CESENG62008-5 Mathematical Techniques

Introduction to Maple

1. What is Maple?

Maple is a GUI driven computer algebra system and has the ability to manipulate algebraic expressions. It is used to help with many of the mathematical tasks we often encounted such as defining and plotting functions, solving equations, evaluating integrals and working with matrices.

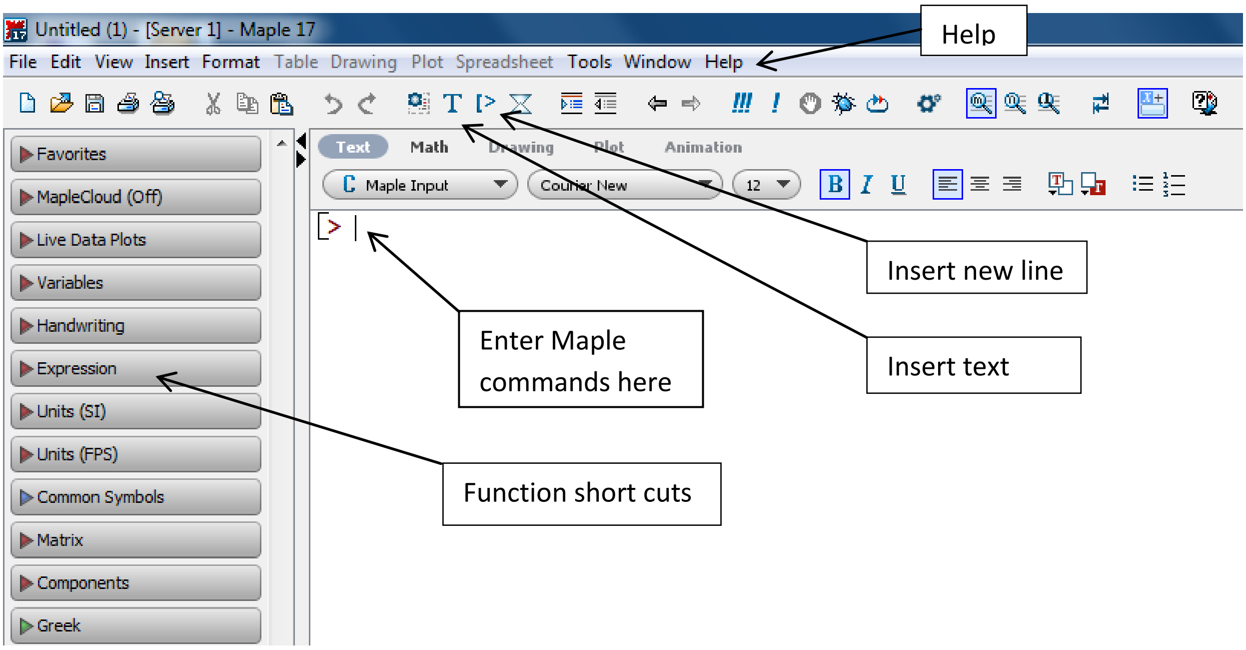
2. Starting Maple

Log on to a machine using your student id and password, then click on Start -> Programs -> Maths & Stats -> MAPLE 17 -> MAPLE 17. After a short time Maple will load and you will see the Maple screen.

Start Maple now and type in the commands as they appear in this document (split into numbered sections using text). You will find it helpful to investigate and explore the various topics on your own as well as in the timetabled sessions.

3. Maple Window

You should see the window below. Most of the buttons are self-explanatory. Try clicking some at random to see what they do…



Usually a worksheet opens by default. If not, or you need another one, then you can open a new worksheet by selecting: File -> New -> Worksheet Mode.

Each time you open Maple you should apply the following settings:

- Under Tools->Options->Display ensure that 'Input display' is set to 'MAPLE Notation' and 'Output display' is set to '2-D Math Notation'.

- Under Tools->Options->Interface ensure that 'Default format for new worksheets' is set to 'Worksheet'.

- Insert a new line by clicking on the symbol [> (in the tool bar at the top of the screen), and ensure that the commands you enter are in red type.

4. Simple Calculations

Maple can be used simply as a calculator. Noting that each line should end with a semi-colon try the following.

```
> 2+2;
```

$$4 \qquad (1)$$

```
> cos(Pi/6);
```

$$\frac{1}{2}\sqrt{3} \qquad (2)$$

```
> sin(2*Pi);
```

$$0 \qquad (3)$$

```
> cos(Pi);
```

$$-1 \qquad (4)$$

```
> 100!;
```

93326215443944152681699238856266700490715968264381621468592963895217599993229\
91560894146397615651828625369792082722375825118521091686400000000000000000\
0000000 (5)

```
> sqrt(99);
```

$$3\sqrt{11} \qquad (6)$$

Note that Maple returns the exact answer to any calculation. To produce a floating point answer, either input floating point numbers initially or request floating point. For example try

```
> sqrt(99.0);
```

$$9.949874371 \qquad (7)$$

In this case the square root is given as a floating point number because a floating point number was typed in as the argument to the square root function.

Alternatively 'evalf' can be used to compute the numerical value as follows:

```
> sin(Pi/3);
```

$$\frac{1}{2}\sqrt{3} \qquad (8)$$

```
> evalf(%);
```

$$0.8660254040 \qquad (9)$$

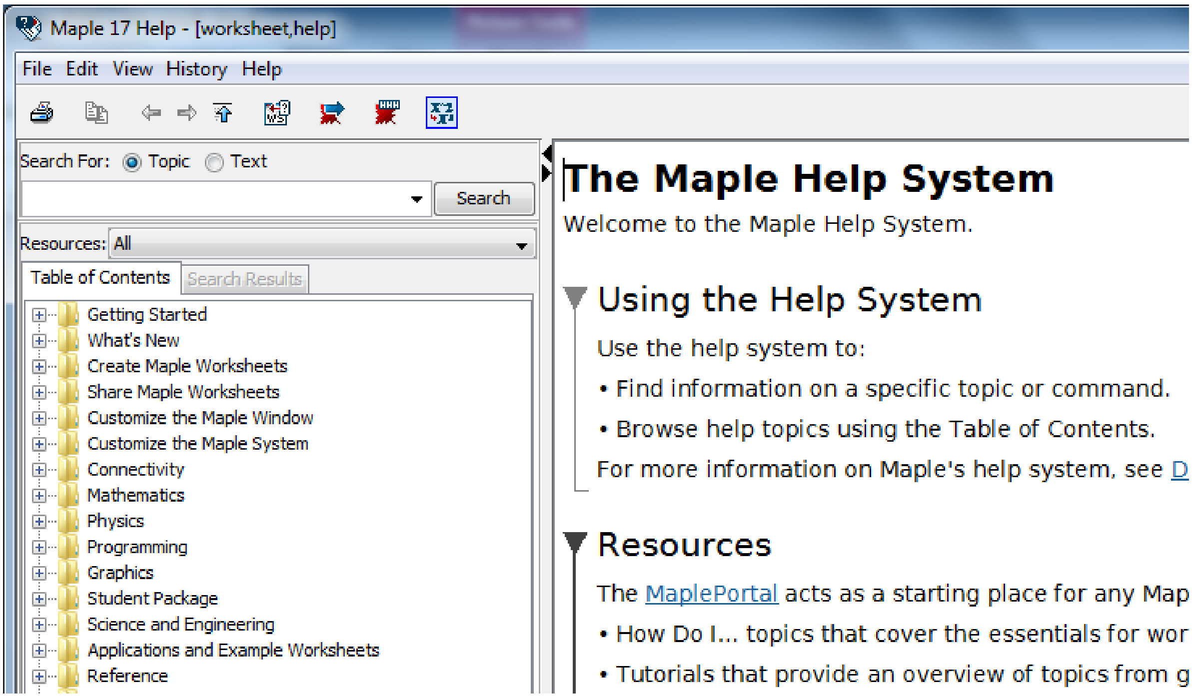
Note that the % character always refers to the line immediately preceding.

Note also that Maple lets you go back and amend any previous line. For example go back and amend any one of the commands entered so far. The new line can be executed by typing 'Return' or using the ! button on the bar at the top. Several new lines can be executed using either method repeatedly.

Note that an entire worksheet can be executed automatically by using the !!! button.

5. Help

Help is available clicking on Help and choosing Maple Help from the menu or Ctrl+F1.



Particular topics can be found using the Search facility – try looking for help on the plot command, and the exponential function, for example.

6. Saving your work

Your work can be saved as a Maple worksheet by using File>Save. It's advisable to save your work frequently in case of unexpected problems. When submitting assignments you should save as pdf, which is an option under File>Export.

Its also useful to include comments about what you are doing. Text can be inserted using the T button or Insert>Text. You can go back to typing maths with Insert>Maple Input or by using the [> button.

7. Basic Algebra

Maple is good at algebra. It can solve equations for example:

```
> eqn:=a*x^2+b*x+c=0;
```

$$eqn := a x^2 + b x + c = 0 \qquad (1)$$

```
> solve(eqn,x);
```

$$\frac{1}{2}\,\frac{-b+\sqrt{-4\,a\,c+b^2}}{a},\ -\frac{1}{2}\,\frac{b+\sqrt{-4\,a\,c+b^2}}{a} \qquad (2)$$

```
> solution:=solve(eqn,x);
```

$$solution := \frac{1}{2}\,\frac{-b+\sqrt{-4\,a\,c+b^2}}{a},\ -\frac{1}{2}\,\frac{b+\sqrt{-4\,a\,c+b^2}}{a} \qquad (3)$$

```
> solution[1];solution[2];
```

$$\frac{1}{2}\,\frac{-b+\sqrt{-4\,a\,c+b^2}}{a}$$

$$-\frac{1}{2}\,\frac{b+\sqrt{-4\,a\,c+b^2}}{a} \qquad (4)$$

The output from solve is given as a set. Each solution (or the only solution if there is just one) can be extracted using the [ ] syntax with the appropriate number.

Note that the 'equals' sign has been used in two different ways here. If you type y=x, you create an equation object that can be used later. On the other hand if you type y:=x (with the colon) then the expression x has been assigned to the variable y, and Maple will substitute x for y whenever it is used later. Try this for example.

```
> a:=b^3;
```

$$a := b^3 \qquad (5)$$

```
> eqn;
```

$$b^3 x^2 + b x + c = 0 \qquad (6)$$

```
> solution[1];
```

$$\frac{1}{2}\,\frac{-b+\sqrt{-4\,b^3\,c+b^2}}{b^3} \qquad (7)$$

Notice that the *a* in the 'eqn' object has been replaced by *b*^3.

The value of a variable can be cleared as follows:

```
> a:='a';
```

$$a := a \qquad (8)$$

```
> eqn;
```

$$a x^2 + b x + c = 0 \qquad (9)$$

```
> solution[1];
```

$$\frac{1}{2}\,\frac{-b+\sqrt{-4\,a\,c+b^2}}{a} \qquad (10)$$

Maple can also do algebra using complex numbers. The $\sqrt{-1}$ is denoted by I.

```
> 1/(a+I*b)+1/(2*a-I*b);
```

$$\frac{1}{a+\mathrm{I}b}+\frac{1}{2a-\mathrm{I}b}$$

The instruction evalc expresses the complex number in $a+ib$ form

```
> evalc(%);
```

$$\frac{a}{a^2+b^2}+\frac{2a}{4a^2+b^2}+\mathrm{I}\left(-\frac{b}{a^2+b^2}+\frac{b}{4a^2+b^2}\right)$$

Complex numbers can be converted to polar form

```
> z:=1+2*I;
```

$$z:=1+2\,\mathrm{I}$$

```
> polarz:=convert(z,polar);
```

$$polarz:=\mathrm{polar}\left(\sqrt{5},\arctan(2)\right)$$

Note the need to use the *, so that 2i is input as 2*I. We can extract the modulus and argument using the op command:

```
> modz:=op(1,polarz);argz:=op(2,polarz);
```

$$modz:=\sqrt{5}$$
$$argz:=\arctan(2)$$

Alternatively we can obtain the modulus and argument using

```
> abs(z); argument(z);
```

$$\sqrt{5}$$
$$\arctan(2)$$

If all variables are to be cleared use the 'restart' command, which resets Maple completely.

It is useful to start any new problem with 'restart', and use it to start the execution of any sequence of commands (particularly when output looks odd or is not what you are expecting).

8. Defining Functions

Functions are defined as follows:

```
> y:=x->2*x+3;
```

$$y:=x\rightarrow 2x+3 \qquad \textbf{(1)}$$

A function can be evaluated at any point using either of the following:

```
> y(4);
```

$$11 \quad (2)$$

```
> subs(x=4,y(x));
```

$$11 \quad (3)$$

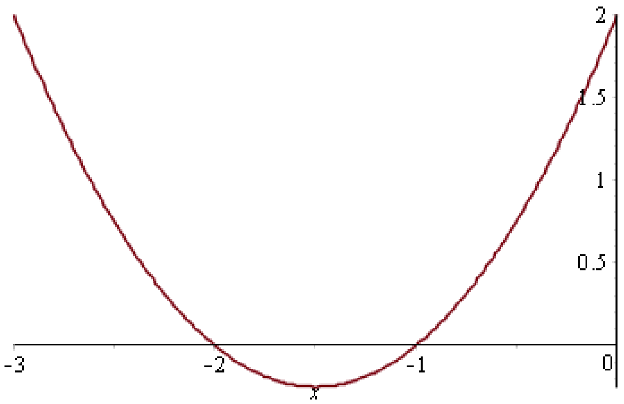
9. Plotting Functions

Once a function has been defined it can be plotted for a chosen range using the plot command:

```
> y:=x->x^2+3*x+2;
```

$$y := x \rightarrow x^2 + 3\,x + 2 \quad (1)$$

```
> plot(y(x),x=-3..0);
```



Several functions can be plotted on the same axes and the viewing area modified if needed:

```
> y1:=x->x^2+3*x+2;
```

$$y1 := x \rightarrow x^2 + 3\,x + 2 \quad (2)$$

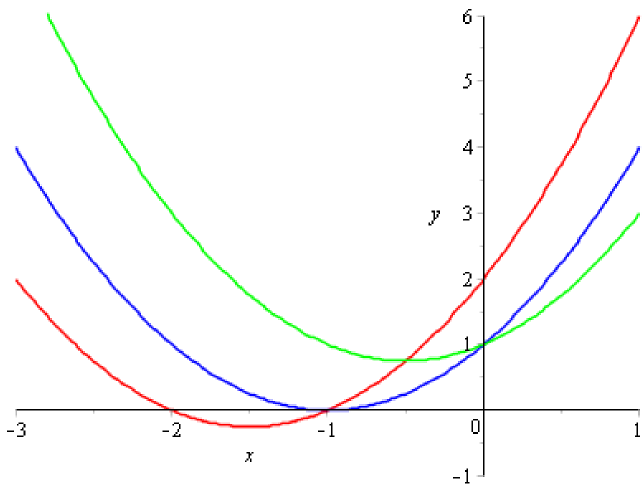
```
> y2:=x->x^2+2*x+1;
```

$$y2 := x \rightarrow x^2 + 2\,x + 1 \quad (3)$$

```
> y3:=x->x^2+x+1;
```

$$y3 := x \rightarrow x^2 + x + 1 \quad (4)$$

```
> plot([y1(x),y2(x),y3(x)],x=-3..1,y=-1..6,colour=[red,blue,green]);
```



10. Solving Equations

The roots of the quadratic equations defined in the previous section can be found using the solve command.

```
> solve(y1(x)=0,x);
```

$$-1,\ -2 \tag{5}$$

```
> solve(y2(x)=0,x);
```

$$-1,\ -1 \tag{6}$$

```
> solve(y3(x)=0,x);
```

$$-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\sqrt{3},\ -\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{3} \tag{7}$$

```
> evalf(%);
```

$$-0.5000000000+0.8660254040\,\mathrm{I},\ -0.5000000000-0.8660254040\,\mathrm{I} \tag{8}$$

These equations represent respectively the three cases: 2 real distinct roots; one real repeated root; and complex roots. You should be able to identify each case from the plot in Section 9. In the complex roots case the symbol 'I' is used for $\sqrt{-1}$, and the numerical values obtained by using 'evalf' as seen in Section 4.

Exercise

Use Maple to solve the following equations and plot each of the quadratics.

(a) $x^2+4x-5=0$ (b) $x^2+6x+9=0$ (c) $x^2+2x+5=0$.

In each case explain what is happening.

11. Basic Calculus

Maple is very good at calculus. Differentiation is achieved like this:

```
> f:=x->4*x^3-3*x^2+2*x-19;
```

$$f := x \rightarrow 4x^3-3x^2+2x-19 \tag{1}$$

```
> diff(f(x),x);
```

$$12x^2-6x+2 \tag{2}$$

Indefinite integration is like this (note the effect of upper/lower case):

```
> Int(12*x^2-6*x+2,x);
```

$$\int \left(12x^2-6x+2\right) dx \tag{3}$$

```
> int(12*x^2-6*x+2,x);
```

$$4x^3-3x^2+2x \tag{4}$$

Definite integration like this:

```
> Int(12*x^2-6*x+2,x=0..2);
```

$$\int_0^2 \left(12x^2 - 6x + 2\right) dx \qquad (4)$$

```
> int(12*x^2-6*x+2,x=0..2);
```

$$24 \qquad (5)$$

Exercises

1. For each of the following functions: (a) plot on a suitable interval and find its roots, (b) find where the derivative (ie gradient) has value +1, 0 and -1.

   (i) $-x^2+4$ (ii) x^3-6x^2+8*x (iii) x^4-9x^2

2. Evaluate each of the following integrals:

   (i) $\int(16x^4+2x^2+5x+3)dx$ (ii) $\int\frac{1}{1+x}dx$ (iii) $\int_0^1\frac{1}{1+x}dx$

12. Loops

A section of code that is executed several times is called a loop. A basic loop in Maple has the form:

for i from m to n do
statement sequence
end do

For example a loop for adding the numbers 1 to 10 would have the form:

```
> total:=0;
```

$$total := 0 \qquad (1)$$

```
> for i from 1 to 10 do
> total:=total+i
> end do;
```

$$total := 1$$
$$total := 3$$
$$total := 6$$
$$total := 10$$
$$total := 15$$
$$total := 21$$
$$total := 28$$
$$total := 36$$
$$total := 45$$
$$total := 55 \qquad (2)$$

Note that the total must be defined and initialised before entering the loop (in this case to zero). The total after performing the loop is shown at the end (55).

Example
The following code indicates whether the integers between 15 and 19 are prime, and gives the factors if any.

```
> for i from 15 to 19 do
> print(i,isprime(i),ifactor(i))
> end do;
```

$$15, \mathit{false}, (3)\,(5)$$
$$16, \mathit{false}, (2)^4$$
$$17, \mathit{true}, (17)$$
$$18, \mathit{false}, (2)\,(3)^2$$
$$19, \mathit{true}, (19) \tag{1}$$

The counter can be made to increment in steps of 2 as follows:

```
> for i from 15 to 19 by 2 do
> print(i,isprime(i),ifactor(i))
> end do;
```

$$15, \mathit{false}, (3)\,(5)$$
$$17, \mathit{true}, (17)$$
$$19, \mathit{true}, (19) \tag{1}$$

To count backwards a negative step can be used.

```
> for i from 19 to 15 by -2 do
> print(i,isprime(i),ifactor(i))
> end do;
```

$$19, \mathit{true}, (19)$$
$$17, \mathit{true}, (17)$$
$$15, \mathit{false}, (3)\,(5) \tag{1}$$

Exercise
Write a loop to generate the sequence of factorials of odd positive integers below fifty. Repeat the sequence but in reverse order.

Recall: "3 factorial" = 3! = 3×2×1=6. It can be achieved by the following:

```
> print(3,3!);
```

$$3, 6$$

General controlled loop: while
The most general type of controlled loop has the form:

> while *condition* do
> *statement_sequence*
> end do

Note that:

- The sequence of statements forming the body of the loop is executed repeatedly while the condition remains true.

- The loop must be preceded by code that sets the initial value of the condition (known as loop initialisation)

- The loop must contain code that changes the value of the condition.

- Ensure that the initialisation statements are in the same execution group as the loop itself, so that initialisation occurs each time the loop is executed. (The execution group is indicated by the brackets to the left of the statements). Do this by highlighting the lines you wish to include together and select "Join Execution Groups" from the "Split or Join" option under the "Edit" menu.

Example

The following is a simple illustration.

```
> i:=0:
> while i<10 do
> i:=i+2;
> end do;
```

$$i := 2$$
$$i := 4$$
$$i := 6$$
$$i := 8$$
$$i := 10 \tag{1}$$

Exercise

Write a while loop which uses the syntax isprime(i) to find the first value of *i* for which the expression $i^2 + i + 41$ is not prime. Check your answer with the function ifactor(i).

13. Newton-Raphson Iteration

Some equations can be solved easily and exactly by hand, such as quadratic equations. More difficult equations cannot be solved in this way and an approximate solution has to be found using an iterative method. The idea of a loop is essential for this kind of process.

As an example consider the equation $x^4 - 3x^3 + 2x - 4 = 0$. We can try using the solve command seen earlier:

```
> solve(x^4-3*x^3+2*x-4=0,x);
```

$$RootOf(\_Z^4 - 3\_Z^3 + 2\_Z - 4, index = 1), RootOf(\_Z^4 - 3\_Z^3 + 2\_Z - 4, index = 2),$$
$$RootOf(\_Z^4 - 3\_Z^3 + 2\_Z - 4, index = 3), RootOf(\_Z^4 - 3\_Z^3 + 2\_Z - 4, index = 4) \tag{1}$$

However Maple is telling us that it can't find the roots. In this case we need to use a numerical (or computational) method. Maple does this using 'fsolve':

```
> fsolve(x^4-3*x^3+2*x-4=0,x);
                    -1.149271804, 2.926070175                    (2)
```

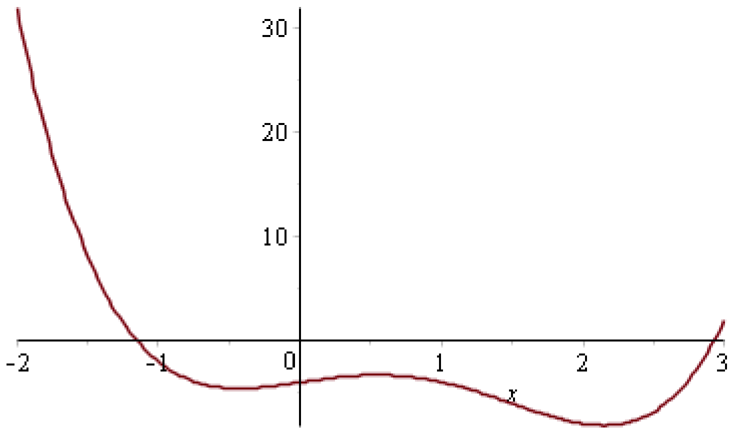
Note that fsolve only produces the real roots. If we use

```
> evalf(solve(x^4-3*x^3+2*x-4=0,x));
2.92607017454950, 0.611600814669297 + 0.903001511241446 I, -1.14927180388810,
    0.611600814669297 - 0.903001511241446 I
```

then all (real and complex) roots are produced.

How do the numerical methods work? The first step is to find an initial approximation to the root, and the easiest way to do this is to plot the graph:

```
> plot(x^4-3*x^3+2*x-4,x=-2..3);
```



The graph clearly shows that there is a root near -1 and another near 3. Once we have an initial approximation one way of finding the root is to use Newton-Raphson iteration.

Suppose we look for the root near 3. The initial approximation is then $x_0 = 3$. The Newton-Raphson method sets up an iterative process which says that if x_n is an approximation to the root then a better one is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

There's a simple geometric explanation of why this is likely to work.

Example

This can be implemented in Maple as follows. Firstly define the function f(x), its derivative df, the initial approximation x[0] and N to be the number of iterations.

```
> restart;
> f:=x->x^4-3*x^3+2*x-4;
```

$$f := x \to x^4 - 3x^3 + 2x - 4 \quad (1)$$

```
> df:=diff(f(x),x);
```

$$df := 4x^3 - 9x^2 + 2 \quad (2)$$

```
> xn[0]:=3;
```

$$xn_0 := 3 \quad (3)$$

```
> N:=5;
```

$$N := 5 \quad (4)$$

Secondly implement the method in a loop:

```
> for i from 0 to N-1 do
> xn[i+1]:=evalf(xn[i]-subs(x=xn[i],f(x)/df))
> end do;
```

$$xn_1 := 2.931034483$$

$$xn_2 := 2.926094547$$

$$xn_3 := 2.926070175$$

$$xn_4 := 2.926070175$$

$$xn_5 := 2.926070175 \quad (5)$$

Note: 1. The use of 'subs' to evaluate f(x)/f'(x) for the current approximation.
2. The use of 'evalf' to convert the results to decimal (try it without and see what happens).

The use of a full colon after any line suppresses output. Hence the initialisation lines could be written as

```
> f:=x->x^4-3*x^3+2*x-4:
> df:=diff(f(x),x):
> xn[0]:=3:
> N:=5:
```

Sometimes this is useful to prevent large amounts of unwanted information. When first starting off however it is best to be able to see the effect of each line and this wouldn't be recommended.

Exercises

1. Find √2 by defining $f(x) = x^2 - 2$ and use Newton Raphson to solve $f(x) = 0$.
2. Find the root near $x = 1$ for $f(x) = x^3 + 2x^2 + 10x - 20$.
3. How many roots of $f(x) = 2x^3 - 15x + 3$ lie in the range $0 \le x \le 3$? Find the largest of them.

14. Heaviside Step Function

It is common in applied mathematics for example to wish to model 'switch-on' or 'switch-off' behaviour. This can be achieved by using the Heaviside step function, denoted by $u(x)$ which is defined to be 1 if its argument is positive and 0 if its argument is negative. That is

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

The Heaviside function can be thought of as switching on if its argument is positive, or switching off if negative.

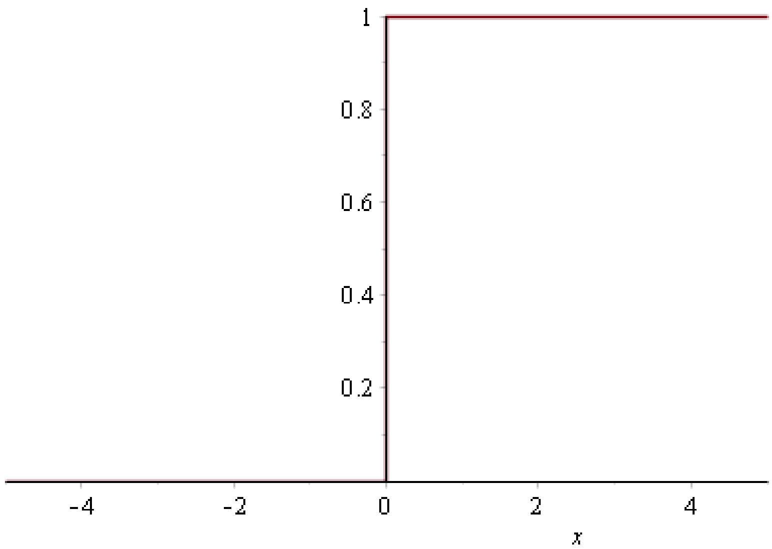
Other notations for the Heaviside function are $U(x)$ and $H(x)$, and it is often referred to as the *unit step function*.

The function can be defined and plotted in Maple as follows:

```
> u:=x->Heaviside(x);
```

$$u := x \to \mathrm{Heaviside}(x) \quad (1$$

```
> plot(u(x),x=-5..5);
```



The location of the switching point can be moved by appropriate modification of the argument. For example while $u(x)$ switches on at $x = 0$, $u(x+2)$ and $u(x-2)$ switch on at $x = -2$ and $x = 2$ respectively. If $u(x)$ is replaced by $u(-x)$ a mirror image is produced, corresponding to a switch off.

Restricting the range on which a function is defined

Heaviside functions can be used to restrict the range of values for which a function is defined. Consider the following functions and their plots:

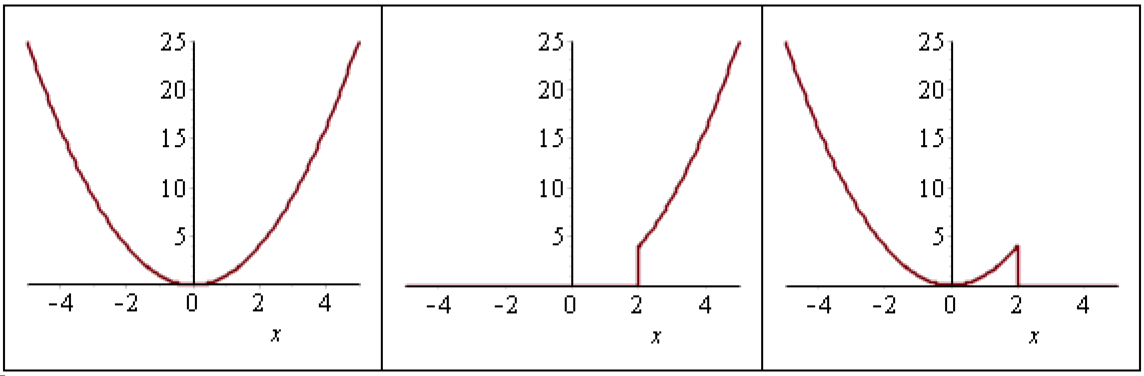
$$f(x) = x^2 \qquad\qquad f(x) = x^2 u(x-2) \qquad\qquad f(x) = x^2 u(2-x)$$

To produce graphs in a format that can be plotted side by side we first set up the individual plots:

```
> p1:=plot(x^2,x=-5..5):
> p2:=plot(x^2*u(x-2),x=-5..5):
> p3:=plot(x^2*u(2-x),x=-5..5):
```

In order to plot the functions side by side we need to load the plots library. Libraries are loaded using the “with” command. To produce the plot use:

```
> with(plots):
> display(array([p1,p2,p3]));
```

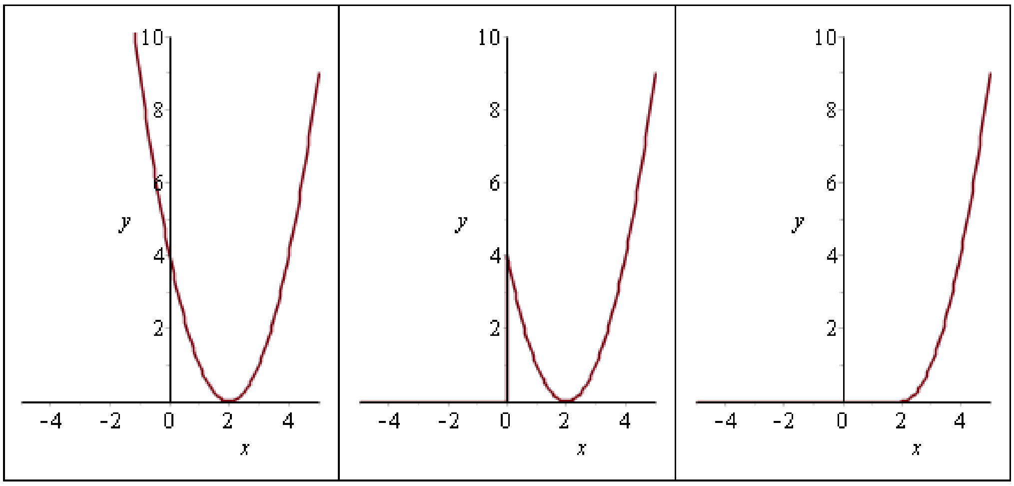


$f(x) = x^2$ $\qquad$ $f(x) = x^2 u(x-2)$ $\qquad$ $f(x) = x^2 u(2-x)$

This illustrates that if we have a function $f(x)$ multiplied by a Heaviside function we can think of the Heaviside function switching $f(x)$ on (or off) at specific locations. Thus $f(x) = x^2 u(x-2)$ switches on the function $f(x) = x^2$ at $x = 2$, while $f(x) = x^2 u(2-x)$ switches it off at $x = 2$.

Delay functions

The previous examples can be adapted to produce a delayed function. Suppose we wanted the function $f(x) = x^2$ delayed by 2 units. This could be achieved by drawing $g(x) = f(x-2) = (x-2)^2$. Consider the differences between the graphs of the following functions:

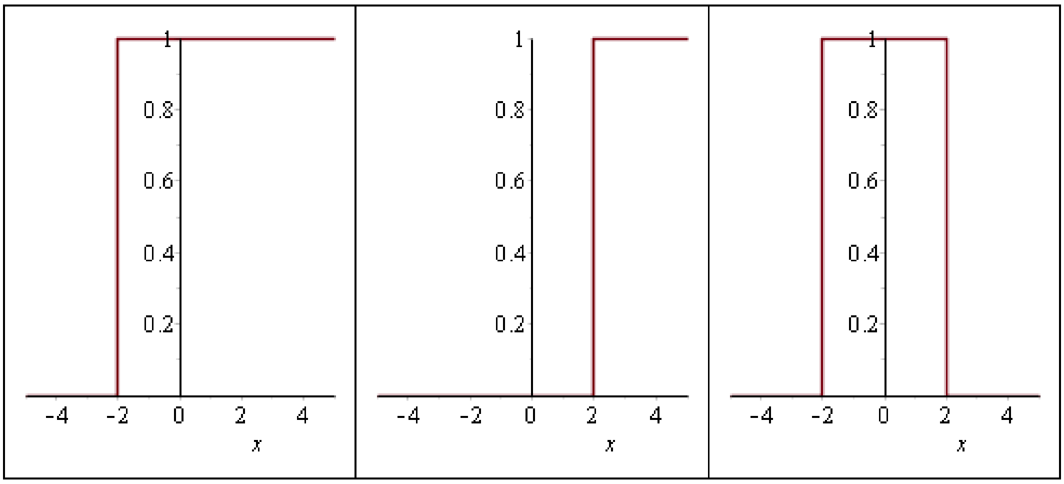


$f(x-2)$ $\qquad$ $f(x-2)u(x)$ $\qquad$ $f(x-2)u(x-2)$

We can see that $f(x-2)$ delays the whole function by 2 units, $f(x-2)u(x)$ switches the delayed function on at $x=0$, while $f(x-2)u(x-2)$ switches the delayed function on at $x=2$.

Combinations of Heaviside Functions

Consider $u(x+2)-u(x-2)$. The graphs below show that this combination is only 'on' between -2 and +2.



$u(x+2)$ $\qquad$ $u(x-2)$ $\qquad$ $u(x+2)-u(x-2)$

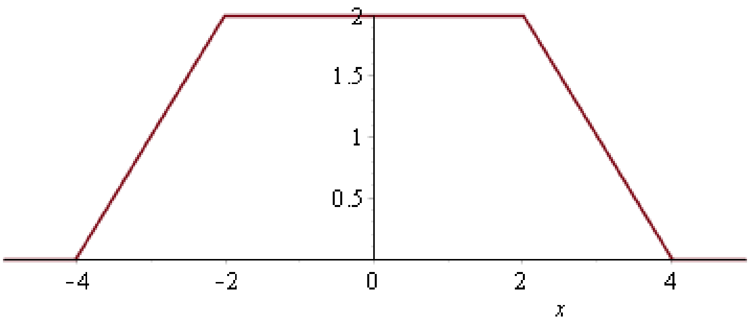
This can be used to restrict a function to a range of values. For example $(2x+1)(u(x+3)-u(x-1))$ corresponds to the function $(2x+1)$ being 'on' in the range $-3<x<1$.

More complicated functions can easily be achieved. The function

$$f(x)=\begin{cases} x+4 & -4<x<-2 \\ 2 & -2<x<2 \\ 4-x & 2<x<4 \end{cases}$$

is defined and plotted as follows:

```
> plot((x+4)*(u(x+4)-u(x+2))+2*(u(x+2)-u(x-2))+(4-x)*(u(x-2)-u(x-4)),
  x=-5..5);
>
```



Exercises:

Draw graphs of the following functions:

1. (i) u(x+2) (ii) u(-x) (iii) u(2x+1) (iv) u(3-x)

2. (i) e^{-x} (ii) $e^{-x}\,u(x-1)$ (iii) $e^{-x}\,u(1-x)$

   (iv) $e^{-(x-1)}$ (v) $e^{-(x-1)}u(x)$ (vi) $e^{-(x-1)}u(x-1)$

   in the range $-2 \le x \le 2$.

3. (i) $u(x+3)-u(x-1)$ (ii) $u(x+2)-u(x)+2u(x)-2u(x-4)$

   (iii) $(2x+1)(u(x)-u(x-3))$ (iv) $x^2\,\,(u(x+1)-u(x-1))$

15. Matrices

Defining matrices

This needs a prior call to the Linear Algebra package

```
> with(LinearAlgebra):
```

To define a column matrix (vector):

```
> vcol:=Matrix([[x],[y],[z]]);
```

$$vcol := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1}$$

to define a row matrix:

```
> vrow:=Matrix([x,y,z]);
```

$$vrow := \begin{bmatrix} x & y & z \end{bmatrix} \tag{2}$$

and to define a general matrix:

```
> M:=Matrix([[a,b,c],[d,e,f],[g,h,i]]);
```

$$M := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{3}$$

To define the identity matrix, you need to specify the size, here we set up the 3 by3 identity matrix:

```
> IdentityMatrix(3);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4)$$

Exercises

Define the following matrices

(i) $\begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -3 & 0 & 3 \\ -4 & 0 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Alternatively the matrix can be declared and its individual elements then specified. To specify the size:

```
> M1:=Matrix(2,2);
```

$$M1 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (5)$$

and to specify individual elements:

```
> M1[1,1]:=4:M1[1,2]:=5:M1[2,1]:=-3:M1[2,2]:=6:
```

Note the colon separating each entry suppresses its output. The resulting matrix can be viewed using:

```
> M1;
```

$$\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix} \qquad (6)$$

Exercises

Use this approach to define the following matrices.

(i) $\begin{bmatrix} -1 & 2 & -3 & 4 & -5 \\ 0 & 3 & -2 & 1 & 7 \end{bmatrix}$ (i) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Rules can be used to generate the matrix, where the indices i and j are used to represent the rows and columns respectively.

```
> Matrix(4,4,(i,j)->1/(i+j));
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix} \quad (7)$$

Exercise

1. The matrix above is known as a Hilbert matrix and has its own generating function. For example type HilbertMatrix(10).
2. The following is an example of a Vandermonde matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}.$$

Generate it using the Matrix function with an appropriate rule.

Matrix Operations

Matrices can be added, subtracted and multiplied using the usual rules:

```
> M1:=Matrix([[1,2],[3,4]]);M2:=Matrix([[2,-3],[-1,1]]);M3:=
  Matrix([[2],[5]]);
```

$$M1 := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$M2 := \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$$

$$M3 := \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad (8)$$

```
> M1+M2;
```

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \quad (9)$$

```
> M1.M2;
```

$$\begin{bmatrix} 0 & -1 \\ 2 & -5 \end{bmatrix} \quad (10)$$

```
> M3.M1;
Error, (in LinearAlgebra:-Multiply) first matrix column 
dimension (1) <> second matrix row dimension (2)
```

Determinant

The determinant of a matrix can be found as follows:

```
> Determinant(M1);
```

$$-2 \tag{11}$$

Matrix Inverse

The matrix inverse can be found as follows:

```
> MatrixInverse(M1);
```

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \tag{12}$$

Exercise

For $A = \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$ use Maple to show $A^2 - 11A + 10I = 0$. By hand multiply this equation by A^{-1} and deduce that $A^{-1} = \frac{1}{10}(11I - A)$. Confirm using Maple.

Solving Equations - Systems of Two Equations

The pair of linear equations

$$x + 3y = 5$$
$$2x - y = 3$$

can be written in matrix form

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

In general we can write a system of linear equations as $A\underline{x} = \underline{b}$.

The system is solved by multiplying both sides of the equation by A^{-1} (provided it exists), giving

$$A^{-1}A\underline{x} = A^{-1}\underline{b}$$

But since $A^{-1}A = I$, this gives the solution vector $\underline{x}$ as

$$\underline{x} = A^{-1}\underline{b}.$$

In the example above $A^{-1} = \frac{-1}{7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}$, so the solution is given by

$$\underline{x} = \frac{-1}{7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \frac{-1}{7}\begin{bmatrix} -14 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \quad \text{ie } x = 2,\ y = 1.$$

This is achieved in Maple as follows:

```
> A:=Matrix([[1,3],[2,-1]]);
```

$$A := \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad (12)$$

```
> MatrixInverse(A);
```

$$\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix} \qquad (13)$$

```
> b:=Matrix([[5],[3]]);
```

$$b := \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad (14)$$

```
> MatrixInverse(A).b;
```

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad (15)$$

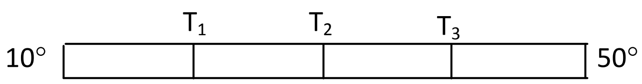
Exercises

1. Use Maple to find the inverse of the matrix A = $\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & -2 \\ -3 & -2 & 1 \end{bmatrix}$ and hence find the solution to the following equations:

$$\begin{aligned} x + 3y - 2z &= 2 \\ -x + 2y - 2z &= 1 \\ -3x - 2y + z &= 6. \end{aligned}$$

(Answer: $x = -5, y = 11, z = 13$)

2. The following system of equations arises from the simulation of steady heat flow in a one-dimensional bar with the ends held at constant temperatures. The values of T_1, T_2 and T_3 are the temperatures at equally spaced points as shown.

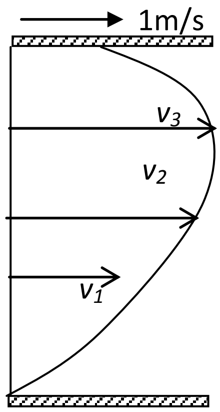


$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 50 \end{bmatrix}$$

Solve the system using Maple.

(Answer: $T_1 = 20, T_2 = 30, T_3 = 40$)

3. The three equations below represent a simple model of steady fluid flow between two plates under a pressure gradient. The lower plate is stationary while the upper plate moves with unit velocity. The values of v_1, v_2 and v_3 are the velocities at equally spaced points between the plates as shown.

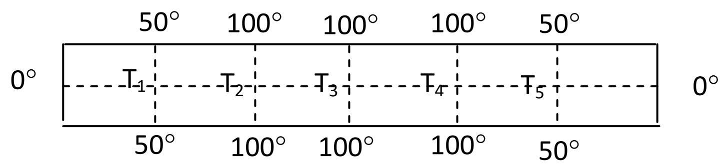


$$
\begin{aligned}
3v_1 - v_2 \quad &= 2 \\
-v_1 + 2v_2 - v_3 &= 2 \\
-v_2 + 3v_3 &= 4
\end{aligned}
$$

Solve the system using MAPLE and comment on the solution found.

(Answer: $v_1 = 5/3, v_2 = 3, v_3 = 7/3$)

4. The following ‘tridiagonal’ system arises from thermal conduction where the unknown variables are temperatures at points equally spaced along a heated bar as shown:



$$
\begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix}
=
\begin{bmatrix} 100 \\ 200 \\ 200 \\ 200 \\ 100 \end{bmatrix}
$$

Solve the system using Maple.

(Answer: $T_1 = 46.1536, T_2 = 84.6154, T_3 = 92.3077, T_4 = 84.6154, T_5 = 46.1538$)