Derived Seminar

Contents

Hard Facts:

What is this about?

The primary focus of this seminar is to read and study Gaitsgory and Rozenblyum's *A Study in Derived Algebraic Geometry* (see [5] and [4]). But why did we decide upon these books[1] specifically? A good answer is provided by consulting the introduction of the first book [4].

Organization

We will use Discord as our main organizational tool. Specifically, we have our very own Discord Server - if you want to join the seminar, send me an email here: *alexander.zahrer(at)proton.me*

Further details regarding the seminar, in particular meeting times etc. will be announced on the discord server.

Complementary Resources

As already said, the main references will be [5, 4], but the following list might provide for some nice complementary reading:

Derived Algebraic Geometry

- **Main reference:** *A Study in Derived Algebraic Geometry Book I & II* - see [5, 4].
- Past Winter school on local geometric Langlands theory
- Ben-Zvi Notes with the possibility to download talks on the subject (by Ben-Zvi) here: Videos (the video links on the Ben-Zvi Notes website seem to be broken).
- Gaitsgory's Website
- Geometric ∞-function theory

[1] As a precursor you could of course first ask: *Why* derived algebraic geometry? One possible answer (among many others) could be that one might want to understand the *Geometric Langlands Program*.

Some relevant lecture series in Regensburg:

- Lectures on both Stable Homotopy Theory and Higher Algebra taught by Cnossen.
- Lecture by Hekking on Derived Algebraic Geometry.

Geometric Langlands

- Expository interview of Edward Frenkel Part I and Part II.

Seminar Outline

We shall meet weekly online (on Zoom), with the following structure:

- We hope for a large group of participants and a sizable fraction of willing speakers.
- Here is the scheme[2] you should expect if you choose to be a presenter:
  - You will be assigned[3] a specific page range in e.g. [4] on which your talk should be based upon.
  - In principal, we will try to match the talk topic to the speaker's expertise.
  - You are not expected to flesh out all the details in some of the proofs. It is completely fine if you sometimes skip proofs, or just give the main idea and intuitions. The *nitty gritty details* will be left to the eager individuals.
  - Optimally, we would like to record the talks and upload the speakers notes to our discord server - of course only if the speaker allows it.
- We will try to also invite prominent mathematicians in the area to speak in our seminar.

If you've ever wanted an *entry* into derived algebraic geometry (and perhaps the Geometric Langlands Program), this is your cue.

[2] Pun intended.
[3] Of course, if you are very eager to present a chosen portion/topic of the text, just let us know.

Week-by-Week Breakdown

The following is a rough outline of what the first few weeks of the seminar might look like. Note that there might be many changes.

Week 1: Guest Talk - Why this book, why $(\infty, 1)$-categories, why derived algebraic geometry, ...?

Reading: Preface (xi–xx), Introduction (xxix–xxxiv)
Keywords: Why DAG? This talk was given by Dennis Gaitsgory.

Week 2: Basics of ∞-categories

Reading: Pages 7–21 (14 pages)
Keywords: $(\infty, 1)$-categories, ∞-category of spaces, homotopy category, functor properties, homotopy (co)limits, (co)Cartesian morphisms, and (co)Cartesian fibrations.

Week 3: Basics Continued

Reading: Pages 21–31 (10 pages)
Keywords: Straightening-Unstraightening, adjoint functors, Kan extensions, cofinality, and passing to adjoints.

Week 4: Basics of Higher Algebra I

Reading: Pages 31–42 (11 pages)
Keywords: Colimits in presentable categories, monoidal categories, lax functors, associative algebras, symmetric monoidal categories, and commutative monoids.

Week 5: Basics of Higher Algebra II

Reading: Pages 43–52 (9 pages)
Keywords: Monads, Barr-Beck-Lurie theorem, geometric realizations, tensor products of modules, and stable categories.

Week 6: Basics of Higher Algebra III

Reading: Some necessary parts of the appendix on 2-categories + Pages 52–60
Keywords: Overview of 2-categories, generation, Lurie tensor product, and spectra.

Week 7: Basics of Higher Algebra IV

Reading: Pages 60–70 (10 pages)
Keywords: Duality of stable categories, generation of tensor products, compactly generated stable categories, and compact generation.

Week 8: Basics of Higher Algebra V

Reading: Pages 71–82 (11 pages)
Keywords: Algebra in stable categories, inner hom and tensor products, residual 2-categorical features, and rigid monoidal categories.

Week 9: Basics of Higher Algebra VI

Reading: Pages 82–93 (11 pages)
Keywords: Modules over rigid categories, DG categories, derived categories, and compact objects.

Week 10: Guest Talk - What to expect from the DAG parts, e.g. why Ind-Coherent Sheaves?

Week 11: Basics of DAG I

Reading: Pages 95–106 (11 pages)
Keywords: Prestacks, connective commutative DG algebras, (derived) affine schemes, coconnectivity, convergence.

Week 12: Basics of DAG II

Reading: Pages 106–116
Keywords: Descent, flat morphisms, Cech nerve, and affine schemes.

Week 13: Basics of DAG III

Reading: Pages 116–126
Keywords: Stacks, truncatedness, descent, derived schemes, and connectivity.

Week 14: Basics of DAG IV

Reading: Pages 126–132
Keywords: Derived Artin stacks, quasicompactness, and quasiseparatedness.

Week 15: Basics of DAG V

Reading: Pages 133–140
Keywords: Artin stacks, descent properties, and connectivity.

Week 16: Quasi-Coherent Sheaves I

Reading: Pages 141–149
Keywords: quasi-coherent sheaves, basic properties of QCoh, quasi-coherent sheaves on stacks locally of finite type, flat descent of QCoh, quasi-coherent sehaves on n-coconnective stacks, quasi-coherent sheaves on Artin stacks.

Week 17: Quasi-Coherent Sheaves II

Reading: Pages 149–157
Keywords: t-structure on quasi-coherent sheaves, Direct image for QCoh, direct image of schematic morphisms, direct image for a map between Artin stacks, classical algebraic stacks

Week 18: Quasi-Coherent Sheaves III

Reading: 158–165
Keywords: symmetric monoidal structure on QCoh as a functor, quasi-affine case, when is QCoh rigid?, passable prestacks, perfect subcategory, perfect prestacks.

Week 19: Ind-Coherent Sheaves I

... details for this week and the following ones will be updated in the future. There will be a slight change in how we go about this: many of the later chapter require heavy machinery from $(\infty,2)$-category theory. The plan is therefore to have some sessions on the basics of $(\infty,2)$-categories sprinkled in (i.e. by giving talks on the appendix). In order to account for this, we will have a 3 weeks rhythm, where there will be two regular sessions (following the book in the natural order) and one $(\infty,2)$-session (focusing on the appendix of the book).

References

[1] Bastiaan Cnossen. *Lectures on Higher Algebra*. Accessed: 2025-01-28. URL: https://sites.google.com/view/bastiaan-cnossen/teaching/so25-introduction-to-higher-algebra?authuser=0.

[2] Bastiaan Cnossen. *Lectures on Stable Homotopy Theory*. Accessed: 2025-01-28. URL: https://sites.google.com/view/bastiaan-cnossen/teaching/wi24-introduction-to-stable-homotopy-theory?authuser=0.

[3] *Formalization of Higher Category Theory*. Accessed: 2025-01-28. URL: https://drive.google.com/file/d/1lKaq7watGGl3xvjqw9qHjm6SDPFJ2-0o/view?usp=sharing.

[4] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*. Vol. 221. American Mathematical Society, 2019.

[5] Dennis Gaitsgory and Nick Rozenblyum. "A study in derived algebraic geometry: Volume ii: Deformations, lie theory and formal geometry". In: *American Mathematical Society* (2017).

[6] Jeroen Hekking. *Lecture on Derived Algebraic Geometry*. Accessed: 2025-01-28. URL: https://www.jeroenhekking.nl/teaching/introduction-to-derived-algebraic-geometry.

[7] Jacob Lurie. *Higher Algebra*. Accessed: 2025-01-28. Unpublished. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[8] Jacob Lurie. *Higher Topos Theory*. Accessed: 2025-01-28. Princeton University Press, 2009. URL: https://people.math.harvard.edu/~lurie/papers/highertopoi.pdf.

[9] Jacob Lurie. *Kerodon*. Accessed: 2025-01-28. URL: https://kerodon.net/.

[10] Jacob Lurie. "Spectral algebraic geometry". In: *preprint* (2018).