Lecture 13: Proof Theory II

Syntax of the Existential Quantifier

MATH230

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Outline

1. Rules of Inference: Existential Quantifier
2. Examples
3. Extended Example
4. Gödel's Completeness Theorem

Existential Introduction

If ${}^{\Sigma}_{\alpha[x/y]}\mathcal{D}$ is a derivation, where y is free for x in α, then

$$\dfrac{\begin{array}{c}\Sigma\\ \mathcal{D}\\ \alpha[x/y]\end{array}}{\exists x\alpha}\ \exists I$$

is a derivation of $\exists x\alpha$ from hypotheses Σ.

Example

Show $\forall x P x \ \vdash \ \exists P x$

Example

Show $\{\forall x(Px \to Qx), \neg Qa\} \vdash \exists x \neg Px$

Existential Elimination

If ${}^{\Sigma_1}_{\exists x\alpha}\mathcal{D}_1$, ${}^{\Sigma_2}_{\alpha[x/y]\to\gamma}\mathcal{D}_2$, and y is neither free in γ nor Σ, then

$$\dfrac{\begin{array}{cc}\Sigma_1 & \Sigma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ \exists x\alpha & \alpha[x/y]\to\gamma\end{array}}{\gamma}\ \exists E$$

is a derivation of γ from hypotheses $\Sigma_1 \cup \Sigma_2$.

Example

Show $\{\forall x(Px \to Qx),\ \exists x Px\} \ \vdash\ \exists x Qx$

Example: When Things Go Wrong I

Sherlock is a detective. Therefore, everyone is a detective.

$$\exists x Dx \;\vdash\; \forall x Dx$$

$$\dfrac{\dfrac{\exists x Dx \qquad \dfrac{\dfrac{}{Da}}{Da \to Da} \to I}{Da} \exists E}{\forall x Dx} \forall I$$

Example: When Things Go Wrong II

Sherlock is a human. Gladstone is a dog. Therefore, there exists somthing that is both a human and a dog.

$$Hs,\ \exists x Dx \ \vdash \ \exists x(Hx \wedge Dx)$$

$$\dfrac{\exists x Dx \qquad \dfrac{\dfrac{\dfrac{Hs \quad \overline{Ds}}{Hs \wedge Ds}\wedge I}{\exists x(Hx \wedge Dx)}\exists I}{Ds \to \exists x(Hx \wedge Dx)}\to I}{\exists x(Hx \wedge Dx)}\exists E$$

Proof Strategy

$$\text{Show } \exists x Px \vee \exists x Qx \ \vdash \ \exists x(Px \vee Qx)$$

Note: Our hypothesis is a disjunction. So we will need the disjunction elimination rule to make use of it.

This breaks the proof into two steps:

1. $\exists x Px \to \exists x(Px \vee Qx)$
2. $\exists x Qx \to \exists x(Px \vee Qx)$

Both of these steps will have the same form, so let's focus on the first.

Proof Strategy

We have reduced the problem to showing:

$$\vdash \; \exists x Px \to \exists x(Px \lor Qx)$$

Proof Strategy

We have reduced the problem to showing:

$$\vdash \ \exists x Px \rightarrow \exists x(Px \vee Qx)$$

The deduction theorem ($\rightarrow I$) implies this is equivalent to showing:

$$\exists x Px \ \vdash \ \exists x(Px \vee Qx)$$

Proof Strategy

We have reduced the problem to showing:

$$\vdash \ \exists x Px \to \exists x(Px \vee Qx)$$

The deduction theorem $(\to I)$ implies this is equivalent to showing:

$$\exists x Px \ \vdash \ \exists x(Px \vee Qx)$$

Now our hypothesis has an existential quantifier in it, so we need to use the corresponding elimination rule. Our proof will have a line of the form:

$$\dfrac{\exists x Px \qquad \dfrac{}{Pa \to \exists x(Px \vee Qx)}\,?}{\exists x(Px \vee Qx)}\,\exists E$$

How can we complete the proof?

Proof Strategy

We want to show $\vdash \exists x Px \to \exists x(Px \lor Qx)$ and so far our proof looks like this. What can we do to finish this proof?

$$
\cfrac{\overline{\exists x Px} \qquad \cfrac{\cfrac{\cfrac{\overline{Pa}}{Pa \lor Qa}\ \lor I}{\exists x(Px \lor Qx)}\ \exists I}{Pa \to \exists x(Px \lor Qx)}\ \to I}{\exists x(Px \lor Qx)}\ \exists E
$$

Proof Strategy

We have proven the first of our subgoals. Go through the same process to prove the second subgoal.

1. $\exists x Px \to \exists x(Px \vee Qx)$
2. $\exists x Qx \to \exists x(Px \vee Qx)$

Use the disjunction elimination to tie these goals together into a proof of the original statement

$$\exists x Px \vee \exists x Qx \ \vdash \ \exists x(Px \vee Qx)$$

Metalogical Theorems

Gödel proved that first-order predicate logic is both sound and complete.

$$\Sigma \vDash \alpha \quad \text{if and only if} \quad \Sigma \vdash \alpha$$

Thus we can provide deductions for all, and only those, wff that are true in every model satisfying the hypotheses Σ.

We will not prove this. But it's worth noting that the proofs follow the same form as the proofs for the soundness and completeness of propositional logic.

Further Reading

Below is a selection of resources that I used to prepare the lecture. You might like to read over them yourself to help get a more complete picture of the topics discussed.

- L∃∀N, *Logic and Proof*. Sections: 7 and 8.
- Logic Matters, *An Introduction to Formal Logic*: Not so helpful for our version of deduction. But you might like to read Chapter 31, 32, and 33.