Chapter I

Preliminaries: Set theory and categories

I.1 Naive Set Theory

Problem I.1.1. Locate a discussion of Russell's paradox, and understand it.

Solution. Consider the set of all sets which do not contain themselves. Does this set contain itself? If it is an element of itself, then clearly it contains itself. Thus it fails to satisfy its defining property and does not contain itself. If it does not contain itself, then it satisfies its defining property and does contain itself. The paradox demonstrates that not all properties can define a set. $\square$

Problem I.1.2. Prove that if $\sim$ is an equivalence relation on a set S, then the corresponding family $\mathscr{P}_\sim$ defined in §1.5 is indeed a partition of S: that is, its elements are nonempty, disjoint, and their union is S.

Solution. Let S be a set with the equivalence relation $\sim$. Consider $\mathscr{P}_\sim = \{[a]_\sim \mid a \in S\}$. Let $[a]_\sim \in \mathscr{P}_\sim$. Since $\sim$ is reflexive, $a \sim a$ so $[a]_\sim$ is nonempty.

Now suppose $a, b \in S$ and $a \nsim b$. Suppose $x \in [a]_\sim \cap [b]_\sim$. Then, since $\sim$ is transitive, $x \sim a$ and $x \sim b$ so $a \sim b$, a contradiction. Thus, each $[a]_\sim$ is disjoint.

Finally, consider $\bigcup_{[a]_\sim \in \mathscr{P}_\sim} [a]_\sim$. If $a \in S$, then $a \in [a]_\sim$. Thus, $\bigcup [a]_\sim = S$. $\square$

Problem I.1.3. Given a partion $\mathscr{P}$ on a set S, show how to define a relation $\sim$ on S such that $\mathscr{P}$ is the corresponding partition.

Solution. Let $a \sim b$ if and only if $\exists X \in \mathscr{P}$ such that $a \in X$ and $b \in X$ and let $\mathscr{P}_\sim$ be the corresponding partition.

Let $X \in \mathscr{P}$. Certainly X is nonempty, so let $a \in X$ and consider $[a]_\sim \in \mathscr{P}_\sim$. We must show that $X = [a]_\sim$. Suppose $a' \in X$ (it may be the case that $a' = a$). Since $a, a' \in X$, we have $a \sim a'$, so $a' \in [a]_\sim$. Now suppose $a' \in [a]_\sim$. Then $a' \sim a$ so $a' \in X$. Thus, $X = [a]_\sim \in \mathscr{P}_\sim$, so $\mathscr{P} \subseteq \mathscr{P}_\sim$.

Now let $[a]_\sim \in \mathscr{P}_\sim$. We know that $[a]_\sim$ is nonempty, so choose $a' \in [a]_\sim$. Then $a' \sim a$ and there exists $X \in \mathscr{P}$ such that $a, a' \in X$. Hence, $[a]_\sim \subseteq X$. Furthermore, if $a, a' \in X$ then $a \sim a'$. Therefore, $\mathscr{P}_\sim \subseteq \mathscr{P}$ and we have that $\mathscr{P} = \mathscr{P}_\sim$. □

Problem I.1.4. How many different equivalence relations may be defined on the set $\{1, 2, 3\}$?

Solution. The number of equivalence relations is in bijection with the number of partitions. We can count these by hand:

$$\begin{aligned}
\mathscr{P}_0 &= \{\{1, 2, 3\}\} \\
\mathscr{P}_1 &= \{\{1\}, \{2\}, \{3\}\} \\
\mathscr{P}_2 &= \{\{1, 2\}, \{3\}\} \\
\mathscr{P}_3 &= \{\{1\}, \{2, 3\}\} \\
\mathscr{P}_4 &= \{\{1, 3\}, \{2\}\}
\end{aligned}$$

There are 5 equivalence relations defined on $\{1, 2, 3\}$. □

Problem I.1.5. Give an example of a relation that is reflexive and symmetric but not transitive. What happens if you attempt to use this relation to define a partition on the set?

Solution. Consider the set of integers $\mathbb{Z}$ and define $a \sim b$ if and only if $|a-b| \leq 1$. Certainly this is reflexive since $a \sim a$ if and only if $|a - a| = 0 \leq 1$, which holds for all integers. It is also symmetric because if $a \sim b$ then $|a - b| \leq 1$, but $|a-b| = |b-a|$ so $|b-a| \leq 1$, implying that $b \sim a$. However, it is not transitive. For example, consider $a = 0, b = 1, c = 2$. Then $a \sim b$ and $b \sim c$, but $a \nsim c$.

Attempting to define a partition using a relation which is not transitive means that partitions are not necessarily disjoint. For example, $[2]_\sim = \{1, 2, 3\}$, but $[3]_\sim = \{2, 3, 4\}$. Hence $\mathscr{P}_\sim$ is not a partition of $\mathbb{Z}$. □

Problem I.1.6. Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $R/\sim$. Do the same for the relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$.

Solution. Let $a, b, c \in \mathbb{R}$. Then $a - a = 0 \in \mathbb{R}$ so $a \sim a$ and $\sim$ is reflexive. If $a \sim b$ then $b - a = n \in \mathbb{Z}$. Then $a - b = -n \in \mathbb{Z}$ so $b \sim a$ and $\sim$ is symmetric. If $a \sim b$ and $b \sim c$ then $b - a = m \in \mathbb{Z}$ and $c - b = n \in \mathbb{Z}$. Then $c - a = (c - b) + (b - a) = n + m \in \mathbb{Z}$, so $a \sim c$ and $\sim$ is transitive. Thus, $\sim$ is an equivalence relation.

$\mathbb{R}/\sim$ is the set of equivalence classes under the given relation. It may be interpreted as the set of integers shifted by a real number $\epsilon \in [0, 1)$. That is, for every set $X \in \mathbb{R}/\sim$, there is a real number $\epsilon \in [0, 1)$ such that every $x \in X$ is of the form $n + \epsilon$ for some $n \in \mathbb{Z}$.

We use a similar procedure to show that $\approx$ is an equivalence relation. Let $(a_1, a_2) \in \mathbb{R} \times \mathbb{R}$. Then we have $a_1 - a_1 = a_2 - a_2 = 0 \in \mathbb{Z}$. Thus, $(a_1, a_2) \approx (a_1, a_2)$ and $\approx$ is reflexive. Let $(b_1, b_2), (c_1, c_2) \in \mathbb{R} \times \mathbb{R}$. If we have $(a_1, a_2) \approx (b_1, b_2)$, then $b_1 - a_1 = m_1 \in \mathbb{Z}$ and $b_2 - a_2 = m_2 \in \mathbb{Z}$. Hence $a_1 - b_1 = -m_1 \in \mathbb{Z}$ and $a_2 - b_2 = -m_2 \in \mathbb{Z}$ so $(b_1, b_2) \approx (a_1, a_2)$ and $\approx$ is symmetric. Finally, suppose $(a_1, a_2) \approx (b_1, b_2)$ and $(b_1, b_2) \approx (c_1, c_2)$. Then $b_1 - a_1 = m_1 \in \mathbb{Z}$, $b_2 - a_2 = m_2 \in \mathbb{Z}$, $c_1 - b_1 = n_1 \in \mathbb{Z}$, and $c_2 - b_2 = n_2 \in \mathbb{Z}$. Therefore, $c_1 - a_1 = (c_1 - b_1) + (b_1 - a_1) = n_1 + m_1 \in \mathbb{Z}$ and $c_2 - a_2 = (c_2 - b_2) + (b_2 - a_2) = n_2 + m_2 \in \mathbb{Z}$. Thus, $(a_1, a_2) \approx (c_1, c_2)$ and $\approx$ is transitive. Then $\approx$ is an equivalence relation over $\mathbb{R} \times \mathbb{R}$.

$\mathbb{R} \times \mathbb{R}/\approx$ is the set of equivalence classes under the given relation. Every element is the 2-dimensional integer lattice shifted by a pair of real numbers $(\epsilon_1, \epsilon_2) \in [0, 1) \times [0, 1)$. $\square$

I.2 Functions between sets

Problem I.2.1. How many different bijections are there between a set S with n elements and itself?

Solution. A function $f : S \to S$ is a subset $\Gamma_f \subseteq S \times S$. Since f is bijective, then for all $y \in S$, there exists a unique $x \in S$ such that $(x, y) \in \Gamma_f$. Certainly $|\Gamma_f| = n$. Since each x is unique, every element $x \in S$ must be present in the first component of exactly one element in Γ_f. Similarly, each element $y \in S$ must be present in the second component of exactly one element in Γ_f. Then each bijection is merely a permutation of S, and there are $n!$ permutations. Thus, there are $n!$ bijections from S to itself. $\square$

Problem I.2.2. Prove statement (2) in Proposition 2.1. You may assume that given a family of disjoint subsets of a set, there is a way to choose one element in each member of the family.

Proposition 2.1. *Assume $A \neq \emptyset$, and let $f : A \to B$ be a function. Then (1) f has a left-inverse if and only if f is injective; and (2) f has a right-inverse if and only if f is surjective.*

Solution. Assume $A \neq \emptyset$ and let $f : A \to B$ be a function.

($\Longrightarrow$) Suppose there exists a function g that is a right-inverse of f. Then $f \circ g = \mathrm{id}_B$. Let $b \in B$. Then $g(b) \in A$ and $f(g(b)) = b$. Thus for all $b \in B$, there exists $a = g(b)$ such that $f(a) = b$. Hence, f is surjective.

($\Longleftarrow$) Suppose that f is surjective. We want a function $g : B \to A$ such that $f(g(b)) = b$ for all $b \in B$. Since f is surjective, for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Construct a set $\Gamma = \{(b, a) \mid f(a) = b\} \subseteq B \times A$. Note that Γ is not necessarily unique since there may be several a such that $f(a) = b$. However, its existence is guaranteed since f is surjective. Then this set may be used to define g where $g(b) = a$ if and only if $(a, b) \in \Gamma$. Now let $b \in B$. Then there exists an $a \in A$ such that $f(a) = b$. Therefore, $(a, b) \in \Gamma$ so $g(b) = a$. We get that $f(g(b)) = f(a) = b$ so g is a right-inverse of f. $\square$

Problem I.2.3. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is bijection.

Solution. Let $f : A \to B$ be a bijection. Consider $f^{-1} : B \to A$. We have that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. Then f is the left- and right-inverse of f^{-1}, so f^{-1} is also a bijection.

Let $f : A \to B$ and $g : B \to C$ be bijections and consider $g \circ f$. Suppose $a, a' \in A$ such that $(g \circ f)(a) = (g \circ f)(a')$. Since g is bijective, and in particular it is injective, we have $(g \circ f)(a) = (g \circ f)(a') \Longrightarrow f(a) = f(a')$. Similarly, f is injective so $f(a) = f(a') \Longrightarrow a = a'$. Thus, $g \circ f$ is injective. Now let $c \in C$. Since g is surjective, there exists a $b \in B$ such that $g(b) = c$. Similarly, since f is surjective, there exists an $a \in A$ such that $f(a) = b$. Then $(g \circ f)(a) = g(b) = c$ so $g \circ f$ is surjective. Hence, $g \circ f$ is bijective. $\square$

Problem I.2.4. Prove that 'isomorphism' is an equivalence relation (on any set of sets).

Solution. Let A be a set. Then id_A is a bijection so $A \cong A$. Let B be another set such that $A \cong B$. That is, there exists a bijection $f : A \to B$. Since f is bijective, it has an inverse $f^{-1} : B \to A$, so $B \cong A$. If C is another set such that $B \cong C$, then there exists a bijection $g : B \to C$. The composition of bijections is a bijection so $g \circ f : A \to C$ is bijective. Hence $A \cong C$ and $\cong$ is an equivalence relation. $\square$

Problem I.2.5. Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism* seen in §2.6, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections.

Proposition 2.3. *A function is injective if and only if it is a monomorphism.*

Solution. A function $f : A \to B$ is an epimorphism if for all sets Z and all functions $\beta, \beta' : B \to Z$ we have $\beta \circ f = \beta' \circ f \implies \beta = \beta'$. Now we show that a function is surjective if and only if it is an epimorphism.

($\Longrightarrow$) Suppose that $f : A \to B$ is surjective. Then f has a right-inverse $g : B \to A$. Let β, β' be functions from B to another set Z such that $\beta \circ f = \beta' \circ f$. Compose on the right by g and use associativity of composition:

$$\beta \circ (f \circ g) = (\beta \circ f) \circ g = (\beta' \circ f) \circ g = \beta' \circ (f \circ g)$$

Since g is a right-inverse of f, we have

$$\beta \circ \mathrm{id}_B = \beta' \circ \mathrm{id}_B$$

and thus $\beta = \beta'$ and f is an epimorphism.

($\Longleftarrow$) Now suppose that $f : A \to B$ is an epimorphism. Let $Z = \{0, 1\}$ and consider the morphisms $\beta, \beta' : B \to Z$ where $\beta(b) = 0$ for all $b \in B$ and $\beta'(b) = 0$ if $b \in \mathrm{im}(f)$ or $\beta'(b) = 1$ otherwise. By construction, $\beta \circ f = \beta' \circ f$. This implies that $\beta = \beta'$, which is only the case if every element $b \in B$ is sent to the same element of Z. β sends every element of B to 0, and β' sends every element of $\mathrm{im}(f)$ to 0, so $\mathrm{im}(f) = B$ and f is surjective. $\square$

Problem I.2.6. With notation as in Example 2.4, explain how any function $f : A \to B$ determines a section of π_A.

Solution. We know f corresponds to a subset $\Gamma_f = \{(a, b) \mid f(a) = b\} \subseteq A \times B$. The projection $\pi_A : A \times B \to A$ is defined such that $\pi_A(a, b) = a$. Let $g : A \to A \times B$ be a function such that $g(a) = (a, f(a)) \in \Gamma_f$. Since $(\pi_A \circ g)(a) = \pi_A(a, f(a)) = a$ for all $a \in A$, g is a section of π_A which is determined by f. $\square$

Problem I.2.7. Let $f : A \to B$ be any function. Prove that the graph Γ_f of f is isomorphic to A.

Solution. Recall that $\Gamma_f = \{(a, b) \mid b = f(a)\} \subseteq A \times B$. Let $g : A \to \Gamma_f$ be defined as $g(a) = (a, f(a))$. For all $(a, b) \in \Gamma_f$, we have $g(a) = (a, f(a)) = (a, b)$ so g is surjective. If $g(a) = g(a')$, then $(a, f(a)) = (a', f(a'))$. That is, $a = a'$ so g is injective, hence it is a bijection. Therefore, $\Gamma_f \cong A$. $\square$

Problem I.2.8. Describe as explicitly as you can all terms in the canonical decomposition of the function $\mathbb{R} \to \mathbb{C}$ defined by $r \mapsto e^{2\pi i r}$. (This exercise matches one assigned previously. Which one?)

Solution. Let $f : \mathbb{R} \to \mathbb{C}$ be the function defined above. The first part of the decomposition is defined by letting $\sim$ be an equivalence relation on $\mathbb{R}$ such that $a \sim b \iff f(a) = f(b)$. That is, $[a]_\sim$ is the set of elements in $\mathbb{R}$ that are mapped to the same element as a in $\mathbb{C}$. Then we have a projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\sim$ which sends each element $a \in \mathbb{R}$ to its equivalence class $[a]_\sim$. Note that $f(x) = f(x+1)$. That is, the function is periodic about the integers so real numbers which differ by an integer amount belong to the same equivalence class. Then $\mathbb{R}/\sim = \{\{r + k \mid k \in \mathbb{Z}\} \mid r \in [0, 1)$ which is identical to the quotient set in Exercise 1.1.6.

The function $\tilde{f} : \mathbb{R} \to \operatorname{im}(f)$ maps each equivalence class to the complex number that f maps the representative to. Certainly if $\tilde{f}([a]_\sim) = \tilde{f}([a']_\sim)$ then $f(a) = f(a')$ and $a \sim a'$ by definition. Thus $[a]_\sim = [a']_\sim$ so $\tilde{f}$ is injective. Similarly, let $b \in \operatorname{im}(f)$. Then there is an element $a \in \mathbb{R}$ such that $f(a) = b$. Then $\tilde{f}([a]_\sim) = f(a) = b$ so $\tilde{f}$ is surjective and hence a bijection. Finally, we have the inclusion $\operatorname{im}(f) \hookrightarrow \mathbb{C}$ which embeds the image of f into its codomain. $\square$

Problem I.2.9. Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \coprod B$ is well-defined *up to isomorphism*.

Solution. There exist bijections $f : A' \to A''$ and $g : B' \to B''$. Then we can define $h : A' \cup B' \to A'' \cup B''$ where

$$h(x) = \begin{cases} f(x) \text{ if } x \in A' \\ g(x) \text{ if } x \in B' \end{cases}$$

Let $y \in A'' \cup B''$. Since $A'' \cap B'' = \emptyset$, we have either $y \in A''$ or $y \in B''$. WLOG, suppose that $y \in A''$. Note that since f is surjective, there exists $x \in A'$ such that $f(x) = y$. Then $h(x) = f(x) = y$ so h is surjective. Suppose $x \neq x'$ for $x, x' \in A' \cup B'$. If $x, x' \in A'$ then since f is injective and $h(x) = f(x)$ for all $x \in A'$, we have $h(x) \neq h(x')$. A similar reasoning shows that if $x, x' \in B'$, then $h(x) \neq h(x')$. WLOG, suppose that $x \in A'$ and $x' \in B'$. Then $h(x) = f(x) \neq g(x') = h(x')$ since $A'' \cap B'' = \emptyset$. Thus h is surjective and hence a bijection, showing that $A' \cup B' \cong A'' \cup B''$.

The constructions of A', A'', B', B'' are equivalent to creating "copies" of sets A and B to use in the disjoint union. Thus, the disjoint union $A \coprod B$ is well-defined up to isomorphism. $\square$

Problem I.2.10. Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$.

Solution. Recall that $|B^A|$ is the number of functions from A to B. Each functions assigns a single element of A to a single element of B. There are $|B|$ choices for each of the $|A|$ elements. This is equivalent to $|B|^{|A|}$ total choices. Thus, $|B^A| = |B|^{|A|}$. $\square$

Problem I.2.11. In view of Exercise 2.10, it is not unreasonable to use 2^A to denote the set of functions from an arbitrary set A to a set with 2 elements (say $\{0,1\}$). Prove that there is a bijection between 2^A and the *power set* of A.

Solution. Consider $f : \mathscr{P}(A) \to 2^A$ defined as

$$f(X) = \{(a,1) \text{ if } a \in X, \text{ and } (a,0) \text{ otherwise}\}$$

Let $g \in 2^A$. Then g is a function from A to $\{0,1\}$. Let $A_1 = \{a \in A \mid g(a) = 1$. Then $A_1 \in \mathscr{P}(A)$ and $f(A_1) = g$, so f is surjective. Now suppose that $X, Y \subseteq A$ such that $f(X) = f(Y)$. That is, for all $a \in A$, $a \in X \iff (a,1) \in f(X) \iff (a,1) \in f(Y) \iff a \in Y$. Thus, $X = Y$ so f is injective and a bijection. Therefore, $2^A \cong \mathscr{P}(A)$. $\square$

I.3 Categories

Problem I.3.1. Let C be a category. Consider a structure C^{op} with

- $\text{Obj}(\mathsf{C}^{op}) := \text{Obj}(\mathsf{C})$;
- for A, B objects of C^{op} (hence objects of C), $\text{Hom}_{\mathsf{C}^{op}}(A,B) := \text{Hom}_{\mathsf{C}}(B,A)$.

Show how to make this into a category (that is, define composition of moprhisms in C^{op} and verify the properties listed in §3.1).

Intuitively, the 'opposite' category C^{op} is simply obtained by 'reversing all the arrows' in C.

Solution. For objects $A, B, C \in \text{Obj}(\mathsf{C}^{op})$, the set of morphisms from A to B, $\text{Hom}_{\mathsf{C}^{op}}(A,B)$, is defined as $\text{Hom}_{\mathsf{C}}(B,A)$. For morphisms $f \in \text{Hom}_{\mathsf{C}^{op}}(A,B)$ and $g \in \text{Hom}_{\mathsf{C}^{op}}(B,C)$, define composition as follows:

$$\circ_{\mathsf{C}^{op}} : \text{Hom}_{\mathsf{C}^{op}}(A,B) \times \text{Hom}_{\mathsf{C}^{op}}(B,C) \to \text{Hom}_{\mathsf{C}^{op}}(A,C)$$

such that

$$\circ_{\mathsf{C}^{op}}(g,f) = \circ_{\mathsf{C}}(f,g)$$

Then if $f \in \text{Hom}_{\mathsf{C}^{op}}(A,B), g \in \text{Hom}_{\mathsf{C}^{op}}(B,C), h \in \text{Hom}_{\mathsf{C}^{op}}(C,D)$, then

$$(h \circ_{\mathsf{C}^{op}} g) \circ_{\mathsf{C}^{op}} f = f \circ_{\mathsf{C}} (g \circ_{\mathsf{C}} h) = (f \circ_{\mathsf{C}} g) \circ_{\mathsf{C}} h = h \circ_{\mathsf{C}^{op}} (g \circ_{\mathsf{C}^{op}} f)$$

so composition is associative. Furthermore, define the identity morphism $1_{A_{\mathsf{C}^{op}}} = 1_{A_{\mathsf{C}}}$. Then for all $f \in \text{Hom}_{\mathsf{C}^{op}}(A,B)$ we have

$$f \circ_{\mathsf{C}^{op}} 1_{A_{\mathsf{C}^{op}}} = 1_{A_{\mathsf{C}}} \circ_{\mathsf{C}} f = f$$
$$1_{B_{\mathsf{C}^{op}}} \circ_{\mathsf{C}^{op}} f = f \circ_{\mathsf{C}} 1_{B_{\mathsf{C}}} = f$$

so identities preserve morphisms. Finally, let $A, B, C, D \in \text{Obj}(\mathsf{C}^{op})$ where $A \neq C$ and $B \neq D$. Consider the sets $\text{Hom}_{\mathsf{C}^{op}}(A,B)$ and $\text{Hom}_{\mathsf{C}^{op}}(C,D)$. These are equal to the sets $\text{Hom}_{\mathsf{C}}(B,A)$ and $\text{Hom}_{\mathsf{C}}(D,C)$ respectively, which are disjoint since C is a category. Thus, C^{op} forms a category. $\square$

Problem I.3.2. If A is a finite set, how large is $\text{End}_{\mathsf{Set}}(A)$?

Solution. Recall that $\text{End}_{\mathsf{Set}}(A)$ is the set of functions from A to A. By Exercise 2.10, we have $|B^A| = |B|^{|A|}$. Thus, $|\text{End}_{\mathsf{Set}}(A)| = |A|^{|A|}$. □

Problem I.3.3. Formulate precisely what it means to say that 1_a is an identity with respect to composition in Example 3.3, and prove this assertion.

Solution. Let S be a set and $\sim$ be a reflexive and transitive relation on S. Consider a category C where

- $\text{Obj}(\mathsf{C})$ are the elements in S
- If a, b are objects, then let $\text{Hom}(a, b) = (a, b) \in S \times S$ if $a \sim b$ and let $\text{Hom}(a, b) = \emptyset$ otherwise.

This forms a category and composition is defined as follows. Let a, b, c be objects and $f \in \text{Hom}(a, b)$, $g \in \text{Hom}(b, c)$. Then $g \circ f = (a, c) \in \text{Hom}(a, c)$ by the transitivity of $\sim$.

Now we verify that the identity preserves morphisms in this category. Let $a, b \in S$ and $f \in \text{Hom}(a, b)$. A morphism $1_a = (a, a) \in \text{End}(a)$ is an identity with respect to composition if

$$f \circ 1_a = f$$

Indeed, we have $f = (a, b)$ and $1_a = (a, a)$. Then by definition we have

$$f \circ 1_a = (a, b)(a, a) = (a, b) = f$$

Thus 1_a is an identity with respect to composition as required. □

Problem I.3.4. Can we define a category in the style of Example 3.3 using the relation $<$ on the set $\mathbb{Z}$.

Solution. No, since the relation $<$ is not reflexive. That is, $a < a$ does not hold for any $a \in \mathbb{Z}$. There is no reasonable way to define an identity morphism. □

Problem I.3.5. Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3.

Solution. Let S be a set and consider the category $\hat{S}$ where

- $\text{Obj}(\hat{S}) = \mathscr{P}(S)$
- For $A, B \in \text{Obj}(\hat{S})$, let $\text{Hom}_{\hat{S}}(A, B)$ be the pair (A, B) if $A \subseteq B$, and let $\text{Hom}_{\hat{S}}(A, B) = \emptyset$ otherwise.

Composition is obtained by using the transitivity of inclusion.

This is equivalent to the category in Example 3.3 by considering the relation $\sim$ defined on $\mathscr{P}(S)$ where $A \sim B$ if and only if $A \subseteq B$. Indeed, this relation is both reflexive and transitive so we may construct the category considered in Example 3.3, and the two are equivalent. □

Problem I.3.6. (Assuming some familiarity with linear algebra.) Define a category V by taking $\mathrm{Obj}(\mathsf{V}) = \mathbb{N}$ and letting $\mathrm{Hom}_{\mathsf{V}}(n, m) =$ the set of $m \times n$ matrices with real entries, for all $n, m \in \mathbb{N}$. (We will leave the reader the task of making sense of a matrix with 0 rows or columns.) Use products of matrices to define composition. Does this category 'feel' familiar?

Solution. First of all, the identity morphism for the object n is the set of $n \times n$ matrices. Let $l, m, n \in \mathbb{N}$ and

$$f \in \mathrm{Hom}(l, m), \quad g \in \mathrm{Hom}(m, n)$$

Then fg is an $l \times n$ matrix and is in $\mathrm{Hom}(l, n)$. Furthermore, matrix multiplication is associative.

This category is another instance of Example 3.3 where the set is $\mathbb{N}$ and the relation $\sim$ is defined as follows: $m \sim n$ if and only if $\mathrm{Hom}(m, n)$ is nonempty. Certainly this relation is both reflexive and transitive so it is an instance of Example 3.3. □

Problem I.3.7. Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition.

Solution. Given a category C and an object $A \in \mathrm{Obj}(\mathsf{C})$, consider the category C^A where

- $\mathrm{Obj}(\mathsf{C}^A) =$ all morphisms from A to any object of C;
- Let f_1, f_2 be objects of C^A, or two arrows

$$\begin{tikzcd} A \arrow[d, "f_1"'] & A \arrow[d, "f_2"] \\ Z_1 & Z_2 \end{tikzcd}$$

  in C. Morphisms $f_1 \to f_2$ are *commutative diagrams*

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Z_1 \arrow[rr, "\sigma"'] & & Z_2 \end{tikzcd}$$

in the category C.

That is, morphisms $\sigma \in \mathrm{Hom}_{\mathsf{C}^A}(f_1, f_2)$ are precisely the morphisms $\sigma : Z_1 \to Z_2$ in C such that $f_2 = \sigma \circ f_1$.

If $\sigma \in \mathrm{Hom}(f, g)$ and $\tau \in \mathrm{Hom}(g, h)$, then $\tau \circ \sigma \in \mathrm{Hom}(f, h)$ is the morphism in C making the following diagram commute:

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\
Z_1 \arrow[r, "\sigma"'] \arrow[rr, bend right, "\tau\circ\sigma"'] & Z_2 \arrow[r, "\tau"'] & Z_3
\end{tikzcd}$$

□

Problem I.3.8. A *subcategory* C' of a category C consists of a collection of objects of C, with morphisms $\mathrm{Hom}_{\mathsf{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects A, B in $\mathrm{Obj}(\mathsf{C}')$, such that identities and compositions in C make C' into a category. A subcategory C' is *full* if $\mathrm{Hom}_{\mathsf{C}'}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all A, B in $\mathrm{Obj}(\mathsf{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of Set.

Solution. Let Set^∞ be a category whose objects are infinite sets and whose morphisms are set functions between them. That is, for infinite sets A, B we let $\mathrm{Hom}_{\mathsf{Set}^\infty}(A, B)$ be the set of set functions from A to B. Certainly this is equivalent to $\mathrm{Hom}_{\mathsf{Set}}(A, B)$ so the subcategory is full. □

Problem I.3.9. An alternative to the notion of *multiset* introduced in §2.2 is obtained by considering sets endowed with equivalence relations; equivalent elements are taken to be multiple instances of elements 'of the same kind'. Define a notion of morphism between such enhanced sets, obtaining a category MSet containing (a 'copy' of) Set as a full subcategory. (There may be more than one reasonable way to do this! This is intentionally an open-ended exercise.) Which objects in MSet determine ordinary multisets as defined in §2.2 and how? Spell out what a morphism of multisets would be from this point of view. (There are several natural notions of morphisms of multisets. Try to define morphisms in MSet so that the notion you obtain for ordinary multisets captures your intuitive understanding of these objects.)

Solution. Consider the category MSet where

- $\mathrm{Obj}(\mathsf{MSet})$ = sets endowed with equivalence relations;

- If $A, B \in \mathrm{Obj}(\mathsf{MSet})$ then $\mathrm{Hom}_{\mathsf{MSet}}(A, B)$ is the collection of functions from A to B which preserve equivalence classes. That is, if $\sim$ is an equivalence relation on A and $\approx$ is an equivalence relation on B then for $a, b \in A$ and $f \in \mathrm{Hom}_{\mathsf{MSet}}(A, B)$ we have $a \sim b \implies f(a) \approx f(b)$.

Composition is naturally defined as it is Set. For objects A, B, C, let $f \in \mathrm{Hom}_{\mathsf{MSet}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{MSet}}(B, C)$. If $a, b \in A$ and $a \sim_A b$ then, since f is a morphism, $f(a) \sim_B f(b)$. Furthermore, g is a morphism so $g(f(a)) \sim_C g(f(b))$ so $g \circ f \in \mathrm{Hom}_{\mathsf{MSet}}(A, C)$. The identity morphism has a natural definition where $1_S : S \to S$ is the identity function Set. It obviously preserves equivalence classes. Associativity is similarly inherited from Set.

In §2.2, multisets are defined as a set A along with a function $m : A \to \mathbb{N}^*$ which takes each element of A to the number denoting its multiplicity. We define the equivalence relation $\sim$ on A which partitions A into its distinct elements, or those elements which are not equal. In other words, $m(a) \neq m(b) \implies a \nsim b$. Morphisms between these objects as defined above can intuitively be expressed as the functions which allow elements to be renamed and naturally mapped to other multisets which preserve multiplicity. $\square$

Problem I.3.10. Since the objects of a category C are not (necessarily) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it *does* make sense to talk about subobjects, and the subobjects of any given object A in C are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed special object Ω of C, called a *subobject classifier*. Show that Set has a subobject classifier.

Solution. Consider the set $\Omega = \{0, 1\}$. Let A be any set. The subsets $X \subseteq A$ induce morphisms $f : A \to \Omega$ where

$$f(x) = \begin{cases} 1 \text{ if } x \in X \\ 0 \text{ if } x \notin X \end{cases}$$

Certainly these morphisms are in bijection with subsets of A. Thus $\{0, 1\}$ is a subobject classifier of Set, though any set with 2 elements works. $\square$

Problem I.3.11. Draw the relevant diagrams and define composition and identities for the category $\mathsf{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10.

Solution. Consider the category $\mathsf{C}^{A,B}$ where

- $\mathrm{Obj}(\mathsf{C}^{A,B})$ = diagrams

  $$\begin{tikzcd}
  A \arrow[dr, "f"] & \\
  & Z \\
  B \arrow[ur, "g"'] &
  \end{tikzcd}$$

  in C

- Morphisms between objects (Z_1, f_1, g_1) and (Z_2, f_2, g_2) are commutative diagrams

  $$\begin{tikzcd}
  A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
  & Z_1 \arrow[r, "\sigma"] & Z_2 \\
  B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & &
  \end{tikzcd}$$

  That is, we have a morphism $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$ such that $f_2 = \sigma \circ f_1$ and $g_2 = \sigma \circ g_1$.

Composition has a natural definition. Given a third object (Z_3, f_3, g_3) with a morphism $\tau : Z_2 \to Z_3$ we define $\tau \circ \sigma : Z_1 \to Z_3$ such that $f_3 = \tau \circ \sigma(f_1)$ and $g_3 = \tau \circ \sigma(g_1)$. Given an object (Z, f, g), the identity morphism $1_Z \in \mathrm{End}_{\mathsf{C}}(Z)$ serves as an identity in $\mathsf{C}^{A,B}$ as well. Specifically, we have $f = 1_Z \circ f$ and $g = 1_Z \circ g$.

Now consider the category $\mathsf{C}^{\alpha,\beta}$ where $\alpha : C \to A$ and $\beta : C \to B$. Then we have

- $\mathrm{Obj}(\mathsf{C}^{\alpha,\beta})$ = commutative diagrams

  $$\begin{tikzcd}
  & A \arrow[dr, "f"] & \\
  C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \\
  & B \arrow[ur, "g"'] &
  \end{tikzcd}$$

  where Z is an object in C

- Morphisms between objects (Z_1, f_1, g_1) and (Z_2, f_2, g_2) are commutative diagrams

$$\begin{tikzcd} & A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\ C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \\ & B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_2"'] & & \end{tikzcd}$$

That is, we have a morphism $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$ such that the diagram commutes.

Composition again has a natural definition. Given a third object (Z_3, f_3, g_3) and a morphism $\tau : Z_2 \to Z_3$, we can define a morphism $\tau \circ \sigma : Z_1 \to Z_3$ such that the corresponding diagram commutes. Finally, given an object (Z, f, g) we inherit the identity morphism 1_Z from C. Certainly the corresponding diagram commutes. $\square$

I.4 Morphisms

Problem I.4.1. Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E$$

then one may compose them in several ways, for example:

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad \text{etc.}$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses. (Hint: Use induction on n to show that any such choice for $f_n f_{n-1} \cdots f_1$ equals

$$((\cdots((f_n f_{n-1}) f_{n-2}) \cdots) f_1).$$

Carefully working out the case $n = 5$ is helpful.)

Solution. For $n = 3$, we have $(fg)h = f(gh)$ by the associativity of composition in a category. Suppose $n \geq 4$ and that for $n-1$ morphisms we have shown that composition is independent of the placement of the parentheses. Let $f_1, \ldots, f_n$ be morphisms in a category:

$$Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} Z_n \xrightarrow{f_n} Z_{n+1}$$

Suppose that a parenthesization of $f_n, f_{n-1}, \ldots, f_1$ is f and that $f = hg$ where h is some parenthesization of $f_n, f_{n-1}, \ldots, f_{i+1}$, and g is some parenthesization of $f_i, f_{i-1}, \dots, f_1$, where $1 \leq i \leq n$. Applying the inductive to h and g, we see that

$$h = ((\cdots((f_n f_{n-1})f_{n-2})\cdots)f_{i+1})$$
$$g = (f_i(f_{i-1}(\cdots(f_2 f_1)\cdots)) = f_i g'$$

hence $f = hg = h(f_i g') = (hf_i)g'$. Effectively, we remove morphisms f_i from the left side of g' and attach them to the right side of h to obtain the form

$$f = ((\cdots((f_n f_{n-1})f_{n-2})\cdots)f_1)$$

□

Problem I.4.2. In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)?

Solution. Recall that a groupoid is a category in which every morphism is an isomorphism and hence has a two-sided inverse. The corresponding category is a groupoid when the relation is also symmetric and hence an equivalence relation. Indeed, if $(x, y) \in \text{Hom}(x, y)$ then $x \sim y$. If $\sim$ is reflexive then this implies that $y \sim x$ so $(y, x) \in \text{Hom}(y, x)$. Then $(x, y)(y, x) = (x, x)$ and $(y, x)(x, y) = (y, y)$, both of which are the identity morphisms of their respective objects. Thus, (x, y) is an isomorphism and the category is a groupoid. □

Problem I.4.3. Let A, B be objects of a category C, and let $f \in \text{Hom}_{\mathsf{C}}(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicity example of a category and an epimorphism without a right-inverse.

Solution. Suppose f has a right-inverse. That is, there exists a morphism $g \in \text{Hom}_{\mathsf{C}}(B, A)$ such that $f \circ g = 1_B$. Then if we consider two morphisms $\beta, \beta' \in \text{Hom}_{\mathsf{C}}(B, Z)$ such that $\beta \circ f = \beta' \circ f$ we have

$$\begin{aligned}
&(\beta \circ f) \circ g = (\beta' \circ f) \circ g \\
\Longrightarrow &\beta \circ (f \circ g) = \beta' \circ (f \circ g) \\
\Longrightarrow &\beta \circ 1_B = \beta' \circ 1_B \\
\Longrightarrow &\beta = \beta'
\end{aligned}$$

Thus, f is an epimorphism.

However, consider the category C where

- $\mathrm{Obj}(\mathsf{C}) = \mathbb{Z}$
- For objects $a, b \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathsf{C}}(a, b) = \{(a, b)\}$ if $a \leq b$ and $\emptyset$ otherwise.

The reflexivity and transitivity of $\leq$ makes this a category. Given morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(a, b)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(b, c)$ we define composition as $g \circ f = (b, c) \circ (a, b) = (a, c) \in \mathrm{Hom}_{\mathsf{C}}(a, c)$. Consider two objects $a, b \in \mathbb{Z}$ such that $a < b$ and let $f : a \to b = (a, b)$ be the morphism from a to b. Consider two morphisms $\beta, \beta' \in \mathrm{Hom}_{\mathsf{C}}(b, c)$ such that $\beta \circ f = \beta' \circ f$. Then we have $\beta = \beta'$ since each Hom set has at most one morphism. Thus f is an epimorphism. However, it does not have a right-inverse. Indeed, suppose $\mathrm{Hom}_{\mathsf{C}}(b, a)$ is nonempty. Then it can only contain (b, a) which would imply that $b \leq a$, a contradiction since we assumed $a < b$. Thus, we have a category where epimorphisms do not necessarily have right-inverses. □

Problem I.4.4. Prove that the composition of two monomorphims is a monomorphism. Deduce that one can define a subcategory C_{mono} of a category C by taking the same objects as in C and defining $\mathrm{Hom}_{\mathsf{C}_{\text{mono}}}(A, B)$ to be subset of $\mathrm{Hom}_{\mathsf{C}}(A, B)$ consisting of monomorphisms, for all objects A, B. (Cf. Exercise 3.8; of course, in general C_{mono} is not full in C.) Do the same for epimorphisms. Can you define a subcategory $\mathsf{C}_{\text{nonmono}}$ of C by restricting to morphisms that are *not* monomorphisms?

Solution. Suppose that $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ are two monomorphisms. Let $\alpha, \alpha' \in \mathrm{Hom}_{\mathsf{C}}(Z, A)$ be two morphisms such that $(g \circ f) \circ \alpha = (g \circ f) \circ \alpha'$. Then we have

$$
\begin{aligned}
&(g \circ f) \circ \alpha = (g \circ f) \circ \alpha' && \\
\Longrightarrow\; & g \circ (f \circ \alpha) = g \circ (f \circ \alpha') && \text{by the associativity of composition} \\
\Longrightarrow\; & f \circ \alpha = f \circ \alpha' && \text{since } g \text{ is a monomorphism} \\
\Longrightarrow\; & \alpha = \alpha' && \text{since } f \text{ is a monomorphism}
\end{aligned}
$$

Hence, $g \circ f$ is a monomorphism. Therefore, the subcategory C_{mono} is closed with respect to composition.

We use a similar proof to show that the composition of two epimorphisms is an epimorphism. Suppose that $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ are epimorphisms. Let $\beta, \beta' \in \mathrm{Hom}_{\mathsf{C}}(C, Z)$ be two morphisms such that $\beta \circ (g \circ f) = \beta' \circ (g \circ f)$. Then we have

$$
\begin{aligned}
&(\beta \circ g) \circ f = (\beta' \circ g) \circ f && \text{by the associativity of composition} \\
\Longrightarrow\; & \beta \circ g = \beta' \circ g && \text{since } f \text{ is an epimorphism} \\
\Longrightarrow\; & \beta = \beta' && \text{since } g \text{ is an epimorphism}
\end{aligned}
$$

Thus, $g \circ f$ is an epimorphism so we can define a similar subcategory C_{epi} which is closed with respect to compositon.

We can also define a category $\mathsf{C}_{\text{nonmono}}$ whose morphisms are restricted to those of C which are not monomorphisms. Indeed, suppose $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is not a monomorphism. That is, there exist morphisms $\alpha, \alpha' \in \mathrm{Hom}_{\mathsf{C}}(Z, A)$ such that $f \circ \alpha = f \circ \alpha'$ but $\alpha \neq \alpha'$. Let $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ be a non-monomorphism. Then we have $(g \circ f) \circ \alpha = (g \circ f) \circ \alpha'$ but $\alpha \neq \alpha'$. Thus, $(g \circ f)$ is not a monomorphism so the category $\mathsf{C}_{\text{nonmono}}$ is closed under composition. Interestingly, this only relies on the fact that f is not a monomorphism. $\square$

Problem I.4.5. Give a concrete description of monomorphims and epimorphisms in the category MSet you constructed in Exercise 3.9. (Your answer will depend on the notion of morphism you defined in that exercise!)

Solution. Recall that we defined multisets to be sets equipped with equivalence relations. A morphism between two multisets is a set function which preserves the equivalence relation. The notions of monomorphism and epimorphism are naturally inherited from Set.

- A morphism $f \in \mathrm{Hom}_{\mathsf{MSet}}(A, B)$ is a monomorphism if for all $a_1, a_2 \in A$ we have $f(a_1) \sim_B f(a_2) \implies a_1 \sim_A a_2$. We call these morphisms *injective.*
- A morphism $f \in \mathrm{Hom}_{\mathsf{MSet}}(A, B)$ is an epimorphism if for all $b \in B$ there exists an $a \in A$ such that $f(a) = b$. We call these morphisms *surjective.*

We will prove that these definitions satisfy the category theoretical definitions of monomorphisms and epimorphisms. We start by proving an analogue of Proposition 2.1 in MSet.

Lemma. *Assume $A \neq \emptyset$ and let $f : A \to B$ be a morphism of multisets. Then*

1. *f has a left-inverse if and only if it is injective.*
2. *f has a right-inverse if and only if it is surjective.*

Proof. First we prove (1). If f has a left-inverse, then there exists a morphism $g \in \mathrm{Hom}_{\mathsf{MSet}}(B, A)$ such that $g \circ f = 1_A$. Let $a_1 \nsim_A a_2$ be elements in A not eqivalent under the relation. Then

$$g \circ f(a_1) = 1_A(a_1) = a_1 \nsim_A a_2 = 1_A(a_2) = g \circ f(a_2)$$

That is, $a_1 \nsim_A a_2 \implies f(a_1) \nsim_B f(a_2)$ which is the contrapositive of the definition for an injective morphism. Thus, if f has a left-inverse it must be injective.

Now suppose $f : A \to B$ is injective. We will construct a left-inverse $g : B \to A$. Choose one fixed element $s \in A$. Now set

$$g(b) = \begin{cases} a \text{ if } b = f(a) \text{ for some } a \in A, \\ s \text{ if } b \notin \mathrm{im} f \end{cases}$$

This definition guarantees that every b that is in the image of f maps to a unique element since f is injective. We can verify that g is a left-inverse of f. If $a \in A$, then $g \circ f(a) = a = 1_A(a)$.

A highly similar proof follows for (2). If $f : A \to B$ has a right-inverse, then there exists a morphism $g : B \to A$ such that $f \circ g = 1_B$. Let $b \in B$. Then $g(b) \in A$ and $f \circ g(b) = b$ for all such b. Thus f is surjective.

For the reverse direction, suppose that $f : A \to B$ is surjective. We will construct a right-inverse $g : B \to A$. Let $S = \{(a, b) \mid f(a) = b\}$. Certainly S contains elements for each $b \in B$ since f is surjective. Then define $g : B \to A$, $g(b) = a$ where a is the least element such that $(a, b) \in S$. This definition guarantees that every element of b is mapped to only one element since there may be several a which are mapped to b. We can verify that g is a right-inverse of f. Let $b \in B$. Then $f \circ g(b) = b = 1_B(b)$. $\square$

With this lemma, we show that our definition of injective and surjective morphisms is precisely equivalent to monomorphisms and epimorphisms in the category MSet.

First suppose that $f : A \to B$ is injective. Then it has a left-inverse $g : B \to A$. Let $\alpha, \alpha' \in \mathrm{Hom}_{\mathsf{MSet}}(Z, A)$ be morphisms such that $f \circ \alpha = f \circ \alpha'$. Then we find

$$\begin{aligned} &(g \circ f) \circ \alpha = (g \circ f) \circ \alpha' && \text{by associativity of composition} \\ \Longrightarrow &1_A \circ \alpha = 1_A \circ \alpha' && \text{since } g \text{ is a left-inverse of } f \\ \Longrightarrow &\alpha = \alpha' \end{aligned}$$

Thus, f is a monomorphism in the category theoretical sense.

Now suppose that $f : A \to B$ is a monomorphism. We will show it is injective. Consider the set $Z = \{p\}$ and let $\alpha, \alpha' \in \mathrm{Hom}_{\mathsf{MSet}}(Z, A)$ be morphisms such that $f \circ \alpha = f \circ \alpha'$. Since f is a monomorphism, this forces $\alpha = \alpha'$. In turn, this means $\alpha(p) \sim_A \alpha'(p)$. Letting $a_1 = \alpha(p)$ and $a_2 = \alpha'(p)$, we have

$$f(a_1) \sim_A f(a_2) \Longrightarrow a_1 \sim_A a_2$$

Thus, f is injective. A nearly identical proof follows for epimorphisms and surjective morphisms. $\square$

I.5 Universal Properties

Problem I.5.1. Prove that a final object in a category C is initial in the opposite category C^{op}.

Solution. Let A be a final object in C. That is, for every object Z of C, there exists exactly one morphism $f \in \mathrm{Hom}_{\mathsf{C}}(Z, A)$. Recall that the opposite category C^{op} is formed by 'reversing' all arrows. More formally, we set $\mathrm{Hom}_{\mathsf{C}^{op}}(Z, B) = \mathrm{Hom}_{\mathsf{C}}(B, Z)$. In particular, for every object Z of C^{op}, there exists exactly one morphism $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, Z)$. Thus, A is initial in C^{op}. $\square$

Problem I.5.2. Prove that $\emptyset$ is the *unique* initial object in Set.

Solution. Note that the empty set $\emptyset$ is initial in Set with the only morphism to other sets being the empty mapping. Now let I be any other initial object in Set. Then $I \cong \emptyset$. Recall that isomorphic sets are those which have the same order (so that a bijection exists between them). Thus, $|I| = |\emptyset| = 0$ and I is necessarily the empty set $\emptyset$ since it is the only set with no elements. $\square$

Problem I.5.3. Prove that final objects are unique up to isomorphism.

Solution. First note that if F is a final object in a category C, then there is a unique morphism $F \to F$, namely the identity 1_F. Now assume F_1 and F_2 are both final in C. Since F_2 is final, there is a unique morphism $f : F_1 \to F_2$. We will show that f is an isomorphism. Since F_1 is final, there is a unique morphism $g : F_2 \to F_1$. Consider the composition $g \circ f : F_1 \to F_1$. As noted earlier, this is necessarily the identity morphism 1_{F_1}. Similarly, $f \circ g : F_2 \to F_2$ is necessarily the identity morphism 1_{F_2}. Thus, f is an isomorphism and $F_1 \cong F_2$. $\square$

Problem I.5.4. What are initial and final objects in the category of 'pointed sets'? Are they unique?

Solution. Recall that the category of pointed sets Set^* is defined as follows:

- $\mathrm{Obj}(\mathsf{Set}^*) =$ morphisms $f : \{*\} \to S$ in Set where S is any set. Note that objects may be denoted as pairs (S, s) where S is the set the morphism maps to and s is the element that f sends $*$ to.
- Given two objects (S, s) and (T, t), a morphism $f : (S, s) \to (T, t)$ corresponds to a set-function $\sigma : S \to T$ such that $\sigma(s) = t$.

Then the pointed singleton sets $(\{s\}, s)$ are the initial and final objects of Set^*. Indeed, let (T, t) be any object in Set^*. Then there is only one morphism $\sigma : S \to T$ such that $\sigma(s) = t$. Similarly, there is only one morphism $\sigma' : T \to S$ such that $\sigma(t) = s$. Thus, pointed singleton sets are both initial and final. They are also clearly not unique as both $(\{a\}, a)$ and $(\{b\}, b)$ where $a \neq b$ are distinct pointed singleton sets. $\square$

Problem I.5.5. What are the final objects in the category considered in §5.3?

Solution. The category considered in §5.3 is defined as follows: Let $\sim$ be an equivalence relation defined on a set A. Consider the category C_A where

- $\mathrm{Obj}(\mathsf{C}_A) =$ morphisms $\varphi : A \to Z$ where Z is an arbitrary set such that $a \sim a' \Longrightarrow \varphi(a) = \varphi(a')$. Objects are frequently denoted (φ, Z).

- Morphisms $(\varphi_1, Z_1) \to (\varphi_2, Z_2)$ are commutative diagrams

$$\begin{tikzcd}
Z_1 \arrow[rr, "\sigma"] & & Z_2 \\
& A \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

Then the objects $(\varphi^*, \{*\})$ are final in this category, where φ^* is the morphism mapping every element of A to $*$. To verify, let (φ, Z) be an object. Then there exists a unique morphism $\sigma : Z \to \{*\}$, namely the one mapping every element of Z to $*$. Certainly this morphism makes the diagram commute, and since it exists for all objects, $\varphi^*, \{*\}$ is final. $\square$

Problem I.5.6. Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ of positive integers with the *divisibility* relation. Thus there is exactly one morphism $d \to m$ in this category if and only if d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their conventional names?

Solution. Given two positive integers m, n, their categorical product $m \times n$ is the positive integer such that, given any positive integer z, the diagram

$$\begin{tikzcd}
& & m \\
z \arrow[urr, bend left, "f_m"] \arrow[r, "\sigma"'] \arrow[drr, bend right, "f_n"'] & m \times n \arrow[ur, "\pi_m"'] \arrow[dr, "\pi_n"] & \\
& & n
\end{tikzcd}$$

commutes.

Note that the existence of projections π_m, π_n implies $m \times n$ divides m and $m \times n$ divides n. Thus, we have $m \times n$ divides $\gcd(m, n)$. Furthermore, consider $z = \gcd(m, n)$. Certainly there exist morphisms $f_m : z \to m$ and $f_n : z \to n$. Then by the definition of categorical products, there exists a unique morphism $\sigma : z \to m \times n$. That is, we have $\gcd(m, n)$ divides $m \times n$. Combined with the earlier observation, we find $m \times n = \gcd(m, n)$.

Now let us consider the categorical coproduct $m \coprod n$. This is a positive integer such that, given any positive integer z, the diagram

$$\begin{tikzcd}
m \arrow[dr, "i_m"] \arrow[drr, bend left, "f_m"] & & \\
& m \coprod n \arrow[r, "\sigma"] & z \\
n \arrow[ur, "i_n"'] \arrow[urr, bend right, "f_n"'] & &
\end{tikzcd}$$

commutes.

The existence of the inclusion morphisms imply that both m and n divide $m \coprod n$, so $\operatorname{lcm}(m,n)$ divides $m \coprod n$. Furthermore, take z to be $\operatorname{lcm}(m,n)$. Then there certainly exist morphisms $f_m : m \to z$ and $f_n : n \to z$. By the definition of the categorical coproduct, there exists a unique morphism $\sigma : m \coprod n \to z$, so $m \coprod n$ divides $\operatorname{lcm}(m,n)$. Thus, we have $m \coprod n = \operatorname{lcm}(m,n)$. □

Problem I.5.7. Redo Exercise 2.9, this time using Proposition 5.4.

Solution. Exercise 2.9 asks that we show if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. We can conclude that $A \coprod B$ is well-defined up to isomorphism.

First consider $i_{A'} : A' \to A' \cup B', i_{A'}(a) = a$ for all $a \in A'$. Define a similar function $i_{B'}$. If Z is a set with morphisms $f_{A'} : A' \to Z$ and $f_{B'} : B' \to Z$, we have a unique morphism $\sigma : A' \coprod B' = A' \cup B' \to Z$ where

$$\sigma(x) = \begin{cases} f_{A'}(x) \text{ if } x \in A' \\ f_{B'}(x) \text{ if } x \in B' \end{cases}$$

This shows that the disjoint union is a coproduct.

We define entirely analagous morphisms for A'' and B''. Then we have a second coproduct $A'' \coprod B'' = A'' \cup B''$.

Proposition 5.4 states that in any category C, two initial objects I_1 and I_2 are isomorphic. Note that the coproducts $A' \coprod B'$ and $A'' \coprod B''$ we have defined are initial in the category $\mathsf{Set}_{A,B}$. Thus, they are isomorphic. □

Problem I.5.8. Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic, if they exist. (Hint: Observe that they both satisfy the universal property for the product of A and B; then use Proposition 5.4.)

Solution. Let $A \times B$ and $B \times A$ be products in a category C. Certainly $A \times B$ satisfies the universal property for products. That is, given an object Z and morphisms $f_A : Z \to A$ and $f_B : Z \to B$, we can construct a unique morphism $\sigma : Z \to A \times B$.

Now consider the morphism $\tau : A \times B \to B \times A, \tau(a,b) = (b,a)$. Certainly this morphism is an isomorphism since it has an inverse $\tau^{-1}(b,a) = (a,b)$. Then for any object Z and morphisms f_A, f_B as defined above, we consider the morphism $\varphi : Z \to B \times A, \varphi = \tau \circ \sigma$. It is unique since it is determined by the product $A \times B$. Therefore, $B \times A$ also satisfies the universal property for the product of A and B. By Proposition 5.4, the two objects are isomorphic. Admittedly, we already observed that an isomorphism exists between the two objects. □

Problem I.5.9. Let C be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of *three* objects of C ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Solution. Given three objects A, B, C of a category C, we can consider the product $A \times B \times C$ with three natural projections π_A, π_B, π_C. The reasonable definition of the universal property is as follows: For every object Z and morphisms $f_A : Z \to A$, $f_B : Z \to B$, and $f_C : Z \to C$, there exists a unique morphism $\sigma : Z \to A \times B \times C$ such that the diagram

$$
\begin{tikzcd}
 & & A & \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[drr, "f_B"'] \arrow[ddrrr, bend right, "f_A"'] & A \times B \times C \arrow[ur, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[ddrr, bend left, "\pi_C"] & & \\
 & & B & \\
 & & & C
\end{tikzcd}
$$

commutes.

First we will show that $(A \times B) \times C$ satisfies this universal property. For every object Z, we have a unique morphism $\tau : Z \to A \times B, \tau(z) = (f_A(z), f_B(z))$. Now we define $\sigma : Z \to (A \times B) \times C$,

$$\sigma(z) = (\tau(z), f_C(z)) = ((f_A(z), f_B(z)), f_C(z))$$

We define a natural projection $\pi'_A : (A \times B) \times C \to A, \pi'_A = \pi_A \circ \pi_{A \times B}$ along with an analogous projection π'_B and the typical π_C. These morphisms make the diagram commute because for all $z \in Z$ we have

$$\pi'_A \circ \sigma(z) = \pi_A \circ \pi_{A \times B}((f_A(z), f_B(z)), f_C(z)) = \pi_A(f_A(z), f_B(z)) = f_A(z)$$

and similarly for f_B and f_C. Thus, $(A \times B) \times C$ satisfies the universal property for the product $A \times B \times C$.

An entirely analogous construction shows that $A \times (B \times C)$ also satisfies this universal property. By Proposition 5.4, we must have $(A \times B) \times C \cong A \times (B \times C)$. $\square$

Problem I.5.10. Push the envelope a little further still, and define products and coproducts for *families* (i.e., indexed sets) of objects of a category. Do these exist in Set? It is common to denote the product $\underbrace{A \times \cdots \times A}_{n \text{ times}}$ by A^n.

Solution. Given a family of objects $\{A_i\}_{i\in I}$ for some set I in a category C, the product $\Pi_{i\in I}A_i$ with natural projections $\{\pi_{A_i}\}_{i\in I}$ should satisfy the universal property that for all objects Z and morphisms $\{f_{A_i}\}_{i\in I}, f_{A_i} : Z \to A_i$, there exists a unique morphism $\sigma : Z \to \Pi_{i\in I}A_i$ such that $\pi_{A_i} \circ \sigma = f_{A_i}$ for all $i \in I$.

Similarly, the coproduct $\coprod_{i\in I} A_i$ with natural inclusions $\{i_{A_i}\}_{i\in I}$ should satisfy the following universal property: for all objects Z and morphisms $\{f_{A_i}\}_{i\in I}, f_{A_i} : A_i \to Z$, there exists a unique morphism $\sigma : \coprod_{i\in I} A_i \to Z$ such that $\sigma \circ i_{A_i} = f_{A_i}$ for all $i \in I$.

The product for finite families of sets exists. However, we require the Axiom of Choice to ensure that the infinite product of nonempty sets is nonempty. The coproduct should exist for any family of sets since the family is indexed so we can just take the coproduct to be $\bigcup\{i\} \times \{A_i\}$ but I'm not positive. $\square$

Problem I.5.11. Let A, resp. B, be a set endowed with an equivalence relation $\sim_A$, resp. $\sim_B$. Define a relation $\sim$ on $A \times B$ by setting

$$(a_1, b_1) \sim (a_2, b_2) \Longleftrightarrow a_1 \sim_A a_2 \text{ and } b_1 \sim_B b_2.$$

(This is immediately seen to be an equivalence relation.)

- Use the universal property for quotients (§5.3) to establish that there are functions $(A \times B)/{\sim} \to A/{\sim_A}$, $(A \times B)/{\sim} \to B/{\sim_B}$.
- Prove that $(A \times B)/{\sim}$, with these two functions, satisfies the universal property for the product of $A/{\sim_A}$ and $B/{\sim_B}$.
- Conclude (without further work) that $(A \times B)/{\sim} \;\cong (A/{\sim_A}) \times (B/{\sim_B})$.

Solution. Let $\pi_A : A\times B \to A$ and $\pi_B : A\times B \to B$ be the canonical projections for A and B. Let $\pi^Z_{\sim} : Z \to Z/{\sim}$ be the canonical quotient mapping for all objects Z and equivalence relations $\sim$. Consider the morphism $\varphi_A : A \times B \to A/{\sim_A}$,

$$\varphi_A = \pi^Z_{\sim_A} \circ \pi_A$$

We then use the universal property of quotients to see that there exists a unique morphism $\overline{\varphi}_A : (A\times B)/{\sim} \to A/{\sim_A}$. By analogous means, there exists a unique morphism $\overline{\varphi}_B : (A \times B)/{\sim} \to B/{\sim_B}$.

Now we will show that these morphisms act as natural projections from the product of $A/{\sim_A}$ and $B/{\sim_B}$. Let Z be a set with morphisms $f_A : Z \to A/{\sim_A}$ and $f_B : Z \to B/{\sim_B}$. Then there exists a unique morphism $\sigma : Z \to (A\times B)/{\sim}$ such that the diagram

$$\begin{tikzcd}
 & & A/{\sim_A} \\
Z \arrow[urr, bend left, "f_A"] \arrow[r, "\sigma"] \arrow[drr, bend right, "f_B"'] & (A \times B)/{\sim} \arrow[ur, "\overline{\varphi}_A"'] \arrow[dr, "\overline{\varphi}_B"] & \\
 & & B/{\sim_B}
\end{tikzcd}$$

commutes. Define a function $\tau : Z \to A/{\sim_A} \times B/{\sim_B}, \tau(z) = (f_A(z), f_B(z))$. Note that by the universal property of the quotient there exists a unique function $\overline{1}_A : A/{\sim_A} \to A$, $\overline{1}_A([a]_{\sim_A}) = a$. We define a similar function $\overline{1}_B$. Then we construct a morphism $\overline{1}_{A\times B} : A/{\sim_A} \times B/{\sim_B} \to A \times B$,

$$\overline{1}_{A\times B}([a]_{\sim_A}, [b]_{\sim_B}) = (\overline{1}_A([a]_{\sim_A}), \overline{1}_B([b]_{\sim_B}))$$

We now finally define $\sigma = \pi_{\sim}^{A\times B} \circ \overline{1}_{A\times B} \circ \tau$. Then we have

$$\begin{aligned}
\overline{\varphi}_A \circ \sigma(z) &= \overline{\varphi}_A \circ \pi_{\sim}^{A\times B}(\overline{1}_{A\times B}(f_A(z), f_B(z))) \\
&= \overline{\varphi}_A(f_A(z), f_B(z)) \\
&= f_A(z)
\end{aligned}$$

Similarly, $\overline{\varphi}_B \circ \sigma(z) = f_B(z)$. Thus, $(A \times B)/{\sim}$ satisfies the universal property for the product of $A/{\sim_A}$ and $B/{\sim_B}$. Therefore, $(A \times B)/{\sim} \cong A/{\sim_A} \times B/{\sim_B}$. $\square$

Problem I.5.12. Define the notions of *fibered products* and *fibered coproducts*, as terminal objects of the categories $\mathsf{C}_{\alpha,\beta}$, $\mathsf{C}^{\alpha,\beta}$ considered in Example 3.10 (cf. also Exercise 3.11), by stating carefullly the corresponding universal properties. As it happens, Set has both fibered products and coproducts. Define these objects 'concretely', in terms of naive set theory.

Solution. Recall that given two morphisms $\alpha : A \to C$ and $\beta : B \to C$, the category $\mathsf{C}_{\alpha,\beta}$ is defined as follows:

- $\mathrm{Obj}(\mathsf{C}_{\alpha,\beta})$ = commutative diagrams

$$\begin{tikzcd}
 & A \arrow[dr, "\alpha"] & \\
Z \arrow[ur, "f"] \arrow[dr, "g"'] & & C \\
 & B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

  where Z is an object in C

- Morphisms between objects (Z_1, f_1, g_1) and (Z_2, f_2, g_2) are commutative diagrams

$$\begin{tikzcd}
 & & A \arrow[dr, "\alpha"] & \\
Z_1 \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "g_1"'] \arrow[r, "\sigma"] & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & & C \\
 & & B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

  That is, we have a morphism $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$ such that the diagram commutes.

The fibered product $A \times_C B$ is a final object in this category. In other words, for every object Z with morphisms $f : Z \to A$ and $g : Z \to B$ where $\alpha \circ f = \beta \circ g$, there exists a unique morphism $\sigma : Z \to A \times_C B$ such that the diagram

$$\begin{tikzcd}
 & & A \arrow[dr, "\alpha"] & \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times_C B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & & C \\
 & & B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

commutes.

We claim that the fibered product in Set is defined as

$$A \times_C B = \{(a, b) \mid \alpha(a) = \beta(b)\}$$

with the natural projections π_A and π_B. Let Z be an arbitrary object with appropriate morphisms f and g. Define $\sigma : Z \to A \times_C B$ as $\sigma(z) = (f_A(z), f_B(z))$. Then we have $\pi_A \circ \sigma = f$ and $\pi_B \circ \sigma = g$. Combined with the condition that $\alpha \circ f = \beta \circ g$, it becomes clear that these definitions make the diagram commute.

We define the fibered coproduct analogously. Recall that given morphisms $\alpha : C \to A$ and $\beta : C \to B$, the category $\mathsf{C}^{\alpha,\beta}$ is defined as:

- $\mathrm{Obj}(\mathsf{C}^{\alpha,\beta})$ = commutative diagrams

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \\
 & B \arrow[ur, "g"'] &
\end{tikzcd}$$

  where Z is an object in C

- Morphisms between objects (Z_1, f_1, g_1) and (Z_2, f_2, g_2) are commutative diagrams

$$\begin{tikzcd}
 & A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \\
 & B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}$$

  That is, we have a morphism $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$ such that the diagram commutes.

The fibered coproduct $A \coprod_C B$ is initial in this category. Thus, for every object Z with morphisms $f : A \to Z$ and $g : B \to Z$ where $f \circ \alpha = g \circ \beta$, the diagram

$$\begin{tikzcd}
 & A \arrow[dr, "i_A"] \arrow[drr, bend left, "f"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & A \coprod_C B \arrow[r, "\sigma"] & Z \\
 & B \arrow[ur, "i_B"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

commutes.

To construct the fibered coproduct $A \coprod_C B$ in Set, first consider the disjoint union $(\{0\} \times A) \cup (\{1\} \times B)$. We define an equivalence relation $\sim$ on this set, setting

$$\begin{aligned}
&(0, a) \sim (0, a') \iff a = a', \\
&(1, b) \sim (1, b') \iff b = b', \\
&(0, a) \sim (1, b) \iff \exists c \in C : \alpha(c) = a \text{ and } \beta(c) = b
\end{aligned}$$

Interestingly, note that equivalence classes have at most 2 elements.

We claim that $A \coprod_C B/\sim$ is a fibered coproduct in Set with the maps $i_A(a) = [(0, a)]_\sim$ and $i_B(b) = [(1, b)]_\sim$. Let Z be a set with functions $f : A \to Z$ and $g : B \to Z$ such that $f \circ \alpha = g \circ \beta$. By the universal property of the coproduct, there is a unique morphism $\sigma' : A \coprod B \to Z$. Now we use the universal property of the quotient to construct a unique function $\sigma : A \coprod B/\sim \to Z$. We can verify that

$$\sigma \circ i_A(a) = \sigma([(0, a)]_\sim) = \sigma'(0, a) = f(a)$$

Similarly, we have $\sigma \circ i_B(b) = g(b)$. Combined with the condition that $f \circ \alpha = g \circ \beta$, it becomes clear that the diagram commutes. $\square$