ASPECTS OF AREA FORMULAS BY WAY OF LUZIN, RADÓ, AND REICHELDERFER ON METRIC MEASURE SPACES

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ABSTRACT. We consider some measure-theoretic properties of functions belonging to a Sobolev-type class on metric measure spaces that admit a Poincaré inequality and are equipped with a doubling measure. The properties we have selected to study are those that are related to area formulas.

Mathematics Subject Classification (2010): Primary: 46E35, 46E40; Secondary: 30L99, 28A99.
Key words and phrases: Area formula, condition N, doubling measure, Luzin's condition, metric space, Newtonian space, Poincaré inequality, Sobolev space, upper gradient.

1. INTRODUCTION

We investigate some measure-theoretic properties of functions belonging to the Banach or vector space-valued Newtonian space $N^{1,p}(X)$ and compare these properties in the more general setting with the classical Euclidean ones. Newtonian space is a metric space analogue of the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$ and was first introduced and studied by Shanmugalingam in [29]; here X refers to a complete metric measure space with a measure μ that satisfies a volume doubling condition and the space is assumed to support a Poincaré inequality. Under these rather standard conditions on the space, we give a metric space version of Luzin's condition for the graph mapping similar to one in Malý et al. [27], we study absolute continuity as defined by Malý [23] for functions in the Newtonian class, and we also discuss the condition due to Radó and Reichelderfer [28].

We provide a version of the area formula for Newtonian functions. In particular, we extend the Euclidean results of Hajłasz [10] and Malý et al. [27] to Newton–Sobolev functions in the aforementioned setting of general metric spaces. We provide another view to a recent result by Magnani [22] which is related to the area formula in general metric measure spaces.

Under rather general assumptions on X (see Section 2) the following area formula will be shown to be valid for the graph mapping $\bar{u}$ of $u \in$

$N^{1,p}_{\mathrm{loc}}(X;\mathbb{R}^m)$, where $p > m$ or $p \geq m = 1$,

$$\mathcal{H}^Q(\bar{u}(A)) = \int_A \mathcal{J}\bar{u}\, d\mu,$$

whenever A is a μ-measurable subset and $\mathcal{J}\bar{u}$ denotes the generalized Jacobian of $\bar{u}$. In particular, $\mathcal{H}^Q(\bar{u}(A))$ whenever $\mu(A) = 0$. Here the exponent Q serves as a substitute for the dimension of X, and it is associated with the doubling constant of the underlying measure μ (see Section 2).

Althought the proofs for these formulas are rather standard, our general setting causes some unexpected difficulties. To overcome these, we carefully consider some local properties of so-called generalized Jacobian of a function and couple them with the aforementioned measure-theoretic properties of Newton–Sobolev functions.

There is a rich supply of examples of complete metric spaces with a volume doubling measure that support a Poincaré inequality and where our results are applicable. To name but a few, we list Carnot–Carathéodory spaces, thus including the Heisenberg group and more general Carnot groups, as well as Riemannian manifolds with non-negative Ricci curvature.

In outline, the paper is organized as follows: In Section 2 we introduce the necessary background material such as the doubling condition for the measure, upper gradients, Poincaré inequality, Newtonian spaces, and capacity. In Section 3 we establish a general criterion for a version of Luzin's condition in the spirit of Radó and Reichelderfer [28, V.3.6], see also Malý et al. [27]. Then we close Section 3 by proving, with the aid of estimates between the capacity ant the Hausdorff content, that the graph mapping of a vector-valued Newtonian function satisfies a version of the Luzin condition. In Section 4 we deal with the area formula. In Section 5 we study the Radó–Reichelderfer condition and absolute continuity of Newtonian functions in the spirit of Malý [23].

2. METRIC MEASURE SPACES: DOUBLING AND POINCARÉ

We briefly recall the basic definitions and collect some well-known results needed later. For a thorough treatment we refer the reader to a monograph by A. and J. Björn [3] and Heinonen [13].

Throughout the paper, if not otherwise stated, $X := (X, d, \mu)$ is a complete metric space endowed with a metric d and a positive complete Borel regular measure μ such that $0 < \mu(B(x,r)) < \infty$ for all balls $B(x,r) := \{y \in X : d(x,y) < r\}$; and if $B = B(x,r)$, then we denote $\tau B = B(x, \tau r)$ for each $\tau > 0$. We also denote the metric ball $B(x,r)$ by $B_X(x,r)$ if necessary. Also throughout the paper, if not otherwise stated, let $Y := (Y, \tilde{d}, \nu)$ be a complete separable metric measure space with a positive complete Borel regular measure ν. A function $f : X \to Y$ is called

L-Lipschitz if for all $x, y \in X$, $\tilde{d}(f(x), f(y)) \le Ld(x, y)$. We let $\mathrm{Lip}(f)$ be the infimum of such L.

In our treatment, it is natural to assume some connection between the measure and the metric. Also by dimension we mean some quantity which relates the measure of a metric ball to its radius. We shall clarify these concepts below. Our standing assumptions on the metric space X are as follows.

(D) The measure μ is doubling, i.e., there exists a constant $C_\mu \ge 1$, called the *doubling constant* of μ, such that

$$\mu(B(x, 2r)) \le C_\mu \mu(B(x, r)).$$

for all $x \in X$ and $r > 0$.

(PI) The space X supports a weak $(1, p)$-Poincaré inequality for some $p \ge 1$ (see below).

We note the doubling condition (D) implies that for every $x \in X$ and $r > 0$, we have for $\lambda \ge 1$

$$\mu(B(x, \lambda r) \le C\lambda^Q \mu(B(x, r)), \tag{2.1}$$

where $Q = \log_2 C_\mu$, and the constant depends only on C_μ. The exponent Q serves as a dimension of the doubling measure μ; we emphasize that it need not be an integer. *When it is necessary to emphasize the relationship between Q and X, we will use the notation X^Q.* Complete metric spaces verifying condition (D) are precisely those that have finite Assouad dimension [13]. This notion of dimension, however, need not to be uniform in space. In what follows, we assume further that there exists a constant $C > 0$, depending only on C_μ, such that the measure μ satisfies the lower mass bound

$$Cr^Q \le \mu(B(x, r)) \tag{2.2}$$

for all $x \in X$ and $0 < r < \mathrm{diam}(X)$. It follows from (D) that μ satisfies the following local version of (2.2): For a fixed $x_0 \in X$ and a scale $r_D > 0$ we have

$$\tilde{C}r^Q \le \mu(B(x, r)) \tag{2.3}$$

for all balls $B(x, r) \subset X$ with $x \in B(x_0, r_D)$ and $0 < r < r_D$, where $\tilde{C} = Cr_D^{-Q}\mu(B(x_0, r_D))$ and C is from (2.1).

Let $s \ge 0$. We define the (spherical) Hausdorff s-measure in X as in Federer [8, 2.10.2] (see also [13]) and will denote it by $\mathcal{H}^s$. We also denote by $\mathcal{H}^s_\infty$ the Hausdorff s-content in X defined as

$$\mathcal{H}^s_\infty(E) = \inf\left\{ \sum_{i=1}^{\infty} r_i^s : \ E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ x_i \in E \right\},$$

where the infimum is taken over all countable covers of E by balls $B(x_i, r_i)$. We note here that if X is a proper, i.e. boundedly compact, metric space, then Hausdorff content is inner regular in the following sense

$$\mathcal{H}^s_\infty(E) = \sup\{\mathcal{H}^s_\infty(K) : K \subset E,\ K \text{ compact}\}$$

whenever $E \subset X$ is a Borel set. See Federer [8, Corollary 2.10.23]. We shall also need the concept of the *Hausdorff measure of codimension* s of $E \subset X$ which we define by applying the Carathéodory construction to the function

$$h(B(x,r)) = \frac{\mu(B(x,r))}{r^s}.$$

Above, we use the convention $h(B(x,0)) := h(\emptyset) = 0$. We thus define the restricted Hausdorff content of codimension s as follows

$$\widetilde{\mathcal{H}}^s_R(E) = \inf\left\{\sum_{i=1}^\infty h(B(x_i,r_i)) : E \subset \bigcup_{i=1}^\infty B(x_i,r_i),\ x_i \in E,\ r_i \le R\right\},$$

where $0 < R < \infty$. When $R = \infty$, we have the corresponding Hausdorff content of E and denote it by $\widetilde{\mathcal{H}}^s_\infty(E)$. Finally, the Hausdorff measure of codimention s is defined as

$$\widetilde{\mathcal{H}}^s(E) = \lim_{R\to 0} \widetilde{\mathcal{H}}^s_R(E).$$

We remark that if the measure μ is Q-regular, i.e., $\mu(B(x,r)) \approx r^Q$, for some $Q \ge 1$, $\widetilde{\mathcal{H}}^s(E) \approx \mathcal{H}^{Q-s}(E)$. Let us mention that the lower mass bound (2.2) for the measure μ implies that $\mathcal{H}^Q$ is absolutely continuous with respect to μ and that $\mathcal{H}^{Q-s}(E) \le C\widetilde{\mathcal{H}}^s(E)$.

The *upper s-density* of a finite Borel regular measure ζ at x is defined by

$$\Theta^*_s(\nu, x) = \limsup_{r\to 0+} \frac{\zeta(B(x,r))}{\omega_s r^s},$$

where ω_s is the Lebesgue measure of the unit ball in $\mathbb{R}^s$ when s is a positive integer, and $\omega_s = \Gamma(1/2)^s/\Gamma(s/2+1)$ otherwise. We record that if for all x in a Borel set $E \subset X$, $\Theta^*_s(\zeta, x) \ge \alpha$, $0 < \alpha < \infty$, then

$$\zeta \ge \alpha C \mathcal{H}^s \llcorner E,$$

where a positive constant C depends only on s. On the other hand, if $\Theta^*_s(\zeta, x) \le \alpha$ we obtain

$$\zeta \llcorner E \le \alpha C \mathcal{H}^s \llcorner E,$$

where a positive constant C depends only on s. See Federer [8, 2.10.19].

Recall that the following genral covering theorem is valid in our setting. From a given family of balls $\mathcal{B}$ with $\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty$ covering a set $E \subset X$ we can select a pairwise disjoint subfamily $\mathcal{B}'$ of balls such that

$$E \subset \bigcup_{B \in \mathcal{B}'} 5B,$$

see [8, Corollary 2.8.5]. If X is separable, then $\mathcal{B}'$ is countable and $\mathcal{B}' = \{B_i\}_{i \geq 1}$.

In this note, a *curve* γ in X is a continuous mapping from a compact interval $[0, L]$ to X. We recall that each curve can be parametrized by 1-Lipschitz map $\tilde{\gamma} : [0, L] \to X$. A nonnegative Borel function g on X is an *upper gradient* of a function $f : X \to Y$ if for all rectifiable curves γ, we have

$$\tilde{d}(f(\gamma(L)), f(\gamma(0))) \leq \int_{\gamma} g \, ds. \tag{2.4}$$

See Cheeger [5] and Shanmugalingam [29] for a discussion on upper gradients. If g is a nonnegative measurable function on X and if (2.4) holds for p-almost every curve, $p \geq 1$, then g is a *weak upper gradient* of f. By saying that (2.4) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus (see, e.g., [29]). If u has an upper gradient in $L^p(X)$, then it has a *minimal weak upper gradient* $g_f \in L^p(X)$ in the sense that for every weak upper gradient $g \in L^p(X)$ of f, $g_f \leq g$ μ-almost everywhere (a.e.), see Corollary 3.7 in Shanmugalingam [30]. While the results in [29] and [30] are formulated for real-valued functions and their upper gradients, they are applicable for metric space valued functions and their upper gradients; the proofs of these results require only the manipulation of upper gradients, which are always real-valued.

We define Sobolev spaces on metric spaces following Shanmugalingam [29]. Let $\Omega \subseteq X$ be nonempty and open. Whenever $u \in L^p(\Omega)$ and $p \geq 1$, let

$$\|u\|_{N^{1,p}(\Omega)} := \|u\|_{1,p} := \left(\int_{\Omega} |u|^p \, d\mu + \int_{\Omega} g_u^p \, d\mu \right)^{1/p}. \tag{2.5}$$

The *Newtonian space* on Ω is the quotient space

$$N^{1,p}(\Omega) = \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$. The space $N^{1,p}(\Omega)$ is a Banach space and a lattice. If $\Omega \subset \mathbb{R}^n$ is open, then $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ as Banach spaces. For these and other properties of Newtonian spaces we refer to [29]. The class $N^{1,p}(\Omega; \mathbb{R}^m)$ consists of those mappings $u : \Omega \to \mathbb{R}^m$ whose component functions each belong to $N^{1,p}(\Omega) = N^{1,p}(\Omega; \mathbb{R})$. Qualitative properties like Lebesgue points, density of Lipschitz functions, quasicontinuity, etc. may be investigated componentwise.

A function belongs to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(V)$ for all bounded open sets V with $\bar{V} \subset \Omega$, the latter space being defined by considering V as a metric space with the metric d and the measure μ restricted to it.

Newtonian spaces share many properties of the classical Sobolev spaces. For example, if $u, v \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then $g_u = g_v$ μ-a.e. in $\{x \in \Omega : u(x) = v(x)\}$, furthermore, $g_{\min\{u,c\}} = g_u \chi_{\{u \neq c\}}$ for $c \in \mathbb{R}$.

We shall also need a *Newtonian space with zero boundary values*. For a measurable set $E \subset \Omega$, let

$$N^{1,p}_0(E) = \{f|_E : f \in N^{1,p}(E) \text{ and } f = 0 \text{ on } \Omega \setminus E\}.$$

This space equipped with the norm inherited from $N^{1,p}(\Omega)$ is a Banach space.

We say that X supports a *weak* $(1,p)$*-Poincaré inequality* if there exist constants $C > 0$ and $\tau \geq 1$ such that for all balls $B(z,r) \subset X$, all measurable functions f on X and for all weak upper gradients g_f of f,

$$\fint_{B(z,r)} |f - f_{B(z,r)}| \, d\mu \leq Cr \Big(\fint_{B(z,\tau r)} g_f^p \, d\mu \Big)^{1/p}, \tag{2.6}$$

where $f_{B(z,r)} := \fint_{B(z,r)} f \, d\mu := \int_{B(z,r)} f \, d\mu / \mu(B(z,r))$.

It is well known that the embedding $N^{1,p}(X) \to L^p(X)$ is not surjective if and only if there exists a curve family in X with a positive p-modulus. Moreover, the validity of a Poincaré inequality can sometimes be stated in terms of p-modulus. More precisely, to require that (2.6) holds in X is to require that the p-modulus of curves between every pair of distinct points of the space is sufficiently large, see Theorem 2 in Keith [15].

It is noteworthy that by a result of Keith and Zhong [16] in a complete metric space equipped with a doubling measure and supporting a weak $(1,p)$-Poincaré inequality there exists $\varepsilon_0 > 0$ such that the space admits a weak $(1,p')$-Poincaré inequality for each $p' > p - \varepsilon_0$.

The following Luzin-type approximation theorem shall be of use later in the paper. We refer to Shanmugalingam [29, Theorem 4.1] for the proof which, in turn, is a modification of an idea due to S. Semmes. See also Hajłasz [9, Theorem 5].

Theorem 2.1. *Suppose X satisfies (D) and (PI) for some $1 < p < \infty$. Let $u \in N^{1,p}(X)$. Then for every $\varepsilon > 0$ there is a Lipschitz function $f_\varepsilon : X \to \mathbb{R}$ such that*

$$\mu(\{x \in X : u(x) \neq f_\varepsilon(x)\}) < \varepsilon$$

and $\|u - f_\varepsilon\|_{1,p} < \varepsilon$. In other words, with $F_\varepsilon := \{x \in X : u(x) \neq f_\varepsilon(x)\}$, we have $u|_{X \setminus F_\varepsilon}$ is Lipschitz.

Capacity. There are several equivalent definitions for capacities, and the following are the ones we find most suitable for our purposes. Let $1 \le p < \infty$ and $\Omega \subset X$ bounded.

- The variational p-capacity of a set $E \subset X$ is the number
$$\operatorname{cap}_p(E) = \inf \|g_u\|^p_{L^p(X)},$$
where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E; recall that g_u is the minimal p-weak upper gradient of u.
- The relative p-capacity of $E \subset \Omega$ is the number
$$\operatorname{Cap}_p(E, \Omega) = \inf \|g_u\|^p_{L^p(\Omega)},$$
where the infimum is taken over all $u \in N^{1,p}_0(\Omega)$ such that $u \ge 1$ on E.
- The Sobolev p-capacity of $E \subset X$ is the number
$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$
where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E.

Observe that if $\mu(X) < \infty$ the constant function will do as a test function, thus all sets are of zero variational p-capacity. However, this is not true for the relative p-capacity whenever $X \setminus \Omega$ is "large", say, $C_p(X \setminus \Omega) > 0$.

Under our assumptions, these capacities enjoy the standard properties of capacities. For instance, when $p > 1$ they are are Choquet capacities, i.e., the capacity of a Borel set can be obtained by approximating with compact sets from inside and open sets from outside. It is noteworthy, however, that the Choquet property fails for $p = 1$ in the general metric setting. This does not cause any problems for us as we mainly deal with compact sets in this note. In a recent paper by Kinnunen–Hakkarainen [12] the BV-capacity was proved to be a Choquet capacity. See, e.g., Kinnunen–Martio [18], [19] for a discussion on capacities on metric spaces.

The Sobolev capacity is the correct gauge for distinguishing between Newtonian functions: if $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ p-quasieverywhere, i.e., outside a set of zero Sobolev p-capacity. Moreover, by Shanmugalingam [29] if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$. A function $u \in N^{1,p}(X)$ is said to be *quasicontinuous*, if there exists an open set $G \subset X$ with arbitrarily small Sobolev p-capacity such that the restriction of u to $X \setminus G$ is continuous. A mapping in $N^{1,p}(X; \mathbb{R}^m)$ is said to be quasicontinuous if each of its component functions is quasicontinuous. Recall that *all* functions in $N^{1,p}(X)$ are quasicontinuous, see Björn et al. [4]. Since Newtonian functions have Lebesgue points outside a set of zero Sobolev capacity, in what follows we may assume that every Newtonian function is precisely represented.

3. GRAPHS OF NEWTONIAN FUNCTIONS: LUZIN'S CONDITION

Let $Q > 0$. Recall that a mapping $f : X \to Y$ is said to satisfy *Luzin's condition* (N_Q) if $\mathcal{H}^Q(f(E)) = 0$ whenever $E \subset X$ satisfies $\mu(E) = 0$. By way of motivation, the validity of Luzin's condition implies certain change of variable formulas, thus it is of independent interest in analysis.

Let $E \subset X$. We denote by $\bar{f} : X \to X \times Y$ the *graph mapping of f*

$$\bar{f}(x) = (x, f(x)), \quad x \in X,$$

and $\mathcal{G}_f(E)$ is the *graph* of f over E defined by

$$\mathcal{G}_f(E) = \{(x, f(x)) : x \in E\} \subset X \times Y.$$

It is well known that if the mapping f is Borel measurable, then the graph $\mathcal{G}_f(X)$ is Borel measurable as well, see, e.g., [10, Lemma 18]. We, furthermore, denote by $\mathrm{pr}_X : X \times Y \to X$ the projection $\mathrm{pr}_X(x, y) = x$, and by $\mathrm{pr}_Y : X \times Y \to Y$ the projection $\mathrm{pr}_Y(x, y) = y$. Observe that $\mathrm{Lip}(\mathrm{pr}_X) = \mathrm{Lip}(\mathrm{pr}_Y) = 1$. Also it is well-known that if $f : X \to Y$ is continuous, then $\mathcal{G}_f(X)$ is homeomorphic to X.

Lemma 3.1. *Let $f : X \to \mathbb{R}^m$, $m \geq 1$, be measurable. Then $\mathrm{pr}_X(\mathcal{G}_f(X) \cap E)$ is measurable for every Borel measurable subset $E \subset X \times \mathbb{R}^m$.*

Proof. Let f^* and f_* be Borel measurable representatives of f; Borel regularity of the measure μ implies that if f is measurable, then there exist Borel measurable functions f_*, f^* such that $f_* \leq f \leq f^*$ and $f_*(x) = f^*(x)$ for μ-a.e. $x \in X$. Thus the graph $\mathcal{G}_{f_*}(X)$ of f_* and the graph $\mathcal{G}_{f^*}(X)$ of f^* are Borel subsets of $X \times \mathbb{R}^m$. Then Kuratowski [20, Theorem 2, p. 385] implies that the projections $\mathrm{pr}_X(\mathcal{G}_{f_*}(X) \cap E)$ and $\mathrm{pr}_X(\mathcal{G}_{f^*}(X) \cap E)$ are Borel measurable for every Borel measurable set $E \subset X \times \mathbb{R}^m$. Since f_* and f^* agree up to a set of μ-measure zero, so do sets $\mathrm{pr}_X(\mathcal{G}_{f^*}(X) \cap E)$ and $\mathrm{pr}_X(\mathcal{G}_f(X) \cap E)$, implying that $\mathrm{pr}_X(\mathcal{G}_f(X) \cap E)$ is μ-measurable. $\square$

We now state a general criterion for the condition (N_Q) similar to that of Radó and Reichelderfer, see [28, V.3.6] and Malý [23]. In Euclidean spaces this result was obtained by Malý et al. [27].

In what follows, we suppose that $1 \leq m < Q$, where m is related to $\mathbb{R}^m$.

Theorem 3.2. *Suppose X satisfies condition (D) and the lower mass bound* (2.2) *is satisfied. Let $f : X^Q \to \mathbb{R}^m$ be a measurable function. Denote*

$$\Xi_r = \mathcal{G}_f(X^Q) \cap B(z, r),$$

where $z \in X^Q \times \mathbb{R}^m$ and $0 < r < \operatorname{diam}(X^Q)$. Suppose that there exists a weight $\Phi \in L^1_{\mathrm{loc}}(X^Q)$ such that

$$\mathcal{H}^{Q-m}_\infty(\mathrm{pr}_X(\Xi_r)) \leq \frac{1}{\operatorname{diam}(\Xi_r)^m} \int_{\mathrm{pr}_X(\Xi_{4r})} \Phi \, d\mu \tag{3.1}$$

for all $z \in X^Q \times \mathbb{R}^m$ and all $0 < r < \operatorname{diam}(X^Q)/4$. Then there exists a positive constant $C < \infty$, depending on C_μ and m, such that

$$\mathcal{H}^Q(\bar{f}(E)) \leq C \int_E \Phi \, d\mu \tag{3.2}$$

for each Borel measurable set $E \subset X^Q$. In particular, $\bar{f}$ satisfies Luzin's condition (N_Q).

Proof. Define a set function σ on the Cartesian product $X^Q \times \mathbb{R}^m$ by

$$\sigma(E) = \int_{\mathrm{pr}_X(\mathcal{G}_f(X^Q) \cap E)} \Phi \, d\mu, \quad E \subset X^Q \times \mathbb{R}^m.$$

By a Vitali-type covering theorem there is a pairwise disjoint countable subfamily of balls $\{B_i\} := \{B(x_i, r_i)\}$ such that we may cover $\mathrm{pr}_X(\Xi_r)$ as follows

$$\mathrm{pr}_X(\Xi_r) \subset \bigcup_i B(x_i, 5r_i) =: \bigcup_i 5B_i.$$

For each i let M_i denote the greatest integer satisfying

$$(M_i - 1) r_i < \operatorname{diam}(\Xi_r).$$

Since $\Xi_r \cap \mathrm{pr}_X^{-1}(5B_i)$ is bounded in $X^Q \times \mathbb{R}^m$, it can be contained in a large enough cylinder of the form $B(x_i, 5r_i) \times \mathcal{R}_i$, where $\mathcal{R}_i$ is a cube in $\mathbb{R}^m$ with side-length $\operatorname{diam}(\Xi_r)$. Since $M_i r_i \geq \operatorname{diam} \Xi_r$, $\mathcal{R}_i$ may be covered by M_i^m cubes $\{\mathcal{R}_i^j\}$ with side r_i. We hence obtain

$$\begin{aligned}\mathcal{H}^Q_\infty(\Xi_r \cap \mathrm{pr}_X^{-1}(5B_i)) &\leq C M_i^m r_i^Q \leq C (M_i r_i)^m r_i^{Q-m} \\ &\leq C(\operatorname{diam}(\Xi_r) + r_i)^m \mu(5B_i)(5r_i)^{-m}.\end{aligned}$$

As $r_i \approx \operatorname{diam}(5B_i) \leq \operatorname{diam} \mathrm{pr}_X(\Xi_r) \leq \operatorname{diam}(\Xi_r)$ summing over i shows that

$$\mathcal{H}^Q_\infty(\Xi_r) \leq C \operatorname{diam}(\Xi_r)^m \sum_{i=1}^\infty \frac{\mu(5B_i)}{(5r_i)^m}.$$

Hence by taking the infimum over all coverings we have obtained the following estimate

$$\mathcal{H}^Q_\infty(\Xi_r) \leq C \operatorname{diam}(\Xi_r)^m \widetilde{\mathcal{H}}^m_\infty(\mathrm{pr}_X(\Xi_r)),$$

where the constant C depends only on C_μ and m. Assumption (3.1) together with this estimate gives for each $z \in X \times \mathbb{R}^m$ and $0 < r < \operatorname{diam}(X^Q)/4$

$$\begin{aligned} \mathcal{H}^Q_\infty(\Xi_r) &\le C \operatorname{diam}(\Xi_r)^m \widetilde{\mathcal{H}}^m_\infty(\mathrm{pr}_X(\Xi_r)) \\ &\le C \int_{\mathrm{pr}_X(\Xi_{4r})} \Phi \, d\mu \le C\sigma(B(z,4r)). \end{aligned} \tag{3.3}$$

Since for $\mathcal{H}^Q$-almost every $z \in \mathcal{G}_f(X^Q)$, see Federer [7, Lemma 10.1],

$$\limsup_{r\to 0+} \frac{\mathcal{H}^Q_\infty(\Xi_r)}{\omega_Q r^Q} \ge C, \tag{3.4}$$

it follows from (3.3) that

$$\limsup_{r\to 0+} \frac{\sigma(B(z,r))}{\omega_Q r^Q} \ge C$$

for $\mathcal{H}^Q$-almost every $z \in \mathcal{G}_f(X^Q)$. Lemma 3.1 implies that σ is a measure on the Borel sigma algebra of $X^Q \times \mathbb{R}^m$, and it may be extended to a regular Borel outer measure σ^* on all of $X^Q \times \mathbb{R}^m$ in the usual way

$$\sigma^*(A) := \inf\{\sigma(E) : \ A \subset E, \ E \text{ is a Borel set}\}.$$

Since $\Phi \in L^1_{\mathrm{loc}}(X^Q)$ it follows that σ^* is a Radon measure on $X^Q \times \mathbb{R}^m$. Therefore, by (3.4)

$$\mathcal{H}^Q(E) \le C\sigma^*(E)$$

for all $E \subset \mathcal{G}_f(X^Q)$. Finally, given a μ measurable set $E \subset X^Q$, choose a Borel set G with $E \subset G$. Then $\bar{f}(E) \subset G \times \mathbb{R}^m$, $G \times \mathbb{R}^m$ is a Borel set, and

$$\mathcal{H}^Q(\bar{f}(E)) \le C\sigma^*(\bar{f}(E)) \le C\sigma(G \times \mathbb{R}^m) = C \int_G \Phi \, d\mu.$$

The proof is completed by taking the infimum over all such G. If $E \subset X^Q$ such that $\mu(E) = 0$ then it readily follows that $\mathcal{H}^Q(\overline{f}(E)) = 0$. This completes the proof. $\square$

In (3.1) we may replace the Hausdorff content $\mathcal{H}^{Q-m}_\infty(\mathrm{pr}_X(\Xi_r))$ with an inequality involving $\widetilde{\mathcal{H}}^m_\infty(\mathrm{pr}_X(\Xi_r))$ on the left hand side.

We shall show, as an application of Theorem 3.2, that the graph mapping of a Newtonian function satisfies a version of Luzin's condition (N_Q). We start with a few auxiliary estimates. We shall need the following relation between the p-capactity and the Hausdorff content when $p \ge 1$. For the proof of the next lemma the reader should consult Costea [6, Thoerem 4.4] and Kinnunen et al. in [17, Theorem 3.5] for the case (I) and (II), respectively.

Lemma 3.3. *Suppose X satisfies conditions (D) and (PI), and the lower mass bound* (2.2) *is satisfied.*

(I) *Let $1 < p \le Q$ and $E \subset X$ and suppose $Q - p < t \le Q$. Then*

$$\mathcal{H}^t_\infty(E \cap B(x,r)) \le C r^{t-Q+p} \operatorname{Cap}_p(E \cap B(x,r), B(x,2r)),$$

where $x \in X$, $r > 0$, and C depends on C_μ, p, t, and the constants in the weak $(1,p)$-Poincaré inequality.

(II) *Let $p = 1$ and $E \subset X$ compact. Then*

$$\widetilde{\mathcal{H}}^1_\infty(E) \le C \operatorname{cap}_1(E),$$

where the constant C depends only on the doubling constant C_μ and the constants in the weak $(1,1)$-Poincaré inequality.

Remark 3.4. If $u \in N^{1,p}_0(B(x,2r);\mathbb{R}^m)$ such that $u \ge 1$ on $E \cap B(x,r)$, g_u is a minimal p-weak upper gradient of u, and m, where $1 \le m < \min\{p,Q\}$, we obtain

$$\mathcal{H}^{Q-m}_\infty(E \cap B(x,r)) \le C r^{p-m} \int_{B(x,2r)} g_u^p \, d\mu,$$

where the constant C is as in Lemma 3.3 (I).

If $u \in N^{1,1}(X;\mathbb{R})$ such that $u \ge 1$ on E and g_u is a minimal 1-weak upper gradient of u, Lemma 3.3 (II) implies that

$$\widetilde{\mathcal{H}}^1_\infty(E) \le C \int_X g_u \, d\mu,$$

where the constant C is from Lemma 3.3 (II).

The preceding estimates imply the following. Observe also that the graph mapping is always one-to-one.

Theorem 3.5. *Suppose that X satisfies conditions (D) and (PI) with some $1 \le p \le Q$, and the lower mass bound* (2.2) *is satisfied. Let $u \in N^{1,p}(X^Q;\mathbb{R}^m)$, where either $p > m$ or $p \ge m = 1$. Then the graph mapping $\overline{u}$ satisfies Luzin's condition (N_Q).*

The assumption that $p > m$ or $p \ge m = 1$ is necessary already in the Euclidean case. We refer to a discussion in Malý et al. [27].

Proof of Theorem 3.5. It is sufficient to verify the hypothesis of Theorem 3.2 with some locally integrable function Φ on X^Q.

Assume first $p > m$ and, to this end, fix a point $z = (\tilde{x},\tilde{y}) \in X^Q \times \mathbb{R}^m$ and $r > 0$. We observe the following

$$\Xi_r = \mathcal{G}_u(X^Q) \cap B(z,r) \subset (\mathcal{G}_u(X^Q) \cap (B_X(\tilde{x},r) \times B(\tilde{y},r))).$$

Hence we have that

$$\operatorname{pr}_X(\Xi_r) \subset (B_X(\tilde{x},r) \cap u^{-1}(B(\tilde{y},r))),$$

moreover $u(x) \in B(\tilde{y}, r)$ for μ-a.e. $x \in B_X(\tilde{x}, r) \cap u^{-1}(B(\tilde{y}, r))$. Let us define the function $v : X^Q \to \mathbb{R}$ by

$$v(x) = \max\left\{2 - \frac{|u(x) - u(\tilde{x})|}{r}, 0\right\},$$

and consider an open subset $O \subset X^Q$ such that $\{x \in X^Q : v(x) > 0\} \subset O$. Then $g_u/r\chi_O$ is a p-weak upper gradient of v [29, Lemma 4.3], where g_u is a minimal p-weak upper gradient of u. Let $\eta : X^Q \to \mathbb{R}$ be a Lipschitz cut-off function so that $\eta = 1$ on $B_X(\tilde{x}, r)$, $\eta = 0$ in $X^Q \setminus B_X(\tilde{x}, 2r)$, $0 \le \eta \le 1$, and $g_\eta \le 2/r$. Then $v\eta \ge 1$ on $B_X(\tilde{x}, r) \cap u^{-1}(B(\tilde{y}, r))$, and $v\eta \in N_0^{1,p}(B(\tilde{x}, 2r))$. Moreover, the product rule for upper gradients gives us the following $g_{v\eta} \le g_v + 2v/r$ μ-a.e. Thus $v\eta$ is admissible for the relative p-capacity and Lemma 3.3 (I) implies that

$$\begin{aligned}
\mathcal{H}^{Q-m}_\infty(\mathrm{pr}_X(\Xi_r)) &\le \mathcal{H}^{Q-m}_\infty(B_X(\tilde{x}, r) \cap u^{-1}(B(\tilde{y}, r))) \\
&\le Cr^{p-m} \int_{B_X(\tilde{x},2r)\cap O} g_{v\eta}^p \, d\mu \\
&\le Cr^{p-m} \int_{B_X(\tilde{x},2r)\cap O} \left(\frac{v^p}{r^p} + g_v^p\right) d\mu \\
&\le Cr^{-m} \int_{B_X(\tilde{x},2r)\cap u^{-1}(B(\tilde{y},2r))} (1 + g_u^p) \, d\mu.
\end{aligned}$$

Since

$$B_X(\tilde{x}, 2r) \cap u^{-1}(B(\tilde{y}, 2r)) \subset \mathrm{pr}_X(\Xi_{4r}),$$

above reasoning gives us that

$$\mathcal{H}^{Q-m}_\infty(\mathrm{pr}_X(\Xi_r)) \le \frac{C}{r^m} \int_{\mathrm{pr}_X(\Xi_{4r})} (1 + g_u^p) \, d\mu.$$

This verifies the assumptions of Theorem 3.2 with $\Phi = C(1 + g_u^p)$, and thus concludes the proof when $p > m$. The case $p \ge m = 1$ is dealt with by a similar argument together with the estimate in Lemma 3.3 (II). $\square$

4. Aspects of area formulas for Newtonian functions

In this section we shall prove versions of the area formula for Newtonian functions. In the metric measure space setting these formulas have been studied previously by Ambrosio–Kirchheim [1], Magnani [21, 22], and Malý [24, 25], to name but a few. In particular, in [24] coarea properties and coarea formula, which is considered as dual to the area formula, are thoroughly studied in metric spaces. We also refer to Hajłasz [10] for a very nice discussion on the topic in Euclidean spaces.

We define the generalized Jacobian of a continuous map $f : X \to Y$ at x as follows

$$\mathcal{J}f(x) := \limsup_{r\to 0} \frac{\nu(f(B(x,r)))}{\mu(B(x,r))},$$

where, we recall, ν measures Y. It follows from [8, 2.2.13] applied to the pull-back measure $\nu_f(E) := \nu(f(E))$, that $f(E)$ is ν-measurable for every Borel set $E \subset X$. Moreover, for μ-a.e. x, the generalized Jacobian $\mathcal{J}f(x)$ is finite, see Federer [8, 2.9]. It is also easy to see that if $g : X \to Y$ is another continuous map such that $g = f$ on an open subset $A \subset X$, then $\mathcal{J}f(x) = \mathcal{J}g(x)$ for μ-a.e. $x \in A$.

An alternative, but maybe less tractable, way to define a generalized Jacobian of f at x could be as follows. Set

$$\widetilde{\mathcal{J}}f(x) := \limsup_{r\to 0} \frac{f^*\nu(B(x,r))}{\mu(B(x,r))},$$

where $f^*\nu$ is a measure which results by Carathéodory's construction from $\zeta(A) = \nu(f(A))$, $A \subset X$, on the family of all Borel subsets of X, see [8, 2.10.1]. Hence if A is a Borel subset of X, then

$$f^*\nu(A) = \sup\left\{\sum_{B\in\mathcal{H}} \zeta(B) : \mathcal{H} \text{ is a Borel partition of } A\right\}$$

cf. [8, Theorem 2.10.8]; for any Borel set $A \subset X$ the following identity will be satisfied [8, Theorem 2.10.10]

$$f^*\nu(A) = \int_Y N(f|_A, y)\, d\nu(y),$$

where the *multiplicity function* of f relative to a subset A is written as $N(f|_A, y) = \#(A \cap f^{-1}(y))$ for each $y \in Y$.

To compare these two notions, we have that

$$\mathcal{J}f(x) = \widetilde{\mathcal{J}}f(x) = \mathcal{J}f|_D(x)$$

for μ-a.e. $x \in D$, where $D \subset X$ is closed and $f|_D$ is assumed to be one-to-one. Here we denote

$$\mathcal{J}f|_D(x) := \limsup_{r\to 0} \frac{\nu(f(B(x,r)\cap D))}{\mu(B(x,r))}.$$

Let us clarify this. Clearly, $\mathcal{J}f|_D(x) \le \mathcal{J}f(x) \le \widetilde{\mathcal{J}}f(x)$ for μ-a.e. $x \in D$. On the other hand, since f is one-to-one on D we have that $\zeta(A) := \nu(f(A))$ is, in fact, a measure on D, and that $\zeta(A) = f^*\nu(A)$ for every Borel subset of D. Thus we obtain as in Magnani [22, proof of Theorem 2] for every (density point) $x \in D$

$$\begin{aligned}\widetilde{\mathcal{J}} f(x) &\leq \limsup_{r\to 0} \frac{\nu(f(B(x,r)\cap D))}{\mu(B(x,r))} + \limsup_{r\to 0} \frac{f^*\nu(B(x,r)\setminus D)}{\mu(B(x,r))} \\ &= \mathcal{J} f|_D(x),\end{aligned}$$

where the last equality follows form [8, Corollary 2.9.9] applied to $\widetilde{\mathcal{J}} f(x)\chi_D$, where χ_D is the characteristic function of the set D.

Magnani [22] has recently presented a unified approach to the area formula for merely continuous mappings between metric spaces, and thus without any notion of differentiability. We remark that in the present paper a function in $N^{1,p}_{\mathrm{loc}}(X^Q;\mathbb{R}^m)$ although having some "differentiability" properties, need not to be even continuous as all Newtonian functions are, a priori, only quasicontinuous. Let us state the following area formula.

Theorem 4.1. *Suppose X satisfies conditions (D) and (PI) with some $1 \leq p \leq Q$, and the lower mass bound* (2.2) *is satisfied. Let $u \in N^{1,p}_{\mathrm{loc}}(X^Q;\mathbb{R}^m)$, where $p > m$ or $p \geq m = 1$. Then the following area formula is valid*

$$\mathcal{H}^Q(\bar{u}(A)) = \int_A \mathcal{J}\bar{u}(x)\, d\mu(x), \tag{4.1}$$

whenever $A \subset X$ is μ-measurable.

Proof. By Theorem 3.5 the graph mapping $\bar{u}$ satisfies Luzin's condition (N_Q) and is, moreover, one-to-one on X. Thus the pull-back measure $\mathcal{H}^Q(\bar{u}(A))$, $A \subset X^Q$ arbitrary μ-measurable subset, is absolute continuous with respect to the doubling measure μ.

Let $\{f_i\}_{i\geq 1}$, $f_i : X^Q \to \mathbb{R}^m$, be a sequence of Lipschitz functions and $E_1 \subset E_2 \subset \ldots \subset X^Q$ associated closed sets such that $u_i := u|_{E_i} = f_i|_{E_i}$ and $\mu(X^Q \setminus \bigcup_i E_i) = 0$. The existence of such sets and functions follows from Theorem 2.1. Then the following identity is valid by the area formula obtained in [22]

$$\int_{E_i} \mathcal{J}\bar{f}_i(x)\, d\mu(x) = \mathcal{H}^Q(\bar{f}_i(E_i)). \tag{4.2}$$

Since $u_i(x) = f_i(x)$ for $x \in E_i$, E_i closed, it follows that $\mathcal{J}\bar{u}_i(x) = \mathcal{J}\bar{f}_i(x)$ for μ-a.e. $x \in E_i$. The equality (4.2) remains true for measurable $A \subset E_\infty$, where $E_\infty = \bigcup_{i=1}^\infty E_i$, and moreover, (4.2) will also be valid whenever $\mu(A) = 0$. Thus (4.1) holds for all μ-measurable set $A \subset X^Q$. □

Let us discuss an alternative formulation of the area formula which can be obtained by using Theorem 2 in Magnani [22]. Assume X satisfies conditions (D) and (PI) with some $1 \leq p < \infty$, and assume further that there exist disjoint μ-measurable sets $\{A_j\}_{j\geq 1}$ such that they occupy μ-a.a. of X, i.e. $\mu(X \setminus \bigcup_j A_j) = 0$. Let $u \in N^{1,p}_{\mathrm{loc}}(X^Q;\mathbb{R}^N)$, where $Q \leq N$. Assume further that u satisfies Luzin's condition (N_Q) and $u|_{A_j}$ is one-to-one for each $i = 1, 2, \ldots$. Then the following area formula is valid

$$\int_A \theta(x)\mathcal{J}u(x)\, d\mu(x) = \int_{\mathbb{R}^N} \sum_{x \in u^{-1}(y)} \theta(x)\, d\mathcal{H}^N(y),$$

whenever $A \subset X$ is μ-measurable and $\theta : A \to [0, \infty]$ is a measurable function. In particular,

$$\int_A \mathcal{J}u(x)\, d\mu(x) = \int_{\mathbb{R}^N} N(u|_A, y)\, d\mathcal{H}^N(y)$$

is valid whenever $A \subset X$ is μ-measurable.

5. Newtonian functions: absolute continuity, Radó, Reichelderfer, and Malý

Absolutely continuous functions on the real line satisfy Luzin's condition, are continuous, and differentiable almost everywhere. It is well-known that these properties for the Sobolev class $W^{1,p}(\mathbb{R}^m)$ depend on p. For instance, functions in $W^{1,m}(\mathbb{R}^m)$ may be nowhere differentiable and nowhere continuos whereas functions in $W^{1,p}(\mathbb{R}^m)$, $p > m$, have Hölder continuous representatives and are differentiable almost everywhere. We consider Luzin's condition, absolute continuity, and differentiability for the Banach space valued Newtonian space $N^{1,p}(X^Q; \mathcal{V})$, when $p \geq Q$, and thus extend some related results studied in Heinonen et al. [14]. Here $\mathcal{V} := (\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is an arbitrary Banach space of positive dimension. We refer the reader to [14] for a detailed discussion on the Banach space valued Newtonian functions. Suppose X satisfies conditions (D) and (PI) with some $1 \leq p < \infty$; the following is known:

- Let $p > Q$. In this case each function $u \in N^{1,p}(X^Q; \mathbb{R})$ is locally $(1 - Q/p)$-Hölder continuous (Shanmugalingam [29]), moreover u is differentiable μ-a.e. with respect to the strong measurable differentiable structure (see Cheeger [5]). For the latter result we refer to Balogh et al. [2].
- Let $p = Q$. Then every continuous pseudomonotone mapping in $N^{1,Q}_{\text{loc}}(X^Q; \mathcal{V})$ satisfies Luzin's condition (N_Q) (Heinonen et al. [14, Theorem 7.2]).

It would be interesting to generalize Calderon's differentiability theorem to Banach space valued Newtonian functions.

Recall that following Malý–Martio [26], a map $f : X \to \mathcal{V}$ is *pseudomonotone* if there exists a constant $C_M \geq 1$ and $r_M > 0$ such that

$$\operatorname{diam}(f(B(x,r))) \leq C_M \operatorname{diam}(f(\partial B(x,r)))$$

for all $x \in X$ and all $0 < r < r_M$. Note that we denote $\partial B(x,r) := \{y \in X : d(y,x) = r\}$.

Let Ω be open such that $\overline{\Omega} \subset X^Q$. We show next that $u \in N^{1,p}(\Omega; \mathcal{V})$, $p \geq Q$, is absolutely continuous in the following sense. Following Malý [23] we say that a mapping $f : \Omega \to \mathcal{V}$ is *Q-absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every pairwise disjoint finite family $\{B_i\}_{i=1}^{\infty}$ of (closed) balls in Ω we have that

$$\sum_{i=1}^{\infty} \operatorname{diam}(f(B_i))^Q < \varepsilon,$$

whenever $\sum_{i=1}^{\infty} \mu(B_i) < \delta$. Furthermore, we say that a mapping $f : X \to \mathcal{V}$ satisfies the *Q-Radó–Reichelderfer condition*, *condition (RR)* for short, if there exists a non-negative control function $\Phi_f \in L^1_{\mathrm{loc}}(X)$ such that

$$\operatorname{diam}(f(B(x,r)))^Q \leq \int_{B(x,r)} \Phi_f \, d\mu \tag{5.1}$$

for every ball $B(x,r) \subset X$ with $0 < r < R$. A condition similar to this was used by Radó and Reichelderfer in [28, V.3.6] as a sufficient condition for the mappings with the condition (RR) to be differentiable a.e. and to satisfy Luzin's condition, see also Malý [23]. A function f is said to satisfy condition (RR) weakly if (5.1) holds true with a dilated ball $B(x, \alpha r)$, $\alpha > 1$, on the right-hand side of the equation.

It readily follows that condition (RR) implies (local) Q-absolute continuity of f. Indeed, let $\varepsilon > 0$ and $\{B(x_i, r_{x_i})\}$, $0 < r_{x_i} < R$, a pairwise disjoint finite family of balls in Ω such that $E = \bigcup_i B(x_i, r_{x_i})$, and $\mu(E) < \delta$. Then condition (RR) and pairwise disjointness of $\{B(x_i, r_{x_i})\}$ imply

$$\sum_i \operatorname{diam}(f(B(x_i, r_{x_i})))^Q \leq \sum_i \int_{B(x_i, r_{x_i})} \Phi_f \, d\mu = \int_E \Phi_f \, d\mu < \varepsilon.$$

Local absolute continuity of a function follows even if the functions satisfies condition (RR) weakly.

Condition (RR) also implies that the map f has finite pointwise Lipschitz constant almost everywhere, see Wildrick–Zürcher [31, Proposition 3.4]. Combined with a Stepanov-type differentiability theorem [2], this has implications for differentiability [5]. We also refer to a recent paper [32].

For the next proposition, we recall that the noncentered Hardy–Littlewood maximal function restricted to Ω, denoted $\mathcal{M}_\Omega$, is defined for an integrable (real-valued) function f on Ω by

$$\mathcal{M}_\Omega f(x) := \sup_B \fint_{B(x,r)} |f| \, d\mu,$$

where the supremum is taken over all balls $B \subset \Omega$ containing x. Consider further the restrained noncentered maximal function $\mathcal{M}_{\Omega,R}$ in which the supremum is taken only over balls in Ω with radius less than R. Then $\mathcal{M}_\Omega f = \sup_{R>0} \mathcal{M}_{\Omega,R} f$. It is standard also in the metric space setting, we refer to Heinonen [13], that for $1 < p \leq \infty$ the operator $\mathcal{M}_\Omega$ is bounded on L^P, i.e., there exists a constant C, depending on C_μ and p, such that for all $f \in L^p$

$$\|\mathcal{M}f\|_{L^p} \leq C\|f\|_{L^p}.$$

We have the following generalization.

Proposition 5.1. *Suppose X satisfies conditions (D) and (PI) with*

(I) *$p = Q$. If $u \in N^{1,Q}_{\mathrm{loc}}(X^Q; \mathcal{V})$ is continuous and pseudomonotone, then u satisfies condition (RR), and thus is (locally) Q-absolutely continuous.*
(II) *some $p > Q$. Then $u \in N^{1,p}_{\mathrm{loc}}(X^Q; \mathcal{V})$ satisfies condition (RR) weakly, and thus is (locally) Q-absolutely continuous.*

Proof. Let $\Omega \Subset X^Q$ be open, and fix $x \in \Omega$.

(I): Let $B(x, r_x)$, $0 < r_x < \min\{r_D, r_M\}$, be a ball such that $B(x, 12\tau r_x) \subset \Omega$; $\tau \geq 1$ is the dilatation constant appearing in the Poincaré inequality. By a Sobolev embedding theorem Hajłasz–Koskela [11, Theorem 7.1] there exists a constant C, depending on C_μ and the constants in the weak $(1, Q)$-Poincaré inequality, and a radius $r_x < r < 2r_x$ such that

$$\|u(z) - u(y)\|^p_{\mathcal{V}} \leq C d(z, y)^{p/Q} r_x^{p(1-1/Q)} \fint_{B(x,5\tau r_x)} g_u^p \, d\mu \tag{5.2}$$

for each $z, y \in \Omega$ with $d(y, x) = r = d(z, x)$, where $p \in (Q - \varepsilon_0, Q)$. In fact, [11, Theorem 7.1] is stated and proved only for real-valued functions, but the argument is valied also when the target is a Banach space as we may make use of the Lebesgue differentation theorem for Banach space valued maps as in [14, Proposition 2.10]. Since u is pseudomonotone we obtain from (5.2)

$$\operatorname{diam}(u(B(x, r_x)))^p \leq C_M^p \operatorname{diam} u(\partial B(x, r)))^p \leq C r_x^p \fint_{B(x,5\tau r_x)} g_u^p \, d\mu,$$

where C depends on C_μ, C_M, and the constants in the weak $(1, Q)$-Poincaré inequality. For each $y \in B(x, r_x)$ we have

$$\fint_{B(x,5\tau r_x)} g_u^p \, d\mu \leq \fint_{B(y,10\tau r_x)} g_u^p \, d\mu \leq \mathcal{M}_{\Omega,12\tau r_x} g_u^p(y).$$

Compining the preceding two estimates and integrating over $y \in B(x, r_x)$ we get

$$\operatorname{diam}(u(B(x, r_x)))^p \leq C r_x^p \fint_{B(x,r_x)} \mathcal{M}_{\Omega,12\tau r_x} g_u^p \, d\mu.$$

Recall that $Q - \varepsilon_0 < p < Q$; we get

$$\begin{aligned} \operatorname{diam}(u(B(x, r_x)))^p &\leq C r_x^p \mu(B(x, r_x))^{-p/Q} \\ &\qquad \left(\int_{B(x,r_x)} \left(\mathcal{M}_{\Omega,12\tau r_x} g_u^p\right)^{Q/p} \, d\mu \right)^{p/Q} \\ &\leq C r_x^p \mu(B(x, r_x))^{-p/Q} \left(\int_{B(x,r_x)} g_u^Q \, d\mu \right)^{p/Q}, \end{aligned}$$

which implies together with (2.3) that

$$\operatorname{diam}(u(B(x, r_x)))^Q \leq C \tilde{C} \int_{B(x,r_x)} g_u^Q \, d\mu,$$

where C depends on C_μ, C_M, and the constants in the weak $(1, Q)$-Poincaré inequality, and $\tilde{C}$ is from (2.3). As $g_u^Q \in L^1_{\mathrm{loc}}(X)$ this verifies the fact that u satisfies condition (RR), and thus is locally Q-absolutely continuous.

(II): Let $B(x, r_x)$, $0 < r_x < r_D$, be a ball such that $B(x, 5\tau r_x) \subset \Omega$. Theorem 5.1 (3) in Hajłasz–Koskela [11, Theorem 5.1] implies that there exist a constant C, depending on C_μ, p, and the constants appearing in the weak $(1, p)$-Poincaré inequality, such that

$$\|u(z) - u(y)\|_{\mathcal{V}} \leq C d(z, y)^{1-Q/p} r_x^{Q/p} \left(\fint_{B(x,5\tau r_x)} g_u^p \, d\mu \right)^{1/p}$$

for all $z, y \in B(x, r_x)$. In fact, [11, Theorem 5.1] is stated and proved only for real-valued functions, but the argument is valied also when the target is a Banach space. Young's inequality $ab \leq a^p/p + b^{p'}/p'$ and (2.3) imply

$$\begin{aligned} \operatorname{diam}(u(B(x, r_x)))^Q &\leq \frac{C r_x^Q}{\mu(B(x, r_x))^{Q/p}} \left(\int_{B(x,5\tau r_x)} g_u^p \, d\mu \right)^{Q/p} \\ &\leq C \left(\tilde{C}^{-1} \mu(B(x, r_x)) + \int_{B(x,5\tau r_x)} g_u^p \, d\mu \right) \\ &\leq C \left(\int_{B(x,\alpha r_x)} \left(\tilde{C}^{-1} + g_u^p \right) \, d\mu \right). \end{aligned}$$

Hence u satisfies condition (RR) weakly with $\alpha = 5\tau$ and with $\Phi_u = C(\tilde{C}^{-1} + g_u^p)$, $\tilde{C}$ is from (2.3). $\square$

The fact that a continuous pseudomonotone function $u \in N^{1,Q}_{\mathrm{loc}}(X^Q;\mathcal{V})$ verifies Luzin's condition (N_Q) would easily follow also from Proposition 5.1 (I).

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