.1 Homomorphisms of free modules, II

Exercise .1.1. Use Gaussian elimination to find all integer solutions of the system of equations

$$\begin{cases} 7x - 36y + 12z = 1, \\ -8x + 42y - 14z = 2. \end{cases}$$

Solution. Transforming the system of equations into a matrix and applying Gaussian elimination yields the factorization

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ -8 & -7 \end{pmatrix} \cdot \begin{pmatrix} 7 & -36 & 12 \\ -8 & 42 & -14 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 4 & -3 \end{pmatrix},$$

or $D = M \cdot A \cdot N$. As we are trying to solve $A\boldsymbol{x} = \boldsymbol{b}$, we now can now solve $D\boldsymbol{y} = M\boldsymbol{b}$. Finally, we solve $\boldsymbol{x} = N\boldsymbol{y}$ for a solution of

$$\boldsymbol{x} = \begin{pmatrix} 19 \\ -11 - z \\ -44 - 3z \end{pmatrix}$$

so there are infinitely many solutions based on the free variable z. □

Exercise .1.2. Provide details for the proof of Lemma 3.2.

Lemma 3.2. *Let A be a square matrix with entries in an integral domain R.*

- *Let A' be obtained from A by switching two rows or two columns. Then* $\det(A') = -\det(A)$.
- *Let A' be obtained from A by adding to a row (column) a multiple of another row (column). Then* $\det(A') = \det(A)$.
- *Let A' be obtained from A by multiplying a row (column) by an element $c \in R$. Then* $\det(A') = c\det(A)$.

In other words, the effect of an elementary operation on $\det A$ *is the same as multiplying* $\det A$ *by the determinant of the corresponding matrix.*

Solution. Switching two rows is equivalent to multiplying each $\sigma \in S_n$ by a fixed transposition. Then the sign of each permutation is switched so we have

$$\det(A') = \sum_{\sigma \in S_n} (-1)^{\sigma+1} \prod_{i=1}^{n} a_{i\sigma(i)} = -\det(A)$$

yielding the desired result.

For the third point, each product has exactly one c in it so we find

$$\det(A') = \sum_{\sigma \in S_n} (-1)^\sigma c \prod_{i=1}^{n} a_{i\sigma(i)} = c \det(A)$$

yielding the desired result.

For the second point, note that $A' = (a_1, a_2, \dots, a_i + ka_j, \dots, a_n)$ so A and A' differ at only one row. Then we have

$$\begin{aligned}\det(A') &= \det(A) + \det(a_1, a_2, \dots, ka_j, \dots, a_n)\\ &= \det(A) + k\det(a_1, a_2, \dots, a_j, \dots, a_n).\end{aligned}$$

But then rows i and j are identical in the second matrix so it follows that the determinant of that matrix is 0. Thus, we are left with $\det(A') = \det(A)$. □

Exercise .1.3. Redo Exercise II.8.8.

Exercise II.8.8. Prove that $\mathrm{SL}_n(\mathbb{R})$ is a *normal subgroup* of $\mathrm{GL}_n(\mathbb{R})$, and 'compute' $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$ as a well-known group.

Solution. Recall that $\mathrm{SL}_n(\mathbb{R})$ is the set of $n \times n$ matrices with determinant 1. Certainly this is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$ since it is the kernel of the homomorphism induced by det from $\mathrm{GL}_n(\mathbb{R})$ to $\mathbb{R}^\times$. Then by the first isomorphism theorem, we find that $\mathbb{R}^\times \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$. □

Exercise .1.4. Formalize the discussion of 'universal identities': by what cocktail of universal properties is it true that if an identity holds in $\mathbb{Z}[x_1, \dots, x_r]$, then it holds over every commutative ring R, for every choice of $x_i \in R$? (Is the commutativity of R necessary?)

Solution. This holds because $Z[x_1, \dots, x_r]$ is a free object in the category of commutative rings, or commutative $\mathbb{Z}$-algebras. In particular, for every commutative ring R and set function $f : A \to R$, there exists a unique $\mathbb{Z}$-algebra homomorphism from $\mathbb{Z}[x_1, \dots, x_r]$ to R. If the identity is preserved by homomorphisms, then it will hold in every commutative ring. Furthermore, the commutativity of R is not necessary but it is necessary that given a set-function f, we have $f(a)$ commutes with every element of R for all $a \in A$. □

Exercise .1.5. Let A be an $n \times n$ square invertible matrix with entries in a field, and consider the $n \times (2n)$ matrix $B = (A \mid I_n)$ obtained by placing the identity matrix to the side of A. Perform elementary row operations on B so as to reduce A to I_n (cf. Exercise 2.15). Prove that this transforms B into $(I_n \mid A^{-1})$.

(This is a much more efficient way to compute the inverse of a matrix than by using determinants as in §3.2.)

Solution. Each elementary row operation on B can be encoded as an elementary matrix whose product reduces A to I_n. That is, we have $PA = I_n$. But then $P = A^{-1}$ and since $PI_n = P$, it must be the case that $B = (I_n \mid P) = (I_n \mid A^{-1})$. $\square$

Exercise .1.6. Let R be a commutative ring and $M = \langle m_1, \ldots, m_r \rangle$ a finitely generated R-module. Let $A \in \mathcal{M}_r(R)$ be a matrix such that $A \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Prove that $\det(A)m = 0$ for all $m \in M$. (Hint: Multiply by the adjoint.)

Solution. Denote the adjoint matrix of A by A'. Recall that $A'A = \det(A)I_n$. Multiplying both sides of the equation by the adjoint yields

$$A'A \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = A' \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$\det(A)I_n \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

so $\det(A)m_i = 0$ for all $m_i \in \langle m_1, \ldots, m_r \rangle$. Since this is a generating set for M, all $m \in M$ are linear combinations of m_i. Thus, we have $\det(A)m = 0$ for all $m \in M$. $\square$

Exercise .1.7. Let R be a commutative ring, M a finitely generated R-module, and let J be an ideal of R. Assume $JM = M$. Prove that there exists an element $b \in J$ such that $(1 + b)M = 0$. (Let $m_1, \ldots, m_r$ be generators for M. Find an $r \times r$ matrix B with entries in J such that $\begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = B \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix}$. Then use Exercise 3.6.)

Solution. Let $\langle m_1, \ldots, m_r \rangle$ be a set of generators for M. Since $JM = M$, for all m_j in the generating set, there exists a finite sum

$$m_j = \sum_{i=0}^{r} b_i m_i.$$

Thus, we can construct a matrix B with entries in J such that

$$\begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = B \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix}$$

which can be rearranged to $(I_r - B)(m_i)^T = 0$. Let $d = \det(I_r - B)$. Then $d \in 1 + J$ since $B \equiv 0 \mod J$. By Exercise 3.7, $dm = 0$ for all $m \in M$. That is, there exists $b \in J$ such that $(1 + b)M = 0$. □

Exercise .1.8. Let R be a commutative ring, M be a finitely generated R-module, and let J be an ideal of R *contained in the Jacobson radical* of R (Exercise V.3.14). Prove that $M = 0 \iff JM = M$. (Use Exercise 3.7. This is *Nakayama's lemma*, a result with important applications in commutative algebra and algebraic geometry. A particular case was given as Exercise III.5.16.)

Solution. If $M = 0$, then clearly for all $b \in J$, we have $bM = 0 = M$ so $JM = M$. Now suppose $JM = M$. Recall that the Jacobson radical of a ring is the intersection of its maximal ideals. By Exercise 3.7, there exists some $b \in J$ such that $(1 + b)M = 0$. Then $1 + b$ is a unit in R so multiplying both sides by its inverse yields $M = 0$. □

Exercise .1.9. Let R be a commutative local ring, that is, a ring with a single maximal ideal $\mathfrak{m}$, and let M, N be finitely generated R-modules. Prove that if $M = \mathfrak{m}M + N$, then $M = N$. (Apply Nakayama's lemma, that is, Exercise 3.8, to M/N. Note that the Jacobson radical of R is $\mathfrak{m}$.)

Solution. If $M = \mathfrak{m}M + N$, then $M/N = \mathfrak{m}M/N$. By Nakayama's lemma, $M/N = 0$ so $M = N$. □

Exercise .1.10. Let R be a commutative local ring, and let M be a finitely generated R-module. Note that $M/\mathfrak{m}M$ is a finite-dimensional vector space over the field $R/\mathfrak{m}$; let $m_1, \ldots, m_r \in M$ be elements whose cosets mod $\mathfrak{m}M$ form a basis of $M/\mathfrak{m}M$. Prove that $m_1, \ldots, m_r$ generate M.

(Show that $\langle m_1, \ldots, m_r\rangle + \mathfrak{m}M = M$; then apply Nakayama's lemma in the form of Exercise 3.9.)

Solution. We have $\langle \bar{m_1}, \ldots, \bar{m_r}\rangle = M/\mathfrak{m}M$, where $\bar{m_i} = m_i \mod \mathfrak{m}M$. That is, $\langle m_1, \ldots, m_r\rangle + \mathfrak{m}M = M$. Then, by Exercise 3.9, $\langle m_1, \ldots, m_r\rangle = M$. □

Exercise .1.11. Explain how to use Gaussian elimination to find bases for the row space and the column space of a matrix over a field.

Solution. Recall that Gaussian elimination does not change the row space of the matrix. Then reducing a matrix to reduced echelon form yields a matrix whose rows have the same span as the row space of the original matrix and are linearly independent. Thus, they form a basis for the row space. Similarly, applying Gaussian elimination to the transpose of the matrix yields a basis for the column space. □

Exercise .1.12. Let R be an integral domain, and let $M \in \mathcal{M}_{m,n}(R)$, with $m < n$. Prove that the columns of M are linearly dependent over R.

Solution. Recall that M represents a homomorphism $f : R^n \to R^m$ and the column space of M is equal to the span of im f. If the columns of M are linearly independent, then the standard basis vectors of R^n map to a linearly independent set in R^m. That is, the rank of R^m must be greater than or equal to that of R^n, or $m \geq n$. Thus, if $n < m$, it must be the case that the columns of M are linearly dependent. $\square$

Exercise .1.13. Let k be a field. Prove that a matrix $M \in \mathcal{M}_{m,n}(k)$ has rank $\leq r$ if and only if there exist matrices $P \in \mathcal{M}_{m,r}(k), Q \in \mathcal{M}_{r,n}(k)$ such that $M = PQ$. (Thus the rank of M is the smallest such integer.)

Solution. Suppose there exist $P \in \mathcal{M}_{m,r}(k), Q \in \mathcal{M}_{r,n}(k)$ such that $M = PQ$. Let $s = \text{rank } P, t = \text{rank } Q$. Then

$$P = \left(\begin{array}{c|c} I_s & 0 \\ \hline 0 & 0 \end{array}\right), \quad Q = \left(\begin{array}{c|c} I_t & 0 \\ \hline 0 & 0 \end{array}\right).$$

Including zero rows (columns) to the smaller identity matrix to make block multiplication possible yields

$$M = \left(\begin{array}{c|c} I_{\min(s,t)} & 0 \\ \hline 0 & 0 \end{array}\right)$$

and since $\min(s,t) \leq r$, the rank of $M \leq r$.

Now suppose M has rank $r' \leq r$. Consider the matrices

$$P = \left(\begin{array}{c|c} I_{r'} & 0 \\ \hline 0 & 0 \end{array}\right), \quad Q = \left(\begin{array}{c|c} I_{r'} & 0 \\ \hline 0 & 0 \end{array}\right).$$

Note that P and Q can be defined for all $r \geq r'$. Then their product is equivalent to M (up to multiplication by invertible matrices on the left and right, both of which preserve the rank of M). $\square$

Exercise .1.14. Generalize Proposition 3.11 to the case of finitely generated free modules over any integral domain. (Embed the integral domain in its field of fractions.)

Proposition 3.11. *Let*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

be a short exact *sequence of finite-dimensional vector spaces. Then*

$$\dim(V) = \dim(U) + \dim(W).$$

Solution. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of finitely generated free modules over an integral domain R. Embed R in its field of fractions K. By Exercise 1.7, each module is naturally mapped to a vector space over K. In particular, if A' denotes the vector space corresponding to the module A, we have $\operatorname{rank}(A) = \dim(A')$. Then, by Proposition 3.11, we have $\dim(B') = \dim(A') + \dim(C')$ which translates into $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$. □

Exercise .1.15. Prove Proposition 3.13 for the case $N = 1$.

Proposition 3.13. *With notation as above,*

$$\chi(V_\bullet) = \sum_{i=0}^{N} (-1)^i \dim(H_i(V_\bullet)).$$

In particular, if $V_\bullet$ is exact, then $\chi(V_\bullet) = 0$.

Solution. Let

$$V_\bullet :\ 0 \longrightarrow V_1 \xrightarrow{\alpha_1} V_0 \longrightarrow 0$$

be a complex of finite-dimensional vector spaces and linear maps. By definition, we have $\chi(V_\bullet) = \dim(V_0) - \dim(V_1)$. Furthermore, we find

$$H_0(V_\bullet) = \frac{V_0}{\operatorname{im}(\alpha_1)}, \quad H_1(V_\bullet) = \ker(\alpha_1).$$

By Proposition 3.11,

$$\begin{aligned} \dim(H_0(V_\bullet)) &= \dim(V_0) - \dim(\operatorname{im}(\alpha_1)), \\ \dim(H_1(V_\bullet)) &= \dim(\ker(\alpha_1)), \\ \dim(V_1) &= \dim(\ker(\alpha_1)) + \dim(\operatorname{im}(\alpha_1)) \end{aligned}$$

so we find

$$\begin{aligned} \sum_{i=0}^{1} (-1)^i \dim(H_i(V_\bullet)) &= \dim(H_0(V_\bullet)) - \dim(H_1(V_\bullet)) \\ &= \dim(V_0) - \dim(\operatorname{im}(\alpha_1) - \dim(\ker(\alpha_1)) \\ &= \dim(V_0) - \dim(V_1) \\ &= \chi(V_\bullet) \end{aligned}$$

proving the desired result. □

Exercise .1.16. Prove Claim 3.14.

Claim 3.14. *With notation as above, we have the following:*

- χ_K *'is an Euler characteristic', in the sense that it satisfies the formula given in Proposition 3.13:*

$$\chi_K(V_\bullet) = \sum_i (-1)^i [H_i(V_\bullet)].$$

- χ_K *is a 'universal Euler characteristic', in the following sense. Let G be an abelian group, and let δ be a function associating an element of G to each finite-dimensional vector space, such that $\delta(V) = \delta(V')$ if $V \cong V'$ and $\delta(V/U) = \delta(V) - \delta(U)$. For $V_\bullet$ a complex, define*

$$\chi_G(V_\bullet) = \sum_i (-1)^i \delta(V_i).$$

  Then δ induces a (unique) group homomorphism

$$K(k\text{-}\mathsf{Vect}^f) \to G$$

  mapping $\chi_K(V_\bullet)$ to $\chi_G(V_\bullet)$.

- *In particular, $\delta = \dim$ induces a group homomorphism*

$$K(k\text{-}\mathsf{Vect}^f) \to \mathbb{Z}$$

  such that $\chi_K(V_\bullet) \mapsto \chi(V_\bullet)$.

- *This is in fact an isomorphism.*

Solution. Recall that we define $F(k\text{-}\mathsf{Vect}^f)$ to be the set of isomorphism classes of finite-dimensional vector spaces $[V]$ over a field k. We let E be the subgroup generated by the elements $[V] - [U] - [W]$ for all short exact sequences

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

and define

$$K(k\text{-}\mathsf{Vect}^f) := \frac{F(k\text{-}\mathsf{Vect})}{E}$$

to be the Grothendieck group of the category $k\text{-}\mathsf{Vect}^f$. We also define

$$\chi_K(V_\bullet) := \sum_i (-1)^i [V_i] \in K$$

where summation is the direct sum.

First we prove that χ_k is an Euler characteristic. We adapt the proof by induction used to prove Propositoin 3.11, starting with the case $N = 1$. Again, let

$$V_\bullet : \ 0 \longrightarrow V_1 \xrightarrow{\alpha_1} V_0 \longrightarrow 0$$

be a complex of finite-dimensional vector spaces and linear maps. By definition, we have $\chi_K(V_\bullet) = [V_0] - [V_1]$. Recall that, by the definition of homology,

$$H_0(V_\bullet) = \frac{V_0}{\mathrm{im}(\alpha_1)}, \quad H_1(V_\bullet) = \ker(\alpha_1)$$

so we have two exact sequences in $k\text{-}\mathsf{Vect}^f$:

$$0 \longrightarrow H_1(V_\bullet) \longrightarrow V_1 \longrightarrow \mathrm{im}(\alpha_1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{im}(\alpha_1) \longrightarrow V_0 \longrightarrow H_0(V_\bullet) \longrightarrow 0$$

so we have the relations $[H_1(V_\bullet)] = [V_1] - [\mathrm{im}(\alpha_1)]$ and $[H_0(V_\bullet)] = [V_0] - [\mathrm{im}(\alpha_1)]$. Thus, we find

$$\begin{aligned}\sum_{i=0}^{1}[H_i(V_\bullet)] = [H_0(V_\bullet)] - [H_1(V_\bullet)]&\\ &= ([V_0] - [\mathrm{im}(\alpha_1)]) - [V_1] - [\mathrm{im}(\alpha_1)]\\ &= [V_0] - [V_1]\\ &= \chi_K(V_\bullet)\end{aligned}$$

so the statement holds in the base case. Now we prove the inductive step. Given a complex

$$V_\bullet:\ 0 \longrightarrow V_N \xrightarrow{\alpha_N} V_{N-1} \xrightarrow{\alpha_{N-1}} \cdots \xrightarrow{\alpha_2} V_1 \xrightarrow{\alpha_1} V_0 \longrightarrow 0$$

we can consider the truncated complex

$$V'_\bullet:\ 0 \longrightarrow V_{N-1} \xrightarrow{\alpha_{N-1}} \cdots \xrightarrow{\alpha_2} V_1 \xrightarrow{\alpha_1} V_0 \longrightarrow 0$$

where the result is known to hold for $V'_\bullet$. Then

$$\chi_K(V_\bullet) = \chi_K(V'_\bullet) + (-1)^N [V_N]$$

and

$$H_i(V_\bullet) = H_i(V'_\bullet) \text{ for } 0 \le i \le N-2$$

while

$$H_{N-1}(V'_\bullet) = \ker(\alpha_{N-1}), \quad H_{N-1}(V_\bullet) = \frac{\ker(\alpha_{N-1}}{\mathrm{im}(\alpha_N)}, \quad H_N(V_\bullet) = \ker(\alpha_N).$$

Then we have exact sequences

$$0 \longrightarrow \ker(\alpha_N) \longrightarrow V_N \longrightarrow \mathrm{im}(\alpha_N) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{im}(\alpha_N) \longrightarrow \ker(\alpha_{N-1}) \longrightarrow H_{N-1}(V_\bullet) \longrightarrow 0$$

which yield the relations $[V_N] = [\ker(\alpha_N)] + [\mathrm{im}(\alpha_N)]$ and $[H_{N-1}(V_\bullet)] = [\ker(\alpha_{N-1})] - [\mathrm{im}(\alpha_N)]$. Then we have

$$[H_{N-1}(V'_\bullet)] - [V_N] = [H_{N-1}(V_\bullet)] - [H_N(V_\bullet)]$$

so we find

$$\begin{aligned}
\chi_K(V_\bullet) &= \chi_K(V'_\bullet) + (-1)^N[V_N] \\
&= \sum_{i=0}^{N-1} (-1)^i [H_i(V'_\bullet)] + (-1)^N [V_N] \\
&= \sum_{i=0}^{N-2} (-1)^i [H_i(V'_\bullet)] + (-1)^{N-1} \left([H_{N-1}(V'_\bullet)] - [V_N]\right) \\
&= \sum_{i=0}^{N-2} (-1)^i [H_i(V_\bullet)] + (-1)^{N-1} ([H_{N-1}(V_\bullet)] - [H_N(V_\bullet)]) \\
&= \sum_{i=0}^{N} (-1)^i [H_i(V_\bullet)]
\end{aligned}$$

which proves the desired result.

For the second part, let $\varphi : K(k\text{-}\mathsf{Vect}^f) \to G$ be the unique group homomorphism induced by δ. We claim that $\varphi([V]) = \delta(V)$ satisfies this universal property. First we check that it is well defined; suppose $[V] = [V']$. Then, since $V \cong V'$, we have $\delta(V) = \delta(V')$. Now we show that this is a group homomorphism. Let $[U], [V] \in K(k\text{-}\mathsf{Vect}^f)$. Then

$$\varphi([V] - [U]) = \varphi([V/U]) = \delta(V/U) = \delta(V) - \delta(U) = \varphi([V]) - \varphi([U])$$

which verifies that this is a group homomorphism. Finally, let $V_\bullet$ be a complex of finite-dimensional vector spaces. Then

$$\begin{aligned}
\varphi(\chi_K(V_\bullet)) &= \varphi\left(\sum_i (-1)^i [V_i]\right) \\
&= \sum_i (-1)^i \varphi([V_i]) \\
&= \sum_i (-1)^i \delta(V_i) \\
&= \chi_G(V_\bullet)
\end{aligned}$$

where the second equality follows from φ being a group homomorphism.

The third point follows naturally from the second. Indeed, letting $\delta = \dim$ induces a group homomorphism from $K(k\text{-}\mathsf{Vect}^f)$ to $\mathbb{Z}$ such that $\chi_K(V_\bullet) = \chi(V_\bullet)$, where χ is the natural definition of the Euler characteristic.

To show that this is an isomorphism, we prove it is both injective and surjective. First note that for any non-negative integer n, we may consider the vector space $V = k^n$. Then $\varphi([V]) = n$. If n is negative, consider $V = k^{-n}$ such that $\varphi(-[V]) = -\varphi([V]) = n$. Thus, φ is surjective. Now suppose $\varphi([U]) = \varphi([V])$. That is, $\dim(U) = \dim(V)$. Then $U \cong V$ so $[U] = [V]$ and φ is injective. Thus, φ is an isomorphism and

$$K(k\text{-}\mathsf{Vect}^f) \cong \mathbb{Z}.$$

□

Exercise .1.17. Extend the definition of Grothendieck group of vector spaces given in §3.4 to the category of vector spaces of *countable* (possibly infinite) dimension, and prove that it is the trivial group.

Solution. Consider the sequence

$$0 \longrightarrow k^{\oplus\mathbb{N}} \longrightarrow k^{\oplus\mathbb{N}} \longrightarrow k^n \longrightarrow 0$$

where $n \in \mathbb{N}$. Certainly, this sequence is exact because $k^{\oplus\mathbb{N}} \cong k^{\oplus\mathbb{N}} \oplus k^n$. But this implies that $[k^{\oplus\mathbb{N}}] = [k^{\oplus\mathbb{N}}] + [k^n]$ or $[k^n] = 0$. Since this also holds for $[k^{\oplus\mathbb{N}}]$, the group $K(k\text{-}\mathsf{Vect})$ is the trivial group. □

Exercise .1.18. Let Ab^{fg} be the category of finitely generated abelian groups. Define a Grothendieck group of this category in the style of the construction of $K(k\text{-}\mathsf{Vect}^f)$, and prove that $K(\mathsf{Ab}^{fg}) \cong \mathbb{Z}$.

Solution. Note that every object G of Ab^{fg} determines an isomorphism class $[G]$. Let $F(\mathsf{Ab}^{fg})$ be the free abelian group on the set of these isomorphism classes. Furthermore, let E be the subgroup generated by the elements $[B] - [A] - [C]$ for all short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in Ab^{fg}. Let

$$K(\mathsf{Ab}^{fg}) := \frac{F(\mathsf{Ab}^{fg})}{E}$$

be the Grothendieck group of this category.

Recall that every finitely generated abelian group is isomorphic to a direct sum of cyclic groups □

Exercise .1.19. Let Ab^f be the category of finite abelian groups. Prove that assigning to every finite abelian group its order extends to a homomorphism from the Grothendieck group $K(\mathsf{Ab}^f)$ to the multiplicative group $(\mathbb{Q}^*, \cdot)$.

Solution. To do. □

Exercise .1.20. Let $R\text{-}\mathsf{Mod}^f$ be the category of modules of finite *length* (cf. Exercise 1.16) over a ring R. Let G be an abelian group, and let δ be a function assigning an element of G to every *simple* R-module. Prove that δ extends to a homomorphism from the Grothendieck group of $R\text{-}\mathsf{Mod}^f$ to G.

Explain why Exercise 3.19 is a particular case of this observation.

(For another example, letting $\delta(M) = 1 \in \mathbb{Z}$ for every simple module M shows that length itself extends to a homomorphism from the Grothendieck group of $R\text{-}\mathsf{Mod}^f$ to $\mathbb{Z}$.)

Solution. To do. $\square$