The idea

Mean Curvature of level sets in Metric Measure Spaces

The idea here is rather simple and leads to several immediate questions as well as several computational explorations. The inspirational starting place is the observation that in $\mathbb{R}^n$,

$$\frac{d}{d\epsilon}\int |\nabla f + \epsilon \nabla \phi| \; dx \bigg|_{\epsilon=0} = \int \nabla \cdot \frac{\nabla f}{|\nabla f|}\phi \; dx \approx \left(\nabla \cdot \frac{\nabla f}{|\nabla f|}\right)(x)\int \phi \; dx.$$

where the last approximation gets better as the support of ϕ gets smaller and smaller (all of this assuming the level sets of f are smooth).

The idea:

- Assume that the metric space we are working in, X, has a doubling measure μ.
- We will use upper gradients. For every ϵ in some neighborhood of 0, we compute "the" upper gradient of $f + \epsilon\phi$. This gives us $|\nabla f + \epsilon \nabla \phi|$.
- Compute $a(\epsilon) = \int |\nabla f + \epsilon \nabla \phi| d\mu$
- Compute $d(\epsilon) \equiv \frac{da}{d\epsilon}$.
- Assume (1) the level sets of f are "smooth" in the support of ϕ and (2) the support of ϕ is small in comparison to the curvatures of the level sets in that support. (If not, presmooth the level sets: see below.)
- Evaluate d at 0, to get $d(0)$.
- Define the curvature of the level set of f at x to be

  $$\mathbf{H}(x) \equiv \lim_{\delta \to 0} \frac{d(0)}{\int \phi d\mu}$$

  where δ is the diameter of the support of ϕ and x is in the support of all the ϕ's. Again, The idea here is that if the support of ϕ is small enough, the mean curvature should be almost constant and can be taken out of the integral:

$$\frac{1}{\int \phi d\mu}\left\{\frac{d}{d\epsilon}\int |\nabla f + \epsilon\nabla\phi|\, d\mu\Big|_{\epsilon=0}\right\} = \frac{1}{\int \phi d\mu}\left\{\int \nabla\cdot\frac{\nabla f}{|\nabla f|}\phi\, d\mu\right\} \tag{1}$$

$$\approx \frac{1}{\int \phi d\mu}\left\{\left(\nabla\cdot\frac{\nabla f}{|\nabla f|}\right)(x)\int \phi\, d\mu\right\} \tag{2}$$

$$= \left(\nabla\cdot\frac{\nabla f}{|\nabla f|}\right)(x) \tag{3}$$

Smoothing:

- Smoothing level sets of f: here we simply take averages, which are the metric space version of convolution.[1] Note that smoothing f does not eliminate all high curvature of level sets – close to a peak, we will always get high curvature.
- Define the r-mullification (I.e. scale $= r$) to be

$$f^*(x) \equiv \int \chi_{B(x,r)} f d\mu.$$

- One can do this multiple times to get higher order smoothing.

All this brings up several questions:

- Does the answer above depend on which minimal upper gradient we choose? A quick look at Bjorn and Bjorn [1] says it doesn't: there is a theorem that says minimal upper gradients are unique. But maybe we don't want to use minimal upper gradients, for some reason?? Anyway, look at this more closely.
- When does the derivative d exist? What are the conditions that we need for it to exist if it doesn't always exist?

[1] See what stuff Coifman and Weiss [3] did on densities, maximal functions, etc, in metric spaces. (Relevant part starts on page 587.) They did this in the 1970's. The paper I cite here is an English overview of work that appeared in French in 1971.

- When is it the case that there is an $h : X \to R$ such that

$$\frac{d}{d\epsilon} \int |\nabla f + \epsilon \nabla \phi| \, d\mu \bigg|_{\epsilon=0} = \int h\phi \, d\mu.$$

  for all ϕ in some class of test functions?

- The theory of BV functions on metric spaces should have information that is useful to us. See Chris Camfield's thesis [2].

- Each of the above steps can be done computationally. Outline a variety of ways to do each step – think about the practical and theoretical issues that these ideas bring up.

Notes

Euler-Lagrange for $\int |\nabla u| \, dx$

Computing

$$\frac{d}{du} \int_\Omega |\nabla u| \, dx$$

via a directional derivative in the direction of ϕ, a test function compactly supported in Ω:

$$\begin{aligned}
\frac{d}{dt} \int_\Omega |\nabla (u + t\phi)| \, dx \bigg|_{t=0} &= \frac{d}{dt} \int_\Omega \sqrt{(\nabla u + t\nabla\phi) \cdot (\nabla u + t\nabla\phi)} \, dx \bigg|_{t=0} && (4)\\
&= \int_\Omega \frac{\nabla\phi \cdot \nabla u}{\sqrt{\nabla u \cdot \nabla u}} \, dx && (5)\\
&= -\int_\Omega \nabla \cdot \frac{\nabla u}{|\nabla u|} \, \phi \, dx \quad \text{(integration by parts)} && (6)
\end{aligned}$$

Since $\frac{\nabla u}{|\nabla u|}$ is the unit normal vectorfield point **into** the superlevel set, $-\frac{\nabla u}{|\nabla u|}$ is the unit normal vectorfield pointing out of the super-level set. Thus $-\nabla \cdot \frac{\nabla u}{|\nabla u|}$ is the mean curvature of the boundary of the superlevel set on which it is evaluated.

Simple Observations Involving the p-Laplacian

Note that in the case that u is a distance function, $|\nabla u| = 1$ and we have:

$$\nabla \cdot \frac{\nabla u}{|\nabla u|} = \nabla \cdot \nabla u \tag{7}$$

$$= \Delta u \tag{8}$$

so the mean curvature of level sets is obtained by the Laplacian operator. Seems like this might be convenient.

More generally, we can always use the method we are introducing in this thread to compute any quantity that is obtainable as a derivative of some function of $|\nabla u|$. For example, since

$$\Delta u = \frac{d}{dt} \int |\nabla u + t\nabla\phi|^2 \, dx \bigg|_{t=0},$$

we can get the Lapacian of functions on a metric space. More generally, we can get the p-Laplacian for any p:

$$\Delta_p u = \frac{d}{dt} \int |\nabla u + t\nabla\phi|^p \, dx \bigg|_{t=0}.$$

Misc

- We could compare mean curvature results for curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$ for various different metrics. This is a very nice controlled sort of experimental approach to this. I suggest starting with the L^p metrics on $\mathbb{R}^2$, working first with L^1, comparing it with the L^2 (Euclidean) metric on $\mathbb{R}^2$.

- Notice that while, for any given function u, the gradient functional ∇u does not change as we change metrics on $\mathbb{R}^{\ltimes}$, the norm of the gradient does. This norm is the operator norm: the maximum value the functional take on the unit ball in the tangent space. (Remember that our notion of the gradient as a vector depends on the fact that we can use the inner product to turn a gradient function into a row vector.)

- Thinking about our calculation above: the calculation depends on the smoothness of the test function ϕ. To see this try the calculation a linear function

$u(x,y) = ax + by$ and (1) $\phi = \chi_{B(0,\epsilon)}$ and (2) $\phi =$ a slightly smoothed version of $\chi_{B(0,\epsilon)}$ (for which the formula's above are correct and so the result is zero since the mean curvature of the level sets of u is zero everywhere). **All the action turns out to be in sloping parts of ϕ.** To get that smoothed version of $\chi_{B(0,\epsilon)}$ you can simply convolve with $\chi_{B(0,\delta)}$, where $\delta << \epsilon$.

Concrete tasks

Circles Do the computation in $\mathbb{R}^2$ with the usual euclidean norm to see how much work it is to get the right answer when the function is $f(x) = |x| = \sqrt{x_1^2 + x_2^2}$. You can construct some family of bump functions that integrate to one and have a radius of support getting very small ... see how hard it is to get the right answer.

Josh gets a Surface or pad of some sort so we can draw things when we are working together. I already have a Microsoft Surface that works quite well. And we can use an iDroo account.

book? Here is one to consider: `http://assets.cambridge.org/97811070/92341/toc/9781107092341_toc.pdf` Here is the reference: [4]. Another is the Bjorn and Bjorn book. There are a few other references too.

To be added to soon ...

References

[1] Anders Bjorn and Jana Bjorn. *Nonlinear Potential Theory on Metric Spaces*. Number 17 in EMS Tracts in Mathemaics. European Mathematical Society, 2011.

[2] Christopher S. Camfield. *Comparison of BV Norms in Weighted Euclidean Spaces and Metric Measure Spaces*. PhD thesis, University of Cincinnati, 2008.

[3] Ronald R Coifman and Guido Weiss. Extensions of hardy spaces and their use in analysis. *Bulletin (New Series) of the American Mathematical Society*, 83(4):569–645, 1977.

[4] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T Tyson. *Sobolev spaces on metric measure spaces*. Number 27. Cambridge University Press, 2015.