# Lecture 13: Proof Theory II

## Syntax of the Existential Quantifier

MATH230

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# Outline

# Existential Introduction

If $\begin{smallmatrix}\Sigma \\ \alpha[x/y]\end{smallmatrix}\mathcal{D}$ is a derivation, where $y$ is free for $x$ in $\alpha$, then

$$\dfrac{\begin{matrix}\Sigma \\ \mathcal{D} \\ \alpha[x/y]\end{matrix}}{\exists x\alpha}\ \exists I$$

is a derivation of $\exists x\alpha$ from hypotheses $\Sigma$.

# Example

Show $\forall x Px \ \vdash \ \exists Px$

# Example

Show $\{\forall x(Px \to Qx), \neg Qa\} \vdash \exists x\, \neg Px$

# Existential Elimination

If ${}^{\Sigma_1}_{\exists x\alpha}\mathcal{D}_1$, ${}^{\Sigma_2}_{\alpha[x/y]\to\gamma}\mathcal{D}_2$, and $y$ is neither free in $\gamma$ nor $\Sigma$, then

$$\dfrac{\begin{matrix}\Sigma_1 \\ \mathcal{D}_1 \\ \exists x\alpha\end{matrix} \qquad \begin{matrix}\Sigma_2 \\ \mathcal{D}_2 \\ \alpha[x/y] \to \gamma\end{matrix}}{\gamma}\ \exists E$$

is a derivation of $\gamma$ from hypotheses $\Sigma_1 \cup \Sigma_2$.

# Example

Show $\{\forall x(Px \to Qx),\ \exists x Px\} \ \vdash \ \exists x Qx$

# Example: When Things Go Wrong I

Sherlock is a detective. Therefore, everyone is a detective.

$$\exists x Dx \ \vdash \ \forall x Dx$$

$$\dfrac{\dfrac{\exists x Dx \qquad \dfrac{\dfrac{}{Da}}{Da \to Da} \to I}{Da} \exists E}{\forall x Dx} \forall I$$

# Example: When Things Go Wrong II

Sherlock is a human. Gladstone is a dog. Therefore, there exists somthing that is both a human and a dog.

$$Hs,\ \exists x Dx\ \vdash\ \exists x(Hx \land Dx)$$

$$\dfrac{\exists x Dx \qquad \dfrac{\dfrac{\dfrac{Hs \qquad \overline{Ds}}{Hs \land Ds}\,\land I}{\exists x(Hx \land Dx)}\,\exists I}{Ds \to \exists x(Hx \land Dx)}\,\to I}{\exists x(Hx \land Dx)}\,\exists E$$

## Proof Strategy

Show $\exists x Px \vee \exists x Qx \ \vdash \ \exists x(Px \vee Qx)$

**Note:** Our hypothesis is a disjunction. So we will need the disjunction elimination rule to make use of it.

This breaks the proof into two steps:

1. $\exists x Px \rightarrow \exists x(Px \vee Qx)$
2. $\exists x Qx \rightarrow \exists x(Px \vee Qx)$

Both of these steps will have the same form, so let's focus on the first.

## Proof Strategy

We have reduced the problem to showing:

$$\vdash \ \exists x Px \rightarrow \exists x (Px \vee Qx)$$

# Proof Strategy

We have reduced the problem to showing:

$$\vdash \; \exists x Px \to \exists x (Px \vee Qx)$$

The deduction theorem ($\to I$) implies this is equivalent to showing:

$$\exists x Px \; \vdash \; \exists x (Px \vee Qx)$$

# Proof Strategy

We have reduced the problem to showing:

$$\vdash \ \exists x Px \to \exists x(Px \vee Qx)$$

The deduction theorem ($\to I$) implies this is equivalent to showing:

$$\exists x Px \ \vdash \ \exists x(Px \vee Qx)$$

Now our hypothesis has an existential quantifier in it, so we need to use the corresponding elimination rule. Our proof will have a line of the form:

$$\dfrac{\exists x Px \qquad \dfrac{}{Pa \to \exists x(Px \vee Qx)}\,?}{\exists x(Px \vee Qx)}\,\exists E$$

How can we complete the proof?

## Proof Strategy

We want to show $\vdash \exists x Px \rightarrow \exists x(Px \vee Qx)$ and so far our proof looks like this. What can we do to finish this proof?

$$
\dfrac{\overline{\exists x Px} \qquad \dfrac{\dfrac{\dfrac{\overline{Pa}}{Pa \vee Qa}\ \vee I}{\exists x(Px \vee Qx)}\ \exists I}{Pa \rightarrow \exists x(Px \vee Qx)}\ \rightarrow I}{\exists x(Px \vee Qx)}\ \exists E
$$

# Proof Strategy

We have proven the first of our subgoals. Go through the same process to prove the second subgoal.

1. $\exists x Px \to \exists x(Px \vee Qx)$
2. $\exists x Qx \to \exists x(Px \vee Qx)$

Use the disjunction elimination to tie these goals together into a proof of the original statement

$$\exists x Px \vee \exists x Qx \ \vdash \ \exists x(Px \vee Qx)$$

# Metalogical Theorems

Gödel proved that first-order predicate logic is both sound and complete.

$$\Sigma \vDash \alpha \quad \text{if and only if} \quad \Sigma \vdash \alpha$$

Thus we can provide deductions for all, and only those, wff that are true in every model satisfying the hypotheses $\Sigma$.

Gödel's original proof is different in style to the one we gave for propositional logic. However, Leon Henkin gave a different proof of the completeness theorem of first-order logic. We followed that style in proving completeness and soundness of propositional logic.

# Further Reading

Below is a selection of resources that I used to prepare the lecture. You might like to read over them yourself to help get a more complete picture of the topics discussed.

- L∃∀N, *Logic and Proof*. Sections: 7 and 8.
- Logic Matters, *An Introduction to Formal Logic*: Not so helpful for our version of deduction. But you might like to read Chapter 31, 32, and 33.