Lecture 4: Natural Deduction

MATH230

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Outline

Validity

In order to show an argument for γ from Σ is valid, it is necessary to show there are no counterexamples. At present the only way to determine this is to write down the entire truth table and check for counterexamples.

In order to show an argument is *not valid* it suffices to find a counterexample. Thus the entire table is not required to show an argument is not valid.

Both of these problems are decidable by an algorithm. However truth tables grow exponentially with the number of propositional variables: If there are n variables, then there are 2^n rows.

Satisfiability

If we are only interested in the truth values of one propositional formula, then there are some different possibilities.

Definition (Tautology)

a sentence γ is a tautology if it is true in every case.

Definition (Satisfiable)

a sentence γ is satisfiable if it is true in some case.

Definition (Contradiction)

a sentence γ is a contradiction if it is false in every case.

Examples

Determine whether the following propositional formulae are either
(i) tautologies, (ii) satisfiable, or (iii) contradictions.

1. $A \wedge \neg A$
2. $P \vee \neg P$
3. Q
4. $A \rightarrow (B \rightarrow A)$
5. $A \wedge (B \vee C)$
6. $(A \vee \neg B \vee C) \wedge (A \vee \neg C) \wedge (A \vee C)$

Valuations

Instead of carrying truth tables around all the time, we will often use the more compact notation of *valuations*.

Definition

Given a finite set of propositional variables $S = \{A_1, A_2, \ldots, A_n\}$ a valuation function (or valuation, or assignment) $V : S \to \{T, F\}$ assigns each A_i a unique truth value. Given a well-formed formula α whose propositional variables are in S, a valuation assigns

$$V(\alpha) = \begin{cases} T & \text{if } \alpha \text{ is true in the case determined by } V \\ F & \text{if } \alpha \text{ is false in the case determined by } V \end{cases}$$

Note: specifying a valuation is equivalent to specifying a row in a truth table. See Tutorial 2 for more on valuation functions.

Key Theorem

Theorem: $\Sigma \vDash \alpha \rightarrow \beta$ if and only if $\Sigma \cup \{\alpha\} \vDash \beta$

Proof:

i) First suppose $\Sigma \vDash \alpha \rightarrow \beta$ and $v(\Sigma \cup \{\alpha\}) = T$.

ii) Now suppose $\Sigma \cup \{\alpha\} \vDash \beta$ and $v(\Sigma) = T$.

Satisfiable

Definition

We say a set of wff Σ is satisfiable if there exists some valuation that satisfies each $\sigma \in \Sigma$.

Example:

- $\Sigma = \{A, \ \neg B, \ \neg A \vee \neg B\}$
- $\Sigma = \{A, \ \neg A, \ B, \ C\}$

Valid Arguments

Definition

We say an argument from Σ for γ is valid if each valuation that satisfies Σ satisfies γ.

This is the same definition, but now written in terms of valuations.
Again, if $\Sigma \vDash \gamma$, then we say γ is a semnatic consequence of Σ.

Natural Deduction

Given an argument with hypotheses Σ and conclusion γ, we judge
the argument to be valid by checking all of the rows in the
corresponding truth table.

1. This is slow (Later impossible!).
2. It does not reflect how we prove things in mathematics.

We will now introduce another method of deduction: natural
deduction.

Syntactic Meaning of Logical Connectives

Rather than using the semantic (truth valued) meaning of the propositional variables, we focus on the syntactic meaning of the logical connectives.

Example: What would it require to deduce $A \wedge B$ in the course of a proof?

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Rather than using the semantic (truth valued) meaning of the propositional variables, we focus on the syntactic meaning of the logical connectives.

Example: What would it require to deduce $A \wedge B$ in the course of a proof?

$$\frac{\begin{array}{cc} \Sigma_1 & \Sigma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ A & B \end{array}}{A \wedge B}$$

If we have a deduction for A and a deduction for B, then together we should consider those deductions a proof for $A \wedge B$.

Example: Suppose $A \wedge B$ were a premise in a proof. What can we conclude from such a premise?

∧ **Elimination**

Example: Suppose $A \wedge B$ were a premise in a proof. What can we conclude from such a premise?

$$\frac{\begin{array}{c} \Sigma \\ \mathcal{D} \\ A \wedge B \end{array}}{A} \qquad\qquad \frac{\begin{array}{c} \Sigma \\ \mathcal{D} \\ A \wedge B \end{array}}{B}$$

Deductions

Definition

We define deductions (or derivations, or proofs) inductively by the rules:

- *For each formula α,*

$$\alpha$$

  is a deduction with conclusion α and premises $\{\alpha\}$.

- *From a given deduction, an application of a rule of inference yields a new deduction.*

- *Anything that is not a deduction by virtue of the above is not a deduction.*

If there exists a deduction $\frac{\Sigma}{\alpha}\mathcal{D}$ of α from Σ, then we say α is a *syntactic consequence of* (*derivable from*, or *provable from*) Σ and denote this $\Sigma \vdash \alpha$.

Propositional Calculus

If we follow this idea for all of the logical connectives in propositional logic, then we can develop a method for writing proofs based on the *syntactic* structre of the logical connectives alone. We call this method of proof *Natural Deduction* and we are following Gerhard Gentzen's notation.

We need to know how to (i) deduce and (ii) conclude from, each logical connective. In other words, for each logical connective we need to develop rules from introducing the logical connective and eliminating the logical connective.

So we will spend some time writing down the **Rules of Inference** for our logical connectives.

Report: There are other proof theories - these could be explored for your report.

$\rightarrow$ **Elimination (MP)**

If $\frac{\Sigma_1}{\alpha \rightarrow \beta} \mathcal{D}_1$ and $\frac{\Sigma_2}{\alpha} \mathcal{D}_2$ are deductions, then

$$\frac{\begin{array}{cc} \Sigma_1 & \Sigma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ \alpha \rightarrow \beta & \alpha \end{array}}{\beta} \rightarrow E$$

is a deduction of β from $\Sigma_1 \cup \Sigma_2$.

Note: This is modus ponens, which we showed to be a semantically valid deduction.

Show that $\{P \to (\neg S \to L), \quad P \to \neg S, \quad P\} \vdash L$

Example

Show that $\{P \to (\neg S \to L), \quad P \to \neg S, \quad P\} \vdash L$

$$\dfrac{\dfrac{P \to (\neg S \to L) \quad P}{\neg S \to L} \; MP \qquad \dfrac{P \to \neg S \quad P}{\neg S} \; MP}{L} \; MP$$

Further Reading

Here are some recommended reading to follow up on the lecture content. They are all freely available online.

- L∃∀N, *Logic and Proof*. Sections: 3.1, 6.1, and 6.2
- Logic Matters, *An Introduction to Formal Logic*: p172 - 216