Homework 3 done by Ivan Akinfiev

Problem 1

In [292…

```
import numpy as np
import pandas as pd
from scipy.io import arff
import matplotlib.pyplot as plt
import seaborn as sns
from sklearn.model_selection import train_test_split
```

In [293…

```
data = arff.loadarff('data.arff')
df = pd.DataFrame(data[0])
df['class'] = df['class'].replace({b'tested_positive': 1, b'tested_negative': -1})
df.head()
```

Out[293…

	preg	plas	pres	skin	insu	mass	pedi	age	class
0	6.0	148.0	72.0	35.0	0.0	33.6	0.627	50.0	1
1	1.0	85.0	66.0	29.0	0.0	26.6	0.351	31.0	-1
2	8.0	183.0	64.0	0.0	0.0	23.3	0.672	32.0	1
3	1.0	89.0	66.0	23.0	94.0	28.1	0.167	21.0	-1
4	0.0	137.0	40.0	35.0	168.0	43.1	2.288	33.0	1

Composing

$$\mathcal{B} = \{\sigma \cdot (2\mathbb{I}(x_i \leq t) - 1) : \sigma \in \{\pm 1\}, i \in \{1, \dots, d\}, t \in \mathbb{R}\}$$

In [294…

```
train, test  = train_test_split(df, train_size=100, shuffle=True)
test.shape
```

Out[294…

```
(668, 9)
```

This is my first attempt to implement in $O(dm^2)$. It works pretty slowly. Completely impractical at large t

In [295…

```
def ERM_DS_naive(train: pd.DataFrame, D: np.ndarray):
    theta_star = 0
    F_star = np.inf
    columns = train.columns
    j_star = -1
    for j in range(len(columns) - 1):
        parsed_j = train[columns[j]].values
        parsed_j = np.append(parsed_j, parsed_j[-1] + 1)
        theta = parsed_j[0] - 1
```

```python
        for i in range(train.shape[0]):
            F = 0
            theta = (parsed_j[i] + parsed_j[i+1]) / 2
            for k in range(train.shape[0]):
                if theta > parsed_j[k] and train.iloc[k, -1] == 1:
                    F += D[k]
                if theta < parsed_j[k] and train.iloc[k, -1] == -1:
                    F += D[k]
            if F < F_star:
                F_star = F
                j_star = j
                theta_star = theta

    return j_star, theta_star
```

This is algorithm from Shai, UML. It works in $O(dm)$

In [296...

```python
def ERM_DS(train: pd.DataFrame, D: list):
    theta_star = 0
    F_star = np.inf
    columns = train.columns[:-1]
    j_star = -1
    m = train.shape[0]
    D = np.array(D)

    for j in range(len(columns)):
        sorted_idx = train[columns[j]].argsort()
        sorted_x = train.iloc[sorted_idx, j].values
        sorted_y = train.iloc[sorted_idx, -1].values
        sorted_D = D[sorted_idx]

        F = np.sum(sorted_D[sorted_y == 1])
        if F < F_star:
            F_star = F
            j_star = j
            theta_star = sorted_x[0] - 1
        for i in range(m - 1):
            F += sorted_D[i] * (1 if sorted_y[i] == -1 else -1)
            if F < F_star and sorted_x[i] != sorted_x[i + 1]:
                F_star = F
                j_star = j
                theta_star = (sorted_x[i] + sorted_x[i + 1]) / 2
    return j_star, theta_star
```

A couple of helper functions.

In [297...

```python
def indicator(y_true, y_pred):
    return np.where(y_true != y_pred, 1, 0)

def h(x, theta):
    return np.where(x <= theta, 1, -1)
```

Algo itself.

In [316...

```
def ADA_BOOST(train: pd.DataFrame, T: int):
    D = np.array([1 / train.shape[0]] * train.shape[0])
    weights = []
    h_s = []
    for t in range(T):
        j_t, theta_t = ERM_DS(train, D)
        y_true = train.iloc[:, -1].values
        y_pred = h(train.iloc[:, j_t].values, theta_t)
        epsilon_t = np.sum(D * indicator(y_true, y_pred))
        # ADD SMALL NUMBER TO AVOID DIVISION BY ZERO.
        epsilon_t = max(epsilon_t, 1e-10)
        w_t = 0.5 * np.log((1 - epsilon_t) / epsilon_t)

        D *= np.exp(-w_t * y_true * y_pred)
        D /= D.sum()

        weights.append(w_t)
        h_s.append((j_t, theta_t))

    return weights, h_s




weights, h_s = ADA_BOOST(train, 3000)
```

In [304...

```
def ada_boost_eval(train: pd.DataFrame, test:pd.DataFrame, weights, h_s):
    X_train = train.iloc[:, :-1].values
    X_test = test.iloc[:,:-1].values
    y_train = train.iloc[:, -1].values
    y_test = test.iloc[:,-1].values
    predictions_test = np.zeros(shape=(len(weights,)))
    predictions_train = np.zeros(shape=(len(weights,)))
    exp_loss_of_t = np.zeros(shape=(len(weights,)))
    for t in range(len(h_s)):
        prediction_number_train = np.zeros(X_train.shape[0])
        prediction_number_test = np.zeros(X_test.shape[0])
        for s, (j_s, theta_s) in enumerate(h_s[:t]):
            d = h(X_train[:, j_s],theta_s)
            prediction_number_train += weights[s] * h(X_train[:, j_s], theta_s)
            prediction_number_test += weights[s] * h(X_test[:, j_s], theta_s)
        predictions_train_h = np.sign(prediction_number_train)
        predictions_test_h = np.sign(prediction_number_test)
        accuracy_train = np.mean(y_train == predictions_train_h)
        accuracy_test = np.mean(y_test == predictions_test_h)
        predictions_train[t] = accuracy_train
        predictions_test[t] = accuracy_test
        exp_loss = np.sum(np.exp(-(prediction_number_train*y_train)))
        exp_loss_of_t[t] = exp_loss


    return predictions_train, predictions_test, exp_loss_of_t
```

```
predictions_train, predictions_test, exp_loss_of_t = ada_boost_eval(train,test, wei
```

In [305…

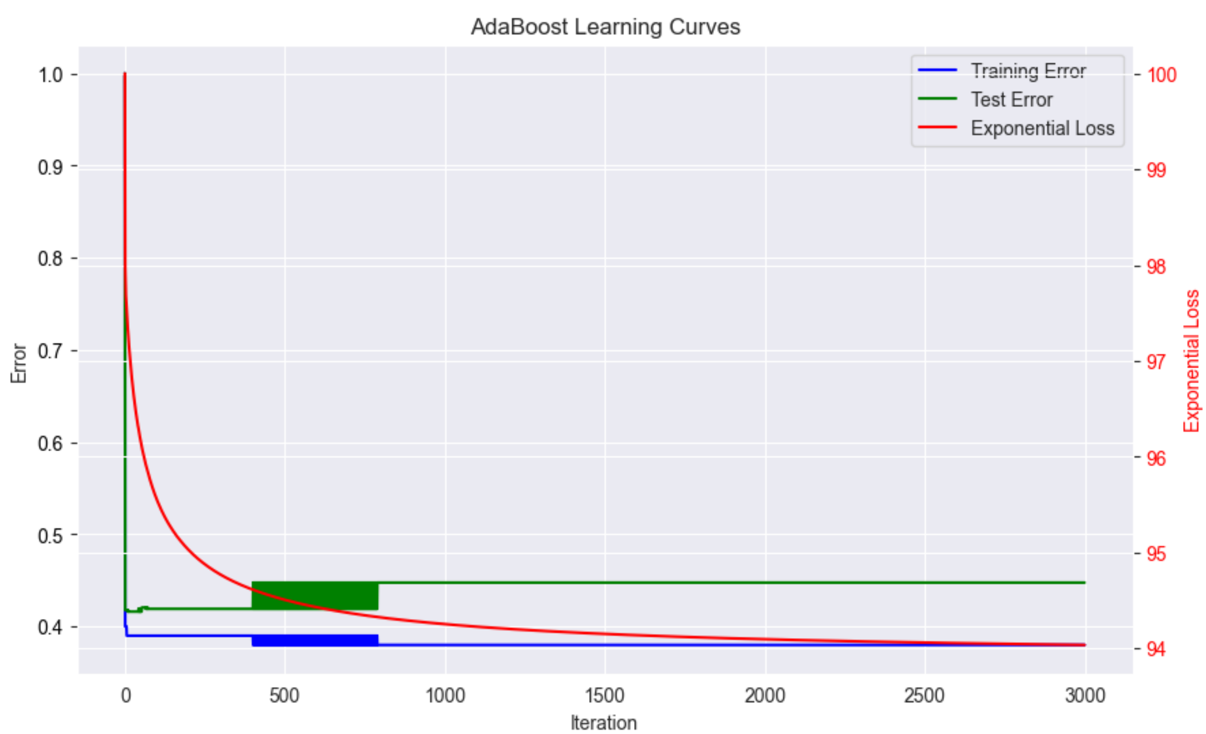
```
predictions_train
```

Out[305…

```
array([0.  , 0.6 , 0.6 , ..., 0.62, 0.62, 0.62])
```

In [315…

```
t = np.arange(len(weights))
fig, ax1 = plt.subplots(figsize=(10, 6))
train_error_line, = ax1.plot(t, 1 - predictions_train, color='blue', label='Trainin
test_error_line, = ax1.plot(t, 1 - predictions_test, color='green', label='Test Err
ax1.set_xlabel('Iteration')
ax1.set_ylabel('Error')
ax1.tick_params(axis='y', labelcolor='black')
ax2 = ax1.twinx()
exp_loss_line, = ax2.plot(t, exp_loss_of_t, color='red', label='Exponential Loss')
ax2.set_ylabel('Exponential Loss', color='red')
ax2.tick_params(axis='y', labelcolor='red')
plt.title('AdaBoost Learning Curves')
lines = [train_error_line, test_error_line, exp_loss_line]
ax1.legend(lines, [line.get_label() for line in lines])
plt.show()
```



We see classical learning curves, at some point test error starts increasing, while training error goes down as expected, so the exponential loss. However, interesting to note, that at 400 < t < 500 iterations, the AdaBoost stopped learning. We can see it from the weights that almost repeating themselves at the end.

In [317…

```
weights[-10:]
```

Out[317…

```
[0.0009346360304307173,
 0.0009345444813734688,
 0.0009336719231768263,
 0.0009340853823730036,
 0.000933213680861494,
 0.0009337594283504078,
 0.0009328883348191226,
 0.0009327971300232735,
 0.0009319278301658351,
 0.0009323397540796247]
```

b)

In [322…

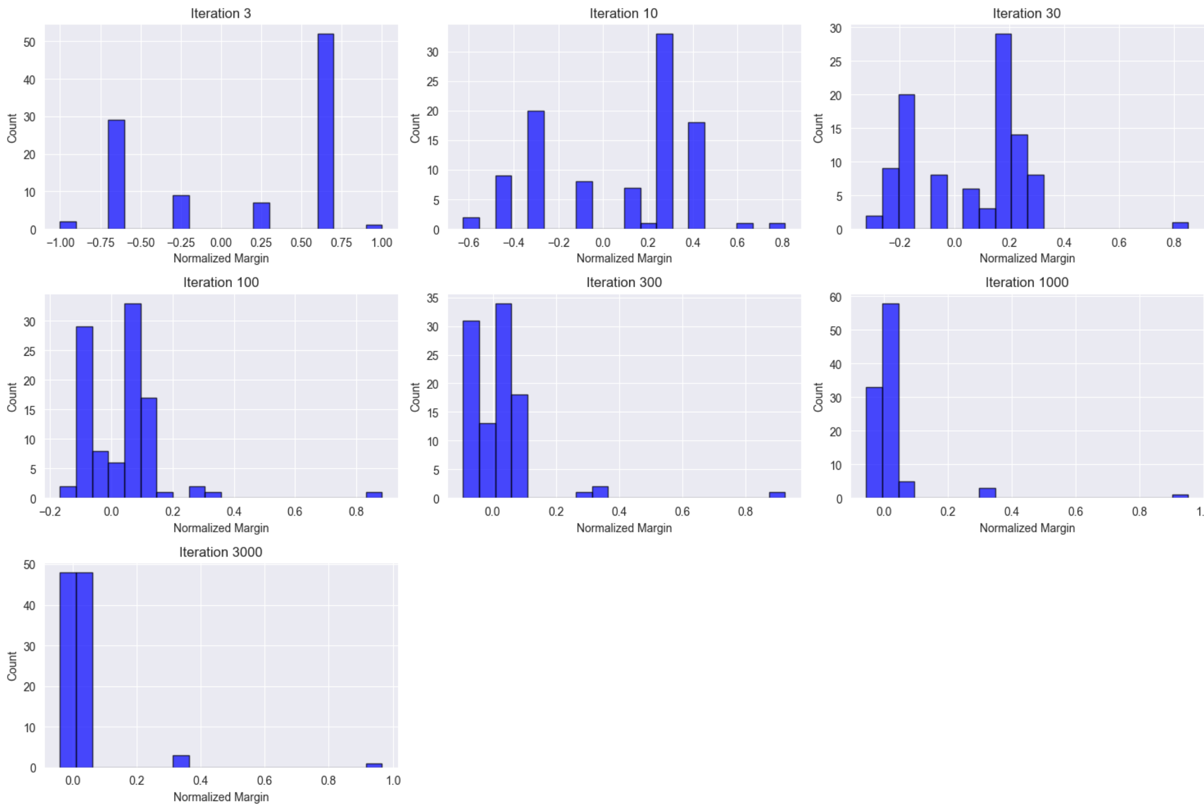
```
def norm_margin(train, weights, h_s, t):
    X_train = train.iloc[:, :-1].values
    y_train = train.iloc[:, -1].values
    f_t = np.zeros(X_train.shape[0])
    alpha = np.sum(weights[:t])
    for s, (j_s, theta_s) in enumerate(h_s[:t]):
        f_t += weights[s] * h(X_train[:, j_s], theta_s)
    n_f_t = f_t / alpha
    n_margin = y_train * n_f_t
    return n_margin
```

In [321…

```
iters = [3, 10, 30, 100, 300, 1000, 3000]
plt.figure(figsize=(15, 10))

for i, t in enumerate(iters, 1):
    normalized_margin = norm_margin(train, weights, h_s, t)
    plt.subplot(3, 3, i)
    plt.hist(normalized_margin, bins=20, alpha=0.7, color='blue', edgecolor='black'
    plt.title(f'Iteration {t}')
    plt.xlabel('Normalized Margin')
    plt.ylabel('Count')

plt.tight_layout()
plt.show()
```



Indeed, we see a trend that Normalized Margin tends to get skewed to 0 as we increase t.

Problem 2

$$\sum_{t+1}^{m} D_{t+1} I(h_t(x_i) \neq y_i) = \frac{1}{2}$$

Proof:

$$\sum_{t+1}^{m} D_{t+1} I(h_t(x_i) \neq y_i) = \frac{\sum_{i=1} m D_t e^{-wh_t(x_i)y_i} \{h_t(x_i) \neq y_i\}}{\sum_{i=1}^{m} D_t e^{-wh_t(x_i)y_i}}$$

Let's divide it using the definition of indicator function:

$$\frac{\sum_{K} D_t e^{-wh_t(x_i)y_i} \cdot 1}{\sum_{i=1}^{m} D_t e^{-wh_t(x_i)y_i}} + \frac{\sum_{E} D_t e^{-wh_t(x_i)y_i} \cdot 0}{\sum_{i=1}^{m} D_t e^{-wh_t(x_i)y_i}} = \frac{\sum_{K} D_t e^{-wh_t(x_i)y_i}}{\sum_{i=1}^{m} D_t e^{-wh_t(x_i)y_i}}$$

Where $w = \frac{1}{2} ln(\frac{1-\epsilon}{\epsilon})$ Here, since $h_t(x_i) \neq y_i$

$$\frac{\sum_{K} D_t e^{-wh_t(x_i)y_i}}{\sum_{i=1}^{m} D_t e^{-wh_t(x_i)y_i}} = \frac{\sum_{K} D_t e^{-w}}{\sum_{i=1}^{m} D_t e^{-wh_t(x_i)y_i}} =$$

$$ \frac{\sum_{K}D_te^{-w}}{\sum_{i=1}^{m}D_te^{-wh_t(x_i)y_i}}= \frac{\sum_{K}D_t}
{\sum_{i=1}^{m}\frac{D_te^{-wh_t(x_i)y_i}}{e^{w}}=

$$Usingthesametrick,$$

\frac{\sum_{K}D_t}{\sum_{i=1}^{m}\frac{D_te^{-wh_t(x_i)y_i}}{e^{w}}}=\frac{\sum_{K}D_t}{\sum_{K}D_t+e^{-2w}\sum_{E}D_t} = $$ Reversing the log function: $e^{2w} = \frac{1-\epsilon}{\epsilon}$ So,

$$\frac{\sum_K D_t}{\sum_K D_t + \sum_E D_t} = w \Rightarrow \sum_E D_t = \frac{\sum_K D_t}{w} - \sum_K D_t$$

Now, replacing $\sum_E D_t$, we have

$$\frac{\sum_K D_t}{\sum_K D_t + e^{-2w}\sum_E D_t} = \frac{\sum_K D_t}{\sum_K D_t + e^{-2w}\frac{\sum_K D_t}{w} - \sum_K D_t} =$$
$$\frac{\sum_K D_t}{\sum_K D_t + \frac{w}{1-w}\frac{\sum_K D_t}{w} - \sum_K D_t}$$
$$\frac{\sum_K D_t}{\sum_K D_t + \frac{\sum_K D_t}{1-w} - \frac{w}{1-w}\sum_K D_t}.$$
$$= \frac{1}{1+\frac{1}{1-w} - \frac{w}{1-w}} = 1/1+1 = \frac{1}{2}$$

Problem 3

To solve this problem we need to collect multiple fact. Since, we are dealing with regression problem, is it quite hard to use VC-dimension. There are attempts to discreditize a response space or find a class of indicator functions that can be formed by thresholding that class of real-valued functions. However, there is far more useful tool - Rademacher Complexity. First of all, let's build our skeleton.

From "UML" by Shai et.al, p. 378, the theorem is provided:

$$L_{\mathcal{D}}(ERM_{\mathcal{H}}(S)) - min_{h\in\mathcal{H}}L_{\mathcal{D}} \leq 2R(l\circ\mathcal{H}\circ S) + 5c\cdot\sqrt{\frac{2ln(8/\delta)}{m}}$$

for any $h \in \mathcal{H}$. We need to figure out c and R. First of all, let's use Contraction Lemma, which states that,

$$\mathbb{E}\left[sup_{h\in\mathcal{H}}\frac{1}{m}\sum_{i=1}^{m}\sigma_i\phi(h(x_i))\right] \leq L\cdot\mathbb{E}\left[sup_{h\in\mathcal{H}}\frac{1}{m}\sum_{i=1}^{m}\sigma_i h(x_i)\right]$$

Notice that square loss is a Lipschitz function $l = (\hat{y}-y)^2$, which is bounded by $[-Y, Y]$. Therefore, we automatically get $c = 2Y$. Therefore, the bound becomes

$$L_{\mathcal{D}}(ERM_{\mathcal{H}}(S)) - min_{h\in\mathcal{H}}L_{\mathcal{D}} \leq 2R(l\circ\mathcal{H}\circ S) + 10Y\cdot\sqrt{\frac{2ln(8/\delta)}{m}}$$

Now, let's figure out the Rademacher complexity. By definition,

$$R(l \circ \mathcal{H} \circ S) = \mathbb{E}_\sigma \left[\sup \frac{1}{m} \sum_{i=1}^{m} \sigma_i h(x_i) \right]$$

Given by the problem, $h(x) = \langle w, x \rangle$, s.t $||x||_\infty \leq R$ and $||w||_1 \leq B$. Let's plug in directly

$$\mathbb{E}_\sigma \left[\sup \frac{1}{m} \sum_{i=1}^{m} \sigma_i h(x_i) \right] = \mathbb{E}_\sigma \left[\sup \frac{1}{m} \sum_{i=1}^{m} \sigma_i \langle x_i, w \rangle \right]$$

By the linearity of expectation,

$$\mathbb{E}_\sigma \left[\sup \frac{1}{m} \sum_{i=1}^{m} \sigma_i \langle x_i, w \rangle \right] = \sup \frac{1}{m} \mathbb{E}_\sigma \left[\sum_{i=1}^{m} \sigma_i \langle x_i, w \rangle \right]$$

Here, the Holder inequality comes perfectly with satisfied property that $1/p + 1/q = 1$, since, $1/\infty + 1/1 = 1$. Both norms are already bounded, so

$$\sup \frac{1}{m} \mathbb{E}_\sigma \left[\sum_{i=1}^{m} \sigma_i \langle x_i, w \rangle \right] \leq \sup \frac{1}{m} \mathbb{E}_\sigma \left[\sum_{i=1}^{m} \sigma_i \; B \cdot R \right] =$$

$$\sup \frac{BR}{m} \mathbb{E}_\sigma \left[\sum_{i=1}^{m} \sigma_i \right]$$

The part of this step is taken from Shai p.382 lemma 26.10. Here is a conceptual trick, since the original expected value of Rademacher RV is zero. However, we are interested in the minimum deviation, so we take the absolute value,

$$\mathbb{E} \left| \sum_{i=1}^{m} \sigma_i \right|$$

which is equals $\sqrt{m}$ Combining everything, we get the resulting bound inequality:

$$L_\mathcal{D}(ERM_\mathcal{H}(S)) - min_{h \in \mathcal{H}} L_\mathcal{D} \leq \frac{2BR}{\sqrt{m}} + 10Y \cdot \sqrt{\frac{2ln(8/\delta)}{m}}$$