Quasilinear Models

Ed Herbst

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What We've Done So Far

- Linear(ized) models
  - Solve a System of Linear Rational Expectations System
  - Estimate the model using Kalman Filter

- Nonlinear models
  - Solve model via higher order perturbation or projection
  - Estimate model using Particle Filter

Today: Models with occasionally binding constraints, piecewise linear solution.

Extended Perfect-Foresight Path (EPFP)

I'm going to talk about solutions based on the *Extended Perfect-Forecast Path* algorithm from Fair and Taylor (1983).

There are many papers using this framework: Eggertsson and Woodford (2003), Christiano, Eichenbaum, and Trabandt (2015),Guerrieri and Iacoviello (2015), Kulish, Morley, and Robinson (2017), Holden (2019), and Boehl (2019).

The Guerrieri and Iacoviello (2015) variant is called `OccBin`, and it's available in Dynare.

The Idea of the Algorithm

- Assume in period $t + H$ system reverts back to the steady state in which the constraint is no longer binding.
- Initial guess about whether the constaint in binding in periods $t + h, h = 1, \ldots, H$
- Solve the model backwards from the steady state in period $t + H$
- Check whether you guess was correct, if not update your guess and try again.

The OccBin Algorithm

Let's suppose there is just one constraint.

Further, let's called the regime where the economy isn't binding the reference regime.

Prerequisites

1. Locally unique rational expectations solution hold at reference regime
2. If shocks move the model away from the reference regime to the alternative regime, the model will return to the reference regime in finite under the assumption that no future shocks occur.

Writing the Equilibrium Conditions

Let x_t be the variables of the model and let ϵ_t be the (iid) exogenous shocks.

Let's write the equilibrium conditions of the model in the reference regime as:

$$\Gamma_{+1}\mathbb{E}_t\left[x_{t+1}\right] + \Gamma_0 x_t + \Gamma_{-1}x_{t-1} + \Gamma_\epsilon \epsilon_t = 0. \tag{1}$$

Note that this has a (locally unique) solution:

$$x_t = Tx_{t-1} + R\epsilon_t. \tag{2}$$

Now, let's write the equilibrium conditions for the model when the constraint is binding:

$$\Gamma^*_{+1}\mathbb{E}_t\left[x_{t+1}\right] + \Gamma^*_0 x_t + \Gamma^*_{-1}x_{t-1} + \Gamma^*_c + \Gamma_\epsilon \epsilon_t = 0. \tag{3}$$

An Example

Consider a simple model:

$$
\begin{aligned}
q_t &= \beta(1-\rho)E_t[q_{t+1}] + \rho q_{t-1} - \sigma r_t + u_t, \\
r_t &= \max\{\underline{r}, \phi q_t\}, \\
u_t &= \rho u_{t-1} + \sigma_\epsilon \epsilon_t
\end{aligned}
$$

Where $\underline{r} = -1/\beta + 1$.
Then:

$$
\Gamma_{+1} = \begin{bmatrix} -\beta\left(-\rho+1\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 1 & \sigma & -1 \\ -\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\Gamma_{-1} = \begin{bmatrix} -\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho \end{bmatrix} \quad \Gamma_\epsilon = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad (4)
$$

When the regime is binding

When the regime is binding we simply replace the second equilibrium condition with $r_t = \underline{r}$.

$$\Gamma^*_{+1} = \begin{bmatrix} -\beta(-\rho+1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Gamma^*_0 = \begin{bmatrix} 1 & \sigma & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma^*_{-1} = \begin{bmatrix} -\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho \end{bmatrix} \quad \Gamma^*_c = \begin{bmatrix} 0 \\ -\underline{r} \\ 0 \end{bmatrix} \quad \Gamma^*_\epsilon = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad (5)$$

Note we don't have a unique solution to this system.

The Solution

- Given an initial state x_0 and shock ϵ_1, we want to solve for the trajectory $\{x_1, \ldots, x_T\}$. Consider the *perfect foresight solution*, with $\epsilon_t = 0$ for $t > 1$.

- This solution will lead to mapping:

$$x_1 = C_1 + T_1 x_0 + R_1 \epsilon_1 \tag{6}$$
$$x_t = C_t + T_t x_{t-1} \quad \text{for } t = 2, \ldots, T. \tag{7}$$

- Thus the dynamics of the state variable are linear with a time-varying intercept and autoregressive coefficent matrix.

- Now we just need to find these matrices $\{C_t, T_t\}_{t=1}^{T}$ and R_1.

The Algorithm

- Assume that in period T the model is back in the reference regime. Thus

$$x_T = Tx_{T-1} + R\epsilon_T = Tx_{T-1}. \tag{8}$$

  So $T_T = T$ and $C_T = 0$.
- Now, suppose that in period $T-1$ are in the binding regime. We'll our system will have to satisfy:

$$\begin{aligned} \Gamma^*_{+1}\mathbb{E}_{T-1}\left[x_T\right] + \Gamma^*_0 x_{T-1} + \Gamma^*_{-1}x_{T-2} + \Gamma^*_c + \Gamma^*_\epsilon \epsilon_t = \\ \Gamma^*_{+1}\mathbb{E}_{T-1}(T_T x_{T-1} + C_T) + \Gamma^*_0 x_{T-1} + \Gamma^*_{-1}x_{T-2} + \Gamma^*_c = 0 \end{aligned} \tag{9}$$

  where the second equation follows from (8) and our perfect foresight assumption.
- By matching coefficients

$$x_{T-1} = \underbrace{-(\Gamma^*_{+1}T_T + \Gamma^*_0)^{-1}\Gamma^*_{-1}}_{T_{T-1}} x_{T-2} + \underbrace{-(\Gamma^*_{+1}T_T + \Gamma^*_0)^{-1}(\Gamma^*_c + T_T C_T)}_{C_{T-1}}$$

Iteration

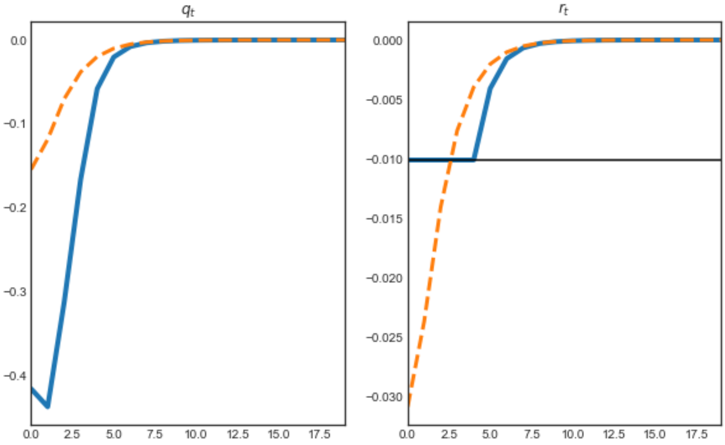
- We proceed in the fashion under time $T = 1$. At which point we deduce that

$$R_1 = -(\Gamma^*_{+1} T_2 + \Gamma^*_0)^{-1} \Gamma^*_{\epsilon}.$$

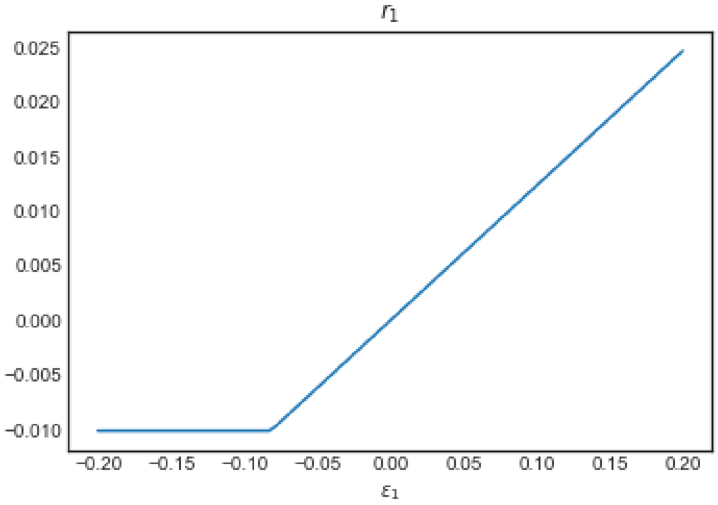
- We have produced a sequence $\{x_t\}_{t=1}^{T}$ based on guesses whether the constraint bound in each period.
- The last thing to do is check whether the sequence $\{x_t\}_{t=1}^{T}$ is consistent with our guesses about whether the constraint bound.
- If so, great, we are done! If not, update our guesses as start again.

Returning to Our Example

Let $x_0 = [0, 0, 0]'$ and $\epsilon_1 = -0.2$. Let's set $T = 20$.



The Policy Function



Caution

- There's not necessarily a unique solution associated with this algorithm.
- Moreover, the solution relies on certainty equivalence.
  - Agents don't expect to return to the bound once they leave!
  - But that an assumption underpinning all linear models.
- For big models, this could take a long time, but it's much faster than a global solution!

Estimating this kind of model:

- Suppose that the number of shocks in your model was equal to the number of observables.

- One could attempt to use the *inversion filter*, to solve for the shocks to satisfy the measurement equation. No measurement error needed (or wanted)! *(Note, we need the jacobian from y to* ϵ*)*

- But, this kind of filter essentially approximates the integral:

$$\int p(Y_{1:T}|s_0)p(s_0)ds_0$$

- It can be very noisy!

Conditionally Optimal Particle Filter for Piecewise Linear Models

Remember our friend the conditionally optimal particle filter.

$$g_t^* \left(\tilde{s}_t \mid s_{t-1}^j, \theta \right) = p \left(\tilde{s}_t \mid y_t, s_{t-1}^j, \theta \right) \propto p \left(y_t \mid \tilde{s}_t, \theta \right) p \left(\tilde{s}_t \mid s_{t-1}^j, \theta \right)$$

with

$$\tilde{\omega}_t^j = \frac{p \left(y_t \mid \tilde{s}_t^j, \theta \right) p \left(\tilde{s}_t^j \mid s_{t-1}^j, \theta \right)}{p \left(\tilde{s}_t^j \mid y_t, s_{t-1}^j, \theta \right)} = p \left(y_t \mid s_{t-1}^j \right)$$

Sketch out the conditionally optimal particle filter for piecewise linear models in Aruoba et al. (2020)

Aruoba et al. (2020)

Derive the solution to a piecewise linear model as:

$$s_t = \begin{cases} \Phi_0(n) + \Phi_1(n)s_{t-1} + \Phi_\eta(n)\eta_t & \text{if } \eta_{1,t} < \zeta(s_{t-1}) \\ \Phi_0(b) + \Phi_1(b)s_{t-1} + \Phi_\eta(b)\eta_t & \text{otherwise} \end{cases}$$

"n" is nonbinding, "b" is binding, and η is a linear combination of the structural shocks.

This solution doesn't have to obey certainty equivalence!

Observation equation:

$$y_t^o = A_0 + A_s s_t + u_t, \quad u_t \sim N(0, \zeta\Sigma_u)$$

Some Definitions

$$\nu_t^j(\cdot) = y_t - A_0 - A_s\left(\Phi_0(\cdot) - \Phi_1(\cdot)s_{t-1}^j\right) \tag{10}$$

$$\bar{\eta}_t^j(\cdot) = \left(\zeta I + \Phi_\eta'(\cdot)A_s'\Sigma_u^{-1}A_s\Phi_\eta(\cdot)\right)^{-1}\Phi_\eta'(\cdot)A_s'\Sigma_u^{-1}\nu_t^j(\cdot) \tag{11}$$

$$\bar{\Omega}(\cdot) = \zeta\left(\zeta I + \Phi_\eta'(\cdot)A_s'\Sigma_u^{-1}A_s\Phi_\eta(\cdot)\right)^{-1} \tag{12}$$

Here $\nu_t^j(\cdot)$ is the error made in forecasting y_t based on s_{t1}^j.

$\bar{\eta}_t^j(\cdot)$ and $\bar{\Omega}(\cdot)$ are the posterior mean vector and covariance matrix of $\eta_t|(y_t, s_{t1}^j)$ absent any truncation
(that is, for $\zeta(s_{t1}^j)$ being $+\infty$ or $-\infty$)

The density of $p(y_t|s^j_{t-1})$

$$
\begin{aligned}
D^j_t(n) =&(2\pi)^{-n_y/2}\left|\Sigma_u\right|^{-1/2}\left|\zeta I+\Phi_\eta(n)'A_s'\Sigma_u^{-1}A_s\Phi_\eta(n)\right|^{1/2}\\
&\times \exp\left\{-\frac{1}{2}\nu^j_t(n)'\left(\zeta\Sigma_u+A_s\Phi_\eta(n)\Phi_\eta'(n)A_s'\right)^{-1}\nu^j_t(n)\right\}\\
&\times \Phi_N\left(\left(\zeta\left(s_{t-1}\right)-\bar{\eta}^j_{1,t}(n)/\sqrt{\bar{\Omega}_{11}(n)}\right)\right.\\
D^j_t(b) =&(2\pi)^{-n_y/2}\left|\Sigma_u\right|^{-1/2}\left|\zeta I+\Phi_\eta(b)'A_s'\Sigma_u^{-1}A_s\Phi_\eta(b)\right|^{1/2}\\
&\times \exp\left\{-\frac{1}{2}\nu^j_t(b)'\left(\zeta\Sigma_u+A_s\Phi_\eta(b)\Phi_\eta'(b)A_s'\right)^{-1}\nu^j_t(b)\right\}\\
&\left(1-\Phi_N\left(\left(\zeta\left(s_{t-1}\right)-\bar{\eta}^j_{1,t}(b)\right)/\sqrt{\bar{\Omega}_{11}(b)}\right)\right)
\end{aligned}
\tag{13}
$$

It can be shown that $D^j_t(n)+D^j_t(b)=p(y_t|s^j_{t-1})$.

Proposition I

Suppose that $\eta_t is N(0, I)$ Draws from the conditional optimal proposal density can be generated by:

1. Let

$$\xi_t^j = \begin{cases} n' \text{ with prob. } \lambda_t^j \\ b' \text{ with prob. } 1 - \lambda_t^j, \quad \text{where} \quad \lambda_t^j = \frac{D_t^j(n)}{D_t^j(n)+D_t^j(b)} \end{cases}$$

2. If $\xi_t^j =' n'$, then generate η_t from the distribution:

$$\eta_{1,t}^j \sim N\left(\bar{\eta}_{1,t}^j(n), \bar{\Omega}_{11}(n)\right) \mathbb{I}\left\{\eta_{1,t}^j \leq \zeta\left(s_{t-1}^j\right)\right\},$$
$$\eta_{2,t}^j \mid \eta_{1,t}^j \sim N\left(\bar{\eta}_{2|1}^j\left(n, \eta_{1,t}^j\right), \bar{\Omega}_{2|1}(n)\right)$$

and let

$$\tilde{s}_t^j = \Phi_0(n) + \Phi_1(n)s_{t-1}^j + \Phi_\eta(n)\eta_t^j$$

Proposition II

If $\xi_t^j =' b'$, then generate η_t from the distribution:

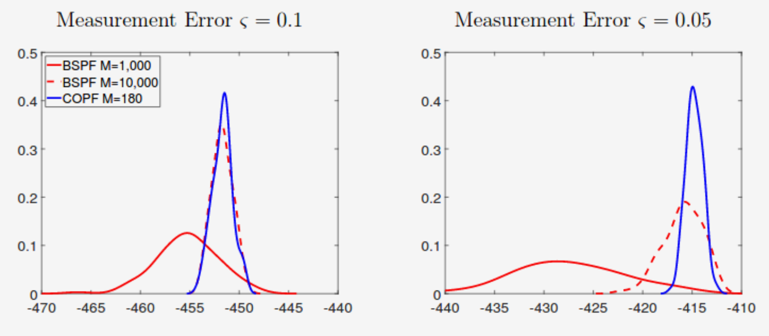
$$\eta_{1,t}^j \sim N\left(\bar{\eta}_1^j(b), \bar{\Omega}_{11}(b)\right) \mathbb{I}\left\{\eta_{1,t}^j > \zeta\left(s_{t-1}^j\right)\right\},$$
$$\eta_{2,t}^j \mid \eta_{1,t}^j \sim N\left(\bar{\eta}_{2|1}^j\left(b, \eta_{1,t}^j\right), \bar{\Omega}_{2|1}(b)\right)$$

and let

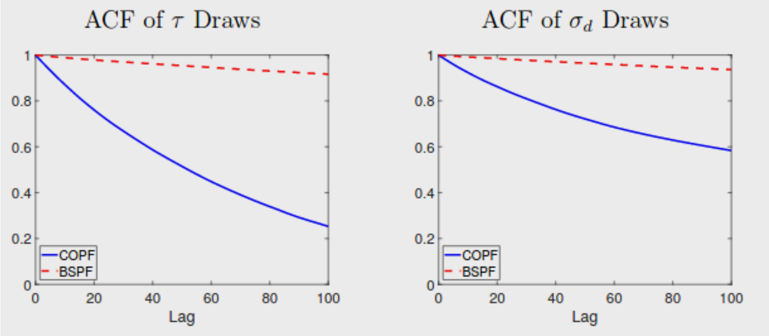
$$\tilde{s}_t^j = \Phi_0(b) + \Phi_1(b) s_{t-1}^j + \Phi_\eta(b) \eta_t^j.$$

3. The incremental particle weight is $D_t^j(n) + D_t^j(b)$.

Likelihood Approximation on a Small Scale DSGE



Autocorrelation Function From a PFMH Run



Conclusion

Overall this is an still an active area of research on both the solution and estimation front.

Key challenges: how to estimate large models with any kind of nonlinearities

Thanks for a great class!

References

ARUOBA, S. B., P. CUBA-BORDA, K. HIGA-FLORES, F. SCHORFHEIDE, AND S. VILLALVAZO (2020): "Piecewise-Linear Approximations and Filtering for DSGE Models with Occasionally Binding Constraints," .

GUERRIERI, L. AND M. IACOVIELLO (2015): "OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily," *Journal of Monetary Economics*, 70, 22 38.