$$\frac{\overset{-}{\alpha}\ 1}{\alpha \to \alpha} \to I, 1$$

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \to E$$

$$\frac{\dfrac{\alpha \quad \alpha \to (\neg\beta \to \gamma)}{\neg\beta \to \gamma} \to E \quad \dfrac{\alpha \quad \alpha \to \neg\beta}{\neg\beta} \to E}{\gamma} \to E$$

Gentzen style proof of **Modus Tollens** $\{\neg\beta, \alpha \to \beta\} \vdash \neg\alpha$

$$\dfrac{\dfrac{\overset{-}{\not{\alpha}} \quad \alpha \to \beta}{\beta} \qquad \neg\beta}{\dfrac{\bot}{\alpha \to \bot} \to I} \to E$$

Gentzen style proof of **Law of Contraposition** with Modus Tollens

$$\dfrac{\dfrac{\alpha \to \beta \quad \overline{\not{\beta}}}{\neg\alpha}\,\mathrm{MT}}{\neg\beta \to \neg\alpha}\to I$$

Gentzen style proof of **Law of Contraposition** $\alpha \to \beta \vdash \neg\beta \to \neg\alpha$

$$\dfrac{\dfrac{\dfrac{\alpha \to \beta \quad \overline{\not{\alpha}}}{\beta}\to I \quad \overline{\not{\beta}}}{\bot}\to E}{\dfrac{\alpha \to \bot}{\neg\beta \to \neg\alpha}\to I}\to I$$

Gentzen style proof of the **Commutativity of Disjunction** $(\alpha \vee \beta) \vdash (\beta \vee \alpha)$

$$\dfrac{\alpha \vee \beta \qquad \dfrac{\dfrac{\overline{\alpha}}{\beta \vee \alpha}\vee I}{\alpha \to (\beta \vee \alpha)}\to I \qquad \dfrac{\dfrac{\overline{\beta}}{\beta \vee \alpha}\vee I}{\beta \to (\beta \vee \alpha)}\to I}{\beta \vee \alpha}$$

Gentzen style proof of the **Associativity of Disjunction**

$$(\alpha \vee \beta) \vee \gamma \vdash \alpha \vee (\beta \vee \gamma)$$

$$\dfrac{(\alpha \vee \beta) \vee \gamma \quad \dfrac{\dfrac{}{\gamma}}{\gamma \to (\alpha \vee (\beta \vee \gamma))} \to I \qquad \dfrac{\dfrac{}{\alpha \vee \beta} \quad \dfrac{\dfrac{\dfrac{}{\alpha}}{\alpha \vee (\beta \vee \gamma)} \vee I}{\alpha \to (\alpha \vee (\beta \vee \gamma))} \to I \quad \dfrac{\dfrac{\dfrac{}{\beta}}{\beta \vee \gamma} \vee I}{\alpha \vee (\beta \vee \gamma)} \vee I}{\beta \to ((\alpha \vee (\beta \vee \gamma))} \to I}{\alpha \vee (\beta \vee \gamma)} \vee E}{(\alpha \vee \beta) \to (\alpha \vee (\beta \vee \gamma))} \to I}{\alpha \vee (\beta \vee \gamma)} \vee E$$