.1 Algebraic closure, Nullstellensatz, and a little algebraic geometry

Exercise .1.1. Prove Lemma 2.1.

Lemma 2.1. *For a field K, the following are equivalent:*

- *K is algebraically closed.*
- *K has no nontrivial algebraic extensions.*
- *If $K \subseteq L$ is any extension and $\alpha \in L$ is algebraic over K, then $\alpha \in K$.*

Solution. First we show that the first point implies the third point. Suppose K is algebraically closed and let $K \subseteq L$ be an extension and consider $\alpha \in L$ to be algebraic over K. That is, there is a polynomial $p(x) \in K[x]$ such that $p(\alpha) = 0$. But since K is algebraically closed, p factors as a product of linear terms. Then $p(\alpha) = 0$ implies that p has a factor of $(x - \alpha)$, which in turn implies that $\alpha \in K$.

To see that the third point implies the second, let L be an algebraic extension of K. That is, every $\alpha \in L$ is algebraic over K, so $\alpha \in K$. But then $L \subseteq K$, and obviously $K \subseteq L$, hence the two are equal and K has no nontrivial algebraic extensions.

Finally, we show that the second point implies the first by proving the contrapositive. Suppose K is not algebraically closed. That is, there is an irreducible polynomial which is not linear, say $p(x)$. Then $K[x]/(p(x)) = L$ is a nontrivial algebraic extension of K (since the coset of x is a root of p in L). Therefore, if K has no nontrivial algebraic extensions, it must be algebraically closed. □

Exercise .1.2. Let $k \subseteq \bar{k}$ be an algebraic closure, and let L be an intermediate field. Assume that every polynomial $f(x) \in k[x] \subseteq L[x]$ factors as a product of linear terms in $L[x]$. Prove that $L = \bar{k}$.

Solution. Certainly $L \subseteq \bar{k}$ so we only need to show the reverse inclusion. Let $\alpha \in \bar{k}$. Then α is algebraic over k, so there is a polynomial $p(x) \in k[x]$ such that $p(\alpha) = 0$. Then $p(x)$ factors as a product of linear terms in $L[x]$, so $p(x)$ contains a factor of $(x - \alpha)$ in $L[x]$. That is, $\alpha \in L$, proving that $L = \bar{k}$. □

Exercise .1.3. Prove that if k is a countable field, then so is $\bar{k}$.

Solution. Recall that $\bar{k}$ may be considered as the field containing all elements which are algebraic over k. Then there is a surjection from $F[x] \times \mathbb{N}$ onto $\bar{k}$ which sends a polynomial to its roots (the second component counts the number of roots, where excess points are sent to 0). The product of countable sets is countable (simply consider a table formed from the product and count diagonally), so it suffices to show that $F[x]$ is countable. A polynomial of degree n in $F[x]$ can be identified with a sequence of length n with elements in F. Thus, $F[x]$ is equal to the countable union $F \cup F^2 \cup F^3 \cup \cdots$. We know each F^k is countable since the cartesian product of countable sets is countable. Finally, a countable union of countable sets is countable. Simply form a table where a_{ij} is the j-th element of the set S_i. Thus, $F[x]$ is countable, so $F[x] \times \mathbb{N}$ is countable, so $\bar{k}$ is countable. $\square$

Exercise .1.4. Let k be a field, let $c_1, \ldots, c_m \in k$ be distinct elements, and let $\lambda_1, \ldots, \lambda_m$ be nonzero elements of k. Prove that

$$\frac{\lambda_1}{x - c_1} + \cdots + \frac{\lambda_1}{x - c_m} \neq 0.$$

(This fact is used in the proof of Theorem 2.9.)

Solution. The left side can be rewritten with a common denominator as

$$\frac{\lambda_1(x - c_2)\cdots(x - c_m) + \cdots + \lambda_m(x - c_1)\cdots(x - c_{m-1})}{(x - c_1)\cdots(x - c_m)}.$$

Let $f(x)$ be the numerator and $g(x)$ be the denominator. It suffices to show that $f(x) \neq 0$ (I think?) Certainly $f(c_1) = \lambda_1(c_1 - c_2)\cdots(c_1 - c_m)$ and this must be nonzero since all of the c_i are distinct and the λ_i are nonzero. $\square$

Exercise .1.5. Let K be a field, let A be a subset of $K[x_1, \ldots, x_n]$ and let I be the ideal generated by A. Prove that $\mathscr{V}(A) = \mathscr{V}(I)$ in $\mathbb{A}^n_K$.

Solution. It is obvious that $\mathscr{V}(I) \subseteq \mathscr{V}(A)$. Now let $p \in \mathscr{V}(A)$. Let $f, g \in A$. Then $(f + g)(p) = f(p) + g(p) = 0$. Similarly, let $h \in K[x_1, \ldots, x_n]$. Then $(h \cdot f)(p) = h(p) \cdot f(p) = 0$. Thus, $p \in \mathscr{V}(I)$, showing the reverse inclusion. $\square$

Exercise .1.6. Let K be your favorite infinite field. Find examples of subsets $S \subseteq \mathbb{A}^n_K$ which cannot be realized as $V(I)$ for any ideal $I \subseteq K[x_1, \ldots, x_n]$. Prove that if K is a finite field, then every subset $S \subseteq \mathbb{A}^n_K$ equals $V(I)$ for some ideal $I \subseteq K[x_1, \ldots, x_n]$.

Solution. Consider the set $S = \mathbb{A}^2_K - \{(0, 0\}$. Suppose this can be realized as $V(I)$ for some ideal $I \subseteq K[x, y]$. That is, for all $f(x) \in I$, we have $f(p) = 0$ for all $p \in S$. But since S is infinite, $f(x) = 0$ for infinitely many points but has finite degree. Therefore, we must have $f(x) = 0$. However, then $f(x) \notin I$ since $f(0, 0) = 0$ and $0 \notin S$.

Now suppose K is a finite field and let $S \subseteq \mathbb{A}^n_K$. Note that S must be finite. Consider the polynomial

$$f(x_1, \ldots, x_n) = (x - p_1)(x - p_2)\cdots(x - p_r)$$

for all $p_i \in S$. Let I be the ideal generated by f. We claim that $S = V(I)$. Certainly if $p \in V(I)$, then $f(p) = 0$, so $p = p_i$ for some $p_i \in S$. That is, $V(I) \subseteq S$. The reverse inclusion follows from Exercise 2.5. Thus, the two are equal. □

Exercise .1.7. Let K be a field and n a nonnegative integer. Prove that the set of algebraic subsets of $\mathbb{A}^n_K$ is the family of closed sets of a topology on $\mathbb{A}^n_K$.

Solution. We simply show that $\mathbb{A}^n_K$ is a topological space with the closed sets defined as $V(I)$ for some ideal $I \subseteq K[x_1, \ldots, x_n]$. Indeed, $\emptyset = V(1)$ and $\mathbb{A}^n_K = V(0)$ so the empty set and the whole set are closed.

If $V(I)$ and $V(J)$ are closed, then $V(I) \cup V(J) = V(IJ)$. Indeed, if $p \in V(I) \cup V(J)$, then $f(p) = 0$ for all $f \in I$ or $g(p) = 0$ for all $g \in J$, so $(f_1 g_1 + \cdots f_k g_k)(p) = 0$. For the other other direction, suppose $f(p)g(p) = 0$ for all $f \in I$ and $g \in J$. If $f(p) = 0$ for all $f \in I$, then $p \in V(I)$ by definition. Suppose now that there exists some $f_0 \in I$ such that $f_0(p) \neq 0$. Then for any $g \in J$, since we assumed that $f_0(p)g(p) = 0$, it must be the case that $g(p) = 0$. Thus, $p \in V(J)$, so $p \in V(I) \cup V(J)$. That is, the union of two closed sets is closed.

Finally, if $V(I)$ and $V(J)$ are closed, then $V(I) \cap V(J) = V(I + J)$. Indeed, if $p \in V(I) \cap V(J)$, then $f(p) = 0$ for all $f \in I$ and $g(p) = 0$ for all $g \in J$. Then $f(p)+g(p) = 0$ so $p \in V(I+J)$. For the other direction, suppose $f(p)+g(p) = 0$ for all $f \in I, g \in J$. In particular, we have $f(p) = 0$ since $0 \in J$. Similarly, we have $g(p) = 0$ for all $g \in J$. Thus, $p \in V(I) \cap V(J)$ and the two sets are equal. That is, the intersection of two closed sets is closed.

This is the Zariski topology on $\mathbb{A}^n_K$. □

Exercise .1.8. With notation as in Definition 2.13:

- Prove that the set $\sqrt{I}$ is an ideal of R.
- Prove that $\sqrt{I}$ corresponds to the nilradical of R/I via the correspondence between ideals of R/I and ideals of R containing I.
- Prove that $\sqrt{I}$ is in fact the intersection of all prime ideals of R containing I.
- Prove that I is radical if and only if R/I is reduced.

Solution. Recall that the set $\sqrt{I} = \{r \in R \mid \exists k \geq 0, r^k \in I\}$. Certainly $0 \in \sqrt{I}$ as $0^1 = 0 \in I$. Suppose $a, b \in \sqrt{I}$. That is, $a^k \in I$ and $b^l \in I$ for some $k, l \geq 0$. Then $(a+b)^{k+l}$ can be expanded via the binomial theorem, and each term in the expansion contains a factor of either a^k or b^l. Thus, each of the terms in the expansion is in I so the entire expression is in I, showing that $a + b \in \sqrt{I}$. Now suppose $a \in \sqrt{I}$. That is, $a^k \in I$ for some $k \geq 0$. Then for all $r \in R$, we have $(ra)^k = r^k \cdot a^k \in I$ since $a^k \in I$. Thus, $ra \in \sqrt{I}$, proving that $\sqrt{I}$ is an ideal.

By the third isomorphism theorem, the ideals of R/I are in bijection with the ideals of R containing I. Since $\sqrt{I}$ is an ideal of R which contains I, it corresponds to an ideal of R/I. In particular, it corresponds to the ideal containing the elements $x + I \in R/I$ such that $(x+I)^n = x^n + I = I$, which is precisely the nilradical of R/I.

Let $x \in \sqrt{I}$. That is, $x^k \in I$ for some $k \geq 0$. Now let $\mathfrak{p}$ be a prime ideal of R containing I. Then $x^k \in \mathfrak{p}$. But then $x \cdot x^{k-1} \in \mathfrak{p}$, so either $x \in \mathfrak{p}$ or $x^{k-1} \in \mathfrak{p}$. In the first case we are done; otherwise we repeat inductively, showing that $x \in \mathfrak{p}$. Thus, $\sqrt{I}$ is in the intersection of all prime ideals containing I. For the other direction, suppose $x \notin \sqrt{I}$. That is, $x^k \notin I$ for any $k \geq 0$. Then the set $S = \{1, x, x^2, x^3, \ldots\}$ is multiplicatively closed and the localization $(R/I)_S$ is nonzero. Thus, it contains a prime ideal and the preimage of this prime ideal in R/I is a prime ideal of R/I which does not contain x. Hence, the preimage of this is a prime ideal of R containing I, which does not contain x, so x is not in the intersection of prime ideals of R containing I.

Finally, suppose I is radical. That is, $I = \sqrt{I}$. In particular, since $\sqrt{I}$ is the preimage of the nilradical in R/I, the niradical is trivial in R/I. Then, R/I has no nilpotents. The other direction works exactly the same way. □

Exercise .1.9. Prove that every affine algebraic set equals $\mathscr{V}(I)$ for a *radical* ideal I.

Solution. Let $S \subseteq \mathbb{A}^n_K$ be an affine algebraic set. We know $\mathscr{I}(S)$ is radical, so let $I = \mathscr{I}(S)$. We claim that $S = \mathscr{V}(I)$. Suppose $p \in S$. Then $f(p) = 0$ for all $f \in I$, so $p \in \mathscr{V}(I)$, showing that $S \subseteq \mathscr{V}(I)$. Now suppose $p \notin S$. That is, there exists some $f \in I$ such that $f(p) \neq 0$. Then $p \notin \mathscr{V}(I)$, hence $\mathscr{V}(I) \subseteq S$ and the two sets are equal. That is, $S = \mathscr{V}(I)$ for a radical ideal I of $K[x_1, \ldots, x_n]$. □

Exercise .1.10. Prove that every ideal in a Noetherian ring contains a power of its radical.

Solution. Let I be an ideal in a Noetherian ring R. Since $\sqrt{I}$ is an ideal of R, it is finitely generated: say $\sqrt{I} = (a_1, \ldots, a_n)$. Since $a_i \in \sqrt{I}$, there exists some $k_i \geq 0$ such that $a_i^{k_i} \in I$. Let $K = k_1 + k_2 + \cdots + k_n$. Note that $\sqrt{I}^K$ is generated by elements of the form $a_1^{r_1} \cdots a_n^{r_n}$ where $r_1 + r_2 + \cdots + r_n = K$. Then we must have $r_i \geq k_i$ for some i, or else we contradict the fact that the sum of the r_i is K. That is, each element $a_1^{r_1} \cdots a_n^{r_n} \in I$, so $\sqrt{I}^K \subseteq I$. □

Exercise .1.11. Assume a field is *not* algebraically closed. Find a reduced finite-type K-algebra which is not the coordinate ring of any affine algebraic set.

Solution. Since K is not algebraically closed, there exists a nonlinear irreducible polynomial $p(x) \in K[x]$ which has a root not contained in K. Consider $L = K[x]/(p(x))$. Certainly L is a finite-type K-algebra. To see that it is reduced, suppose note that since $p(x)$ is irreducible, the ideal $(p(x))$ is prime, hence radical, hence L is reduced. Finally, L is not the coordinate ring of an algebraic set $S \subseteq \mathbb{A}^1_K$. Indeed, suppose $(p(x)) = \mathscr{I}(S)$ for some algebraic set S. In particular, $p(\alpha) = 0$ for all $\alpha \in S$. But then $p(x)$ factors into linear terms in $K[x]$, contradicting our choice of $p(x)$. $\square$

Exercise .1.12. Let K be an infinite field. A *polynomial function* on an affine algebraic set $S \subseteq \mathbb{A}^n_K$ is the restriction to S of (the evaluation function of) a polynomial $f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$. Polynomial function on an algebraic S manifestly form a ring and in fact a K-algebra. Prove that this K-algebra is isomorphic to the coordinate ring of S.

Solution. Consider the map sending a polynomial $f \in K[x_1, \dots, x_n]$ to $f|_S$. Since the codomain is a ring under pointwise addition and multiplication, this becomes a surjective ring homomorphism. The kernel of this map consists of the set of polynomials which vanish when restricted to S. In particular, these maps are precisely the ideal $\mathscr{I}(S)$. Thus, we find

$$K[S] \cong \frac{K[x_1, \dots, x_n]}{\mathscr{I}(S)}$$

where $K[S]$ denotes the ring of polynomials on S. $\square$

Exercise .1.13. Let K be an algebraically closed field. Prove that every reduced commutative K-algebra of finite type is the coordinate ring of an algebraic set S in some affine space $\mathbb{A}^n_K$.

Solution. Let L be a reduced commutative K-algebra of finite type. That is, there is a surjection $\pi : K[x_1, \dots, x_n] \to L$ so

$$L \cong \frac{K[x_1, \dots, x_n]}{\ker(\pi)}.$$

Furthermore, since L is reduced, we find $\ker(\pi)$ is radical. Then, since K is algebraically closed, $\mathscr{V}(\ker(\pi))$ is an algebraic subset $S \subseteq \mathbb{A}^n_K$. That is, L is the coordinate ring of S. $\square$

Exercise .1.14. Prove that, over an algebraically closed field K, the points of an algebraic set S correspond to the maximal ideals of the coordinate ring $K[S]$ of S, in such a way that if p corresponds to the maximal ideal $\mathfrak{m}_p$, then the value of the function $f \in K[S]$ at p equals the coset of f in $K[S]/\mathfrak{m}_p \cong K$.

Solution. By the correspondence theorems, the maximal ideals of $K[S]$ are of the form $\mathfrak{m}_p + \mathscr{I}(S)$ for some maximal ideal $\mathfrak{m}_p \subseteq K[x_1, \ldots, x_n]$. Then by the Nullstellensatz, $\mathfrak{m}_p = (x_1 - c_1, \ldots, x_n - c_n)$, so it is natural to consider the map

$$(c_1, \ldots, c_n) \mapsto (\overline{x_1 - c_1}, \ldots, \overline{x_n - c_n})$$

where $\overline{x_i - c_i}$ denotes the image of $x_i - c_i$ in $K[S]$.

Consider the map which sends $f \in K[S]$ to $f(p) \in K$. It is quick to verify that this map is indeed well-defined (since f is technically an equivalence class of polynomials). The kernel of this map is the set of polynomials g on $K[S]$ such that $g(p) = 0$. But $g(p) = 0$ if and only if $g \in \mathfrak{m}_p$. In particular, we have an isomorphism

$$\frac{K[S]}{\mathfrak{m}_p} \cong K$$

via the evaluation map, so $f(p)$ is equal to the coset of f in this quotient. □

Exercise .1.15. Let K be an algebraically closed field. An algebraic subset S of $\mathbb{A}^n_K$ is *irreducible* if it *cannot* be written as the union of two algebraic subsets properly contained in it. Prove that S is irreducible if and only if its ideal $\mathscr{I}(S)$ is prime, if and only if its coordinate ring $K[S]$ is an integral domain.

An irreducible algebraic set is 'all in one piece', like $\mathbb{A}^n_K$ itself, and unlike (for example) $\mathscr{V}(xy)$ in the affine plane $\mathbb{A}^n_K$ with coordinates x, y. Irreducible affine algebraic sets are called (affine algebraic) *varieties*.

Solution. The second and third statements are equivalent by the definition of the coordinate ring and a prime ideal. Thus, we only need to show that the first is equivalent to these. If $\mathscr{I}(S)$ is not prime, then there exist $f, g \notin \mathscr{I}(S)$ such that $fg \in \mathscr{I}(S)$. We claim that

$$S = (S \cap \mathscr{V}(f)) \cup (S \cap \mathscr{V}(g)).$$

The direction $\supseteq$ is clear. Now suppose $p \in S$. Then, since $(fg)(p) = 0$, we must have either $f(p) = 0$ or $g(p) = 0$. WLOG, suppose $f(p) = 0$. Then $p \in S \cap \mathscr{V}(f)$, proving $\subseteq$. Thus, the sets are equal and S is reducible.

Now suppose V is reducible so $V = V_1 \cup V_2$ with $V_i \subset V$. Then $\mathscr{I}(V_i) \supset \mathscr{I}(V)$. That is, there exist $f_i \in \mathscr{I}(V_i) \setminus \mathscr{I}(V)$ such that $f_1 f_2 \in \mathscr{I}(V)$, hence $\mathscr{I}(V)$ is not prime. □

Exercise .1.16. Let K be an algebraically closed field. The field of rational functions $K(x_1, \dots, x_n)$ is the field of fractions of $K[\mathbb{A}^n_K] = K[x_1, \dots, x_n]$; every rational function $\alpha = \frac{F}{G}$ (with $G \neq 0$ and F, G relatively prime) may be viewed as defining a function on the open set $\mathbb{A}^n_K \setminus \mathscr{V}(G)$; we say that α is 'defined' for all points in the complement of $\mathscr{V}(G)$.

Let $G \in K[x_1, \dots, x_n]$ be irreducible. The set of rational functions that are defined in the complement of $\mathscr{V}(G)$ is a subring of $K(x_1, \dots, x_n)$. Prove that this subring may be identified with the *localization* of $K[\mathbb{A}^n_K]$ at the multiplicative set $\{1, G, G^2, G^3, \dots\}$. (Use the Nullstellensatz.)

The same considerations may be carried out for any irreducible algebraic set S, adopting as field of 'rational functions' $K(S)$ the field of fractions of the integral domain $K[S]$.

Solution. Let $S = \mathscr{V}(G)$ so that the ring of rational functions defined in the complement of $\mathscr{V}(G)$ is the field of fractions of $K[S]$. That is, we define $K(S) = \{\frac{f}{g} \mid f, g \in K[S], g \neq 0\}/\sim$ where $\frac{f_1}{g_1} \sim \frac{f_2}{g_2}$ iff $f_1 g_2 - f_2 g_1 = 0 \in K[S]$. Since G is irreducible, and $K[\mathbb{A}^n_K]$ is a UFD, (G) is prime and S is irreducible. Consider now the condition $g \neq 0$ in the ring of rational functions, which implies that g does not vanish anywhere on the complement of S. We claim that the set of all functions which do not vanish in the complement of S is equal to the set $\{1, G, G^2, \dots\}$. The $\supseteq$ direction follows easily. Now suppose g is a function which does not vanish on the complement of S. If g vanishes everywhere on S, then $g = G^k$. If g vanishes on a subset of S, then g must be constant $\neq 0$ because S is irreducible. Thus, the two sets are equal and the identification via localization follows naturally. $\square$

Exercise .1.17. Let K be an algebraically closed field, and let $\mathfrak{m}$ be a maximal ideal of $K[x_1, \dots, x_n]$, corresponding to a point p of $\mathbb{A}^n_K$. A *germ* of a function at p is determined by an open set containing p and a function defined on that open set; in our context (dealing with rational functions and where the open set may be taken to be the complement of a function that does not vanish at p) this is the same information as a rational function defined at p, in the sense of Exercise 2.16.

Show how to identify the ring of germs with the localization $K[\mathbb{A}^n_K]_{\mathfrak{m}}$ (defined in Exercise V.4.11).

As in Exercise 2.16, the same discussion can be carried out for any algebraic set. This is the origin of the name 'localization': localizing the coordinate ring of a variety V at the maximal ideal corresponding to a point p amounts to considering only functions defined in a neighborhood of p, thus studying V 'locally', 'near p'.

Solution. The complement of $\mathfrak{m}$, that is, the set of polynomials which do not vanish at p, is a multiplicatively closed set. Let A denote the ring of germs at p. This is equivalent to the ring of rational functions defined at p, or the set of functions $\frac{f}{g}$ such that $g(p) \neq 0$. The localization $K[\mathbb{A}^n_K]_{\mathfrak{m}}$ is the ring of fractions whose denominators are functions not in $\mathfrak{m}$, or functions which do not vanish at p. There is a natural surjection from $A \to K[\mathbb{A}^n_K]_\mathfrak{m}$. To see that it is an isomorphism, note that a germ f can be identified by its value at p. In particular, if $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ in the localization, then there exists some h which vanishes at p such that $p(f_1g_2 - f_2g_1) = 0$. If this is the case, then $f_1g_2 - f_2g_1$ vanish in a neighborhood U around p, so $\frac{f_1}{g_1} = \frac{g_2}{g_2}$ are equivalent as germs. □

Exercise .1.18. Let K be an algebraically closed field. Consider the two 'curves' $C_1 : y = x^2, C_2 : y^2 = x^3$ in $\mathbb{A}^2_K$ (pictures of the real points of these algebraic sets are shown in Example 2.12).

- Prove that $K[C_1] \cong K[t] = K[\mathbb{A}^1_K]$, while $K[C_2]$ may be identified with the subring $K[t^2, t^3]$ of $K[t]$ consisting of polynomials $a_0+a_2t^2+\cdots+a_dt^d$ with zero t-coefficient. (Note that every polynomial in $K[x, y]$ may be written as $f(x) + g(x)y + h(x, y)(y^2 - x^3)$ for uniquely determined polynomials $f(x), g(x), h(x, y)$.)
- Show that C_1, C_2 are both irreducible.
- Prove that $K[C_1]$ is a UFD, while $K[C_2]$ is not.
- Show that the Krull dimension of both $K[C_1]$ and $K[C_2]$ is 1. (This is why these sets would be called 'curves'. You may use the fact that maximal chains of prime ideals in $K[x, y]$ have length 2.)
- The origin $(0, 0)$ is in both C_1, C_2 and corresponds to the maximal ideals $\mathfrak{m}_1$, resp., $\mathfrak{m}_2$, in $K[C_1]$, resp., $K[C_2]$, generated by the classes of x and y.
- Prove that the localization $K[C_1]_{\mathfrak{m}_1}$ is a DVR. Prove that the localization $K[C_2]_{\mathfrak{m}_2}$ is *not* a DVR. (Note that the relation $y^2 = x^3$ still holds in this ring; prove that $K[C_2]_{\mathfrak{m}_2}$ is not a UFD.)

The fact that a DVR admits a local parameter, that is, a single generator for its maximal ideal, is a good algebraic translation of the fact that a curve such as C_1 has a single, smooth branch through $(0, 0)$. The maximal ideal of $K[C_2]_{\mathfrak{m}_2}$ cannot be generated by just one element, as the reader may verify.

Solution. We find that $\mathscr{V}(C_1) = (y - x^2)$ so

$$K[C_1] = \frac{K[x, y]}{(y - x^2)} \cong K[t]$$

by mapping $x \mapsto t$ and $y \mapsto t^2$. Similarly, we find that $\mathscr{V}(C_2) = (y^2 - x^3)$ so

$$K[C_2] = \frac{K[x, y]}{(y^2 - x^3)} \cong K[t^2, t^3].$$

To see the latter isomorphism, note that every polynomial in $K[x, y]$ may be written as $f(x) + g(x)y + h(x, y)(y^2 - x^3)$ where $\deg(g) < 3$. Consider the map which sends $x \mapsto t^2$ and $y \mapsto t^3$. Clearly t is not in the image of this map, and the kernel of this map is generated by $(t^3)^2 - (t^2)^3 = y^2 - x^3$.

Recall that an algebraic set is irreducible if and only if its ideal $\mathsf{I}(S)$ is prime, if and only if $K[S]$ is an integral domain. Since both $K[C_1]$ and $K[C_2]$ are quotients by prime ideals, they are integral domains.

Note that $K[C_1] = K[t]$ is a Euclidean domain (where the valuation is given by the degree of a polynomial), so it is a UFD. However, we find that in $K[C_2] = K[t^2, t^3]$ we have $(t^2)^3 = (t^3)^2$ so we have two factorizations of t^6, hence the ring is not a UFD.

Since $K[C_1]$ is a PID and is not a field, it must have Krull dimension 1. Similarly, since $K[C_2]$ is not a field, it must have Krull dimension at least 1. Suppose it has dimension $n \geq 2$. Then there is a chain of prime ideals of length $n+1$ in $K[x, y]$. But this implies that the dimension of $K[x, y]$ is at least 3, a contradiction. Thus, the Krull dimension of $K[C_2]$ is 1.

It is easy to see that $(0, 0)$ is in both C_1 and C_2 since $0 = 0^2$ and $0^2 = 0^3$. The corresponding maximal ideals generated by this point are $\mathfrak{m}_1 = (t)$ and $\mathfrak{m}_2 = (t^2, t^3)$.

The localization $K[C_1]_{\mathfrak{m}_1} = \{\frac{f}{g} \mid g \notin (t)\}$. Since $K[t]$ is a UFD and t is irreducible, we can express $f = t^k \cdot h$ for some $k \geq 0$ and $h \in K[t]$ such that $t \nmid h$. Then we define a valuation $v(\frac{f}{g}) = k$ and it easy to check that this is indeed a discrete valuation. On the other hand, the localization $K[C_2]_{\mathfrak{m}_2} = \{\frac{f}{g} \mid g \notin (t^2, t^3)\}$ is not a DVR. Indeed, a DVR must be a UFD, but as noted earlier, $t^6 = (t^2)^3 = (t^3)^2$, both of which are irreducible in the localization. Thus, $K[C_2]_{\mathfrak{m}_2}$ is not a DVR. □

Exercise .1.19. Prove that the fields of rational functions (Exercise 2.16) of the curves C_1 and C_2 of Exercise 2.18 are isomorphic and both have transcendence degree 1 over k (cf. Exercise 1.27).

This is another reason why we should think of C_1 and C_2 as 'curves'. In fact, it can be proven that the Krull dimension of the coordinate ring of a variety equals the transcendence degree of its field of rational functions. This is a consequence of *Noether's normalization theorem*, a cornerstone of commutative algebra.

Solution. Recall that the field of rational functions of the curve $C_1 : y = x^2$ can be identified with the localization of $K[x, y]$ at the set $S = \{1, f, f^2, f^3, \ldots\}$ where $f = y - x^2$. Similarly, the ring of rational functions of the curve C_2 is identified with the localization of $K[x, y]$ at the set $T = \{1, g, g^2, g^3, \ldots\}$ where $g = y^2 - x^3$. We must show that $S^{-1}K[x, y] \cong T^{-1}K[x, y]$. The map

$$\frac{h(x, y)}{f^k} \mapsto \frac{h(x, y)}{g^k}$$

is a ring homomorphism. To do. □

Exercise .1.20. Recall from Exercise VI.2.13 that $\mathbb{P}_K^n$ denotes the 'projective space' parametrizing lines in the vector space K^{n+1}. Every such line consists of multiples of a nonzero vector $(c_0, \ldots, c_n) \in K^{n+1}$, so that $\mathbb{P}_K^n$ may be identified with the quotient of $K^{n+1} \setminus \{(0, \ldots, 0)\}$ by the equivalence relation $\sim$ defined by

$$(c_0, \ldots, c_n) \sim (c'_0, \ldots, c'_n) \iff (\exists \lambda \in K^*), (c'_0, \ldots, c'_n) = (\lambda c_0, \ldots, \lambda c_n).$$

The 'point' in $\mathbb{P}_K^n$ determined by the vector $(c_0, \ldots, c_n)$ is denoted $(c_0 : \ldots : c_n)$; these are the 'projective coordinates' of the point. Note that there is no 'point' $(0 : \ldots : 0)$.

Prove that the function $\mathbb{A}_K^n \to \mathbb{P}_K^n$ defined by

$$(c_1, \ldots, c_n) \mapsto (1 : c_1 : \ldots : c_n)$$

is a bijection. This function is used to realize $\mathbb{A}_K^n$ as a subset of $\mathbb{P}_K^n$. By using similar functions, prove that $\mathbb{P}_K^n$ can be covered with $n + 1$ copies of $\mathbb{A}_K^n$, and relate this fact to the cell decomposition obtained in Exercise VI.2.13. (Suggestion: Work out carefully the case $n = 2$.)

Solution. Note that the given function is merely a function between sets. It is clearly injective because if $(1 : c_1 : \ldots : c_n) = (1 : c'_1 : \ldots : c'_n)$ then the corresponding scalar $\lambda \in K*$ is simply 1, hence $c_1 = c'_1, \ldots, c_n = c'_n$ so the affine coordinates are equal. Similarly, it is easily seen to be surjective since the projective coordinate $(1 : c_1 : \ldots : c_n)$ is mapped to by $(c_1, \ldots, c_n)$. Thus, the function is a bijection.

To cover $\mathbb{P}_K^n$ with $n+1$ copies of $\mathbb{A}_K^n$, simply consider the images of the bijections $f_i : \mathbb{A}_K^n \to \mathbb{P}_K^n$ which sends $(c_1, \ldots, c_n) \mapsto (c_1 : \ldots : 1 \ldots : c_n)$ with a 1 in the i-th coordinate for $0 \leq i \leq n$. By this convention, the above map is f_0. Then every projective point is equal to a point in the image of one of the f_i since not all of the projective coordinates are equal to 0.

Working this out explicitly for $n = 2$, the functions are

$$f_0 : (a, b) \mapsto (1 : a : b)$$
$$f_1 : (a, b) \mapsto (a : 1 : b)$$
$$f_2 : (a, b) \mapsto (a : b : 1)$$

For any projective point $(c_0 : c_1 : c_2)$, it is equivalent to at least one of the following: $(1 : \frac{c_1}{c_0} : \frac{c_2}{c_0})$, $(\frac{c_0}{c_1} : 1 : \frac{c_2}{c_1})$, or $(\frac{c_0}{c_2} : \frac{c_1}{c_2} : 1)$ as at least one of the $c_i \neq 0$. In any case, these are all contained in the image of the f_i.

Recall that the cell decomposition in Exercise VI.2.13 states that $\mathbb{P}_K^{n-1} = k^{n-1} \cup k^{n-2} \cup \cdots k^1 \cup k^0$. The connection is that the map f_n identifies the subset $k^{n-1} \subset \mathbb{P}_K^{n-1}$ since by the realization of projective space as a union, the n-th projective coordinate is 1 for all points in the hyperplane lying above the origin. Then the remaining f_i can be realized as identifying $k^{i-1} \subset \mathbb{P}_K^{i-1}$ inductively. Returning to our explicit example, f_2 identifies $\mathbb{A}_K^2$ with the plane $z = 1$ in K^3.

Then f_1 restricted to points of the form $(a, 0)$ maps to projective coordinates in the plane $z = 0$ where lines are identified with a unique point on the line $y = 1$. Finally, there is a single point identified by the restriction of f_0 to k since all lines in this space pass throught the point $x = 1$. □

Exercise .1.21. Let $F(x_0, \dots, x_n) \in K[x_0, \dots, x_n]$ be a *homogeneous* polynomial. With notation as in Exercise 2.20, prove that the condition '$F(c_0, \dots, c_n) = 0$' for a point $(c_0 : \dots : c_n) \in \mathbb{P}^n_K$ is well-defined: it does not depend on the representative $(c_0, \dots, c_n)$ chosen for the points $(c_0 : \dots : c_n)$. We can then define the following subset of $\mathbb{P}^n_K$:

$$\mathscr{V}(F) := \{(c_0 : \dots : c_n) \in \mathbb{P}^n_K \mid F(c_0, \dots, c_n) = 0\}.$$

Prove that this 'projective algebraic set' can be covered with $n+1$ affine algebraic sets.

The basic definitions in 'projective algebraic geometry' can be developed along essentially the same path taken in this section for affine algebraic geometry, using 'homogeneous ideals' (that is, ideals generated by homogeneous polynomials; see §VIII.4.3) rather than ordinary ideals. This problem shows one way to relate projective and affine algebraic sets, in one template example.

Solution. Recall that a homogeneous polynomial is one whose nonzero terms all have the same degree. Suppose F is homogeneous and let $(c_0 : \dots : c_n) \in \mathbb{P}^n_K$ be such that $F(c_0, \dots, c_n) = 0$. Choose another representative, say $(\lambda c_0, \dots, \lambda c_n)$. Then since F is homogeneous (say of degree i), every nonzero term has a factor λ^i. Thus, we may factor this out, yielding

$$F(\lambda c_0, \dots, \lambda c_n) = \lambda^i F(c_0, \dots, c_n) = 0$$

proving that the choice of representative is irrelevant. Thus, the notion of a 'projective algebraic set' is well-defined.

To cover this set with affine algebraic sets, we effectively use the same process as in the previous exercise. In particular, consider the $n + 1$ polynomials $F_i := F(x_0, \dots, 1, \dots, x_n)$ defined by setting the i-th argument of F to be 1. Letting f_i denote the same injection as in the previous exercise, we find that for a point $(c_0 : \dots : c_n) \in \mathscr{V}(F)$,

$$F(c_0, \dots, c_n) = F\left(\frac{c_0}{c_i}, \dots, 1, \dots, \frac{c_n}{c_i}\right) = F(f_i(c_1, \dots c_n)) = F_i(c_1, \dots, c_n) = 0.$$

Thus, every point of the projective algebraic set is in the image of one of the F_i. □