This is a homework 1 for CSC588 Learning Theory course

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Problem 1. (a)

First of all, let's specify the range of X, since the table will depend on it. From (c) we can assume that this is $0, 1, 2, 3$

Having in mind basic probability, we will use $Bin(x, n, p)$ which equals,

$$f_B(x, n, p) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

Now, we will complete the table using $\mathbb{P}(X, Y) = \mathbb{P}(X|Y)\mathbb{P}(Y)$ Considering that $\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = \frac{1}{2}$ we will just divide every output from $f_B(x, n, p)$ by 2

Then, let's fill the table when $Y = +1$ and, therefore, $f_B(x = x, n = 3, p = 2/3)$
$\mathbb{P}(X = 0, Y = -1) = 1 \cdot (1 - 2/3)^3 = (1/3)^3/2 = 1/27/2 = 1/54$
$\mathbb{P}(X = 1, Y = -1) = 3 \cdot (2/3)^1(1 - 2/3)^2/2 = 2/9/2 = 1/9$
$\mathbb{P}(X = 2, Y = -1) = 3 \cdot (2/3)^2(1 - 2/3)^1 = 4/9/2 = 2/9$
$\mathbb{P}(X = 3, Y = -1) = 1 \cdot (2/3)^3(1 - 2/3)^0 = 8/27/2 = 4/27$
$f_B(x = x, n = 2, 1/3)$
$\mathbb{P}(X = 0, Y = +1) = 1 \cdot (1 - 1/3)^3 = (2/3)^2/2 = 4/9/2 = 2/9$
$\mathbb{P}(X = 1, Y = +1) = 2 \cdot (1/3)^1(2/3)^1/2 = 2 * 2/9/2 = 2/9$
$\mathbb{P}(X = 2, Y = +1) = 1 \cdot (1/3)^2(1 - 2/3)^0 = 4/9/2 = 1/9/2 = 1/18$
$\mathbb{P}(X = 3, Y = +1) = 0$

Can be easily checked that sum over everything is 1.

(b)

Using law of alternatives:

$$\mathbb{P}(Y = -1|X = 1) = \frac{\mathbb{P}(X, Y)}{\mathbb{P}(X)} = \frac{\mathbb{P}(X, Y)}{\sum_{y \in Y} \mathbb{P}(X = 1|Y = y)\mathbb{P}(Y = y)} =$$

$$= \frac{\mathbb{P}(X, Y)}{\sum_{y \in Y} \mathbb{P}(X = 1, Y = y)} = \boxed{\frac{1/9}{(2/9 + 1/9)} = \frac{1}{3}}$$

(c)

We need to minimize the error under $\mathbb{P}(Y|X)$. If so, all the information is already given in table, except we also need to find where $\mathbb{P}(Y|X)$ is the biggest in every possible X and choose where probability is bigger.

$$\mathbb{P}(Y=-1|X=0) =\sim 0.07$$

$$\mathbb{P}(Y=-1|X=1) = 1/3$$

$$\mathbb{P}(Y=-1|X=2) = 0.8$$

$$\mathbb{P}(Y=-1|X=3) = 1$$

$$\mathbb{P}(Y=+1|X=0) =\sim 0.92$$

$$\mathbb{P}(Y=+1|X=1) = 2/3$$

$$\mathbb{P}(Y=+1|X=2) = 0.2$$

$$\mathbb{P}(Y=+1|X=3) = 0$$

By calculating that, function should output {1,1,-1,-1}

The function that estimates that is

$$ \boxed{f(x)=sign(-(x-2)) ∧ {-1 \ if \ x = 2} $$

The following function will have the following classification error:

$$(0.07 + 1/3 + 0.2 + 0)/4 =\sim 0.6/4 = \boxed{\sim 0.15}$$

Problem 2 (a)

Given set of examples $(x_1, x_2, x_3. \dots x_n) \in \mathbb{R}^d$ and having fact that normal distribution has additivity in its parameters, we have the following. Given,

$$y_i = \langle \theta, x_i \rangle + \epsilon_i, \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

and the fact that if

$$\epsilon \sim \mathcal{N}(0, \sigma^2), a + \epsilon \sim \mathcal{N}(a, \sigma^2)$$

we have

$$y_i \sim \mathcal{N}(\langle \theta, x_i \rangle, \sigma^2)$$

or, in general terms,

$$\boxed{y \sim \mathcal{N}(X\theta, \sigma^2 I)}$$

where $X = \Sigma$

We need to note that since all ϵ_i are independent, the $y = (y_1, y_2, \ldots y_n)$ will have multivariate Gaussian distribution with trivial covariance matrix, i.e. they are uncorrelated.

(b)

We will stick to the matrix form. Firstly, the $\hat{\theta}$ is stated

$$\hat{\theta} = \left(X^\top X\right)^{-1} X^\top y$$

Then, we are willing to use the theorem that states that if

$$\epsilon \sim \mathcal{N}(\mu, \Sigma)$$

then

$$A\epsilon + m \sim \mathcal{N}(A\mu + m, A\Sigma A^\top)$$

Using this theorem, we consider A to be $\left(X^\top X\right)^{-1} X^\top$

Then, if $y \sim \mathcal{N}(X\theta, \sigma^2 I)$

$$\hat{\theta} \sim \mathcal{N}\left(AX\theta, A\sigma^2 I A^\top\right) = \mathcal{N}\left(\left(X^\top X\right)^{-1} X^\top X\theta, \left(X^\top X\right)^{-1} X^\top \sigma^2 I \left(X^\top X\right)^{-1} X^\top\right) =$$

$$\boxed{\mathcal{N}\left(\theta, \sigma^2 \left(X^\top X\right)^{-1}\right)}$$

(c)

Considering the vector on the right side of bracket $\left\langle v, \hat{\theta} - \theta \right\rangle$, using the arguments above, it would be still be gaussian with shifted mean by θ. Now, note that dot product squeezes two vectors into a number, so the output should be One-Dimension Gaussian distribution.

From question 2,

$$\mu = \left(X^\top X\right)^{-1} X^\top X\theta$$

Then the distribution equals

$$\left\langle v, \hat{\theta} - \theta \right\rangle \sim \mathcal{N}\left(v \cdot (\mu - \theta), v^\top \sigma^2 \left(X^\top X\right)^{-1} v\right)$$

But, the $\mu = \theta$. Then,

$$\mathcal{N}\left(v \cdot 0, v^\top \sigma^2 \left(X^\top X\right)^{-1} v\right) =$$

$$\mathcal{N}\left(0, v^\top \sigma^2 \left(X^\top X\right)^{-1} v\right)$$

This is one-dimensional normal distribution and $\mathbb{E}[X] = 0$ and finite variance $Var[X] = \sigma^2 \left(X^\top X\right)^{-1}$ Let's call it V

Now, we need to construct a function that

$$\forall \delta \in (0, 1], \mathbb{P}(\left(\left|\left\langle v, \hat{\theta} - \theta\right\rangle\right| \geqslant f(\delta)\right) \leq \delta$$

Let's use Chebyshev inequality, which states that

$$\mathbb{P}\left(|X - \mu| \leq \sqrt{V} k\right) \geq \frac{1}{k^2}$$

We want to find $f(\delta)$ such that

$$\mathbb{P}\left(\left|\left\langle v, \hat{\theta} - \theta\right\rangle\right| \geq f(\delta)\right) \leq \delta)$$

Setting

$$\frac{1}{k^2} = \delta$$

, we solve for k:

$$k = \frac{1}{\sqrt{\delta}}$$

Then, substituting δ into k we have,

$$\mathbb{P}\left(\left|\left\langle v, \hat{\theta} - \theta\right\rangle\right| \geq \frac{\sqrt{V}}{\sqrt{\delta}}\right) \leq \delta$$

Hence,

we can define f as this:

$$\boxed{f(\delta) = \frac{\sqrt{Var[X]}}{\sqrt{\delta}}}$$

given that $Var[X]$ is some constant, so $f(\delta)$ only depends on δ. Also, $\frac{1}{\sqrt{\delta}}$ is a decreasing function as desired.

Problem 3 (a)

In [ ]:

```

```

In [1]:

```
import random as rn
import numpy as np
import pandas as pd
from typing import List
import math
import matplotlib.pyplot as plt
import seaborn as sns

%matplotlib inline
```

In [268...

```
def generate_data(n, lam):
    data = pd.DataFrame([], columns=['x_1', 'x_2', 'y'])
    count = 0
    w_star = (1/math.sqrt(2), 1/math.sqrt(2))
    while True:
        if count == n:
            break
        x_1 = np.random.uniform(-1,1)
        x_2 = np.random.uniform(-1,1)
        x_i = np.array([x_1,x_2])
        norm = np.linalg.norm(x_i, ord=2)
        if norm <= 1:
            wx_dot = np.dot(x_i, w_star)
            if np.abs(wx_dot) >= lam:
                data.loc[count] = (x_i[0], x_i[1], np.sign(wx_dot))
                count += 1
        else:
            continue

    return data
```

In [269...

```
sample_1000 = generate_data(1000, 1/32 )
sample_1000
```

Out[269…

	x_1	x_2	y
0	0.009787	0.485603	1.0
1	-0.344589	-0.444828	-1.0
2	0.448348	0.112515	1.0
3	-0.290947	0.755389	1.0
4	-0.060750	-0.009066	-1.0
...	...	...	...
995	0.761210	0.475867	1.0
996	0.554750	0.661231	1.0
997	-0.422267	-0.622717	-1.0
998	-0.062733	0.006968	-1.0
999	-0.558891	0.226253	-1.0

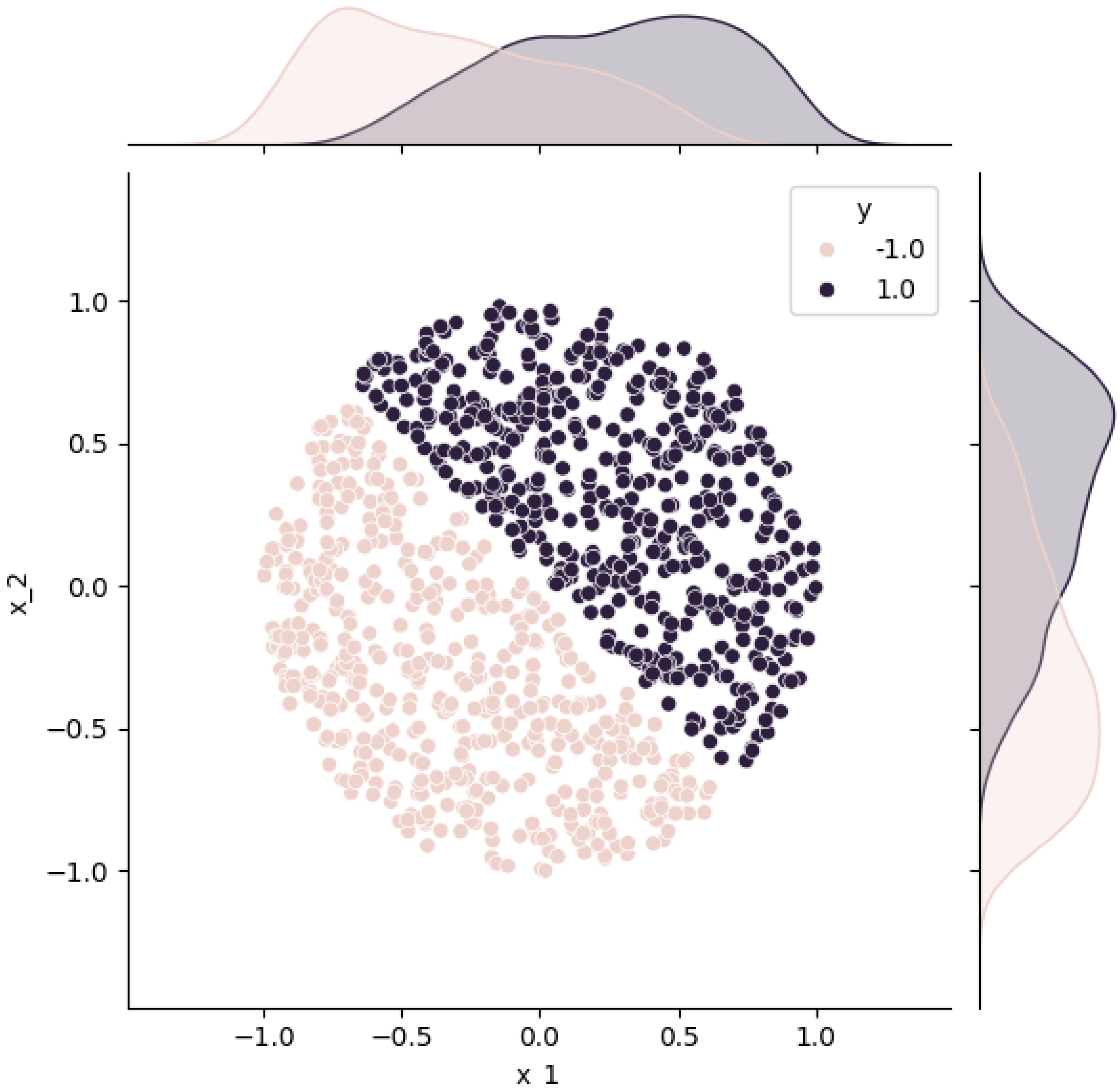
1000 rows × 3 columns

In [270…

```
sns.jointplot(data=sample_1000, x='x_1', y='x_2', hue='y')
```

Out[270…

```
<seaborn.axisgrid.JointGrid at 0x20c5a3ec9e0>
```



(b)

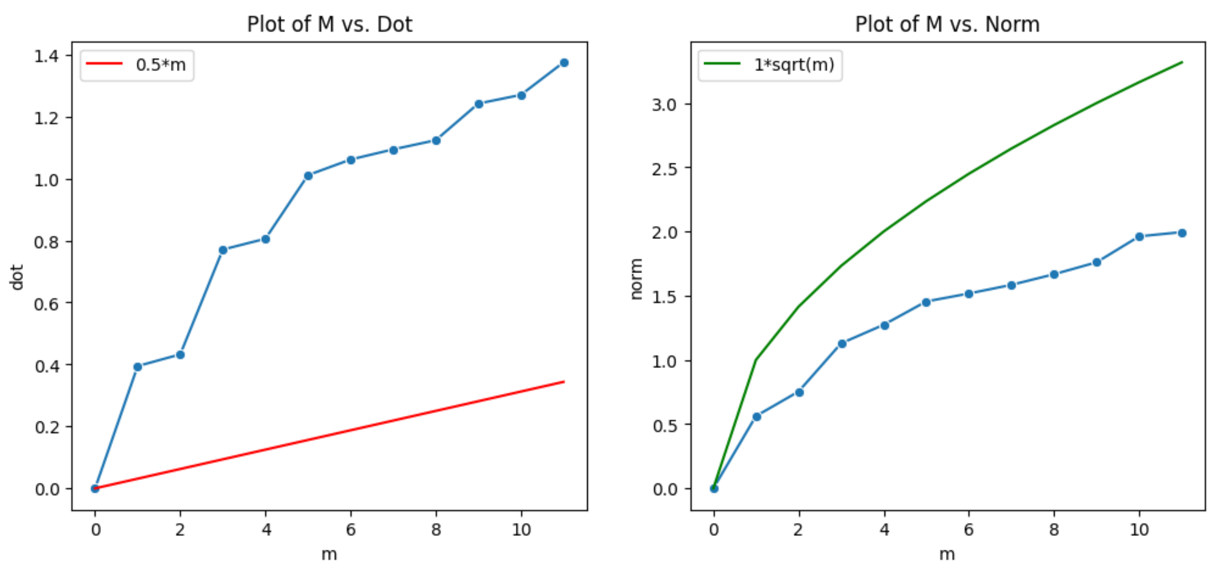
In [271…

```
def sim_perceptron(data: pd.DataFrame, lam = 1/32, draw=True, plot=True):
    data_for_plot = pd.DataFrame([], columns=['m', 'dot', 'norm'])
    w_iter = np.array([0,0])
    w_star = np.array([0.5, 0.5])
    M = 0
    for iteration in range(len(data)):
        y_true = data.iloc[iteration,2]
        y_hat = np.sign(np.dot((data.iloc[iteration,0],data.iloc[iteration,1]), w_i
        if y_hat == 0: y_hat = 1
        if y_hat == y_true:
            continue
        else:
            norma = np.linalg.norm(w_iter,2)
            dots = np.dot(w_iter,w_star)
            w_iter = w_iter + np.array([y_true*data.iloc[iteration,0], y_true*data.
            data_for_plot.loc[M] = [M, dots, norma]
            M += 1
```

```python
    if plot:
        fig, axes = plt.subplots(1, 2, figsize=(12, 5))
        sns.lineplot(ax=axes[0], x='m', y='dot', data=data_for_plot, marker='o')
        axes[0].set_title('Plot of M vs. Dot')
        axes[0].plot(data_for_plot['m'], lam * data_for_plot['m'], label='0.5*m', c
        axes[0].legend()
        sns.lineplot(ax=axes[1], x='m', y='norm', data=data_for_plot, marker='o')
        axes[1].set_title('Plot of M vs. Norm')
        axes[1].plot(data_for_plot['m'], np.sqrt(data_for_plot['m']), label='1*sqrt
        axes[1].legend()
    if draw:
        print("TOTAL M VALUE IS: " + str(M))
        print("FINAL WEIGHTS ARE: " + str(w_iter[0]) + "    " + str(w_iter[1]))


    return M








sim_perceptron(sample_1000)
```

```
TOTAL M VALUE IS: 12
FINAL WEIGHTS ARE: 1.3796161659813737    1.4919113180729662
```

Out[271… 12



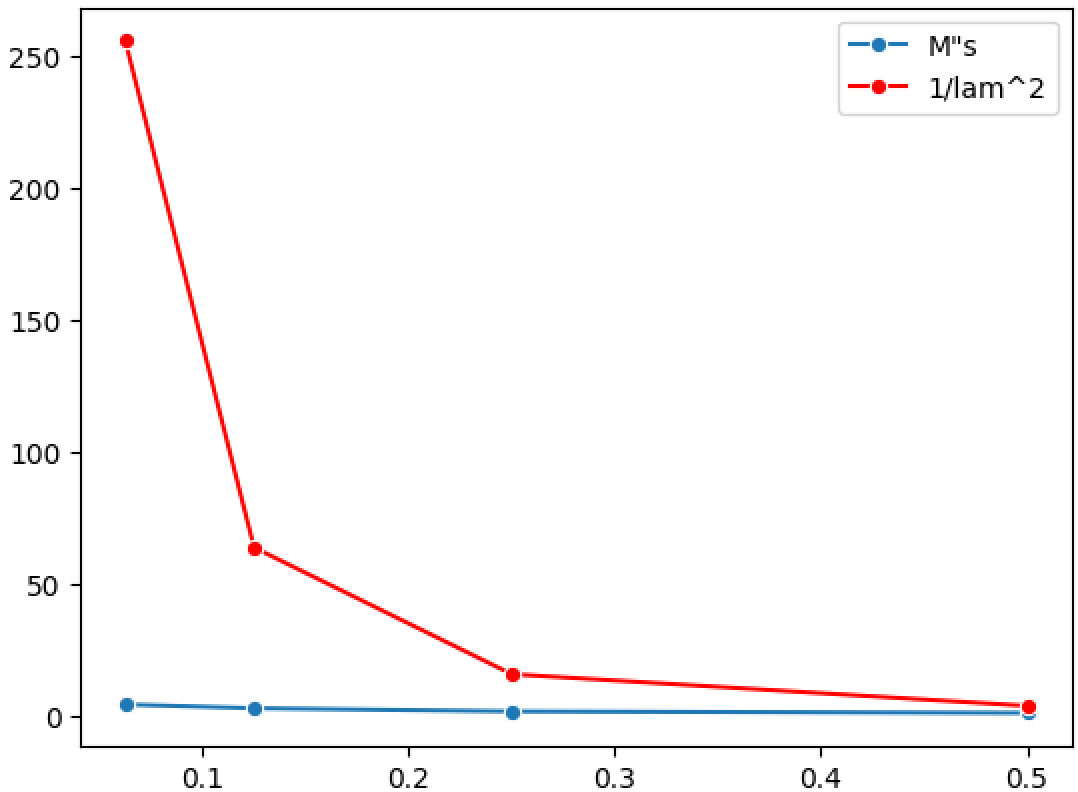
Indeed, as we see, the graphs give us the correct bounds for both dot products and norms.

(c)

We will plot only $1/\lambda^2$ up untill $\lambda = \frac{1}{2^{-4}}$ in order the plot not explode in the skies, because $1/(1/4098)^2$ already is a very big number

In [277…

```
def problem_c():
    lams = [2**(-i) for i in range(1,7)]
    M_i = []
    for i in lams:
        list_of_datasets = [generate_data(n=100,lam=i) for j in range(10)]
        sim_percs = [sim_perceptron(data=ds, draw=False, plot=False) for ds in list
        M_i.append(np.mean(sim_percs))

    lams_squared = []
    for i in lams:
        lams_squared.append(1/(i**2))
    sns.lineplot(x=lams[:-2], y=M_i[:-2], marker='o', label='M"s')
    sns.lineplot(x=lams[:-2], y=lams_squared[:-2], label='1/lam^2', color='red', ma
problem_c()
```



It is always far below $1/\lambda^2$ due to the Perceptron Convergence Theorem.

Problem 4

Approximately 12-14 hours, including reading material.