The Particle Filter

Ed Herbst

November 17, 2020

From Linear to Nonlinear DSGE Models

- While DSGE models are inherently nonlinear, the nonlinearities are often small and decision rules are approximately linear.

- One can add certain features that generate more pronounced nonlinearities:
  - stochastic volatility;
  - markov switching coefficients;
  - asymmetric adjustment costs;
  - occasionally binding constraints.

From Linear to Nonlinear DSGE Models

- Linear DSGE model leads to

$$\begin{aligned} y_t &= \Psi_0(\theta) + \Psi_1(\theta)t + \Psi_2(\theta)s_t + u_t, \quad u_t \sim N(0, \Sigma_u), \\ s_t &= \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_\epsilon). \end{aligned}$$

- Nonlinear DSGE model leads to

$$\begin{aligned} y_t &= \Psi(s_t, t; \theta) + u_t, \quad u_t \sim F_u(\cdot; \theta) \\ s_t &= \Phi(s_{t-1}, \epsilon_t; \theta), \quad \epsilon_t \sim F_\epsilon(\cdot; \theta). \end{aligned}$$

Some Prominent Examples

- Fernndez-Villaverde et al. (2011)
- Fernndez-Villaverde et al. (2015)
- Aruoba et al. (2018)
- Gust et al. (2017)

Particle Filters

- There are many particle filters...

- We will focus on four types:
  - Bootstrap PF
  - A generic PF
  - A conditionally-optimal PF
  - Tempered Particle Filter

Filtering - General Idea

- State-space representation of nonlinear DSGE model

$$\begin{aligned} \text{Measurement Eq.} &: \quad y_t = \Psi(s_t, t; \theta) + u_t, \quad u_t \sim F_u(\cdot;\theta) \\ \text{State Transition} &: \quad s_t = \Phi(s_{t-1}, \epsilon_t; \theta), \quad \epsilon_t \sim F_\epsilon(\cdot;\theta). \end{aligned}$$

- Likelihood function:

$$p(Y_{1:T}|\theta) = \prod_{t=1}^{T} p(y_t|Y_{1:t-1}, \theta)$$

- A filter generates a sequence of conditional distributions $s_t|Y_{1:t}$.

Filtering - General Idea

- Iterations:
  - Initialization at time $t-1$: $p(s_{t-1}|Y_{1:t-1},\theta)$
  - Forecasting t given $t-1$:
    1. Transition equation:
       $p(s_t|Y_{1:t-1},\theta) = \int p(s_t|s_{t-1}, Y_{1:t-1},\theta)p(s_{t-1}|Y_{1:t-1},\theta)ds_{t-1}$
    2. Measurement equation:
       $p(y_t|Y_{1:t-1},\theta) = \int p(y_t|s_t, Y_{1:t-1},\theta)p(s_t|Y_{1:t-1},\theta)ds_t$
  - Updating with Bayes theorem. Once y_t becomes available:

$$p(s_t|Y_{1:t},\theta) = p(s_t|y_t, Y_{1:t-1},\theta) = \frac{p(y_t|s_t, Y_{1:t-1},\theta)p(s_t|Y_{1:t-1},\theta)}{p(y_t|Y_{1:t-1},\theta)}$$

Bootstrap Particle Filter

- Initialization. Draw the initial particles from the distribution $s_0^j \stackrel{iid}{\sim} p(s_0)$ and set $W_0^j = 1$, $j = 1, \ldots, M$.
- Recursion. For $t = 1, \ldots, T$:
  1. Forecasting s_t. Propagate the period $t-1$ particles $\{s_{t-1}^j, W_{t-1}^j\}$ by iterating the state-transition equation forward:
  $$\tilde{s}_t^j = \Phi(s_{t-1}^j, \epsilon_t^j; \theta), \quad \epsilon_t^j \sim F_\epsilon(\cdot; \theta). \tag{1}$$
  An approximation of $\mathbb{E}[h(s_t)|Y_{1:t-1}, \theta]$ is given by
  $$\hat{h}_{t,M} = \frac{1}{M} \sum_{j=1}^{M} h(\tilde{s}_t^j) W_{t-1}^j. \tag{2}$$

Bootstrap Particle Filter

- **Initialization.**
- **Recursion.** For $t = 1, \ldots, T$:
  1. **Forecasting s_t.**
  2. **Forecasting y_t.** Define the incremental weights

$$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j, \theta). \tag{3}$$

The predictive density $p(y_t|Y_{1:t-1}, \theta)$ can be approximated by

$$\hat{p}(y_t|Y_{1:t-1}, \theta) = \frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{4}$$

If the measurement errors are $N(0, \Sigma_u)$ then the incremental weights take the form

$$\tilde{w}_t^j = (2\pi)^{-n/2}|\Sigma_u|^{-1/2}\exp\left\{-\frac{1}{2}\big(y_t - \Psi(\tilde{s}_t^j, t; \theta)\big)'\Sigma_u^{-1}\big(y_t - \Psi(\tilde{s}_t^j, t; \theta)\big) \tag{5}$$

where n here denotes the dimension of y_t.

Bootstrap Particle Filter

- Initialization.
- Recursion. For $t = 1, \ldots, T$:
  1. Forecasting s_t.
  2. Forecasting y_t. Define the incremental weights

$$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j, \theta). \tag{6}$$

  3. Updating. Define the normalized weights

$$\tilde{W}_t^j = \frac{\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M}\sum_{j=1}^M \tilde{w}_t^j W_{t-1}^j}. \tag{7}$$

  An approximation of $\mathbb{E}[h(s_t)|Y_{1:t}, \theta]$ is given by

$$\tilde{h}_{t,M} = \frac{1}{M}\sum_{j=1}^M h(\tilde{s}_t^j)\tilde{W}_t^j. \tag{8}$$

Bootstrap Particle Filter

- Initialization.
- Recursion. For $t = 1, \ldots, T$:
  1. Forecasting s_t.
  2. Forecasting y_t.
  3. Updating.
  4. Selection (Optional). Resample the particles via multinomial resampling. Let $\{s_t^j\}_{j=1}^M$ denote M iid draws from a multinomial distribution characterized by support points and weights $\{\tilde{s}_t^j, \tilde{W}_t^j\}$ and set $W_t^j = 1$ for $j =, 1 \ldots, M$. An approximation of $\mathbb{E}[h(s_t)|Y_{1:t}, \theta]$ is given by

$$\bar{h}_{t,M} = \frac{1}{M} \sum_{j=1}^{M} h(s_t^j) W_t^j. \tag{9}$$

Likelihood Approximation

- The approximation of the log likelihood function is given by

$$\ln \hat{p}(Y_{1:T}|\theta) = \sum_{t=1}^{T} \ln \left(\frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j \right). \tag{10}$$

- One can show that the approximation of the likelihood function is unbiased.

- This implies that the approximation of the log likelihood function is downward biased.

The Role of Measurement Errors

- Measurement errors may not be intrinsic to DSGE model.
- Bootstrap filter needs non-degenerate $p(y_t|s_t, \theta)$ for incremental weights to be well defined.
- Decreasing the measurement error variance Σ_u, holding everything else fixed, increases the variance of the particle weights, and reduces the accuracy of Monte Carlo approximation.

Generic Particle Filter – Recursion

- **Forecasting s_t.** Draw $\tilde{s}_t^j$ from density $g_t(\tilde{s}_t|s_{t-1}^j,\theta)$ and define

$$\omega_t^j = \frac{p(\tilde{s}_t^j|s_{t-1}^j,\theta)}{g_t(\tilde{s}_t^j|s_{t-1}^j,\theta)}. \tag{11}$$

- An approximation of $\mathbb{E}[h(s_t)|Y_{1:t-1},\theta]$ is given by

$$\hat{h}_{t,M} = \frac{1}{M}\sum_{j=1}^{M} h(\tilde{s}_t^j)\omega_t^j W_{t-1}^j. \tag{12}$$

- **Forecasting y_t.** Define the incremental weights

$$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j,\theta)\omega_t^j. \tag{13}$$

  The predictive density $p(y_t|Y_{1:t-1},\theta)$ can be approximated by

$$\hat{p}(y_t|Y_{1:t-1},\theta) = \frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{14}$$

- **Updating / Selection.** Same as BS PF

Asymptotics

- The convergence results can be established recursively, starting from the assumption

$$\begin{aligned} \bar{h}_{t-1,M} &\stackrel{a.s.}{\longrightarrow} \mathbb{E}[h(s_{t-1})|Y_{1:t-1}], \\ \sqrt{M}\big(\bar{h}_{t-1,M} - \mathbb{E}[h(s_{t-1})|Y_{1:t-1}]\big) &\Longrightarrow N\big(0, \Omega_{t-1}(h)\big). \end{aligned}$$

- Forward iteration: draw s_t from $g_t(s_t|s^j_{t-1}) = p(s_t|s^j_{t-1})$.
- Decompose

$$\begin{aligned} & \hat{h}_{t,M} - \mathbb{E}[h(s_t)|Y_{1:t-1}] \qquad (15) \\ &= \frac{1}{M}\sum_{j=1}^{M}\left(h(\tilde{s}^j_t) - \mathbb{E}_{p(\cdot|s^j_{t-1})}[h]\right)W^j_{t-1} \\ &\quad + \frac{1}{M}\sum_{j=1}^{M}\left(\mathbb{E}_{p(\cdot|s^j_{t-1})}[h]W^j_{t-1} - \mathbb{E}[h(s_t)|Y_{1:t-1}]\right) \\ &= I + II, \end{aligned}$$

- Both I and II converge to zero (and potentially satisfy CLT).

Asymptotics

- Updating step approximates

$$\mathbb{E}[h(s_t)|Y_{1:t}] = \frac{\int h(s_t)p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t}{\int p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t} \approx \frac{\frac{1}{M}\sum_{j=1}^{M} h(\tilde{s}_t^j)\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j} \tag{16}$$

- Define the normalized incremental weights as

$$v_t(s_t) = \frac{p(y_t|s_t)}{\int p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t}. \tag{17}$$

- Under suitable regularity conditions, the Monte Carlo approximation satisfies a CLT of the form

$$\begin{aligned} &\sqrt{M}\big(\tilde{h}_{t,M} - \mathbb{E}[h(s_t)|Y_{1:t}]\big) \qquad (18) \\ &\quad \Longrightarrow \quad N\big(0, \tilde{\Omega}_t(h)\big), \quad \tilde{\Omega}_t(h) = \hat{\Omega}_t\big(v_t(s_t)(h(s_t) - \mathbb{E}[h(s_t)|Y_{1:t}])\big). \end{aligned}$$

- Distribution of particle weights matters for accuracy! $\Longrightarrow$ Resampling!

Adapting the Generic PF

- Conditionally-optimal importance distribution:

$$g_t(\tilde{s}_t|s^j_{t-1}) = p(\tilde{s}_t|y_t, s^j_{t-1}).$$

  This is the posterior of s_t given s^j_{t-1}. Typically infeasible, but a good benchmark.

- Approximately conditionally-optimal distributions: from linearize version of DSGE model or approximate nonlinear filters.

- Conditionally-linear models: do Kalman filter updating on a subvector of s_t. Example:

$$\begin{aligned} y_t &= \Psi_0(m_t) + \Psi_1(m_t)t + \Psi_2(m_t)s_t + u_t, \quad u_t \sim N(0, \Sigma_u), \\ s_t &= \Phi_0(m_t) + \Phi_1(m_t)s_{t-1} + \Phi_\epsilon(m_t)\epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_\epsilon), \end{aligned}$$

  where m_t follows a discrete Markov-switching process.

More on Conditionally-Linear Models

- State-space representation is linear conditional on m_t.
- Write
$$p(m_t, s_t|Y_{1:t}) = p(m_t|Y_{1:t})p(s_t|m_t, Y_{1:t}), \tag{19}$$
where
$$s_t|(m_t, Y_{1:t}) \sim N\big(\bar{s}_{t|t}(m_t), P_{t|t}(m_t)\big). \tag{20}$$
- Vector of means $\bar{s}_{t|t}(m_t)$ and the covariance matrix $P_{t|t}(m)_t$ are sufficient statistics for the conditional distribution of s_t.
- Approximate $(m_t, s_t)|Y_{1:t}$ by $\{m_t^j, \bar{s}_{t|t}^j, P_{t|t}^j, W_t^j\}_{i=1}^N$.
- The swarm of particles approximates

$$\begin{aligned}
&\int h(m_t, s_t)p(m_t, s_t, Y_{1:t})d(m_t, s_t) \qquad (21)\\
&= \int \left[\int h(m_t, s_t)p(s_t|m_t, Y_{1:t})ds_t\right] p(m_t|Y_{1:t})dm_t\\
&\approx \frac{1}{M}\sum_{j=1}^{M}\left[\int h(m_t^j, s_t^j)p_N\big(s_t|\bar{s}_{t|t}^j, P_{t|t}^j\big)ds_t\right] W_t^j.
\end{aligned}$$

More on Conditionally-Linear Models

- We used Rao-Blackwellization to reduce variance:

$$\begin{aligned}\mathbb{V}[h(s_t, m_t)] &= \mathbb{E}\big[\mathbb{V}[h(s_t, m_t)|m_t]\big] + \mathbb{V}\big[\mathbb{E}[h(s_t, m_t)|m_t]\big] \\ &\geq \mathbb{V}\big[\mathbb{E}[h(s_t, m_t)|m_t]\big]\end{aligned}$$

- To forecast the states in period generate $\tilde{m}_t^j$ from $g_t(\tilde{m}_t|m_{t-1}^j)$ and define:

$$\omega_t^j = \frac{p(\tilde{m}_t^j|m_{t-1}^j)}{g_t(\tilde{m}_t^j|m_{t-1}^j)}. \tag{22}$$

- The Kalman filter forecasting step can be used to compute:

$$\begin{aligned}\tilde{s}_{t|t-1}^j &= \Phi_0(\tilde{m}_t^j) + \Phi_1(\tilde{m}_t^j)s_{t-1}^j \\ P_{t|t-1}^j &= \Phi_\epsilon(\tilde{m}_t^j)\Sigma_\epsilon(\tilde{m}_t^j)\Phi_\epsilon(\tilde{m}_t^j)' \\ \tilde{y}_{t|t-1}^j &= \Psi_0(\tilde{m}_t^j) + \Psi_1(\tilde{m}_t^j)t + \Psi_2(\tilde{m}_t^j)\tilde{s}_{t|t-1}^j \\ F_{t|t-1}^j &= \Psi_2(\tilde{m}_t^j)P_{t|t-1}^j\Psi_2(\tilde{m}_t^j)' + \Sigma_u.\end{aligned} \tag{23}$$

More on Conditionally-Linear Models

- Then,

$$\int h(m_t, s_t)p(m_t, s_t|Y_{1:t-1})d(m_t, s_t) \tag{24}$$
$$= \int \left[\int h(m_t, s_t)p(s_t|m_t, Y_{1:t-1})ds_t \right] p(m_t|Y_{1:t-1})dm_t$$
$$\approx \frac{1}{M}\sum_{j=1}^{M} \left[\int h(m_t^j, s_t^j)p_N(s_t|\tilde{s}_{t|t-1}^j, P_{t|t-1}^j)ds_t \right] \omega_t^j W_{t-1}^j$$

- The likelihood approximation is based on the incremental weights

$$\tilde{w}_t^j = p_N(y_t|\bar{y}_{t|t-1}^j, F_{t|t-1}^j)\omega_t^j. \tag{25}$$

- Conditional on $\tilde{m}_t^j$ we can use the Kalman filter once more to update the information about s_t in view of the current observation y_t:

$$\begin{aligned}
\tilde{s}_{t|t}^j &= \tilde{s}_{t|t-1}^j + P_{t|t-1}^j \Psi_2(\tilde{m}_t^j)' \left(F_{t|t-1}^j\right)^{-1}(y_t - \bar{y}_{t|t-1}^j) \\
\tilde{P}_{t|t}^j &= P_{t|t-1}^j - P_{t|t-1}^j \Psi_2(\tilde{m}_t^j)' \left(F_{t|t-1}^j\right)^{-1} \Psi_2(\tilde{m}_t^j) P_{t|t-1}^j.
\end{aligned}$$

Particle Filter For Conditionally Linear Models

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
   2.1 **Forecasting s_t.** Draw $\tilde{m}_t^j$ from density $g_t(\tilde{m}_t | m_{t-1}^j, \theta)$, calculate the importance weights ω_t^j in (22), and compute $\tilde{s}_{t|t-1}^j$ and $P_{t|t-1}^j$ according to (23). An approximation of $\mathbb{E}[h(s_t, m_t) | Y_{1:t-1}, \theta]$ is given by (25).

   2.2 **Forecasting y_t.** Compute the incremental weights $\tilde{w}_t^j$ according to (25). Approximate the predictive density $p(y_t | Y_{1:t-1}, \theta)$ by

$$\hat{p}(y_t | Y_{1:t-1}, \theta) = \frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{27}$$

   2.3 **Updating.** Define the normalized weights

$$\tilde{W}_t^j = \frac{\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j} \tag{28}$$

   and compute $\tilde{s}_{t|t}^j$ and $\tilde{P}_{t|t}^j$ according to (26). An approximation of $\mathbb{E}[h(m_t, s_t) | Y_{1:t}, \theta]$ can be obtained from

Nonlinear and Partially Deterministic State Transitions

- Example:

$$s_{1,t} = \Phi_1(s_{t-1}, \epsilon_t), \quad s_{2,t} = \Phi_2(s_{t-1}), \quad \epsilon_t \sim N(0,1).$$

- Generic filter requires evaluation of $p(s_t|s_{t-1})$. Define $\varsigma_t = [s_t', \epsilon_t']'$ and add identity $\epsilon_t = \epsilon_t$ to state transition. Factorize the density $p(\varsigma_t|\varsigma_{t-1})$ as

$$p(\varsigma_t|\varsigma_{t-1}) = p^{\epsilon}(\epsilon_t)p(s_{1,t}|s_{t-1}, \epsilon_t)p(s_{2,t}|s_{t-1}).$$

  where $p(s_{1,t}|s_{t-1}, \epsilon_t)$ and $p(s_{2,t}|s_{t-1})$ are pointmasses.
- Sample innovation ϵ_t from $g_t^{\epsilon}(\epsilon_t|s_{t-1})$.
- Then

$$\omega_t^j = \frac{p(\tilde{\varsigma}_t^j|\varsigma_{t-1}^j)}{g_t(\tilde{\varsigma}_t^j|\varsigma_{t-1}^j)} = \frac{p^{\epsilon}(\tilde{\epsilon}_t^j)p(\tilde{s}_{1,t}^j|s_{t-1}^j, \tilde{\epsilon}_t^j)p(\tilde{s}_{2,t}^j|s_{t-1}^j)}{g_t^{\epsilon}(\tilde{\epsilon}_t^j|s_{t-1}^j)p(\tilde{s}_{1,t}^j|s_{t-1}^j, \tilde{\epsilon}_t^j)p(\tilde{s}_{2,t}^j|s_{t-1}^j)} = \frac{p^{\epsilon}(\tilde{\epsilon}_t^j)}{g_t^{\epsilon}(\tilde{\epsilon}_t^j|s_{t-1}^j)}$$

Degenerate Measurement Error Distributions

- Our discussion of the conditionally-optimal importance distribution suggests that in the absence of measurement errors, one has to solve the system of equations

$$y_t = \Psi\big(\Phi(s^j_{t-1}, \tilde{\epsilon}^j_t)\big),$$

to determine $\tilde{\epsilon}^j_t$ as a function of s^j_{t-1} and the current observation y_t. Then define

$$\omega^j_t = p^{\epsilon}(\tilde{\epsilon}^j_t) \quad \text{and} \quad \tilde{s}^j_t = \Phi(s^j_{t-1}, \tilde{\epsilon}^j_t).$$

- Difficulty: one has to find all solutions to a nonlinear system of equations. While resampling duplicates particles, the duplicated particles do not mutate, which can lead to a degeneracy.

Next Steps

- We will now apply PFs to linearized DSGE models.
- This allows us to compare the Monte Carlo approximation to the “truth.”
- Small-scale New Keynesian DSGE model
- Smets-Wouters model

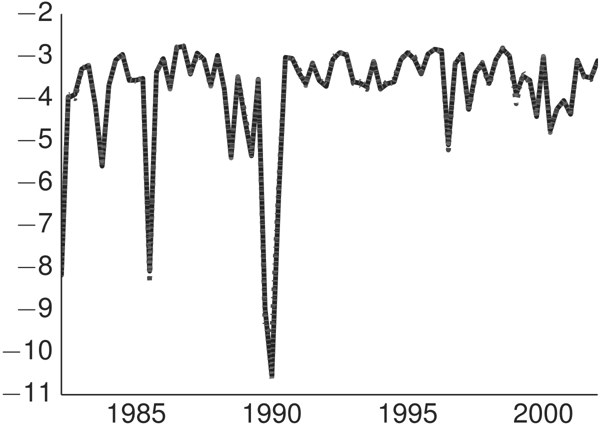
Illustration 1: Small-Scale DSGE Model

Parameter Values For Likelihood Evaluation

Parameter	θ^m	θ^l	Parameter	θ^m	θ^l
τ	2.09	3.26	κ	0.98	0.89
ψ_1	2.25	1.88	ψ_2	0.65	0.53
ρ_r	0.81	0.76	ρ_g	0.98	0.98
ρ_z	0.93	0.89	$r^{(A)}$	0.34	0.19
$\pi^{(A)}$	3.16	3.29	$\gamma^{(Q)}$	0.51	0.73
σ_r	0.19	0.20	σ_g	0.65	0.58
σ_z	0.24	0.29	$\ln p(Y\|\theta)$	-306.5	-313.4

Likelihood Approximation

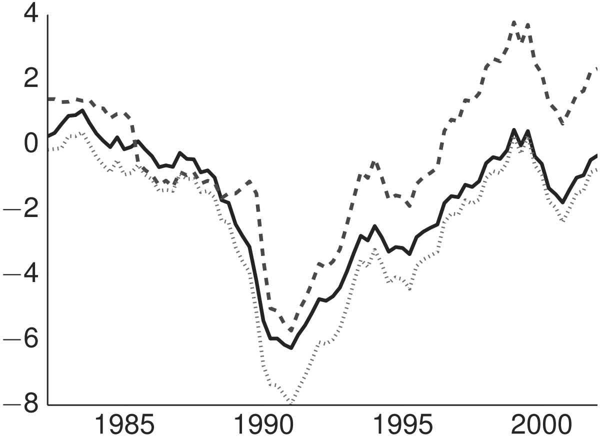
$\ln \hat{p}(y_t|Y_{1:t-1}, \theta^m)$ vs. $\ln p(y_t|Y_{1:t-1}, \theta^m)$



Notes: The results depicted in the figure are based on a single run of the bootstrap PF (dashed, $M = 40,000$), the conditionally-optimal PF (dotted, $M = 400$), and the Kalman filter (solid).

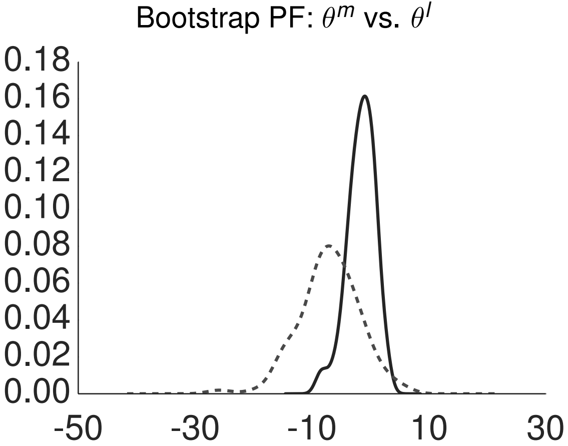
Filtered State

$\widehat{\mathbb{E}}[\hat{g}_t | Y_{1:t}, \theta^m]$ vs. $\mathbb{E}[\hat{g}_t | Y_{1:t}, \theta^m]$



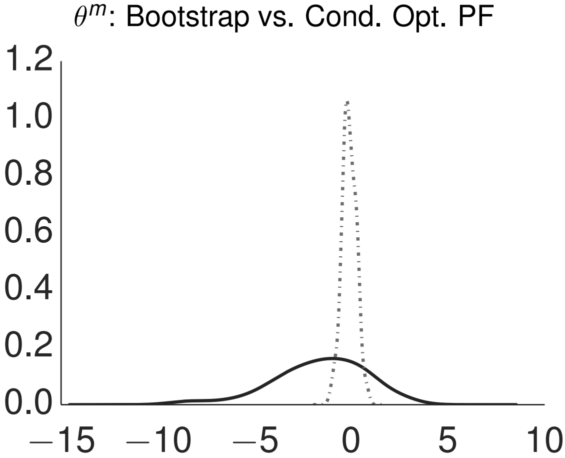
Notes: The results depicted in the figure are based on a single run of the bootstrap PF (dashed, $M = 40,000$), the conditionally-optimal PF (dotted, $M = 400$), and the Kalman filter (solid).

Distribution of Log-Likelihood Approximation Errors



Notes: Density estimate of $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ based on $N_{run} = 100$ runs of the PF. Solid line is $\theta = \theta^m$; dashed line is $\theta = \theta^l$ ($M = 40,000$).

Distribution of Log-Likelihood Approximation Errors



Notes: Density estimate of $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ based on $N_{run} = 100$ runs of the PF. Solid line is bootstrap particle filter ($M = 40,000$); dotted line is conditionally optimal particle filter ($M = 400$).

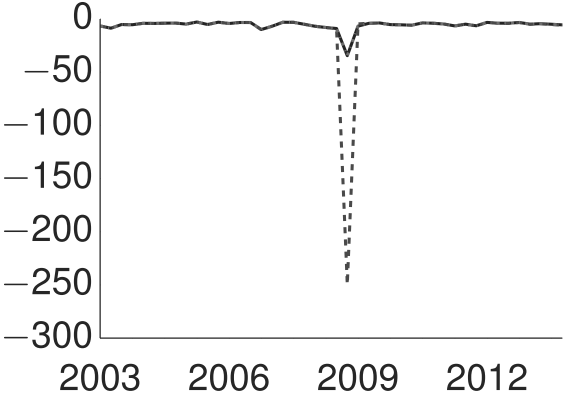
Summary Statistics for Particle Filters

	Bootstrap	Cond. Opt.	Auxiliary
Number of Particles M	40,000	400	40,000
Number of Repetitions	100	100	100
High Posterior Density: $\theta = \theta^m$			
Bias $\hat{\Delta}_1$	-1.39	-0.10	-2.83
StdD $\hat{\Delta}_1$	2.03	0.37	1.87
Bias $\hat{\Delta}_2$	0.32	-0.03	-0.74
Low Posterior Density: $\theta = \theta^l$			
Bias $\hat{\Delta}_1$	-7.01	-0.11	-6.44
StdD $\hat{\Delta}_1$	4.68	0.44	4.19
Bias $\hat{\Delta}_2$	-0.70	-0.02	-0.50

Notes: $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ and $\hat{\Delta}_2 = \exp[\ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)] - 1$. Results are based on $N_{run} = 100$ runs of the particle filters.

Great Recession and Beyond

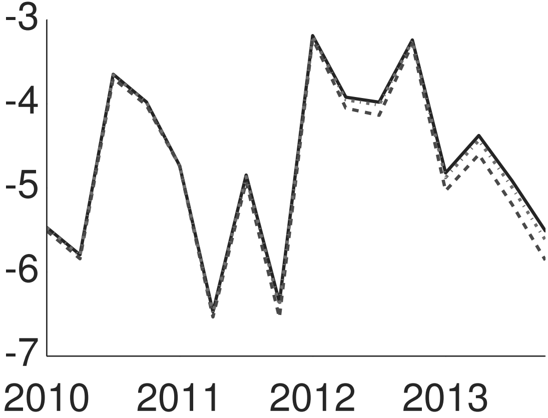
Mean of Log-likelihood Increments $\ln \hat{p}(y_t|Y_{1:t-1}, \theta^m)$



Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

Great Recession and Beyond

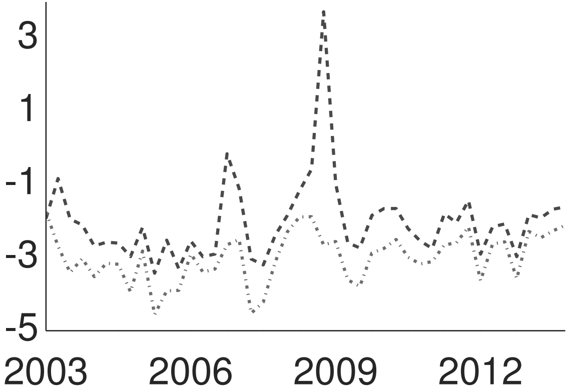
Mean of Log-likelihood Increments $\ln \hat{p}(y_t | Y_{1:t-1}, \theta^m)$



Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

Great Recession and Beyond

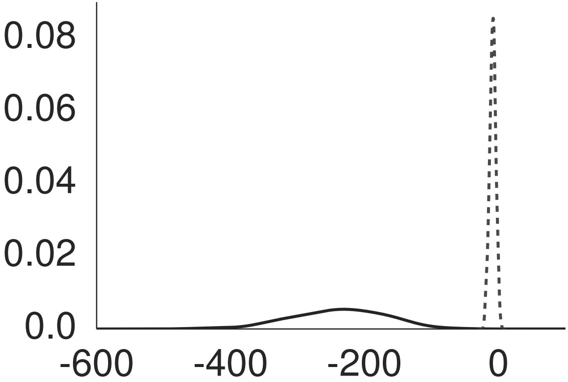
Log Standard Dev of Log-Likelihood Increments



Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

SW Model: Distr. of Log-Likelihood Approximation Errors

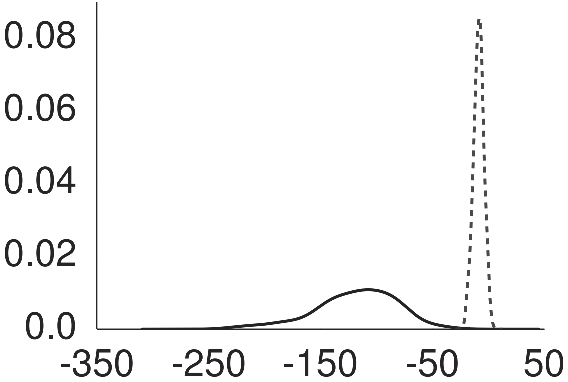
BS ($M = 40,000$) versus CO ($M = 4,000$)



Notes: Density estimates of $\hat{\Delta}_1 = \ln \hat{p}(Y|\theta) - \ln p(Y|\theta)$ based on $N_{run} = 100$. Solid densities summarize results for the bootstrap (BS) particle filter; dashed densities summarize results for the conditionally-optimal (CO) particle filter.

SW Model: Distr. of Log-Likelihood Approximation Errors

BS ($M = 400,000$) versus CO ($M = 4,000$)



Notes: Density estimates of $\hat{\Delta}_1 = \ln \hat{p}(Y|\theta) - \ln p(Y|\theta)$ based on $N_{run} = 100$. Solid densities summarize results for the bootstrap (BS) particle filter; dashed densities summarize results for the conditionally-optimal (CO) particle filter.

SW Model: Summary Statistics for Particle Filters

	Bootstrap		Cond. Opt.	
Number of Particles M	40,000	400,000	4,000	40,000
Number of Repetitions	100	100	100	100
High Posterior Density: $\theta = \theta^m$				
Bias $\hat{\Delta}_1$	-238.49	-118.20	-8.55	-2.88
StdD $\hat{\Delta}_1$	68.28	35.69	4.43	2.49
Bias $\hat{\Delta}_2$	-1.00	-1.00	-0.87	-0.41
Low Posterior Density: $\theta = \theta^l$				
Bias $\hat{\Delta}_1$	-253.89	-128.13	-11.48	-4.91
StdD $\hat{\Delta}_1$	65.57	41.25	4.98	2.75
Bias $\hat{\Delta}_2$	-1.00	-1.00	-0.97	-0.64

Notes: $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ and $\hat{\Delta}_2 = \exp[\ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)] - 1$. Results are based on $N_{run} = 100$.

Tempered Particle Filter

- Use sequence of distributions between the forecast and updated state distributions.
- Candidates? Well, the PF will work arbitrarily well when $\Sigma_u \to \infty$.
- Reduce measurement error variance from an inflated initial level $\Sigma_u(\theta)/\phi_1$ to the nominal level $\Sigma_u(\theta)$.

The Key Idea

Define

$$p_n(y_t|s_t,\theta) \quad \propto \quad \phi_n^{d/2}|\Sigma_u(\theta)|^{-1/2}\exp\left\{-\frac{1}{2}(y_t-\Psi(s_t,t;\theta))' \times\phi_n\Sigma_u^{-1}(\theta)(y_t-\Psi(s_t,t;\theta))\right\},$$

where:

$$\phi_1 < \phi_2 < \ldots < \phi_{N_\phi} = 1.$$

Bridge posteriors given s_{t-1}:

$$p_n(s_t|y_t,s_{t-1},\theta) \propto p_n(y_t|s_t,\theta)p(s_t|s_{t-1},\theta).$$

Bridge posteriors given $Y_{1:t-1}$:

$$p_n(s_t|Y_{1:t}) = \int p_n(s_t|y_t,s_{t-1},\theta)p(s_{t-1}|Y_{1:t-1})ds_{t-1}.$$

Algorithm Overview

- For each t we start with the BS-PF iteration by simulating the state-transition equation forward.

- Incremental weights are obtained based on inflated measurement error variance Σ_u/ϕ_1.

- Then we start the tempering iterations...

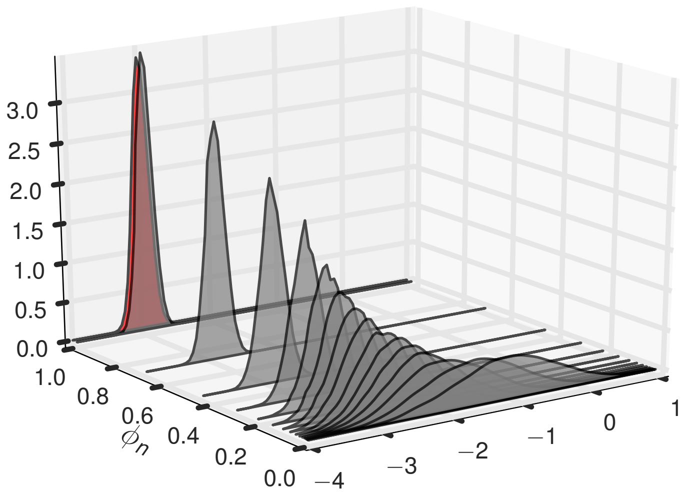
- After the tempering iterations are completed we proceed to $t+1$...

Overview

- If $N_\phi = 1$, this collapses to the Bootstrap particle filter.

- For each time period t, we embed a "static" SMC sampler used for parameter estimation [See Earlier Lectures]: Iterate over $n = 1, \ldots, N_\phi$:
  - Correction step: change particle weights (importance sampling)
  - Selection step: equalize particle weights (resampling of particles)
  - Mutation step: change particle values (based on Markov transition kernel generated with Metropolis-Hastings algorithm
  - Each step approximates the same $\int h(s_t) p_n(s_t|Y_{1:t}, \theta) ds_t$.

An Illustration: $p_n(s_t|Y_{1:t})$, $n = 1, \ldots, N_\phi$.



Choice of ϕ_n

Based on Geweke and Frischknecht (2014). Express post-correction inefficiency ratio as

$$\mathrm{InEff}(\phi_n) = \frac{\frac{1}{M}\sum_{j=1}^{M} \exp[-2(\phi_n - \phi_{n-1})e_{j,t}]}{\left(\frac{1}{M}\sum_{j=1}^{M} \exp[-(\phi_n - \phi_{n-1})e_{j,t}]\right)^2}$$

where

$$e_{j,t} = \frac{1}{2}(y_t - \Psi(s_t^{j,n-1}, t; \theta))'\Sigma_u^{-1}(y_t - \Psi(s_t^{j,n-1}, t; \theta)).$$

Pick target ratio r^* and solve equation $\mathrm{InEff}(\phi_n^*) = r^*$ for ϕ_n^*.

Small-Scale Model: PF Summary Statistics

	BSPF	TPF			
Number of Particles M	40k	4k	4k	40k	40k
Target Ineff. Ratio r^*		2	3	2	3
High Posterior Density: $\theta = \theta^m$					
Bias	-1.4	-0.9	-1.5	-0.3	-.05
StdD	1.9	1.4	1.7	0.4	0.6
$T^{-1}\sum_{t=1}^{T} N_{\phi,t}$	1.0	4.3	3.2	4.3	3.2
Average Run Time (s)	0.8	0.4	0.3	4.0	3.3
Low Posterior Density: $\theta = \theta^l$					
Bias	-6.5	-2.1	-3.1	-0.3	-0.6
StdD	5.3	2.1	2.6	0.8	1.0
$T^{-1}\sum_{t=1}^{T} N_{\phi,t}$	1.0	4.4	3.3	4.4	3.3
Average Run Time (s)	1.6	0.4	0.3	3.7	2.9

Parallel Particle Filtering

- We want (need) to use a lot of particles.
- Use *distributed memory* parallelism to allocate the operations among many processing elements (processors), each processor has its own local memory.
- Forecasting and updating steps can operate independently for each particle. Great news!
- Bad news: resampling phase cannot be executed locally.

Parallel Resampling

- M total particles, K processors.
- Let $M_{local} = M/K$ (assume it's an integer)
- $(s_t^{i,k}, W_t^{i,k})$ denote the ith particle on the kth processor.
- Use a stratified resampling scheme across processors, new particles will have weight

$$\tilde{W}_t^k = {M_{local}}^{-1} \sum_{j=1}^{M_{local}} \tilde{W}_t^{j,k}.$$

(*Distributed resampling with proportial allocation*, Bolic et al. [2005])

Weight Balancing

- Let α_k be the share of the weighted particles associated with processor k.

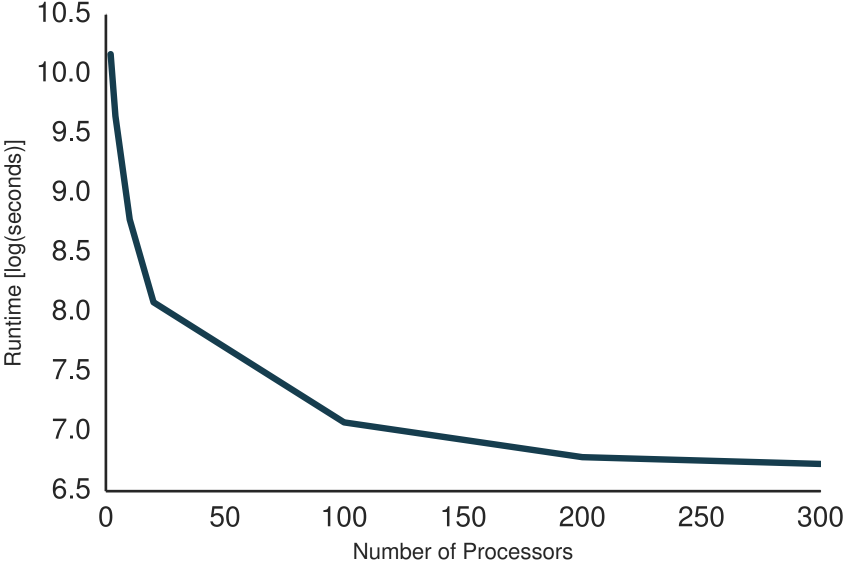
$$\alpha_k = \frac{\sum_{i=1}^{M_{local}} W_t^{i,k}}{\sum_{j=1}^{K} \sum_{i=1}^{M_{local}} W_t^{i,j}}, \qquad (29)$$

- effective number of processors as

$$EP_t = \frac{1}{\sum_{k=1}^{K} \alpha_k^2}. \qquad (30)$$

- If $EP_t < K/2$ shuffle the particles in the following way.
  - Rank processors according to α_k
  - Match largest α_k with smallest, and so on.
  - Exchange $M_{exchange} (< M_{local})$ particles between these processors
- Is it worth it? YES

Speed Gains from Parallelization, 100 lik. eval.



References

ARUOBA, B., P. CUBA-BORDA, AND F. SCHORFHEIDE (2018): "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries," *The Review of Economic Studies*, 85, 87 118.

FERNNDEZ-VILLAVERDE, J., G. GORDON, P. GUERRN-QUINTANA, AND J. F. RUBIO-RAMREZ (2015): "Nonlinear adventures at the zero lower bound," *Journal of Economic Dynamics and Control*, 57, 182204.

FERNNDEZ-VILLAVERDE, J., P. GUERRN-QUINTANA, K. KUESTER, AND J. RUBIO-RAMREZ (2011): "Fiscal Volatility Shocks and Economic Activity," .

GUST, C., E. HERBST, D. LPEZ-SALIDO, AND M. E. SMITH (2017): "The Empirical Implications of the Interest-Rate Lower Bound," *American Economic Review*, 107, 1971 2006.