.1 Geometric impossibilities

Exercise .1.1. Prove that if A, B are constructible, then the midpoint of the segment AB is also constructible. Prove that if two lines ℓ_1, ℓ_2 are constructible and not parallel, then the two lines bisecting the angles formed by ℓ_1 and ℓ_2 are also constructible.

Solution. To construct the midpoint of AB, draw two circles of radius at least half the length of the segment AB (I think you can just use the radius of the entire line segment): one which is centered at A and one which is centered at B. These circles should intersect in two places; draw the line connecting them. This line will intersect the segment AB, and the point at which they intersect is the midpoint of AB.

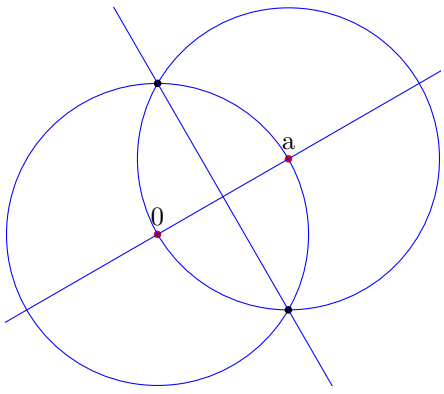
To construct an angle bisector, consider the angle $\angle AOB$. Draw a circle centered at A which intersects both ℓ_1 and ℓ_2 at, say, C and D. Then draw two circles with C and D as centers. These circles should intersect at one point in the region bounded by ℓ_1 and ℓ_2, say E. Finally, draw the line containing both O and E. This line bisects the angle formed by ℓ_1 and ℓ_2. □

Exercise .1.2. Prove that if a, b are constructible numbers, then so is $a - b$.

Solution. First note that $a + b$ is constructible: draw a line through a and b and draw the circle centered at b with radius a. This circle intersects the line at one other point, which is $a + b$. Now note that if b is constructible, $-b$ is also constructible (by the above construction with the circle centered at the origin). Since a and $-b$ are constructible, so is $a + (-b) = a - b$. □

Exercise .1.3. Find an explicit straightedge-and-compass construction for the product of two real numbers.

Solution. Given (constructible) $a, b \in \mathbb{R}$, we have explicit constructions for a and $\frac{1}{b}$. Then we may construct $a/(1/b) = ab$. Note that the below construction does not illustrate this so you should go back and fix it.



□

Exercise .1.4. Show how to square a *triangle* by straightedge and compass.