Chapter I

Groups, first encounter

I.1 Definition of group

Problem I.1.1. Write a careful proof that every group is the group of isomorphisms of a groupoid. In particular, every group is the group of automorphisms of some object in some category.

Solution. Let G be a group. Consider a category C_G with a single object $\bullet$. Morphisms from $\bullet$ to itself are the elements of G. If $g, h \in G$, then the morphism gh is their composition which is also in G. The existence of inverse elements indicates that each morphism has an inverse, so they are all isomorphisms. Hence, each morphism is an isomorphism from $\bullet$ to itself, thus making each morphism an automorphism. Therefore C_G is a grouped with one object. $\square$

Problem I.1.2. Consider the 'sets of numbers' listed in §1.1, and decide which are made into groups by conventional operations such as $+$ and $\cdot$. Even if the answer is negative (for example, $(\mathbb{R}, \cdot)$ is not a group), see if variations on the definition of these sets lead to groups (for example, $(\mathbb{R}^*, \cdot)$ *is* a group; cf. §1.4).

Solution. $\mathbb{Z}$ is a group under normal addition, as are other sets with negatives and 0. Multiplicative groups require reciprocals and exclude zero since it has no multiplicative inverse. $\square$

Problem I.1.3. Prove that $(gh)^{-1} = h^{-1}g^{-1}$ for all elements g, h of a group G.

Solution. First note that

$$(gh)(gh)^{-1} = (gh)(h^{-1}g^{-1}) = g(hh^{-1})g^{-1} = gg^{-1} = e$$

Furthermore, we have

$$(gh)^{-1}(gh) = (h^{-1}g^{-1})(gh) = h^{-1}g^{-1}gh = h^{-1}h = e$$

Thus, $(gh)^{-1} = h^{-1}g^{-1}$ is a two-sided inverse of gh. $\square$

Problem I.1.4. Suppose that $g^2 = e$ for all elements g of a group G; prove that G is commutative.

Solution. First note that $g^2 = e \Longrightarrow g = g^{-1}$. Then we have

$$(gh)^{-1} = h^{-1}g^{-1} \Longrightarrow gh = hg$$

Thus, G is commutative. $\square$