.1 Hom and duals

Exercise .1.1. Prove that if F is a free R-module of finite rank and N is any R-module, then $\mathrm{Hom}_R(F, N) \cong F^\vee \otimes_R N$.

Solution. We have

$$\mathrm{Hom}_R(R^n, N) \cong \mathrm{Hom}_R(R, N)^n \cong N^n \cong R^n \otimes_R N \cong (R^n)^\vee \otimes_R N.$$

□

Exercise .1.2. Let $\alpha : A \to B$ be a homomorphism of R-modules. Prove that α is an epimorphism if the induced map $\alpha^* : \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N)$ is injective for all R-modules N and α is a monomorphism if α^* is surjective for all N.

Prove that the converse to the first statement holds and the converse to the second statement does not hold. However, show that if α admits a left-inverse, then α^* *is* surjective for all N.

Solution. Suppose the induced map $\alpha^* : \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N)$ is injective for all R-modules N. Let $g_1, g_2 : B \to N$ be morphisms such that $g_1 \circ \alpha(a) = g_2 \circ \alpha(a)$ for all $a \in A$. In particular, $(g_1 \circ \alpha - g_2 \circ \alpha)(a) = (g_1 - g_2) \circ \alpha(a) = 0$. But $(g_1 - g_2) \circ \alpha = \alpha^*(g_1 - g_2)$, and α^* is injective, hence $g_1 - g_2 = 0 \implies g_1 = g_2$.

For the second statement, recall that a module homomorphism is monic if and only if it is injective, i.e. its kernel is trivial. Suppose α^* is surjective for all R-modules N. In particular, there exists $g : B \to A$ such that $1_A = g \circ \alpha$. Let $x \in \ker(\alpha)$. Then $1_A(x) = g \circ \alpha(x) = 0$, hence $x = 0$ so $\ker(\alpha)$ is trivial and α is injective.

The converse to the first statement is clear if one recalls that an epimorphism of modules is also a surjection of modules. In particular, if α is surjective, then we have an exact sequence

$$A \xrightarrow{\alpha} B \longrightarrow 0$$

Applying $\mathrm{Hom}_R(-, N)$ and using its left-exactness yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(B, N) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, N)$$

which implies that α^* is injective since its kernel is trivial.

To see that the converse of the second statement is not true in general, consider the monomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}$, $\alpha(n) = 2n$. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ yields the map $\alpha^* : \mathbb{Z}^\vee \to \mathbb{Z}^\vee$ sending a linear functional $\gamma \mapsto \gamma \circ (\cdot 2)$ which precomposes with multiplication by 2. Clearly this map is not surjective because by linearity, we have $\gamma(2n) = 2\gamma(n)$ for all $n \in \mathbb{Z}$, so in particular, $1_{\mathbb{Z}} \notin \mathrm{im}(\alpha^*)$.

However, suppose that α admits a left-inverse β. Let $f : A \to N$ be a morphism. Consider the precomposition $f \circ \beta : B \to N$. Then

$$\alpha^*(f \circ \beta) = f \circ \beta \circ \alpha = f$$

hence $f \in \operatorname{im}(\alpha^*)$ so α^* is surjective. $\square$

Exercise .1.3. Prove that a sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of R-modules is exact if the induced sequence

$$(*) \quad 0 \longrightarrow \operatorname{Hom}_R(C, N) \xrightarrow{\beta^*} \operatorname{Hom}_R(B, N) \xrightarrow{\alpha^*} \operatorname{Hom}_R(A, N) \longrightarrow 0$$

is exact for all R-modules N. (You have done most of this already, in Exercise 5.2. To show $\ker \beta \subseteq \operatorname{im} \alpha$, choose $N = B/\operatorname{im}(\alpha)$.) Remember that the converse does not hold, since in general $\operatorname{Hom}_R(-, N)$ is not exact. What extra hypothesis on α would guarantee the exactness of (*) for all N?

Solution. By Exercise 5.2, if α^* is surjective (that is, the induced sequence is exact at $\operatorname{Hom}_R(A, N)$) then α is injective (hence the sequence is exact at A). Similarly, if β^* is injective (that is, the induced sequence is exact at $\operatorname{Hom}_R(C, N)$) then β is surjective (hence the sequence is exact at B).

To show that exactness at $\operatorname{Hom}_R(B, N)$ implies exactness at B, choose $N = B/\operatorname{im}(\alpha)$ and let $x \in \ker \beta$. Exactness at $\operatorname{Hom}_R(B, N)$ implies that if $f : B \to N$ is a morphism such that $f \circ \alpha(b) = 0$ for all $b \in B$, then there exists $g : C \to N$ such that $g \circ \beta = f$. In particular, consider $\pi : B \to B/\operatorname{im}(\alpha)$ to be the natural projection. Then $\pi \circ \alpha = 0$, hence there exists $g : C \to B/\operatorname{im}(\alpha)$ such that $g \circ \beta = \pi$. In particular, we have $\pi(x) = g \circ \beta(x) = 0$, hence $x \in \ker \pi = \operatorname{im} \alpha$.

As stated in the problem, this does not hold in general though $\operatorname{Hom}_R(-, N)$ is left-exact. By Exercise 5.2, the extra condition needed is that α admits a left-inverse. $\square$

Exercise .1.4. Let I be an ideal of R. As $I^2 \subseteq I$, there is a natural restriction map $\operatorname{Hom}_R(I, R/I) \to \operatorname{Hom}_R(I^2, R/I)$. Prove that the image of this map is 0. Prove that $\operatorname{Hom}_R(I/I^2, R/I) \cong \operatorname{Hom}_R(I, R/I)$. (This module is important in algebraic geometry, as it carries the information of a 'normal bundle' in good situations.)

Solution. Let $f : I \to R/I$ be a homomorphism of modules. The action of the restriction map sends f to $f \circ i$ where i is the inclusion $I^2 \hookrightarrow I$. Let $x \in I^2$. Then $x = ij$ for some $i, j \in I \subseteq R$. But then $f \circ i(x) = f(ij) = i \cdot f(j) = 0 \in R/I$. Thus, the image of the restriction map is 0.

For the second part, we have the exact sequence

$$0 \longrightarrow I^2 \xrightarrow{i} I \xrightarrow{\pi} I/I^2 \longrightarrow 0$$

which, after applying $\mathrm{Hom}_R(-, R/I)$, yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(I/I^2, R/I) \xrightarrow{\pi^*} \mathrm{Hom}_R(I, R/I) \xrightarrow{i^*} \mathrm{Hom}_R(I^2, R/I)$$

We know the image of i^* is 0, hence $\mathrm{Hom}_R(I/I^2, R/I) = \ker(i^*) = \mathrm{im}(\pi^*)$ so π^* is surjective. By exactness on the left, π^* is injective. Thus, it is an isomorphism. □

Exercise .1.5. Prove that the evaluation map $M^\vee \otimes_R F \to \mathrm{Hom}_R(M, F)$ is an isomorphism if F is free of finite rank, providing an alternative proof of Proposition 5.5.

Solution. Recall that the evaluation map is given by $f \otimes x \mapsto \varphi$ where $\varphi(m) = f(m) \cdot x$. We claim that the diagram

$$\begin{tikzcd} M^\vee \otimes_R R^n \arrow[rr, "\epsilon"] \arrow[dr, "\sim"'] & & \mathrm{Hom}_R(M, R^n) \arrow[dl, "\sim"] \\ & (M^\vee)^n & \end{tikzcd}$$

commutes.

Indeed, let $f \otimes x \in M^\vee \otimes_R R^n$. Tracing one direction yields

$$f \otimes x \mapsto (x_1 \cdot f, \ldots, x_n \cdot f)$$

while tracing the other direction yields

$$f \otimes x \mapsto \varphi \mapsto (\varphi_1, \ldots, \varphi_n)$$

where the latter map sends φ to each of its component morphisms. Indeed, evaluating both at an element $m \in M$ yields

$$(x_1 \cdot f, \ldots, x_n \cdot f)(m) = (x_1 \cdot f(m), \ldots, x_n \cdot f(m)),$$
$$(\varphi_1, \ldots, \varphi_n)(m) = \varphi(m) = f(m) \cdot x = (x_1 \cdot f(m), \ldots, x_n \cdot f(m)).$$

Thus, the diagram commutes, hence ε must be an isomorphism. □

Exercise .1.6. Show that $(\mathbb{Z}/2\mathbb{Z})^\vee = \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$.

Solution. Suppose $\varphi : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is a module homomorphism. In particular, we have $\varphi(0) = \varphi(1 + 1) = \varphi(1) + \varphi(1)$. This implies that $\varphi(1)$ is a zero-divisor in $\mathbb{Z}$, but since $\mathbb{Z}$ is an integral domain, we must have $\varphi(1) = 0$. Thus, φ is the zero morphism, hence $(\mathbb{Z}/2\mathbb{Z})^\vee = 0$. □

Exercise .1.7. Prove 'directly' that $(R^{\oplus S})^\vee \cong R^S$: how does an R-linear map $R^{\oplus S} \to R$ determine a function $S \to R$, and what is the inverse of this correspondence?

Solution. Recall that $R^{\oplus S}$ consists of all sequences (r_s) for $s \in S$ such that $r_s = 0$ for all but finitely many s. Then the datum of an element $\varphi \in (R^{\oplus S})^\vee = \mathrm{Hom}_R(R^{\oplus S}, R)$ consists of a collection of maps such that the restriction $\varphi|_{R_s}$ is R-linear. Thus, this yields an R-linear map $(R^{\oplus S})^\vee \to (R^\vee)^S$ given by $\varphi \mapsto (\varphi|_{R_s})_{s\in S}$. Finally, since $R^\vee \cong R$ by $f \mapsto f(1)$, we have the induced map $(R^{\oplus S})^\vee \to R^S$ which sends $\varphi \mapsto (\varphi|_{R_s}(1))_{s\in S}$.

For the inverse correspondence, given $(r_s)_{s\in S} \in R^S$, consider

$$(\varphi_s)_{s\in S} \in (R^\vee)^s, \quad \varphi_s(1) = r_s$$

where each morphism is extended by linearity. Note that this collection of morphisms may be infinite. However, we may now define a map $\varphi : R^{\oplus S} \to R$ given by

$$\varphi((r'_s)_{s\in S}) = \sum_{s\in S} \varphi_s(r'_s)$$

This map is well-defined because $(r'_s)_{s\in S} \in R^{\oplus S}$, hence there are only finitely many non-zero terms. Furthermore, it is R-linear, hence $\varphi \in (R^{\oplus S})^\vee$. Thus, this yields a map $R^S \to (R^{\oplus S})^\vee$.

Finally, we can check that this is a two-sided inverse to the first map. Indeed, given a map $\varphi \in (R^{\oplus S})^\vee$, the first map sends it to $(\varphi|_{R_s}(1))_{s\in S} \in R^S$. Applying the second map sends this to

$$\psi((r_s)_{s\in S}) = \sum_{s\in S} \varphi\mid_{R_s}(r_s) = \varphi((r_s)_{s\in S})$$

by virtue of the fact that φ is R-linear, hence we can evaluate it as a sum of its restriction to the submodules R_s. On the other hand, given an element $(r_s)_{s\in S} \in R^S$, applying the second map yields a morphism $\varphi : R^{\oplus S} \to R$ where

$$\varphi((r'_s)_{s\in S}) = \sum_{s\in S} \varphi_s(r'_s), \quad \varphi_s(1) = r_s$$

Finally, applying the first map to this morphism sends φ to

$$(\varphi|_{R_s}(1))_{s\in S} = (\varphi_s(1))_{s\in S} = (r_s)_{s\in S}$$

proving that this is in fact an isomorphism $(R^{\oplus S})^\vee \cong R^S$. □

Exercise .1.8. Prove that the datum of an R-linear map $M \to M^\vee$ is equivalent to the datum of an R-bilinear map $M \times M \to R$, and explain why this equivalence can be set up in two ways. If $F = R^n$ is a free R-module of finite rank, determine the bilinear map $F \times F \to R$ corresponding to the isomorphism $F \cong F^\vee$ given in Corollary 5.7.

Solution. Let $\varphi : M \to M^\vee = \mathrm{Hom}_R(M, R)$ be an R-linear map. Then φ determines an R-bilinear map $M \times M \to R$ given by $(m_1, m_2) \mapsto \varphi(m_1)(m_2)$. Indeed, the fact that this is bilinear follows from the fact that φ is linear (hence this map is linear in the first argument) and that elements of $\mathrm{Hom}_R(M, R)$ are linear (hence this map is linear in the second argument). Conversely, let $f : M \times M \to R$ be an R-bilinear map. Then we may construct an R-linear map $\varphi : M \to M^\vee$ given by $\varphi(m) = f(m, -) \in \mathrm{Hom}_R(M, R)$.

Of course, the choice of mapping m to the first argument of f is arbitrary, and we could just as easily send it to the second argument, yielding a different element of $M^\vee$.

In the case where $F = R^n$ is a free R-module of finite rank, the isomorphism $F \cong F^\vee$ is given by $(r_n) \mapsto \varphi$ where

$$\varphi((r'_n)) = \sum \varphi_n(r'_n), \quad \varphi_n(1) = r_n.$$

This determines an R-bilinear map $R^n \times R^n \to R$ which sends

$$((r_n), (s_n)) \mapsto \sum \varphi_n(s_n), \quad \varphi_n(1) = r_n$$

but by the R-linearity of φ_n, this is merely

$$\sum r_n \cdot s_n.$$

Note that this is the inner product in $\mathbb{R}^n$. □

Exercise .1.9. An R-bilinear map $\varphi : M \times M \to R$ is *nondegenerate* if the induced maps $M \to M^\vee$ are injective, and it is *nonsingular* if they are isomorphisms. The notions coincide if M is a finite-dimensional vector space. Prove that the 'standard inner product' in $\mathbb{R}^n$ (defined in Exercise VI.6.18) is nondegenerate.

If M is free of rank n, let $(e_1, \dots, e_n)$ be a basis of M, and let $A = (a_{ij})$ be the matrix with entries $a_{ij} = \varphi(e_i, e_j)$. Prove that φ is nondegenerate if and only if $\det A$ is nonzero, and it is nonsingular if and only if $\det A$ is a unit. (Cf. Proposition VI.6.5.)

Solution. The standard inner product in $\mathbb{R}^n$ is given by

$$\langle v, w \rangle = v^t w.$$

The induced map $\mathbb{R} \to \mathbb{R}^\vee$ is $v \mapsto \langle v, - \rangle$ (or alternatively, $v \mapsto \langle -, v \rangle$). Suppose $\langle v, w \rangle = 0$ for all $w \in \mathbb{R}^n$. In particular, for each elementary basis vector e_i, we have $\langle v, e_i \rangle = v_i = 0$. But then $v = 0$, hence the induced map is injective.

This second part won't be formal because I suck at using the algebraic determinant, but intuition prevails. If $\det A = 0$, then one row is a linear combination of the others. In particular, this implies that fixing the elementary basis vector corresponding to that column, there is a linear combination for the second argument of $\varphi(e_i, -)$ which maps to zero, hence the induced map $M \to M^\vee$ is not injective. On the other hand, if φ is degenerate, then there exists some nonzero v such that $\varphi(v, w) = 0$ for for all $w \in R^n$. Expanding in terms of the basis and applying bilinearity, this implies that the rows of A are linearly dependent, hence $\det A = 0$.

For nonsingularity, my intuition tells me that $\det A$ a unit corresponds to A being invertible, hence there is a dual map $M^\vee \to M$ sending $f \mapsto f(1)$ since this is guaranteed to be an isomorphism, or something along those lines. □

Exercise .1.10. Prove Lemma 5.15.

Lemma 5.15. *Let A be the matrix representing a linear map $\alpha : R^n \to R^m$ with respect to the standard bases. Then the dual map $\alpha^\vee : (R^m)^\vee \to (R^n)^\vee$ is represented by the transpose of A with respect to the corresponding dual bases.*

Solution. Recall that the dual map $\alpha^\vee$ sends a map $f \mapsto f \circ \alpha$. We show that A^t agrees with $\alpha^\vee$ on the dual basis. Let e_i^* denote a dual standard basis vector. Suppose $\alpha^\vee(e_i^*) = \sum_j b_{ij} e_j^*$. In particular, $[b_{ij}]$ is the matrix of $\alpha^\vee$ with respect to the dual basis. It follows that for all i, j we have

$$\begin{aligned} b_{ij} = \sum_k b_{ik}\delta_{kj} = \sum_k b_{ik} e_k^* e_j &= \left(\sum_k b_{ik} e_k^* \right) e_j \\ &= \alpha^\vee(e_i^*) e_j \\ &= (e_i^* \circ \alpha)(e_j) \end{aligned}$$

But then

$$\begin{aligned} e_i^*(\alpha(e_j)) = e_i^* \left(\sum_k a_{jk} e_k \right) &= \sum_k a_{jk} e_i^* e_k \\ &= a_{ji} \end{aligned}$$

Thus, $b_{ij} = a_{ji}$, or in other words, $[b_{ij}] = A^t$. □

Exercise .1.11. Let M be a finitely generated module over a PID of rank r. ‘Compute’ the dual $M^\vee$.

Solution. Recall that a finitely generated module over a PID R has the form

$$M = R^r \oplus (R/r_1)^{m_1} \oplus \cdots \oplus (R/r_n)^{m_n}.$$

Then any R-linear map $M \to R$ must send torsion elements to zero, and clearly every element in $(R/r_i)^{m_i}$ is r_i-torsion. Thus, $M^\vee = (R^r)^\vee \cong R^r$. □

Exercise .1.12. Let M, N be R-modules. Show that there is a canonical bijection

$$\mathrm{Hom}_R(N, M^\vee) \cong \mathrm{Hom}_R(M, N^\vee).$$

Choosing $N = M^\vee$, the left-hand side has a distinguished element, namely the identity $M^\vee \to M^\vee$. Prove that the corresponding element on the right is the map $\omega : M \to M^{\vee\vee}$ defined in §5.5.

Solution. The isomorphism follows from an application of tensor-hom adjunction:

$$\mathrm{Hom}_R(N, M^\vee) \cong \mathrm{Hom}_R(N \otimes M, R) \cong \mathrm{Hom}_R(M \otimes N, R) \cong \mathrm{Hom}_R(M, N^\vee).$$

Under this identification, the identity $1_{M^\vee} : M^\vee \to M^\vee$ is first sent to the map $f \otimes m \mapsto f(m)$ by tracing through adjunction. This is then sent to the map $m \otimes f \mapsto f(m)$. Finally, this is sent to the map taking m to evaluation at m (i.e. $f(m)$ as $f \in M^\vee$). This is precisely the map $\omega : M \to M^{\vee\vee}$. $\square$

Exercise .1.13. Let $F \cong R^n$ be a finite-rank free R-module. Verify that the composition of the (noncanonical) isomorphisms $F \cong F^\vee \cong F^{\vee\vee}$ from Corollary 5.7 is the (canonical isomorphism) ω defined in §5.5.

Solution. Let $\{e_1, \dots, e_n\}$ be a basis for F and let $\{e_1^*, \dots, e_n^*\}$ denote the dual basis for $F^\vee$. On the basis, the noncanonical isomorphism $F \to F^\vee$ sends $e_i \mapsto e_i^*$ where

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Since a basis has already been determined for $F^\vee$, this induces a basis $\{e_1^{**}, \dots, e_n^{**}\}$ for $F^{\vee\vee}$. Then the noncanonical isomorphism sends e_i^* to e_i^{**} where

$$e_i^{**}(e_j^*) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

That is, expanding an element m in terms of the basis and taking it through both isomorphisms yields a map $M^\vee \to R$ which is precisely evaluation of an element of the dual space at m, the canonical isomorphism specified earlier. $\square$

Exercise .1.14. Let F be a free R-module (of any rank). Prove that the canonical map $F \to F^{\vee\vee}$ is injective. What is the simplest example you know of a module M such that $M \to M^{\vee\vee}$ is not injective?

Solution. Suppose $f(m) = 0$ for all $f \in F^\vee$. In particular, letting $\{e_i^*\}$ denote the standard basis of the dual space $F^\vee$, we find that $e_i^*(m) = 0$, hence the i-th component of m is 0 for all i. Thus, $m = 0$ and the canonical map $F \to F^{\vee\vee}$ is injective. The simplest module for which the map is not injective is $\mathbb{Z}/2\mathbb{Z}$, for which $(\mathbb{Z}/2\mathbb{Z})^{\vee\vee} = 0$. $\square$

Exercise .1.15. Let V be a vector space, and let $W \subseteq V$ be a subspace. The *annihilator* of W is

$$W^{\perp} := \{v^* \in V^{\vee} \mid (\forall w \in W), v^*(w) = 0\}.$$

Prove that $W^{\perp}$ is a subspace of $V^{\vee}$. If $\dim V = n$ and $\dim W = r$, prove that $\dim W^{\perp} = n - r$.

Assuming V is finite dimensional, prove that, under the canonical isomorphism $V^{\vee\vee} \cong V$, $W^{\perp\perp}$ maps isomorphically to W.

Solution. To show that $W^{\perp}$ is a subspace of $V^{\vee}$, first note that $0 \in W^{\perp}$. Indeed, for all $w \in W$, $0(w) = 0$. Now suppose $f, g \in W^{\perp}$. Then $(f - g)(w) = f(w) - g(w) = 0$, so $f - g \in W^{\perp}$. Thus, $W^{\perp}$ is a subspace of $V^{\vee}$.

To compute $\dim W^{\perp}$, let $T : V^{\vee} \to W^{\vee}$ be the map which sends $f^* \mapsto f^*|_W$. It is easy to see that this is a linear map and it is surjective since the image of any element $W^{\vee}$ is itself. Furthermore, the kernel of this map is the set of functionals on V which vanish on W, but this is precisely $W^{\perp}$. Then by the rank-nullity theorem, we find that $\dim W^{\perp} = \dim V^{\vee} - \dim W^{\vee} = n - r$.

Recall that the isomorphism $V \cong V^{\vee\vee}$ sends a vector v to evaluation of an element of the dual space at v. Note that

$$W^{\perp\perp} = \{v^{**} \in V^{\vee\vee} \mid (\forall w^* \in W^{\perp}), v^{**}(w^*) = 0\}.$$

Under the canonical isomorphism, a vector $w \in W$ is sent to evaluation at w. Given an element $v^* \in W^{\perp}$, we find that $w^{**}(v^*) = v^*(w) = 0$, hence $w^{**} \in W^{\perp\perp}$. Thus, the restriction of the canonical isomorphism to W induces an isomorphism $W \cong W^{\perp\perp}$. □

Exercise .1.16. Let

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

be an exact sequence of free R-modules. Viewing F_1 as a submodule of F_2 and extrapolating the notation introduced for vector spaces in Exercise 5.15, prove that $F_1^{\perp} \cong F_3^{\vee}$.

Solution. Let $\alpha : F_1 \to F_2$ and $\beta : F_2 \to F_3$ be the maps in the above exact sequence. Dualizing and using the left-exactness of Hom yields the exact sequence

$$0 \longrightarrow F_3^{\vee} \xrightarrow{\beta^{\vee}} F_2^{\vee} \xrightarrow{\alpha^{\vee}} F_1^{\vee}$$

where $\alpha^{\vee}(f) = f \circ \alpha$ and $\beta^{\vee}(g) = g \circ \beta$.

We claim that $\ker \beta^{\vee} = F_1^{\perp}$. Indeed,

$$\ker \beta^{\vee} = \{g \in \mathrm{Hom}(F_2, R) : (g \circ \alpha)(m) = 0, \forall m \in F_1\} = F_1^{\perp}$$

where α identifies F_1 as a submodule of F_2. Then the exactness of the above sequence implies that $\mathrm{im}\, \beta^{\vee} = \ker(\alpha^{\vee})$, but by the injectivity of $\beta^{\vee}$ we conclude that $F_3^{\vee} \cong F_1^{\perp}$. □

Exercise .1.17. Let V be a vector space of dimension n. Prove that there is a natural bijection between the Grassmannian $\mathrm{Gr}(r, V)$ of r-dimensional subspaces of V (cf. Exercise VI.2.13) and the Grassmannian $\mathrm{Gr}(n-r, V^\vee)$ of $(n-r)$-dimensional subspaces of the dual of V. (Use Exercise 5.15.)

In particular, the Grassmannian $\mathrm{Gr}_k(n-1, n)$ has the same structure as the projective space $\mathbb{P}V = \mathrm{Gr}_k(1, n)$. We could in fact *define* the projective space associated to a vector space V of dimension n to be the set of subspaces of 'codimension 1' (that is, dimension $n-1$) in V. There are reasons why this would be preferable, but established conventions are what they are.

Solution. Let $W \in \mathrm{Gr}(r, V)$ be an r-dimensional subspace of V. By Exercise 5.15, $W^\perp$ is an $(n-r)$-dimensional subspace of $V^\vee$. Furthermore, given an $(n-r)$-dimensional subspace of $V^\vee$, we can send it to *its* annihilator, which is an n-dimensional subspace of $V^{\vee\vee} \cong V$, and again by Exercise 5.15, this map sends $W^\perp \mapsto W$. Thus, the natural bijection $\mathrm{Gr}(r, V) \to \mathrm{Gr}(n-r, V^\vee)$ sends a subspace to its annihilator (or orthogonal complement, in more standard terminology). □

Exercise .1.18. Let F be a free R-module of finite rank. For any $r \geq 1$, define a multilinear map

$$\delta : \underbrace{F \times \cdots \times F}_{r} \times \underbrace{F^\vee \times \cdots \times F^\vee}_{r} \to R$$

by

$$\delta(v_1, \ldots, v_r, w_1^*, \ldots, w_r^*) = \det(w_i^*(v_j)), \quad 1 \leq i, j \leq r.$$

- Prove that δ is multilinear and alternating in the first r and in the last r entries.
- Deduce that δ induces a bilinear map $\hat{\delta} : \Lambda^r(F) \times \Lambda^r(F^\vee) \to R$.
- Prove that $\hat{\delta}$ induces an isomorphism $(\Lambda^r(F))^\vee \cong \Lambda^r(F^\vee)$.

Solution. To do. □

Exercise .1.19. Let R be a ring, not necessarily commutative, and let M be a *left*-R-module. Prove that $M^\vee$ carries a natural *right*-R-module structure. Prove that $R^\vee$ is isomorphic as a ring to the *opposite* ring R^{op}. (Cf. Exercise III.5.1.)

Solution. Define a right-R-module structure on $M^\vee = \mathrm{Hom}_R(M, R)$ by

$$(f \cdot r)(m) := f(m)r.$$

Indeed, for all $r, s \in R$ and $f, g \in M^\vee$ we have

$$\begin{gathered}
((f+g)\cdot r)(m) = (f(m)+g(m))r = f(m)r + g(m)r = (f\cdot r)(m) + (g\cdot r)(m), \\
(f\cdot(r+s))(m) = f(m)(r+s) = f(m)r + f(m)s = (f\cdot r)(m) + (f\cdot s)(m), \\
(f\cdot(rs))(m) = f(m)(rs) = (f\cdot r\cdot s)(m), \\
(f\cdot 1)(m) = f(m)\cdot 1 = f(m),
\end{gathered}$$

hence this is a right-R-module structure.

Since $R^\vee \cong R$ as R-modules, this action induces a ring structure on $R^\vee$ given by $r\cdot s = sr$, hence $R^\vee \cong R^{op}$ as rings. $\square$