

Lecture 3: Building Blocks for Time Series

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October, 2025

Intro

Common data structures in Economics:

- Cross-sectional data: many units, one time period
 - Example: grades of several students in a given exam, GDP growth of several countries in a given year;
- Time series data: one unit, many time periods
 - Example: inflation over time for a given country, amount on rain in a given area, price of a stock over time;
- Panel data: many units, many time periods for the same units
 - Example: GDP growth of several countries over several years, grades of several students in several exams, prices of several products over time...
- Text, spatial data, images, etc.

- So far: mostly cross-sectional methods (everything about Y_i);
- Next step: time series methods (everything about Y_t);
- Next next step: panel data methods (everything about $Y_{i,t}$) – probably the most prevalent type nowadays;

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- Next step: time series methods (everything about Y_t);
- Next next step: panel data methods (everything about $Y_{i,t}$) – probably the most prevalent type nowadays;
- Next weeks: focus on understanding the challenges of time series data;
- This lecture: how is a time series different from cross-sectional data? + important definitions;
- Main reference: **Time Series Analysis**, by James Hamilton;
- Hansen's book provides a nice introduction, but it is too short on the topic;

What is a time series?

- A *time series* is a sequence of observations on a variable (or several variables) over time on an equally-spaced interval;
- Example: annual population of a country;
- Unlike cross-sectional data, time series data is **ordered**;
- Also assume in this course that time is discrete (i.e., we observe data at specific time intervals, like days, months, years...);
- Continuous time series (e.g., high-frequency financial data) is a more advanced topic, but with a vast literature as well;

Why should you care?

1. Policy evaluation: every policy takes place over time;
 - What's the impact of a reform? Before vs after? Are effects long-lasting? Fast die-outs?
 - A very important building block for panel data methods;
2. Forecasting: how can the future look like?
 - What will the inflation rate be next month? What is the expected number of COVID cases next week? How many students will enroll next semester?
 - How much inventory should a firm hold? What's the likely path of deforestation in a given area?
3. Nowcasting: high(er)-frequency monitoring of low-frequency variable;
 - What is the current state of the economy? How many people are currently unemployed? How many people are currently infected with COVID?

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3. Nowcasting: high(er)-frequency monitoring of low-frequency variable;
 - What is the current state of the economy? How many people are currently unemployed? How many people are currently infected with COVID?
4. It has **very** elegant math behind it!

The Role of Dependence

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Examples:

- If inflation is high this month, it will likely be high next month;
- If a student is doing well on every exam, they will likely do well on the next one;
- The major difference w.r.t. cross-sectional data is that the future might depend on the past;

Building Blocks

Formally, the right way to think about time series is as a **stochastic process**;

First, we define what a *random variable* is:

Definition (Random Variable)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable is a real function $Y : \Omega \rightarrow \mathbb{R}$, such that for all $c \in \mathbb{R}$, $A_c = \{\omega \in \Omega | Y(\omega) \leq c\} \in \mathcal{F}$, $\forall c \in \mathbb{R}$.

- Ω is a sample space. Example: the numbers on a die (1, 2, 3, 4, 5, 6);
- \mathcal{F} is a collection of events. Example: even numbers.
- $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is a probability measure. Example: probability of rolling an even number;

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- $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is a probability measure. Example: probability of rolling an even number;
- Think about an i.i.d sample as different realizations of ω ;
- $Y_1 = Y(\omega_1), Y_2 = Y(\omega_2), \dots$

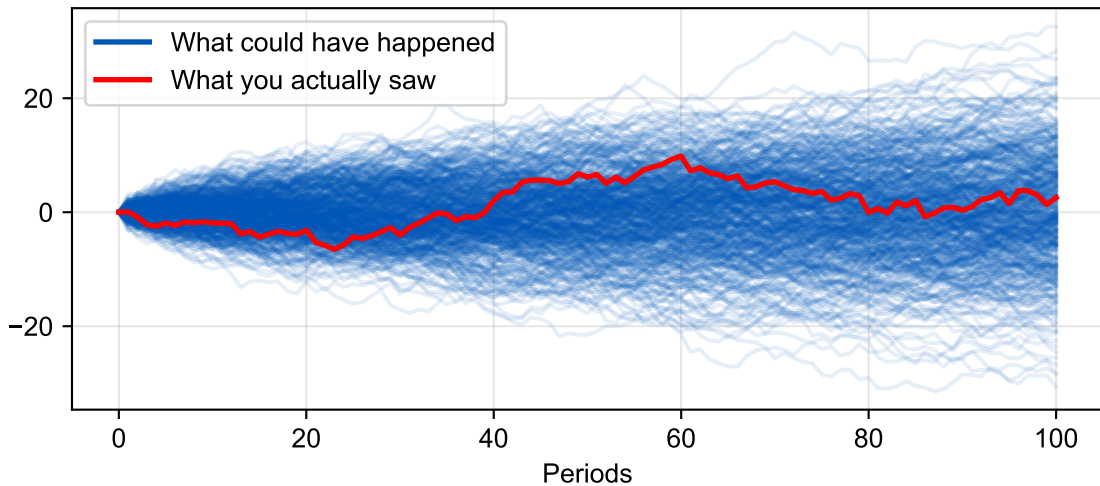
Definition (Stochastic Process – a bit informal)

A *stochastic process* is an ordered sequence (collection) of random variables $\{Y_t(\omega), \omega \in \Omega, t \in \mathcal{T}\}$, such that for all $t \in \mathcal{T}$, $Y_t(\omega)$ **is a random variable** and \mathcal{T} is an ordering set, for example, $\mathbb{Z} = \{-2, -1, 0, 1, 2, \dots\}$.

Loosely speaking:

- A **random variable** is way to model an uncertain number;
- A **stochastic process** is a way to model an uncertain *path*;
- In reality, we only observe one, and only one, realization of the stochastic process;
- Think about the history of the world: it is a single realization of Ω ;
- YOLO: you only live once!

500 paths of the same stochastic process



The Challenge Ahead

- You only observe the red, but the blue paths were equally likely to have happened;
- The main challenge is that you see *one* path and you want to make inference about the *whole process*;

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Example: inflation in Brazil in 1989;

- We know it was super high;
- But we have only one reading of inflation for 1989;
- When can we say something about the “inflation process” in Brazil given **only one** observation?

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We will need to impose a lot of structure! Without structure, we are lost!

Questions?

Stationarity and Ergodicity

Weak Stationarity

Definition (Weak Stationarity)

A stochastic process $\{Y_t\}$ is said to be *weakly stationary* (or *second-order stationary*, or *covariance stationary*) if, and only if, the first two population unconditional moments of $\{Y_t\}$ exists and are constant:

$$\mathbb{E}[Y_t] = \mu, \quad |\mu| < \infty, \quad \forall t \in \mathcal{T} \text{ and}$$

$$\mathbb{E}[(Y_t - \mu)(Y_{t-h} - \mu)] = \gamma_h, \quad |\gamma_h| < \infty, \quad \forall t \in \mathcal{T} \text{ and} \quad h = 0, \pm 1, \pm 2, \dots$$

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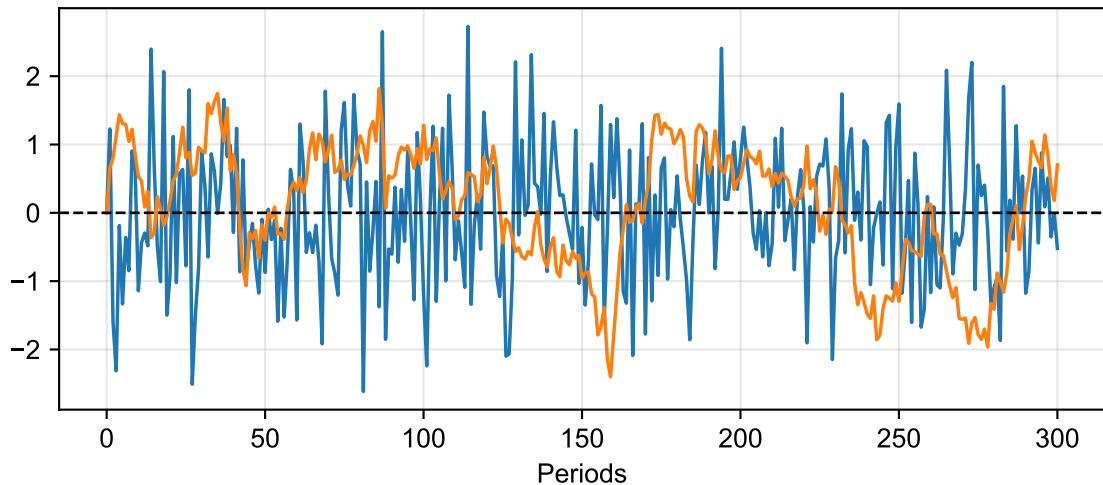
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True or false?

- $\gamma_0 = \text{Var}(Y_t)$ (True or False?)
- $\gamma_h > \gamma_{-h}$ for $h \neq 0$. (True or False?)
- An i.i.d process with finite variance is weakly stationary. (True or False?)

What process has a higher γ_1 ?

Two stationary processes with different autocovariance structures



Strong (or Strict) Stationarity

Definition (Strong Stationarity)

A stochastic process $\{Y_t\}$ is said to be *strongly stationary* (ou *strictly stationary*) if, and only if, the joint distribution of (Y_1, Y_2, \dots, Y_T) is invariant with respect to time shifts:

$$F_Y(Y_1, Y_2, \dots, Y_n) = F_Y(Y_{1+\tau}, Y_{2+\tau}, \dots, Y_{n+\tau}), \quad \forall \tau$$

where $F_Y(\cdot)$ is the joint CDF of the random vector (Y_1, Y_2, \dots, Y_n) .

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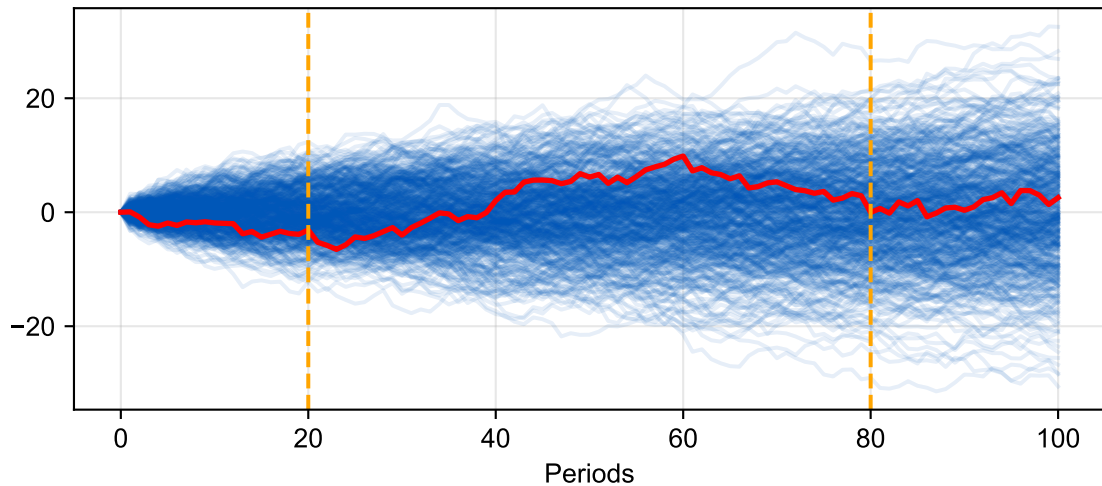
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- Denote by $m_t^{(i)}$ as the realization of m at time t for the i -th path;
- The paths are independent (because I chose so!);

- Let's call the mysterious process from the YOLO simulation m_t ;
- Denote by $m_t^{(i)}$ as the realization of m at time t for the i -th path;
- The paths are independent (because I chose so!);
- Let's say I want to estimate two (potentially different) quantities:
 - $\mathbb{E}[m_{20}]$
 - $\mathbb{E}[m_{80}]$;
- How can I do that?

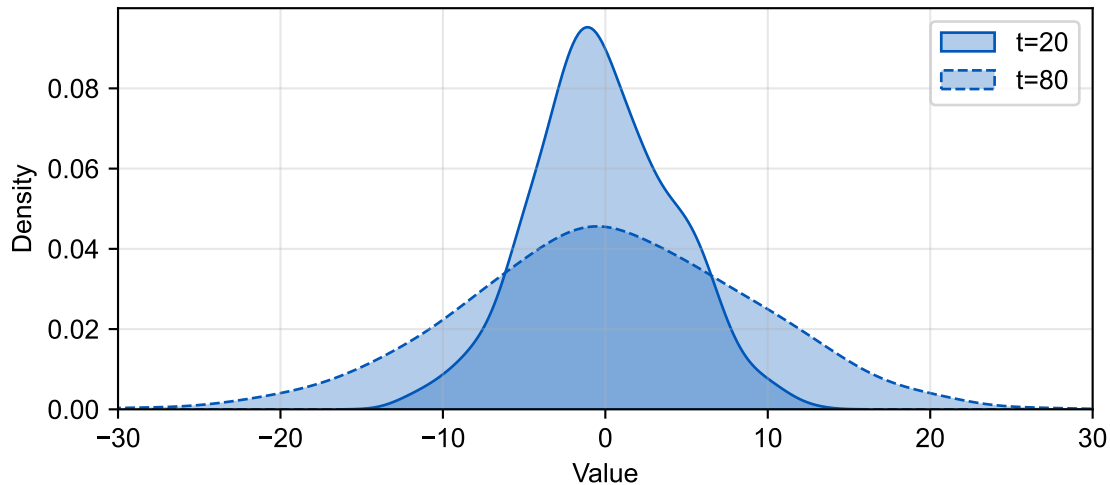
Ergodicity - Motivation

500 paths of the same stochastic process



Ergodicity - Motivation

Kernel Density Estimator at $t=20$ and $t=80$;)



Proposal 1: Estimate $\mathbb{E}[m_{20}]$ by averaging the values at $t = 20$ across all paths;

- Formally: $\hat{m}_{20} = \frac{1}{\text{number of paths } (n)} \sum_{i=1}^n m_{20}^{(i)}$;
- Do the same for $t = 80$;
- Would this yield a consistent estimator?

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Proposal 2: Since I do not have access to the whole process, I will use the one path I have;

- Estimate $\mathbb{E}[m_{20}]$ by averaging the values at $t = 20$ across all time periods in the path;
- Formally: $\tilde{m}_{20} = \frac{1}{T} \sum_{t=1}^T m_t^{(1)}$;
- Would I need stationarity for this to yield a consistent estimator?
- Intuitively, would that be enough?

- Ergodicity is a property that some stochastic processes have;
- Intuitive definition (don't quote me on this):

Definition (Ergodicity - Intuitive)

A stochastic process $\{Y_t\}$ is said to be *ergodic* if its realized paths are "rich enough" with probability 1. By "rich enough", we mean that it **will not** get stuck in a subset of the state space or will get into cyclic trajectories with probability 1.

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Said in another way: if you observe a process for long enough T , you will be able to learn everything about the process;

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- There are ways to test whether a process is stationary. We will get there;
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Important:

- When a process is stationary, there are simple conditions that imply ergodicity;
- Under ergodicity and strict stationarity, \widetilde{m}_{20} is consistent!
- This result is called the **Ergodic Theorem** (Theorem 14.9 on Hansen's book).

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Also important:

- Stationarity does not imply ergodicity: one example in the problem set;

Formal Definition - Ergodicity (Time Allowing)

Trivial Invariant Events

- Let $G \subset \mathbb{R}^\infty$;
- An event A is $A = \{\omega \in \Omega | \tilde{Y}_t \in G\}$, where $\tilde{Y}_t = (\dots, Y_{t-1}, Y_t, Y_{t+1}, \dots)$ is the history of the process;
- The l -th time shift of A is $A_l = \{\omega \in \Omega | \tilde{Y}_{t+l} \in G\}$, where $\tilde{Y}_{t+l} = (\dots, Y_{t-1+l}, Y_{t+l}, Y_{t+1+l}, \dots)$ is the history of the process shifted by l periods;

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- An event A is called **trivial** if $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

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A process Y_t is called **ergodic** if every invariant event is trivial.

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- Suppose $Y_t = Z$, where $Z \sim U[-1, 1]$, for all $t \in \mathbb{Z}$;
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- This means that the process is constant after Z is drawn;
- What's $\mathbb{P}(A)$ here?
- Is this process ergodic?

One useful characterizations of Ergodicity

Theorem (Ergodicity - Characterization)

A strictly stationary series $Y_t \in \mathbb{R}^m$ is ergodic if, and only if, for all events A and B

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n \mathbb{P}(A_l \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- Intuition: as we go back in time, on average, events become nearly independent;

Two Useful Theorems

Stationarity + Ergodicity = LLN

- You will be interested in approximating means and seconds moments of a process;
- Stationarity + ergodicity is the “right set of assumptions” to do that ;

Theorem (Theorem 2 (page 203) in Hannan (1970))

If Y_t is strictly stationary and ergodic with $\mathbb{E}[|Y_t|] < \infty$, then

$$\bar{Y}_T = \frac{1}{T} \sum_{t=1}^T Y_t \xrightarrow{a.s.} \mathbb{E}[Y_t] \quad \text{as } T \rightarrow \infty$$

Also, if $\mathbb{E}[Y_t^2] < \infty$, then

$$\hat{\gamma}_h = \frac{1}{T} \sum_{t=1}^{T-h} (Y_t - \bar{Y}_T)(Y_{t+h} - \bar{Y}_T) \xrightarrow{a.s.} \gamma_h \quad \text{as } T \rightarrow \infty$$

What to do in practice?

- The last theorem is not ideal: it assumes something we cannot test;
- If we are only concerned with estimation, there is an easier way out:

Theorem (Theorem 6 (page 210) in Hannan (1970))

If Y_t is weakly stationary and $\sum_{i=1}^{\infty} |\gamma_i| < \infty$, then we have that

$$\bar{Y}_T = \frac{1}{T} \sum_{t=1}^T Y_t \xrightarrow{L_2} \mathbb{E}[Y_t] \quad \text{as } T \rightarrow \infty$$
$$\hat{\gamma}_h = \frac{1}{T} \sum_{t=1}^{T-h} (Y_t - \bar{Y}_T)(Y_{t+h} - \bar{Y}_T) \xrightarrow{L_2} \gamma_h \quad \text{as } T \rightarrow \infty$$

- Recall that L_2 -convergence is stronger than convergence in probability, but weaker than almost-sure convergence;

Questions?

White Noise and Martingale Difference Sequences

White Noise

- In the cross-section context, it's common to write $y_i = \alpha + \beta x_i + u_i$ where u_i is i.i.d;
- We will need more flexible assumptions on “error terms” now;
- Two very important weaker notions of “error term” are **white noise** processes and **martingale difference sequences**.

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Definition (White Noise Process)

A sequence of random variables $(\varepsilon_t)_{t \in \mathbb{Z}}$ is called a **white noise process** if:

1. $\mathbb{E}[\varepsilon_t] = 0$ for all t ;
2. $\text{Var}(\varepsilon_t) = \sigma^2 < \infty$ for all t ;
3. $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ for all $t \neq s$.

In words, a white noise process is a sequence of uncorrelated random variables with mean zero and constant variance.

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In words, a white noise process is a sequence of uncorrelated random variables with mean zero and constant variance.

Any i.i.d. sequence of variables with mean zero and finite variance is a white noise process.

- Conditioning on past information is common in economic models and in Econometrics;
- The correct mathematical object to use here are σ -**fields** and **filtrations**;
- We will avoid measure-theoretic formalism right now;
- Intuitively: the σ -field (or -algebra) generated by $(Y_t, Y_{t-1}, Y_{t-2}, \dots)$ is the collection of possible histories for this process up to t ;
- We denote this in two equivalent ways: $\mathcal{F}_t = \sigma(Y_t, Y_{t-1}, Y_{t-2}, \dots)$;
- In other areas of Economics, we typically refer to σ -fields as *information sets*;
- \mathcal{F}_t contains all information available up to time t ;
- Example: $\mathbb{E}[Y_t | \mathcal{F}_{t-1}]$ is our best guess for Y_t conditional on information from this process, and only that, up to $t - 1$;

Definition (Martingale Difference Sequence)

A sequence of random variables $(\varepsilon_t)_{t \in \mathbb{Z}}$ is called a **martingale difference sequence** (MDS) with respect to the information set \mathcal{F}_{t-1} if:

1. ε_t is adapted to \mathcal{F}_t ;
2. $\mathbb{E}[|\varepsilon_t|] < \infty$ for all t ;
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True or false?

- Every MDS sequence has (unconditional) mean zero; (True or false?)
- $\text{Cov}(\varepsilon_t, \varepsilon_{t+h}) = 0$ for all $h \neq 0$. (True or false?)
- $\mathbb{E}[\varepsilon_{t+h} | \mathcal{F}_t] = 0$ for all $h > 0$. (True or false?)
- $\mathbb{E}[\varepsilon_{t-h} | \mathcal{F}_t] = 0$ for all $h > 0$. (True or false?)

Example

- Let u_t be an i.i.d sequence with mean zero and variance $\sigma^2 < \infty$.
- Define $v_t = u_t \cdot u_{t-1}$.
 - Is this process i.i.d? Is it a MDS with respect its natural filtration (information set)? Is it white noise?

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- Define $v_t = u_t \cdot u_{t-1}$.
 - Is this process i.i.d? Is it a MDS with respect its natural filtration (information set)? Is it white noise?
- Now, define $m_t = u_t + u_{t-1} \cdot u_{t-2}$;
 - Is this process i.i.d? Is it a MDS with respect its natural filtration (information set)? Is it white noise?

The End

- Hannan, E. J. (1970). *Multiple Time Series*. Wiley.
- Chapters 3 and 4 from Hamilton's book;
- MDS sequences are covered in Chapter 7;
- Chapter 14 from Hansen's book;