Engineering Mathematics 2

Exam

solutions

1. $A - \lambda I = \begin{pmatrix} 2-\lambda & 0 & 3 \\ 0 & -2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{pmatrix}$

$$\det(A-\lambda I) = (-2-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[(2-\lambda)(3-\lambda)-6\right]$$

$$= (-2-\lambda)(\lambda^2-5\lambda)$$

$$= (-2-\lambda)\,\lambda(\lambda-5)$$

$\therefore$ eigenvalues are $\lambda=-2$, $\lambda=0$ & $\lambda=5$.

for $\lambda=-2$:

$$\begin{pmatrix} 4 & 0 & 3 \\ 0 & 0 & 0 \\ 2 & 0 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 4a+3c&=0 \\ 0&=0 \\ 2a+5c&=0 \end{aligned}\right\} \Rightarrow a=c=0, \quad b=\text{anything}$$

Choose $b=1$ $\quad \underline{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda = 0$

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & -2 & 0 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2a + 3c &= 0 \\ -2b &= 0 \quad \Rightarrow \quad b = 0 \qquad 2a + 3c = 0 \\ 2a + 3c &= 0 \end{aligned}$$

Choose $c = -2$ then $a = 3$ $\therefore$ $e_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$.

For $\lambda = 5$

$$\begin{pmatrix} -3 & 0 & 3 \\ 0 & -7 & 0 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -3a + 3c &= 0 \\ -7b &= 0 \quad \Rightarrow \quad b = 0 \qquad a = c \\ 2a - 2c &= 0 \end{aligned}$$

Choose $a = 1$ $\Rightarrow$ $c = 1$ $\therefore$ $e_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

b) In matrix form:

$$\frac{d}{dt}\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & -2 & 0 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}$$

$$\frac{d}{dt}\underline{i} = A\,\underline{i}$$

Let $\underline{i} = P\underline{z}$ $\quad \frac{d}{dt} P\underline{z} = AP\underline{z}$

$$\frac{d}{dt} \underline{z} = P^{-1}AP\underline{z}$$

if we choose $P =$ matrix of eigenvectors

$$= \begin{pmatrix} 0 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

then $P^{-1}AP = D =$ diagonal matrix containing eigenvalues

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\therefore \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\frac{dz_1}{dt} = -2z_1 \quad \Rightarrow \quad z_1 = Ae^{-2t}$$

$$\frac{dz_2}{dt} = 0 \quad \Rightarrow \quad z_2 = B$$

$$\frac{dz_3}{dt} = 5z_3 \quad \Rightarrow \quad z_3 = Ce^{5t}$$

Hence

$$\underline{i} = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} Ae^{-2t} \\ B \\ Ce^{5t} \end{pmatrix} = \begin{pmatrix} 3B + Ce^{5t} \\ Ae^{-2t} \\ -2B + Ce^{5t} \end{pmatrix}$$

$$\left.\begin{aligned} i_1 &= 3B + Ce^{5t} \\ i_2 &= Ae^{-2t} \\ i_3 &= -2B + Ce^{5t} \end{aligned}\right\} \text{ General Solution.}$$

(i) $i_1(0) = 6 \qquad i_2(0) = 3 \qquad i_3(0) = -4.$

$$\begin{aligned} 6 &= 3B + C \\ 3 &= A \\ -4 &= -2B + C. \end{aligned} \qquad \Rightarrow \quad A = 3, \; B = 2 \quad C = 0.$$

$\therefore$ $i_1(t) = 6$ constant behaviour

$i_2(t) = 3e^{-2t}$ exponential decay to zero.

$i_3(t) = -4$ constant behaviour.

(ii) $i_1(0) = 7 \qquad i_2(0) = 3 \qquad i_3(0) = 2.$

$$\begin{aligned} 7 &= 3B + C \quad (1) \\ 3 &= A \\ 2 &= -2B + C. \quad (2) \end{aligned}$$

$(1) - (2)$ $5 = 5B$ $\therefore B = 1$

$\therefore C = 4$

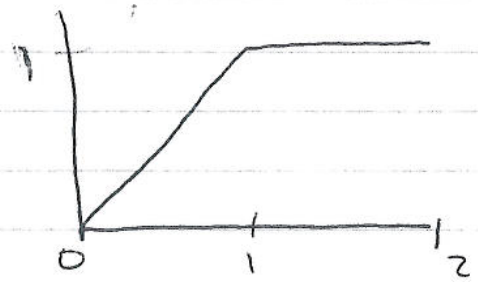
$A = 3$

$i_1(t) = 3 + 4e^{5t}$ exponential growth

$i_2(t) = 3e^{-2t}$ exponential decay to zero.

$i_3(t) = -2 + 4e^{5t}$ exponential growth.

(a) Given that $f(x) = x(u(x) - u(x-1)) + (u(x-1) - u(x-2))$
we have in $0 \le x \le 2$:



The period is 2.

(b)

$$I_n = \int_0^1 x \sin(n\pi x)\,dx + \int_1^2 \sin(n\pi x)\,dx =$$

$$\left[-\frac{x\cos(n\pi x)}{n\pi}\right]_0^1 + \left[-\frac{\cos(n\pi x)}{n\pi}\right]_1^2 \cancelto{0}{+\frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx}$$

$$-\frac{\cos \pi n}{\pi n} + \frac{\cos \pi n}{\pi n} - \frac{\cos 2\pi n}{n\pi} = -\frac{1}{n\pi}$$

(c)

$$J_n = \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 \cos(n\pi x)\,dx$$

$$\left[x\frac{\sin(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin(\pi n x)\,dx}{n\pi} + \left(\frac{\sin(n\pi x)}{n\pi}\right)_1^2$$

$$= 0 + \left(\frac{\cos(n\pi x)}{(n\pi)^2}\right)_0^1 + 0 = \frac{(-1)^n - 1}{n^2\pi^2}.$$

(d) $$J_0 = \int_0^1 x\,dx + \int_1^2 dx = \left[\frac{x^2}{2}\right]_0^1 + [x]_1^2$$

$$= \frac{1}{2} + 1 = 3/2$$

(e) The Fourier series is

$$\frac{3}{4} + \sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n - 1}{n^2\pi^2}\right)\cos(n\pi x) + \left(\frac{-1}{n\pi}\right)\sin(n\pi x)\right].$$

(f) At $x = 0$ the series converges to $\frac{1}{2}(0+1) = 1/2$.

a) $FT(t f(t)) = \int_0^\infty t e^{-2t} e^{-j\omega t} dt = \int_0^\infty t e^{-(2+j\omega)t} dt$

$$= \left[-\frac{t e^{-(2+j\omega)t}}{(2+j\omega)} \right]_0^\infty + \int_0^\infty \frac{e^{-(2+j\omega)t}}{2+j\omega} dt$$

$$= 0 + \left[-\frac{e^{-(2+j\omega)t}}{(2+j\omega)^2} \right]_0^\infty = \frac{1}{(2+j\omega)^2}.$$

$$FT(f(t)) = \int_0^\infty e^{-2t} e^{-j\omega t} dt = \left[-\frac{e^{-(2+j\omega)t}}{2+j\omega} \right]_0^\infty$$

$$= \frac{1}{2+j\omega}.$$

This verifies the result $FT\{t f(t)\} = +j \frac{d}{d\omega} FT\{f(t)\}$

since $+j \frac{d}{d\omega} \frac{1}{(2+j\omega)} = +j \left(\frac{-j}{(2+j\omega)^2} \right) = \frac{1}{(2+j\omega)^2}$

(b) $FT\{g(t)\} = 2 \int_0^h \frac{1}{2h} \cos \omega t \, dt = \frac{2}{2h} \left[\frac{\sin \omega t}{\omega} \right]_0^h$

$$= \frac{2}{2h} \frac{\sin(\omega h)}{\omega} = \operatorname{sinc}(\omega h).$$

$$= G(\omega)$$

(i) $FT\{g(t+2)\} = e^{2j\omega} \operatorname{sinc}(\omega h)$

(ii) $FT(g(t-2)) = e^{-2j\omega} \operatorname{sinc}(\omega h)$

(iii) $FT(g(3t+2)) = FT\{g(3(t+\frac{2}{3}))\} = e^{\frac{2}{3}j\omega} \frac{1}{3} \operatorname{sinc}(\frac{\omega h}{3})$

(iv) $FT^{-1}\{G(2\omega+3)\} = FT^{-1} G(2(\omega+\frac{3}{2})) = e^{-\frac{3}{2}jt} \frac{1}{2} \frac{1}{2h} X(t)$

$$X(t) = u(t/2+h) - u(t/2-h).$$

(v) $FT\{e^{jt}g(t)\} = G(\omega-1)$.

As $h \to 0$ $g(t) \to \delta(t)$

since $\int_{-\infty}^{\infty} g(t)\,dt = 1$

Thus $G(\omega) \to 1$.

(c) Given $F(\omega) = \delta(\omega+3) + \delta'(\omega-2)$

$$\int_{-\infty}^{0} \omega F(\omega)\,d\omega = \int_{-\infty}^{\infty} \omega\,\delta(\omega+3)\,d\omega + \int_{-\infty}^{0} \omega\,\delta'(\omega-2)\,d\omega$$

$$= -3 + 0$$

$$= \underline{-3}$$