Lecture 5: Rules of Inference

MATH230

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Outline

Hypothetical Reasoning

In mathematics we often prove statements of the following hypothetical form: "If ..., then ..."

Example: Suppose f is a smooth function. If f is continuous, then it is differentiable.

The proof of such a statement will assume the hypothesis of continuity and show that it implies differentiability.

The conclusion is the entire implication, not just differentiablity.

$\to$ Introduction

If ${}^{\Sigma}_{\beta}\mathcal{D}$ is a deduction of β from Σ, then

$$\dfrac{\dfrac{\overline{\cancel{\alpha}}}{\begin{matrix}\Sigma\\ \mathcal{D}\\ \beta\end{matrix}}}{\alpha \to \beta} \to I$$

is a deduction of $\alpha \to \beta$ from hypotheses $\Sigma \backslash \{\alpha\}$.

Note: As the assumption α is struck out after this deduction, we are free to use α *even if it is not in* Σ when using implication introduction.

Example

Show $\{\alpha \to \beta,\ \beta \to \gamma\} \ \vdash \ \alpha \to \gamma$

Falsum

We introduce the logical constant $\perp$ (falsum or absurdity) to define the syntactic form of the $\neg$ connective. We make the following definition:

$$\neg\alpha := \alpha \to \perp$$

$$\dfrac{\alpha \qquad \alpha \to \perp}{\perp}\ MP \qquad\qquad \dfrac{\begin{array}{c}\overline{\cancel{\alpha}}\\ \mathcal{D}\\ \perp\end{array}}{\alpha \to \perp} \to I$$

Falsum $\perp$ is to be thought of as "absurdity" or "contradiction".

Example: Modus Tollens

Show $\{\alpha \to \beta, \quad \neg\beta\} \vdash \neg\alpha$

Contradiction Implies Absurdity

Show $A \wedge \neg A \ \vdash \ \perp$

$\wedge$ Introduction

If ${}^{\Sigma_1}_{\alpha}\mathcal{D}_1$ and ${}^{\Sigma_2}_{\beta}\mathcal{D}_2$ are deductions, then

$$\dfrac{\begin{matrix}\Sigma_1 \\ \mathcal{D}_1 \\ \alpha\end{matrix} \qquad \begin{matrix}\Sigma_2 \\ \mathcal{D}_2 \\ \beta\end{matrix}}{\alpha \wedge \beta}\ \wedge I$$

is a deduction of $\alpha \wedge \beta$ from $\Sigma_1 \cup \Sigma_2$.

$\wedge$ Elimination

If ${}^{\Sigma}_{\alpha\wedge\beta}\mathcal{D}$ is a deduction of $\alpha \wedge \beta$ from Σ, then

$$\dfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ \alpha \wedge \beta\end{array}}{\alpha}\ \wedge E_L \qquad\qquad \dfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ \alpha \wedge \beta\end{array}}{\beta}\ \wedge E_R$$

are deductions of α and β from Σ.

Example: Commutativity of $\wedge$

Show $\alpha \wedge \beta \dashv\vdash \beta \wedge \alpha$

Example: Idempotence of $\wedge$

Show $\alpha \wedge \alpha \dashv\vdash \alpha$

$\vee$ Introduction

If ${}^{\Sigma}_{\alpha}\mathcal{D}$ is a derivation of α from Σ, then

$$\dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ \alpha\end{array}}{\alpha \vee \beta}\ \vee I_R \qquad\qquad \dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ \alpha\end{array}}{\beta \vee \alpha}\ \vee I_R$$

are derivations of $\alpha \vee \beta$ and $\beta \vee \alpha$ from Σ.

Note: We are free to choose β as, if we know α to be the case, then $\alpha \vee \beta$ is necessarily the case *for any* β.

$\vee$ Elimination

If ${}^{\Sigma_1}_{\alpha\vee\beta}\mathcal{D}_1$, ${}^{\Sigma_2}_{\alpha\to\gamma}\mathcal{D}_2$, and ${}^{\Sigma_2}_{\beta\to\gamma}\mathcal{D}_3$ are derivations, then

$$\dfrac{\begin{array}{c}\Sigma_1\\ \mathcal{D}_1\\ \alpha\vee\beta\end{array}\qquad\begin{array}{c}\Sigma_2\\ \mathcal{D}_2\\ \alpha\to\gamma\end{array}\qquad\begin{array}{c}\Sigma_2\\ \mathcal{D}_2\\ \beta\to\gamma\end{array}}{\gamma}\ \vee_E$$

is a derivation of γ from $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.

Note: You can't remove one of the arguments from a disjunction. Knowledge of $\alpha \vee \beta$ is not sufficient to conclude either α or β alone.

Minimal Logic

Together the rules of inference that we've given so far define *minimal* logic. They include much, but not all, of the logical inferences that practising mathematicians might use in a proof. What they do include is uncontroversial.

However it is not universally agreed upon how minimal logic should be extended. There are philosophical differences among mathematicians and logicians about what other rules of inference should be included.

- Classical logic
- Intuitionistic logic
- Modal logic

Report: These philosophical differences and different logics would make for an interesting report topic.

Further Reading

Below is a selection of resources that I used to prepare the lecture. You might like to read over them yourself to help get a more complete picture of the topics discussed.

- L∃∀N, *Logic and Proof*. Section: 3 (and 4 if you're interested)
- Logic Matters, *An Introduction to Formal Logic*. Sections: Interlude, 20 – 22