STAFFORDSHIRE
UNIVERSITY

Faculty of Computing, Engineering & Technology

Mathematical

Formulae

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Algebra

Quadratic Equations

Equation: $ax^2 + bx + c = 0$ Solution: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Indices

$a^m \times a^n = a^{m+n}$ $a^m \div a^n = a^{m-n}$

$(a^m)^n = (a^n)^m = a^{mn}$ $a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{1}{q}} = \sqrt[q]{a}$ $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$

$a^0 = 1$

Logarithms

Definition

$y = b^x \Leftrightarrow x = \log_b(y)$ $y = e^x \Leftrightarrow x = \ln(y)$

Laws of Logarithms

$\log_a(x) + \log_a(y) = \log_a(xy)$ $\log_a(x) - \log_a(y) = \log_a\left(\dfrac{x}{y}\right)$

$\log_a(x^n) = n\log_a(x)$ $\log_a\left(\dfrac{1}{x}\right) = -\log_a(x)$

$\log_a(a) = 1$ $\log_a(1) = 0$

Change of Base

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

Series

Arithmetic Progression

For the series

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

the n-th term is $a+(n-1)d$ and S_n, the sum of the first n terms in the series, is

$$S_n = \frac{n}{2}\left(2a+(n-1)d\right) = \frac{n}{2}(a+l)$$

where l is the n-th term

Geometric Progression

For the series

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

the n-th term is ar^{n-1} and S_n, the sum of the first n terms in the series is

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1.$$

For $-1 < r < 1$,

$$S_\infty = \frac{a}{1-r}$$

Binomial Theorem

For any positive integer n

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}a^{n-r}b^r + \ldots + b^n$$

The $(r+1)$-th term is ${}^nC_r a^{n-r}b^r$. An alternative formulation is

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^r b^{n-r}, \qquad \text{where} \qquad {}^nC_r = \frac{n!}{r!(n-r)!}$$

For $-1 < x < 1$ and for any n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Trigonometric Formulae

Definitions

$$\tan A = \frac{\sin(A)}{\cos(A)} \qquad \cot A = \frac{\cos A}{\sin A}$$

$$\sec A = \frac{1}{\cos A} \qquad \operatorname{cosec} A = \frac{1}{\sin A}$$

Identities

$$\cos^2 A + \sin^2 A = 1 \qquad \sec^2 A = 1 + \tan^2 A \qquad \operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(-A) = -\sin(A) \qquad \cos(-A) = \cos(A) \qquad \tan(-A) = -\tan(A)$$

Double Angle Formulae

$$\begin{aligned}
\sin(2A) &= 2\sin(A)\cos(A) \\
\cos(2A) &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A \\
\tan(2A) &= \frac{2\tan(A)}{1 - \tan^2 A}
\end{aligned}$$

Compound Angle Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A \cos B = \frac{1}{2}\big(\sin(A + B) + \sin(A - B)\big)$$

$$\cos A \cos B = \frac{1}{2}\big(\cos(A + B) + \cos(A - B)\big)$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

Half-Angle Tangent Formulae

If $t = \tan\left(\frac{A}{2}\right)$ then

$$\sin A = \frac{2t}{1+t^2} \qquad \cos A = \frac{1-t^2}{1+t^2} \qquad \tan A = \frac{2t}{1-t^2}$$

$$\frac{dA}{dt} = \frac{2}{1+t^2}$$

Euler's Formulae

$$\cos(A) = \frac{e^{jA} + e^{-jA}}{2} \qquad \sin(A) = \frac{e^{jA} - e^{-jA}}{2j}$$

Calculus Properties

Product Rule

If $y = uv$, where u and v are functions of x then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

If $y = \frac{u}{v}$, where u and v are functions of x then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain Rule (or Function of a Function)

If $y = f(g(x))$ and we substitute $t = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Integration by Parts

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

Leibnitz's Theorem

If $y = uv$, where u and v are functions of x then

$$\frac{d^n}{dx^n}(uv) = u\frac{d^n v}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{2!}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \cdots + \frac{d^n u}{dx^n}v$$

Table of Derivatives and Integrals

Function (Integral)	Derivative (Function)
k (constant)	0
x^n	nx^{n-1}
$\frac{x^{n+1}}{n+1}$	$x^n \quad (n \neq -1)$
$\log_e\lvert x\rvert$ or $\ln\lvert x\rvert$	$\frac{1}{x}$
$\ln(ax+b)$	$\frac{a}{ax+b}$
e^{ax}	ae^{ax}
$\sin(x)$	$\cos(x)$
$\sin(ax)$	$a\cos(ax)$
$\cos(x)$	$-\sin(x)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(x)$	$\sec^2(x)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\mathrm{cosec}^2(ax)$
$\mathrm{cosec}(ax)$	$-a\mathrm{cosec}(ax)\cot(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\ln(\sec(ax))$	$a\tan(ax)$
$\ln(\sin(ax))$	$a\cot(ax)$
$-\ln(\mathrm{cosec}(ax)+\cot(ax))$	$a\mathrm{cosec}(ax)$
$\ln(\sec(ax)+\tan(ax))$	$a\sec(ax)$

Numerical Schemes

Numerical Integration

Notation: $x_i = x_0 + ih$ and $y_i = f(x_i)$

Trapezium Rule

$$\int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}\{y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\}$$

Simpson's Rule

$n \geq 2$ must be even

$$\int_{x_0}^{x_n} f(x)dx \approx \frac{h}{3}\{y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2}) + y_n\}$$

Roots of equations

Newton Raphson

If x_n is an approximation to the root of $f(x) = 0$ then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is generally a better approximation.

Series Expansions

Maclaurin's Theorem

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

Series for elementary functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots \qquad -1 < x \le 1$$

Taylor's Theorem

1 Variable

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

2 Variables

$$f(a+h, b+k) = f(a,b) + h\left.\frac{\partial f}{\partial x}\right|_{(a,b)} + k\left.\frac{\partial f}{\partial y}\right|_{(a,b)} + \frac{1}{2!}\left(h^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)} + 2hk \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)} + k^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(a,b)} \right) + \cdots$$

Laplace Transform

Definition

$$L\{f(t)\} = \overline{f}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Properties

1. First Shift Theorem
$$L\{e^{at}f(t)\} = \overline{f}(s-a)$$
$$L^{-1}\{\overline{f}(s-a)\} = e^{at}f(t)$$

2. Derivatives
$$L\left\{\frac{df}{dt}\right\} = s\overline{f}(s) - f(0)$$
$$L\left\{\frac{d^2f}{dt^2}\right\} = s^2\overline{f}(s) - sf(0) - f'(0)$$

3. Change of Scale
$$L\{f(at)\} = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$$

4. Initial Value Theorem
$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}\{s\overline{f}(s)\}$$

5. Final Value Theorem
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\{s\overline{f}(s)\}$$

6. Integration Theorem
$$L\left\{\int_0^t f(x)dx\right\} = \frac{1}{s}\overline{f}(s)$$
$$L^{-1}\left\{\frac{1}{s}\overline{f}(s)\right\} = \int_0^t f(x)dx$$

7. Multiplication by t
$$L\{tf(t)\} = -\frac{d}{ds}\{\overline{f}(s)\}$$

8. Division by t
$$L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \overline{f}(x)dx$$

9. Periodic Function (period a)
$$L\{f(t)\} = \frac{1}{(1-e^{-sa})}\int_0^a f(t)e^{-st}\,dt$$

10. Second Shift Theorem
$$L\{f(t-a)u(t-a)\} = \overline{f}(s)e^{-as}$$
$$L^{-1}\{\overline{f}(s)e^{-as}\} = f(t-a)u(t-a)$$

11. Convolution
$$L\{f*g(t)\} = \overline{f}(s)\overline{g}(s)$$
$$L^{-1}\{\overline{f}(s)\overline{g}(s)\} = f*g(t)$$

where

$$f*g(t) = \int_0^t f(x)g(t-x)dx = \int_0^t f(t-x)g(x)dx$$

A Table of Laplace Transforms

Function	Transform
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$t\sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$\sin(bt)-bt\cos(bt)$	$\frac{2b^3}{(s^2+b^2)^2}$
$\frac{\cos(at)-\cos(bt)}{b^2-a^2} \quad (b^2 \neq a^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$

Note: An alternative notation for $u(t)$ is $H(t)=u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$

Fourier Series

1. Whole-range series

For an interval $(0,T)$

Series: $$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{2n\pi x}{T}\right)+b_n\sin\left(\frac{2n\pi x}{T}\right)\right\}$$

Coefficients: $$a_0=\frac{2}{T}\int_0^T f(x)dx$$

$$a_n=\frac{2}{T}\int_0^T f(x)\cos\left(\frac{2n\pi x}{T}\right)dx \qquad b_n=\frac{2}{T}\int_0^T f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$$

Special case of even function

Series: $$\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2n\pi x}{T}\right)$$

Coefficients: $$a_0=\frac{4}{T}\int_0^{\frac{T}{2}} f(x)dx \qquad a_n=\frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$$

Special case of odd function

Series: $$\sum_{n=1}^{\infty}b_n\sin\left(\frac{2n\pi x}{T}\right)$$ Coefficients: $$b_n=\frac{4}{T}\int_0^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$$

2.Half-range series

	Cosine Series	Sine Series
Series	$\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2n\pi x}{T}\right)$	$\sum_{n=1}^{\infty}b_n\sin\left(\frac{2n\pi x}{T}\right)$
Coefficients	$a_0=\frac{4}{T}\int_0^{\frac{T}{2}} f(x)dx$	
	$a_n=\frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$	$b_n=\frac{4}{T}\int_0^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$

Fourier Transform

Definition

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \text{with inverse} \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

If f(t) is an even function

$$F(\omega) = 2\int_{0}^{\infty} f(t)\cos(\omega t) dt$$

If f(t) is an odd function

$$F(\omega) = -2j\int_{0}^{\infty} f(t)\sin(\omega t) dt$$

Fourier Transform Properties

1.	Transformation	$f(t) \leftrightarrow F(\omega)$
2.	Linearity	$a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$
3.	Symmetry	$F(t) \leftrightarrow 2\pi f(-\omega)$
4.	Scaling	$f(at) \leftrightarrow \frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
5.	Delay	$f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(\omega)$
6.	Modulation	$e^{j\omega_0 t} f(t) \leftrightarrow F(\omega - \omega_0)$
7.	Convolution	$f_1 * f_2(t) \leftrightarrow F_1(\omega) F_2(\omega)$
8.	Multiplication	$f_1(t) f_2(t) \leftrightarrow \frac{1}{2\pi} F_1 * F_2(\omega)$
9.	Time Differentiation	$\frac{d^n}{dt^n} f(t) \leftrightarrow (j\omega)^n F(\omega)$
10.	Time Integration	$\int f(t) dt \leftrightarrow \frac{F(\omega)}{j\omega}$
11.	Frequency Differentiation	$tf(t) \leftrightarrow j\frac{dF}{d\omega}$
12.	Frequency Integration	$\frac{f(t)}{-jt} \leftrightarrow \int F(\omega') d\omega'$
13.	Reversal	$f(-t) \leftrightarrow F(-\omega)$

Useful Fourier Transforms (Energy Signals)

	Time Function, $f(t)$	Fourier Transform, $F(\omega)$	
1.	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2.	$-e^{at}u(-t)$	$\dfrac{1}{j\omega-a}$	$a>0$
3.	$te^{-at}u(t)$	$\left(\dfrac{1}{a+j\omega}\right)^2$	$a>0$
4.	$g_T(t)=\begin{cases}1, & \lvert t\rvert<\dfrac{T}{2}\\ 0, & \text{otherwise}\end{cases}$	$T\text{sinc}\left(\dfrac{\omega T}{2}\right)$	
5.	$\begin{cases}A\left(1-\dfrac{\lvert t\rvert}{T}\right) & \lvert t\rvert<T\\ 0 & \lvert t\rvert>T\end{cases}$	$AT\text{sinc}^2\left(\dfrac{\omega T}{2}\right)$	
6.	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	
7.	$e^{-at}\sin(\omega_0 t)u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	
8.	$e^{-at}\cos(\omega_0 t)u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	
9.	e^{-at^2}	$\sqrt{\dfrac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$	
10.	$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\dfrac{1}{(j\omega+a)^n}$	
11.	$\dfrac{1}{a^2+t^2}$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$	
12.	$\dfrac{\cos(bt)}{a^2+t^2}$	$\dfrac{\pi}{2a}\left\{e^{-a\lvert\omega-b\rvert}+e^{-a\lvert\omega+b\rvert}\right\}$	
13.	$\dfrac{\sin(bt)}{a^2+t^2}$	$\dfrac{\pi}{2aj}\left\{e^{-a\lvert\omega-b\rvert}-e^{-a\lvert\omega+b\rvert}\right\}$	
14.	$\cos(\omega_0 t)\left\{u\left(t+\dfrac{T}{2}\right)-u\left(t-\dfrac{T}{2}\right)\right\}$	$\dfrac{T}{2}\left\{\sin c\left(\dfrac{(\omega-\omega_0)T}{2}\right)+\sin c\left(\dfrac{(\omega+\omega_0)T}{2}\right)\right\}$	

Note: $\text{sinc}(x)=\dfrac{\sin(x)}{x}$

Useful Fourier Transform (Power Signals)

These are transforms used in convolution calculations. The convolution of $g(t)$ and $f(t)$ is

$$g * f(t) = \int_{-\infty}^{\infty} g(t-u) f(u) du = \int_{-\infty}^{\infty} f(t-u) g(u) du$$

For arbitrary $g(t)$ with Fourier Transform $G(\omega)$ we have the transforms:

$$g * f(t) \leftrightarrow G(\omega) F(\omega) \qquad g(t) f(t) \leftrightarrow \frac{1}{2\pi} G * F(\omega)$$

Time Function	Fourier Transform
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\operatorname{sgn}(t) = u(t) - u(-t)$	$\frac{2}{j\omega}$
e^{jat}	$2\pi\delta(\omega - a)$
$\sin(at)$	$\frac{\pi}{j}\left(\delta(\omega - a) - \delta(\omega + a)\right)$
$\cos(at)$	$\pi\left(\delta(\omega - a) + \delta(\omega + a)\right)$
$tu(t)$	$j\pi\delta'(\omega) - \frac{1}{\omega^2}$
t^n	$2\pi j^n \delta^{(n)}(\omega)$
$\lvert t \rvert$	$-\frac{2}{\omega^2}$
$\delta^{(n)}(t)$	$(j\omega)^n$

Z-Transforms

Definition

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$$

Table of transforms

$f(n)$	$F(z)$
a^n	$\dfrac{z}{z-a}$
n	$\dfrac{z}{(z-1)^2}$
n^2	$\dfrac{z(z+1)}{(z-1)^3}$
n^3	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$
$n^{(k)} = n(n-1)\cdots(n-k+1)$	$\dfrac{zk!}{(z-1)^{k+1}}$
$\dfrac{n(n+1)}{2}$	$\dfrac{z^2}{(z-1)^3}$
$\dfrac{n(n+1)(n+2)}{6}$	$\dfrac{z^3}{(z-1)^4}$
e^{an}	$\dfrac{z}{z-e^a}$
$u(n) = 1, \quad n \geq 0$	$\dfrac{z}{z-1}$
$\delta(n)$	1
$\delta(n-k)$	z^{-k}

Properties of Z-transforms

$$Z(f(n+1)) = z(F(z) - f(0))$$

$$Z(f(n+2)) = z^2\left(F(z) - f(0) - \frac{f(1)}{z}\right)$$

$$Z(f(n+k)) = z^k\left(F(z) - \sum_{n=0}^{k-1} f(n)z^{-n}\right)$$

$$Z(f(n-k)u(n-k)) = z^{-k}F(z)$$

$$Z(nf(n)) = -z\frac{d}{dz}F(z)$$

$$Z(a^{-n}f(n)) = F(az)$$

$Z\{f * g(n)\} = F(z)G(z)$, where $f * g(n) = \sum_{n=0}^{m} f(n)g(m-n)$

$$Z(na^n) = \frac{az}{(z-a)^2}$$