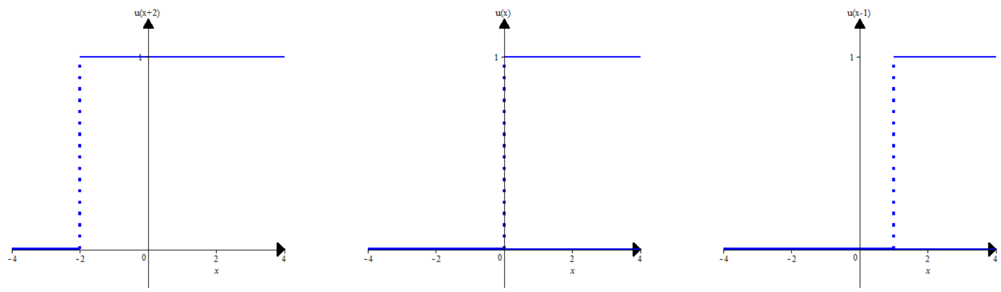
1 Revision and Background Information

1.1 Graphs of functions involving Heavisides

The Heaviside function $u(x)$ is defined as

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

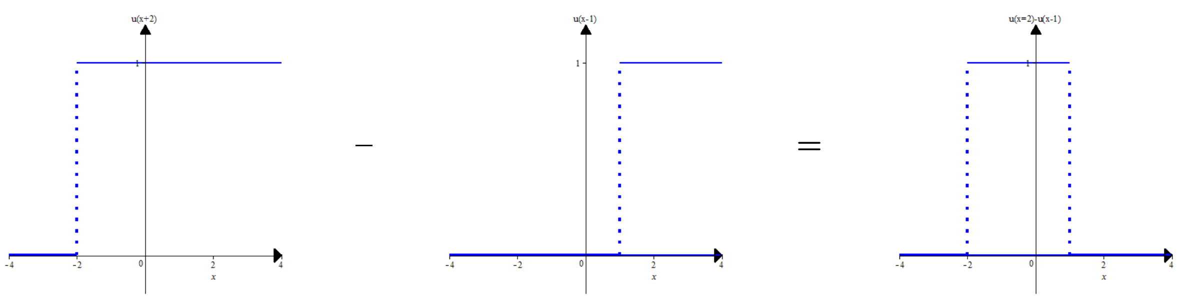
It can be thought of as being a switch - which is either "on" or "off". Consider the graphs of $u(x+2)$, $u(x)$ and $u(x-1)$:



These essentially switch on at $x = -2$, $x = 0$ and $x = 1$ and so in general, $u(x-a)$ switches on at $x = a$.

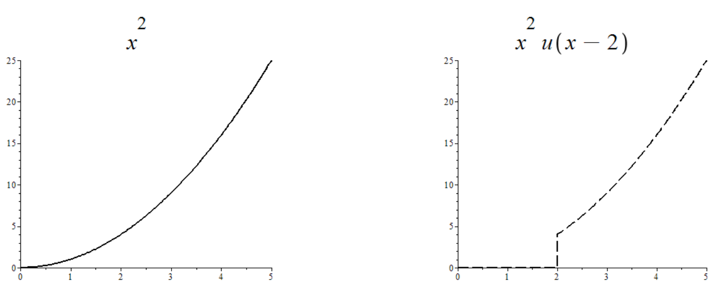
Combinations of Heaviside functions

Consider $u(x+2) - u(x-1)$:



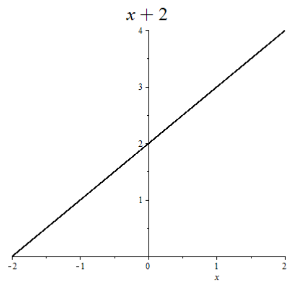
The combination $u(x+2) - u(x-1)$ switches on at $x = -2$ and off at $x = 1$. So in general $u(x-a) - u(x-b)$ switches on at $x = a$ and off at $x = b$ (assuming $b > a$).

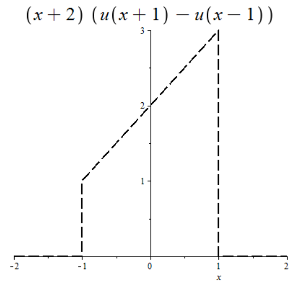
Heaviside functions can be used to switch on and off other functions. For example, if we have $x^2 u(x-2)$ this corresponds to x^2 being switched on at $x = 2$:



Similarly if we have $(x+2)(u(x+1)-u(x-1))$ this corresponds to $x+2$ being switched on between $x=-1$ and $x=1$. This is the same as

$$f(x) = (x+2)(u(x+1)-u(x-1)) = \begin{cases} x+2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$





Example 1

Sketch the graphs of:

(i) $f(x) = 5\,(u(x+3)-u(x+1)) + 2\,(u(x-2)-u(x-4))$

(ii) $f(x) = (x+2)\,(u(x+2)-u(x)) + 2\,(u(x)-u(x-2))$

(iii) $f(x) = (x+2)\,(u(x+2)-u(x)) + (2-x)\,(u(x)-u(x-2))$

Tutorial 1.1

Sketch the graphs of the following functions

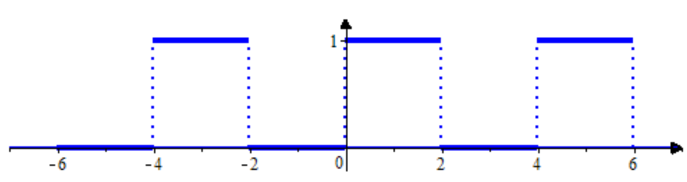
(i) $4u(x+2)$ (ii) $-2u(x-1)$ (iii) $2(u(x+2)-u(x-3))$
(iv) $x(u(x-1)-u(x-3))$ (v) $2(u(x+2)-u(x+1))-2(u(x-1)-u(x-2))$
(vi) $(x+2)(u(x+2)-u(x+1))+u(x+1)-u(x-1)+(2-x)(u(x-1)-u(x-2))$

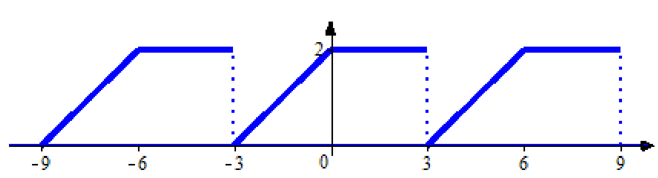
1.2 Revision of properties of trigonometric functions

Periodic Functions

A periodic function is one such that

$$f(t+T) = f(t); \qquad f(t+2T) = f(t+T) = f(t) \quad \text{and in general } f(t+nT) = f(t).$$





The period, T, is the time taken to complete one cycle of the periodic function. The corresponding frequency, f, is

$$f = \frac{1}{T},$$

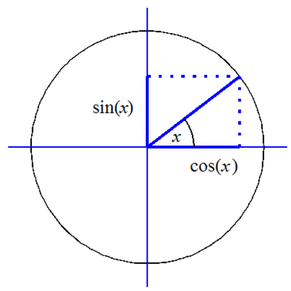
which is measured in cycles/seconds (or Hertz, Hz). The **angular frequency**, ω, is given by

$$\omega = \frac{2\pi}{T}$$

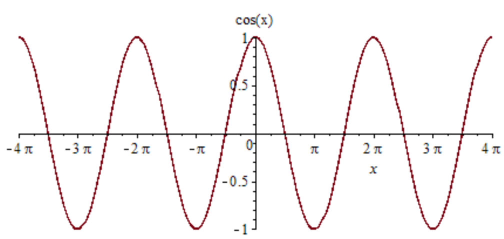
and is measured in radians/unit time.

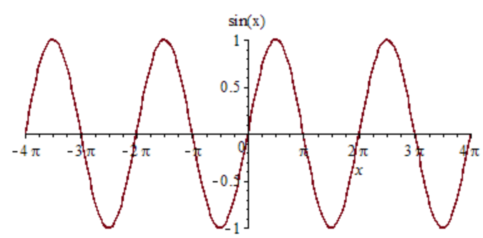
Graphs

The functions $\sin(x)$ and $\cos(x)$ can be thought of as the projections onto the vertical and horizontal axes respectively of the unit circle when the radius makes an angle x with the horizontal:



The graphs of $\cos(x)$ and $\sin(x)$ are:





Outside of the range $(0, 2\pi)$ these functions repeat themselves, so that

$$\sin(x + 2\pi) = \sin(x), \quad \sin(x + 4\pi) = \sin(x) \quad \text{etc.}$$

These functions are periodic with period $T = 2\pi$.

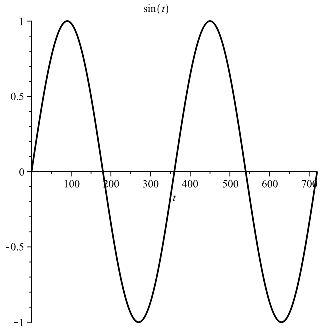
Values

The following values of the sine and cosine functions are important and should be remembered:

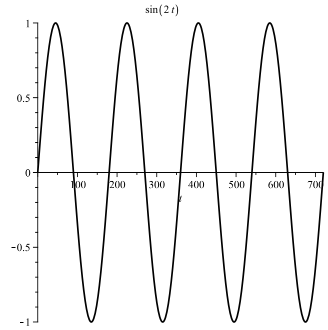
x	0	$\frac{\pi}{2}$	$n\pi$	$2n\pi$
$\sin(x)$	0	1	0	0
$\cos(x)$	1	0	$(-1)^n$	1

Graphs with different periods

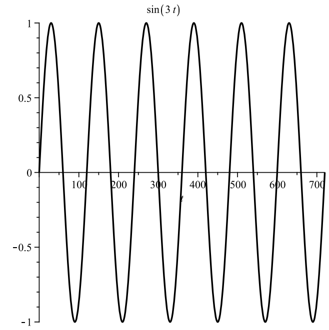
Sine and cosine can be defined on other intervals with different periods. Consider $\sin(\omega x)$, which has **angular frequency** ω. The angular frequency is how many cycles are completed in 2π seconds. For example:



$\sin t$



$\sin 2t$



$\sin 3t$

So for a function of period T:

$$\sin(\omega(x+T)) = \sin(\omega x)$$

and therefore

$$\omega T = 2\pi n,$$

where n is an integer. Hence

$$T = \frac{2\pi n}{\omega} \qquad \text{or} \qquad \omega = \frac{2\pi n}{T}$$

The **fundamental frequency** corresponds to $n = 1$.

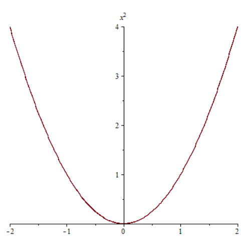
1.3 Odd and Even functions

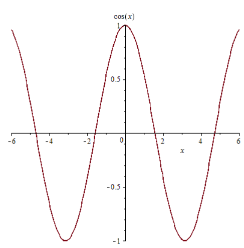
Even functions

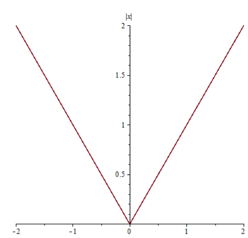
The definition of an **even** function is

$$f(-x) = f(x).$$

From a graphical perspective, an even function has **reflectional symmetry** in the y-axis. Some examples of even functions are x^2; any even power of x; $\cos(x)$; $|x|$ and so on





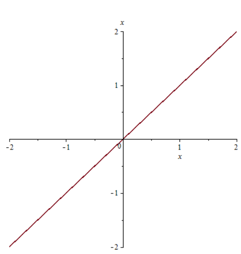


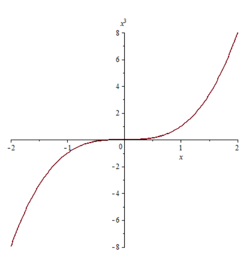
Odd functions

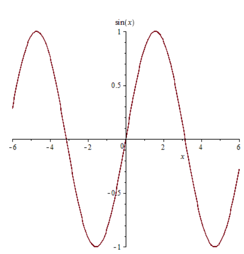
The definition of an **odd** function is

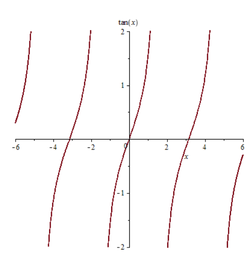
$$f(-x) = -f(x).$$

From a graphical perspective, an odd function has **rotational symmetry about the origin** ie the graph remains unchanged when rotated by 180^{O} about the origin. Some examples of odd functions are x; x^3; any odd power of x; $\sin(x)$; $\tan(x)$ and so on









Note: Being able to recognise odd and even functions is important in Fourier Series and Fourier Transforms because this knowledge can save having to calculate certain integrals.

Properties of odd and even functions

- If $f(x)$ is odd then $\int_{-a}^{a} f(x)\, dx = 0$
- If $f(x)$ is even then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$
- The product of two even functions is **even**

  If $h(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are even then
  $$h(-x) = f(-x)g(-x) = f(x)g(x) = h(x)$$
  so $h(-x) = h(x)$ and hence $h(x)$ an even function
- The product of two odd functions is **even**

  If $h(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are odd then
  $$h(-x) = f(-x)g(-x) = -f(x) \times -g(x) = f(x)g(x) = h(x)$$
  so $h(-x) = h(x)$ and hence $h(x)$ an even function
- The product of an odd and an even function is **odd**

  If $h(x) = f(x)g(x)$, where $f(x)$ is odd and $g(x)$ is even then
  $$h(-x) = f(-x)g(-x) = -f(x) \times g(x) = -h(x)$$
  so $h(-x) = -h(x)$ and hence $h(x)$ an odd function

Tutorial 1.3

State which of the following are odd, even or neither

$$\begin{array}{llllllll} (i) & x + x^3 & (ii) & x^2 + \sin(x) & (iii) & x^2 + \cos(x) & (iv) & (x+2)^2 \\ (v) & x\sin(x) & (vi) & x\cos(x) & (vii) & x^2\sin(x) & (viii) & |x| + 1 \end{array}$$

1.4 Revision of integration

To calculate Fourier Series and Fourier Transforms you need to be able to evaluate integrals involving the trigonometric functions.

Reminder:

$$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) \qquad \int \cos(ax)\, dx = \frac{1}{a}\sin(ax)$$

Example 2

Calculate

$$(i) \int \sin(n\pi x)\, dx \quad (ii) \int \cos\left(\frac{nx}{3}\right) dx \quad (iii) \int_{-\pi}^{\pi} \sin(nx)\, dx \quad (iv) \int_{0}^{3} \cos\left(\frac{2n\pi x}{3}\right) dx$$

Tutorial 1.4

Calculate the following integrals:

$$
\begin{array}{llllllll}
(i) & \displaystyle\int \sin(3x)\,dx & (ii) & \displaystyle\int \cos\left(\frac{x}{2}\right)\,dx & (iii) & \displaystyle\int \sin\left(\frac{2n\pi x}{3}\right)\,dx & (iv) & \displaystyle\int \cos\left(\frac{nx}{4}\right)\,dx \\
(v) & \displaystyle\int_0^{\frac{\pi}{2}} \sin(3x)\,dx & (vi) & \displaystyle\int_0^{\pi} \cos\left(\frac{x}{2}\right)\,dx & (vii) & \displaystyle\int_0^{3} \sin\left(\frac{n\pi x}{3}\right)\,dx & (viii) & \displaystyle\int_0^{2\pi} \cos\left(\frac{nx}{4}\right)\,dx
\end{array}
$$

1.5 Integration by parts

If u and v are functions of x, we know from the product rule that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Transposing this gives

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

If we integrate both sides with respect to x we get

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

This technique is called **integration by parts**. Notice that integration by parts does not find the value of the integration - it simply replaces one integral with a new (hopefully simpler) integral that can be done.

An alternative form of the integration by parts formula is

$$\int u\,v\,dx = u\int vdx - \int\left(\left\{\int vdx\right\}\frac{du}{dx}\right)dx.$$

Note that if you integrate the wrong function, then the integral will get worse rather than better!

Example 3

Evaluate

$$(i)\quad \int x\cos(2x)\,dx, \qquad (ii)\quad \int_0^1 (t-1)\sin(t)\,dt, \qquad (iii)\quad \int_0^1 x\,\cos(2\pi x)\,dx.$$

Tutorial 1.5

Evaluate the following integrals (assume n is an integer):

$$
\begin{array}{llllll}
(i) & \displaystyle\int (t-2)\cos(t)\,dt & (ii) & \displaystyle\int_0^{2\pi} t\sin(nt)\,dt & (iii) & \displaystyle\int_{-1}^{1} t\sin(2\pi nt)\,dt \\
(iv) & \displaystyle\int_0^{\frac{\pi}{\omega}} t\cos(\omega t)\,dt & (v) & \displaystyle\int_0^{1} (1-t)\sin(\omega t)\,dt & (vi) & \displaystyle\int x^2\sin(3x)\,dx
\end{array}
$$

1.6 Other integration techniques

Integrals such as

$$\int \cos(x)e^x\,dx$$

can be calculated using integration by parts. However, there is an alternative, simpler way of doing these particular type of calculations. This technique uses the Euler identity

$$e^{ix} = \cos(x) + i\sin(x) \qquad \text{so that} \qquad \cos(x) = \text{Re }(e^{ix}) \qquad \text{and} \qquad \sin(x) = \text{Im }(e^{ix})$$

Example

Find $\int \cos(x)\, e^x\, dx$

Tutorial 1.6

Evaluate the following integrals (assume n is an integer):

$$(i)\quad \int e^{2t}\sin(t)\,dt \quad (ii)\quad \int e^{-t}\cos(\omega t)\,dt \quad (iii)\quad \int_0^1 e^{-2\omega}\sin(\omega t)\,d\omega \quad (iv)\quad \int_{-1}^1 e^{-3t}\sin(2\pi n t)\,dt$$

2 Fourier Series

In the 1820s Joseph Fourier showed that any periodic function can be represented by a set of harmonics of the basic periodic functions sine and cosine. This discovery provides the basis of Fourier techniques which underlie modern digital signal processing, coding and information theory.

The whole range Fourier Series

Any periodic function $f(x)$, of period T, can be represented as a sum of harmonics of sine and/or cosine. Mathematically this is written as follows:

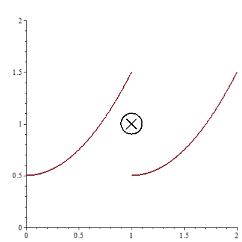
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right),$$

where the constants $\{a_n\}$ and $\{b_n\}$ are known as the **Fourier coefficients** and are given by

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T f(t)\,dt \\ a_n &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi n t}{T}\right)dt, \qquad n = 1, 2, 3, \ldots \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right)dt \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Notes

- The series is periodic with period T
- The interval 0 to T can be replaced by $-\frac{T}{2}$ to $\frac{T}{2}$ if this is more convenient.
- If $f(t)$ is discontinuous so that it has two possible values (that is it has a jump):



  then the series converges (as the number of terms increases) to the **average** or **midpoint** of the two values.
- $\frac{a_0}{2}$ is the **average value of** $f(t)$ over the interval $[0, T]$.

- The first term in the sum, corresponding to $n = 1$ is

$$a_1 \cos\left(\frac{2\pi t}{T}\right) + b_1 \sin\left(\frac{2\pi t}{T}\right)$$

and is the **fundamental frequency** or **first harmonic** and has frequency $\frac{1}{T}$.

The n-th term is

$$a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

and is the **n-th harmonic** and has frequency $\frac{n}{T}$. The n-th harmonic can be written in the form

$$R_n \sin\left(\frac{2\pi n t}{T} + \delta_n\right),$$

where $R_n = \sqrt{a_n^2 + b_n^2}$ is the **amplitude** and $\tan\delta_n = \frac{a_n}{b_n}$ is the **phase**.

Derivation of the Fourier Series coefficients

In the following we use $\omega = \frac{2\pi}{T}$ to simplify the integrations. Consider

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t).$$

- To calculate a_0 we note that :

$$\int_0^T \sin(n\omega t)\,dt = 0, \qquad \text{and} \qquad \int_0^T \cos(n\omega t)\,dt = 0 \qquad \text{for } n > 0$$

If we now integrate the series term by term over the interval $[0, T]$, then

$$\int_0^T f(t)\,dt = \frac{a_0}{2}\int_0^T dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos(n\omega t)\,dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin(n\omega t)\,dt.$$

The second and third integrals on the right are zero, using the results above, and so

$$\int_0^T f(t)\,dt = \frac{a_0}{2}\int_0^T dt = \frac{a_0}{2} \times T$$

and so

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt.$$

- To find a_n we need the results

$$\int_0^T \cos(m\omega t)\cos(n\omega t)\,dt = \begin{cases} 0, & m \neq n \\ \frac{T}{2}, & m = n \end{cases} \qquad \text{and} \qquad \int_0^T \sin(m\omega t)\cos(n\omega t)\,dt = 0.$$

Multiply the series by $\cos(m\omega t)$ and integrate the resulting series term by term over the interval $[0, T]$:

$$\begin{aligned} \int_0^T f(t)\,\cos(m\omega t)\,dt = \frac{a_0}{2}\int_0^T \cos(m\omega t)\,dt \quad &+ \sum_{n=1}^{\infty} a_n \int_0^T \cos(m\omega t)\cos(n\omega t)\,dt \\ &+ \sum_{n=1}^{\infty} b_n \int_0^T \cos(m\omega t)\sin(n\omega t)\,dt. \end{aligned}$$

The first and third integrals on the right are zero, using the above results, and the second integral is zero unless $n = m$ when it has a value of $\frac{T}{2}$. Hence

$$\int_0^T f(t)\,\cos(m\omega t)\,dt = a_m \times \frac{T}{2}$$

and so

$$a_m = \frac{2}{T}\int_0^T f(t)\,\cos(m\omega t)\,dt$$

- To find b_n we need the results

$$\int_0^T \sin(m\omega t)\sin(n\omega t)\,dt = \begin{cases} 0, & m \neq n \\ \frac{T}{2}, & m = n \end{cases} \qquad \text{and} \qquad \int_0^T \sin(m\omega t)\cos(n\omega t)\,dt = 0.$$

Multiply the series by $\sin(m\omega t)$ and integrate the resulting series term by term over the interval $[0, T]$:

$$\begin{aligned}\int_0^T f(t)\,\sin(m\omega t)\,dt = \frac{a_0}{2}\int_0^T \sin(m\omega t)\,dt \quad &+ \sum_{n=1}^{\infty} a_n \int_0^T \sin(m\omega t)\cos(n\omega t)\,dt \\ &+ \sum_{n=1}^{\infty} b_n \int_0^T \sin(m\omega t)\sin(n\omega t)\,dt.\end{aligned}$$

The first and second integrals on the right are zero, using the above results, and the third integral is zero unless $n = m$ when it has a value of $\frac{T}{2}$. Hence

$$\int_0^T f(x)\,\sin(m\omega t)\,dt = b_m \times \frac{T}{2}$$

and so

$$b_m = \frac{2}{T}\int_0^T f(t)\,\sin(m\omega t)\,dt$$

Example 4 - Square Wave

A periodic waveform of period 4 is modelled over the interval $[0, 4]$ by $f(t) = u(t) - u(t-2)$
(i) Sketch the waveform for $-6 < t < 6$.
(ii) Show that the whole-range Fourier Series for $f(t)$ is

$$f(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\left(\frac{n\pi t}{2}\right).$$

(iii) Write down the first 4 non-zero terms in the Fourier Series
(iv) What value does the series converge to at $t = 0$, $t = 1$, $t = 2$ and $t = 3$?

Example 5 - Saw tooth

A periodic waveform of period 4 is modelled over the interval $(0, 4)$ by $f(t) = t(u(t) - u(t-4))$
(i) Sketch the waveform for $-8 < t < 8$.
(ii) Obtain the whole-range Fourier Series for $f(t)$.
(iii) List the whole-range Fourier series of $f(t)$ as far as the fourth harmonic.
(iv) State the value of the whole-range Fourier series when $t = 0$ and when $t = 4$.
(v) By considering the function and the series at $t = 1$ show that

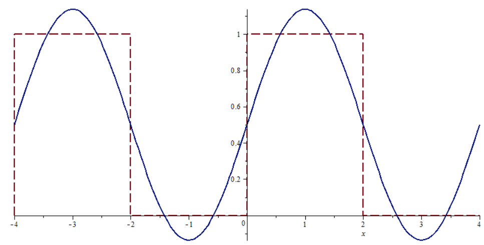
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 4 - Square Wave

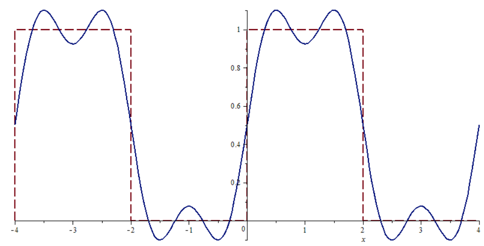
Writing out the terms

$$f(t) \sim \frac{1}{2} + \frac{2}{\pi}\left(\sin\left(\frac{\pi t}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{2}\right) + \dots\right).$$

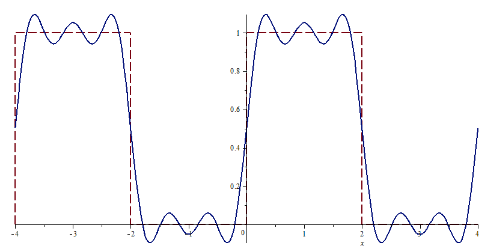
2 Terms:



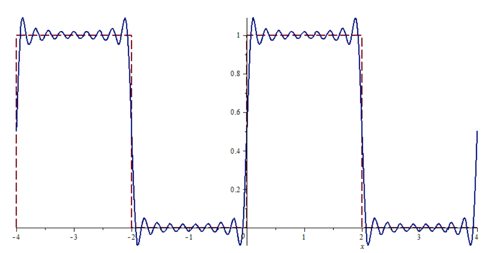
3 Terms:



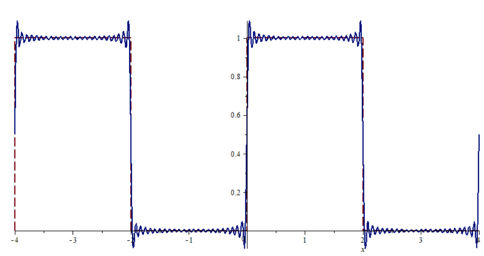
4 Terms:



10 Terms:



25 Terms:

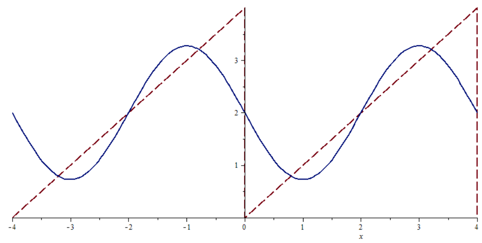


Example 5 - Saw tooth Wave

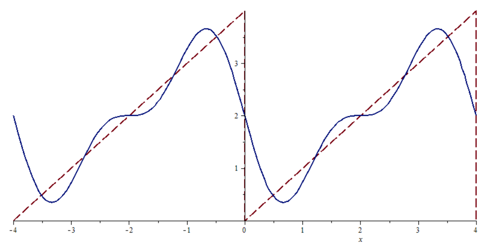
Writing out the terms

$$f(t) \sim 2 - \frac{4}{\pi}\left(\sin\left(\frac{\pi t}{2}\right) - \frac{1}{2}\sin(\pi t) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right) + \dots\right).$$

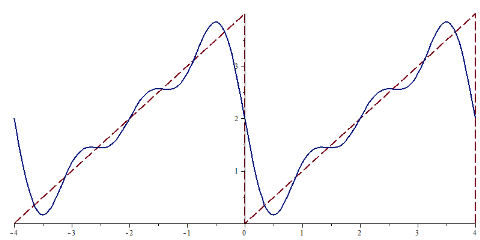
2 Terms:



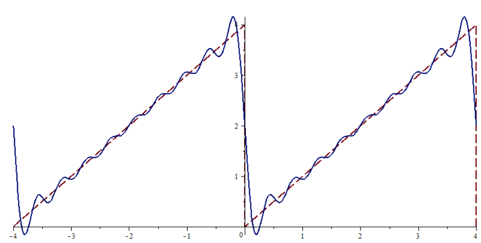
3 Terms:



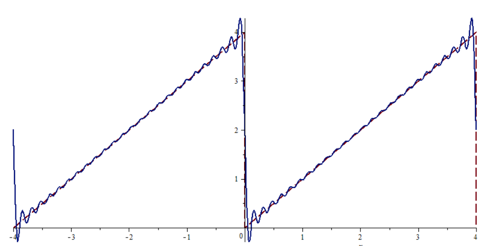
4 Terms:



10 Terms:



25 Terms:



Tutorial 2

1. A signal of period 0.5 is defined by

$$f(t)=\begin{cases} 2, & -0.25\le t<0,\\ -1, & 0\le t<0.25.\end{cases}$$

(i) Sketch the waveform for $-1<t<1$.
(ii) Show that the Fourier Series expansion if $f(t)$ is

$$f(t)=\frac{1}{2}-\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(4n\pi t).$$

(iii) Write down the first 4 non-zero terms.
(iv) Use the first four terms to estimate $f(0.1)$ to three decimal places and compare it with the exact value of $f(0.1)$.

2. A periodic function of period $T=10$ is defined over one period by

$$f(x)=100(u(x+5)-u(x))-100(u(x)-u(x-5))$$

Find the whole-range Fourier series. What does the series converge to at $x=0$?

3. An electro-mechanical device is controlled by a waveform defined over one period by

$$f(x)=\begin{cases} 0, & -1<x\le 0,\\ -x, & 0\le x<1,\end{cases}$$

(i) Sketch the graph of the function for the range $-3<x<3$.
(ii) Show that the Fourier Series is given by

$$-\frac{1}{4}+\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi x)+\frac{(-1)^n}{n\pi}\sin(n\pi x)\right]$$

(iii) Write down the first three terms (ie up to and including terms when $n=1$)
(iv) State the value the series converges to when $x=1$.

4. A periodic waveform has amplitude

$$f(t)=(\pi+t)(u(t+\pi)-u(t))+\pi(u(t)-u(t-\pi))$$

(i) Sketch the function over the interval $[-3\pi,3\pi]$.
(ii) Obtain the whole-range series for $f(t)$.
(iii) Show that the whole-range series can be written as

$$\frac{3\pi}{4}+\frac{2}{\pi}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}\cos((2r+1)t)-\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\sin(rt).$$

Deduce from the series that

$$\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}=\frac{\pi^2}{8}.$$

5. Show that

$$\int_0^{2\pi}x\sin(nx)\,dx=-\frac{2\pi}{n},$$

where n is a positive integer. Hence expand $f(x)=x+\pi$, $0<x<2\pi$ as a whole-range Fourier series. What is the value of the Fourier series when $x=0$ and $x=2\pi$.

6. Obtain the whole-range Fourier series which represents the function

$$f(x) = \begin{cases} -1, & -1 < x < 0, \\ x, & 0 \leq x < 1. \end{cases}$$

7. Find the Fourier Series for the "half-rectified" sine wave of period $T = 2c$, where

$$f(x) = \sin\left(\frac{\pi x}{c}\right)(u(x) - u(x-c)), \qquad \text{for} - c \leq x \leq c.$$

Hint: You should express the product of two sines; or the product of a cosine and a sine in terms of single sine or cosine functions.

8. The periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \cosh\,(x), & 0 \leq x \leq \pi \end{cases}$$

is to be expanded as a Fourier Series.

(i) Obtain expressions for $\{a_n\}$ and $\{b_n\}$.

(ii) By considering $x = 0$ and $x = \pi$ show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2\sinh(\pi)} - \frac{1}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2\tanh(\pi)} - \frac{1}{2}$$

3 Even and odd functions

Recall:

1.
   - If $f(-x) = f(x)$ then $f(x)$ is an **even function**
   - $f(x)$ has **reflectional symmetry** in the y-axis.
   - $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2.
   - If $f(-x) = -f(x)$ then $f(x)$ is an **odd function**
   - $f(x)$ has **rotational symmetry** about the origin
   - $\int_{-a}^{a} f(x)\,dx = 0$
3.
   - **even** function $\times$ **even** function $\Rightarrow$ **even** function
   - **odd** function $\times$ **odd** function $\Rightarrow$ **even** function
   - **even** function $\times$ **odd** function $\Rightarrow$ **odd** function

Fourier Series for an odd function

If $f(x)$ is odd then

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = 0, \qquad a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,\cos\left(\frac{2n\pi x}{T}\right)\,dx = 0$$

and

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,\sin\left(\frac{2n\pi x}{T}\right)\,dx = \frac{4}{T}\int_{0}^{\frac{T}{2}} f(x)\,\sin\left(\frac{2n\pi x}{T}\right)\,dx.$$

The Fourier Series therefore only includes **sine terms** and is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{T}\right).$$

Fourier Series for an even function

If $f(x)$ is even then

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{4}{T}\int_{0}^{\frac{T}{2}} f(x)\,dx,$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,\cos\left(\frac{2n\pi x}{T}\right)\,dx = \frac{4}{T}\int_{0}^{\frac{T}{2}} f(x)\,\cos\left(\frac{2n\pi x}{T}\right)\,dx$$

and

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,\sin\left(\frac{2n\pi x}{T}\right)\,dx = 0.$$

The Fourier Series therefore only includes the **constant** and **cosine terms** and is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{T}\right).$$

Example 6

Consider the function defined over one period by

$$f(t) = -\frac{1}{2}\left(u(t+1) - u\left(t+\frac{1}{2}\right)\right) + t\left(u\left(t+\frac{1}{2}\right) - u\left(t-\frac{1}{2}\right)\right) + \frac{1}{2}\left(u\left(t-\frac{1}{2}\right) - u(t-1)\right)$$

(i) Sketch the waveform for $-3 < t < 3$.
(ii) Find the Fourier Series representation of $f(t)$.

Example 7

A periodic waveform of period 4 is modelled over the interval $[-2, 2]$ by

$$f(t) = \begin{cases} 2+t, & -2 < t \leq -1 \\ 1, & -1 \leq t \leq 1 \\ 2-t, & 1 \leq t \leq 2. \end{cases}$$

(i) Sketch the waveform for $-6 < t < 6$.
(ii) Obtain the whole-range Fourier coefficients for $f(t)$.
(iii) List the whole-range Fourier series of $f(t)$ as far as the fifth harmonic.

Tutorial 3

1. A periodic waveform of period 4 is modelled on the interval $(-2, 2)$ by

$$f(t) = \begin{cases} -2, & -2 < t \leq 0, \\ 2, & 0 < t < 2. \end{cases}$$

(i) Sketch the waveform for $-6 < t < 6$.
(ii) Find the Fourier Series representation of $f(t)$.
(iii) List the first 5 terms of the Fourier Series.

2. A periodic function of period $T = 8$ is defined over one period by

$$f(t) = 2(u(t+4) - u(t+1)) + 3(u(t+1) - u(t-1)) + 2(u(t-1) - u(t-4))$$

Find the Fourier Series representation of this function

3. Find the Fourier Series representation of

$$f(x) = (2+x)(u(x+2) - u(x)) + (2-x)(u(x) - u(x-2)).$$

4. Find the Fourier Series of the wave of period 4 given by the functions

$$f(x) = \begin{cases} -x, & -2 \leq x < -1 \\ 1 & -1 \leq x < 1 \\ x & 1 \leq x < 2 \end{cases}$$

5. Consider the function defined over one period by

$$f(x) = \begin{cases} -1 & -2 < x \leq -1 \\ x & -1 < x \leq 1 \\ 1 & 1 < x \leq 2 \end{cases}$$

(a) Sketch the graph of the function for the range $-6 < x \leq 6$.
(b) Show that the Fourier Series is

$$\sum_{n=1}^{\infty} \left(\frac{4\sin(n\pi/2)}{n^2\pi^2} - \frac{2(-1)^n}{n\pi} \right) \sin\left(\frac{n\pi x}{2}\right).$$

(c) Write down the first three terms of the series.
(d) State the value the series converges to when $x = 2$.

6. The function $f(x)$ is defined over one period by

$$f(x) = \left(1 + \frac{x}{c}\right)(u(x+c) - u(x)) + \left(1 - \frac{x}{c}\right)(u(x) - u(x-c)).$$

(i) Sketch the graph over 3 periods and show that $f(x)$ is even.
(ii) Find the corresponding Fourier Series

4 Half-range series

Suppose $f(t)$ is defined on the interval $0 \leq t \leq \tau$. We may enlarge or extend the function to cover a range $[-\tau, \tau]$ by defining it **arbitrarily** in the negative region. The most useful extensions are the **half-range cosine** of **half-range sine** series, where the function is extended to an even or odd function respectively.

Example 8

For the function $f(t) = t$ defined on the interval $0 < t < 4$, determine
a) the half-range cosine series expansion and
b) the half-range sine series expansion of $f(t)$.

Tutorial 4

1. Determine the half-range cosine series representations of the following functions:

   (a) $f(t) = 2(u(t) - u(t-1)) + u(t-1) - u(t-2), \quad 0 \le t < 2$

   (b) $f(x) = (2x+1)(u(x) - u(x-1)), \quad 0 \le x \le 1$

   (c) $f(t) = 2t - 1, \quad 0 \le t < 1;$

   (d) $f(t) = \sin \pi t, \quad 0 \le t < 1.$

2. Determine the half-range sine series representing the following functions:

   (a) $f(t) = 1, \quad 0 \le t < 1;$

   (b) $f(x) = (2x+1)(u(x) - u(x-1)), \quad 0 \le x \le 1$

   (c) $f(t) = 2 - t, \quad 0 \le t < 2.$

   (d) $f(t) = t^2, \quad 0 \le t < 1;$

3. The function $f(t)$ of period T is defined by

$$\frac{2}{T}\left(u(t) - u\left(t - \frac{T}{2}\right)\right) + \frac{2}{T}(T-t)\left(u\left(t - \frac{T}{2}\right) - u(t-T)\right).$$

   (a) Find the whole range series for this function.

   (b) Find the half-range sine series

5 Complex notation for Fourier Series

The Fourier series is sometimes written in complex form to simplify calculations. The Fourier Series of a function $f(x)$ of period T is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + b_n \sin(n\omega x)$$

where $\omega = \dfrac{2\pi}{T}$. If we use the Euler formulae:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad \text{and} \qquad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

then

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \text{and} \qquad \sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

We can therefore write

$$\begin{aligned} a_n \cos(n\omega x) + b_n \sin(n\omega x) &= \frac{a_n}{2}(e^{in\omega x} + e^{-in\omega x}) + \frac{b_n}{2i}(e^{in\omega x} - e^{-in\omega x}) \\ &= \frac{1}{2}(a_n - ib_n)e^{in\omega x} + \frac{1}{2}(a_n + ib_n)e^{-in\omega x} \\ &= c_n e^{in\omega x} + k_n e^{-in\omega x} \quad \text{say,} \end{aligned}$$

so that

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega x} + k_n e^{-in\omega x},$$

where $c_0 = \dfrac{a_0}{2}$.

It may be easily verified that

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{T}\int_0^T f(x)\ (\cos(n\omega x) - i\sin(n\omega x))\ dx = \frac{1}{T}\int_0^T f(x)\, e^{-in\omega x}\, dx$$

and

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{T}\int_0^T f(x)\ (\cos(n\omega x) + i\sin(n\omega x))\ dx = \frac{1}{T}\int_0^T f(x)\, e^{in\omega x}\, dx$$

so that $k_n = c_{-n}$ and hence

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega x},$$

with

$$c_n = \frac{1}{T}\int_0^T f(x)\, e^{-in\omega x}\, dx \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Alternatively:

$$c_n = \frac{1}{T}\int_0^T f(x)\cos(n\omega x)\, dx - \frac{i}{T}\int_0^T f(x)\sin(n\omega x)\ dx.$$

- If $f(x)$ is an **even function**, the second of these integrals is zero so that the coefficients are real.
- If $f(x)$ is an **odd function**, the first of the integrals is zero so that the coefficients are purely imaginary.
- c_0 must be calculated seperately from c_n.
- The real form of the Fourier Series can be recovered using the fact that

$$c_n = \frac{1}{2}(a_n - ib_n).$$

Example 9

Find the complex Fourier Series of the square wave of period 4 given by

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$

Example 10

Show that the complex Fourier Series of the function of period 4 given by $f(x) = x$ for $0 \leq x < 4$ is

$$f(x) = 2 + \frac{2i}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\, e^{in\pi x/2}.$$

Convert the series to real form.

Tutorial 5

1. Find the complex fourier series of the function defined over one period by

$$f(x) = \begin{cases} 2 & -0.25 \le x < 0 \\ -1 & 0 \le x < 0.25 \end{cases}$$

2. Find the complex fourier series of the function defined over one period by

$$f(x) = \begin{cases} 0 & -1 < x \le 0 \\ -x & 0 \le x < 1 \end{cases}$$

3. Find the complex fourier series of the function defined over one period by

$$f(x) = \begin{cases} -1 & -2 < x \le -1 \\ x & -1 < x \le \ \ 1 \\ 1 & 1 < x \le \ \ 2 \end{cases}$$

4. Find the complex fourier series of the function defined over one period by

$$f(x) = \begin{cases} 2 & -4 < x \le -2 \\ -x & -2 < x \le \ \ 0 \\ x & 0 < x \le \ \ 2 \\ 2 & 2 < x \le \ \ 4 \end{cases}$$

6 Integration of Fourier Series

Theorem

The Fourier Series corresponding to a function $f(t)$ may be integrated term by term from a to t and the resulting series will converge to $\int_a^t f(u)\,du$ provided $f(t)$ is piecewise continuous in $-\frac{T}{2} \le t \le \frac{T}{2}$.

Example 11

Expand $f(x) = x$, $0 < x < 2$ in a half-range sine series. Find a Fourier series for $f(x) = x^2$, $0 < x < 2$ by integrating this series. Hence evaluate the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

Tutorial 6

1. (i) Find the Fourier series for

$$f(x) = -(u(x+\pi) - u(x)) + (u(x) - u(x-\pi))$$

   (ii) By integrating $f(x)$ and the series and choosing the arbitrary constant to be zero, find the Fourier Series for $-x(u(x+\pi) - u(x)) + x(u(x) - u(x-\pi))$.

   (iii) By integrating again, find the Fourier Series for $-x^2(u(x+\pi)-u(x))+x^2(u(x)-u(x-\pi))$.

2. (a) Determine the Fourier Series expansion of the periodic function $f(t) = t, \quad -1 < t < 1$.

(b) Given that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

determine, by integration, the Fourier Series expansion of the periodic function $f(t) = t^2$ $-1 < t < 1$.

(c) Find the Fourier Series expansion of $f(t) = t^3$ for $-1 < t < 1$.

3. Show that for $-\pi < x < \pi$,

$$x\cos(x) = -\frac{1}{2}\sin(x) + 2\left(\frac{2}{1\times 3}\sin(2x) - \frac{3}{2\times 4}\sin(3x) + \frac{4}{3\times 5}\sin(4x) + \ldots\right)$$

Given

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} = \frac{1}{4},$$

use your result to show that

$$x\sin(x) = 1 - \frac{1}{2}\cos(x) - 2\left(\frac{\cos(2x)}{1\times 3} - \frac{\cos(3x)}{2\times 4} + \frac{\cos(4x)}{3\times 5} + \ldots\right)$$

4. If $f(t)$ is a periodic function whose definition over one period is

$$f(t) = \begin{cases} 0 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$$

Find by integration the Fourier Series of

$$g(t) = \begin{cases} 0 & -1 < t < 0 \\ t & 0 < t < 1 \end{cases}$$

You may use the result

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$

7 Parsevals Identity

If $\{a_n\}$ and $\{b_n\}$ are the Fourier coefficients corresponding to $f(x)$ then

$$\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Example 12

Find the half-range cosine series of the function $f(x) = x$, $0 < x < 2$. Write down Parsevals identity corresponding to this Fourier Series. Hence determine the value of the sum of the series

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \frac{1}{n^4} + \ldots$$

Tutorial 7

1. Show that the Fourier Series of $f(t) = l^2 - t^2$ for $(-l, l)$ is

$$f(t) = \frac{2l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{l}\right)$$

Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. Determine the Fourier Series for

$$f(x) = \begin{cases} 0 & -l \leq x < 0 \\ x & 0 \leq x < l \end{cases}$$

Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Hint: You will need to use the result

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$$

3. The Fourier Series of $f(x) = x^3 - 4x$, $-2 \leq x < 2$ is

$$f(x) = \frac{96}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\left(\frac{n\pi x}{2}\right).$$

Use Parsevals Theorem to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

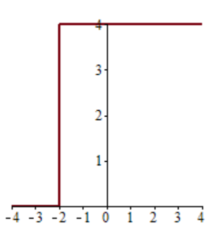
4. Expand $f(x) = \sin(x)$, $0 < x < \pi$ in a Fourier cosine series. Hence show that

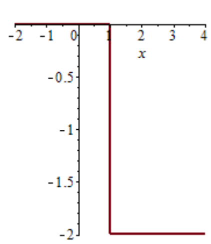
$$\frac{1}{1^2 \times 3^2} + \frac{1}{3^2 \times 5^2} + \frac{1}{5^2 \times 7^2} + \ldots = \frac{\pi^2 - 8}{16}.$$

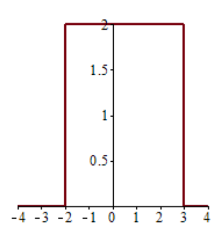
Answers

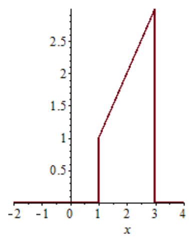
Tutorial 1

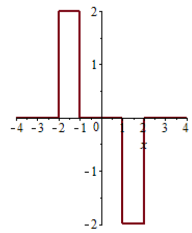
Tutorial 1.1

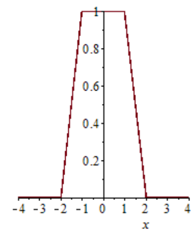












Tutorial 1.3

(i) Odd (ii) Neither (iii) Even (iv) Neither (v) Even (vi) Odd (vii) Odd
(viii) Even

Tutorial 1.4

(i) $-\dfrac{1}{3}\cos(3x)+C$ (ii) $2\sin\left(\dfrac{x}{2}\right)+C$ (iii) $-\dfrac{3}{2\pi n}\cos\left(\dfrac{2\pi nx}{3}\right)+C$ (iv) $\dfrac{4}{n}\sin\left(\dfrac{nx}{4}\right)+C$
(v) $\dfrac{1}{3}$ (vi) 2 (vii) $\dfrac{3}{n\pi}(1-(-1)^n)$ (viii) $\dfrac{4}{n}\sin\left(\dfrac{n\pi}{2}\right)$

Tutorial 1.5

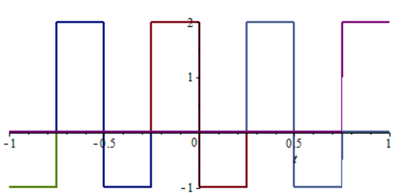
(i) $(t-2)\sin(t)+\cos(t)+C$ (ii) $-\dfrac{2\pi}{n}$ (iii) $-\dfrac{1}{\pi n}$ (iv) $-\dfrac{2}{\omega^2}$ (v) $\dfrac{\omega-\sin(\omega)}{\omega^2}$
(vi) $-\dfrac{x^2}{3}\cos(3x)+\dfrac{2x}{9}\sin(3x)+\dfrac{2}{27}\cos(3x)+C$

Tutorial 1.6

(i) $-\dfrac{1}{5}e^{2t}\cos(t)+\dfrac{2}{5}e^{2t}\sin(t)+C$
(ii) $\dfrac{e^{-t}}{\omega^2+1}(\omega\sin(\omega t)-\cos(\omega t))+C$
(iii) $\dfrac{t-e^{-2}t\cos(t)-2e^{-2}\sin(t)}{t^2+4}$
(iv) $\dfrac{2\pi n(e^3-e^{-3})}{9+4\pi^2n^2}$

Tutorial 2

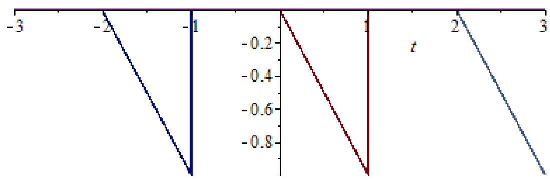
1. (i)



(iii) $\dfrac{1}{2}-\dfrac{6}{\pi}\sin(4\pi t)-\dfrac{2}{\pi}\sin(12\pi t)-\dfrac{6}{5\pi}\sin(20\pi t)$
(iv) Using first 4 terms: $f(0.1)\sim -0.942188$, Exact $f(0.1)=-1$

2. $f(x)=\dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n-1}{n}\sin\left(\dfrac{n\pi x}{5}\right)$.
At $x=0$ series converges to 0.

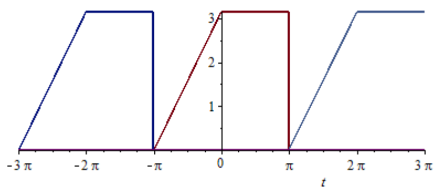
3. (i)



(iii) First 3 terms: $-\dfrac{1}{4} + \dfrac{2}{\pi^2}\cos(\pi x) - \dfrac{1}{\pi}\sin(\pi x)$

(iv) -0.5

4. (i)



(ii) $f(t) = \dfrac{3\pi}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{\pi n^2}\cos(nt) - \dfrac{(-1)^n}{n}\sin(nt).$

5. $f(x) = 2\pi - 2\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n}\sin(nx)$

When $x = 0$ and $x = 2\pi$ series converges to 2π

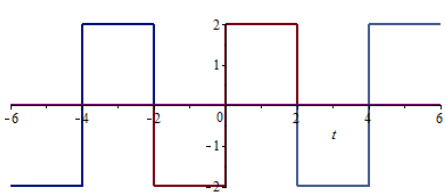
6. $f(x) = -\dfrac{1}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n - 1}{\pi^2 n^2}\cos(n\pi x) + \dfrac{1-2(-1)^n}{\pi n}\sin(n\pi x)$

7. $f(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\sin\left(\dfrac{\pi x}{c}\right) - \dfrac{1}{\pi}\displaystyle\sum_{n=2}^{\infty} \dfrac{(-1)^n + 1}{n^2 - 1}\cos\left(\dfrac{\pi n x}{c}\right)$

8. $a_0 = \dfrac{\sinh(\pi)}{\pi} \qquad a_n = \dfrac{(-1)^n}{\pi(1+n^2)}\sinh(\pi) \qquad b_n = -\dfrac{n}{\pi(1+n^2)}((-1)^n\cosh(\pi) - 1)$

Tutorial 3

1. (i)



(ii) $f(t) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{n}\sin\left(\dfrac{n\pi t}{2}\right)$

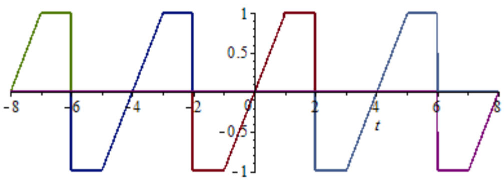
(iii) First 5 terms: $\dfrac{8}{\pi}\left(\sin\left(\dfrac{\pi t}{2}\right) + \dfrac{1}{3}\sin\left(\dfrac{3\pi t}{2}\right) + \dfrac{1}{5}\sin\left(\dfrac{5\pi t}{2}\right) + \dfrac{1}{7}\sin\left(\dfrac{7\pi t}{2}\right) + \dfrac{1}{9}\sin\left(\dfrac{9\pi t}{2}\right)\right)$

2. $f(t) = \dfrac{9}{4} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin(\pi n/4)}{n}\cos\left(\dfrac{n\pi t}{4}\right)$

3. $f(x) = 1 + \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{n^2}\cos\left(\dfrac{\pi n x}{2}\right)$

4. $f(t) = \dfrac{5}{4} + \dfrac{4}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^n - \cos(\pi n/2)}{n^2} \cos\left(\dfrac{n\pi x}{2}\right)$

5. (a)



(c) First 3 terms: $\dfrac{2(\pi+2)}{\pi^2} \sin\left(\dfrac{\pi x}{2}\right) - \dfrac{\sin(\pi x)}{\pi} + \dfrac{2(3\pi-2)}{9\pi^2} \sin\left(\dfrac{3\pi x}{2}\right)$

(d) 0

6. $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{n^2} \cos\left(\dfrac{n\pi x}{c}\right)$

Tutorial 4

1. (a) $f(t) = \dfrac{3}{2} + \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{\sin(\pi n/2)}{n} \cos\left(\dfrac{n\pi t}{2}\right)$

   (b) $f(x) = 2 - \dfrac{4}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{n^2} \cos(n\pi x)$

   (c) $f(t) = -\dfrac{4}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{n^2} \cos(n\pi t)$

   (d) $f(t) = \dfrac{2}{\pi} - \dfrac{2}{\pi} \sum_{n=2}^{\infty} \dfrac{1+(-1)^n}{n^2-1} \cos(n\pi t)$

2. (a) $f(t) = \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{n} \sin(n\pi t)$

   (b) $f(x) = \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1-3(-1)^n}{n} \sin(n\pi x)$

   (c) $f(t) = \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \sin\left(\dfrac{n\pi t}{2}\right)$

   (d) $f(t) = -\dfrac{2}{\pi} \sum_{n=1}^{\infty} \left(\dfrac{(-1)^n}{n} + \dfrac{2(1-(-1)^n)}{\pi^2 n^3}\right) \sin(n\pi t)$

3. Whole range series:

$$f(t) = \frac{1}{T}\left(1 + \frac{T}{4}\right) - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\left(\frac{2\pi n t}{T}\right) - \frac{2}{\pi T} \sum_{n=1}^{\infty} \frac{1}{n}\left((-1)^n - 1 - \frac{T}{2}(-1)^n\right) \sin\left(\frac{2\pi n t}{T}\right)$$

Half-range sine series:

$$f(t) = -\frac{4}{\pi T} \sum_{n=1}^{\infty} \frac{1}{n}\left(\cos\left(\frac{\pi n}{2}\right) - 1 - \frac{T}{2}\cos\left(\frac{\pi n}{2}\right) - \frac{T}{\pi n}\sin\left(\frac{\pi n}{2}\right)\right) \sin\left(\frac{\pi n t}{T}\right)$$

Tutorial 5

1. $c_0 = \frac{1}{2}$; $c_n = \frac{3i(1-(-1)^n)}{2\pi n}$

   $f(x) = \frac{1}{2} + \frac{3i}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{n} e^{4i\pi n x}$

2. $c_0 = -\frac{1}{4}$; $c_n = \frac{(1-(-1)^n)}{2\pi^2 n^2} - \frac{i(-1)^n}{2\pi n}$;

   $f(x) = -\frac{1}{4} + \frac{1}{2\pi n} \sum_{\substack{n=-\infty \\ n\neq 0}} \left(\frac{1-(-1)^n}{\pi n} - i(-1)^n \right) e^{i\pi n x}$

   Real form: $f(x) = -\frac{1}{4} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi^2 n^2} \cos(n\pi x) + \frac{(-1)^n}{\pi n} \sin(n\pi x)$

3. $c_0 = 0$; $c_n = \frac{i((-1)^n \pi n - 2\sin(\pi n/2)}{\pi^2 n^2}$

   $f(x) = \frac{i}{\pi^2 n^2} \sum_{\substack{n=-\infty \\ n\neq 0}} ((-1)^n \pi n - 2\sin(\pi n/2))\, e^{i\pi n x/2}$

4. $c_0 = \frac{3}{2}$; $c_n = \frac{4(\cos(\pi n/2) - 1)}{\pi^2 n^2}$

   $f(x) = \frac{3}{2} + \frac{4}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}} \frac{\cos(\pi n/2) - 1}{n^2} e^{i\pi n x/4}$

   Real form $f(x) = \frac{3}{2} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(\pi n/2) - 1}{n^2} \cos\left(\frac{\pi n x}{4}\right)$

Tutorial 6

1. (i) $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin(nx)$ (ii) $-\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos(nx)$

   (iii) $-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin(nx)$

2. (a) $t = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi t)$ (b) $t^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t)$

   (c) $t^3 = -\frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n(\pi^2 n^2 - 6)}{n^3} \sin(n\pi t)$

4. $f(t) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin(n\pi t)$ $g(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi^2 n^2} \cos(n\pi t) - \frac{(-1)^n}{\pi n} \sin(n\pi t)$

Tutorial 7

2 $f(x) = \frac{l}{4} + l \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi^2} \cos\left(\frac{n\pi x}{l}\right) - \frac{(-1)^n}{\pi n} \sin\left(\frac{n\pi x}{l}\right)$

4 $f(x) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{n^2 - 1} \cos(nx)$