Eigenvalues and Eigenvectors

Revision of matrices

The order of a matrix

A matrix consists of **rows** and **columns**. For example

$$\begin{pmatrix} 4 & 3 \\ 2 & 7 \\ 1 & 5 \end{pmatrix}$$

has 3 rows and 2 columns and is a 3×2 or 3 by 2 matrix. If there are m rows and n columns the **order** of the matrix is $m \times n$ or m by n.

Products of matrices

Matrices can only be multiplied if they are **compatible**. The test for this is as follows: Write down the orders of the two matrices in the order they are to be multiplied. They are compatible if the middle two numbers are the same. The outer two numbers give the order of the resulting matrix. When multiplying matrices, the element ij in the resultant is found by multiplying the i^{th} row by the j^{th} column.

Example

Let

$$A = \begin{pmatrix} 3 & -4 & 2 \\ 2 & 3 & 4 \\ 1 & -1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 0 \end{pmatrix}.$$

The product $A.B$ is a 3 by 3 matrix multiplied by a 3 by 2 matrix. This is compatible and the result is a 3 by 2 matrix:

$$\begin{aligned} A.B = \begin{pmatrix} 3 & -4 & 2 \\ 2 & 3 & 4 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 0 \end{pmatrix} &= \begin{pmatrix} 3\times 1 + (-4)\times 3 + 2\times 4 & 3\times 2 + (-4)\times 1 + 2\times 0 \\ 2\times 1 + 3\times 3 + 4\times 4 & 2\times 2 + 3\times 1 + 4\times 0 \\ 1\times 1 + (-1)\times 3 + 3\times 4 & 1\times 2 + (-1)\times 1 + 2\times 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 27 & 7 \\ 10 & 1 \end{pmatrix} \end{aligned}$$

Note that it is not possible to calculate $B.A$ because we have a 3 by 2 matrix multiplying a 3 by 3 matrix, which is not compatible.

Solutions of systems of equations

Consider an $n \times n$ system of equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots a_{nn}x_n &= b_n \end{aligned}$$

This can be written in matrix form as

$$A\boldsymbol{x} = \boldsymbol{b}, \qquad \text{with solution} \qquad \boldsymbol{x} = A^{-1}\boldsymbol{b},$$

provided the inverse of the matrix, A^{-1}, exists. This requires det $(A) \neq 0$. If the right-hand side is identically zero, $\boldsymbol{b} = \mathbf{0}$, then the **trivial** solution is $\boldsymbol{x} = \mathbf{0}$. However, **non-trivial** results may be obtained provided $\det(A) = 0$.

The indentity matrix

The **identity matrix**, I, is a square matrix which has 1's on its diagonal and 0's elsewhere. For example, the 2 by 2 and 3 by 3 identity matrices are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Transpose

The **transpose** of a matrix A, denoted by A^T, is found by interchanging rows and columns. For example,

$$\begin{pmatrix} 2 & 4 & 2 \\ 3 & 1 & 3 \\ 1 & 2 & 5 \end{pmatrix}^T = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 2 & 3 & 5 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 3 \end{pmatrix}.$$

Determinants

2 by 2 determinants

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{then} \qquad \det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Examples

1. If $A = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}$ then $\det(A) = 5 \times 3 - 2 \times (-1) = 17$

2. $\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 4 \times 1 - 2 \times 2 = 0.$

3 by 3 determinants

The determinant of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

can be found by expanding about **any** row or column. If, for example, we expand about the top row then

$$\det\,(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Using the middle row gives

$$\det\,(A) = -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

The inverse of a 2 by 2 matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the **inverse matrix** A^{-1} is given by

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

provided $\det(A) \neq 0$.

Examples

1. If $A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ then $A^{-1} = -\frac{1}{2} \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$
2. If $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ then A^{-1} does **not** exist because $\det(A) = 0$.

Vectors

Component Form

A vector, $\boldsymbol{v}$ or $\underline{v}$, with components v_1, v_2 and v_3 is written as

$$\boldsymbol{v} = \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Length of a vector

The **length** or **norm** of a vector, denoted by $|\boldsymbol{v}|$ or v is given by

$$|\boldsymbol{v}| = v = \sqrt{v_1^2 + v_2^2 + v_3^3}$$

Example

If $\boldsymbol{v} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ then $|\boldsymbol{v}| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$.

The Dot Product

The dot product of two vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ is defined as

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos(\theta),$$

where θ is the angle between the vectors. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are in component form

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

then

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example

If $\boldsymbol{v} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}$ then $\boldsymbol{v} \cdot \boldsymbol{u} = 4 \times -2 + 2 \times 0 + 1 \times 6 = -2$

Length of a vector using Dot Product

The length of a vector can be calculated using the dot product:

$$v = |\boldsymbol{v}| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$$

Example

If $\boldsymbol{v} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ then $\boldsymbol{v} \cdot \boldsymbol{v} = 4 \times 4 + (-2) \times (-2) + 1 \times 1 = 21$, so $|\boldsymbol{v}| = \sqrt{21}$

Normalised and Orthogonal Vectors

Normalised

A vector is said to be **normalised**, denoted by $\hat{\boldsymbol{v}}$, if its length is 1. For a general vector $\boldsymbol{v}$

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}.$$

Example

If $\boldsymbol{v} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ then $\hat{\boldsymbol{v}} = \frac{1}{\sqrt{21}} \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$.

Orthogonal

Two vectors are **orthogonal** if they are at right angles to each other. In this case the angle between them is 90^{O} and so $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

Example

If $\boldsymbol{v} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ then $\boldsymbol{v} \cdot \boldsymbol{u} = 2 \times 1 + (-1) \times 1 + 1 \times (-1) = 0$ and so $\boldsymbol{v}$ and $\boldsymbol{u}$ are orthogonal.

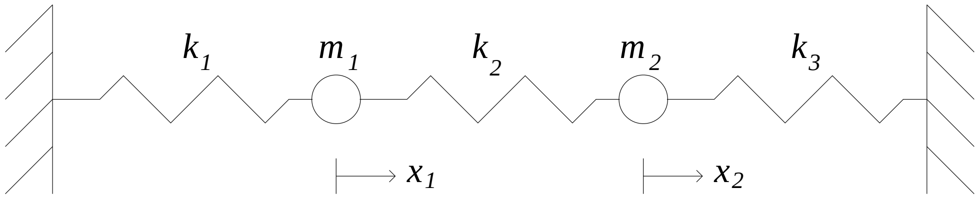
Orthonormal

A set of vectors are **orthonormal** if they are orthogonal and normalised.

The Eigenvalue/Eigenvectors problem

Eigenvalues and eigenvectors arise in various engineering contexts. Consider the following two applications:

Mechanics



Applying Newton's Second Law to each mass gives the following equations of motion:

$$m_1\frac{d^2x_1}{dt^2} = k_2(x_2 - x_1) - k_1x_1 \qquad \text{and} \qquad m_2\frac{d^2x_2}{dt^2} = -k_3x_2 - k_2(x_2 - x_1)$$

If $m_1 = m_2 = 1$ and $k_1 = k_2 = k_3 = 1$ these equations simplify to

$$\frac{d^2x_1}{dt^2} = -2x_1 + x_2 \qquad \text{and} \qquad \frac{d^2x_2}{dt^2} = x_1 - 2x_2$$

or in matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{or} \qquad \frac{d^2\boldsymbol{x}}{dt^2} = A\boldsymbol{x}, \qquad \text{where} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

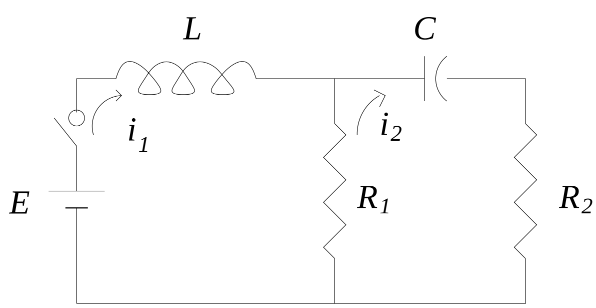
If we seek periodic solutions of the form $\boldsymbol{x} = \boldsymbol{x_0}e^{i\omega t}$ then the differential equation reduces to solving the system of equations

$$(A + \omega^2 I)\boldsymbol{x_0} = \mathbf{0}.$$

Writing $\omega^2 = -\lambda$, we arrive at the standard form of an eigenvalue problem:

$$(A - \lambda I)\boldsymbol{x_0} = \mathbf{0}.$$

Electrical



Applying Kirchhoff's Second Law to each loop gives

$$E = L\frac{di_1}{dt} + R_1(i_1 - i_2) \qquad \text{and} \qquad R_2i_2 + R_1(i_2 - i_1) + \frac{1}{C}\int^t i_2\,dt = 0.$$

If $L = 1$, $C = 0.5$, $R_1 = 1$ and $R_2 = 2$ the equations governing this circuit become

$$3\frac{di_1}{dt} = 2i_1 - 2Q_2 + 3E \qquad \text{and} \qquad 3\frac{dQ_2}{dt} = i_1 - 2Q_2,$$

where we have introduced $Q_2 = \int^t i_2 \, dt$. In matrix form these equations are

$$3\frac{d}{dt}\begin{pmatrix} i_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} i_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 3E \\ 0 \end{pmatrix} \qquad \text{or} \qquad \frac{d\boldsymbol{i}}{dt} = A\boldsymbol{i} + \boldsymbol{c},$$

where $\boldsymbol{i} = \begin{pmatrix} i_1 \\ Q_2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 3E \\ 0 \end{pmatrix}$. If we seek solutions of the form $\boldsymbol{i} = \boldsymbol{x_0}e^{\lambda t}$ then the homogeneous differential equation ($E = 0$) reduces to solving the equation

$$(A - \lambda I)\boldsymbol{x_0} = \boldsymbol{0}.$$

Eigenvalues and Eigenvectors

An eigenvalue problem is one where we look for non-trivial solutions of the matrix system

$$A\boldsymbol{x} = \lambda\boldsymbol{x}, \qquad \text{or} \qquad (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}.$$

These solutions will only exist if

$$\det\,(A - \lambda I) = 0,$$

which is known as the **characteristic equation**. The values of λ which satisfy this equation are known as the **eigenvalues** of the matrix A. The corresponding solutions $\boldsymbol{x}$ are called the **eigenvectors**.

Example 1

Determine the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Example 2

Determine the eigenvalues and normalised eigenvectors of the matrix $A = \begin{pmatrix} 2 & 6 \\ -2 & -5 \end{pmatrix}$.

Example 3

Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}.$$

Example 4

Determine the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -3 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Example 5

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 5 & 3 \\ 2 & 11 & 6 \\ 1 & -5 & 0 \end{pmatrix}.$$

Important Result

Consider any square matrix A. Let D be a diagonal matrix, which contains the corresponding eigenvalues of A and let P be a matrix whose columns are the eigenvectors of A. It may be shown that

$$P^{-1}AP = D.$$

Example

Verify this result for example 1 above, using

$$\boldsymbol{x_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and } \boldsymbol{x_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where the corresponding eigenvalues were $\lambda_1 = 1$ and $\lambda_2 = 3$.

Revision of differential equations

First order differential equations

The first order differential equation

$$\frac{dy}{dt} = ky, \qquad (k \text{ constant}), \qquad y(t_0) = y_0$$

has general solution

$$y(t) = Ae^{kt},$$

where A is determined using the initial conditions.

Example

Find the solution of $\dfrac{dy}{dt} = -3y, \quad y(0) = 2$

Exponential growth and decay

Reminder The behaviour of the solution is governed by its **dominant term**. This is the term which has the **largest** exponential. If the largest exponential is positive, then we have exponential growth and if it is negative we have exponential decay. For example

$x = 5e^{6t} - 7e^{-8t}$	dominant term is e^{6t}	exponential growth
$y = 5^{-4t} + 6e^{-5t}$	dominant term is e^{-4t}	exponential decay
$z = 6 - 6e^{-2t}$	dominant term is 6	constant behavior

Second order differential equations

The second order differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad (\omega \text{ constant}), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

has general solution

$$y(t) = A\cos(\omega t) + B\sin(\omega t),$$

where A and B are determined using the initial conditions.
The second order differential equation

$$\frac{d^2y}{dt^2} - k^2 y = 0, \qquad (k \text{ constant}), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

has general solution

$$y(t) = Ae^{kt} + Be^{-kt} \qquad \text{or} \qquad y = C\cosh(kt) + D\sinh(kt),$$

where A and B (or C and D) are determined using the initial conditions.

Example

Find the solution of $\dfrac{d^2y}{dt^2} = -4y, \quad y(0) = 2,\ y'(0) = 3$.

Example

Find the solution of $\dfrac{d^2y}{dt^2} = 4y, \quad y(0) = 2,\ y'(0) = 3$.

Systems of equations

General Theory

Consider the coupled pair of first order differential equations

$$\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

Writing these equations in matrix form gives

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x},$$

where $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As they stand, these equations cannot be immediately solved. Introduce a change of variable $\boldsymbol{x} = P\boldsymbol{z}$, where P is some constant matrix, so that the differential equation becomes

$$P\frac{d\boldsymbol{z}}{dt} = AP\boldsymbol{z} \qquad \text{or} \qquad \frac{d\boldsymbol{z}}{dt} = P^{-1}AP\boldsymbol{z}.$$

If we **choose** P to be the matrix of eigenvectors, then $P^{-1}AP = D$, where D is a diagonal matrix containing the eigenvalues, so that

$$\frac{d\boldsymbol{z}}{dt} = D\boldsymbol{z},$$

or in component form

$$\frac{dz_1}{dt} = \lambda_1 z_1 \qquad \frac{dz_2}{dt} = \lambda_2 z_2.$$

The system of differential equations has been **decoupled** into two separate differential equations which can be independently solved. Having found the solution for $\boldsymbol{z}$ we can determine the **General Solution** for $\boldsymbol{x}$ using $\boldsymbol{x} = P\boldsymbol{z}$. If any initial conditions are given they are applied at this stage to determine the constants of integration. This analysis is easily extended to include three or more coupled first order differential equations; or coupled systems of second order differential equations.

Example 6

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & -3 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$. Hence find the general solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 3x - 3y \\ \frac{dy}{dt} &= 2x - 4y \\ \frac{dz}{dt} &= -4z \end{aligned}$$

Determine the solutions of the differential equations for the following initial conditions and discuss the differences in the behaviour of the solutions obtained:

$$\begin{array}{llll} (i) & x(0) = 1 & y(0) = 2 & z(0) = 4 \\ (ii) & x(0) = 4 & y(0) = 3 & z(0) = 4 \end{array}$$

Example 7

Solve the pair of second order differential equations

$$\begin{aligned} \frac{d^2x}{dt^2} &= 2x + 6y, \qquad x(0) = 1, \quad x'(0) = 0 \\ \frac{d^2y}{dt^2} &= -2x - 5y, \qquad y(0) = 0, \quad y'(0) = 1 \end{aligned}$$

Simultaneous differential equations

Consider:

$$\begin{aligned} \frac{dx}{dt} - \frac{dy}{dt} &= 4x - 7y, \\ 2\frac{dx}{dt} - 3\frac{dy}{dt} &= 2x - 3y, \end{aligned}$$

These differential equations can be written in matrix form as

$$C\dot{\boldsymbol{x}} = D\boldsymbol{x},$$

where

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 4 & -7 \\ 2 & -3 \end{pmatrix}.$$

We can reduce the system of equations to **normal form** as

$$\dot{\boldsymbol{x}} = C^{-1}D\boldsymbol{x} = \begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix} \boldsymbol{x},$$

which can be solved using the theory presented earlier.

Higher order differential equations

Higher order differential equations can be reduced to coupled systems of first order differential equations

Example 8

Reduce

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

to a coupled system of first order equations.

Example 9

Solve the differential equation

$$y''' - 6y'' + 11y' - 6y = 0$$

Systems of Higher order equations

Let A be a square matrix and assume we have solved the eigenvalue problem so that

$$A\boldsymbol{x}_j = \lambda_j \boldsymbol{x}_j, \qquad \text{and} \qquad P^{-1}AP = D,$$

where P is the matrix whose columns are the eigenvectors and D is the diagonal matrix with elements $D_{jj} = \lambda_j$. This implies that for any power of A we have

$$P^{-1}A^nP = P^{-1}APP^{-1}A^{n-1}P = DP^{-1}A^{n-1}P = \ldots = D^n,$$

where D^n is the diagonal matrix with elements λ_j^n. Furthermore if $f(A)$ is any polynomial in A we have

$$P^{-1}f(A)P = f(D),$$

where $f(D)$ is a diagonal matrix with elements $f(\lambda_j)$.

Thus if we have the differential system

$$g(A)\frac{d^2\boldsymbol{y}}{dt^2} + f(A)\frac{d\boldsymbol{y}}{dt} + h(A)\boldsymbol{y} = \boldsymbol{0},$$

where $f(A), g(A)$ and $h(A)$ are polynomials in A then we can write $\boldsymbol{y} = P\boldsymbol{x}$, and multiply the entire equation by P^{-1} to obtain

$$g(D)\frac{d^2\boldsymbol{x}}{dt^2} + f(D)\frac{d\boldsymbol{x}}{dt} + h(D)\boldsymbol{x} = \boldsymbol{0}.$$

For each component we therefore have

$$g(\lambda_j)\frac{d^2x_j}{dt^2} + f(\lambda_j)\frac{dx_j}{dt} + h(\lambda_j)x_j = 0.$$

This is an equation with constant coefficients and can be solved generally or with initial conditions. If $g = 0, f = 1$ and $h = -A$ we recover the earlier theory.

Example 10

Find the eigenvalues of $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Hence solve the following differential systems.

a. Find the general solution of
$$\frac{d^2\boldsymbol{y}}{dt^2} + A\boldsymbol{y} = \mathbf{0}.$$

b. Find the particular solution in (a) where $\boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{y'}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. Find the general solution of
$$\frac{d^2\boldsymbol{y}}{dt^2} - \frac{1}{2}A\frac{d\boldsymbol{y}}{dt} + A\boldsymbol{y} = \mathbf{0}.$$
and describe the behaviour for large t.

Properties of Eigenvalues

Consider a matrix A with eigenvalues λ_1, λ_2, $\ldots$, λ_n

1. The sum of the eigenvalues is equal to the sum of the diagonal elements of the matrix A. The sum of the diagonal elements of a matrix is known as the **trace** of the matrix, or trace(A).
2. The product of the eigenvalues is equal to $\det(A)$.
3. The eigenvalues of A^{-1} (if it exists) are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.
4. The eigenvalues of A^T are the same as the eigenvalues of A.
5. If k is a scalar, then the eigenvalues of kA are $k\lambda_1$, $k\lambda_2$, $\ldots$, $k\lambda_n$.
6. If k is a scalar and I the identity matrix, then the eigenvalues of $A \pm kI$ are $\lambda_1 \pm k$, $\lambda_2 \pm k$, $\ldots$, $\lambda_n \pm k$.
7. If k is a positive integer, then the eigenvalues of A^k are λ_1^k, λ_2^k, $\ldots$, λ_n^k.

Symmetric matrices

For a symmetric matrix we have $a_{ij} = a_{ji}$ and the eigenvalues and eigenvectors have special properties

1. The eigenvalues are always real
2. The eigenvectors can be chosen to be real
3. If the eigenvalues are distinct then the eigenvectors are **othogonal** (the vectors are perpendicular to each other ie their dot product is zero.)
4. If the eigenvalues are repeated, then it is always possible to determine a distinct (and orthogonal) eigenvector for each repeated root.

The Cayley-Hamilton Theorem

Let A be an $n \times n$ matrix with characteristic polynomial $p(\lambda)$. Then

$$p(A) = 0$$

ie a matrix satisfies its own characteristic polynomial.

Example 11

Verify the Cayley-Hamilton Theorem for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Example 12

For the matrix

$$A = \begin{pmatrix} 2 & 5 & 3 \\ 2 & 11 & 6 \\ 1 & -5 & 0 \end{pmatrix}.$$

Find A^2 and hence calculate A^4 and A^{-1} using the Cayley-Hamilton Theorem.

Eigenvalue Exercises

1. If

$$A = \begin{pmatrix} 3 & -4 & 2 \\ 2 & 3 & 4 \\ 1 & -1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & -3 & -4 \\ 2 & 2 & -1 \\ 4 & 1 & 3 \end{pmatrix}.$$

   Find (i) $D.B$, (ii) $D.C$, (iii) $D.A$.

2. Calculate

$$(i) \quad \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix}, \qquad (ii) \quad \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix}, \qquad (iii) \quad \begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix}.$$

3. Evaluate

$$(i) \quad \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & -1 & 3 \end{vmatrix}, \qquad (ii) \quad \begin{vmatrix} 3 & -1 & 0 \\ 4 & 7 & 0 \\ 0 & 0 & 4 \end{vmatrix}, \qquad (iii) \quad \begin{vmatrix} 4 & 2 & 0 \\ 1 & 3 & 4 \\ 2 & 2 & 0 \end{vmatrix}.$$

4. If $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = -2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{c} = 4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ find

$$(i) \quad \boldsymbol{a} \cdot \boldsymbol{a} \qquad (ii) \quad \boldsymbol{a} \cdot \boldsymbol{c} \qquad (iii) \quad \boldsymbol{b} \cdot \boldsymbol{c}$$

5. Normalise the following vectors:

$$(i) \quad 6\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k} \qquad (ii) \quad -2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} \qquad (iii) \quad -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$$

6. Obtain the eigenvalues and eigenvectors, normalised to unit length, for each of the following matrices:
$$(a)\quad \begin{pmatrix} 1 & -8 \\ 2 & 11 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \qquad (c)\quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$
Verify that the eigenvectors for (c) are orthogonal.

7. Obtain the eigenvalues and a set of normalised eigenvectors for each of the following matrices
$$(a)\quad \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (c)\quad \begin{pmatrix} 3 & 1 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & -4 \end{pmatrix}$$
$$(d)\quad \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix} \qquad (e)\quad \begin{pmatrix} 5 & 0 & 6 \\ 0 & 11 & 6 \\ 6 & 6 & -2 \end{pmatrix} \qquad (f)\quad \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

8. Obtain the eigenvalues and a set of orthonormal eigenvectors for $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

9. Given the matrix
$$\begin{pmatrix} 4 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -4 \end{pmatrix},$$
show that the equation for determining the eigenvalues is $(2-\lambda)(\lambda^2 - 25) = 0$. Hence find the eigenvalues and a set of corresponding normalised eigenvectors.

10. Find the eigenvalues and eigenvectors of
$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$
and write down the matrices P and D. Hence verify the result $P^{-1}AP = D$.

11. A particle moves in a plane so that its displacement (x, y) satisfies the following system of equations:
$$\frac{dx}{dt} = x - 8y \qquad \frac{dy}{dt} = 2x + 11y$$
with $x = y = 1$ when $t = 0$. Use the results of 6(a) above to solve this system.

12. Find the eigenvalues and eigenvectors of
$$\begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}$$
Hence solve the system of equations:
$$\frac{di_1}{dt} = i_1 - i_2, \qquad \frac{di_2}{dt} = -4i_1 + i_2,$$
where $i_1(0) = 1$ and $i_2(0) = 0$.

13. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and hence find the general solution of the system of equations:

$$\begin{aligned} \frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= 2x + 5y \\ \frac{dz}{dt} &= 3z \end{aligned}$$

14. (a) Show that the eigenvalues of

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & -4 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

are $\lambda_1 = -4$, $\lambda_2 = -2$ and $\lambda_3 = 5$ and hence determine a set of eigenvectors.

(b) The currents i_1, i_2 and i_3 in a circuit satisfy

$$\begin{aligned} \frac{di_1}{dt} &= 2i_1 + 3i_3 \\ \frac{di_2}{dt} &= -4i_2 \\ \frac{di_3}{dt} &= 4i_1 + i_3 \end{aligned}$$

Using your answers to (a) determine the general solution for i_1, i_2 and i_3.

(c) Discuss the differences in the behaviour of the solutions if the following initial conditions are imposed:

$$\begin{array}{llll} (i) & i_1(0) = 1 & i_2(0) = 4 & i_3(0) = 8 \\ (ii) & i_1(0) = 0 & i_2(0) = 4 & i_3(0) = 0 \end{array}$$

15. Find the general solutions of the following systems of second order differential equations:

$$(i) \quad \begin{aligned} \frac{d^2x}{dt^2} &= x - 8y \\ \frac{d^2y}{dt^2} &= 2x + 11y \end{aligned} \qquad (ii) \quad \begin{aligned} \frac{d^2x}{dt^2} &= x + 2y \\ \frac{d^2y}{dt^2} &= 3x + 2y \end{aligned} \qquad (iii) \quad \begin{aligned} \frac{d^2x}{dt^2} &= -y \\ \frac{d^2y}{dt^2} &= 2x - 3y \end{aligned}$$

16. Express the following in normal form

$$(i) \quad \begin{aligned} -2\dot{y} + \dot{z} &= y + z \\ 3\dot{y} - \dot{z} &= 4z \end{aligned} \qquad (ii) \quad \begin{aligned} \dot{w} + 2\dot{y} \phantom{{}-\dot{z}} &= w + z \\ 2\dot{w} + \dot{y} - \dot{z} &= y + z \\ 3\dot{w} + \dot{y} + \dot{z} &= w + y \end{aligned}$$

17. Show by reducing the differential equations to a system of first order equations that the general solution of

(a) $y''' + y'' + 3y' - 5y = 0$ is $y = C_1 e^x + e^{-x}\left(C_2 \sin(2x) + C_3 \cos(2x)\right)$ and

(b) $y''' - 8y = 0$ is $y = C_1 e^{2x} + e^{-x}\left(C_2 \sin(\sqrt{3}x) + C_3 \cos(\sqrt{3}x)\right)$

18. For a general matrix A show by considering $A^{-1}A = I$ or otherwise that

$$P^{-1}A^{-1}P = D^{-1}$$

where P is the matrix whose columns are the eigenvectors of A and D is the diagonal matrix whose elements are the corresponding eigenvalues. Hence find the general solution of

$$A\frac{d\boldsymbol{y}}{dt} - \boldsymbol{y} = \boldsymbol{0}$$

where $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

19. Determine the orthonormal eigenvectors of

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Let U be the matrix whose columns are the orthonormal eigenvectors of A and show that $U^T AU$ is a diagonal matrix whose entries are the eigenvalues corresponding to the columns of U.

20. Given that A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ find the eigenvalues of A^2 and $A - kI$ (ie prove properties 6 and 7).

21. Show that the matrices

$$A = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

satisfy their characteristic equations.

22. Use the Cayley-Hamilton Theorem to evaluate A^2, A^3 and A^4 if

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

23. Find B^{-1} using the Cayley-Hamilton Theorem for

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}.$$

24. Given

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

compute A^2 and use the Cayley-Hamilton Theorem to evaluate $A^7 - 3A^6 + A^4 + 3A^3 - 2A^2 + 3I$.

25. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 5 & -6 \\ -2 & 4 & -4 \end{pmatrix}.$$

If the eigenvalues of the 3×3 matrix B are 0, 2 and -2 and $B^3 = A^4$ show that $4B = 5A^2 - 4I$.

Answers

1. (i) $DB = \begin{pmatrix} -13 \\ 5 \\ 6 \end{pmatrix}$ (ii) $DC = \begin{pmatrix} -24 & -1 \\ 4 & 6 \\ 19 & 9 \end{pmatrix}$ (iii) $DA = \begin{pmatrix} -7 & -9 & -22 \\ 9 & -1 & 9 \\ 17 & -16 & 21 \end{pmatrix}$

2. (i) 10 (ii) 17 (iii) 11

3. (i) 30 (ii) 100 (iii) -16

4. (i) 18 (ii) 19 (iii) -3

5. (i) $\frac{1}{\sqrt{61}}(6\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k})$ (ii) $\frac{1}{3}(-2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$ (iii) $\frac{1}{\sqrt{3}}(-\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

6. (a) $\lambda_1 = 3,\ \lambda_2 = 9;\ \boldsymbol{x_1} = \frac{1}{\sqrt{17}}\begin{pmatrix} -4 \\ 1 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

   (b) $\lambda_1 = 2,\ \lambda_2 = 3;\ \boldsymbol{x_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \boldsymbol{x_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

   (c) $\lambda_1 = 0,\ \lambda_2 = 2;\ \boldsymbol{x_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix};\ \boldsymbol{x_1} \cdot \boldsymbol{x_2} = 0$

7. (a) $\lambda_1 = 1,\ \lambda_2 = 2;\ \lambda_3 = 3;\ \boldsymbol{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}\ \boldsymbol{x_3} = \frac{1}{\sqrt{989}}\begin{pmatrix} 29 \\ 12 \\ 2 \end{pmatrix}$

   (b) $\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 4;\ \boldsymbol{x_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\ \boldsymbol{x_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

   (c) $\lambda_1 = -4,\ \lambda_2 = 1,\ \lambda_3 = 4;\ \boldsymbol{x_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix};\ \boldsymbol{x_3} = \frac{1}{\sqrt{137}}\begin{pmatrix} 8 \\ 8 \\ 3 \end{pmatrix};$

   (d) $\lambda_1 = -3,\ \lambda_2 = 3,\ \lambda_3 = 9;\ \boldsymbol{x_1} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix};\ \boldsymbol{x_3} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix};$

   (e) $\lambda_1 = -7,\ \lambda_2 = 7,\ \lambda_3 = 14;\ \boldsymbol{x_1} = \frac{1}{7}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{7}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix};\ \boldsymbol{x_3} = \frac{1}{7}\begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix};$

   (f) $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 5;\ \boldsymbol{x_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix};\ \boldsymbol{x_3} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};$

8. $\lambda_1 = -1,\ \lambda_2 = \lambda_3 = 1;\ \boldsymbol{x_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ \boldsymbol{x_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};\ \boldsymbol{x_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$

9. $\lambda_1 = -5,\ \lambda_2 = 2,\ \lambda_3 = 5;\ \boldsymbol{x_1} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},\ \boldsymbol{x_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};\ \boldsymbol{x_3} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix};$

10. $\lambda_1 = 4,\ \lambda_2 = -1;\ \boldsymbol{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},\ \boldsymbol{x_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

11. General solution: $x = -4Ae^{3t} + Be^{9t}, \quad y = Ae^{3t} - Be^{9t}$

    Specific solution: $x = \frac{8}{3}e^{3t} - \frac{5}{3}e^{9t}, \quad y = -\frac{2}{3}e^{3t} + \frac{5}{3}e^{9t}$

12. $\lambda_1 = -1,\ \lambda_2 = 3;\ \boldsymbol{x_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \boldsymbol{x_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

    General solution: $i_1 = pe^{3t} + qe^{-t}, \quad i_2 = -2pe^{3t} + 2qe^{-t}$

    Specific solution: $i_1 = \frac{1}{2}(e^{-t} + e^{3t}), \quad i_2 = e^{-t} - e^{3t}$

13. $\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 6;\ \boldsymbol{x_1} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},\ \boldsymbol{x_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \boldsymbol{x_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix};$

    General solution: $x = 2Ae^{t} + Ce^{6t}; \quad y = -Ae^{t} + 2Ce^{6t}, \quad z = Be^{3t}$

14. $\boldsymbol{x_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \boldsymbol{x_2} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},\ \boldsymbol{x_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix};$

    General solution: $i_1 = 3Be^{-2t} + Ce^{5t}; \quad i_2 = Ae^{-4t}, \quad i_3 = -4Be^{-2t} + Ce^{5t}$

    (i) $i_1 = -3e^{-2t} + 4e^{5t}; \quad i_2 = 4e^{-4t}, \quad i_3 = 4e^{-2t} + 4e^{5t}$

    (ii) $i_1 = i_3 = 0, \quad i_2 = 4e^{-4t}$

15. (a) $x = -4Ae^{\sqrt{3}t} - 4Be^{-\sqrt{3}t} + Ce^{3t} + De^{-3t}$
    $y = Ae^{\sqrt{3}t} + Be^{-\sqrt{3}t} - Ce^{3t} - De^{-3t}$

    (b) $x = 2Ae^{2t} + 2Be^{-2t} + C\cos(t) + D\sin(t)$
    $y = 3Ae^{2t} + 3Be^{-2t} - C\cos(t) - D\sin(t)$

    (c) $x = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t) + C\cos(t) + D\sin(t)$
    $y = 2A\cos(\sqrt{2}t) + 2B\sin(\sqrt{2}t) + C\cos(t) + D\sin(t)$

16. (a) $\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 3 & 11 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$

    (b) $\begin{pmatrix} \dot{w} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 0 \\ 2 & -1 & 2 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} w \\ y \\ z \end{pmatrix}$

18. $y_1 = Ae^{\frac{t}{2}} + Be^{\frac{t}{4}},\ y_2 = Ae^{\frac{t}{2}} - Be^{\frac{t}{4}},\ y_3 = Ce^{\frac{t}{3}}$

19. $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = 3;\ \boldsymbol{x_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \boldsymbol{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ \boldsymbol{x_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};$

20. Eigenvalues of A^2 are λ_1^2, λ_2^2, $\ldots$, λ_n^2

Eigenvalues of $A - kI$ are $\lambda_1 - k$, $\lambda_2 - k$, $\ldots$, $\lambda_n - k$

22. $A^2 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$; $A^3 = \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix}$; $A^4 = \begin{pmatrix} 17 & 24 \\ 12 & 17 \end{pmatrix}$

23. $B^{-1} = \dfrac{1}{11}\begin{pmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{pmatrix}$

24. $A^7 = \begin{pmatrix} 47231 & 47342 & 47270 \\ 47342 & 47195 & 47306 \\ 47270 & 47306 & 47267 \end{pmatrix}$;

25. $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$

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