.1 Preliminaries, reprise

Exercise .1.1. Let $\mathscr{F} : \mathsf{C} \to \mathsf{D}$ be a covariant functor, and assume that both C and D have products. Prove that for all objects A, B of C, there is a unique morphism $\mathscr{F}(A \times B) \to \mathscr{F}(A) \times \mathscr{F}(B)$ such that the relevant diagram involving natural projections commutes.

If D has *co*products (denoted $\coprod$) and $\mathscr{G} : \mathsf{C} \to \mathsf{D}$ is contravariant, prove that there is a unique morphism $\mathscr{G}(A) \coprod \mathscr{G}(B) \to \mathscr{G}(A \times B)$ (again, such that an appropriate diagram commutes).

Solution. Recall that the product $A \times B$ in C comes equipped with natural projections π_A and π_B to A and B respectively. Then, by the universal property of products, we have the following diagram in D.

$$\begin{tikzcd}
 & & \mathscr{F}(A) \\
\mathscr{F}(A \times B) \arrow[r, "\exists! f"] \arrow[urr, bend left, "\mathscr{F}(\pi_A)"] \arrow[drr, bend right, "\mathscr{F}(\pi_B)"'] & \mathscr{F}(A) \times \mathscr{F}(B) \arrow[ur, "\pi'_A"] \arrow[dr, "\pi'_B"'] & \\
 & & \mathscr{F}(B)
\end{tikzcd}$$

where the morphism from $\mathscr{F}(A \times B) \to \mathscr{F}(A) \times \mathscr{F}(B)$ is unique.

If $\mathscr{G}$ is contravariant, then there are instead morphisms $\mathscr{G}(\pi_A) : \mathscr{G}(A) \to \mathscr{G}(A \times B)$ and similarly for $\mathscr{G}(\pi_B)$. Then the universal property for coproducts induces a unique morphism from $\mathscr{G}(A) \coprod \mathscr{G}(B) \to \mathscr{G}(A \times B)$. $\square$

Exercise .1.2. Let $\mathscr{F} : \mathsf{C} \to \mathsf{D}$ be a fully faithful functor. If A, B are objects in C, prove that $A \cong B$ in C if and only if $\mathscr{F}(A) \cong \mathscr{F}(B)$ in D.

Solution. Recall that $A \cong B$ means there exist morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Furthermore, since functors preserve composition and identity, the forward direction is trivial. Now suppose $\mathscr{F}(A) \cong \mathscr{F}(B)$ in D. Then there exist isomorphisms $f \in \mathrm{Hom}_{\mathsf{D}}(\mathscr{F}(A), \mathscr{F}(B))$ and $g \in \mathrm{Hom}_{\mathsf{D}}(\mathscr{F}(B), \mathscr{F}(A))$. The two Hom sets are in bijection with the sets $\mathrm{Hom}_{\mathsf{C}}(A, B)$ and $\mathrm{Hom}_{\mathsf{C}}(B, A)$ so we have corresponding morphisms f' and g'. In particular, we find

$$f \circ g = \mathscr{F}(f') \circ \mathscr{F}(g') = \mathscr{F}(f' \circ g') = \mathscr{F}(1_B) = 1_{\mathscr{F}(B)}$$

and the bijectivity on morphisms implies that $f' \circ g' = 1_B$. A similar argument holds to show that $g' \circ f' = 1_A$ so these are isomorphisms and $A \cong B$ in C. $\square$

Exercise .1.3. Recall that a group G may be thought of as a groupoid G with a single object. Prove that defining the action of G on an object of a category C is equivalent to defining a functor $\mathsf{G} \to \mathsf{C}$.

Solution. Indeed, recall that a group G can be considered as a category G with one object, X, where $\mathrm{Hom}_{\mathsf{G}}(X, X) = \{g\cdot \mid g \in G\}$. Since every morphism is an isomorphism, this Hom set contains inverses, there is an identity, and composition guarantees associativity. To define a group action of G on an object A of C, let $\mathscr{F} : \mathsf{G} \to \mathsf{C}$ be a functor sending $X \mapsto A$. Similarly, we send each element of $\mathrm{Hom}_{\mathsf{G}}(X, X)$ to an element of $\mathrm{Hom}_{\mathsf{C}}(A, A)$. Since functors preserve identities, $\mathscr{F}(1_X) = 1_A$ which corresponds to $e \cdot a = a$ for all $a \in A$ (if it has some set structure). Similarly, since functors preserve composition, we find $\mathscr{F}(g \circ h) = \mathscr{F}(g) \circ \mathscr{F}(h)$, or $(gh)(a) = g(h(a))$. Thus, we have defined an action. An action can be converted into a functor in a similar manner. □

Exercise .1.4. Let R be a commutative ring, and let $S \subseteq R$ be a *multiplicative subset* in the sense of Exercise V.4.7. Prove that 'localization is a functor': associating with every R-module M the localization $S^{-1}M$ (Exercise V.4.8) and with every R-module homomorphism $\varphi : M \to N$ the naturally induced homomorphism $S^{-1}M \to S^{-1}N$ defines a covariant functor from the category of R-modules to the category of $S^{-1}R$-modules.

Solution. The map assigns every object of R-Mod to an object of $S^{-1}R$-Mod. Furthermore, given a module homomorphism $\varphi : M \to N$, we have an induced homomorphism which maps $\frac{m}{s} \mapsto \frac{\varphi(m)}{s}$. We show that it preserves identities and composition. Let $1_M : M \to M$ be the identity. Then $\mathscr{F}(1_M) : S^{-1}M \to S^{-1}M$ is defined as $\frac{m}{s} \mapsto \frac{m}{s}$ which is equivalent to the identity on $S^{-1}M$. Now let $\alpha : M \to N$ and $\beta : N \to P$ be module homomorphisms. Then $\mathscr{F}(\alpha)$ sends $\frac{m}{s} \mapsto \frac{\alpha(m)}{s}$. Similarly, $\mathscr{F}(\beta)$ sends $\frac{n}{s} \mapsto \frac{\beta(n)}{s}$. Then we find that

$$\mathscr{F}(\beta) \circ \mathscr{F}(\alpha)\left(\frac{m}{s}\right) = \mathscr{F}(\beta)\left(\frac{\alpha(m)}{s}\right) = \frac{\beta(\alpha(m))}{s} = \mathscr{F}(\beta \circ \alpha)\left(\frac{m}{s}\right)$$

so this map preserves composition, hence it is a functor. □

Exercise .1.5. For F a field, denote by F^* the group of nonzero elements of F, with multiplication. The assignment $\mathsf{Fld} \to \mathsf{Grp}$ mapping F to F^* and a homomorphism of fields $\varphi : k \to F$ to the restriction $\varphi|_{k^*} : k^* \to F^*$ is clearly a covariant functor.

On the other hand, a homomorphism of fields $k \to F$ is nothing but a field extension $k \subseteq F$. Prove that the assignment $F \mapsto F^*$ on objects, together with the prescription associating with every $k \subseteq F$ the *norm* $N_{k\subseteq F} : F^* \to k^*$ (cf. Exercise VII.1.12), gives a *contravariant* functor $\mathsf{Fld} \to \mathsf{Grp}$. State and prove an analogous statement for the *trace* (cf. Exercise VII.1.13).

Solution. To do. □

Exercise .1.6. Formalize the notion of presheaf of abelian groups on a topological space T. If $\mathscr{F}$ is a presheaf on T, elements of $\mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ on U. The homomorphism $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ induced by an inclusion $V \subseteq U$ is called the *restriction map.*

Show that an example of a presheaf is obtained by letting $\mathscr{C}(U)$ be the additive abelian group of continuous complex-valued functions on U, with restriction of sections defined by ordinary restrictions of functions.

For this presheaf, prove that one can uniquely glue sections agreeing on overlapping open sets. That is, if U and V are open sets and $s_U \in \mathscr{C}(U), s_V \in \mathscr{C}(V)$ agree after restriction to $U \cap V$, prove that there exists a unique $s \in \mathscr{C}(U \cup V)$ such that s restricts to s_U on U and to s_V on V.

This is essentially the condition making $\mathscr{C}$ a *sheaf.*

Solution. A presheaf of abelian groups on a topological space T is a map $\mathscr{F}$ which assigns each open set U of T to an abelian group. By assumption, the restriction maps ρ_{UV} are homomorphisms of abelian groups. Indeed, we can define the presheaf of continuous complex-valued functions on a topological space. To any open set U of T, let $\mathscr{C}(U)$ be the abelian group of continuous functions on U (under pointwise addition). For an inclusion $U \subseteq V$, we can define the restriction map $\rho_{UV} : \mathscr{C}(V) \to \mathscr{C}(U)$ by sending $f \mapsto f|_U$. Certainly if f is continuous on V, it is continuous on a subset of V. To prove that this is in fact a group homomorphism, we see that

$$\rho_{UV}(f+g) = (f+g)|_U = f|_U + g|_U = \rho_{UV}(f) + \rho_{UV}(g)$$

where the second equality follows from addition being defined as point-wise. Thus, this is in fact a presheaf of abelian groups.

To see that it is also a sheaf, let $s_U \in \mathscr{C}(U)$ and $s_V \in \mathscr{C}(V)$ such that $\rho_{U,U\cap V}(s_U) = \rho_{V,U\cap V}(s_V)$. Define

$$s := \begin{cases} s_U(x) & \text{if } x \in U \\ s_V(x) & \text{if } x \in V \end{cases}$$

Certainly s is continuous on $U \cup V$ since it is continuous on both U and V, as well as on $U \cap V$. It is also unique by construction, as for any function f which agrees with s on $U \cup V$, we find $s - f = 0$ so they are equivalent. $\square$

Exercise .1.7. Define a topology on $\operatorname{Spec} R$ by declaring the closed sets to be the sets $V(I)$, where $I \subseteq R$ is an ideal and $V(I)$ denotes the set of prime ideals *containing* I.

- Verify that this indeed defines a topology on $\operatorname{Spec} R$. (This is the *Zariski topology* on $\operatorname{Spec} R$.)
- Relate this topology to the Zariski topology defined in §VII.2.3.

- Prove that Spec is then a contravariant functor from the category of commutative rings to the category of topological spaces (where morphisms are continuous functions).

Solution. We verify that this is a topology on $\operatorname{Spec} R$. Certainly $\emptyset$ is closed since $R \subseteq R$ is an ideal and no prime ideals contain R. Similarly, $\operatorname{Spec} R$ is closed as $\{0\} \subseteq R$ is an ideal and every ideal contains $\{0\}$. Now let $V(I)$ and $V(J)$ be closed sets. Recall that IJ is the ideal generated by elements of the form ab where $a \in I, b \in J$. We claim that $V(IJ) = V(I) \cup V(J)$. Suppose $\mathfrak{p} \in V(I) \cup V(J)$ and WLOG, assume $\mathfrak{p} \in V(I)$. Then, since $IJ \subseteq I$, we have $IJ \subseteq I \subseteq \mathfrak{p}$ so $\mathfrak{p} \in V(IJ)$. For the other direction, suppose $\mathfrak{p} \in V(IJ)$. Then $\mathfrak{p} \in V(I)$ or $\mathfrak{p} \in V(J)$. Indeed, otherwise we could find an element $ab \in IJ \subseteq \mathfrak{p}$ such that $a, b \notin \mathfrak{p}$, contradicting the assumption that $\mathfrak{p}$ is prime. Thus, $V(IJ) = V(I) \cup V(J)$ and the topology is closed under finite unions. Finally, we claim that $V(I + J) = V(I) \cap V(J)$. Suppose $\mathfrak{p} \in V(I + J)$. That is, $I + J \subseteq \mathfrak{p}$. Since $I \subseteq I + J$ and $J \subseteq I + J$, we find that $\mathfrak{p} \in V(I) \cap V(J)$. Now suppose $\mathfrak{p} \in V(I) \cap V(J)$. That is, $I \subseteq \mathfrak{p}$ and $J \subseteq \mathfrak{p}$. Now let $x \in I + J$. That is, $x = a + b$ for some $a \in I, b \in J$. Then since $a, b \in \mathfrak{p}$, we have $x \in \mathfrak{p}$, thus $I + J \subseteq \mathfrak{p}$ and $\mathfrak{p} \in V(I + J)$. Thus, the intersection of two closed sets is closed, proving that this is in fact a topology on $\operatorname{Spec} R$.

Recall that the Zariski topology defined in §VII.2.3 is defined on $\mathbb{A}^n_K$ by setting algebraic subsets to be the closed sets. Given a set $S \subseteq K[x_1, \ldots, x_n]$, the points of $V(S)$ correspond to the maximal ideals of $K[x_1, \ldots, x_n]$ which contain S. Thus, this is a natural generalization where instead of only using maximal ideals, one extends to prime ideals.

To see that this is indeed a contravariant functor from $\mathsf{CRing} \to \mathsf{Top}$, first note that Spec maps every commutative ring to a topological space (as shown above). Now let $\varphi : R \to S$ be a homomorphism of rings. Then $\operatorname{Spec} R$ induces a morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ which sends $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ (it is easy to verify that the preimage of a prime ideal is prime, which verifies that this is a continuous map). Certainly Spec takes the identity to the identity, and it can quickly be seen that it preserves composition. $\square$

Exercise .1.8. Let K be an algebraically closed field, and consider the category K-Aff defined in Example 1.9.

- Denote by h_S the functor $\operatorname{Hom}_{K-\mathsf{Aff}}(\_, S)$ (as in §1.2), and let $p = \mathbb{A}^0_K$ be a point. Show that there is a natural bijection between S and $h_S(p)$. (Use Exercise VII.2.14.)
- Show how every $\varphi \in \operatorname{Hom}_{K-\mathsf{Aff}}(S, T)$ determines a function *of sets* $S \to T$.
- If $S \subseteq \mathbb{A}^m_K, T \subseteq \mathbb{A}^n_K$, show that the function $S \to T$ determined by a morphism $\varphi \in \operatorname{Hom}_{K-\mathsf{Aff}}(S, T)$ is the restriction of a 'polynomial function' $\mathbb{A}^m_K \to \mathbb{A}^n_K$. (Part of this exercise is to make sense of what this means!)

Solution. Note that $h_S(p) = \mathrm{Hom}_{K-\mathsf{Aff}}(p, S)$. Each map in this set is uniquely determined by the point in q where p is sent to. To formalize this notion, recall that we define $\mathrm{Hom}_{K-\mathsf{Aff}}(p, S) = \mathrm{Hom}_{K-\mathsf{Alg}}(K[S], K)$. By Exercise VII.2.14, there is a natural bijection between the points of S and the maximal ideals of $K[S]$ such that if q corresponds to the ideal $\mathfrak{m}_q$, then the evaluation map from $K[S]$ sending $f \mapsto f(q)$ has kernel $\mathfrak{m}_q$. Thus, each point of S corresponds to a map in the Hom set.

This can be extended to see that every $\varphi \in \mathrm{Hom}_{K-\mathsf{Aff}}(S, T)$ determines a set function $S \to T$. Intuitively, this reflects nothing more than the fact that φ maps points of S to points of T. Formally, we have that $\varphi : K[T] \to K[S]$ is determined by sending $y_i \mapsto f_i(x_1, \ldots, x_m)$. Thus, given a point $p = (x_1, \ldots, x_m) \in S$, we find that φ induces a set function sending $p \mapsto (f_1(p), \ldots, f_n(p))$.

To do. $\square$

Exercise .1.9. Let C, D be categories, and assume C to be small. Define a *functor category* D^{C}, whose objects are covariant functors $\mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations.

Prove that the assignment $X \mapsto h_X := \mathrm{Hom}_{\mathsf{C}}(\_, X)$ defines a covariant functor $\mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathsf{op}}}$. (Define the action on morphisms in the natural way.)

Solution. Note that $\mathsf{Set}^{\mathsf{C}^{\mathsf{op}}}$ is the category whose objects are covariant functors $\mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$. Indeed, since C is small, we find $\mathrm{Hom}_{\mathsf{C}}(A, X)$ is a set for all objects A of C. Given a morphism $f : X \to Y$ in C, we set $\mathscr{F}(f) : h_X \to h_Y$ to be the natural transformation $v_A : \mathrm{Hom}_{\mathsf{C}}(A, X) \to \mathrm{Hom}_{\mathsf{C}}(A, Y)$ which maps $\alpha : A \to X$ to $\beta : A \to Y$ where $\beta = f \circ \alpha$. Verifying that that this is in fact a natural transformation is a brief diagram chase. We check that this functor $\mathscr{F}$ preserves identities. Indeed, consider $\mathscr{F}(1_X) : h_X \to h_X$ to be the natural transformation $v_A : \mathrm{Hom}_C(A, X) \to \mathrm{Hom}_C(A, X)$ which sends $\alpha : A \to X$ to $\alpha = 1_X \circ \alpha$. Then clearly v is the identity on all Hom sets. Similarly, since natural transformations can be composed, it is quick to check that $\mathscr{F}$ preserves compositions. $\square$

Exercise .1.10. Let C be a category, X and object of C, and consider the contravariant functor $h_X := \mathrm{Hom}_{\mathsf{C}}(, X)$. For every contravariant functor $\mathscr{F} : \mathsf{C} \to \mathsf{Set}$, prove that there is a bijection between the set of natural transformations $h_x \rightsquigarrow \mathscr{F}$ and $\mathscr{F}(X)$ as follows. The datum of a natural transformation $h_X \rightsquigarrow \mathscr{F}$ consists of a morphism from $h_X(A) = \mathrm{Hom}_{\mathsf{C}}(A, X)$ to $\mathscr{F}(A)$ for every object A of C. Map h_X to the image of $\mathrm{id}_X \in h_X(X)$ in $\mathscr{F}(X)$. (Hint: Produce an inverse of the specified map. For every $f \in \mathscr{F}(X)$ and every $\varphi \in \mathrm{Hom}_{\mathsf{C}}(A, X)$, how do you construct an element of $\mathscr{F}(A)$?)

This result is called the *Yoneda lemma.*

Solution. Given an element $f \in F(x)$, we construct a natural transform $\alpha_f : h_X \to F$. The associated morphism is

$$\alpha_f(A) : \mathrm{Hom}(A, X) \to F(A), \quad \varphi \mapsto F(\varphi)(f) \in F(A)$$

since F is contravariant. To verify that this is indeed a natural transformation, let $f \in F(x)$, $g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, and $\varphi \in \mathrm{Hom}(B, X)$. Then,

$$(\alpha_f(A) \circ h_x(g))(\varphi) = F(\varphi \circ g)(f)$$
$$(F(g) \circ \alpha_f(B))(\varphi) = F(\varphi)(f) \circ F(g) = F(\varphi \circ g)(f).$$

The two are equal, hence α_f is indeed a natural transformation. Finally, we show that the two constructed functions are inverses. Let $f \in F(X)$. Then we can construct the natural transformation α_f. But then we obtain an element of $F(X)$ by sending this transformation to $\alpha_f(X)(\mathrm{id}_X) = F(\mathrm{id}_X)(f) = \mathrm{id}_X(f) = f$ since functors preserve identity morphisms. Thus, the two are inverses and there is a bijection between the set of natural transformations from $h_x \to F$ and the set $F(X)$. $\square$

Exercise .1.11. Let C be a small category. A contravariant functor $\mathsf{C} \to \mathsf{Set}$ is *representable* if it is naturally isomorphic to a functor h_X. In this case, X 'represents' the functor. Prove that C is equivalent to the subcategory of representable functors in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$.

Thus, every (small) category is equivalent to a subcategory of a functor category.

Solution. Consider the functor $F : C \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ sending $X \mapsto h_X$ and which sends morphisms $f : X \to Y$ to the natural transformation $\alpha : h_X \to h_Y$ such that $\alpha_A(\varphi) = f \circ \varphi$. We prove that F is an equivalence of categories. By the Yoneda lemma, there is a bijection between the set of natural transformations $h_X \to h_Y$ and the elements of $h_Y(X)$. In particular, there is a bijection between $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $\mathrm{Hom}_{\mathsf{C}^{\mathsf{Set}^{\mathrm{op}}}}(h_X, h_Y)$. Thus, F is fully faithful. To show that F is essentially surjective, let G be a representable functor. That is, G is naturally isomorphic to a functor h_X for some object X in C. Then $G \cong h_X = F(X)$ in $\mathsf{C}^{\mathsf{Set}^{\mathrm{op}}}$. Thus, F is an equivalence of categories. $\square$

Exercise .1.12. Let C, D be categories, and let $\mathscr{F} : \mathsf{C} \to \mathsf{D}$, $\mathscr{G} : \mathsf{D} \to \mathsf{C}$ be functors. Prove that $\mathscr{F}$ is left-adjoint to $\mathscr{G}$ if and only if, for every object Y in D, the object $\mathscr{G}(Y)$ represents the functor $h_Y \circ \mathscr{F}$.

Solution. Recall that $\mathscr{F}$ is left-adjoint to $\mathscr{G}$ iff there is a natural isomorphism such that for all objects X of C and Y of D, $\mathrm{Hom}_{\mathsf{C}}(X, \mathscr{G}(Y)) \cong \mathrm{Hom}_{\mathsf{D}}(\mathscr{F}(X), Y)$. In particular, if we fix Y, then there is a natural isomorphism between $h_{\mathscr{G}(Y)}$ and $h_Y \circ \mathscr{F}$ (since $h_Y \circ \mathscr{F}(X) = \mathrm{Hom}_{\mathsf{D}}(\mathscr{F}(X), Y)$). That is, $\mathscr{G}(Y)$ represents $h_Y \circ \mathscr{F}$. The other direction is effectively the same. $\square$

Exercise .1.13. Let Z be the 'Zen' category consisting of no objects and no morphisms. One can contemplate a functor $\mathscr{L}$ from Z to any category C: no datum whatsoever need be specified. What is $\varprojlim \mathscr{L}$ (when such an object exists)?

Solution. Since no datum is specified and by the definition of a limit, the object is final with respect to the property defined by a cone up to isomorphism, $\varprojlim \mathscr{L}$ is a final object of C. $\square$

Exercise .1.14. Verify that the construction described in Example 1.11 indeed recovers the kernel of a homomorphism of R-modules, as claimed.

Solution. The construction describes taking the limit of a functor from a two-object category with parallel morphisms in which one is sent to the zero morphism. In particular, such a limit is determined by the choice of two modules, A_1 and A_2, along with morphisms $\varphi : A_2 \to A_1$ and $0 : A_2 \to A_1$. The limit of this functor, say $\varprojlim \mathscr{K}$, is a module K equipped with morphisms $\lambda_i : K \to A_i$ such that $\lambda_1 = \varphi \circ \lambda_2$ and $\lambda_1 = 0 \circ \lambda_2$. That is, any morphism from $K \to A_1$ is the zero map and uniquely factors through A_2. Since K is final with respect to this property, we recover the definition of the kernel of a module homomorphism. $\square$

Exercise .1.15. Verify that the construction given in the proof of Claim 1.13 is an inverse limit, as claimed.

Solution. Claim 1.13 constructs the limit $\varprojlim A_i$ in $R-\mathsf{Mod}$. Consider the product ΠA_i which consists of arbitrary sequences $(a_i)_{i>0}$ where $a_i \in A_i$. A sequence $(A_i)_{i>0}$ is *coherent* if for all $i > 0$, we have $a_i = \varphi_{i,i+1}(a_{i+1})$. Coherent sequences form an R-submodule A of ΠA_i where the canonical projections restrict to homomorphisms $\varphi_i : A \to A_i$. Indeed, we have $\varphi_i(a) = a_i = \varphi_{i,i+1} \circ \varphi_{i+1}(a)$. Furthermore, suppose M is another module endowed with morphisms satisfying the requirement. Then, since there are morphisms $\lambda_i : M \to A_i$, there is a unique morphism $\lambda : M \to A$ sending $m \mapsto (\lambda_i(m))_{i>0}$. This morphism makes all relevant diagrams to commute and is entirley determined by M, hence A is final with respect to this property, making it a limit. $\square$

Exercise .1.16. Flesh out the sketch of the constructions of colimits in Set and $R-\mathsf{Mod}$ given in §1.4, for an indexing poset. In Set, observe that the construction of the colimit is simpler if the poset I is *directed*; that is, if $\forall i, j \in \mathsf{I}$, there exists a $k \in \mathsf{I}$ such that $i \leq k$, $j \leq k$.

Solution. No, I don't think I will. / To do. $\square$

Exercise .1.17. Let R, S be rings. Prove that an additive covariant functor $\mathscr{F} : R-\mathsf{Mod} \to S-\mathsf{Mod}$ is exact if and only if $\mathscr{F}(A) \longrightarrow \mathscr{F}(B) \longrightarrow \mathscr{F}(C)$ is exact in $S-\mathsf{Mod}$ whenever $A \longrightarrow B \longrightarrow C$ is exact in $R-\mathsf{Mod}$. Deduce that an exact functor sends exact complexes to exact complexes.

Proof. One direction is trivial since if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then so is

$$A \longrightarrow B \longrightarrow C.$$

For the other direction, suppose $\mathscr{F}$ is an additive covariant functor satisfying the specified property. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of R-modules. In particular, each 'sub-sequence' is exact. Then applying $\mathscr{F}$ to each 'sub-sequence' preserves exactness, and since $\mathscr{F}(d)^2 = 0$, we may concatenate the 'sub-sequences' to obtain the necessary exact sequence of S-modules. Thus, $\mathscr{F}$ is an exact functor. The same logic applies to arbitrary exact sequences. □

Exercise .1.18. Let R, S be rings. An additive covariant functor $\mathscr{F} : R-\mathsf{Mod} \to S-\mathsf{Mod}$ is *faithfully* exact if '$\mathscr{F}(A) \longrightarrow \mathscr{F}(B) \longrightarrow \mathscr{F}(C)$ is exact in $S-\mathsf{Mod}$ if and only if $A \longrightarrow B \longrightarrow C$ is exact in $R-\mathsf{Mod}$'. Prove that an exact functor $\mathscr{F} : R-\mathsf{Mod} \to S-\mathsf{Mod}$ is faithfully exact if and only if $\mathscr{F}(M) \neq 0$ for every nonzero R-module M, if and only if $\mathscr{F}(\varphi) \neq 0$ for every nonzero morphism φ in $R-\mathsf{Mod}$.

Solution. Suppose $\mathscr{F}$ is faithfully exact. Suppose $\mathscr{F}(M) = 0$ for some R-module M. Then there is an exact sequence $0 \longrightarrow \mathscr{F}(M) \longrightarrow 0$. But then $0 \longrightarrow M \longrightarrow 0$ is an exact sequence of R-modules, implying that $M = 0$.

Now suppose $\mathscr{F}(M) \neq 0$ for every nonzero R-module M. Let $\varphi : M \to N$ be a homomorphism of R-modules. We obtain a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
\mathscr{F}(M) \arrow[r, "\mathscr{F}(\varphi')"] \arrow[dr, "\mathscr{F}(\varphi)"'] & \mathscr{F}(\operatorname{im}\varphi) \arrow[r] \arrow[d, "\mathscr{F}(i)"] & 0 \\
 & \mathscr{F}(N) &
\end{tikzcd}$$

Suppose $\mathscr{F}(\varphi) = 0$. Then $\mathscr{F}(i) \circ \mathscr{F}(\varphi') = 0$, and by the injectivity of i, we have $\mathscr{F}(\varphi') = 0$. This implies that $\mathscr{F}(\operatorname{im}\varphi) = 0$, hence $\operatorname{im}\varphi = 0$, hence $\varphi = 0$.

Finally, suppose $\mathscr{F}(\varphi) \neq 0$ for every nonzero R-module homomorphism φ. Let $\mathscr{F}(A) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(B) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(C)$ be exact. Since $\mathscr{F}(g) \circ \mathscr{F}(f) = \mathscr{F}(g \circ f) = 0$, we find that $g \circ f = 0$ and $\operatorname{im} f \subseteq \ker g$. Then we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
0 & \mathscr{F}(\operatorname{im} f) \arrow[l] \arrow[d, "\mathscr{F}(i)"'] & \mathscr{F}(A) \arrow[l, "\mathscr{F}(f')"'] \arrow[d, "\mathscr{F}(f)"] & \\
0 \arrow[r] & \mathscr{F}(\ker g) \arrow[r, "\mathscr{F}(j)"'] \arrow[d, "\mathscr{F}(p)"'] & \mathscr{F}(B) \arrow[r, "\mathscr{F}(g)"'] & \mathscr{F}(C) \\
 & \mathscr{F}(\ker g / \operatorname{im} f) \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

We show that $\mathscr{F}(i)$ is surjective. Indeed, let $x \in \mathscr{F}(\ker g)$. Since $\mathscr{F}(g) \circ \mathscr{F}(j) = 0$, $\mathscr{F}(j)(x) \in \ker \mathscr{F}(g) = \operatorname{im} \mathscr{F}(f)$. That is, there exists some $y \in \mathscr{F}(A)$ such that $\mathscr{F}(f)(y) = \mathscr{F}(j)(x)$. Thus, $\mathscr{F}(j)(x) = \mathscr{F}(f)(y) = \mathscr{F}(j) \circ \mathscr{F}(i) \circ \mathscr{F}(f')(y)$. By the injectivity of $\mathscr{F}(j)$, we find that $x = \mathscr{F}(i) \circ \mathscr{F}(f')(y)$. Thus, $\mathscr{F}(i)$ is surjective, as well as injective. Therefore, $\mathscr{F}(p) = 0^*$, hence $p = 0$, hence $\ker g = \operatorname{im} f$, so the corresponding sequence of R-modules is exact. $\square$

Exercise .1.19. Prove that localization is an *exact* functor.

In fact, prove that localization 'preserves homology': if

$$M_\bullet : \ \cdots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

is a complex of R-modules and S is a multiplicative subset of R, then the localization $S^{-1}H_i(M_\bullet)$ of the i-th homology of $M_\bullet$ is the i-th homology $H_i(S^{-1}M_\bullet)$ of the localized complex

$$S^{-1}M_\bullet : \ \cdots \longrightarrow S^{-1}M_{i+1} \xrightarrow{S^{-1}d_{i+1}} S^{-1}M_i \xrightarrow{S^{-1}d_i} S^{-1}M_{i-1} \longrightarrow \cdots$$

Solution. We first prove that localization is an exact functor. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules. That is, $\ker g = \operatorname{im} f$. Localizing yields a complex $S^{-1}A \xrightarrow{S^{-1}f} S^{-1}B \xrightarrow{S^{-1}g} S^{-1}C$ where $\operatorname{im} S^{-1}f \subseteq \ker S^{-1}g$. To see the other inclusion, let $m/s \in \ker S^{-1}g$. That is, $S^{-1}g(m/s) = 0$, so $g(m)/s = 0$, hence there exists $t \in S$ such that $tg(m) = 0$. But $tg(m) = g(tm)$, hence $tm \in \ker g = \operatorname{im} f$. Therefore, there exists $a \in A$ such that $f(a) = tm$. Thus, $m/s = mt/st = f(a)/st \in \operatorname{im} S^{-1}f$. Hence, localization is an exact functor.

The i-th homology of $M_\bullet$ is given by $\frac{\ker d_i}{\operatorname{im} d_{i+1}}$, which inherits the structure of an R-module. Thus, we may localize it to obtain the $S^{-1}R$-module $S^{-1}H_i(M_\bullet)$.

On the other hand, localizing the complex yields induced morphisms $S^{-1}d_i$ such that the homology $H_i(S^{-1}M_\bullet) = \frac{\ker S^{-1}d_i}{\operatorname{im} S^{-1}d_{i+1}}$.

We have the following exact sequences:

$$0 \longrightarrow \ker d_i \xrightarrow{i} M_i \xrightarrow{d_i} M_{i-1}$$

$$M_{i+1} \xrightarrow{d_{i+1}} \operatorname{im} d_{i+1} \longrightarrow 0$$

Localizing both of these yields exact sequences which show that $S^{-1}\ker d_i \cong \ker S^{-1}d_i$ and that $\operatorname{im} S^{-1}d_{i+1} \cong S^{-1}\operatorname{im} d_{i+1}$. Finally, we have the exact sequence

$$0 \longrightarrow \operatorname{im} d_{i+1} \longrightarrow \ker d_i \longrightarrow \frac{\ker d_i}{\operatorname{im} d_{i+1}} \longrightarrow 0$$

Localizing yields the exact sequence

$$0 \longrightarrow S^{-1}\operatorname{im} d_{i+1} \longrightarrow S^{-1}\ker d_i \longrightarrow S^{-1}\frac{\ker d_i}{\operatorname{im} d_{i+1}} \longrightarrow 0$$

Finally, combining all of the above yields

$$S^{-1}H_i(M_\bullet) \cong S^{-1}\frac{\ker d_i}{\operatorname{im} d_{i+1}} \cong \frac{S^{-1}\ker d_i}{S^{-1}\operatorname{im} d_{i+1}} \cong \frac{\ker S^{-1}d_i}{\operatorname{im} S^{-1}d_{i+1}} \cong H_i(S^{-1}M_\bullet).$$

$\square$

Exercise .1.20. Prove that localization is faithfully exact in the following sense: let R be a commutative ring and let

$$(*) \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a sequence of R-modules. Then (*) is exact if and only if the induced sequence of $R_\mathfrak{p}$-modules

$$0 \longrightarrow A_\mathfrak{p} \longrightarrow B_\mathfrak{p} \longrightarrow C_\mathfrak{p} \longrightarrow 0$$

is exact for evey prime ideal $\mathfrak{p}$ of R, if and only if it is exact for every maximal ideal $\mathfrak{p}$.

Solution. Suppose (*) is exact and let $\mathfrak{p}$ be a prime ideal of R. Then $S = R \setminus \mathfrak{p}$ so we may consider the localization $R_\mathfrak{p}$. Since $(*)$ is exact, each homology group vanishes. Furthermore, since localization preserves homology, the homology of the induced sequence of $R_\mathfrak{p}$-modules also vanishes, hence it is exact.

Now suppose the induced sequence is exact for every prime ideal of R. In particular, since maximal ideals are prime, the sequence is exact for every maximal ideal of R.

Finally, suppose the induced sequence of $R_{\mathfrak{p}}$-modules is exact for every maximal ideal $\mathfrak{p}$ of R. Consider a homology group H of (*) with the inherited R-module structure and suppose that $m \neq 0$ is in H. Then the ideal $\{r \in R | rm = 0\}$ is a proper ideal of R (since $1 \cdot m \neq 0$). In particular, it is contained in some maximal ideal $\mathfrak{m}$. But then we may consider the localized homology group $H_{\mathfrak{m}}$, which is nonempty in this case. This implies that the sequence of $R_{\mathfrak{m}}$-modules is not exact. Thus, the contrapositive yields the desired result. □

Exercise .1.21. Let R, S be rings. Prove that right-adjoint functors $R-\mathsf{Mod} \to S-\mathsf{Mod}$ are left-exact and left-adjoint functors are right-exact.

Solution. Let $\mathscr{F} : R-\mathsf{Mod} \to S-\mathsf{Mod}$ be a right-adjoint functor, say it is right-adjoint to $\mathscr{G}$. Consider an exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

We want to show that the sequence

$$\mathscr{F}(0) \longrightarrow \mathscr{F}(A) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(B) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(C)$$

is exact. First, recall that 0 as a zero object, hence a final object, is a limit. Furthermore, right-adjoints commute with limits. Therefore, $\mathscr{F}$ preserves this object; that is, $\mathscr{F}(0) = 0$. Thus, it suffices to show that $\ker \mathscr{F}(f) = 0$ and $\ker \mathscr{F}(g) = \operatorname{im} \mathscr{F}(f)$.

Recall that the kernel is a categorical limit, so $\mathscr{F}$ preserves kernels. Since $\ker f = 0$, we find that

$$\ker \mathscr{F}(f) = \mathscr{F}(\ker f) = \mathscr{F}(0) = 0.$$

Furthermore, we find that $\ker \mathscr{F}(g) = \mathscr{F}(\ker g) = \mathscr{F}(\operatorname{im} f)$. Since $\ker \mathscr{F}(f) = 0$, $\mathscr{F}(f)$ is injective, hence $\mathscr{F}(A) \cong \operatorname{im} \mathscr{F}(f)$. But from this, we find that

$$\ker \mathscr{F}(g) = \mathscr{F}(\ker g) = \mathscr{F}(\operatorname{im} f) \cong \mathscr{F}(A) \cong \operatorname{im} \mathscr{F}(f).$$

To tell the truth, this probably doesn't verify what's necessary but I'm stuck at this point so idk. I figure it's a similar strategy for showing that left-adjoints are right-exact. Note to come back and finish missing problems / To do. □