# THE ÉTALE FUNDAMENTAL GROUP OF $\mathbb{N}$-SCHEMES

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ABSTRACT.

CONTENTS



## 1. INTRODUCTION

**Acknowledgments:** Most of this work was done during Robert Culling's time as James Borger's PhD student at the Australian National University, from 2015 through 2019. This work is presented in much more detail in Culling's thesis which can be found online []. Culling would like to acknowledge the support of the Mathematical Sciences Institute at ANU and the support from the Australian People through the Australian Government Research Training Program Stipend and Scholarship. Without this kindness the work would not have been completed.

## 2. Commutative Algebra of the Natural Numbers

We present the following section on the commutative algebra of the natural numbers for sake of completeness. We follow the presentation of Bertrand Toën and Michel Vaquié [3] and James Borger [1]. The reader may consult these authors for further details. One may also like to consult Johnathon Golan [2] for a different perspective on monoids and semirings.

**Definition 1** A commutative monoid is a triple $(M, +_M, 0_0)$ which consists of: a set, $M$; a commutative and associative binary operation $+ : M \times M \to M$ called addition, and the distinguished element $0 \in M$ such that $+(0, m) = m$ called the identity. If $(M, +_M, 0_M)$ and $(N, +_N, 0_N)$ are commutative monoids, then a homomorphism is a function of sets $\varphi : M \to N$ which maps $\varphi(0_M) = 0_N$ and commutes with the binary operations $+_M$ and $+_N$.

We will denote a commutative monoid by its set and drop the subscripts from the binary operation and identity. Similarly we will write $a + b$ instead of $+(a, b)$. Throughout this paper commutative monoids will be referred to as $\mathbb{N}$-modules and the category of $\mathbb{N}$-modules will be denoted $\mathbf{Mod}_{\mathbb{N}}$.

The category $\mathbf{Mod}_{\mathbb{N}}$ has products and coproducts. If $I$ is a set indexing a family of $\mathbb{N}$-modules $\{M_i\}_I$, then the set theoretic product $\prod_I M_i$ with component wise addition and the distinguished element $0 = (..., 0, ...)$ is the product in $\mathbf{Mod}_{\mathbb{N}}$. It has the sub-$\mathbb{N}$-module $\bigoplus_I M_i \subseteq \prod_I M_i$ of elements that have only finitely many components different from 0, this forms the coproduct of the family in $\mathbf{Mod}_{\mathbb{N}}$.

If $M, N$ are $\mathbb{N}$-modules, then the set $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{N}}}(M, N)$ is an $\mathbb{N}$-module under point-wise addition of homorphisms. Thus for a fixed $\mathbb{N}$-module $M$, there is an endofunctor $\mathrm{Hom}_{\mathbf{Mod}_{\mathbb{N}}}(M, -) : \mathbf{Mod}_{\mathbb{N}} \to \mathbf{Mod}_{\mathbb{N}}$ on the category of $\mathbb{N}$-modules. This functor has a left adjoint, $M \otimes -$. Thus for each $M, N$ in $\mathbf{Mod}_{\mathbb{N}}$ there is an $\mathbb{N}$-module $M \otimes N$ defined by the universal property of the left adjoint. This can be constructed in the same way as for $\mathbb{Z}$-modules, as long as one is careful to avoid negatives. See [1] for details.

**Definition 2** A commutative $\mathbb{N}$-algebra is defined to be a commutative monoid $(A, +_A, 0_A)$ equipped with a commutative binary operation denoted $\times_A$ that distributes over $+_A$, with the property that for each $a \in A$ the equation $0_A \times_A a = 0_A$ is true, and has identity $1_A$. If $(A, +_A, \times_A, 0_A, 1_A)$ and $(B, +_B, \times_B, 0_B, 1_B)$ are commutative $\mathbb{N}$-algebras, then a homomorphism of $\mathbb{N}$-algebras is a homomorphism of monoids $\varphi : A \to B$ that commutes with the binary operations $\times_A$ and $\times_B$. If there exists a homomorphism $\varphi : A \to B$, then we say that $B$ is an $A$-algebra.

Again, we will drop all subscripts and denote the $\mathbb{N}$-algebras by their underlying set. All $\mathbb{N}$-algebrs are assumed to be commutative for the rest of the paper, so we will drop the use of commutative. In the literature $\mathbb{N}$-algebras are more commonly referred to as semirings. Notice that an $\mathbb{N}$-algebra structure on a monoid is equivalent to a commutative monoid object in the monoidal category $(\mathbf{Mod}_{\mathbb{N}}, \otimes)$.

## 3. Arithmetic Geometry of the Natural Numbers

## 4. Galois Category of Finite Étale Morphisms

## 5. Non-Negative Reals and Real Number Fields

## 6. Natural Numbers

## 7. Non-Trivial Quadratic Finite Étale Cover

## References

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[2] Johnathon S Golan. *Semirings and their Applications*. Kluwer Academic Publishers, 1999.

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