Discrete Math Notes

B.Tech. CSE

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Contents

1 UNIT 1

1.1 Set Theory

Schaum series- Lipscitz

1.1.1 Sets

Sets are <u>well defined</u> collection of mathematical objects.

Example:

The collection of best mathematicians in the world is not a set as there is no fixed criteria for being the best mathematicians.

Notation: Sets are denoted by capital letters such as A, B, X, Y.
the elements are denoted by small letters such as a, b, x, y.

$\underline{Def^n}$:
A set A is called to be a subset of B iff

$$a \in A \implies a \in B$$

It is denoted by $A \subseteq B$.

1.1.2 Empty and Universal set

$\underline{Def^n}$:
An empty set is a set which contains no elements. It is either denoted by empty braces or the greek letter ϕ.

$\underline{Def^n}$:
A Universal set is a set which contains all the elements (in the context).

$\underline{Def^n}$:
A set which contains only one element is called a singleton set.
for example: $\{5\}$.

Note:
for any set A, ϕ and A are always subsets called improper subsets.

1.1.3 Power Set

$\underline{Def^n}$:
A power set of a set is the collection of all the subsets of A. It is denotes by 2^A.

1.2 Representation of Sets

There are 2 ways to represent sets:

1. Set builder form

2. Roaster form

1.2.1 Set builder form

$\underline{Def^n:}$
It is based on the unique property of the collection. The iterator is set and a property is defined in curly braces

Example:

$$A = \{x : x = 2y, y \in \mathbb{Z}\}$$

OR

$$A = \{2x : x \in \mathbb{Z}\}$$

1.2.2 Roaster form

$\underline{Def^n:}$
In this representation we list the elements in curly braces seperated by commas.

Example:

$$A = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

1.3 Operations on sets

We have defined the following functions on sets

1. Union
2. Intersection
3. Difference
4. Symmetric Difference

1.3.1 Union of Sets($\cup$)

$\underline{Def^n:}$
Collection of all the elements of the sets

Example:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

1.3.2 Intersection of Sets($\cap$)

$\underline{Def^n:}$
Collection of all the elements in both the sets

Example:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

1.3.3 Difference of Sets(-)

Def

$\underline{Def^n}$:
Collection of all the elements one set but not the other

Example:

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

1.3.4 Symmetric difference of Sets(Δ)

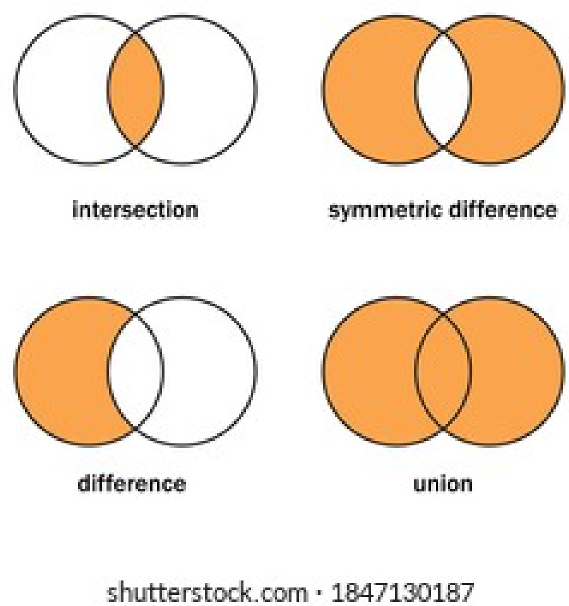
$\underline{Def^n}$:
Collection of all the elements which exist in exactly one of the sets

Example:

$$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

1.4 Venn diagram

A pictorial representation of sets is called a venn diagram



1.5 De-morgan's Law

Let A and B be two sets then

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Proof: Let $x \in (A \cup B)^c$

$$\begin{aligned}
&\implies x \notin A \cup B \\
&\implies x \notin A, x \notin B \\
&\implies x \in A^c, x \in B^c \\
&\implies x \in A^c \cap B^c
\end{aligned}$$

Thus we can say that $(A \cup B)^c \subseteq A^c \cap B^c$
Similarly Let $x \in A^c \cap B^c$.

$$\begin{aligned}
&\implies x \in A^c, x \in B^c \\
&\implies x \notin A, x \notin B \\
&\implies x \notin A \cup B \\
&\implies x \in (A \cup B)^c
\end{aligned}$$

Thus we can say that $A^c \cap B^c \subseteq (A \cup B)^c$
This is possible iff $(A \cup B)^c = A^c \cap B^c$

Q.E.D.

1.6 Partition of sets

Let S be a non-empty set. Then S has the partition if it has a collection of subsets A_i such that:

1. $\forall a \in S, \exists$ unique i such that $a \in A_i$
2. $A_i \cup A_j = \phi, i \neq j$

Example:
Consider the set $S = \{1, 2, \ldots, 9\}$

1. $[\{1, 3, 5\}, \{2, 6\}, \{4, 9\}]$
2. $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

then 1 is not a partition of S as the element 7 is missing. However 2 is a partition of the set S.

1.7 Relations

A relation R from set A to set B is subset of $A \times B$ where :

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$R \subseteq A \times B = \{(a, b) : a \in A, b \in B\}$$

- Domain $\rightarrow$ All the elements of set A
- Codomain $\rightarrow$ All the elements of set B
- Range $\rightarrow$ All the second elements of R

Example:

$$\begin{aligned}
R =& \{(a,b) : b = 2a+1, a \in [1,10], b \in [1,10]\} \\
A =& [1,10] \\
R \subseteq& A \times A \\
=& \{(1,3),(2,5),(3,7),(4,9)\}
\end{aligned}$$

- Domain $= \{1,2,3,4\}$
- Range $= \{3,5,7,9\}$

1.7.1 Composition of Relations

Let R be a relation from A to B
Let S be a relation from B to C
then

$$R \circ S = \{(a,c) : \exists b \in B \ s.t. \ (a,b) \in R, (b,c) \in S\}$$

Example:

$$\begin{aligned}
\text{Let } A &= \{1,2,3,4\} \\
B &= \{a,b,c,d\} \\
C &= \{x,y,z\} \\
R &= \{(1,a),(2,a),(3,a),(4,d)\} \\
S &= \{(a,y),(b,x),(c,z)\} \\
\implies R \circ S &= \{(1,y),(2,y),(3,z)\}
\end{aligned}$$

1.7.2 Equivalence Relations

A relation R from A to A is said to be equivalence if it satisfies the following conditions:

1. Reflexivity
2. Transivity
3. Symmetricity

<u>*Def^n* :</u>
A relation is said to be reflexive iff:

$$(a,a) \in R \ \forall a \in A$$

<u>*Def^n* :</u>
A relation is said to be Transitive iff:

$$(a,b) \in R, (b,c) \in R \implies (a,c) \in R$$

***Def**^n* :
A relation is said to be Symmetric iff:

$$(a, b) \in R \implies (b, a) \in R$$

Example:
Consider the relation R on $A = \{1, 2, 3, 4\}$

$$\begin{aligned}
R_1 &= \{(1,1),(1,2),(2,3),(1,3),(4,4)\} \\
R_2 &= \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\} \\
R_3 &= \{(1,3),(2,1)\} \\
R_4 &= A \times A
\end{aligned}$$

Then

- R_1 is Transitive only
- R_2 is Equivalence
- R_3 is neither Symmetric, Reflexive or Transitive only
- R_4 is Equivalence

Def^n :
A relation on a set A is said to be Anti-symmetric if and only if:

$$(a, b) \in A, (b, a) \in A \implies a = b$$

1.8 Functions

A relation from set A to B such that each element of A has a unique mapping in B then it is called a function and is denoted as

$$f : A \to B$$

All functions are relations but not vice-versa.

1.8.1 Domin, Range and Codomain

for $f : A \to B$

- Domain: A
- Codomain: B
- Range: $\{b \in B : \exists a \in A \text{ such that } f(a) = b\}$

Note: Domain$(f) \subseteq A$
Range$(f) \subseteq B$

One-one	Each horizontal line cuts the graph at atmost one point
Onto	Each horizontal line cuts the graph at one or more points

1.8.2 One-one and Onto function

Let $f : A \to B$ be a funciton then f is called one-one if distinct elements of A have different image.
Mathematically:

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

f is said to be onto if each element of B has a preimage in A.
Mathematically

$$\forall b \in B, \exists a \in A \text{ such that } f(a) = b$$

Def^n :
f is bijective iff it is both one-one and onto

1.8.3 Geometrical characterisation of one-one and onto functions

Example:
Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$
One-one:

$$\begin{aligned}
&\text{Let } f(x_1) = f(x_2) \\
&\implies x_1^2 = x_2^2 \\
&\implies x_1^2 - x_2^2 = 0 \\
&\implies (x_1 - x_2)(x_1 + x_2) = 0 \\
&\implies x_1 = x_2 \text{ or } x_1 = -x_2
\end{aligned}$$

Onto: We have $f(x) = x^2 \geq 0$
$\nexists x < 0 \in \mathbb{R}$ such that $f(x) = x^2$

1.9 Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two functions then
$g \circ f : A \to C$ is called composition of f and g.

$$g \circ f(x) = g(f(x))$$

Result:
Let $f : A \to B$ and $g : B \to C$ be two functions then

1. $g \circ f$ is one-one if both f and g are one-one
2. $g \circ f$ is onto if both f and g are onto

Proof:

1. Let $(g \circ f)(x_1) = (g \circ f)(x_2)$

$$\begin{aligned} &\implies g(f(x_1)) = g(f(x_2)) \\ &\implies f(x_1) = f(x_2) \text{ as g is one-one} \\ &\implies x_1 = x_2 \text{ as f is one-one} \end{aligned}$$

2. $g \circ f : A \to C$
   Let $z \in C$, we will show that $\exists x \in A$ such that $g \circ f(x) = z$
   Since g is onto $\implies \exists y \in B$ such that $g(y) = z$
   Since f is onto and $y \in B \implies \exists x \in A$ such that $f(x) = y$

Q.E.D.

1.10 Induction

Used to prove that a statement is true for all integers or natural numbers.

$$P(n) \text{ holds } \forall n$$

1.10.1 Principle of Mathematical Induction

Let $P(n)$ be the given statement.

1. Basic Step: $P(1)$ is true
2. Induction Step: $P(k) \implies P(k+1)$ is true

This causes a domino effect and effectively proves that $P(n)$ is true $\forall n$ *Example:*
Using Induction, Prove that:

$$1 + 2 + 3 + 4 + 5 + \cdots + n = \frac{n(n+1)}{2}, n \in \mathbb{Z}^+$$

1. Base Case:

$$1 = \frac{1(1+1)}{2} = 1$$

Q.E.D.

2. Induction Step: Let $P(k)$ is true, then we have to show that $P(k+1)$ is true.

$$\begin{aligned} 1 + 2 + \cdots + k &= \frac{k(k+1)}{2} \\ \implies 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ \implies 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ \implies 1 + 2 + \cdots + k + (k+1) &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Q.E.D.

1.11 Recursion

This is used when we cannot explicitly represent any object or mathematical term. so we use recursion.

<u>*Def*^{n} :</u>
<u>Recursive function</u>
Let $\{a\}_n, n \in \mathbb{Z}_0^+$ such that $a_n \in \mathbb{Z} \forall n$ then:

1. Basic Step: a_n is given at $n = 0$
2. Recursive step: $a_n = f(a_{n-1}, a_{n-2} \dots)$

Well known examples of recursive expressions are:

1. Arithemetic progression
$$a_n = a_{n-1} + d$$
2. Geometric progression
$$a_n = ra_{n-1}$$
3. Factorial function
$$f(n) = n \times f(n-1)$$
$$f(0) = 1$$
4. Fibonacci function
$$f(n) = f(n-1) + f(n-2)$$
$$f(0) = 0$$
$$f(1) = 1$$

1.11.1 Linear recurrence relations with constant coefficients

Let $a_n = \phi(a_{n-1}, a_{n-2}, \dots, a_0, m)$
It can be written as:

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_{n-1}a_1 + c_na_0 + f(m)$$

If $f(m) = 0$ then the function is called homogeneous
If $f(m) \neq 0$ then the function is called non-homogeneous

HOMOGENEOUS SOLUTION
Note: kth order recurrence relation is one such that

$$a_n = \phi(a_{n-1}, a_{n-2}, \dots, a_{n_k}, m)$$

or

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} + f(m)$$

so a second order recurrence relation will look like:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 \tag{1}$$

Then characteristic equation of (1) is given by:

$$x^2 - c_1 x - c_2 = 0, c_1, c_2 \in \mathbb{R}$$

Come forth 3 cases:

1. When roots are real and distinct:

$$x^2 - c_1 x - c_2 = 0 \implies \begin{cases} x = r_1 \\ x = r_2 \end{cases} \quad r_1 \neq r_2$$

Then the solution of (1) is given by:

$$a_n = p_1 r_1^n + p_2 r_2^n$$

2. Roots are real and equal:

$$r_1 = r_2 = r$$

Then the solution of (1) is given by:

$$a_n = (p_1 + np_2) r^n$$

Example:
Solve $a_n = 2a_{n_1} + 3a_{n-2}$

$$\begin{aligned} a_n - 2a_{n-1} - 3a_{n-2} &= 0 \\ \implies x^2 - 2x - 3 &= 0 \\ \implies (x-3)(x+1) &= 0 \\ \implies x &= -1, 3 \end{aligned}$$

Solution:

$$a_n = p_1(-1)^n + p_2(3)^n$$

Suppose: $a_0 = 1, a_1 = 2$

$$\begin{aligned} n = 0 &\implies a_0 = p_1(-1)^0 + p_2(3)^0 \\ &\implies 1 = p_1 + p_2 \\ n = 1 &\implies a_1 = p_1(-1)^1 + p_2(3)^1 \\ &\implies 2 = -p_1 + 3p_2 \end{aligned}$$

From the above 2 equations we get:

$$\begin{aligned} p_1 &= \frac{1}{4} \\ p_2 &= \frac{3}{4} \\ \implies a_n &= \frac{1}{4}(-1)^n + \frac{3}{4}(3)^n \end{aligned}$$

Example:
Solve: $a_n = 6a_{n-1} - 9a_{n-2}$
with $a_1 = 3, a_2 = 27$

NON-HOMOGENEOUS SOLUTION

2 Lattice

3 GROUP THEORY yeah imma kms