Lecture 12: Proof Theory I

Syntax of First Order Logic

MATH230

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Outline

Syntax v. Semantics

Given a first order language $\mathcal{L}$ we have determined how to assign meaning to well-formed formulae. As the quantifers and predicates increase the scope of the language, the truth and falsity of formulae in $\mathcal{L}$ are difficult to determine.

As in the case of propositional logic, we have another method for analysing whether some formulae follow from others; the syntactic, proof theoretic, point of view.

We will now extend the natural deduction calculus to include rules of inference for the quantifiers.

Universal Quantification

Suppose we had, in the course of a proof, deduced the statement $\forall x \alpha$, where α is a well-formed formula that (may) contain the variable x. What can we conclude from that?

$$\dfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ \forall x \alpha\end{array}}{?}\ \forall E$$

Following our model theory interpretation of $\forall$ we should then be able to substitute any term, t, in place of each instance of the variable x in α.

Substitution Rules

If we don't take care when subsitituting for variables, then we can go astray.

Definition

If a variable, x, is used to quantify over α — as in $\gamma = \forall x \alpha$ — then we say any instance of x in α is bound in γ by the quantifer. Similarly for the existential quantifier.

Definition

If a variable, x, is not bound in α, then we say that it is free in α.

Bound and free variables act differently when interpreted; they play different roles.

Free vs. Bound Variables

We can translate the first-order expression: $\forall x(x < y)$ as "every x is less than ____". The interpretation of this statement depends on the interpretation of y; because y is free. It's truth will depend on the interpretation of y.

On the other hand the interpretation of $\exists y(\forall x(x < y))$ doesn't depend on y. As, in a model, there either exists something that satisfies this sentence, or there doesn't.

For this reason we should not make free variables bound, or vice versa, when substituting.

Examples

x_2 is free for x_0 in $\exists x_3(P(x_0, x_3))$,

$f(x_0, x_1)$ is not free for x_0 in $\exists x_1(P(x_0, x_3))$,

x_5 is free for x_1 in $P(x_1, x_3) \to \exists x_1(Q(x_1, x_2))$.

What Can Go Wrong?

Consider the formula

$$\forall x \exists y (y > x).$$

In a model with domain the natural numbers, this asserts that there is always a larger natural number.

Notice that x is free in the formula $\exists y(y > x)$. So if we aren't careful and substitute $x \mapsto y + 1$, then we obtain the statement

$$\exists y(y > y + 1)$$

Which is not true in this interpretation. **This problem arose because y is bound in the formula, so we can't substitute $t = y + 1$ for x.**

Example from Section 7.3 L∃∀N

Universal Elimination

If $\overset{\Sigma}{\underset{\forall x \alpha}{\mathcal{D}}}$ is a derivation, where y is free for x in α, then

$$\dfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ \forall x \alpha\end{array}}{\alpha[x/y]}\ \forall E$$

is a derivation of $\alpha[x/y]$ from hypotheses Σ.

Example

Every human is mortal. Sherlock is Human. Therefore, Sherlock is mortal.

Proof of For All

What does it mean to prove something "for all x"?

A proof of the irrationality of $\sqrt{2}$ starts as follows:

> Assume there are some integers A, B such that $\sqrt{2} = A/B$....
> Eventually deriving a contradiction from this assumption.

The proof itself, in the end, assumes *nothing* of the integers A, B apart from their (possible) existence. Thus it shows that no matter which integers are in place of x, y, the formula

$$(x^2 = 2y^2)$$

can have no solutions and hence we may conclude

$$\forall x, y \; \neg(x^2 = 2y^2)$$

Universal Introduction

If ${}^{\Sigma}_{\alpha[x/y]}\mathcal{D}$ is a derivation, where y does not appear free in Σ nor α, then

$$\dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ \alpha[x/y]\end{array}}{\forall x\alpha}\ \forall I$$

is a derivation of $\forall x\alpha$ from hypotheses Σ.

Properties of y can't play a role in the proof of $\alpha[x/y]$. They can't appear in the hypotheses Σ of the proof.
Nor can any y be free in alpha. You should be replacing all instances of y with an x when using the $\forall$ introduction.

Example

Show $\{\forall x(Px \to Qx),\ \forall xPx\} \ \vdash \ \forall xQx$

Example

Show $\{\forall x(Px \to Qx),\ \forall x(Qx \to Rx)\} \ \vdash\ \forall x(Px \to Rx)$

Example: When Things Go Wrong

Sherlock is a person. Gladstone is not a person. Therefore, Moriarty is the Queen.

Show $\{Ps,\ \neg Pg\} \ \vdash \ \ Q$

$$
\begin{prooftree}
\AxiomC{$\neg Pg$}
\AxiomC{Ps}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x Px$}
\RightLabel{$\forall E$}
\UnaryInfC{Pg}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\bot$}
\UnaryInfC{Q}
\end{prooftree}
$$

Example: When Things Go Wrong

Sherlock is a person. Gladstone is not a person. Therefore, Moriarty is the Queen.

Show $\{Ps, \neg Pg\} \vdash Q$

$$\dfrac{\dfrac{\neg Pg \qquad \dfrac{\dfrac{Ps}{\forall x Px}\,\forall I}{Pg}\,\forall E}{\bot}\,MP}{Q}\,\bot$$

From one instance (Sherlock's Personhood) we cannot assume all objects are people.

Further Reading

Below is a selection of resources that I used to prepare the lecture. You might like to read over them yourself to help get a more complete picture of the topics discussed.

- L∃∀N, *Logic and Proof*. Section: 8.
- Logic Matters, *An Introduction to Formal Logic*: 32 and 33.