$$\frac{\overline{\alpha}^{\;1}}{\alpha \to \alpha} \to I, 1$$

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \to E$$

$$\frac{\dfrac{\alpha \quad \alpha \to (\neg\beta \to \gamma)}{\neg\beta \to \gamma} \to E \quad \dfrac{\alpha \quad \alpha \to \neg\beta}{\neg\beta} \to E}{\gamma} \to E$$

Gentzen style proof of **Modus Tollens** $\{\neg\beta, \alpha \to \beta\} \vdash \neg\alpha$

$$\frac{\dfrac{\overline{\cancel{\alpha}}\quad \alpha \to \beta}{\dfrac{\beta \qquad \neg\beta}{\bot} \to E}}{\alpha \to \bot} \to I$$

Gentzen style proof of **Law of Contraposition** with Modus Tollens

$$\cfrac{\cfrac{\alpha \to \beta \quad \overline{\not\to \beta}}{\neg\alpha} \text{ MT}}{\neg\beta \to \neg\alpha} \to I$$

Gentzen style proof of **Law of Contraposition** $\alpha \to \beta \vdash \neg\beta \to \neg\alpha$

$$\cfrac{\cfrac{\cfrac{\cfrac{\alpha \to \beta \quad \overline{\not\alpha}}{\beta} \to I \quad \overline{\not\to\beta}}{\bot} \to E}{\alpha \to \bot} \to I}{\neg\beta \to \neg\alpha} \to I$$

Gentzen style proof of the **Commutativity of Disjunction** $(\alpha \vee \beta) \vdash (\beta \vee \alpha)$

$$\cfrac{\alpha \vee \beta \quad \cfrac{\cfrac{\overline{\alpha}}{\beta \vee \alpha} \vee I}{\alpha \to (\beta \vee \alpha)} \to I \quad \cfrac{\cfrac{\overline{\beta}}{\beta \vee \alpha} \vee I}{\beta \to (\beta \vee \alpha)} \to I}{\beta \vee \alpha}$$

3

Gentzen style proof of the **Associativity of Disjunction**

$$(\alpha \lor \beta) \lor \gamma \vdash \alpha \lor (\beta \lor \gamma)$$

$$
\cfrac{
  (\alpha \lor \beta) \lor \gamma
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{\overline{\gamma}}{\beta \lor \gamma}\ \lor I
    }{\alpha \lor (\beta \lor \gamma)}\ \lor I
  }{\gamma \to (\alpha \lor (\beta \lor \gamma))}\ \to I
  \qquad
  \cfrac{
    \cfrac{
      \overline{\alpha \lor \beta}
      \qquad
      \cfrac{
        \cfrac{\overline{\alpha}}{\alpha \lor (\beta \lor \gamma)}\ \lor I
      }{\alpha \to (\alpha \lor (\beta \lor \gamma))}\ \to I
      \qquad
      \cfrac{
        \cfrac{
          \cfrac{\overline{\beta}}{\beta \lor \gamma}\ \lor I
        }{\alpha \lor (\beta \lor \gamma)}\ \lor I
      }{\beta \to ((\alpha \lor (\beta \lor \gamma))}\ \to I
    }{
      \cfrac{\alpha \lor (\beta \lor \gamma)}{(\alpha \lor \beta) \to (\alpha \lor (\beta \lor \gamma))}\ \to I
    }\ \lor E
  }
}{\alpha \lor (\beta \lor \gamma)}\ \lor E
$$