Section A: Pure Mathematics [40 marks]

1   Prove by mathematical induction that $\displaystyle\sum_{r=1}^{n} \cos r\theta = \frac{\sin\left(n+\dfrac{1}{2}\right)\theta - \sin\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}}$, $n \in \square^{+}$. [5]

2   A patient in a hospital receives a course of treatment. At the start of the treatment, 80 mg of drug A is injected into the blood stream of the patient. It is known that x mg, the amount of drug A remaining in the blood stream at time t hours after an injection decreases at a rate proportional to the amount of drug A present at that instant in the blood stream. It is also found that 3 hours after an injection, only a quarter of the drug A injected remains in the blood stream of the patient.

  (i)   Write down a differential equation involving x and t. Hence find an expression for x in terms of t, where $0 \le t < 6$. [6]

The treatment is such that the same amount of drug A is injected to the patient at regular intervals of 6 hours.
Let u_n mg be the amount of drug A present immediately after the nth injection, $n \ge 1$.

  (ii)   Find constants a and b such that $u_{n+1} = a + bu_n$. [3]

   If the patient continues with this treatment indefinitely, use a calculator to determine what you can expect about the amount of drug A present in the blood stream of the patient immediately after each injection in the long run. [1]

3   The points A and B have coordinates $(0, 9, c)$ and $(d, 5, -2)$ respectively, where c and d are constants. The line l has equation $\dfrac{x+3}{-1} = \dfrac{y-1}{4} = \dfrac{z-5}{3}$.

  (a)   Given that $d = \dfrac{22}{7}$ and the line AB intersects l, find the value of c. [3]

   Find also, the coordinates of the foot of perpendicular from A to l. [3]

  (b)   Given instead that the lines AB and l are parallel, state the value of c and d. [2]

  (i)   By using a cross product, or otherwise, find the shortest distance between the lines AB and l. [3]

  (ii)   The plane p_1 contains A, B and l. Find an equation of p_1 in scalar product form. [2]

  (iii)   The plane p_2 is obtained by first translating p_1 2 units in the positive z-direction and then reflecting in the xz-plane. Obtain an equation of p_2 in scalar product form. [2]

4(a) Solve the equation $z^6 - 1 = 0$, giving the roots in the form
$$re^{i\alpha}, \text{ where } r > 0 \text{ and } -\pi < \alpha \le \pi.$$
[2]

(b) Let the points P_1, P_2, ..., P_n represent complex numbers satisfying $z^n - 1 = 0$, $n \ge 5$, as shown on the Argand diagram below. The points Q_2, Q_3, ..., Q_n are formed by dropping perpendiculars from P_1 to OP_2, and Q_k to OP_{k+1} for $k = 2, 3, ..., n-1$.
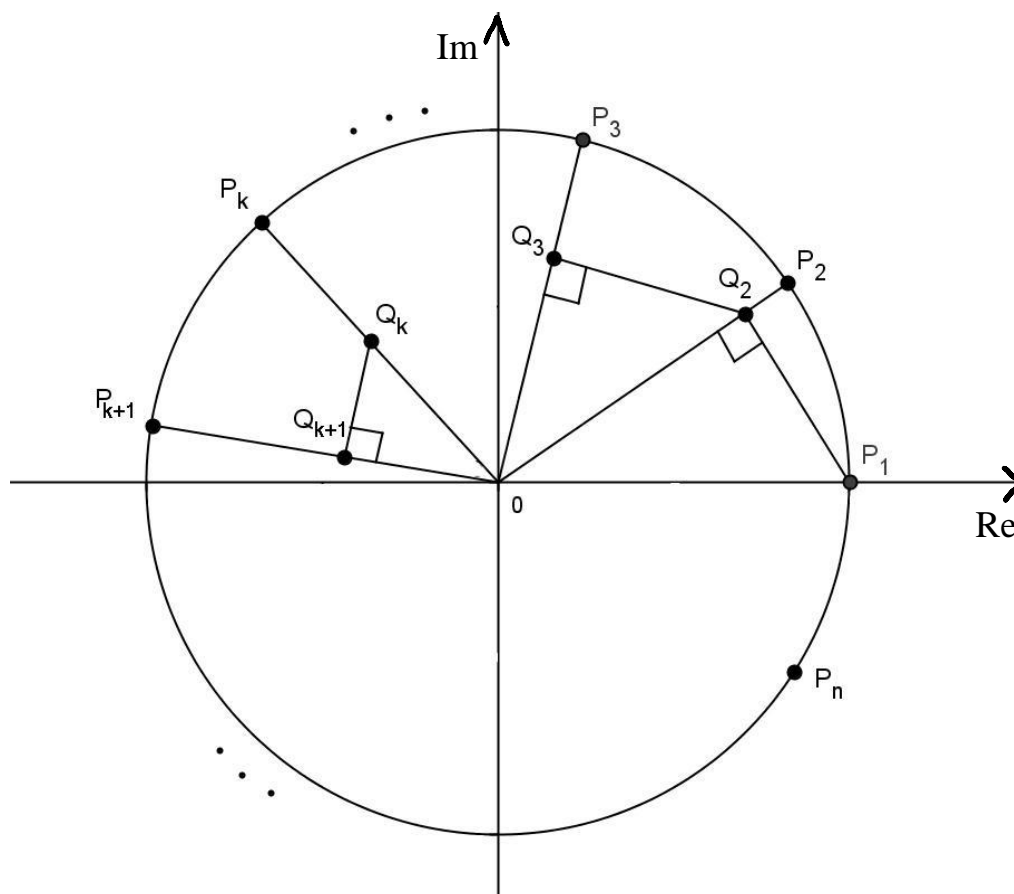
(i) State $\angle P_k O P_{k+1}$, where $1 \le k \le n-1$.
Find the length of OQ_2 and P_1Q_2. [2]

(ii) Show that $OQ_2, OQ_3, ..., OQ_n$ are in geometric progression.
Show that $Q_2Q_3, Q_3Q_4, ..., Q_{n-1}Q_n$ are also in geometric progression with the same common ratio. [2]

(iii) Show that the sum of the lengths,
$$P_1Q_2 + Q_2Q_3 + \cdots + Q_{n-1}Q_n$$
does not exceed $\cot\left(\dfrac{\pi}{n}\right)$. [4]

Section B: Statistics [60 marks]

5 The monthly rainfall in a region is modeled by a random variable with mean 159 mm and standard deviation 45 mm. Find the probability that the average monthly rainfall over five years is less than 150 mm. State an assumption you have used in your calculations. [3]

6 The National Transport Authority intends to conduct a survey to find out users' perception on the efficiency of the train system. Describe how a quota sample of 120 users can be taken. State a disadvantage of quota sampling in this context.

State, with a reason, whether stratified sampling can be conducted in this case. [4]

7 A committee of three people is to be chosen from 6 men and n women.

(i) Show that the number of different committees consisting of more women than men is $an(n-1)(n+16)$, where a is a constant to be determined. [2]

(ii) Given that the number of different committees consisting of more women than men exceeds 300, find the range of values of n. [2]

Let the random variable X be the number of men in the committee. State, with a reason, whether X follows a binomial distribution. [1]

8 A rapidly growing bacteria has been discovered and its growth was observed every hourly for 5 hours in a particular sample. The results are given in the table.

Hours since observation began (t)	0	1	2	3	4	5
Number of bacteria in the sample, in thousands (x)	20	40	75	150	297	510

(i) Calculate the product moment correlation coefficient between x and t and explain whether your answer suggests that a linear model is appropriate. [2]

(ii) Draw a scatter diagram for the data and explain why a model of the form $\ln x = a + bt$ would be a better fit. [2]

(iii) Find the equation of the regression line of $\ln x$ on t and the product moment correlation coefficient. [2]

(iv) By using a suitable regression line, estimate the value of t, correct to 1 decimal place, when $x = 200$. Comment on the reliability of your estimate. [2]

9   Red and green grapes are sold in boxes by weight. The masses, in grams, of a box of red grapes and a box of green grapes are modeled as having independent normal distributions with means and standard deviations as shown in the table

	Mean Mass	Standard Deviation
Box of red grapes	300	40
Box of green grapes	150	20

(i)   A fruit-seller packs and weighs each box of red grapes. Find the probability that the 10th box of red grapes he weighs is the second box that is at least 310g.   [2]

(ii)   Find the probability that the mass of 3 boxes of red grapes differs from 3 times the mass of a box of green grapes by at least 500 g.   [3]

(iii)   Red grapes are sold at \$5 per kg and green grapes are sold at \$6 per kg. Find the probability that 10 boxes of red grapes cost at most \$7 more than 10 boxes of green grapes.   [4]

10   In the gold market, there are 4 trading days, Monday to Thursday, in a week. The gold price can only rise or fall on any day. If the gold price rises on a day, there is a probability of 0.7 that it will rise on the next day; and if the gold price falls on a day, there is a probability of 0.1 that it will rise on the subsequent day.

In a particular week, the gold price falls on the Monday, the events A and B are defined as follows.

   A : the gold price rises on Tuesday.
   B : the gold price falls on Thursday.

Find

(i)   P($A \cap B$),   [2]

(ii)   P(B),   [3]

(iii)   P($A' \cup B$),   [2]

(iv)   P($B \mid A$).   [2]

State, with a reason, whether A and B are independent.   [1]

11   As part of a promotion, a large number of scratch-and-win cards are printed and one card is given out with every value meal purchase in a restaurant. A restaurant sells 20 value meals per day and the proportion of scratch-and-win cards with prizes is 0.1.

(i)   Show that the probability that more than 2 prizes are given out in 2 days is 0.777.   [2]

(ii)   Using a suitable approximation, find the largest k such that the probability that at most k prizes are given out in 7 days is less than 0.4.   [4]

10 restaurants took part in the promotion and each sold 20 value meals per day.

(iii)   Find the probability that at most 2 restaurants give out fewer than 3 prizes in 2 days.   [2]

Another restaurant came up with a similar promotion and sells 50 value meals a day but the proportion of winning cards it printed is only 0.05. Using a suitable approximation, find the probability that there is at least 1 prize given out each day for the first 3 days.   [3]

12   A coffee machine is set to dispense 250 ml of coffee into cups. Prior to the annual maintenance of the machine, it is intended to investigate how much coffee is dispensed in a cup. A sample of 8 cups of coffee is collected and the quantities of coffee dispensed, in ml, are shown below:

250 251 254 249 253 248 252 253

(i)   Carry out an appropriate test, using a 5% significance level, to test whether the mean quantity of coffee in a cup is more than 250 ml. State an assumption that you need to make, in order to use the test that you have chosen.   [4]

(ii)   Explain, in context of the question, the meaning of "a 5% significance level".   [1]

Given that the population standard deviation is 5 ml, a random sample of another 8 cups of coffee gives a mean quantity of c ml. A test at the 5% level of significance indicates that the mean quantity of coffee dispensed is not 250 ml. Find the range of values of c. State, with a reason, whether you need to assume that the quantity of coffee dispensed in a cup is normally distributed.   [5]