Lecture 7: Metalogic

MATH230

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Outline

Logical Equivalence

Definition

We say well-formed formulae are syntactically equivalent if both

$$\alpha \vdash \beta \quad \text{and} \quad \beta \vdash \alpha$$

Examples

- $A \vee B \dashv\vdash B \vee A$
- $A \rightarrow B \ \dashv\vdash \ \neg A \vee B$

Theorems

Definition

We say a well-formed formula α is a theorem if there exists a natural deduction $\mathcal{D}$ from no assumptions i.e. $\Sigma = \emptyset$ and we denote this as $\vdash \alpha$.

Example: Law of the Excluded Middle

Example: $\vdash \alpha \to (\beta \to \alpha)$

Example of Equivalence

If α and β are logically equivalent, then $\vdash \alpha \leftrightarrow \beta$.

Equivalence of Theorems

If $\vdash \alpha$ and $\vdash \beta$, then $\vdash \alpha \leftrightarrow \beta$

Deduction Theorem

Theorem: $\Sigma \vdash \alpha \to \beta$ if and only if $\Sigma \cup \{\alpha\} \vdash \beta$

Proof

Semantics and Syntax

If Σ is a set of hypotheses and α is a well-formed formula, then we have two methods for analysing whether α follows, in some manner, from Σ. We can ask the following:

- Is α a semantic consequence of Σ?
- Is α a syntactic consequence of Σ?

This leads us to the following question: How do these two notions relate to one another?

Metalogical Definitions

Definition

a formal system of deduction is **sound** *if it only allows derivations of valid arguments.*

Definition

a formal system of deduction is (semantically) **complete** *if it allows derivations of every valid argument.*

Soundness

Theorem: If $\Sigma \vdash \alpha$, then $\Sigma \models \alpha$

Proof

Proof Continued

Consistent

Definition

a set of wff Σ is consistent if and only if for no α does $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$.

Definition

a set of wff Σ is maximally consistent if and only if it is consistent and for every α either $\Sigma \cup \{\alpha\}$ is inconsistent or $\Sigma \vdash \alpha$.

Gödel Numbering

Gödel numbering is a method for encoding wff and sequences of wff i.e. proofs. Kurt Gödel used this encoding to prove famous results about the limits of the methods of mathematics.

$\neg$	$\vee$	$\wedge$	$\rightarrow$	(	)	A_1	A_2	A_3	$\cdots$
1	2	3	4	5	6	7	8	9	$\cdots$

For now we will use this encoding to show there are only *countably* many wff. Which means we can, in theory, think of the (infinite) list of wff $\{\alpha_n\}_{n=1}^{\infty}$.

Gödel Numbering Example

Metalogical Lemmata

Lemma: If Σ is consistent, then there exists a maximally consistent $\overline{\Sigma}$ such that $\Sigma \subseteq \overline{\Sigma}$.

Proof

Proof continued

Metalogical Lemmata

Lemma: If $\overline{\Sigma}$ is maximally consistent, then for each wff α either $\overline{\Sigma} \vdash \alpha$ or $\overline{\Sigma} \vdash \neg\alpha$.

Proof

Note: This is called *syntactic completeness*.

Metalogical Lemmata

Lemma If Σ is satisfiable, then Σ is consistent.

Proof: We will prove the contrapositive of this statement.

If Σ is inconsistent, then there exists a wff α such that $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$.

Thus, by soundness, we have $\Sigma \models \alpha$ and $\Sigma \models \neg\alpha$.

Therefore there can be no valuation that satisfies Σ, as it it would necessarily statisfy both α and $\neg\alpha$, which is impossible.

Thus we see Σ is not satisfiable.

Metalogical Lemmata

Lemma If Σ is consistent, then Σ is satisfiable.

Proof: We will prove this directly by constructing a valuation that satisfies Σ. This proof will use induction on the length of well-formed formulae. We provide a valuation on the maximal consistent extension of Σ which will restrict to a valuation on Σ.

Proof continued

Proof continued

Completeness

Theorem If $\Sigma \models \alpha$, then $\Sigma \vdash \alpha$

Proof: We will prove the contrapositive of this statement.

If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{\neg\alpha\}$ is consistent.

Hence $\Sigma \cup \{\neg\alpha\}$ is satisfiable.

So there is a valuation, v, which satisfies Σ and $\neg\alpha$.

Which means this valuation, v, satisfies Σ but does not satisfy α.

Therefore $\Sigma \not\models \alpha$.

Further Reading

Below is a selection of resources that I used to prepare the lecture. You might like to read over them yourself to help get a more complete picture of the topics discussed.

- L∃∀N, *Logic and Proof*. Sections: 6.3 and 6.4
- Logic Matters, *An Introduction to Formal Logic*. Sections 24.5, 24.6, and 24.7.