Sequential Monte Carlo Methods for DSGE Models [1]

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[1]Material at `http://edherbst.net/teaching/indiana-minicourse`.
The views expressed in this presentation are those of the presenters and do not necessarily reflect the views of the Federal Reserve Board of Governors or the Federal Reserve System.

Some References

These lectures use material from our joint work:

- "Tempered Particle Filtering," 2016, *PIER Working Paper*, 16-017
- *Bayesian Estimation of DSGE Models*, 2015, Princeton University Press
- "Sequential Monte Carlo Sampling for DSGE Models," 2014, *Journal of Econometrics*

Sequential Monte Carlo (SMC) Methods

SMC can help to

Lecture 1

- approximate the posterior of θ: Chopin (2002) ... Durham and Geweke (2013) ... Creal (2007), Herbst and Schorfheide (2014)

Lecture 2

- approximate the likelihood function (particle filtering): Gordon, Salmond, and Smith (1993) ... Fernandez-Villaverde and Rubio-Ramirez (2007)
- or both: SMC^2: Chopin, Jacob, and Papaspiliopoulos (2012) ... Herbst and Schorfheide (2015)

Lecture 2

Approximating the Likelihood Function

- DSGE models are inherently nonlinear.
- Sometimes linear approximations are sufficiently accurate...
- but in other applications nonlinearities may be important:
  - asset pricing;
  - borrowing constraints;
  - zero lower bound on nominal interest rates;
  - ...
- Nonlinear state-space representation requires nonlinear filter:

$$\begin{aligned} y_t &= \Psi(s_t, t; \theta) + u_t, \quad u_t \sim F_u(\cdot; \theta) \\ s_t &= \Phi(s_{t-1}, \epsilon_t; \theta), \quad \epsilon_t \sim F_\epsilon(\cdot; \theta). \end{aligned}$$

Particle Filters

- There are many particle filters...
- We will focus on three types:
    - Bootstrap PF
    - A generic PF
    - A conditionally-optimal PF

Filtering - General Idea

- State-space representation of nonlinear DSGE model

$$\begin{aligned} \text{Measurement Eq.} \quad &: \quad y_t = \Psi(s_t, t; \theta) + u_t, \quad u_t \sim F_u(\cdot; \theta) \\ \text{State Transition} \quad &: \quad s_t = \Phi(s_{t-1}, \epsilon_t; \theta), \quad \epsilon_t \sim F_\epsilon(\cdot; \theta). \end{aligned}$$

- Likelihood function:

$$p(Y_{1:T}|\theta) = \prod_{t=1}^{T} p(y_t|Y_{1:t-1}, \theta)$$

- A filter generates a sequence of conditional distributions $s_t|Y_{1:t}$.
- Iterations:
  - Initialization at time $t-1$: $p(s_{t-1}|Y_{1:t-1}, \theta)$
  - Forecasting t given $t-1$:
    1. Transition equation: $p(s_t|Y_{1:t-1}, \theta) = \int p(s_t|s_{t-1}, Y_{1:t-1}, \theta) p(s_{t-1}|Y_{1:t-1}, \theta) ds_{t-1}$
    2. Measurement equation: $p(y_t|Y_{1:t-1}, \theta) = \int p(y_t|s_t, Y_{1:t-1}, \theta) p(s_t|Y_{1:t-1}, \theta) ds_t$
  - Updating with Bayes theorem. Once y_t becomes available:

$$p(s_t|Y_{1:t}, \theta) = p(s_t|y_t, Y_{1:t-1}, \theta) = \frac{p(y_t|s_t, Y_{1:t-1}, \theta) p(s_t|Y_{1:t-1}, \theta)}{p(y_t|Y_{1:t-1}, \theta)}$$

Bootstrap Particle Filter

1. **Initialization.** Draw the initial particles from the distribution $s_0^j \stackrel{iid}{\sim} p(s_0)$ and set $W_0^j = 1$, $j = 1, \dots, M$.
2. **Recursion.** For $t = 1, \dots, T$:
    1. **Forecasting s_t.** Propagate the period $t-1$ particles $\{s_{t-1}^j, W_{t-1}^j\}$ by iterating the state-transition equation forward:

    $$\tilde{s}_t^j = \Phi(s_{t-1}^j, \epsilon_t^j; \theta), \quad \epsilon_t^j \sim F_\epsilon(\cdot; \theta). \tag{1}$$

    An approximation of $\mathbb{E}[h(s_t)|Y_{1:t-1}, \theta]$ is given by

    $$\hat{h}_{t,M} = \frac{1}{M} \sum_{j=1}^{M} h(\tilde{s}_t^j) W_{t-1}^j. \tag{2}$$

Bootstrap Particle Filter

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.**
   2. **Forecasting y_t.** Define the incremental weights

$$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j, \theta). \tag{3}$$

The predictive density $p(y_t|Y_{1:t-1}, \theta)$ can be approximated by

$$\hat{p}(y_t|Y_{1:t-1}, \theta) = \frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{4}$$

If the measurement errors are $N(0, \Sigma_u)$ then the incremental weights take the form

$$\tilde{w}_t^j = (2\pi)^{-n/2} |\Sigma_u|^{-1/2} \exp\left\{ -\frac{1}{2} \big(y_t - \Psi(\tilde{s}_t^j, t; \theta)\big)' \Sigma_u^{-1} \big(y_t - \Psi(\tilde{s}_t^j, t; \theta)\big) \right\}, \tag{5}$$

where n here denotes the dimension of y_t.

Bootstrap Particle Filter

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.**
   2. **Forecasting y_t.** Define the incremental weights

$$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j, \theta). \tag{6}$$

   3. **Updating.** Define the normalized weights

$$\tilde{W}_t^j = \frac{\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M}\sum_{j=1}^M \tilde{w}_t^j W_{t-1}^j}. \tag{7}$$

An approximation of $\mathbb{E}[h(s_t)|Y_{1:t}, \theta]$ is given by

$$\tilde{h}_{t,M} = \frac{1}{M}\sum_{j=1}^M h(\tilde{s}_t^j)\tilde{W}_t^j. \tag{8}$$

Bootstrap Particle Filter

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
    1. **Forecasting s_t.**
    2. **Forecasting y_t.**
    3. **Updating.**
    4. **Selection (Optional).** Resample the particles via multinomial resampling. Let $\{s_t^j\}_{j=1}^M$ denote M iid draws from a multinomial distribution characterized by support points and weights $\{\tilde{s}_t^j, \tilde{W}_t^j\}$ and set $W_t^j = 1$ for $j =, 1 \ldots, M$.
    An approximation of $\mathbb{E}[h(s_t)|Y_{1:t}, \theta]$ is given by

$$\bar{h}_{t,M} = \frac{1}{M} \sum_{j=1}^{M} h(s_t^j) W_t^j. \tag{9}$$

Likelihood Approximation

- The approximation of the log likelihood function is given by

$$\ln \hat{p}(Y_{1:T}|\theta) = \sum_{t=1}^{T} \ln \left(\frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j \right). \tag{10}$$

- One can show that the approximation of the likelihood function is unbiased.
- This implies that the approximation of the log likelihood function is downward biased.

The Role of Measurement Errors

- Measurement errors may not be intrinsic to DSGE model.
- Bootstrap filter needs non-degenerate $p(y_t|s_t,\theta)$ for incremental weights to be well defined.
- Decreasing the measurement error variance Σ_u, holding everything else fixed, increases the variance of the particle weights, and reduces the accuracy of Monte Carlo approximation.

Generic Particle Filter – Recursion

1. **Forecasting s_t.** Draw $\tilde{s}_t^j$ from density $g_t(\tilde{s}_t|s_{t-1}^j, \theta)$ and define

$$\omega_t^j = \frac{p(\tilde{s}_t^j|s_{t-1}^j, \theta)}{g_t(\tilde{s}_t^j|s_{t-1}^j, \theta)}. \tag{11}$$

An approximation of $\mathbb{E}[h(s_t)|Y_{1:t-1}, \theta]$ is given by

$$\hat{h}_{t,M} = \frac{1}{M}\sum_{j=1}^{M} h(\tilde{s}_t^j)\omega_t^j W_{t-1}^j. \tag{12}$$

2. **Forecasting y_t.** Define the incremental weights

$$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j, \theta)\omega_t^j. \tag{13}$$

The predictive density $p(y_t|Y_{1:t-1}, \theta)$ can be approximated by

$$\hat{p}(y_t|Y_{1:t-1}, \theta) = \frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{14}$$

3. **Updating / Selection.** Same as BS PF

- The convergence results can be established recursively, starting from the assumption

$$
\begin{aligned}
\bar{h}_{t-1,M} &\stackrel{a.s.}{\longrightarrow} \mathbb{E}[h(s_{t-1})|Y_{1:t-1}], \\
\sqrt{M}\big(\bar{h}_{t-1,M} - \mathbb{E}[h(s_{t-1})|Y_{1:t-1}]\big) &\Longrightarrow N\big(0, \Omega_{t-1}(h)\big).
\end{aligned}
$$

- Forward iteration: draw s_t from $g_t(s_t|s^j_{t-1}) = p(s_t|s^j_{t-1})$.
- Decompose

$$
\begin{aligned}
&\hat{h}_{t,M} - \mathbb{E}[h(s_t)|Y_{1:t-1}] \qquad (15)\\
&= \frac{1}{M}\sum_{j=1}^{M}\left(h(\tilde{s}^j_t) - \mathbb{E}_{p(\cdot|s^j_{t-1})}[h]\right) W^j_{t-1} \\
&\quad + \frac{1}{M}\sum_{j=1}^{M}\left(\mathbb{E}_{p(\cdot|s^j_{t-1})}[h] W^j_{t-1} - \mathbb{E}[h(s_t)|Y_{1:t-1}]\right) \\
&= I + II,
\end{aligned}
$$

- Both I and II converge to zero (and potentially satisfy CLT).

Asymptotics

- Updating step approximates

$$\mathbb{E}[h(s_t)|Y_{1:t}] = \frac{\int h(s_t)p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t}{\int p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t} \approx \frac{\frac{1}{M}\sum_{j=1}^{M} h(\tilde{s}_t^j)\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j} \tag{16}$$

- Define the normalized incremental weights as

$$v_t(s_t) = \frac{p(y_t|s_t)}{\int p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t}. \tag{17}$$

- Under suitable regularity conditions, the Monte Carlo approximation satisfies a CLT of the form

$$\begin{aligned} &\sqrt{M}\big(\tilde{h}_{t,M} - \mathbb{E}[h(s_t)|Y_{1:t}]\big) \\ &\quad \Longrightarrow \quad N\big(0, \tilde{\Omega}_t(h)\big), \quad \tilde{\Omega}_t(h) = \hat{\Omega}_t\big(v_t(s_t)(h(s_t) - \mathbb{E}[h(s_t)|Y_{1:t}])\big). \end{aligned} \tag{18}$$

- Distribution of particle weights matters for accuracy! $\Longrightarrow$ Resampling!

Adapting the Generic PF

- Conditionally-optimal importance distribution:

$$g_t(\tilde{s}_t|s^j_{t-1}) = p(\tilde{s}_t|y_t, s^j_{t-1}).$$

  This is the posterior of s_t given s^j_{t-1}. Typically infeasible, but a good benchmark.

- Approximately conditionally-optimal distributions: from linearize version of DSGE model or approximate nonlinear filters.

- Conditionally-linear models: do Kalman filter updating on a subvector of s_t. Example:

$$\begin{aligned} y_t &= \Psi_0(m_t) + \Psi_1(m_t)t + \Psi_2(m_t)s_t + u_t, \quad u_t \sim N(0, \Sigma_u), \\ s_t &= \Phi_0(m_t) + \Phi_1(m_t)s_{t-1} + \Phi_\epsilon(m_t)\epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_\epsilon), \end{aligned}$$

  where m_t follows a discrete Markov-switching process.

More on Conditionally-Linear Models

- State-space representation is linear conditional on m_t.
- Write

$$p(m_t, s_t|Y_{1:t}) = p(m_t|Y_{1:t})p(s_t|m_t, Y_{1:t}), \tag{19}$$

where

$$s_t|(m_t, Y_{1:t}) \sim N\big(\bar{s}_{t|t}(m_t), P_{t|t}(m_t)\big). \tag{20}$$

- Vector of means $\bar{s}_{t|t}(m_t)$ and the covariance matrix $P_{t|t}(m)_t$ are sufficient statistics for the conditional distribution of s_t.
- Approximate $(m_t, s_t)|Y_{1:t}$ by $\{m_t^j, \bar{s}_{t|t}^j, P_{t|t}^j, W_t^j\}_{i=1}^N$.
- The swarm of particles approximates

$$\begin{aligned} & \int h(m_t, s_t)p(m_t, s_t, Y_{1:t})d(m_t, s_t) \qquad (21) \\ = \quad & \int \left[\int h(m_t, s_t)p(s_t|m_t, Y_{1:t})ds_t\right] p(m_t|Y_{1:t})dm_t \\ \approx \quad & \frac{1}{M}\sum^{M} \left[\int h(m_t^j, s_t^j)p_N\big(s_t|\bar{s}_{t|t}^j, P_{t|t}^j\big)ds_t\right] W_t^j. \end{aligned}$$

More on Conditionally-Linear Models

- We used Rao-Blackwellization to reduce variance:

$$\begin{aligned} \mathbb{V}[h(s_t, m_t)] &= \mathbb{E}\big[\mathbb{V}[h(s_t, m_t)|m_t]\big] + \mathbb{V}\big[\mathbb{E}[h(s_t, m_t)|m_t]\big] \\ &\geq \mathbb{V}\big[\mathbb{E}[h(s_t, m_t)|m_t]\big] \end{aligned}$$

- To forecast the states in period t, generate $\tilde{m}_t^j$ from $g_t(\tilde{m}_t|m_{t-1}^j)$ and define:

$$\omega_t^j = \frac{p(\tilde{m}_t^j|m_{t-1}^j)}{g_t(\tilde{m}_t^j|m_{t-1}^j)}. \tag{22}$$

- The Kalman filter forecasting step can be used to compute:

$$\begin{aligned} \tilde{s}_{t|t-1}^j &= \Phi_0(\tilde{m}_t^j) + \Phi_1(\tilde{m}_t^j)s_{t-1}^j \\ P_{t|t-1}^j &= \Phi_\epsilon(\tilde{m}_t^j)\Sigma_\epsilon(\tilde{m}_t^j)\Phi_\epsilon(\tilde{m}_t^j)' \\ \tilde{y}_{t|t-1}^j &= \Psi_0(\tilde{m}_t^j) + \Psi_1(\tilde{m}_t^j)t + \Psi_2(\tilde{m}_t^j)\tilde{s}_{t|t-1}^j \\ F_{t|t-1}^j &= \Psi_2(\tilde{m}_t^j)P_{t|t-1}^j\Psi_2(\tilde{m}_t^j)' + \Sigma_u. \end{aligned} \tag{23}$$

More on Conditionally-Linear Models

- Then,

$$
\begin{aligned}
&\int h(m_t, s_t) p(m_t, s_t | Y_{1:t-1}) d(m_t, s_t) \\
&\quad = \int \left[\int h(m_t, s_t) p(s_t | m_t, Y_{1:t-1}) ds_t \right] p(m_t | Y_{1:t-1}) dm_t \\
&\quad \approx \frac{1}{M} \sum_{j=1}^{M} \left[\int h(m_t^j, s_t^j) p_N\big(s_t | \bar{s}_{t|t-1}^j, P_{t|t-1}^j\big) ds_t \right] \omega_t^j W_{t-1}^j
\end{aligned} \tag{24}
$$

- The likelihood approximation is based on the incremental weights

$$
\tilde{w}_t^j = p_N\big(y_t | \bar{y}_{t|t-1}^j, F_{t|t-1}^j\big) \omega_t^j. \tag{25}
$$

- Conditional on $\tilde{m}_t^j$ we can use the Kalman filter once more to update the information about s_t in view of the current observation y_t:

$$
\begin{aligned}
\bar{s}_{t|t}^j &= \bar{s}_{t|t-1}^j + P_{t|t-1}^j \Psi_2(\tilde{m}_t^j)' \big(F_{t|t-1}^j\big)^{-1} (y_t - \bar{y}_{t|t-1}^j) \\
\tilde{P}_{t|t}^j &= P_{t|t-1}^j - P_{t|t-1}^j \Psi_2(\tilde{m}_t^j)' \big(F_{t|t-1}^j\big)^{-1} \Psi_2(\tilde{m}_t^j) P_{t|t-1}^j.
\end{aligned} \tag{26}
$$

Particle Filter For Conditionally Linear Models

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.** Draw $\tilde{m}_t^j$ from density $g_t(\tilde{m}_t | m_{t-1}^j, \theta)$, calculate the importance weights ω_t^j in (22), and compute $\tilde{s}_{t|t-1}^j$ and $P_{t|t-1}^j$ according to (23). An approximation of $\mathbb{E}[h(s_t, m_t)|Y_{1:t-1}, \theta]$ is given by (25).
   2. **Forecasting y_t.** Compute the incremental weights $\tilde{w}_t^j$ according to (25). Approximate the predictive density $p(y_t|Y_{1:t-1}, \theta)$ by

$$\hat{p}(y_t|Y_{1:t-1}, \theta) = \frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{27}$$

   3. **Updating.** Define the normalized weights

$$\tilde{W}_t^j = \frac{\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j} \tag{28}$$

   and compute $\tilde{s}_{t|t}^j$ and $\tilde{P}_{t|t}^j$ according to (26). An approximation of $\mathbb{E}[h(m_t, s_t)|Y_{1:t}, \theta]$ can be obtained from $\{\tilde{m}_t^j, \tilde{s}_{t|t}^j, \tilde{P}_{t|t}^j, \tilde{W}_t^j\}$.
   4. **Selection.**

Nonlinear and Partially Deterministic State Transitions

- Example:

$$s_{1,t} = \Phi_1(s_{t-1}, \epsilon_t), \quad s_{2,t} = \Phi_2(s_{t-1}), \quad \epsilon_t \sim N(0,1).$$

- Generic filter requires evaluation of $p(s_t|s_{t-1})$.
- Define $\varsigma_t = [s_t', \epsilon_t']'$ and add identity $\epsilon_t = \epsilon_t$ to state transition.
- Factorize the density $p(\varsigma_t|\varsigma_{t-1})$ as

$$p(\varsigma_t|\varsigma_{t-1}) = p^{\epsilon}(\epsilon_t) p(s_{1,t}|s_{t-1}, \epsilon_t) p(s_{2,t}|s_{t-1}).$$

where $p(s_{1,t}|s_{t-1}, \epsilon_t)$ and $p(s_{2,t}|s_{t-1})$ are pointmasses.

- Sample innovation ϵ_t from $g_t^{\epsilon}(\epsilon_t|s_{t-1})$.
- Then

$$\omega_t^j = \frac{p(\tilde{\varsigma}_t^j|\varsigma_{t-1}^j)}{g_t(\tilde{\varsigma}_t^j|\varsigma_{t-1}^j)} = \frac{p^{\epsilon}(\tilde{\epsilon}_t^j) p(\tilde{s}_{1,t}^j|s_{t-1}^j, \tilde{\epsilon}_t^j) p(\tilde{s}_{2,t}^j|s_{t-1}^j)}{g_t^{\epsilon}(\tilde{\epsilon}_t^j|s_{t-1}^j) p(\tilde{s}_{1,t}^j|s_{t-1}^j, \tilde{\epsilon}_t^j) p(\tilde{s}_{2,t}^j|s_{t-1}^j)} = \frac{p^{\epsilon}(\tilde{\epsilon}_t^j)}{g_t^{\epsilon}(\tilde{\epsilon}_t^j|s_{t-1}^j)}.$$

Degenerate Measurement Error Distributions

- Our discussion of the conditionally-optimal importance distribution suggests that in the absence of measurement errors, one has to solve the system of equations

$$y_t = \Psi\big(\Phi(s^j_{t-1}, \tilde{\epsilon}^j_t)\big),$$

to determine $\tilde{\epsilon}^j_t$ as a function of s^j_{t-1} and the current observation y_t.

- Then define

$$\omega^j_t = p^{\epsilon}(\tilde{\epsilon}^j_t) \quad \text{and} \quad \tilde{s}^j_t = \Phi(s^j_{t-1}, \tilde{\epsilon}^j_t).$$

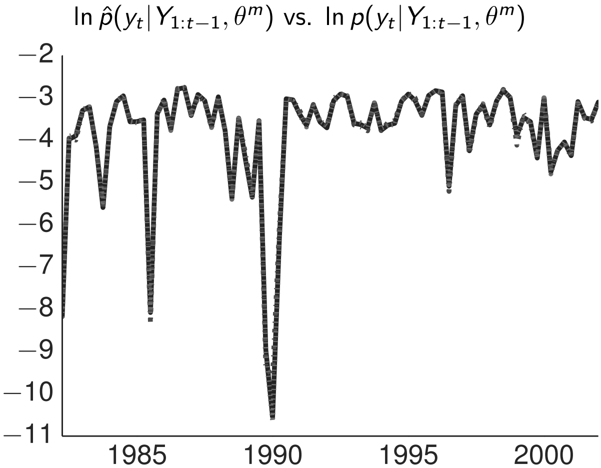
- Difficulty: one has to find all solutions to a nonlinear system of equations.
- While resampling duplicates particles, the duplicated particles do not mutate, which can lead to a degeneracy.

Next Steps

- We will now apply PFs to linearized DSGE models.
- This allows us to compare the Monte Carlo approximation to the “truth.”
- Small-scale New Keynesian DSGE model
- Smets-Wouters model

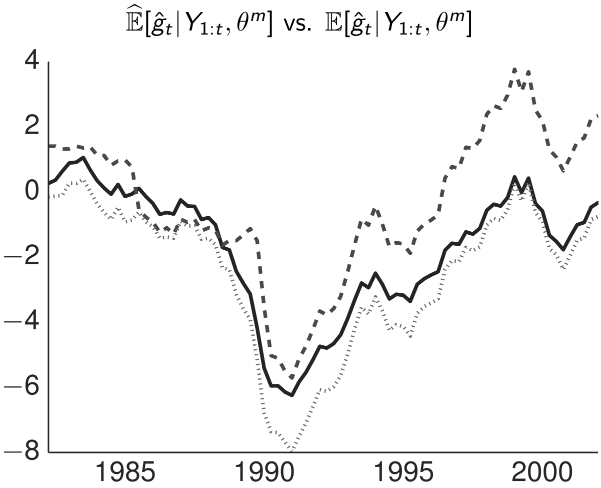
Parameter Values For Likelihood Evaluation

Parameter	θ^m	θ^l	Parameter	θ^m	θ^l
τ	2.09	3.26	κ	0.98	0.89
ψ_1	2.25	1.88	ψ_2	0.65	0.53
ρ_r	0.81	0.76	ρ_g	0.98	0.98
ρ_z	0.93	0.89	$r^{(A)}$	0.34	0.19
$\pi^{(A)}$	3.16	3.29	$\gamma^{(Q)}$	0.51	0.73
σ_r	0.19	0.20	σ_g	0.65	0.58
σ_z	0.24	0.29	$\ln p(Y\|\theta)$	-306.5	-313.4



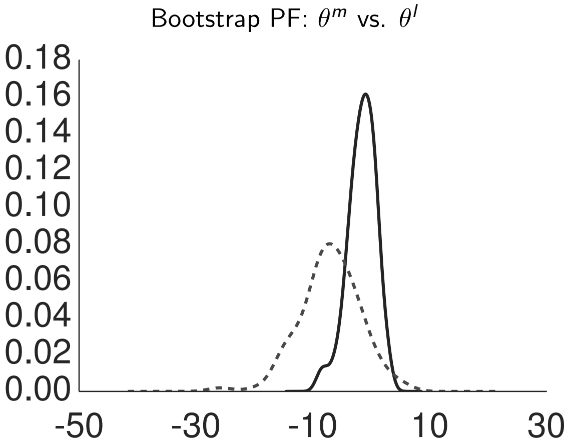
Notes: The results depicted in the figure are based on a single run of the bootstrap PF (dashed, $M = 40,000$), the conditionally-optimal PF (dotted, $M = 400$), and the Kalman filter (solid).

Filtered State



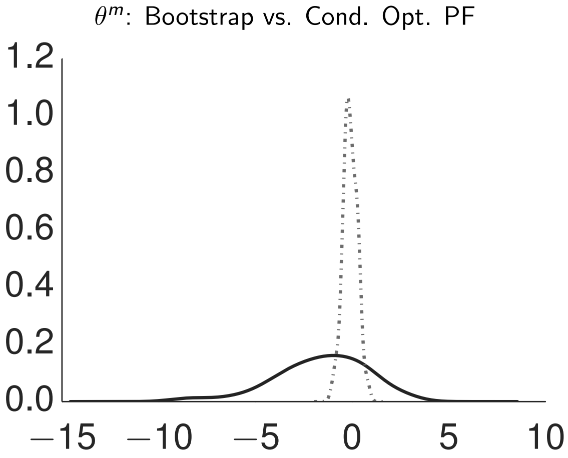
Notes: The results depicted in the figure are based on a single run of the bootstrap PF (dashed, $M = 40,000$), the conditionally-optimal PF (dotted, $M = 400$), and the Kalman filter (solid).

Distribution of Log-Likelihood Approximation Errors



Notes: Density estimate of $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ based on $N_{run} = 100$ runs of the PF. Solid line is $\theta = \theta^m$; dashed line is $\theta = \theta^l$ ($M = 40,000$).

Distribution of Log-Likelihood Approximation Errors



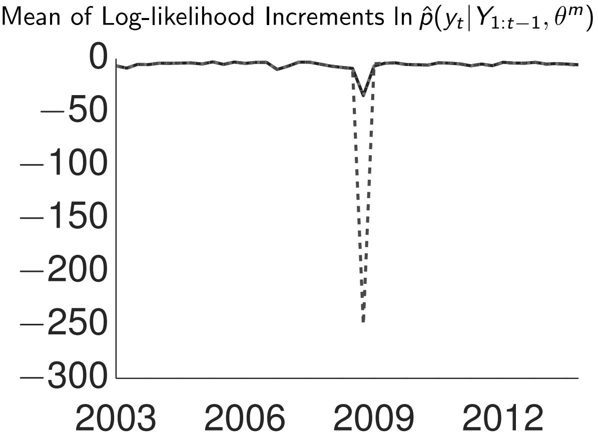
Notes: Density estimate of $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ based on $N_{run} = 100$ runs of the PF. Solid line is bootstrap particle filter ($M = 40,000$); dotted line is conditionally optimal particle filter ($M = 400$).

Summary Statistics for Particle Filters

	Bootstrap	Cond. Opt.	Auxiliary
Number of Particles M	40,000	400	40,000
Number of Repetitions	100	100	100
High Posterior Density: $\theta = \theta^m$			
Bias $\hat{\Delta}_1$	-1.39	-0.10	-2.83
StdD $\hat{\Delta}_1$	2.03	0.37	1.87
Bias $\hat{\Delta}_2$	0.32	-0.03	-0.74
Low Posterior Density: $\theta = \theta^l$			
Bias $\hat{\Delta}_1$	-7.01	-0.11	-6.44
StdD $\hat{\Delta}_1$	4.68	0.44	4.19
Bias $\hat{\Delta}_2$	-0.70	-0.02	-0.50

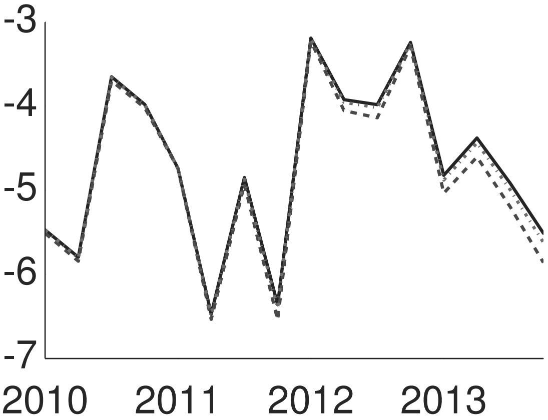
Notes: $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ and $\hat{\Delta}_2 = \exp[\ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)] - 1$. Results are based on $N_{run} = 100$ runs of the particle filters.

Great Recession and Beyond



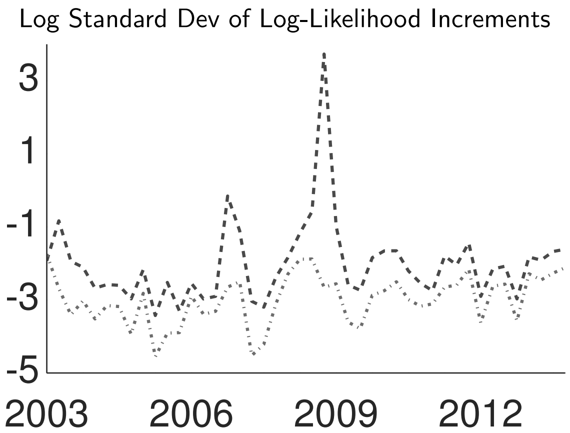
Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

Mean of Log-likelihood Increments $\ln \hat{p}(y_t|Y_{1:t-1}, \theta^m)$



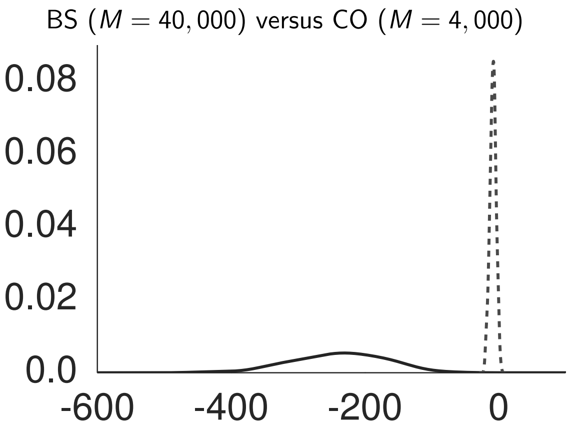
Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

Great Recession and Beyond



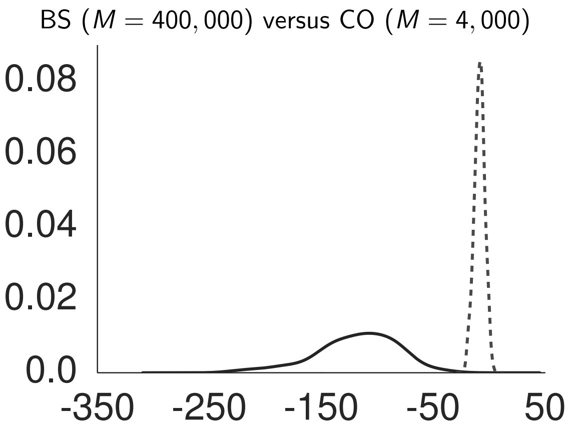
Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

SW Model: Distr. of Log-Likelihood Approximation Errors



Notes: Density estimates of $\hat{\Delta}_1 = \ln \hat{p}(Y|\theta) - \ln p(Y|\theta)$ based on $N_{run} = 100$. Solid densities summarize results for the bootstrap (BS) particle filter; dashed densities summarize results for the conditionally-optimal (CO) particle filter.

SW Model: Distr. of Log-Likelihood Approximation Errors



Notes: Density estimates of $\hat{\Delta}_1 = \ln \hat{p}(Y|\theta) - \ln p(Y|\theta)$ based on $N_{run} = 100$. Solid densities summarize results for the bootstrap (BS) particle filter; dashed densities summarize results for the conditionally-optimal (CO) particle filter.

SW Model: Summary Statistics for Particle Filters

	Bootstrap		Cond. Opt.	
Number of Particles M	40,000	400,000	4,000	40,000
Number of Repetitions	100	100	100	100
High Posterior Density: $\theta = \theta^m$				
Bias $\hat{\Delta}_1$	-238.49	-118.20	-8.55	-2.88
StdD $\hat{\Delta}_1$	68.28	35.69	4.43	2.49
Bias $\hat{\Delta}_2$	-1.00	-1.00	-0.87	-0.41
Low Posterior Density: $\theta = \theta^l$				
Bias $\hat{\Delta}_1$	-253.89	-128.13	-11.48	-4.91
StdD $\hat{\Delta}_1$	65.57	41.25	4.98	2.75
Bias $\hat{\Delta}_2$	-1.00	-1.00	-0.97	-0.64

Notes: $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ and $\hat{\Delta}_2 = \exp[\ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)] - 1$. Results are based on $N_{run} = 100$.

Tempered Particle Filter

- Use sequence of distributions between the forecast and updated state distributions.
- Candidates? Well, the PF will work arbitrarily well when $\Sigma_u \to \infty$.
- Reduce measurement error variance from an inflated initial level $\Sigma_u(\theta)/\phi_1$ to the nominal level $\Sigma_u(\theta)$.

The Key Idea

- Define

$$p_n(y_t|s_t,\theta) \quad \propto \quad \phi_n^{d/2}|\Sigma_u(\theta)|^{-1/2}\exp\left\{ -\frac{1}{2}(y_t - \Psi(s_t,t;\theta))' \right.$$
$$\left. \times \phi_n \Sigma_u^{-1}(\theta)(y_t - \Psi(s_t,t;\theta)) \right\},$$

  where:

$$\phi_1 < \phi_2 < \ldots < \phi_{N_\phi} = 1.$$

- Bridge posteriors given s_{t-1}:

$$p_n(s_t|y_t,s_{t-1},\theta) \propto p_n(y_t|s_t,\theta)p(s_t|s_{t-1},\theta).$$

- Bridge posteriors given $Y_{1:t-1}$:

$$p_n(s_t|Y_{1:t}) = \int p_n(s_t|y_t,s_{t-1},\theta)p(s_{t-1}|Y_{1:t-1})ds_{t-1}.$$

Algorithm Overview

- For each t we start with the BS-PF iteration by simulating the state-transition equation forward.
- Incremental weights are obtained based on inflated measurement error variance Σ_u/ϕ_1.
- Then we start the tempering iterations...
- After the tempering iterations are completed we proceed to $t+1$...

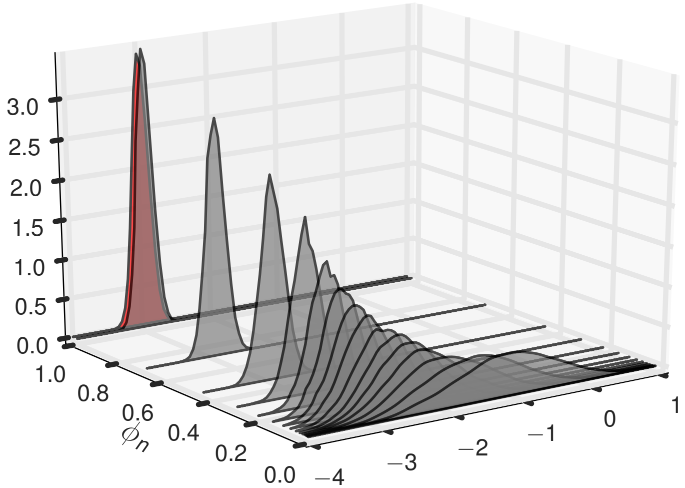
Overview

- If $N_\phi = 1$, this collapses to the Bootstrap particle filter.
- For each time period t, we embed a "static" SMC sampler used for parameter estimation [See Lecture 1]:

  Iterate over $n = 1, \ldots, N_\phi$:

  - **Correction step**: change particle weights (importance sampling)
  - **Selection step**: equalize particle weights (resampling of particles)
  - **Mutation step**: change particle values (based on Markov transition kernel generated with Metropolis-Hastings algorithm)
  - Each step approximates the same $\int h(s_t) p_n(s_t | Y_{1:t}, \theta) ds_t$.

An Illustration: $p_n(s_t|Y_{1:t})$, $n = 1, \ldots, N_\phi$.



Choice of ϕ_n

- Based on Geweke and Frischknecht (2014).
- Express post-correction inefficiency ratio as

$$\text{InEff}(\phi_n) = \frac{\frac{1}{M}\sum_{j=1}^{M} \exp[-2(\phi_n - \phi_{n-1})e_{j,t}]}{\left(\frac{1}{M}\sum_{j=1}^{M} \exp[-(\phi_n - \phi_{n-1})e_{j,t}]\right)^2}$$

where

$$e_{j,t} = \frac{1}{2}(y_t - \Psi(s_t^{j,n-1}, t; \theta))'\Sigma_u^{-1}(y_t - \Psi(s_t^{j,n-1}, t; \theta)).$$

- Pick target ratio r^* and solve equation $\text{InEff}(\phi_n^*) = r^*$ for ϕ_n^*.

Small-Scale Model: PF Summary Statistics

	BSPF	TPF			
Number of Particles M	40k	4k	4k	40k	40k
Target Ineff. Ratio r^*		2	3	2	3
High Posterior Density: $\theta = \theta^m$					
Bias	-1.4	-0.9	-1.5	-0.3	-.05
StdD	1.9	1.4	1.7	0.4	0.6
$T^{-1}\sum_{t=1}^{T} N_{\phi,t}$	1.0	4.3	3.2	4.3	3.2
Average Run Time (s)	0.8	0.4	0.3	4.0	3.3
Low Posterior Density: $\theta = \theta^l$					
Bias	-6.5	-2.1	-3.1	-0.3	-0.6
StdD	5.3	2.1	2.6	0.8	1.0
$T^{-1}\sum_{t=1}^{T} N_{\phi,t}$	1.0	4.4	3.3	4.4	3.3
Average Run Time (s)	1.6	0.4	0.3	3.7	2.9

Embedding PF Likelihoods into Posterior Samplers

- Likelihood functions for nonlinear DSGE models can be approximated by the PF.
- We will now embed the likelihood approximation into a posterior sampler: PFMH Algorithm (a special case of PMCMC).
- The book also discusses SMC^2.

Embedding PF Likelihoods into Posterior Samplers

- Distinguish between:
  - $\{p(Y|\theta), p(\theta|Y), p(Y)\}$, which are related according to:
  $$p(\theta|Y) = \frac{p(Y|\theta)p(\theta)}{p(Y)}, \quad p(Y) = \int p(Y|\theta)p(\theta)d\theta$$
  - $\{\hat{p}(Y|\theta), \hat{p}(\theta|Y), \hat{p}(Y)\}$, which are related according to:
  $$\hat{p}(\theta|Y) = \frac{\hat{p}(Y|\theta)p(\theta)}{\hat{p}(Y)}, \quad \hat{p}(Y) = \int \hat{p}(Y|\theta)p(\theta)d\theta.$$
- Surprising result (Andrieu, Docet, and Holenstein, 2010): under certain conditions we can replace $p(Y|\theta)$ by $\hat{p}(Y|\theta)$ and still obtain draws from $p(\theta|Y)$.

PFMH Algorithm

For $i = 1$ to N:

1. Draw ϑ from a density $q(\vartheta|\theta^{i-1})$.
2. Set $\theta^i = \vartheta$ with probability

$$\alpha(\vartheta|\theta^{i-1}) = \min\left\{1,\ \frac{\hat{p}(Y|\vartheta)p(\vartheta)/q(\vartheta|\theta^{i-1})}{\hat{p}(Y|\theta^{i-1})p(\theta^{i-1})/q(\theta^{i-1}|\vartheta)}\right\}$$

and $\theta^i = \theta^{i-1}$ otherwise. The likelihood approximation $\hat{p}(Y|\vartheta)$ is computed using a particle filter.

Why Does the PFMH Work?

- At each iteration the filter generates draws $\tilde{s}_t^j$ from the proposal distribution $g_t(\cdot|s_{t-1}^j)$.
- Let $\tilde{S}_t = \left(\tilde{s}_t^1, \ldots, \tilde{s}_t^M\right)'$ and denote the entire sequence of draws by $\tilde{S}_{1:T}^{1:M}$.
- Selection step: define a random variable A_t^j that contains this ancestry information. For instance, suppose that during the resampling particle $j = 1$ was assigned the value $\tilde{s}_t^{10}$ then $A_t^1 = 10$. Let $A_t = \left(A_t^1, \ldots, A_t^N\right)$ and use $A_{1:T}$ to denote the sequence of A_t's.
- PFMH operates on an enlarged probability space: θ, $\tilde{S}_{1:T}$ and $A_{1:T}$.

Why Does the PFMH Work?

- Use $U_{1:T}$ to denote random vectors for $\tilde{S}_{1:T}$ and $A_{1:T}$. $U_{1:T}$ is an array of *iid* uniform random numbers.

- The transformation of $U_{1:T}$ into $(\tilde{S}_{1:T}, A_{1:T})$ typically depends on θ and $Y_{1:T}$, because the proposal distribution $g_t(\tilde{s}_t | s^j_{t-1})$ depends both on the current observation y_t as well as the parameter vector θ.

- E.g., implementation of conditionally-optimal PF requires sampling from a $N(\bar{s}^j_{t|t}, P_{t|t})$ distribution for each particle j. Can be done using a prob integral transform of uniform random variables.

- We can express the particle filter approximation of the likelihood function as

$$\hat{p}(Y_{1:T}|\theta) = g(Y_{1:T}|\theta, U_{1:T}).$$

where

$$U_{1:T} \sim p(U_{1:T}) = \prod_{t=1}^{T} p(U_t).$$

Why Does the PFMH Work?

- Define the joint distribution

$$p_g\big(Y_{1:T}, \theta, U_{1:T}\big) = g(Y_{1:T}|\theta, U_{1:T})p\big(U_{1:T}\big)p(\theta).$$

- The PFMH algorithm samples from the joint posterior

$$p_g\big(\theta, U_{1:T}\big|Y_{1:T}\big) \propto g(Y|\theta, U_{1:T})p\big(U_{1:T}\big)p(\theta)$$

  and discards the draws of $\big(U_{1:T}\big)$.

- For this procedure to be valid, it needs to be the case that PF approximation is unbiased:

$$\mathbb{E}[\hat{p}(Y_{1:T}|\theta)] = \int g(Y_{1:T}|\theta, U_{1:T})p\big(U_{1:T}\big)d\theta = p(Y_{1:T}|\theta).$$

Why Does the PFMH Work?

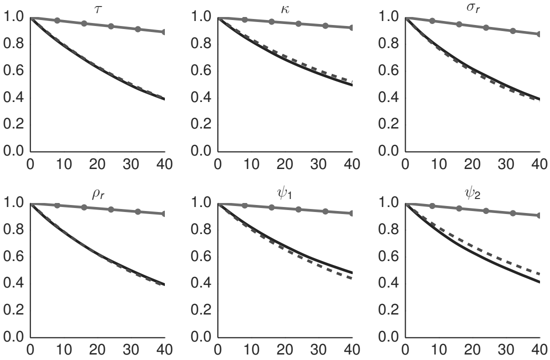
- We can express acceptance probability directly in terms of $\hat{p}(Y_{1:T}|\theta)$.
- Need to generate a proposed draw for both θ and $U_{1:T}$: ϑ and $U^*_{1:T}$.
- The proposal distribution for $(\vartheta, U^*_{1:T})$ in the MH algorithm is given by $q(\vartheta|\theta^{(i-1)})p(U^*_{1:T})$.
- No need to keep track of the draws $(U^*_{1:T})$.
- MH acceptance probability:

$$
\begin{aligned}
\alpha(\vartheta|\theta^{i-1}) &= \min\left\{1, \frac{\frac{g(Y|\vartheta,U^*)p(U^*)p(\vartheta)}{q(\vartheta|\theta^{(i-1)})p(U^*)}}{\frac{g(Y|\theta^{(i-1)},U^{(i-1)})p(U^{(i-1)})p(\theta^{(i-1)})}{q(\theta^{(i-1)}|\theta^*)p(U^{(i-1)})}}\right\} \\
&= \min\left\{1, \frac{\hat{p}(Y|\vartheta)p(\vartheta)\big/q(\vartheta|\theta^{(i-1)})}{\hat{p}(Y|\theta^{(i-1)})p(\theta^{(i-1)})\big/q(\theta^{(i-1)}|\vartheta)}\right\}.
\end{aligned}
$$

Small-Scale DSGE: Accuracy of MH Approximations

- Results are based on $N_{run} = 20$ runs of the PF-RWMH-V algorithm.
- Each run of the algorithm generates $N = 100,000$ draws and the first $N_0 = 50,000$ are discarded.
- The likelihood function is computed with the Kalman filter (KF), bootstrap particle filter (BS-PF, $M = 40,000$) or conditionally-optimal particle filter (CO-PF, $M = 400$).
- "Pooled" means that we are pooling the draws from the $N_{run} = 20$ runs to compute posterior statistics.

Autocorrelation of PFMH Draws



Notes: The figure depicts autocorrelation functions computed from the output of the 1 Block RWMH-V algorithm based on the Kalman filter (solid), the conditionally-optimal particle filter (dashed) and the bootstrap particle filter (solid with dots).

Small-Scale DSGE: Accuracy of MH Approximations

	Posterior Mean (Pooled)			Inefficiency Factors			Std Dev of Means		
	KF	CO-PF	BS-PF	KF	CO-PF	BS-PF	KF	CO-PF	BS-PF
τ	2.63	2.62	2.64	66.17	126.76	1360.22	0.020	0.028	0.091
κ	0.82	0.81	0.82	128.00	97.11	1887.37	0.007	0.006	0.026
ψ_1	1.88	1.88	1.87	113.46	159.53	749.22	0.011	0.013	0.029
ψ_2	0.64	0.64	0.63	61.28	56.10	681.85	0.011	0.010	0.036
ρ_r	0.75	0.75	0.75	108.46	134.01	1535.34	0.002	0.002	0.007
ρ_g	0.98	0.98	0.98	94.10	88.48	1613.77	0.001	0.001	0.002
ρ_z	0.88	0.88	0.88	124.24	118.74	1518.66	0.001	0.001	0.005
$r^{(A)}$	0.44	0.44	0.44	148.46	151.81	1115.74	0.016	0.016	0.044
$\pi^{(A)}$	3.32	3.33	3.32	152.08	141.62	1057.90	0.017	0.016	0.045
$\gamma^{(Q)}$	0.59	0.59	0.59	106.68	142.37	899.34	0.006	0.007	0.018
σ_r	0.24	0.24	0.24	35.21	179.15	1105.99	0.001	0.002	0.004
σ_g	0.68	0.68	0.67	98.22	64.18	1490.81	0.003	0.002	0.011
σ_z	0.32	0.32	0.32	84.77	61.55	575.90	0.001	0.001	0.003
$\ln \hat{p}(Y)$	-357.14	-357.17	-358.32				0.040	0.038	0.949

Computational Considerations

- We implement the PFMH algorithm on a single machine, utilizing up to twelve cores.
- For the small-scale DSGE model it takes 30:20:33 [hh:mm:ss] hours to generate 100,000 parameter draws using the bootstrap PF with 40,000 particles. Under the conditionally-optimal filter we only use 400 particles, which reduces the run time to 00:39:20 minutes.