.1 Presentations and resolutions

Exercise .1.1. Prove that if R is an integral domain and M is an R-module, then $\mathrm{Tor}(M)$ is a submodule of M. Give an example showing that the hypothesis that R is an integral domain is necessary.

Solution. Clearly $\mathrm{Tor}(M) \neq \emptyset$ since $0 \in \mathrm{Tor}(M)$. Now suppose $a, b \in \mathrm{Tor}(M)$. Then $\exists r, s \in R$ such that $ra = sb = 0$. Therefore, $rs(a + b) = s(ra) + r(sb) = 0$ so $a + b \in \mathrm{Tor}(M)$. Similarly, for all $s \in R$, we have $r(sa) = s(ra) = 0$ so $sa \in \mathrm{Tor}(M)$. Thus, $\mathrm{Tor}(M)$ is a submodule of M.

To see that R is an integral domain is necessary, consider $R = M = \mathbb{Z}/6\mathbb{Z}$. Then $\mathrm{Tor}(M) = \{0, 2, 3, 4\}$. But then $2 + 3 = 5 \notin \mathrm{Tor}(M)$ so $\mathrm{Tor}(M)$ is not a submodule of M. □

Exercise .1.2. Let M be a module over an integral domain R, and let N be a torsion-free module. Prove that $\mathrm{Hom}_R(M, N)$ is torsion-free. In particular, $\mathrm{Hom}_R(M, R)$ is torsion-free. (We will run into this fact again; see Proposition VIII.5.16.)

Solution. Let $f \in \mathrm{Hom}_R(M, N)$ and suppose $r \cdot f = 0$ for some $r \in R$. That is, for all $m \in M$,

$$r \cdot f(m) = 0.$$

But since $f(m) \in N$, $f(m)$ is not a torsion element and $r = 0$. Thus, $\mathrm{Hom}_R(M, N)$ is torsion-free. □

Exercise .1.3. Prove that an integral domain R is a PID if and only if every submodule of R itself is free.

Solution. Note that the submodules of R are its ideals. If R is a PID, then every submodule of R is generated by a single element. That is, every submodule of R has a basis, making it free. Now suppose every submodule of R is free. Recall that if M is a submodule of R, then $\dim(M) \leq \dim(R)$. In particular, $\dim(M) \leq 1$. Thus, every ideal of R is generated by at most one element so R is a PID. □

Exercise .1.4. Let R be a commutative ring and M an R-module.

- Prove that $\mathrm{Ann}(M)$ is an ideal of R.
- If R is an integral domain and M is finitely generated, prove that M is torsion if and only if $\mathrm{Ann}(M) \neq 0$.
- Give an example of a torsion module M over an integral domain, such that $\mathrm{Ann}(M) = 0$. (Of course this example cannot be finitely generated!)

Solution. Let $a, b \in \mathrm{Ann}(M)$. That is, for all $m \in M$, we have $am = bm = 0$. Then $(a+b)m = am + bm = 0$ so $a + b \in \mathrm{Ann}(M)$. Similarly, for all $r \in R$, we find $(ra) \cdot m = r \cdot (am) = r \cdot 0 = 0$ so $ra \in \mathrm{Ann}(M)$, proving that it is an ideal.

If $\mathrm{Ann}(M) \neq 0$, there exists an $r \in R$ such that $rm = 0$ for all $m \in M$. Thus, every element of M is torsion. Now suppose M is torsion. That is, for every element $m_i \in M$, there exists an $r_i \in R, r_i \neq 0$ such that $r_i m_i = 0$. In particular, there is such an r_i for each generator of M. Then we may consider s to be the product of these r_i. Since R is an integral domain, $s \neq 0$. Furthermore, since all $m \in M$ is a linear combination of these generators, we have $sm = 0$ for all $m \in M$. Thus, $s \in \mathrm{Ann}(M)$.

Let $R = \mathbb{Z}$ and consider the $\mathbb{Z}$-module

$$M = \bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}}{2^i \mathbb{Z}}.$$

Then each element of M has the form

$$a = (a_1 + \mathbb{Z}/2\mathbb{Z}, a_2 + \mathbb{Z}/2^2\mathbb{Z}, \ldots, a_k + \mathbb{Z}/2^k\mathbb{Z}, 0, 0, \ldots)$$

so $2^k a = 0$ which makes M a torsion module. Now suppose $r \in \mathrm{Ann}(M)$. Choose $k \in \mathbb{Z}$ such that $r < 2^k$ and consider the element

$$a = (0, 0, \ldots, 1 + \mathbb{Z}/2^k\mathbb{Z}, 0, 0, \ldots).$$

Then $ra = 0$, but since $r < 2^k$, it must be the case that $r = 0$. Thus, $\mathrm{Ann}(M) = 0$. □

Exercise .1.5. Let M be a module over a commutative ring R. Prove that an ideal I of R is the annihilator of an element of M if and only if M contains an isomorphic copy of R/I (viewed as an R-module).

The *associated primes* of M are the prime ideals among the ideals $\mathrm{Ann}(m)$, for $m \in M$. The set of the associated primes of a module M is denoted $\mathrm{Ass}_R(M)$. Note that every prime in $\mathrm{Ass}_R(M)$ contains $\mathrm{Ann}_R(M)$.

Solution. Let I be the annihilator of an element $m \in M$. That is, for all $r \in I$, $rm = 0$. Consider the map $\varphi : R \to M$ which sends r to rm. The kernel of this map is the set of r such that $rm = 0$. That is, $\ker(\varphi) = I$ so, by the isomorphism theorem,

$$\frac{R}{I} \cong \mathrm{im}(\varphi) \subseteq M.$$

Now suppose M contains a submodule $N \cong R/I$ for an ideal $I \subseteq R$ and let $\varphi : R \to M$ be the composition of the natural projection and inclusion. We claim that I is the annihilator of $m = \varphi(1)$. Indeed, if $r \in I$ then

$$rm = r\varphi(1) = \varphi(r) = i(\pi(r)) = i(0) = 0$$

so $r \in \mathrm{Ann}(m)$ and $I \subseteq \mathrm{Ann}(m)$. Similarly, if $r \in \mathrm{Ann}(m)$ then

$$rm = 0 \Longrightarrow \varphi(r) = 0 \Longrightarrow \pi(r) = 0$$

so $r \in I$ and $\mathrm{Ann}(m) = I$. $\square$

Exercise .1.6. Let M be a module over a commutative ring R, and consider the family of ideals $\mathrm{Ann}(m)$, as m ranges over the nonzero elements of M. Prove that the maximal elements in this family are prime ideals of R. Conclude that if R is Noetherian, then $\mathrm{Ass}_R(M) \neq \emptyset$ (cf. Exercise 4.5).

Solution. Let $\mathfrak{m}$ be a maximal element in this family of ideals, say $\mathfrak{m} = \mathrm{Ann}(m)$. Suppose $rs \in \mathfrak{m}$. If $r \in \mathfrak{m}$ then there is nothing to prove so suppose otherwise. We know $rs \cdot m = 0$ but $rm \neq 0$. Thus, $s \in \mathrm{Ann}(rm)$. Furthermore, it is clear that $\mathrm{Ann}(m) \subseteq \mathrm{Ann}(rm)$ since if $am = 0$ then $a(rm) = 0$. Then, by the maximality of $\mathrm{Ann}(m)$, we have $\mathrm{Ann}(m) = \mathrm{Ann}(rm)$ so $s \in \mathfrak{m}$ and the ideal is prime.

If R is Noetherian, then every family of ideals has a maximal element. In particular, given a module M, the family of ideals $\mathrm{Ann}(m)$ as m ranges over the nonzero elements of M has a maximal element which is a prime ideal. Such prime ideals are elements of $\mathrm{Ass}_R(M)$, meaning the set is nonempty. $\square$

Exercise .1.7. Let R be a commutative Noetherian ring, and let M be a finitely generated module over R. Prove that M admits a finite series

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_m = \langle 0 \rangle$$

in which all quotients M_i/M_{i+1} are of the form $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R. (Hint: Use Exercises 4.5 and 4.6 to show that M contains an isomorphic copy M' of $R/\mathfrak{p}_1$ for some prime $\mathfrak{p}_1$. Then do the same with M/M', producing an $M'' \supseteq M'$ such that $M''/M' \cong R/\mathfrak{p}_2$ for some prime $\mathfrak{p}_2$. Why must this process stop after finitely many steps?)

Solution. By Exercise 4.6, $\mathrm{Ass}_R(M) \neq \emptyset$ so let $\mathfrak{p}_1 \in \mathrm{Ass}_R(M)$. Then by Exercise 4.5, M contains a submodule $M' \cong R/\mathfrak{p}_1$. Now consider M/M', which is also an R-module. Thus, $\mathrm{Ass}_R(M/M') \neq \emptyset$ and there is a submodule $M'' \supseteq M'$ of M such that $M''/M' \cong R/\mathfrak{p}_2$ for some prime $\mathfrak{p}_2$. That is, we have a chain

$$M \supsetneq M'' \supsetneq M' \supsetneq \langle 0 \rangle$$

such that $M''/M' \cong R/\mathfrak{p}_2$ and $M'/0 \cong R/\mathfrak{p}_1$ for prime ideals of R. Since M is finitely generated over a Noetherian ring, it is a Noetherian module and all chains of submodules eventually stabilize. Thus, iterating this process yields a finite series whose quotients are isomorphic to $R/\mathfrak{p}$ for prime ideals. $\square$

Exercise .1.8. Let R be a commutative Noetherian ring, and let M be a finitely generated module over R. Prove that every prime in $\operatorname{Ass}_R(M)$ appears in the list of primes produced by the procedure presented in Exercise 4.7. (If $\mathfrak{p}$ is an associated prime, then M contains an isomorphic copy N of $R/\mathfrak{p}$. With notation as in the hint in Exercise 4.7, prove that either $\mathfrak{p}_1 = \mathfrak{p}$ or $N \cap M' = 0$. In the latter case, N maps isomorphically to a copy of $R/\mathfrak{p}$ in M/M'; iterate the reasoning.)

In particular, if M is a finitely generated module over a Noetherian ring, then $\operatorname{Ass}(M)$ is *finite*.

Solution. Let $\mathfrak{p} \in \operatorname{Ass}_R(M)$ and suppose $R/\mathfrak{p} \cong N \subseteq M$. In particular, if $x \in M$ such that $\operatorname{Ann}_R(x) = \mathfrak{p}$, then $N = Rx$. If $Rx \cap M' \neq 0$, say $rx = m$ is a nonzero element, then $\operatorname{Ann}_R(m) \subseteq \mathfrak{p}$. But by definition, $\operatorname{Ann}_R(m) = \mathfrak{p}_1$ so $\mathfrak{p}_1 \subseteq \mathfrak{p}$. The reverse inclusion can be shown similarly. Thus, if M' and N have nontrivial intersection, $\mathfrak{p} = \mathfrak{p}_1$. Otherwise, $M' \cap N = 0$. In the latter case, N is isomorphic to some $R/\mathfrak{p}$ in $M/M' \cong R/\mathfrak{p}_2$. Thus, we may repeat the above reasoning which eventually terminates. $\square$

Exercise .1.9. Let M be a module over a commutative Noetherian ring R. Prove that the union of all annihilators of nonzero elements equals the union of all associated primes of M. (Use Exercise 4.6)

Deduce that the *union* of the associated primes of a Noetherian ring R (viewed as a module over itself) equals the set of zero-divisors of R.

Solution. Certainly every associated prime is the annihilator of some element $m \in M$, so we only need to show the other direction. If $I \in \operatorname{Ann}_R(m)$ for some $m \in M$, then $I \subseteq \mathfrak{p}$ for some maximal element in the family of annihilators of elements of M. By Exercise 4.6, $\mathfrak{p}$ is prime in R so I is in the union of all associated primes, proving the result. $\square$

Exercise .1.10. Let R be a commutative Noetherian ring. One can prove that the minimal primes of $\operatorname{Ann}(M)$ (cf. Exercise V.1.9) are in $\operatorname{Ass}(M)$. Assuming this, prove that the *intersection* of the associated primes of a Noetherian ring R (viewed as a module over itself) equals the nilradical of R.

Solution. Recall that the nilradical of R is the set of elements $r \in R$ such that $r^n = 0$ for some $n > 0$. If $x \in \operatorname{nil}(R)$ then x is in the intersection of all prime ideals of R, particularly the intersection of associated primes of R. Now suppose x is in the intersection of the associated primes of R. Then it is in the minimal primes of $\operatorname{Ann}(R)$. Since every prime ideal contains a minimal prime ideal, the intersection of all prime ideals equals the intersection of all minimal prime ideals. Thus, $x \in \operatorname{nil}(R)$. $\square$

Exercise .1.11. Review the notion of presentation *of a group*, and relate it to the notion of presentation introduced in §4.2.

Solution. Recall that a presentation of a group G is an explicit isomorphism

$$G \cong \frac{F(A)}{R}$$

for a set A and a subgroup R of relations. A presentation of an R-module M is an exact sequence

$$R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$$

In particular, if G is an abelian group, then we have the exact sequence

$$R \longrightarrow F(A) \longrightarrow G$$

where R is also a free module since it is a submodule of $F(A)$. $\square$

Exercise .1.12. Let $\mathfrak{p}$ be a prime ideal of a polynomial ring $k[x_1, \ldots, x_n]$ over a field k, and let $R = k[x_1, \ldots, x_n]/\mathfrak{p}$. Prove that every finitely generated module over R has a finite presentation.

Solution. Let M be a finitely generated module over R. Then there is a surjection $\pi : R^a \to M$ for some $a \in \mathbb{Z}$ where $\ker(\pi)$ is a submodule of R^a. Since k is a field, by Hilbert's basis theorem, $k[x_1, \ldots, x_n]$ is also Noetherian. But then R is a quotient of a Noetherian ring and is Noetherian itself. Thus, $\ker(\pi)$ is finitely generated and there is an exact sequence

$$R^b \longrightarrow \ker(\pi) \longrightarrow 0$$

which yields the exact sequence

$$R^b \longrightarrow R^a \longrightarrow M \longrightarrow 0$$

so M is finitely presented. $\square$

Exercise .1.13. Let R be a commutative ring. A tuple $(a_1, a_2, \ldots, a_n)$ of elements of R is a *regular sequence* if a_1 is a non-zero-divisor in R, a_2 is a non-zero-divisor modulo (a_1), a_3 is a non-zero-divisor modulo (a_1, a_2), and so on.

For a, b in R, consider the following complex of R-modules:

$$(*) \qquad 0 \longrightarrow R \xrightarrow{d_2} R \oplus R \xrightarrow{d_1} R \xrightarrow{\pi} \frac{R}{(a,b)} \longrightarrow 0$$

where π is the canonical projection, $d_1(r, s) = ra + sb$, and $d_2(t) = (bt, -at)$. Put otherwise, d_1 and d_2 correspond, respectively, to the matrices

$$\begin{pmatrix} a & b \end{pmatrix}, \quad \begin{pmatrix} b \\ -a \end{pmatrix}.$$

- Prove that this is indeed a complex, for every a and b.
- Prove that if (a, b) is a regular sequence, this complex is *exact*.

The complex (*) is called the *Koszul complex* of (a, b). Thus, when (a, b) is a regular sequence, the Koszul complex provides us with a free resolution of the module $R/(a, b)$.

Solution. First we verify that this is a complex for all a and b. Certainly the image of the zero map is a subset of $\ker(d_2)$. Let $(r, s) \in \operatorname{im}(d_2)$. Then $(r, s) = (bt, -at)$ for some $t \in R$ and

$$d_1(bt, -at) = bta - bta = 0$$

so $\operatorname{im}(d_2) \subseteq \ker(d_1)$. Furthermore, let $ra + sb \in \operatorname{im}(d_1)$. Then $\pi(ra + sb) = 0 \in R/(a, b)$ so $\operatorname{im}(d_1) \subseteq \ker(\pi)$. Finally, the image of π is clearly a subset of the kernel of the zero map. Thus, we have verified that this is in fact a complex.

Now suppose (a, b) is a regular sequence. Let $t \in \ker(d_2)$. That is, $(bt, -at) = (0, 0)$. Since $a \neq 0$, it must be the case that $t = 0$ so t is in the image of the zero map, proving the two are equal.

Now suppose $(r, s) \in \ker(d_1)$. Then $ra + sb = 0$. Consider the equation mod a: $sb = 0$. Since b is not a zero-divisor in $R/(a)$, $s \in (a)$ so $s = at$ for some $t \in R$. Then we have $ra + atb = 0$, or $(r + tb)a = 0$. Since a is not a zero-divisor in R, it must be the case that $r + tb = 0$, or $r = -tb$. That is, $(r, s) = (-tb, at) \in \operatorname{im}(d_2)$ so the two sets must be equal.

Now let $x \in \ker(\pi)$ so $\pi(x) = 0 \implies x = ra + sb$ for $r, s \in R$. Then $x = d_1(r, s) \in \operatorname{im}(d_1)$ and the two sets are equal.

Finally, the projection is surjective and the kernel of the zero map is all of its domain so the last map is exact. $\square$

Exercise .1.14. A Koszul complex may be defined for any sequence $a_1, \dots, a_n$ of elements of a commutative ring R. The case $n = 2$ seen in Exercise 4.13 and the case $n = 3$ reviewed here will hopefully suffice to get a gist of the general construction; the general case will be given in Exercise VIII.4.22.

Let $a, b, c \in R$. Consider the following complex:

$$0 \longrightarrow R \xrightarrow{d_3} R \oplus R \oplus R \xrightarrow{d_2} R \oplus R \oplus R \xrightarrow{d_1} R \xrightarrow{\pi} \frac{R}{(a,b,c)} \longrightarrow 0$$

where π is the canonical projection and the matrices for d_1, d_2, d_3 are, respectively,

$$\begin{pmatrix} a & b & c \end{pmatrix}, \quad \begin{pmatrix} 0 & -c & -b \\ -c & 0 & a \\ b & a & 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ -b \\ c \end{pmatrix}.$$

- Prove that this is indeed a complex, for every a, b, c.

- Prove that if (a, b, c) is a regular sequence, this complex is *exact*.

Koszul complexes are very important in commutative algebra and algebraic geometry.

Solution. Clearly the image of the zero map is in the kernel of d_3. Let $(ar, -br, cr) \in \operatorname{im}(d_3)$. Then

$$\begin{pmatrix} 0 & -c & -b \\ -c & 0 & a \\ b & a & 0 \end{pmatrix} \cdot \begin{pmatrix} ar \\ -br \\ cr \end{pmatrix} = \begin{pmatrix} bcr - bcr \\ -acr + acr \\ abr - abr \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $(ar, -br, cr) \in \ker(d_2)$. Now let $(-cs - bt, -cr + at, br + as) = d_2(r, s, t) \in \operatorname{im}(d_2)$. Then

$$\begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} -cs - bt \\ -cr + at \\ br + as \end{pmatrix} = -acs - abt - bcr + abt + bcr + acs = 0$$

so $\operatorname{im}(d_2) \subseteq \ker(d_1)$. Now consider $ra + sb + ct = d_1(r, s, t) \in \operatorname{im}(d_1)$. We have

$$\pi(ra + sb + ct) = 0$$

by definition of the projection to a quotient so $\operatorname{im}(d_1) \subseteq \ker(\pi)$. The image of projection is obviously a subset of the kernel of the zero map. Thus, this is indeed a complex.

Now suppose (a, b, c) is a regular sequence. If $r \in \ker(d_3)$ then $d_3(r) = (0, 0, 0)$. In particular, $ar = 0$ and since a is not a zero-divisor, we must have $r = 0$ so r is in the image of the zero map, hence it equals the image of d_3.

If $(r_1, r_2, r_3) \in \ker(d_2)$, then

$$\begin{pmatrix} 0 & -c & -b \\ -c & 0 & a \\ b & a & 0 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -cr_2 - br_3 \\ -cr_1 + ar_3 \\ br_1 + ar_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The third equation mod a yields $br_1 = 0$ in $R/(a)$. Since b is not a zero-divisor in this ring, we must have $r_1 = at$ for some $t \in R$. Substituting this back into the third equation, we have $abt + ar_2 = 0$, or $r_2 = -bt$ (since a is not a zero-divisor in R). Substituting this into the second equation yields $-act + ar_3 = 0$ so $r_3 = ct$ by the same reasoning as above. But then

$$(r_1, r_2, r_3) = (at, -bt, ct) = d_3(t)$$

so $\operatorname{im}(d_3) = \ker(d_2)$.

If $(r_1, r_2, r_3) \in \ker(d_1)$, then

$$\begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = ar_1 + br_2 + cr_3 = 0.$$

Considering this equation mod (a, b) yields $cr_3 = 0$ in $R/(a, b)$ and since c is not a zero-divisor in this ring, we must have $r_3 \in (a, b)$ or $r_3 = ar + bs$ for $r, s \in R$. Substituting this into the equation yields

$$ar_1 + br_2 + acr + bcs = 0$$

which we can consider mod a to yield $br_2 + bcs = 0$ in $R/(a)$, or $r_2 + cs = at$ for some $t \in R$. That is, $r_2 = at - cs$, which we can again substitute into the equation to obtain

$$ar_1 + abt - bcs + acr + bcs = 0$$

which yields $a(r_1 + bt + cs) = 0$ so $r_1 = -bt - cs$. But then

$$(r_1, r_2, r_3) = (-bt - cs, at - cs, ar + bs) = d_2(r, s, t)$$

so $\operatorname{im}(d_2) = \ker(d_1)$.

Finally, suppose $x \in \ker(\pi)$. That is, $x \in (a, b, c)$. Then $x = ra + bs + ct = d_1(r, s, t)$ and $\operatorname{im}(d_1) = \ker(\pi)$. The last equality is obvious. Thus, the complex is exact. ☐

Exercise .1.15. View $\mathbb{Z}$ as a module over the ring $R = \mathbb{Z}[x, y]$, where x and y act by 0. Find a free resolution of $\mathbb{Z}$ over R.

Solution. Recall that a free resolution of an R-module M is an exact complex

$$\cdots \longrightarrow R^{m_3} \longrightarrow R^{m_2} \longrightarrow R^{m_1} \longrightarrow R^{m_0} \longrightarrow M \longrightarrow 0.$$

Consider the complex

$$0 \longrightarrow R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$

where d_1 and d_2 correspond to the matrices

$$\begin{pmatrix} x & y \end{pmatrix}, \quad \begin{pmatrix} y \\ -x \end{pmatrix}$$

and π is the natural projection to the constant term. It is easy to see that this is in fact a complex. To see that it is exact, let $f(x, y) \in \ker(\pi)$. That is, f has no constant term, so it may be written as

$$f = \begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} f_1(y) \\ f_2(x) \end{pmatrix}$$

so $f \in \operatorname{im}(d_1)$. Similarly, if $(f, g) \in \ker(d_1)$ then $fx + gy = 0$. Gathering terms, this is only possible if $f = hy$ and $g = -hx$ for some $h \in R$. That is, $(f, g) = d_2(h)$ so $\ker(d_1) = \operatorname{im}(d_2)$ and the sequence is exact. Thus, this is a free resolution of $\mathbb{Z}$ over R. ☐

Exercise .1.16. Let $\varphi : R^n \to R^m$ and $\psi : R^p \to R^q$ be two R-module homomorphisms, and let

$$\varphi \oplus \psi : R^n \oplus R^p \to R^m \oplus R^q$$

be the morphism induced on direct sums. Prove that

$$\operatorname{coker}(\varphi \oplus \psi) = \operatorname{coker} \varphi \oplus \operatorname{coker} \psi.$$

Solution. First note that

$$\operatorname{im}(\varphi \oplus \psi) = \operatorname{im}(\varphi) \oplus \operatorname{im}(\psi).$$

Now consider the map

$$R^m \oplus R^q \to \frac{R^m}{\operatorname{im} \varphi} \oplus \frac{R^q}{\operatorname{im} \psi}.$$

The kernel of this map is $\operatorname{im}(\varphi) \oplus \operatorname{im}(\psi)$ so by the first isomorphism theorem, we have

$$\frac{R^m \oplus R^q}{\operatorname{im}(\varphi \oplus \psi)} \cong \frac{R^m}{\operatorname{im} \varphi} \oplus \frac{R^q}{\operatorname{im} \psi}$$

and $\operatorname{coker}(\varphi \oplus \psi) = \operatorname{coker}(\varphi) \oplus \operatorname{coker}(\psi)$. □

Exercise .1.17. Determine (as a better known entity) the module represented by the matrix

$$\begin{pmatrix} 1+3x & 2x & 3x \\ 1+2x & 1+2x-x^2 & 2x \\ x & x^2 & x \end{pmatrix}$$

over the polynomial ring $k[x]$ over a field.

Solution. We perform Gaussian elimination to reduce the matrix to a simpler but equivalent form. Subtracting three times the third row from the first yields a unit in the $1,1$ position so we are reduced to the 2×2 matrix

$$\begin{pmatrix} 1+2x-x^2 & 2x \\ x^2 & x \end{pmatrix}.$$

Adding the second row to the first and subtracting $\frac{2}{3}$ times the second column from the first yields another unit in the $1,1$ position so we have reduced the matrix to

$$\begin{pmatrix} x \end{pmatrix}.$$

The module represented by the original matrix is isomorphic to the cokernel of the homomorphism

$$\varphi : k[x] \to k[x]$$

which maps 1 to x. That is,

$$M \cong \operatorname{coker} \varphi \cong \frac{k[x]}{(x)} \cong k.$$

□