Lecture 2: Propositional Logic

MATH230

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Outline

Arguments and Proofs

Analysis of a proof in mathematics, or an argument in a natural language, relies on how particular statements are connected to each other. How one or more statements are said to entail others.

Example:
If p divides ab, then p divides a or p divides b.[1]

It is the connective words, as much as the mathematical content words, that we have to analyse when deciding whether this is a good argument.

[1] This is an example of an argument: this statement is not necessarily true!

Example: Natural Language

- If Watson moves in with Holmes, then Holmes will be forever annoyed. Watson moved in with Holmes. Therefore Holmes will be forever annoyed.

- If Watson can trap Moriarty, then Holmes can. Holmes can't trap Moriarty. Therefore Watson can't.

- Either Holmes catches Moriarty or the world will fall into chaos. The world has fallen into chaos. Therefore Holmes did not catch Moriarty.

Argument

Definition

An argument is a set of declaritive sentences (propositions), one of which is singled out as the conclusion, while the rest are considered premises.

Premises are the evidence to support the conclusion.

Example: Natural Language

- If Watson moves in with Holmes, then Holmes will be forever annoyed. Watson moved in with Holmes. Therefore, Holmes will be forever annoyed.

  Let's break this up into premises and conclusion:

Propositional Structures

Definition

An atomic proposition has no propositional substructure.

We saw above that some propositions do have extra structure: "If... , then...." and "Either .... or ... " and "can't" are important to the nature of the argument.

Such connectives are used to join atomic propositions into compound propositions.

Example: Natural Language

- Either Holmes catches Moriarty or the world will fall into chaos. The world has fallen into chaos. Therefore Holmes did not catch Moriarty.

  Let's break this up into premises and conclusion and determine the atomic propositions.

Comment on Natural Language

It can be subjective as to what one considers atomic. Natural language is very rich and open to many interpretations. As we are hoping for a precise language on which to found mathematics, we need to remove these ambiguities.

For this reason we introduce the following notations, our first formal logic, collectively known as (classical) propositional logic.

Binary Connectives

To express the same syntactic structure of an *argument* without the ambiguities of a natural language we use capital (English) letters to denote atomic propositions, called *propositional variables*. We use the following symbols to express compound propositions:

- $\neg$: “It is not the case that... ” or ”Not... ”
- $\wedge$: “Both... and ... ”
- $\vee$: “Either... or ... ”
- $\rightarrow$: “If... , then ... ”
- $\leftrightarrow$: “ ... if and only if ... ”

Example: Translating from NL to PL

- If Watson can trap Moriarty, then Holmes can. Holmes Can't trap Moriarty. Therefore Watson can't.

Grammar

Our language is further made up of *well-formed formulae* which we define inductively as follows:

Definition (Well-Formed Formulae)

- **Atomic Formulae:** *If* α *is a single propositional variable, then* α *is a wff.*
- **Negation:** *If* α *is a wff, then* $\neg\alpha$ *is a wff.*
- **Binary Connective:** *If* α *and* β *are wff and* $*$ *is a binary connective, then* $(\alpha * \beta)$ *is a wff.*

Note: Any expression that is not a wff by virtue of these three constructions is not a wff.

Examples

Which of the following are wff in propositional logic?

1. A
2. AB
3. $(A \to B)$
4. $A \to B \to C$
5. $((A \to B) \to C)$
6. $\neg Q$
7. $A \vee Q$
8. $A \to \neg B \vee C$

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7. $A \vee Q$
8. $A \to \neg B \vee C$

Binding Conventions:

- $\neg$ binds most tightly,
- $\vee$ and $\wedge$ bind more tightly than $\to$,
- $\to$ binds more tightly than $\leftrightarrow$.

Further Reading

Here are some recommended reading to follow up on the lecture content. They are all freely available online.

- L∃∀N, *Logic and Proof*. Sections: 3.1, 6.1, and 6.2
- Logic Matters, *An Introduction to Formal Logic*: p172 - 216