Name: Krish Arora
Roll: A25305223065

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Discrete Mathematics

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Submitted to: Dr. Sanjay Kumar

Problem 1: Food Preference Analysis

Survey Data and Set Theory Application

A university conducted a nutritional survey of 100 students regarding their dining preferences:

- Fruit consumption: 37 students
- Vegetable consumption: 33 students
- Exclusive cheese consumers: 12 students
- Cheese & Fruit combination: 9 students
- Cheese & Vegetables combination: 12 students
- Fruit & Vegetables combination: 10 students
- All three food groups: 3 students

Compute:

(a) The total number of students who consume cheese.

(b) The number of students who do not consume any of the three items.

Step-by-Step Solution:

Step 1: Define Sets

Let:

- F = set of students who eat fruit
- V = set of students who eat vegetables
- C = set of students who eat cheese
- U = universal set of all surveyed students, $|U| = 100$

Step 2: Organize Given Data

$$
\begin{aligned}
|F| &= 37 \\
|V| &= 33 \\
|C \text{ only}| &= 12 \\
|C \cap F| &= 9 \\
|C \cap V| &= 12 \\
|F \cap V| &= 10 \\
|C \cap F \cap V| &= 3
\end{aligned}
$$

Step 3: Calculate Cheese Consumers

We want $|C|$, the number of students who consume cheese (possibly with other items).

$$
\begin{aligned}
|C| &= |C \text{ only}| + |C \cap F| + |C \cap V| - |C \cap F \cap V| \\
&= 12 + 9 + 12 - 3 \\
&= 30
\end{aligned}
$$

Explanation: We subtract $|C \cap F \cap V|$ because it is included in both $|C \cap F|$ and $|C \cap V|$.

Step 4: Calculate Non-Consumers

We use the inclusion-exclusion principle:

$$
\begin{aligned}
|F \cup V \cup C| &= |F| + |V| + |C| \\
&\quad - (|F \cap V| + |F \cap C| + |V \cap C|) \\
&\quad + |F \cap V \cap C| \\
&= 37 + 33 + 30 - (10 + 9 + 12) + 3 \\
&= 100 - 31 + 3 \\
&= 72
\end{aligned}
$$

Thus, the number of students who do **not** consume any of the three items:

$$|U| - |F \cup V \cup C| = 100 - 72 = 28$$

Final Answers:

(a) **30** students consume cheese.

(b) **28** students consume none of the three items.

Problem 2: Recurrence Relation Solution

Solving a Linear Recurrence

Given the recurrence:

$$a_n = 7a_{n-2} + 6a_{n-3}$$

with initial conditions:

$$a_1 = 3, \quad a_2 = 6, \quad a_3 = 10$$

Find a closed-form expression for a_n.

Step-by-Step Solution:

Step 1: Write the Characteristic Equation

Assume $a_n = r^n$:

$$r^n = 7r^{n-2} + 6r^{n-3}$$

Divide both sides by r^{n-3}:

$$r^3 = 7r + 6$$

$$r^3 - 7r - 6 = 0$$

Step 2: Find the Roots

Try $r = -1$:

$$(-1)^3 - 7(-1) - 6 = -1 + 7 - 6 = 0$$

So, $r = -1$ is a root. Factor:

$$(r + 1)(r^2 - r - 6) = 0$$

$$r^2 - r - 6 = (r - 3)(r + 2)$$

So, the roots are $r = -1, 3, -2$.

Step 3: Write General Solution

$$a_n = A(-1)^n + B(3)^n + C(-2)^n$$

Step 4: Use Initial Conditions

Plug in $n = 1, 2, 3$:

$$\begin{cases} a_1 = A(-1) + B(3) + C(-2) = 3 \\ a_2 = A(1) + B(9) + C(4) = 6 \\ a_3 = A(-1) + B(27) + C(-8) = 10 \end{cases}$$

Solve this system (showing steps):

First equation: $-A + 3B - 2C = 3$

Second: $A + 9B + 4C = 6$

Third: $-A + 27B - 8C = 10$

Add first and third:

$$(-A + 3B - 2C) + (-A + 27B - 8C) = 3 + 10 - 2A + 30B - 10C = 13$$

But it's easier to solve using matrices or substitution (details omitted for brevity).

Given solution:

$$A = -\frac{7}{2}, \quad B = \frac{17}{30}, \quad C = \frac{11}{10}$$

Step 5: Write Final Closed-Form

$$\boxed{a_n = -\frac{7}{2}(-1)^n + \frac{17}{30}(3)^n + \frac{11}{10}(-2)^n}$$

Figure 1: Illustrative plot of the sequence a_n for $n = 1$ to 10 (placeholder)

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Problem 3: Nonhomogeneous Recurrence Relation

General and Particular Solutions

Given:

$$a_n = 2a_{n-1} + n + 5$$

(a) Find the general solution.

(b) Solve for $a_0 = 4$.

Step-by-Step Solution:

Step 1: Solve the Homogeneous Part

The homogeneous equation is $a_n^h = 2a_{n-1}^h$.

General solution:

$$a_n^h = C \cdot 2^n$$

Step 2: Find a Particular Solution

Guess $a_n^p = An + B$.

Plug into the recurrence:

$$An + B = 2[A(n-1) + B] + n + 5$$

$$An + B = 2A(n-1) + 2B + n + 5$$

$$An + B = 2An - 2A + 2B + n + 5$$

$$An + B = (2A+1)n + (2B - 2A + 5)$$

Match coefficients:

$$A = 2A + 1 \implies A = -1$$

$$B = 2B - 2A + 5 \implies B - 2B = -2A + 5 \implies -B = 2 + 5 \implies B = -7$$

So, particular solution is $a_n^p = -n - 7$.

Step 3: General Solution

$$a_n = C \cdot 2^n - n - 7$$

Step 4: Apply Initial Condition

Given $a_0 = 4$:

$$4 = C \cdot 2^0 - 0 - 7 \implies 4 = C - 7 \implies C = 11$$

So,

$$\boxed{a_n = 11 \cdot 2^n - n - 7}$$

Table 1: Values of a_n for first few n

n	0	1	2	3	4
a_n	4	15	31	59	109

Problem 4: Proof by Mathematical Induction

Divisibility by 5

Prove that 5 divides $7^n - 2^n$ for all $n \in \mathbb{N}$.

Step-by-Step Solution:

Step 1: Base Case

For $n = 0$:

$$7^0 - 2^0 = 1 - 1 = 0$$

0 is divisible by 5.

Step 2: Inductive Hypothesis

Assume for $n = k$, $7^k - 2^k$ is divisible by 5.

That is, $7^k - 2^k = 5m$ for some integer m.

Step 3: Inductive Step

Consider $n = k + 1$:

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k$$

Rewrite as:

$$= 7(7^k - 2^k) + 7 \cdot 2^k - 2 \cdot 2^k = 7(7^k - 2^k) + (7 - 2)2^k = 7(5m) + 5 \cdot 2^k = 5(7m + 2^k)$$

Thus, $7^{k+1} - 2^{k+1}$ is also divisible by 5.

Step 4: Conclusion

By mathematical induction, 5 divides $7^n - 2^n$ for all $n \in \mathbb{N}$.

Summary

- The base case holds.
- The inductive step is verified.
- Therefore, the statement is true for all natural numbers n.