Engineering Maths Apps 2

Exam solutions

1. $$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & -2-\lambda & 2 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)\begin{vmatrix} -2-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-2-\lambda)(1-\lambda)+2\right]$$

$$= (1-\lambda)\left[-2+2\lambda-\lambda+\lambda^2+2\right]$$

$$= (1-\lambda)(\lambda^2+\lambda) = 0$$
$$= (1-\lambda)\lambda(\lambda+1)$$

$\therefore$ eigenvalues are $\lambda = 1$, $\lambda = 0$ & $\lambda = -1$

$\lambda = 1$
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -3 & 2 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$c = 0$

$-3b + 2c = 0$ $\quad\therefore$ $b = c = 0$

$-b = 0$ $\quad$ a = anything

Choose $a = 1$ $\quad\therefore$ $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 0$
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} a + c = 0 \\ -2b + 2c = 0 \\ -b + c = 0 \end{array}\right\} \begin{array}{l} \Rightarrow a = -c \\ \Rightarrow b = c \\ \end{array}$$

choose $c = 1$ then $a = -1$, $b = 1$

$$\underline{e}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$ $$\begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 2a + c = 0 \\ -b + 2c = 0 \\ -b + 2c = 0 \end{array}\right] \Rightarrow \begin{array}{l} a = -c/2 \\ b = 2c \\ \end{array}$$

choose $c = 2$ $a = -1$, $b = 4$ $\underline{e}_3 = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$.

b) $$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\frac{d}{dt}\underline{v} = A\underline{v}$$

Let $\underline{v} = P\underline{z}$ where $P = \begin{pmatrix} 1 & -1 & -4 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

so $\dfrac{d}{dt}P\underline{z} = AP\underline{z}$ $\qquad \dfrac{d\underline{z}}{dt} = P^{-1}AP\underline{z} = D\underline{z}$

where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\frac{dz_1}{dt} = z_1 \Rightarrow z_1 = Ae^{t}$$

$$\frac{dz_2}{dt} = 0 \Rightarrow z_2 = B$$

$$\frac{dz_3}{dt} = -z_3 \Rightarrow z_3 = Ce^{-t}$$

$$\underline{v} = P\,\underline{z} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} Ae^{t} \\ B \\ Ce^{-t} \end{pmatrix}$$

$$\therefore \quad u = Ae^{t} - B - Ce^{t}$$

$$v = B + 4Ce^{-t}$$

$$w = B + 2Ce^{-t}.$$

u : exponential growth unless $A=0$
if $A=0$ dominant term is $-B$ constan
if $A=0$ & $B=0$ exponential decay

v & w : dominant term is B - constant
so exponential decay to constant
if $B=0$ exponential decay.

$$I_n = \int_{-1}^{0} (1+x)\cos(\pi n x)\,dx = \left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{0} + \int_{-1}^{0} x\cos(n\pi x)\,dx$$

$$= 0 + \left[\frac{x\sin n\pi x}{n\pi}\right]_{-1}^{0} - \int_{-1}^{0} \frac{\sin(n\pi x)}{n\pi}\,dx$$

$$= 0 + \left[\frac{\cos(n\pi x)}{n^2\pi^2}\right]_{-1}^{0} = \frac{1-(-1)^n}{n^2\pi^2} \quad n \neq 0$$

(3 marks)

$$J_n = \int_0^1 (1-x)\cos(n\pi x)\,dx = \left[\frac{\sin n\pi x}{n\pi}\right]_0^1 - \int_0^1 x\cos(n\pi x)\,dx$$

$$= 0 + \left[\frac{-x\sin(\pi n x)}{\pi n}\right]_0^1 + \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx$$

$$= 0 + \left[\frac{-\cos(n\pi x)}{n^2\pi^2}\right]_0^1 = \frac{(-(-1)^n + 1)}{\pi^2 n^2} \quad n \neq 0$$

(3 marks)

$$K_n = \int_{-1}^{0} (1+x)\sin(n\pi x)\,dx = \left[\frac{-\cos(n\pi x)}{n\pi}\right]_{-1}^{0} + \int_{-1}^{0} x\sin(n\pi x)\,dx$$

$$= -\frac{1}{n\pi} + \frac{(-1)^n}{n\pi} + \left[\frac{-x\cos n\pi x}{n\pi}\right]_{-1}^{0} + \int_{-1}^{0} \frac{\cos(n\pi x)}{n\pi}\,dx$$

$$= -\frac{1}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{(-1)^n}{n\pi} + 0 = -\frac{1}{n\pi}.$$

(3 marks)

$$H_n = \int_0^1 (1+x)\sin(\pi n x)\,dx = \left[\frac{-\cos\pi n x}{\pi n}\right]_0^1 + \int_0^1 x\sin(n\pi x)\,dx$$

$$= -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi} + \left[\frac{-x\cos(n\pi x)}{n\pi}\right]_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx$$

$$H_n = +\frac{1}{n\pi} - \frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} + 0 - 0 = \frac{1}{n\pi}$$

(3 marks)

For the F.S. we require also

$$I_0 = \int_{-1}^{0} (1+x)\,dx = \left(x + \frac{x^2}{2}\right)_{-1}^{0} = 1 - \frac{1}{2} = \frac{1}{2}$$

and $$J_0 = \int_0^1 (1-x)\,dx = \left(x - \frac{x^2}{2}\right)_0^1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(\pi n x) + b_n \sin(n\pi x)\right]$$

$$b_n = \int_{-1}^{0} (1+x)\sin(n\pi x)\,dx + \mu\int_0^1 (1-x)\sin(\pi n x)\,dx$$

$$= -\frac{1}{n\pi} + \frac{\mu}{n\pi}$$

$$a_n = \mu\int_0^1 (1-x)\cos n\pi x\,dx + \int_{-1}^{0} (1+x)\cos \pi n x\,dx$$

$$= \mu\left(\frac{-(-1)^n + 1}{n^2\pi^2}\right) + \frac{\left(1 - (-1)^n\right)}{n^2\pi^2} \qquad n \neq 0$$

$$a_0 = \int_{-1}^{0} (1+x)\,dx + \mu\int_0^1 (1-x)\,dx = \frac{1}{2} + \frac{\mu}{2} = \frac{1+\mu}{2}$$

$$f = \frac{1+\mu}{4} + \sum_{n=1}^{\infty} \frac{\left(1-(-1)^n\right)(1+\mu)\cos \pi n x}{\pi^2 n^2} + \frac{\mu - 1}{n\pi}\sin(n\pi x)$$ (6 marks)

When $\mu = 1$ ($f(x)$ even) and we have $f = \frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{\left(1-(-1)^n\right)\cos \pi n x}{\pi^2 n^2}$

When $\mu = -1$ ($f(x)$ odd), $f = -2\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n\pi}$.

2 marks

$$FT\{e^{-(t-b)^2}\} = \int_{-\infty}^{\infty} e^{-(t-b)^2} e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} e^{-x^2} e^{-jw(x+b)} dx$$

using $t-b = x$.

Thus $FT\{e^{-(t-b)^2}\} = e^{-jwb}\int_{-\infty}^{\infty} e^{-x^2} e^{-jwx} dx$

$$= e^{-jwb} FT\{e^{-x^2}\} = \pi e^{-w^2/4} e^{-jwb}$$

(6 marks)

(a) $FT\{f(t-3)\} = e^{-3jw}\pi e^{-w^2/4}$

(b) $FT\{f(3t-5)\} = FT\{f(3(t-5/3))\}$

$$= e^{-\frac{5}{3}jw} FT\{f(3t)\} = e^{-\frac{5}{3}jw}\frac{1}{3}F\left(\frac{w}{3}\right)$$

$$= \frac{\pi}{3} e^{-\frac{w^2}{36}} e^{-\frac{5}{3}jw}$$

(c) $FT^{-1}\{jw\pi e^{-w^2/4}\} = \left\{\frac{d}{dt} e^{-t^2}\right\}$

$$= \underline{-2t e^{-t^2}}$$

(d) $FT^{-1}(F(w-3)) = e^{3jt} e^{-t^2}$

(8 marks)

$Y(w) = 3\,\delta(w-4)X(w) \Rightarrow$

$$y(t) = \frac{3}{2\pi}\int_{-\infty}^{\infty}\delta(w-4)X(w)e^{jwt}dw = \frac{3}{2\pi}e^{4jt}X(4) = \frac{3}{2\pi}e^{4jt}\pi e^{-4}$$

$$= \underline{\frac{3}{2}e^{-4}e^{4jt}}$$

(6 marks)