Lecture 6: Classical Logic

MATH230

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Outline

Example: Minimal Logic

Show $\{A \vee B,\ (A \vee C) \to D,\ B \to D\} \vdash D$

Ex Falso Sequitur Quadlibet

So far, we have not made much mention of how to deal with the derivation of $\perp$ absurdity.

If ${}^{\Sigma}_{\perp}\mathcal{D}$ is a deduction of $\perp$ from Σ, then

$$\dfrac{\begin{array}{c}\Sigma \\ \mathcal{D} \\ \perp\end{array}}{\alpha}\ \perp$$

is a derivation of α from the assumptions Σ.

Anything you want follows from a falsehood. Recall this is valid.

Disjunctive Syllogism

Show $\{\alpha \vee \beta, \neg\beta\} \vdash \alpha$

Double Negation Elimination

Show $\neg\neg\alpha \vdash \alpha$

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Show $\neg\neg\alpha \vdash \alpha$

Ex falso does not give us a proof. In fact we have shown the following: $\{\neg\neg\alpha, \neg\alpha\} \vdash \alpha$.

Reductio Ad Absurdum

If ${}^{\Sigma}_{\perp}\mathcal{D}$ is a deduction of $\perp$ from Σ, then

$$\dfrac{\dfrac{\dfrac{}{\cancel{\neg\alpha}}}{\begin{matrix}\Sigma\\ \mathcal{D}\\ \perp\end{matrix}}}{\alpha}\ RAA$$

is a derivation of α from the assumptions $\Sigma \backslash \{\neg\alpha\}$.

If absurdity follows from $\neg\alpha$, then we may conclude α **and discharge** $\neg\alpha$ **from our assumptions.**

Double Negation

Show $\neg\neg\alpha \vdash \alpha$

Law of Excluded Middle

Show $\vdash \alpha \vee \neg\alpha$

Example

Show $\{\alpha \to \beta, \;\; \alpha \to \neg\beta\} \vdash \neg\alpha$

Derived Rules of Inference

Proofs can be simplified by using results already proved. You may, in the course of a proof, use any result that has been proven in class or previously in a tutorial. However, when substituting previous proofs, you must bring all of the premises with the conclusion.

We have already seen this with the use of *modus tollens* (MT) in some examples.

This can help keep proofs manageable and neat.

Example: Substituting LEM

Show $\alpha \to \beta \vdash \neg\alpha \vee \beta$

Consistent

Definition

a set of wff Σ is consistent if and only if for no α does $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$.

Definition

a set of wff Σ is maximally consistent if and only if it is consistent and for every α either $\Sigma \cup \{\alpha\}$ is inconsistent or $\Sigma \vdash \alpha$.

Lemma: If Σ is consistent, then there exists a maximally consistent $\overline{\Sigma}$ such that $\Sigma \subseteq \overline{\Sigma}$.

Lemma: If $\overline{\Sigma}$ is maximally consistent, then for each wff α either $\overline{\Sigma} \vdash \alpha$ or $\overline{\Sigma} \vdash \neg\alpha$.

Further Reading

Below is a selection of resources that I used to prepare the lecture. You might like to read over them yourself to help get a more complete picture of the topics discussed.

- L∃∀N, *Logic and Proof*. Section: 5
- Logic Matters, *An Introduction to Formal Logic*: Sections 17, 23, and 24.8