.1 Functions between sets

Problem .1.1. How many different bijections are there between a set S with n elements and itself?

Solution. A function $f : S \to S$ is a subset $\Gamma_f \subseteq S \times S$. Since f is bijective, then for all $y \in S$, there exists a unique $x \in S$ such that $(x, y) \in \Gamma_f$. Certainly $|\Gamma_f| = n$. Since each x is unique, every element $x \in S$ must be present in the first component of exactly one element in Γ_f. Similarly, each element $y \in S$ must be present in the second component of exactly one element in Γ_f. Then each bijection is merely a permutation of S, and there are $n!$ permutations. Thus, there are $n!$ bijections from S to itself. □

Problem .1.2. Prove statement (2) in Proposition 2.1. You may assume that given a family of disjoint subsets of a set, there is a way to choose one element in each member of the family.

Proposition 2.1. *Assume $A \neq \emptyset$, and let $f : A \to B$ be a function. Then (1) f has a left-inverse if and only if f is injective; and (2) f has a right-inverse if and only if f is surjective.*

Solution. Assume $A \neq \emptyset$ and let $f : A \to B$ be a function.

($\Longrightarrow$) Suppose there exists a function g that is a right-inverse of f. Then $f \circ g = \mathrm{id}_B$. Let $b \in B$. Then $g(b) \in A$ and $f(g(b)) = b$. Thus for all $b \in B$, there exists $a = g(b)$ such that $f(a) = b$. Hence, f is surjective.

($\Longleftarrow$) Suppose that f is surjective. We want a function $g : B \to A$ such that $f(g(b)) = b$ for all $b \in B$. Since f is surjective, for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Construct a set $\Gamma = \{(b, a) \mid f(a) = b\} \subseteq B \times A$. Note that Γ is not necessarily unique since there may be several a such that $f(a) = b$. However, its existence is guaranteed since f is surjective. Then this set may be used to define g where $g(b) = a$ if and only if $(a, b) \in \Gamma$. Now let $b \in B$. Then there exists an $a \in A$ such that $f(a) = b$. Therefore, $(a, b) \in \Gamma$ so $g(b) = a$. We get that $f(g(b)) = f(a) = b$ so g is a right-inverse of f. □

Problem .1.3. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is bijection.

Solution. Let $f : A \to B$ be a bijection. Consider $f^{-1} : B \to A$. We have that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. Then f is the left- and right-inverse of f^{-1}, so f^{-1} is also a bijection.

Let $f : A \to B$ and $g : B \to C$ be bijections and consider $g \circ f$. Suppose $a, a' \in A$ such that $(g \circ f)(a) = (g \circ f)(a')$. Since g is bijective, and in particular it is injective, we have $(g \circ f)(a) = (g \circ f)(a') \Longrightarrow f(a) = f(a')$. Similarly, f is injective so $f(a) = f(a') \Longrightarrow a = a'$. Thus, $g \circ f$ is injective. Now let $c \in C$.

Since g is surjective, there exists a $b \in B$ such that $g(b) = c$. Similarly, since f is surjective, there exists an $a \in A$ such that $f(a) = b$. Then $(g \circ f)(a) = g(b) = c$ so $g \circ f$ is surjective. Hence, $g \circ f$ is bijective. $\square$

Problem .1.4. Prove that 'isomorphism' is an equivalence relation (on any set of sets).

Solution. Let A be a set. Then id_A is a bijection so $A \cong A$. Let B be another set such that $A \cong B$. That is, there exists a bijection $f : A \to B$. Since f is bijective, it has an inverse $f^{-1} : B \to A$, so $B \cong A$. If C is another set such that $B \cong C$, then there exists a bijection $g : B \to C$. The composition of bijections is a bijection so $g \circ f : A \to C$ is bijective. Hence $A \cong C$ and $\cong$ is an equivalence relation. $\square$

Problem .1.5. Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism* seen in §2.6, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections.

Proposition 2.3. *A function is injective if and only if it is a monomorphism.*

Solution. A function $f : A \to B$ is an epimorphism if for all sets Z and all functions $\beta, \beta' : B \to Z$ we have $\beta \circ f = \beta' \circ f \implies \beta = \beta'$. Now we show that a function is surjective if and only if it is an epimorphism.

($\Longrightarrow$) Suppose that $f : A \to B$ is surjective. Then f has a right-inverse $g : B \to A$. Let β, β' be functions from B to another set Z such that $\beta \circ f = \beta' \circ f$. Compose on the right by g and use associativity of composition:

$$\beta \circ (f \circ g) = (\beta \circ f) \circ g = (\beta' \circ f) \circ g = \beta' \circ (f \circ g)$$

Since g is a right-inverse of f, we have

$$\beta \circ \mathrm{id}_B = \beta' \circ \mathrm{id}_B$$

and thus $\beta = \beta'$ and f is an epimorphism.

($\Longleftarrow$) Now suppose that $f : A \to B$ is an epimorphism. Let $Z = \{0, 1\}$ and consider the morphisms $\beta, \beta' : B \to Z$ where $\beta(b) = 0$ for all $b \in B$ and $\beta'(b) = 0$ if $b \in \mathrm{im}(f)$ or $\beta'(b) = 1$ otherwise. By construction, $\beta \circ f = \beta' \circ f$. This implies that $\beta = \beta'$, which is only the case if every element $b \in B$ is sent to the same element of Z. β sends every element of B to 0, and β' sends every element of $\mathrm{im}(f)$ to 0, so $\mathrm{im}(f) = B$ and f is surjective. $\square$

Problem .1.6. With notation as in Example 2.4, explain how any function $f : A \to B$ determines a section of π_A.

Solution. We know f corresponds to a subset $\Gamma_f = \{(a,b) \mid f(a) = b\} \subseteq A \times B$. The projection $\pi_A : A \times B \to A$ is defined such that $\pi_A(a,b) = a$. Let $g : A \to A \times B$ be a function such that $g(a) = (a, f(a)) \in \Gamma_f$. Since $(\pi_A \circ g)(a) = \pi_A(a, f(a)) = a$ for all $a \in A$, g is a section of π_A which is determined by f. $\square$

Problem .1.7. Let $f : A \to B$ be any function. Prove that the graph Γ_f of f is isomorphic to A.

Solution. Recall that $\Gamma_f = \{(a,b) \mid b = f(a)\} \subseteq A \times B$. Let $g : A \to \Gamma_f$ be defined as $g(a) = (a, f(a))$. For all $(a,b) \in \Gamma_f$, we have $g(a) = (a, f(a)) = (a,b)$ so g is surjective. If $g(a) = g(a')$, then $(a, f(a)) = (a', f(a'))$. That is, $a = a'$ so g is injective, hence it is a bijection. Therefore, $\Gamma_f \cong A$. $\square$

Problem .1.8. Describe as explicitly as you can all terms in the canonical decomposition of the function $\mathbb{R} \to \mathbb{C}$ defined by $r \mapsto e^{2\pi i r}$. (This exercise matches one assigned previously. Which one?)

Solution. Let $f : \mathbb{R} \to \mathbb{C}$ be the function defined above. The first part of the decomposition is defined by letting $\sim$ be an equivalence relation on $\mathbb{R}$ such that $a \sim b \iff f(a) = f(b)$. That is, $[a]_\sim$ is the set of elements in $\mathbb{R}$ that are mapped to the same element as a in $\mathbb{C}$. Then we have a projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\sim$ which sends each element $a \in \mathbb{R}$ to its equivalence class $[a]_\sim$. Note that $f(x) = f(x+1)$. That is, the function is periodic about the integers so real numbers which differ by an integer amount belong to the same equivalence class. Then $\mathbb{R}/\sim = \{\{r + k \mid k \in \mathbb{Z}\} \mid r \in [0,1)$ which is identical to the quotient set in Exercise 1.1.6.

The function $\tilde{f} : \mathbb{R} \to \mathrm{im}(f)$ maps each equivalence class to the complex number that f maps the representative to. Certainly if $\tilde{f}([a]_\sim) = \tilde{f}([a']_\sim)$ then $f(a) = f(a')$ and $a \sim a'$ by definition. Thus $[a]_\sim = [a']_\sim$ so $\tilde{f}$ is injective. Similarly, let $b \in \mathrm{im}(f)$. Then there is an element $a \in \mathbb{R}$ such that $f(a) = b$. Then $\tilde{f}([a]_\sim) = f(a) = b$ so $\tilde{f}$ is surjective and hence a bijection. Finally, we have the inclusion $\mathrm{im}(f) \hookrightarrow \mathbb{C}$ which embeds the image of f into its codomain. $\square$

Problem .1.9. Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \coprod B$ is well-defined *up to isomorphism*.

Solution. There exist bijections $f : A' \to A''$ and $g : B' \to B''$. Then we can define $h : A' \cup B' \to A'' \cup B''$ where

$$h(x) = \begin{cases} f(x) \text{ if } x \in A' \\ g(x) \text{ if } x \in B' \end{cases}$$

Let $y \in A'' \cup B''$. Since $A'' \cap B'' = \emptyset$, we have either $y \in A''$ or $y \in B''$. WLOG, suppose that $y \in A''$. Note that since f is surjective, there exists $x \in A'$ such that $f(x) = y$. Then $h(x) = f(x) = y$ so h is surjective. Suppose $x \neq x'$ for $x, x' \in A' \cup B'$. If $x, x' \in A'$ then since f is injective and $h(x) = f(x)$ for all $x \in A'$, we have $h(x) \neq h(x')$. A similar reasoning shows that if $x, x' \in B'$, then $h(x) \neq h(x')$. WLOG, suppose that $x \in A'$ and $x' \in B'$. Then $h(x) = f(x) \neq g(x') = h(x')$ since $A'' \cap B'' = \emptyset$. Thus h is surjective and hence a bijection, showing that $A' \cup B' \cong A'' \cup B''$.

The constructions of A', A'', B', B'' are equivalent to creating "copies" of sets A and B to use in the disjoint union. Thus, the disjoint union $A \coprod B$ is well-defined up to isomorphism. □

Problem .1.10. Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$.

Solution. Recall that $|B^A|$ is the number of functions from A to B. Each functions assigns a single element of A to a single element of B. There are $|B|$ choices for each of the $|A|$ elements. This is equivalent to $|B|^{|A|}$ total choices. Thus, $|B^A| = |B|^{|A|}$. □

Problem .1.11. In view of Exercise 2.10, it is not unreasonable to use 2^A to denote the set of functions from an arbitrary set A to a set with 2 elements (say $\{0, 1\}$). Prove that there is a bijection between 2^A and the *power set* of A.

Solution. Consider $f : \mathscr{P}(A) \to 2^A$ defined as

$$f(X) = \{(a, 1) \text{ if } a \in X, \text{ and } (a, 0) \text{ otherwise}\}$$

Let $g \in 2^A$. Then g is a function from A to $\{0, 1\}$. Let $A_1 = \{a \in A \mid g(a) = 1$. Then $A_1 \in \mathscr{P}(A)$ and $f(A_1) = g$, so f is surjective. Now suppose that $X, Y \subseteq A$ such that $f(X) = f(Y)$. That is, for all $a \in A$, $a \in X \iff (a, 1) \in f(X) \iff (a, 1) \in f(Y) \iff a \in Y$. Thus, $X = Y$ so f is injective and a bijection. Therefore, $2^A \cong \mathscr{P}(A)$. □