EMA2 - Examination - solutions

1 a) $A - \lambda I = \begin{pmatrix} 3-\lambda & -1 & 0 \\ 4 & -2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{pmatrix}$

$$\det(A - \lambda I) = (-2-\lambda) \begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix}$$
$$= (-2-\lambda)\left[(3-\lambda)(-2-\lambda) + 4\right]$$
$$= (-2-\lambda)\left[\lambda^2 + 2\lambda - 3\lambda - 6 + 4\right]$$
$$= (-2-\lambda)(\lambda^2 - \lambda - 2)$$
$$= (-2-\lambda)(\lambda-2)(\lambda+1)$$

$\therefore$ Eigenvalues are $\lambda = -2$; $\lambda = 2$ and $\lambda = -1$ 4

$\lambda = -2$ $\begin{pmatrix} 5 & -1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$5a - b = 0$ ①
$4a = 0$ ② $\therefore a = 0$ ① $\Rightarrow b = 0$
$0 = 0$ ③ c - anything

Choose $c = 1$ $\underline{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 3

$\lambda = -1$ $\begin{pmatrix} 4 & -1 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{array}{l} 4a - b = 0 \\ 4a - b = 0 \\ -c = 0 \end{array}\right\} \Rightarrow \begin{array}{l} b = 4a \\ c = 0 \end{array}$$

choose $a=1 \Rightarrow b=4 \quad \therefore \underline{e}_2 = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ 3

$\lambda = 2.$ $\begin{pmatrix} 1 & -1 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} a-b &= 0 \\ 4a-4b &= 0 \\ -4c &= 0 \end{aligned}\right\} \Rightarrow a=b \quad c=0$$

choose $a=1$ $\quad \underline{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$ 3

[10]

(b) In matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{d}{dt}\underline{x} = A\underline{x}$$

[1]

Let $\underline{x} = P\underline{z}$ $\quad \frac{d}{dt}P\underline{z} = AP\underline{z}$

$$\frac{d}{dt}\underline{z} = P^{-1}AP\underline{z}$$

if we chose P = matrix of eigenvectors

$$= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$ [1]

then $P^{-1}AP = D$ = diagonal matrix containing eigenvalues

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad [1]$$

$$\therefore \frac{d}{dt}\underline{z} = D\underline{z} \qquad \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$\frac{dz_1}{dt} = -2z_1 \qquad z_1 = Ae^{-2t}$$

$$\frac{dz_2}{dt} = -z_2 \qquad z_2 = Be^{t} \qquad [1]$$

$$\frac{dz_3}{dt} = 2z_3 \qquad z_3 = Ce^{2t}.$$

To recover $\underline{x}$

$$\underline{x} = P\underline{z} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} Ae^{-2t} \\ Be^{-t} \\ Ce^{2t} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} Be^{-t} + Ce^{2t} \\ 4Be^{-t} + Ce^{2t} \\ Ae^{-2t} \end{pmatrix} \qquad [1]$$

$\therefore x = Be^{-t} + Ce^{2t}$

if $C \neq 0$ e^{2t} is dominant term – exp grow

$C = 0$ e^{-t} is dominant – exp. deca

$y = 4Be^{-t} + Ce^{2t}$

if $C \neq 0$ e^{2t} is dominant – exp growth

$C = 0$ e^{-t} is dominant – exp deca

$z = Ae^{-2t}$ – exp decay, unless $A = 0$ in which case $z = 0$ (constant)

(i) $x(0)=1 \quad y(0)=1 \quad z(0)=1$.

$$\therefore \left.\begin{aligned} 1 &= B+C \\ 1 &= 4B+C \\ 1 &= A \end{aligned}\right\} \Rightarrow A=1,\ B=0,\ C=1$$

$x=e^{2t}$ $y=e^{2t}$ $z=e^{-t}$ [1]

exp growth   exp growth   exp decay

(ii) $x(0)=1 \quad y(0)=4 \quad z(0)=0$

$$\left.\begin{aligned} 1 &= B+C \\ 4 &= 4B+C. \\ 0 &= A \end{aligned}\right\} \Rightarrow A=0;\ C=0;\ B=1$$

[1]

$\therefore\ x=e^{-t}$ $y=4e^{-t}$ $z=0$

exp decay   exp decay   constant (0)

comments: 3

(a) (i) $\int_{-1}^{1} x^2 \cos(\pi n x)\,dx = 2\int_0^1 x^2 \cos(\pi n x)\,dx$

$$= 2\left\{ \left[x^2 \frac{\sin(\pi n x)}{\pi n}\right]_0^1 - 2\int_0^1 \frac{x \sin(\pi n x)\,dx}{\pi n} \right\}$$

$$= \left(-\frac{4}{\pi n}\right)\int_0^1 x \sin(\pi n x)\,dx \qquad \text{(5 marks)}$$

(ii) $\int_0^1 x \sin(\pi n x)\,dx = \left[-x\frac{\cos(\pi n x)}{\pi n}\right]_0^1 + \int_0^1 \frac{\cos(\pi n x)\,dx}{\pi n}$

$$= -\frac{\cos \pi n}{\pi n} + \left[\frac{\sin(\pi n x)}{\pi^2 n^2}\right]_0^1 = -\frac{(-1)^n}{\pi n}$$

$$= \underline{\underline{(-1)^{n+1}}} \qquad \text{(5 marks)}$$

(b) Since $f(x) = x^2+1$ is even the Fourier series in the interval $-1 < x < 1$ (period 2) is

$$x^2+1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi n x)/dx$$

$$a_0 = \int_{-1}^{1}(x^2+1)\,dx = 2\int_0^1 (x^2+1)\,dx = 2\left[\frac{x^3}{3}+x\right]_0^1$$

$$= 8/3.$$

$$a_n = \int_{-1}^{1}(x^2+1)\cos(\pi n x)\,dx = \int_{-1}^{1} x^2\cos(\pi n x)\,dx + \left[\frac{\sin(\pi n x)}{\pi n}\right]_{-1}^{1}$$

$$= 2\int_0^1 x^2\cos(\pi n x) = \left(-\frac{4}{\pi n}\right)\int_0^1 x\sin(\pi n x)\,dx$$

$$= \frac{4}{\pi n}\frac{(-1)^n}{\pi n} = \frac{4(-1)^n}{\pi^2 n^2} \qquad \text{(6 marks)}$$

Thus the Fourier series is

$$x^2+1 = 4/3 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos(\pi n x)$$

At $x=0$,

$$1 = 4/3 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-1/3 = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\underline{-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}}$$

(4 marks)

2.

$$FT\{f(t)\} = \int_{-1}^{1} \cos(\pi t)\, e^{-j\omega t}\, dt = 2\int_{0}^{1} \cos(\pi t)\cos(\omega t)\, dt$$

$$= \int_{0}^{1} \left(\cos(\pi-\omega)t + \cos(\pi+\omega)t \right) dt$$

$$= \left[\frac{\sin(\pi-\omega)t}{\pi-\omega} + \frac{\sin(\pi+\omega)t}{\pi+\omega} \right]_0^1$$

$$= \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega} = F(\omega)$$

(6 Marks)

(a) $$FT\{f(t-3)\} = e^{-3j\omega}\left\{ \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega} \right\}$$

(1 Mark)

(b) $$FT\{f(3t+2)\} = FT\{f(3(t+2/3))\}$$

$$= \frac{e^{\frac{2}{3}j\omega}}{3}\left\{ \frac{\sin(\pi-\omega/3)}{\pi-\omega/3} + \frac{\sin(\pi+\omega/3)}{\pi+\omega/3} \right\}$$ (2 Marks)

(c) $$FT\{e^{3jt}f(t)\} = F(\omega+3)$$

$$= \frac{\sin(\pi-\omega-3)}{\pi-\omega-3} + \frac{\sin(\pi+3+\omega)}{\pi+3+\omega}$$

(2 Marks)

(d) $$FT\{t f(t)\} = j\frac{dF}{d\omega} =$$

$$j\left\{ -\frac{\cos(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi-\omega)}{(\pi-\omega)^2} + \frac{\cos(\pi+\omega)}{\pi+\omega} - \frac{\sin(\pi+\omega)}{(\pi+\omega)^2} \right\}$$

$$FT\{F(t)\} = 2\pi f(-w)$$

$$= 2\pi\{\{u(-w+1) - u(-w-1)\}\cos(-\pi w)\}$$

$$= 2\pi\{u(-w+1) - u(-w-1)\}\cos(\pi w)$$

$$u(-w+1) = \begin{cases} 1 & -w+1 \geq 0 \quad w \leq 1 \\ 0 & -w+1 < 0 \quad w > 1 \end{cases}$$

$$u(-w-1) = \begin{cases} 1 & -w-1 \geq 0, \quad w \leq -1 \\ 0 & -w-1 < 0 \quad w > -1 \end{cases}$$

Thus $u(-w+1) - u(-w+1)$ is non-zero in $-1 \leq w \leq 1$. (3 Marks)

$$\int_{-\infty}^{\infty} e^{-t^2}\delta(t)\, e^{-jwt} = e^{0} = 1 = FT\{e^{-t^2}\delta(t)\}$$

$$\int_{-\infty}^{\infty}\left(u(t+1) - u(t+0.5)\right)\delta(t)\, e^{-jwt}\,dt$$

$$= \int_{-1}^{-0.5}\delta(t)\, e^{-jwt}\,dt = 0 = FT\{(u(t+1) - u(t+0.5))\delta(t)\}$$

(3 Marks)