Lecture 15: First-Order Theories of Arithmetic II

MATH230

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Outline

1. Peano Arithmetic
2. Theorem Provers and Proof Checkers
3. Answering Hilbert
4. Robinson Arithmetic

Axioms of PA

Peano Arithmetic has signature PA: $\{0, s, +, \times\}$ and axioms

1. $\forall x \neg(s(x) = 0)$
2. $\forall x\ \forall y((s(x) = s(y)) \rightarrow (x = y))$
3. $\forall x\ (x + 0 = x)$
4. $\forall x\ \forall y\ (x + s(y) = s(x + y))$
5. $\forall x\ (x \times 0 = 0)$
6. $\forall x\ \forall y\ (x \times s(y) = (x \times y) + x)$
7. $(P(0) \wedge \forall x\ (P(x) \rightarrow P(x + 1))) \rightarrow \forall y(P(y))$

This is Presburger arithmetic with multiplication. It is accepted as *the* first-order theory of arithmetic.

Abbreviations

We make the following abuse of notation

$$\begin{aligned} 1 &= s(0) \\ 2 &= s(1) = s(s(0)) \\ 3 &= s(2) = s(s(1)) = s(s(s(0))) \\ &\vdots \end{aligned}$$

It is important to distinguish well-formed formulae in our formal language from statements in the intended model. Remember the well-formed formulae in our language are just strings of symbols. Some authors force this with their notation; we will just be mindful of the difference.

Example

$$PA \vdash 2 + 1 = 3$$

Example

$$PA \vdash 2 \neq 1$$

Example

$$PA \vdash 0 + 1 = 1$$

Induction Example

$$PA \vdash \forall x \; (0 + x = x)$$

Example

$$PA \vdash 7 \times 1 = 7$$

Is This Feasible?

We appear to have a precise formal language in which we can prove real number theoretic statements - great!

Unfortunately, it takes slides and slides to prove basic statements about arithmetic - let alone something non-trivial like the fundamental theorem of arithmetic.

Happily we now have computers! These computers can store sub-proofs to make writing new proofs much easier. The **Xena Project** is building a database of undergraduate math with proofs verified by the proof checker L∃∀N.

Tutorial 6 will have you work in L∃∀N. This way you get an idea for how these ideas have had an effect today, on modern mathematics. Plus, proving basic properties like commutativity of addition is much nicer in L∃∀N, than writing it on paper.

Non-Standard Models

There are non-standard models of Peano Arithmetic! Thoralf Skolem and Abraham Robinson are the pioneers of non-standard models.

Example: One can make a model with an element, x, larger than every natural number. (Consequence of the compactness theorem).

Example: Hyperreal numbers (the hypernaturals) were constructed by Abraham Robinson in the 1960s. These provide another non-standard model of Peano Arithmetic.

This already suggests that Peano Arithmetic is *incomplete*.

Gödel's First Incompleteness Theorem

In 1931 Gödel published the following theorem:

Any consistent formal theory F within which Peano arithmetic can be carried out, is necessarily incomplete. That is, there are statements, ϕ, in F such that neither $F \vdash \phi$ nor $F \vdash \neg\phi$.

Thus, Peano arithmetic is incomplete.

Perhaps we can add axioms to PA to make it complete? No!

Peano arithmetic is incompletable!

Gödel's Second Incompleteness Theorem

In 1931 Gödel published the following theorem:

For any consistent formal theory F within which Peano arithmetic can be carried out, the consistency of F cannot be proved in F itself.

Thus, Peano arithmetic can't prove it's own consistency. Nor can any extension of it.

Gödel showed the consistency of PA can be written in PA and is one of those Gödelian sentences!

Axioms of Q

Robinson Arithmetic has signature $\mathcal{Q} : \{0, s, +, \times\}$ and axioms

1. $\forall x \neg(s(x) = 0)$
2. $\forall x \, \forall y((s(x) = s(y)) \rightarrow (x = y))$
3. $\forall x \, ((x = 0) \vee \exists y(x = s(y)))$
4. $\forall x \, (x + 0 = x)$
5. $\forall x \, \forall y \, (x + s(y) = s(x + y))$
6. $\forall x \, (x \times 0 = 0)$
7. $\forall x \, \forall y \, (x \times s(y) = (x \times y) + x)$

Without the induction schema $\mathcal{Q}$ is *fintely* axiomatized, and much weaker than Peano arithmetic. However, Raphael Robinson introduced $\mathcal{Q}$ in 1950 to show that although $\mathcal{Q}$ is much weaker than PA, it (and any extension of $\mathcal{Q}$) is still incomplete and undecidable. There is enough number theory to carry out Gödel's proof.

Further Reading

Here are some recommended reading to follow up on the lecture content. They are all freely available online.

- L∃∀N, *Logic and Proof*. Sections: 17, 18 (+ 23 optional).
- Logic Matters, *Gödel Without Too Many Tears*.
- SEP, *Gödel's Incompleteness Theorem* (Optional!).