var}(\hat{\Delta}_1(x)) = o\left(\frac{1}{nh}\right)$$

- This will imply that $\text{Var}(\sqrt{nh}\hat{\Delta}_1(x)) = o(1)$
- The main result is that:

$$\sqrt{nh}\left(\hat{\Delta}_1(x) - h^2\kappa_2 f(x)B(x)\right) \xrightarrow{p} 0$$

as long as $nh^5 = O(1)$ (why do we need this?);

Asymptotic Theory - Main Result

Please see all the technical conditions on Hansen's book (Chapter 19);

Theorem (Asymptotic Distribution of the NW Estimator)

Under regularity conditions, for interior x*,*

$$\sqrt{nh}\left(\hat{m}(x) - m(x) - h^2\kappa_2 B(x)\right) \xrightarrow{d} N\left(0, \frac{\sigma^2(x)R(K)}{f(x)}\right)$$

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where $B(x) = \left(m'(x)\frac{f'(x)}{f(x)} + \frac{1}{2}m''(x)\right)$.

- Why is the convergence happening at the rate $\sqrt{nh}$ and not just $\sqrt{n}$?
- True or false: is there always a finite sample bias here?

Questions?

The Bias-Variance Trade-off

- If n is very large:

$$(\hat{m}(x) - m(x)) \approx N\left(h^2\kappa_2 B(x), \frac{\sigma^2(x)R(K)}{nh \cdot f(x)}\right)$$

- What is the effect of h on the mean and variance?

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- Or you can get low variance... and a huge bias!

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- Fundamental trade-off: we can reduce the bias, at the expense of variance;
- Or you can get low variance... and a huge bias!
- The Epanechnikov kernel is the one the minimizes the mean-squared-error of this estimation across a large class of kernels;
- The efficiency loss when using the Gaussian kernel is minimal;
- Honestly, let the Gaussian kernel be your default in empirical research;

How to pick the bandwidth?

- The optimal bandwidth will depend on moments of the data and derivatives of m;
- That is unknown in practice;
- Ideally, your results should be robust to different bandwidths (within reason);
- A popular way to choose a bandwidth is leave-one-out cross-validation;
- Exactly as the Machine Learning literature does!
- See Hansen's book for details and the thoery behind it;

What about the boundary?

- Suppose X_i comes from a bounded distribution, for example;
- Let's say $X \sim U[0, 10]$ and $Y|X = x \sim N(x, 1)$;

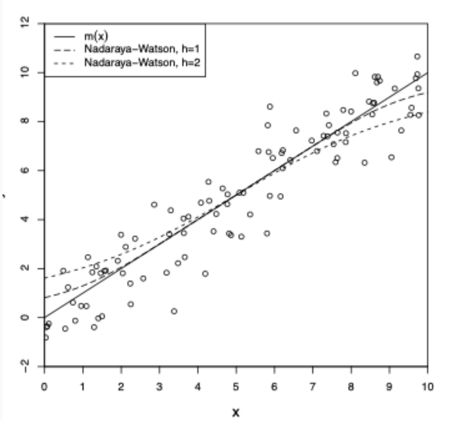

Figure 2: Bias at the boundary

The Local Linear Estimator

- Notice that the Nadaraya-Watson estimator also satisfies:

$$\hat{m}(x) = \arg\min_{c \in \mathbb{R}} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) \cdot (Y_i - c)^2$$

- It's also called the local-constant estimator;

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- It's also called the local-constant estimator;
- What is stopping us from fitting a linear function here?

Definition (Local-Linear Estimator)

For each x, solve the following optimization problem:

$$\left(\hat{\beta}_0(x), \hat{\beta}_1(x)\right) = \arg\min_{(b_0, b_1)} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right) (Y_i - b_0 - b_1(X_i - x))^2$$

The local-linear estimator of $m(x)$, denoted by $\hat{m}(x)_{LL}$, is the local intercept $\hat{\beta}_0(x)$.

How do they compare?

- Deriving the asymptotic distribution requires similar tricks. See Hansen's book;
- **Important**: both have the same asymptotic variance;
- **Also Important**: the bias term for the local-linear estimator is $\frac{h^2\kappa_2}{2}m''(x)$

$$\sqrt{nh}\left(\hat{m}(x)_{LL} - m(x) - h^2\kappa_2\frac{m''(x)}{2}\right) \xrightarrow{d} N\left(0, \frac{\sigma^2(x)R(K)}{f(x)}\right)$$

- What does it imply if $m(.)$ is, in fact, linear?
- The local-linear estimator is much better close to the boundary! Why?

Questions?

Curse of Dimensionality

- So far, we dealt with scalar x.
- What if $X_i \in \mathbb{R}^p$ with $p > 1$?
- In that case we use multivariate kernels and measure "distances" as

$$K\left(\frac{X_1 - x}{h_1}\right) \cdot K\left(\frac{X_1 - x}{h_2}\right) \cdots K\left(\frac{X_p - x}{h_p}\right)$$

where $(h_1, ..., h_p)$ are potentially different bandwidths;

- Math gets more involved but the same type of results are obtained;

Curse of Dimensionality

- So far, we dealt with scalar x.
- What if $X_i \in \mathbb{R}^p$ with $p > 1$?
- In that case we use multivariate kernels and measure "distances" as

$$K\left(\frac{X_1 - x}{h_1}\right) \cdot K\left(\frac{X_1 - x}{h_2}\right) \cdots K\left(\frac{X_p - x}{h_p}\right)$$

where $(h_1, ..., h_p)$ are potentially different bandwidths;

- Math gets more involved but the same type of results are obtained;
- There is a super important caveat: convergence will happen at rate $\sqrt{n \cdot h_1 \cdot h_2 \cdots h_p}$
- In case you use the same h, we will need that $\sqrt{nh^p} \to \infty$. This is **very** slow;
- In practice, if $p > 4$ you are screwed $\implies$ open the Machine Learning toolbox then;

Confidence Bands

- You might be interested in making inference about $m(x)$ since you did all the math to get the distribution...
- Keep in mind: any sort of confidence band you draw is **pointwise**;
- Usually, we cheat compute the 95% confidence interval as

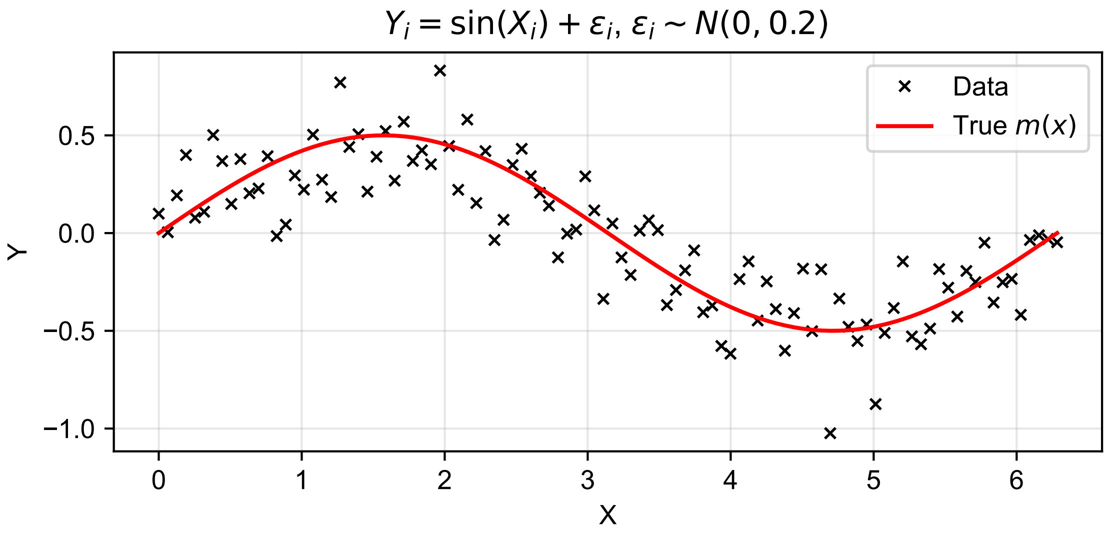
$$\hat{m}(x) \pm 1.96 \cdot \sqrt{\frac{\hat{\sigma}^2(x) R(K)}{nh\hat{f}_n(x)}}$$

where you can use the residuals from your fit and the same kernel to compute $\hat{\sigma}^2(x)$;

- Why is this cheating?

Example

A Quick Simulation



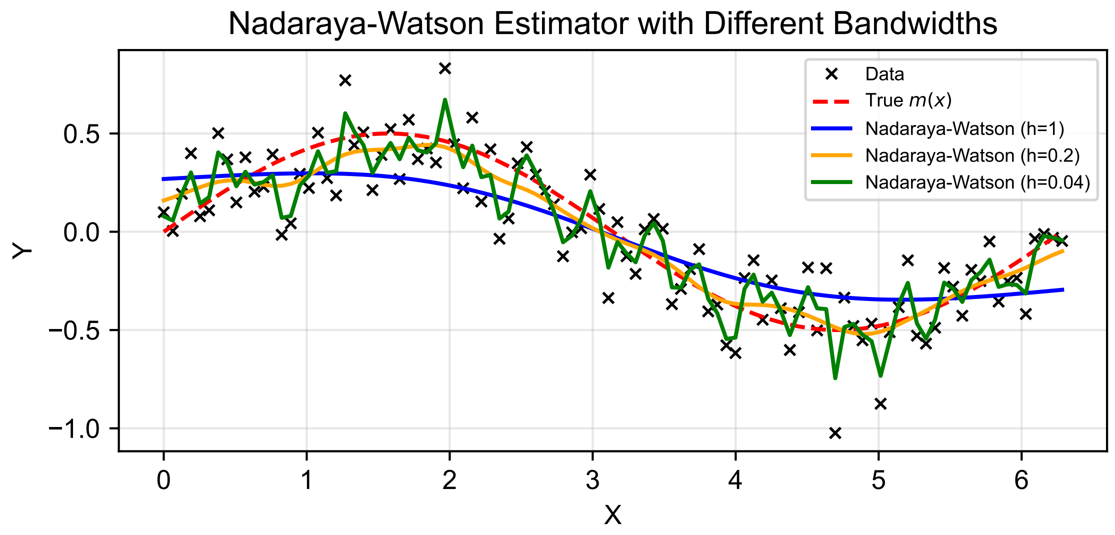
Kernel Regression 101

It's super easy to implement the Nadaraya-Watson estimator:

```
def gaussian_kernel(x, h):
    return (1 / (h * np.sqrt(2 * np.pi))) * np.exp(-0.5 * (x / h) ** 2)

def nw_estimator(X, Y, x, h):
    K = gaussian_kernel((X - x) / h, 1)
    return np.sum(K * Y) / np.sum(K)
```

Different Bandwidths



Questions?

References

- Chapter 19 from *Econometrics*
- Chapter 17 from *Probability and Statistics for Economists*