ECON 616: Lecture Three: The Spectrum

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Last Updated: March 13, 2025

Background

- *Overview:* Chapters 6 from Hamilton (1994).
- *Technical Details*: Chapter 4 from Brockwell and Davis (1987).
- *Other stuff*: You might want to look at a digital signals processing textbook, for example: here.

Cycles as Frequencies

Starting In the 19th Century, economists and others recognized cyclical patterns in economic activity.
Schmupeter distinguished between cycles at different frequencies

- *Kondratieff Cycles* – Longwave cycles lasting 50 years (caused by fundamental innovations.)
- *Juglar Cycles* – medium cycle (8 years) associated with changes in credit condition.
- *Kitchin Cycles* – short run cycles (40 months) associated with information diffusion.

=> model economic activity as a linear combination of periodic function with different frequencies.

A model of frequencies

Consider the following model for quarterly observations

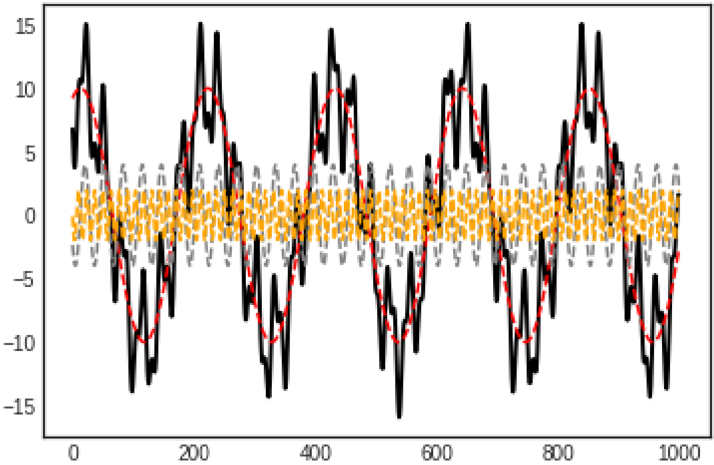
$$X_t = 2\sum_{j=1}^{m} a_j cos(\omega_j t + \theta_j)$$

where θ_j is $\sim iidU[-\pi, \pi]$ and $-\pi \leq \omega_j < \omega_{j+1} \leq \pi$. The random variables θ_j are determined in the infinite past and simply cause a phase shift. According to Schumpeter's hypothesis m should be equal to three. The frequencies ω_j can be determined as follows.

Cycle	Duration	Frequency
Kondratieff	200 quarters	$\omega_1 = (2\pi)/200 = 0.03$
Juglar	32 quarters	$\omega_2 = (2\pi)/32 = 0.20$
Kitchin	13.3 quarters	$\omega_3 = (2\pi)/13.3 = 0.47$

A Time Series of this process

$$a = [5, 2, 1], \quad \omega = [0.03, 0.20, 047].$$



The Spectrum

- The coefficients a_1 to a_3 are the amplitudes of the different cycles
- If a_1 and a_2 are small then most of the variation in X_t is due to the Kitchin cycles.
- The plot of a_j^2 versus ω is called the <u>spectrum</u> of X_t.

Some math

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y && (1)\\
\sin x \sin y &= \frac{1}{2}[\cos(x-y) - \cos(x+y)] && (2)\\
\cos x \cos y &= \frac{1}{2}[\cos(x-y) + \cos(x+y)] && (3)\\
2\sin^2 x &= 1 - \cos(2x) && (4)\\
\sin x \cos x &= \frac{1}{2}\sin(2x) && (5)
\end{aligned}$$

Moreover, $\sin^2 x + \cos^2 x = 1$.
We consider real-valued stochastic processes X_t, complex numbers will help us summarize sine and cosine expressions using exponential functions.

More Math

Let $i = \sqrt{-1}$.

Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

The formula becomes less mysterious if you rewrite $e^{i\varphi}$, $\sin\varphi$, and $\cos\varphi$ as power series.

The Plan

- Rewrite Schumpeter Model
- Define spectral distribution / density function
- Examine relationship between autocovariances $\{\gamma_h\}_{h=-\infty}^{\infty}$ and the spectrum.
- Discuss very general spectral representation for a stationary stochastic process X_t.

Schumpeter Model

$$X_t = 2\sum_{j=1}^{m} a_j \cos\theta_j \cos(\omega_j t) - a_j \sin\theta_j \sin(\omega_j t)$$

where $a_j \cos\theta_j$ and $a_j \sin\theta_j$ can be regarded as random coefficients. Eulers formula implies

$$X_t = \sum_{j=-m}^{m} A(\omega_j) e^{i\omega_j t}$$

where $\omega_{-j} = -\omega_j$. Let $a_{-j} = a_j$ and

This means that

$$A(\omega_j) = \begin{cases} a_j(\cos\theta_{|j|} + i\sin\theta_{|j|}) & \text{if } j > 0 \\ a_j(\cos\theta_{|j|} - i\sin\theta_{|j|}) & \text{if } j < 0 \end{cases}$$

We can verify that:

$$A(\omega_j)e^{i\omega_j t} + A(\omega_{-j})e^{-i\omega_j t} = 2\left[a_j\cos\theta_j\cos(\omega_j t) - a_j\sin\theta_j\sin(\omega_j t)\right]$$

Moments of Linear Cyclical Models

$$\mathbb{E}[\cos\theta_j] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\theta_j d\theta_j = 0 \quad (6)$$

$$\mathbb{E}[\sin\theta_j] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\theta_j d\theta_j = 0 \quad (7)$$

Result: The expectation of X_t in the linear cyclical model is equal to zero. $\square$

Autocovariances

To obtain the autocovariances $\gamma_h = \mathbb{E}[X_t X_{t-h}]$ we have to calculate the moments $\mathbb{E}[A(\omega_j)A(\omega_k)]$.
Let $j \neq k$, $j \neq -k$. Suppose that $j, k > 0$.

$$
\begin{aligned}
\mathbb{E}[A(\omega_j)A(\omega_k)] &= a_j a_k \mathbb{E}[(\cos\theta_j + i\sin\theta_j)(\cos\theta_k + i\sin\theta_k)] \\
&= a_j a_k \mathbb{E}[\cos\theta_j \cos\theta_k + i\cos\theta_j \sin\theta_k i\cos\theta_k \sin\theta_j - \sin \\
&= 0
\end{aligned}
$$

Since θ_j and θ_k are independent. Similar arguments can be made if j and k have different signs.

Covariance

Let $j = k$. Suppose that $j, k > 0$.

$$\begin{aligned}
\mathbb{E}[A(\omega_j)A(\omega_k)] &= a_j^2\mathbb{E}[(\cos\theta_j + i\sin\theta_j)^2] \\
&= a_j^2\mathbb{E}[(\cos^2\theta_j - \sin^2\theta_j + i2\cos\theta_j\sin\theta_j] \\
&= a_j^2\mathbb{E}[1 - 2\sin^2\theta_j + i2\cos\theta_j\sin\theta_j] \\
&= a_j^2\mathbb{E}[\cos(2\theta_j) + i\sin(2\theta_j)] \\
&= 0 \qquad (9)
\end{aligned}$$

In the last step we use the fact that sine and cosine integrate to zero over two cycles. A similar argument can be made for the case $j, k < 0$

Let $j = -k$. Now $A(\omega_j)$ and $A(\omega_k)$ are complex conjugates.

$$\mathbb{E}[A(\omega_j)A(\omega_{-j})] = a_j^2\mathbb{E}[\cos^2\theta_j + \sin^2\theta_j] = a_j^2$$

The upshot

Result: The autocovariances of the process X_t generated by the linear cyclical model are given by

$$\begin{aligned}
\gamma_h &= \mathbb{E}[X_t X_{t-h}] \\
&= \sum_{j=-m}^{m} \sum_{k=-m}^{m} \mathbb{E}[A(\omega_j)A(\omega_k)] e^{i\omega_j t} e^{i\omega_k (t-h)} \\
&= \sum_{j=-m}^{m} \mathbb{E}[A(\omega_j)\overline{A(\omega_j)}] e^{i\omega_j h} = \sum_{j=-m}^{m} a_j^2 e^{i\omega_j h} \qquad (10)
\end{aligned}$$

Since X_t is a real valued process the autocovariances can also be written as

$$\gamma_h = 2 \sum_{j=1}^{m} a_j^2 \cos(\omega_j h) \quad \square$$

The Spectral Distribution

The spectral distribution function for the process X_t, defined on the interval $\omega \in (-\pi, \pi)$, is

$$S(\omega) = \sum_{j=-m}^{m} \mathbb{E}[A(\omega_j)\overline{A(\omega_j)}]\{\omega_j \leq \omega\}$$

where $\{\omega_j \leq \omega\}$ denotes the indicator function that is one if $\omega_j \leq \omega$. $\square$

Remarks

The spectral distribution is non-negative and continuous from the right.
If the spectral distribution function is evaluated at $\omega = \pi$ we obtain

$$S(\pi) = \sum_{j=-m}^{m} \mathbb{E}[A(\omega_j)\overline{A(\omega_j)}] = \sum_{j=-m}^{m} a_j^2 = \mathbb{E}[X_t^2] \tag{11}$$

The spectral distribution function is symmetric in the sense that for $\omega > 0$

$$S(-\omega) = S(\pi) - \lim_{n\to\infty} S((\omega - 1/n)) \tag{12}$$

Autocovariances, again

The representation of the autocovariances can be expressed as a Riemann-Stieltjes integral. Define a sequence of grids

$$[\omega]^{(n)} = \{\omega_k^{(n)} = 2\pi k/n - \pi\}$$

and $\Delta_n \omega = \omega_{k+1}^{(n)} - \omega_k^{(n)} = 2\pi/n$. Moreover, let

$$\Delta_n S(\omega) = S(\omega) - S(\omega - \Delta_n \omega)$$

Roughly,

$$\sum_{k=0}^{n} e^{i\omega_k^{(n)} h} \Delta_n S(\omega_k^{(n)}) \longrightarrow \sum_{j=-m}^{m} a_j^2 e^{i\omega_j h}$$

as $n \to \infty$.

The Upshot

Thus, we can express the autocovariance γ_h as the following integral

$$\gamma_h = \int_{(-\pi,\pi]} e^{i\omega h} dS(\omega)$$

By using a similar argument, we can also obtain a integral representation for the stochastic process X_t. Define the stochastic process

$$Z(\omega) = \sum_{j=-m}^{m} A(\omega_j)\{\omega_j \leq \omega\}$$

with orthogonal increments $\Delta_n Z(\omega) = Z(\omega) - Z(\omega - \Delta_n \omega)$. Note that the increments are now random variables.

Very roughly,

$$\sum_{k=0}^{n} e^{i\omega_k^{(n)} t} \Delta_n Z(\omega_k^{(n)}) \longrightarrow \sum_{j=-m}^{m} A(\omega_j) e^{i\omega_j t}$$

almost surely as $n \to \infty$. Thus, we can express the stochastic process X_t, generated from the linear cyclical model, as the stochastic integral

$$X_t = \int_{(-\pi,\pi]} e^{i\omega t} dZ(\omega)$$

Spectral Representation for Stationary Processes

Every zero-mean stationary process has a representation of the form

$$X_t = \int_{(-\pi,\pi]} e^{i\omega h} dZ(\omega)$$

where $Z(\omega)$ is a orthogonal increment process. Correspondingly, its autocovariance function γ_h can be expressed as

$$\gamma_h = \int_{(-\pi,\pi]} e^{i\omega h} dS(\omega)$$

where $S(\omega)$ is a non-decreasing right continuous function with $S(\pi) = \mathbb{E}[X_t^2] = \gamma_0$.

Spectral Density Function

Suppose the spectral distribution function is differentiable with respect to ω on the interval $(-\pi, \pi]$. The spectral density function is defined as

$$s(\omega) = dS(\omega)/d\omega$$

If a process has a spectral density function $s(\omega)$ then the covariances can be expressed as

$$\gamma_h = \int_{(-\pi,\pi]} e^{ih\omega} s(\omega) d\omega$$

The spectral density uniquely determines the entire sequence of autocovariances. Moreover, the converse is also true.

Consider the sum

$$\begin{aligned} s_n(\omega)^* &= \frac{1}{2\pi}\sum_{h=-n}^{n}\gamma_h e^{-i\omega h} \\ &= \frac{1}{2\pi}\sum_{h=-n}^{n}\left[\int_{(-\pi,\pi]} e^{i\tau h}s(\tau)d\tau\right]e^{-i\omega h} \qquad (13) \end{aligned}$$

The sum $s_n^*(\omega)$ is a Fourier series. If the spectral density $s(\omega)$ is piecewise smooth then

$$s_n^*(\omega) \longrightarrow s(\omega)$$

Thus, the spectral density can be obtained by evaluating the autocovariance generating function of X_t at $z = e^{-i\omega}$.

$$s(\omega) = \frac{1}{2\pi}\gamma(e^{-i\omega}) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_h e^{-i\omega h}$$

where

$$\gamma(z) = \sum_{j=-\infty}^{\infty}\gamma_j z^j$$

Filter

Suppose $s_X(\omega)$ is the spectral density function of a process X_t. Filters are used to dampen or amplify the spectral density at certain frequencies. The spectrum of the filtered series Y_t is given by

$$s_Y(\omega) = f(\omega)s_X(\omega).$$

where $f(\omega)$ is the filter function.
Frequency domain trend/cycle analogue
$X_t =$ low frequency component + high frequency component
Example: For Schumpeter, Kitchin cycle was shortest with $\omega = 0.47$. To remove the effects of other cycles from data, we could use the filter

$$f(\omega) = \begin{cases} 0 & \text{if } \omega < 0.4 \\ 1 & \text{otherwise} \end{cases}$$

Hodrick Prescott filter

A popular filter in the real business cycle literature in macro-economics is the so-called Hodrick Prescott filter.

$$f^{HP}(\omega) = \left[\frac{16\sin^4(\omega/2)}{1/1600 + 16\sin^4(\omega/2)}\right]^2.$$

This filter basically kills long cycles and attenuates medium term ones.
(See Soderlind, 1994.)

HP Filter

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f(2pi/64) =  0.016697846612617945 f(2pi/32) =  0.49370145152
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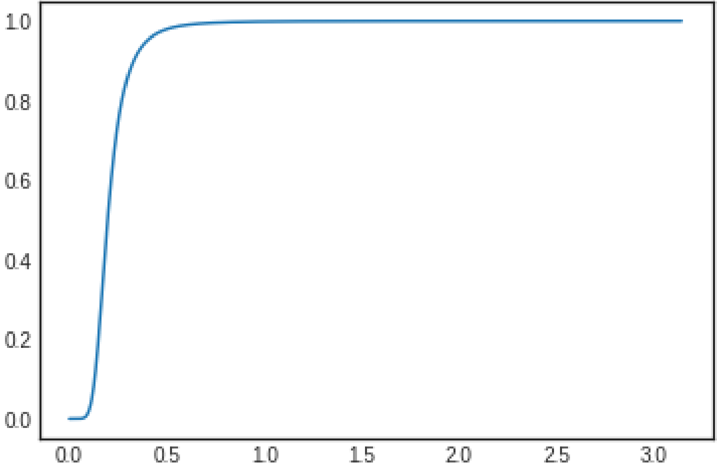

More on Filters

Subsquently we will consider filters that are linear in the time domain, namely, filters of the form,

$$Y_t = \sum_{h=1}^{J} c_h X_{t-h} = C(L) X_t$$

where $C(z)$ is the polynomial function $\sum_{h=1}^{J} c_h z^h$. Recall that

$$X_t = \sum_{j=-m}^{m} A(\omega_j) e^{i\omega_j t}$$

This means that

Hence,

$$X_{t-h} = \sum_{j=-m}^{m} A(\omega_j) e^{i\omega_j t} e^{-i\omega_j h}$$

$$\begin{aligned} Y_t = C(L)X_t &= \sum_{h=1}^{J} c_h X_{t-h} \\ &= \sum_{j=-m}^{m} \left[A(\omega_j) e^{i\omega_j t} \sum_{h=1}^{J} c_h e^{-i\omega_j h} \right] \\ &= \sum_{j=-m}^{m} A(\omega_j) C(e^{-i\omega_j}) e^{i\omega_j t} \\ &= \sum_{j=-m}^{m} \tilde{A}(\omega_j) e^{i\omega_j t} \qquad (14) \end{aligned}$$

Autocovariance

The autocovariances of Y_t can therefore be expressed as

$$\mathbb{E}[Y_t Y_{t-h}] = \sum_{j=-m}^{m} a_j^2 C(e^{-i\omega_j}) C(e^{i\omega_j}) e^{i\omega_j h}$$

Thus, we can define the spectral distribution function of Y_t as

$$S_Y(\omega) = \sum_{j=-m}^{m} a_j^2 C(e^{-i\omega_j}) C(e^{i\omega_j})$$

with increments

$$\Delta S_Y(\omega_j) = \Delta S_X C(e^{-i\omega_j}) C(e^{i\omega_j})$$

Generalization

Result: Suppose that X_t has a spectral density function $s_X(\omega)$ and $Y_t = C(L)X_t$, then the spectral density of the filtered process Y_t is given by

$$s_Y(\omega) = |C(e^{-i\omega})|^2 s_X(\omega)$$

The function $C(e^{-i\omega})$ is called transfer function of the filter, and the filter function $f(\omega) = |C(e^{-i\omega})|^2$ is often called power transfer function. $\square$

Examples of Spectrum

White Noise

$$s(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_h e^{-i\omega h} = \frac{\gamma_0}{2\pi}$$

An AR(1): $Y_t = \phi Y_{t-1} + X_t$
Interpret as a linear filter with $MA(\infty)$ rep: $Y_t = \sum_{h=0}^{\infty} \phi^h X_{t-h}$.
Thus:

$$\begin{aligned} |C(e^{-i\omega})|^2 &= \left|[1-\phi e^{-i\omega}]^{-1}\right|^2 \\ &= \left[|1-\phi\cos\omega + i\phi\sin\omega|^2\right]^{-1} \\ &= \left[(1-\phi\cos\omega)^2 + \phi^2\sin^2\omega\right]^{-1} \\ &= \left[1-2\phi\cos\omega + \phi^2(\cos\omega^2 + \sin^2\omega)\right]^{-1}. \quad (15) \end{aligned}$$

which means $s_Y(\omega) = \frac{\sigma^2/2\pi}{1+\phi^2-2\phi\cos\omega}$. Note $s_Y(0) \longrightarrow \infty$ as $\phi \longrightarrow 1$

More Examples

Stationary ARMA process*: $\phi(L)Y_t = \theta(L)X_t$ with $X_t \sim WN$. The spectral density is given by

$$s_Y(\omega) = \left|\frac{\theta(e^{-i\omega})}{\phi(e^{-i\omega})}\right|^2 \sigma^2$$

Sums of processes. Suppose that $W_t = Y_t + X_t$. The spectrum of the process W_t is simply the sum

$$s_W(\omega) = s_Y(\omega) + s_X(\omega)$$

Visual

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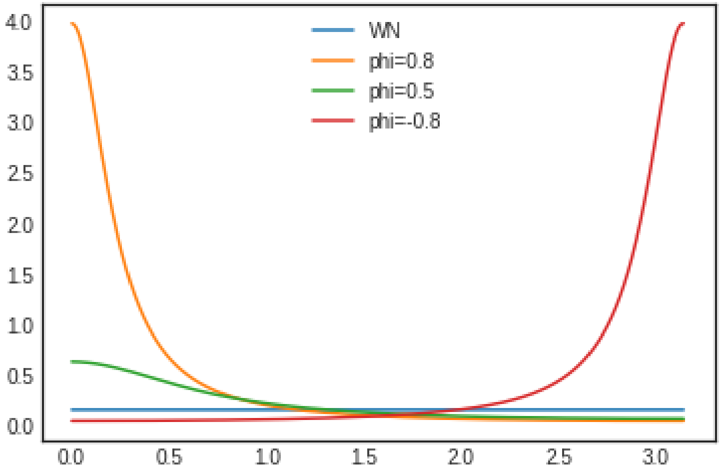


Estimation

1. Parametric – Pick an ARMA process, estimate in time domain, use filtering results to get spectrum.
2. Nonparametric – Estimate autocovariances $\{\hat{\gamma}_h\}$, directly write down spectral density. Let's look at this.

Let $\bar{y} = \frac{1}{T}\sum y_t$ and define the sample covariances

$$\hat{\gamma}_h = \frac{1}{T}\sum_{t=h+1}^{T}(y_t - \bar{y})(y_{t-h} - \bar{y})$$

An intuitively plausible estimate of the spectrum is the sample periodogram

$$\begin{aligned} I_T(\omega) &= \frac{1}{2\pi}\sum_{j=-T+1}^{T-1}\hat{\gamma}_h e^{-i\omega h} \\ &= \frac{1}{2\pi}\left(\hat{\gamma}_0 + 2\sum_{h=1}^{T-1}\hat{\gamma}_h \cos(\omega h)\right) \end{aligned} \tag{16}$$

Result: The sample periodogram is an asymptotically unbiased estimator of the population spectrum, that is,

$$\mathbb{E}[I_T(\omega)] \xrightarrow{p} s(\omega) \tag{17}$$

However, it is inconsistent since the variance $var[I_T(\omega)]$ does not converge to zero as the sample size tends to infinity. □

Smoothed Periodogram

Smoothing: get non-parametric estimators.
To obtain a spectral density estimate at the frequency $\omega = \omega_*$ we will compute the sample periodogram $I_T(\omega)$ for some ω_j's in the neighborhood of ω_* and simply average them. Define the following band around ω_*:

$$B(\omega_*|\lambda) = \left\{\omega : \omega_* - \frac{\lambda}{2} < \omega \leq \omega_* + \frac{\lambda}{2}\right\} \tag{18}$$

The bandwidth is λ, where λ is a parameter. Moreover, define the "fundamental frequencies" (see Hamilton 1994, Chapter 6.2, for a discussion why these frequencies are "fundamental")

$$\omega_j = j\frac{2\pi}{T} \quad j = 1, \ldots, (T-1)/2 \tag{19}$$

iThe number of fundamental frequencies in the band $B(\omega_*)$ is

$$m = \lfloor \lambda T(2\pi)^{-1} \rfloor \tag{20}$$

Smoothed Periodogram

The smoothed periodogram estimator of $s(\omega_*)$ is defined as the average

$$\hat{s}(\omega) = \sum_{j=1}^{(T-1)/2} \frac{1}{m}\{\omega_j \in B(\omega_*|\lambda)\} I_T(\omega_j) \tag{21}$$

where $\{\omega_j \in B(\omega_*|\lambda)\}$ is the indicator function that is equal to one if $\omega_j \in B(\omega_*|\lambda)$ and zero otherwise.

Result: The smoothed periodogram estimator $\hat{s}(\omega_*)$ of $s(\omega_*)$ is consistent, provided that the bandwidth shrinks to zero, that is, $\lambda \to 0$ as $T \to \infty$ and the number of ω_j's in the band $B(\omega_*|\lambda)$ tends to infinity, that is \$ m = λ T/$(2\pi) \to \infty$\$. $\square$

Remarks

- get smoothed estimates => need to get λ. Ultimately subective.
- Most non-parameterics approaches are based on "Kernel estimates"

The expression $\{\omega_j \in B(\omega_*)\}$ can be rewritten as follows

$$\begin{aligned}\{\omega_j \in B(\omega_*)\} &= \left\{\omega_* - \frac{\lambda}{2} < \omega_j \leq \omega_* + \frac{\lambda}{2}\right\} \\ &= \left\{-\frac{1}{2} < \frac{\omega_j - \omega_*}{\lambda} \leq \frac{1}{2}\right\} \end{aligned} \tag{22}$$

Define

$$K\left(\frac{\omega_j - \omega_*}{\lambda}\right) = \left\{-\frac{1}{2} < \frac{\omega_j - \omega_*}{\lambda} \leq \frac{1}{2}\right\} \tag{23}$$

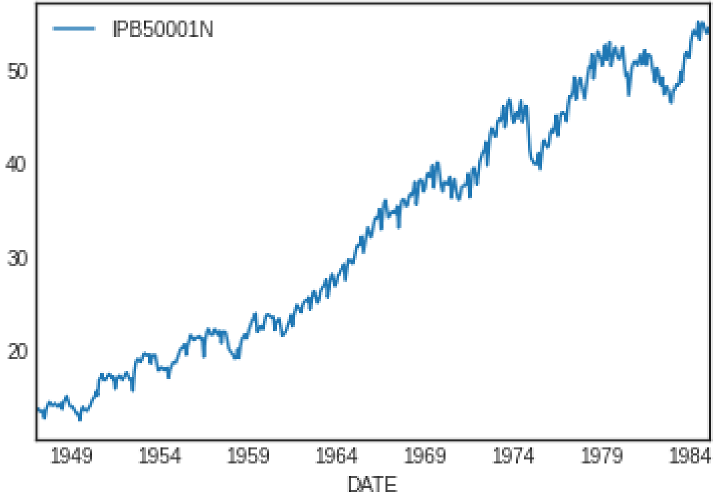
It can be easily verified that

$$\int K\left(\frac{\omega_j - \omega_*}{\lambda}\right) d\omega_* = 1 \tag{24}$$

The function $K\left(\frac{\omega_j - \omega_*}{\lambda}\right)$ is an example of a Kernel function. In general, a Kernel has the property $\int K(x)dx = 1$. Since $m \approx \lambda(T-1)/2$, the spectral estimator can be rewritten as

$$\hat{s}(\omega) = \frac{\pi}{\lambda(T-1)/2} \sum_{j=1}^{(T-1)/2} K\left(\frac{\omega_j - \omega_*}{\lambda}\right) I_T(\omega_j) \tag{25}$$

Application: IP



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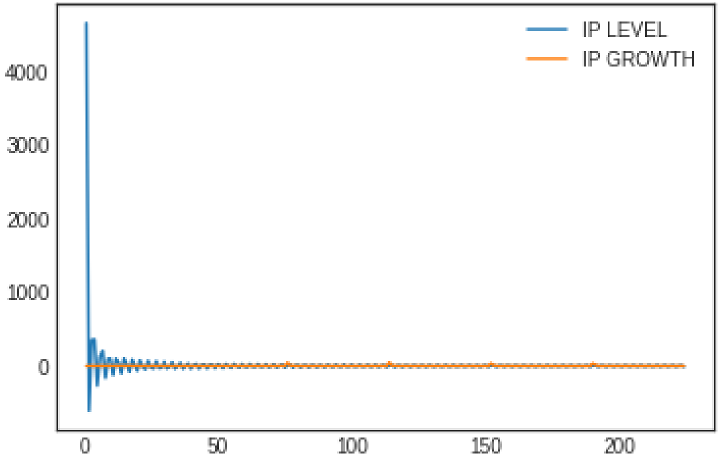


Application: Autocorrelation Consistent Standard Errors

Consider the model

$$y_t = \beta x_t + u_t, \quad u_t = \psi(L)\epsilon_t, \quad \epsilon_t \sim iid(0, \sigma^2) \tag{26}$$

The OLS estimator is given by

$$\hat{\beta} - \beta = \frac{\sum x_t u_t}{\sum x_t^2} \tag{27}$$

The conventional standard error estimates for $\hat{\beta}$ are inconsistent if the u_t's are serially correlated. However, we can construct a consistent estimate based on non-parametric spectral density estimation. Define $z_t = x_t u_t$. We want to obtain an estimate of

$$\text{plim } \Lambda_T = \text{plim } \frac{1}{T} \sum_{t=1}^{T} \sum_{h=1}^{T} E[z_t z_h] \tag{28}$$

It can be verified that

$$\sum_{h=-\infty}^{\infty} \gamma_{zz,h} - \frac{1}{T}\sum_{t=1}^{T}\sum_{h=1}^{T} E[z_t z_h] \xrightarrow{p} 0 \tag{29}$$

Since

$$s(\omega) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} \gamma_{zz,h} e^{-i\omega h} \tag{30}$$

it follows that a consistent estimator of plim Λ_T is

$$\hat{\Lambda}_T = 2\pi \hat{s}(0) \tag{31}$$

where $\hat{s}(0)$ is a non-parametric spectral estimate at frequency zero.

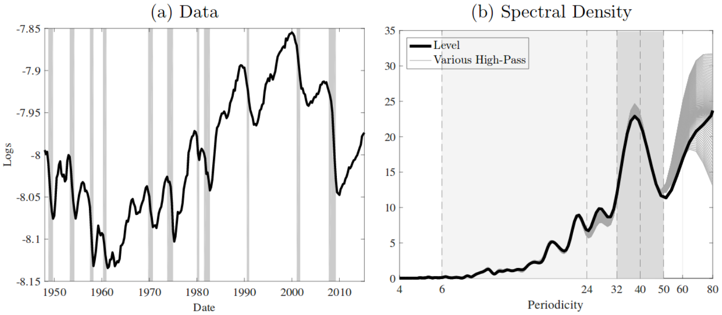
Application: Beaudry, Galizia, and Portier (2020)

Paul Beaudry, Dana Galiza, and Franck Portier (2016): "Putting the Cycle Back into Business Cycle Analysis," *NBER Working Paper*.

- Re-examines the spectral properties of several cyclically sensitive variables such as hours worked, unemployment and capacity utilization.
- Document the presence of an important peak in the spectral density at a periodicity of approximately 36-40 quarters.
- This is cyclical phenomena at the "long end" of the business cycle.
- Suggests a model ("limit cycles") to account for this finding.

The Paper in 1 Picture

Figure 1: Properties of Hours Worked per Capita



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