1. $A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$

$\det(A-\lambda I) = (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix}$

$= (1-\lambda)(-\lambda)(-1-\lambda)$

$\therefore$ eigenvalues are $\lambda = 1$, $\lambda = 0$ and $\lambda = -1$

when $\lambda = -1$: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$a + b = 0$
$0 = 0$
$a + 2c = 0$

choose $a = 2$ then $b = -2$
$c = -1$

$\underline{e}_1 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

$\lambda = 0$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & \phi \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left.\begin{matrix} b = 0 \\ -b = 0 \\ a + c = 0 \end{matrix}\right] \Rightarrow$ $b = 0$ $-a = c$

choose $a = 1$ then $c = -1$ $\underline{e}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = 1$ $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left.\begin{matrix} -a + b = 0 \\ -2b = 0 \\ a = 0 \end{matrix}\right] \Rightarrow$ $a = b = 0$
$c =$ anything
choose $c = 1$

$\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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b) $$\frac{d}{dt}\begin{pmatrix} i \\ v \\ q \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} i \\ v \\ q \end{pmatrix}$$

$$\frac{d}{dt}\underline{i} = A\underline{i}$$

Let $\underline{i} = P\underline{z}$, where $P =$ matrix eigenvectors $= \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}$

$$\frac{d}{dt}P\underline{z} = AP\underline{z} \Rightarrow \frac{d}{dt}\underline{z} = P^{-1}AP\underline{z} = D\underline{z}$$

where $D =$ matrix of eigenvalues $= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\frac{dz_1}{dt} = -z_1 \Rightarrow z_1 = Ae^{-t}$$
$$dz_2/dt = 0 \Rightarrow z_2 = B$$
$$dz_3/dt = z_3 \Rightarrow z_3 = Ce^{t}$$

Hence $$\underline{i} = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} Ae^{-t} \\ B \\ Ce^{t} \end{pmatrix}$$

$$\left.\begin{aligned} i &= 2Ae^{-t} + B \\ v &= -2Ae^{-t} \\ q &= -Ae^{-t} - B + Ce^{t}. \end{aligned}\right]$$ General Solution.

Initial conditions:

(i) $i(0) = -1 \quad v(0) = 2 \quad q(0) = 1$

$$-1 = 2A + B$$
$$2 = -2A \qquad \Rightarrow \quad A = -1 \quad B = 1 \quad C = 1$$
$$1 = -A - B + C$$

$$i = -2e^{-t} + 1 \qquad v = 2e^{-t} \qquad q = e^{-t} - 1 + e^{t}$$

i: dominant behavior is constant (unless $B=0$)
solution exhibits exponential decay to constant
v: exponential decay to zero.

q: dominant behavior is exponential growth, unless $C=0$

(ii) $i(0) = -1 \quad v(0) = 2 \quad q(0) = 0.$

$$-1 = 2A + B \qquad A = -1 \quad B = 1 \quad C = 0$$
$$2 = -2A$$
$$0 = -A - B + C$$

$$i = -2e^{-t} + 1 \qquad v = 2e^{-t} \qquad q = e^{-t} - 1$$

if $C=0$ q exhibits exponential decay to constant

Note if $A = C = 0$ then all solutions are constant

(a) $$\int_0^{\pi} x^2 \cos(nx)\,dx = \left[\frac{x^2 \sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} 2x \frac{\sin(nx)}{n}\,dx$$

$$= 0 - 0 - \frac{2}{n}\int_0^{\pi} x \sin(nx)\,dx$$

$$= -\frac{2}{n}\left\{\left[x\left(-\frac{\cos(nx)}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos(nx)}{n}\right)dx\right\}$$

$$= +\frac{2}{n}\pi\frac{\cos n\pi}{n} - \frac{2}{n^3}\left[\sin(nx)\right]_0^{\pi}$$

$$= \frac{2\pi}{n^2}(-1)^n + 0$$

(b) If $f(x) = x^2 - \pi^2/3$ then $f(-x) = f(x)$ so $f(x)$ is even.

Thus the F.S. contains only cosine and constant terms.

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(x^2 - \frac{\pi^2}{3}\right)dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3} - \frac{\pi^2}{3}x\right]_0^{\pi} = 0 - 0 = 0.$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(x^2 - \frac{\pi^2}{3}\right)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}(-1)^n\right) - \frac{2\pi^2}{3}\cancel{\left[+\sin nx\right]_0^{\pi}}^{\,0}$$

$$= \frac{4\pi}{n^2\pi}(-1)^n = \underline{\frac{4}{n^2}(-1)^n}$$

Thus the F.S is

$$\sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx).$$

(c) The first 3 terms are:

$$-4\cos x + \cos 2x - \frac{4}{9}\cos 3x.$$

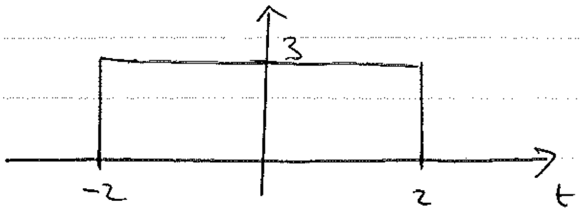
At $x=0$ this sum is

$$-4+1-4/9 = -3.44\dot{4}$$

as compared to the exact value for $f(x)$:

$$f(0) = -\pi^2/3 = -3.2899$$

3 a) $f(t) = 3\left(u(t+2) - u(t-2)\right)$



b) $f(t)$ is even because graph has reflective symmetry in $t=0$

c) $$F(\omega) = 2\int_0^{\infty} f(t)\cos(\omega t)\,dt$$

$$= 2\int_0^{2} 3\cos(\omega t)\,dt$$

$$= 6\left[\frac{\sin(\omega t)}{\omega}\right]_0^2$$

$$= 6\,\frac{\sin(2\omega)}{\omega}.$$

d) (i) $FT\{f(t+2)\} = e^{2j\omega}F(\omega)$ using Delay

$$= 6e^{2j\omega}\frac{\sin(2\omega)}{\omega}.$$

(ii) $FT\{f(4t-5)\} = FT\{f(4(t-5/4))\}$

$$= e^{-\frac{5j\omega}{4}}\,FT\{f(4t)\}$$ using Delay

$$= \frac{1}{4}e^{-\frac{5j\omega}{4}}F\left(\frac{\omega}{4}\right)$$ using scaling

$$= \frac{1}{4}e^{-5j\omega/4}\cdot 6\,\frac{\sin(\omega/2)}{\omega/4}$$

$$= 6\,e^{-5j\omega/4}\,\frac{\sin(\omega/2)}{\omega}$$

(iii) $FT\{e^{2jt} f(t)\} = F(\omega - 2)$

$$= 6 \frac{\sin(2(\omega-2))}{\omega - 2}$$

(iv) $FT\{\frac{df}{dt}\} = j\omega F(\omega)$

$$= 6j\omega \frac{\sin(2\omega)}{\omega} = 6j \sin(2\omega)$$

(v) $FT\{t f(t)\} = j \frac{dF}{d\omega}$

$$= j\, 6 \frac{d}{d\omega} \left\{ \frac{\sin(2\omega)}{\omega} \right]$$

$$= 6j \frac{d}{d\omega} \left\{ \omega^{-1} \sin(2\omega) \right\}$$

$$= 6j \left[-\omega^{-2} \sin(2\omega) + 2\omega^{-1} \cos(2\omega) \right]$$

$$= 6j \left[-\frac{\sin(2\omega)}{\omega^2} + \frac{2\cos(2\omega)}{\omega} \right].$$

e) $\int_0^\infty x^2 \, \delta(x+4) \, dx = 0$

$$\int_{-\infty}^{\infty} x^3 \, \delta(x-4) \, dx = 4^3 = 64$$