Intro to DSGE + State Space Models

Ed Herbst

Last Updated: March 13, 2025

Background

- *Textbook treatments:* Woodford (2003), Galí (2008)

- *Key empirical papers*: Ireland (2004), Christiano, Eichenbaum, and Evans (2005), Smets and Wouters (2007), An and Schorfheide (2007),

- *Frequentist estimation:* Harvey (1991), Hamilton (1994),

- *Bayesian estimation:* Herbst and Schorfheide (2015)

Small-Scale DSGE Model

- Intermediate and final goods producers
- Households
- Monetary and fiscal policy
- Exogenous processes
- Equilibrium Relationships

Final Goods Producers

- Perfectly competitive firms combine a continuum of intermediate goods:

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}.$$

- Firms take input prices $P_t(j)$ and output prices P_t as given; maximize profits

$$\Pi_t = P_t \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}} - \int_0^1 P_t(j) Y_t(j) dj.$$

- Demand for intermediate good j:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\nu} Y_t.$$

- Zero-profit condition implies

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}.$$

Intermediate Goods Producers

- Intermediate good j is produced by a monopolist according to:

$$Y_t(j) = A_t N_t(j).$$

- Nominal price stickiness via quadratic price adjustment costs

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \pi\right)^2 Y_t(j).$$

- Firm j chooses its labor input $N_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits:

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - W_{t+s} N_{t+s}(j) - AC_{t+s}(j)\right)\right].$$

Households

- Household derives disutility from hours worked H_t and maximizes

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(\frac{(C_{t+s}/A_{t+s})^{1-\tau} - 1}{1-\tau} \right.\right.$$
$$\left.\left. + \chi_M \ln \left(\frac{M_{t+s}}{P_{t+s}} \right) - \chi_H H_{t+s} \right) \right].$$

- Budget constraint:

$$\begin{aligned} & P_t C_t + B_t + M_t + T_t \\ & \quad = \quad P_t W_t H_t + R_{t-1} B_{t-1} + M_{t-1} + P_t D_t + P_t SC_t. \end{aligned}$$

Monetary and Fiscal Policy

- Central bank adjusts money supply to attain desired interest rate.
- Monetary policy rule:

$$
\begin{aligned}
R_t &= R_t^{*,1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}} \\
R_t^* &= r\pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^*}\right)^{\psi_2}
\end{aligned}
$$

- Fiscal authority consumes fraction of aggregate output: $G_t = \zeta_t Y_t$.
- Government budget constraint:

$$P_t G_t + R_{t-1} B_{t-1} + M_{t-1} = T_t + B_t + M_t.$$

Exogenous Processes

- Technology:

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \quad \ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t}.$$

- Government spending / aggregate demand: define $g_t = 1/(1 - \zeta_t)$; assume

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}.$$

- Monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated.

Equilibrium Conditions

- Consider the symmetric equilibrium in which all intermediate goods producing firms make identical choices; omit j subscript.
- Market clearing:

$$Y_t = C_t + G_t + AC_t \quad \text{and} \quad H_t = N_t.$$

- Complete markets:

$$Q_{t+s|t} = (C_{t+s}/C_t)^{-\tau}(A_t/A_{t+s})^{1-\tau}.$$

- Consumption Euler equation and New Keynesian Phillips curve:

$$\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right] \\
1 &= \phi(\pi_t - \pi) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\pi}{2\nu} \right] \\
&\quad - \phi\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{Y_{t+1}/A_{t+1}}{Y_t/A_t} (\pi_{t+1} - \pi)\pi_{t+1} \right] \\
&\quad + \frac{1}{\nu} \left[1 - \left(\frac{C_t}{A_t} \right)^{\tau} \right].
\end{aligned}$$

Equilibrium Conditions – Continued

- In the absence of nominal rigidities ($\phi = 0$) aggregate output is given by
$$Y_t^* = (1-\nu)^{1/\tau} A_t g_t,$$
which is the target level of output that appears in the monetary policy rule.

Steady State

- Set $\epsilon_{R,t}$, $\epsilon_{g,t}$, and $\epsilon_{z,t}$ to zero at all times.
- Because technology $\ln A_t$ evolves according to a random walk with drift $\ln \gamma$, consumption and output need to be detrended for a steady state to exist.
- Let
$$c_t = C_t/A_t, \quad y_t = Y_t/A_t, \quad y_t^* = Y_t^*/A_t.$$
- Steady state is given by:
$$\begin{aligned} \pi &= \pi^*, \quad r = \frac{\gamma}{\beta}, \quad R = r\pi^*, \\ c &= (1-\nu)^{1/\tau}, \quad y = gc = y^*. \end{aligned}$$

Solving DSGE Models

- Derive nonlinear equilibrium conditions:
  - System of nonlinear expectational difference equations;
  - transversality conditions.
- Find solution(s) of system of expectational difference methods:
  - Global (nonlinear) approximation
  - Local approximation near steady state
- We will focus on log-linear approximations around the steady state.
- More detail in: Fernandez-Villaverde, Rubio-Ramirez, and Schorfheide (2016): "Solution and Estimation Methods for DSGE Models."

What is a Local Approximation?

- In a nutshell... consider the backward-looking model

$$y_t = f(y_{t-1}, \sigma\epsilon_t).$$

- Guess that the solution is of the form

$$y_t = y_t^{(0)} + \sigma y_t^{(1)} + o(\sigma).$$

- Steady state:

$$y_t^{(0)} = y^{(0)} = f(y^{(0)}, 0)$$

- Suppose $y^{(0)} = 0$. Expand $f(\cdot)$ around $\sigma = 0$:

$$f(y_{t-1}, \sigma\epsilon_t) = f_y y_{t-1} + f_\epsilon \sigma\epsilon_t + o(|y_{t-1}|) + o(\sigma)$$

- Now plug-in conjectured solution:

$$\sigma y_t^{(1)} = f_y \sigma y_{t-1}^{(1)} + f_\epsilon \sigma\epsilon_t + o(\sigma)$$

- Deduce that $y_t^{(1)} = f_y y_{t-1}^{(1)} + f_\epsilon \epsilon_t$

What is a Log-Linear Approximation?

- Consider a Cobb-Douglas production function: $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$.
- Linearization around Y_*, A_*, K_*, N_*:

$$\begin{aligned} Y_t - Y_* \approx\ & K_*^{\alpha} N_*^{1-\alpha}(A_t - A_*) + \alpha A_* K_*^{\alpha-1} N_*^{1-\alpha}(K_t - K_*) \\ & +(1-\alpha)A_* K_*^{\alpha} N_*^{-\alpha}(N_t - N_*) \end{aligned}$$

- Log-linearization: Let $f(x) = f(e^v)$ and linearize with respect to v:

$$f(e^v) \approx f(e^{v_*}) + e^{v_*} f'(e^{v_*})(v - v_*).$$

  Thus:

$$f(x) \approx f(x_*) + x_* f'(x_*)(\ln x/x_*) = f(x_*) + f'(x_*)\tilde{x}$$

- Cobb-Douglas production function (here relationship is exact):

$$\tilde{Y}_t = \tilde{A}_t + \alpha \tilde{K}_t + (1-\alpha)\tilde{N}_t$$

Loglinearization of New Keynesian Model

- Consumption Euler equation:

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\tau}\left(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \mathbb{E}_t[\hat{z}_{t+1}]\right) + \hat{g}_t - \mathbb{E}_t[\hat{g}_{t+1}]$$

- New Keynesian Phillips curve:

$$\hat{\pi}_t = \beta\mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{g}_t),$$

where

$$\kappa = \tau\frac{1-\nu}{\nu\pi^2\phi}$$

- Monetary policy rule:

$$\hat{R}_t = \rho_R\hat{R}_{t-1} + (1-\rho_R)\psi_1\hat{\pi}_t + (1-\rho_R)\psi_2\left(\hat{y}_t - \hat{g}_t\right) + \epsilon_{R,t}$$

Canonical Linear Rational Expectations System

- Define
$$x_t = [\hat{y}_t, \hat{\pi}_t, \hat{R}_t, \epsilon_{R,t}, \hat{g}_t, \hat{z}_t]'.$$
- Augment x_t by $\mathbb{E}_t[\hat{y}_{t+1}]$ and $\mathbb{E}_t[\hat{\pi}_{t+1}]$.
- Define
$$s_t = \left[x_t', \mathbb{E}_t[\hat{y}_{t+1}], \mathbb{E}_t[\hat{\pi}_{t+1}]\right]'.$$
- Define rational expectations forecast errors forecast errors for inflation and output. Let
$$\eta_{y,t} = y_t - \mathbb{E}_{t-1}[\hat{y}_t], \quad \eta_{\pi,t} = \pi_t - \mathbb{E}_{t-1}[\hat{\pi}_t].$$
- Write system in canonical form Sims (2002):
$$\Gamma_0 s_t = \Gamma_1 s_{t-1} + \Psi \epsilon_t + \Pi \eta_t.$$

How Can One Solve Linear Rational Expectations Systems? A Simple Example

- Consider

$$y_t = \frac{1}{\theta}{}_t[y_{t+1}] + \epsilon_t, \tag{1}$$

where $\epsilon_t \sim iid(0, 1)$ and $\theta \in \Theta = [0, 2]$.

- Introduce conditional expectation $\xi_t = \mathbb{E}_t[y_{t+1}]$ and forecast error $\eta_t = y_t - \xi_{t-1}$.

- Thus,

$$\xi_t = \theta\xi_{t-1} - \theta\epsilon_t + \theta\eta_t. \tag{2}$$

A Simple Example

- Determinacy: $\theta > 1$. Then only stable solution:

$$\xi_t = 0, \quad \eta_t = \epsilon_t, \quad y_t = \epsilon_t \tag{3}$$

- Indeterminacy: $\theta \leq 1$ the stability requirement imposes no restrictions on forecast error:

$$\eta_t = \widetilde{M}\epsilon_t + \zeta_t. \tag{4}$$

- For simplicity assume now $\zeta_t = 0$. Then

$$y_t - \theta y_{t-1} = \widetilde{M}\epsilon_t - \theta\epsilon_{t-1}. \tag{5}$$

- General solution methods for LREs: Blanchard and Kahn (1980), King and Watson (1998), Uhlig (1999), Anderson (2000), Klein (2000), Christiano (2002), Sims (2002).

Solving a More General System

- Canonical form:

$$\Gamma_0(\theta)s_t = \Gamma_1(\theta)s_{t-1} + \Psi(\theta)\epsilon_t + \Pi(\theta)\eta_t, \quad (6)$$

- The system can be rewritten as

$$s_t = \Gamma_1^*(\theta)s_{t-1} + \Psi^*(\theta)\epsilon_t + \Pi^*(\theta)\eta_t. \quad (7)$$

- Replace Γ_1^* by $J\Lambda J^{-1}$ and define $w_t = J^{-1}s_t$.
- To deal with repeated eigenvalues and non-singular Γ_0 we use Generalized Complex Schur Decomposition (QZ) in practice.
- Let the i'th element of w_t be $w_{i,t}$ and denote the i'th row of $J^{-1}\Pi^*$ and $J^{-1}\Psi^*$ by $[J^{-1}\Pi^*]_{i.}$ and $[J^{-1}\Psi^*]_{i.}$, respectively.

Solving a More General System

- Rewrite model:

$$w_{i,t} = \lambda_i w_{i,t-1} + [J^{-1}\Psi^*]_{i.}\epsilon_t + [J^{-1}\Pi^*]_{i.}\eta_t. \tag{8}$$

- Define the set of stable AR(1) processes as

$$I_s(\theta) = \left\{ i \in \{1, \dots n\} \,\middle|\, |\lambda_i(\theta)| \leq 1 \right\} \tag{9}$$

- Let $I_x(\theta)$ be its complement. Let Ψ_x^J and Π_x^J be the matrices composed of the row vectors $[J^{-1}\Psi^*]_{i.}$ and $[J^{-1}\Pi^*]_{i.}$ that correspond to unstable eigenvalues, i.e., $i \in I_x(\theta)$.
- Stability condition:

$$\Psi_x^J \epsilon_t + \Pi_x^J \eta_t = 0 \tag{10}$$

for all t.

Solving a More General System

- Solving for η_t. Define

$$
\begin{aligned}
\Pi_x^J &= \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{.1}' \\ V_{.2}' \end{bmatrix} \qquad (11) \\
&= \underbrace{U}_{m\times m} \underbrace{D}_{m\times k} \underbrace{V'}_{k\times k} \\
&= \underbrace{U_{.1}}_{m\times r} \underbrace{D_{11}}_{r\times r} \underbrace{V_{.1}'}_{r\times k}.
\end{aligned}
$$

- If there exists a solution to Eq.~(10) that expresses the forecast errors as function of the fundamental shocks ϵ_t and sunspot shocks ζ_t, it is of the form

$$
\begin{aligned}
\eta_t &= \eta_1 \epsilon_t + \eta_2 \zeta_t \qquad (12) \\
&= (-V_{.1} D_{11}^{-1} U_{.1}' \Psi_x^J + V_{.2} \widetilde{M}) \epsilon_t + V_{.2} M_\zeta \zeta_t,
\end{aligned}
$$

where $\widetilde{M}$ is an $(k-r) \times l$ matrix, M_ζ is a $(k-r) \times p$ matrix, and the dimension of $V_{.2}$ is $k \times (k-r)$. The solution is unique if $k = r$ and $V_{.2}$ is zero.

Proposition

If there exists a solution to Eq. (10) that expresses the forecast errors as function of the fundamental shocks ϵ_t and sunspot shocks ζ_t, it is of the form

$$\begin{aligned} \eta_t &= \eta_1 \epsilon_t + \eta_2 \zeta_t \\ &= (-V_{.1} D_{11}^{-1} U_{.1}' \Psi_x^J + V_{.2} \widetilde{M}) \epsilon_t + V_{.2} M_\zeta \zeta_t, \end{aligned} \tag{13}$$

where $\widetilde{M}$ is an $(k-r) \times l$ matrix, M_ζ is a $(k-r) \times p$ matrix, and the dimension of $V_{.2}$ is $k \times (k-r)$. The solution is unique if $k = r$ and $V_{.2}$ is zero.

At the End of the Day...

- We obtain a transition equation for the vector s_t:

$$s_t = T(\theta)s_{t-1} + R(\theta)\epsilon_t.$$

- The coefficient matrices $T(\theta)$ and $R(\theta)$ are functions of the parameters of the DSGE model.

Measurement Equation

- Relate model variables s_t to observables y_t.
- In NK model:

$$\begin{aligned} YGR_t &= \gamma^{(Q)} + 100(\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) \\ INFL_t &= \pi^{(A)} + 400\hat{\pi}_t \\ INT_t &= \pi^{(A)} + r^{(A)} + 4\gamma^{(Q)} + 400\hat{R}_t. \end{aligned}$$

where

$$\gamma = 1 + \frac{\gamma^{(Q)}}{100}, \quad \beta = \frac{1}{1 + r^{(A)}/400}, \quad \pi = 1 + \frac{\pi^{(A)}}{400}.$$

- More generically:

$$y_t = D(\theta) + Z(\theta)s_t \underbrace{+u_t}_{\text{optional}} .$$

The state and measurement equations define a *State Space Model*.

State Space Models

- State space models form a very general class of models that encompass many of the specifications that we encountered earlier.
- ARMA models and linearized DSGE models can be written in state space form.

A state space model consists of

- a measurement equation that relates an *unobservable* state vector s_t to the *observables* y_t,
- a transition equation that describes the evolution of the state vector s_t.

Measurement Equation

The measurement equation is of the form

$$y_t = Z_{t|t-1}s_t + d_{t|t-1} + u_t, \quad t = 1, \dots, T \tag{14}$$

where y_t is a $n \times 1$ vector of observables, s_t is a $m \times 1$ vector of state variables, $Z_{t|t-1}$ is an $n \times m$ vector, $d_{t|t-1}$ is a $n \times 1$ vector, and u_t are innovations (or often "measurement errors") with mean zero and $\mathbb{E}_{t-1}[u_t u_t'] = H_{t|t-1}$.

- The matrices $Z_{t|t-1}$, $d_{t|t-1}$, and $H_{t|t-1}$ are in many applications constant.
- However, it is sufficient that they are predetermined at $t-1$. They could be functions of $y_{t-1}, y_{t-2}, \dots$.
- To simplify the notation, we will denote them by Z_t, d_t, and H_t, respectively.

Transition Equation

The transition equation is of the form

$$s_t = T_{t|t-1} s_{t-1} + c_{t|t-1} + R_{t|t-1} \eta_t \tag{15}$$

where R_t is $m \times g$, and η_t is a $g \times 1$ vector of innovations with mean zero and variance $\mathrm{E}_{t|t-1}[\eta_t \eta_t'] = Q_{t|t-1}$.

- The assumption that s_t evolves according to an VAR(1) process is not very restrictive, since it could be the companion form to a higher order VAR process.
- It is furthermore assumed that (i) expectation and variance of the initial state vector are given by $[s_0] = A_0$ and $var[s_0] = P_0$;
- u_t and η_t are uncorrelated with each other in all time periods , and uncorrelated with the initial state. [not really necessary]

Adding it all up

If the system matrices $Z_t, d_t, H_t, T_t, c_t, R_t, Q_t$ are non-stochastic and predetermined, then the system is linear and y_t can be expressed as a function of present and past u_t's and η_t's.

1. calculate predictions $y_t|Y^{t-1}$, where $Y^{t-1} = [y_{t-1}, \ldots, y_1]$,
2. obtain a likelihood function

$$p(Y^T|\{Z_t, d_t, H_t, T_t, c_t, R_t, Q_t\})$$

3. back out a sequence

$$\left\{p(s_t|Y^t, \{Z_\tau, d_\tau, H_\tau, T_\tau, c_\tau, R_\tau, Q_\tau\})\right\}$$

The algorithm is called the *Kalman Filter* and was originally adopted from the engineering literature.

A Useful Lemma

Let $(x', y')'$ *be jointly normal with*

$$\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$$

Then the $pdf(x|y)$ is multivariate normal with

$$\begin{aligned} \mu_{x|y} &= \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y) && (16) \\ \Sigma_{xx|y} &= \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} && (17) \end{aligned}$$

$\square$

A Bayesian Interpretation to the Kalman Filter

- Although the idea of the algorithm is based on linear projections, it has a very straightforward Bayesian interpretation.
- We will assume that the conditional distributions of s_t and y_t given time $t-1$ information are Gaussian.
- Since the system is linear, all the conditional and marginal distributions that we calculate when we move from period $t-1$ to period t will also be Gaussian.
- Since the state vector s_t is unobservable, it is natural in Bayesian framework to regard it as a random vector.

Note: The subsequent analysis is conditional on the system matrices $Z_t, d_t, H_t, T_t, c_t, R_t, Q_t$. For notational convenience we will, however, drop the system matrices from the conditioning set.

The calculations will be based on the following conditional distribution, represented by densities:

1. Initialization: $p(s_{t-1}|Y^{t-1})$
2. Forecasting:

$$
\begin{aligned}
p(s_t|Y^{t-1}) &= \int p(s_t|s_{t-1}, Y^{t-1})p(s_{t-1}|Y^{t-1})ds_{t-1} \\
p(y_t|Y^{t-1}) &= \int p(y_t|s_t, Y^{t-1})p(s_t|Y^{t-1})ds_t
\end{aligned}
$$

3. Updating:

$$
p(s_t|Y^t) = \frac{p(y_t|s_t, Y^{t-1})p(s_t|Y_{t-1})}{p(y_t|Y^{t-1})}
$$

- The integrals look troublesome.
- However, since the state space model is linear, and the distribution of the innovations u_t and η_t are Gaussian $\Longrightarrow$ everything is Gaussian!
- Hence, we only have to keep track of conditional means and variances.

Initialization

- In period zero, we will start with a prior distribution for the initial state s_0.
- This prior is of the form $s_0 \sim \mathcal{N}(A_0, P_0)$.
- If the system matrices imply that the state vector has a stationary distribution, we could choose A_0 and P_0 to be the mean and variance of this stationary distribution.

Forecasting

- At $(t-1)^+$, that is, after observing y_{t-1}, the belief about the state vector has the form $s_{t-1}|Y^{t-1} \sim (A_{t-1}, P_{t-1})$.
- Thus, the "posterior" from period $t-1$ turns into a prior for $(t-1)^+$.

Since s_{t-1} and η_t are independent multivariate normal random variables, it follows that

$$s_t|Y^{t-1} \sim \mathcal{N}(\hat{s}_{t|t-1}, P_{t|t-1}) \tag{18}$$

where

$$\begin{aligned} \hat{s}_{t|t-1} &= T_t A_{t-1} + c_t \\ P_{t|t-1} &= T_t P_{t-1} T_t' + R_t Q_t R_t' \end{aligned}$$

Forecasting y_t

The conditional distribution of $y_t|s_t, Y^{t-1}$ is of the form

$$y_t|s_t, Y^{t-1} \sim \mathcal{N}(Z_t s_t + d_t, H_t) \tag{19}$$

Since $s_t|Y^{t-1} \sim \mathcal{N}(\hat{s}_{t|t-1}, P_{t|t-1})$, we can deduce that the marginal distribution of y_t conditional on Y^{t-1} is of the form

$$y_t|Y_{t-1} \sim \mathcal{N}(\hat{y}_{t|t-1}, F_{t|t-1}) \tag{20}$$

where

$$\begin{aligned} \hat{y}_{t|t-1} &= Z_t \hat{s}_{t|t-1} + d_t \\ F_{t|t-1} &= Z_t P_{t|t-1} Z_t' + H_t \end{aligned}$$

Updating

To obtain the posterior distribution of $s_t|y_t, Y^{t-1}$ note that

$$
\begin{aligned}
s_t &= \hat{s}_{t|t-1} + (s_t - \hat{s}_{t|t-1}) && (21)\\
y_t &= Z_t \hat{s}_{t|t-1} + d_t + Z_t(s_t - \hat{s}_{t|t-1}) + u_t && (22)
\end{aligned}
$$

and the joint distribution of s_t and y_t is given by

$$
\left[\begin{array}{c} s_t \\ x_t \end{array} \right] \Big| Y^{t-1} \sim \mathcal{N}\left(\left[\begin{array}{c} \hat{s}_{t|t-1} \\ \hat{y}_{t|t-1} \end{array} \right], \left[\begin{array}{cc} P_{t|t-1} & P_{t|t-1} Z_t' \\ Z_t P_{t|t-1}' & F_{t|t-1} \end{array} \right] \right) \quad (23)
$$

$$
s_t|y_t, Y^{t-1} \sim \mathcal{N}(A_t, P_t) \quad (24)
$$

where

$$
\begin{aligned}
A_t &= \hat{s}_{t|t-1} + P_{t|t-1} Z_t' F_{t|t-1}^{-1} (y_t - Z_t \hat{s}_{t|t-1} - d_t) \\
P_t &= P_{t|t-1} - P_{t|t-1} Z_t' F_{t|t-1}^{-1} Z_t P_{t|t-1}
\end{aligned}
$$

The conditional mean and variance $\hat{y}_{t|t-1}$ and $F_{t|t-1}$ were given above. This completes one iteration of the algorithm. The posterior $s_t|Y^t$ will serve as prior for the next iteration. □

Likelihood Function

We can define the one-step ahead forecast error

$$\nu_t = y_t - \hat{y}_{t|t-1} = Z_t(s_t - \hat{s}_{t|t-1}) + u_t \tag{25}$$

The likelihood function is given by

$$\begin{aligned} p(Y^T|\text{parameters}) &= \prod_{t=1}^{T} p(y_t|Y^{t-1}, \text{parameters}) \\ &= (2\pi)^{-nT/2} \left(\prod_{t=1}^{T} |F_{t|t-1}| \right)^{-1/2} \\ &\quad \times \exp\left\{ -\frac{1}{2} \sum_{t=1}^{T} \nu_t F_{t|t-1} \nu_t' \right\} \end{aligned} \tag{26}$$

This representation of the likelihood function is often called prediction error form, because it is based on the recursive prediction one-step ahead prediction errors ν_t. $\square$

Multistep Forecasting

The Kalman Filter can also be used to obtain multi-step ahead forecasts. For simplicity, suppose that the system matrices are constant over time. Since

$$s_{t+h-1|t-1} = T^h s_{t-1} + \sum_{s=0}^{h-1} T^s c + \sum_{s=0}^{h-1} T^s R \eta_t \qquad (27)$$

it follows that

$$\begin{aligned}
\hat{s}_{t+h-1|t-1} &= [s_{t+h-1|t-1}|Y^{t-1}] = T^h A_{t-1} + \sum_{s=0}^{h-1} T^s c \\
P_{t+h-1|t-1} &= var[s_{t+h-1|t-1}|Y^{t-1}] = T^h P_{t-1} T^h + \sum_{s=0}^{h-1} T^s R Q R' T^{s'}
\end{aligned}$$

which leads to

$$y_{t+h-1}|Y_{t-1} \sim \mathcal{N}(\hat{y}_{t+h-1|t-1}, F_{t+h-1|t-1}) \qquad (28)$$

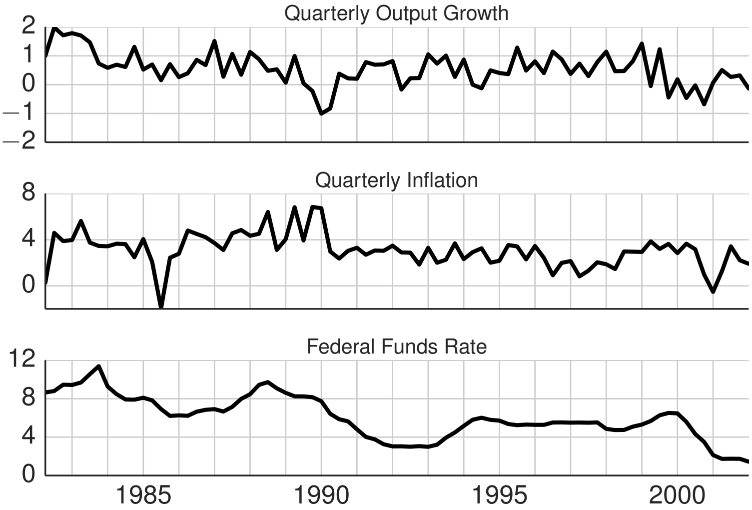
where

$$\hat{y}_{t+h-1|t-1} = Z\hat{s}_{t+h-1|t-1} + d$$

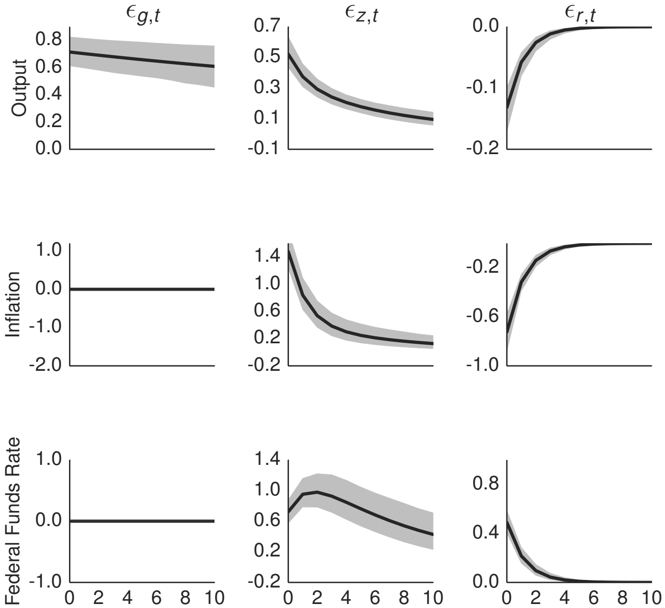
Example 1: New Keynesian DSGE

- We can solve the New Keynesian DSGE model described earlier.
- Obtain state space representation

Observables

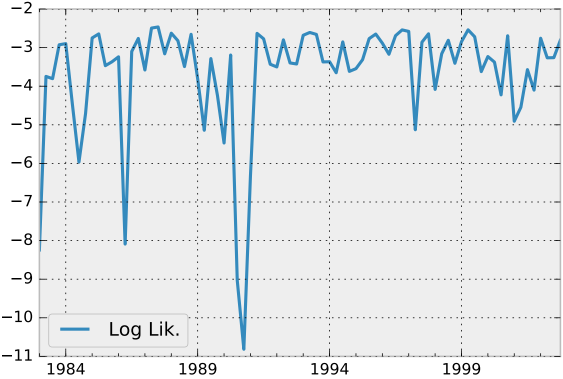


Impulse Responses

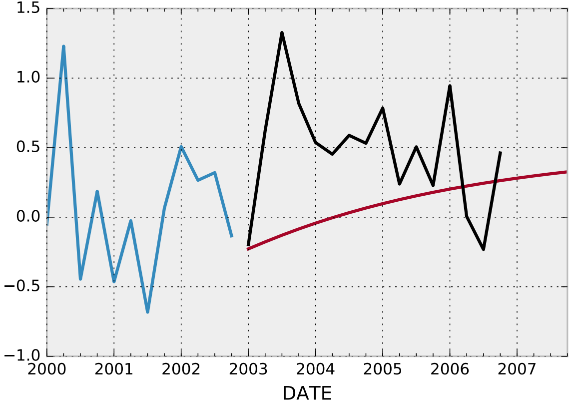


Filtered Technology Shock (Mean)

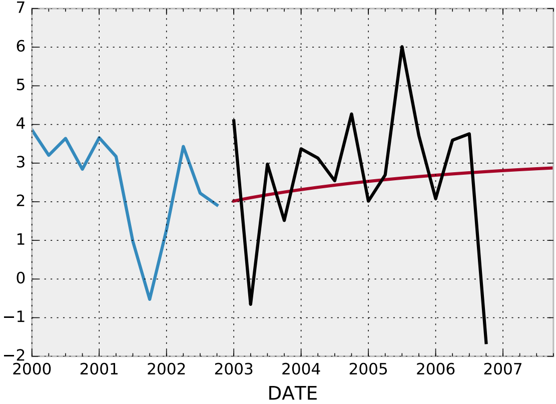
Log Likelihood Increments



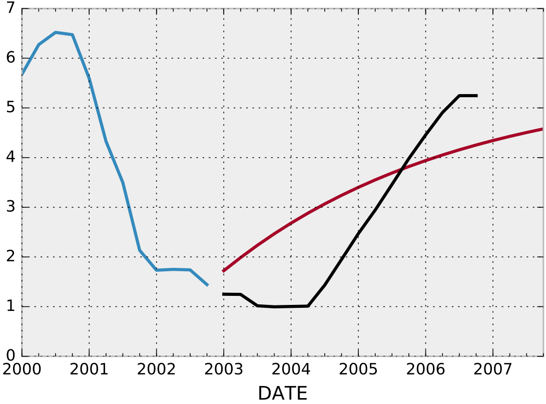
Forecast of Output Growth



Forecast of Inflation



Forecast of Interest Rate



Example 2 – ARMA models

Consider the ARMA(1,1) model of the form

$$y_t = \phi y_{t-1} + \epsilon_t + \theta \epsilon_{t-1} \quad \epsilon_t \sim iid\mathcal{N}(0, \sigma^2) \tag{29}$$

The model can be rewritten in state space form

$$y_t = [1 \ \theta] \begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} + \phi y_{t-1} \tag{30}$$

$$\begin{bmatrix} \epsilon_t \\ \epsilon_{t-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{t-1} \\ \epsilon_{t-2} \end{bmatrix} + \begin{bmatrix} \eta_t \\ 0 \end{bmatrix} \tag{31}$$

where $\eta_t \sim iid\mathcal{N}(0, \sigma^2)$. Thus, the state vector is composed of $\alpha_t = [\epsilon_t, \epsilon_{t-1}]'$ and $d_{t|t-1} = \rho y_{t-1}$. The Kalman filter can be used to compute the likelihood function of the ARMA model conditional on the parameters ϕ, θ, σ^2. A numerical optimization routine has to be used to find the maximum of the likelihood function. $\square$

A Model with Time Varying Coefficients

Consider the following regression model with time varying coefficients

$$y_t = x_t'\beta_t + u_t \quad (32)$$

$$\beta_t = T\beta_{t-1} + c + \eta_t \quad (33)$$

There are many reasons to believe that macroeconomic relationships are not stable over time. An entire branch of the econometrics literature is devoted to tests for structural breaks, that is, tests for changes in the parameter values. However, to be able to predict future changes in the parameter values it is important to model the time variation in the parameters. The state variable α_t corresponds now to the vector of regression parameters β_t. It is often assumed that the regression coefficients follow univariate random walks of the form

$$\beta_{j,t} = \beta_{j,t-1} + \eta_{j,t} \quad (34)$$

Hence, the only unknown parameters are $var[u_t]$ and $var[\eta_{j,t}]$. The Kalman filter can provide us with a sequence of estimates for the time varying coefficients

References I

An, S., and F. Schorfheide. (2007): "Bayesian Analysis of Dsge Models," *Econometric Reviews*, 26, 113–72.

Christiano, L. J., M. Eichenbaum, and C. L. Evans. (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113, 1–45.

Fernandez-Villaverde, J., J. Rubio-Ramirez, and F. Schorfheide. (2016): "Solution and Estimation Methods for Dsge Models," *Handbook of Macroeconomics*, , 527 724.

Galí, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*, 2nd ed.Princeton University Press.

Hamilton, J. (1994): *Time Series Analysis*, Princeton, New Jersey: Princeton University Press.

Harvey, A. C. (1991): *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge, United Kingdom: University of Cambridge Press.

References II

Herbst, E., and F. Schorfheide. (2015): *Bayesian Estimation of Dsge Models*, Princeton: Princeton University Press.

Ireland, P. N. (2004): "A Method for Taking Models to the Data," *Journal of Economic Dynamics and Control*, 28, 1205–26.

Sims, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20, 1–20.

Smets, F., and R. Wouters. (2007): "Shocks and Frictions in Us Business Cycles: A Bayesian Dsge Approach," *American Economic Review*, 97, 586–608.

Woodford, M. (2003): *Interest and Prices*, Princeton University Press.