Geometric Series (Aside)

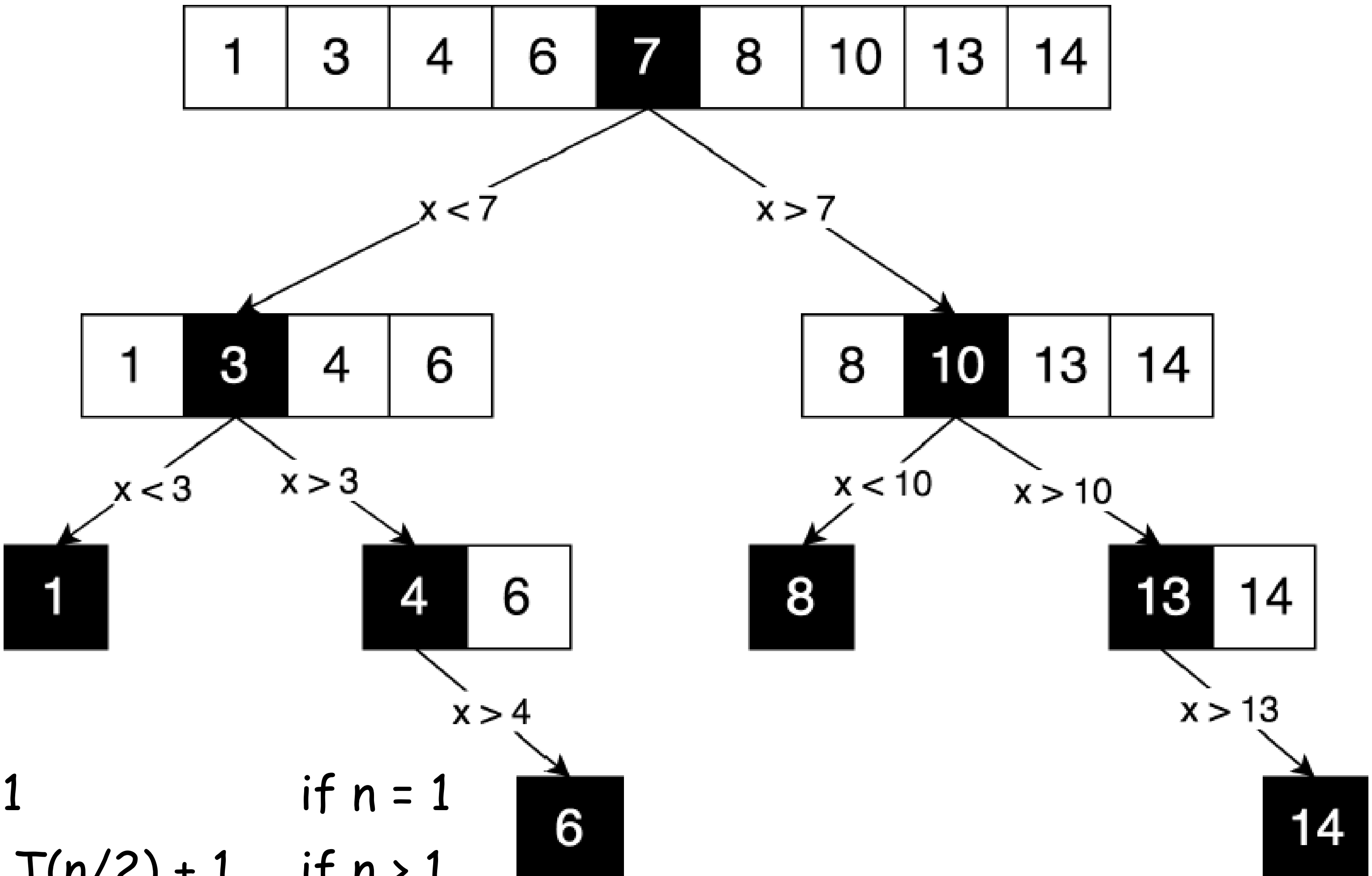
$$1 + x + x^2 + \ \ldots + \ x^n = \frac{1 - \ x^{n+1}}{1 - x} \ for \ x \ \neq 1$$

OR

$$1 + x + x^2 + \ \ldots + \ x^n = \frac{x^{n+1} \ - 1}{x \ - 1} \ for \ x \ \neq 1$$

$$1 + x + x^2 + \ \ldots + \ x^n = \frac{1}{1 - x} \ for \ x < 1$$

Binary search example



$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n > 1 \end{cases}$$

Substitution (Example # 1)

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

$T(n) = n + 2T(n/2)$

$T(n/2) = n/2 + 2T(n/2/2)$

$T(n/2) = n/2 + 2T(n/4)$

$T(n/4) = n/4 + 2T(n/4/2)$

$T(n/4) = n/4 + 2T(n/8)$

For $2^k = n$ => $k = \log n$

By putting 'log' both sides, and

Using: $\log a^b = b.\log a$

$T(n) = n + 2T(n/2)$

$T(n) = n + 2(n/2+2T(n/4))$

$T(n) = n + 2n/2 + 4T(n/4)$

$T(n) = n + 2n/2 + 4(n/4 + 2T(n/8))$

$T(n) = n + n + n + 8T(n/8)$

$T(n) = 3n + 8(n/8 + 2T(n/16))$

$T(n) = 4n + 2^4 T(n/2^4)$

.....

$T(n) = kn + 2^k.T(n/2^k)$

$T(n) = n.\log n + n.T(1)$

$T(n) = O(n\log n)$

Substitution (Example # 2)

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n > 1 \end{cases}$$

$T(n) = n + T(n/2)$

$T(n) = n + (n/2+T(n/4))$

$T(n) = n + n/2 + T(n/4)$

$T(n) = n + n/2 + n/4 + T(n/8))$

$T(n) = n + n/2 + n/4 + T(n/8)$

$T(n) = n (1 + 1/2 + 1/ 4) + T(n/2^3)$

.....

$T(n) = n(1 + ½ + ¼ .... + 1/ 2^k) + T(n/2^k)$

For $2^k = n$ => $k = \log n$

$T(n) = n(1+ ½ + ¼ + ... + 1/n) + T(1)$

$T(n) = n.(1+1) + 1$

$T(n) = \text{Theta}(n)$

Substitution (Example # 3)

Using: $a^{\log b} = b.\log a$

Previously: $\log a^b = b.\log a$

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n > 1 \end{cases}$$

$T(n) = n + 4T(n/2)$

$T(n/2) = n/2 + 4T(n/2/2)$

$T(n/2) = n/2 + 4T(n/4)$

$T(n/4) = n/4 + 4T(n/4/2)$

$T(n/4) = n/4 + 4T(n/8)$

=====================

$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^k$

$= \frac{2^{k+1} - 1}{2 - 1}$

=====================

For $k = \log n \Rightarrow n = 2^k$

$T(n) = n + 4T(n/2)$

$T(n) = n + 4(n/2 + 4T(n/4))$

$T(n) = n + 4n/2 + 16T(n/4)$

$T(n) = n + 2n + 4^2(n/4 + 4T(n/8))$

$T(n) = n + 2n + 4n + 4^3\, T(n/8)$

$T(n) = n + 2n + 4 + 4^3(n/8 + 4T(n/16))$

$T(n) = n + 2n + 4n + 8n + 4^4\, T(n/2^4)$

$T(n) = n\,(1 + 2 + 4 + 8) + 4^4 T(n/\,2^4)$

...

$T(n) = n.\, \frac{2^{k+1} - 1}{2 - 1} + 4^k.T(n/2^k)$

$T(n) = n.\, (2^{\log n+1} - 1) + 4^{\log n}\, .\, T(1)$

$T(n) = O(n^2)$

Substitution (Example # 4)

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n > 1 \end{cases}$$

$T(n) = 1 + T(n/2)$

$T(n) = 1 + (1+T(n/4))$

$T(n) = 2 + T(n/4)$

$T(n) = 3 + T(n/8)$

$T(n) = 3 + T(n/2^3)$

…..

$T(n) = k + T(n/2^k)$

For $k = \log n$, as: $n = 2^k$

$T(n) = \log n + T(1)$

$T(n) = 1 + \log(n)$

i.e. $T(n) = \Theta(\log n)$ (Theta (logn))

Substitution (Example # 5)

$T(n) = 1 + T(n-1)$

$T(n) = 1 + (1+T(n-2))$

$T(n) = 2 + T(n-2)$

$T(n) = 3 + T(n-3)$

$T(n) = 4 + T(n-4)$

…..

$T(n) = k + T(n-k)$

For $k = (n-1)$

$T(n) = n – 1 + T(1)$

$T(n) = n$

i.e. $T(n) = \text{Theta}(n)$

For $n – k = 1$

So, $k = n-1$

Substitution (Example # 6)

$T(n) = n + T(n-1)$

$T(n) = n + (n-1+T(n-2))$

$T(n) = 2n – 1 + T(n-2)$

$T(n) = 2n -1 + ((n-2) + T(n-3))$

$T(n) = 3n -3 + T(n-3)$

$T(n) = 3n -3 + n-3 + T(n-4))$

$T(n) = 4n – 6 + T(n-4)$

…..

$T(n) = kn – c + T(n-k)$

$T(n) = (n – 1).n - n – c + T(1)$

$T(n) = n^2 - n – c + T(1)$

i.e. $T(n) = Theta(n^2)$

For $n – k = 1$

So, $k = n-1$

Substitution (Example # 7)

$T(n) = 2T(n-1)$

$T(n) = 2(2T(n-2))$

$T(n) = 4T(n-2)$

$T(n) = 4(2T(n-3))$

$T(n) = 8T(n-3)$

$T(n) = 8(2T(n-4))$

$T(n) = 16T(n-4)$

$T(n) = 2\^4T(n-4)$

……..

$T(n) = 2\^k.T(n-k)$

$T(n) = 2\^(n-1).T(1)$

$T(n) = O(2\^n)$

For $n – k = 1$

So, $k = n-1$

The master theorem

- Suppose that $a \geq 1, b > 1,$ and d are constants (independent of n).
- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

Three parameters:
a : number of subproblems
b : factor by which input size shrinks
d : need to do n^d work to create all the subproblems and combine their solutions.

We can also take n/b to mean either $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$ and the theorem is still true.

The master theorem (Limitations)

You can not use Master Theorem if:

- T(n) is not monotone, ex: $T(n) = \sin n$
- f(n) is not a polynomial, ex: $T(n) = 2\,T(n/2) + 2^n$
- b cannot be expresses as a contact, ex: $b = 2n$

Examples

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

- An example
  - T(n) = 4 T(n/2) + O(n)
  - T(n) = O(n^2)

a = 4
b = 2
d = 1

a > b^d ✓

- Binary Search
  - T(n) = T(n/2) + c
  - T(n) = O (log(n))

a = 1
b = 2
d = 0

a = b^d ✓

- MergeSort
  - T(n) = 2T(n/2) + O(n)
  - T(n) = O(nlog(n))

a = 2
b = 2
d = 1

a = b^d ✓

- That other one
  - T(n) = T(n/2) + O(n)
  - T(n) = O(n)

a = 1
b = 2
d = 1

a < b^d ✓

Master Theorem Example

- Let T(n) = 2.T(n/4) + sqrt (n) + 42.

  What are the parameters?

  a = 2, b = 4, d = ½

Therefore which condition?

Since $2 = 4^{1/2}$, case 2 applies.

Thus we conclude that:

T (n) ϵ θ (n^d log n) = θ (sqrt(n). log n)

Master Theorem 4th Case

- Fourth Condition:

Recall that we cannot use the Master Theorem if $f(n)$ (the non-recursive cost) is not polynomial.

There is a limited 4-th condition of the Master Theorem that allows us to consider polylogarithmic functions.

Corollary

If $f(n) \in \Theta(n^{\log_b a} \log^k n)$ *for some* $k \geq 0$ *then*

$$T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$$

This final condition is fairly limited and we present it merely for completeness.

MASTER THEOREM

- **$T(n) = 2 \cdot T(n/2) + n \log n$**

a = ?
b = ?
d = ?

a = 2
b = 2
d = n/a
f(n) is not polynomial

i.e. $f(n) = \Theta(n \log n)$

So k = 1 therefore, fourth condition of master theorem

$T(n) = \Theta(n \log^2 n)$

Corollary

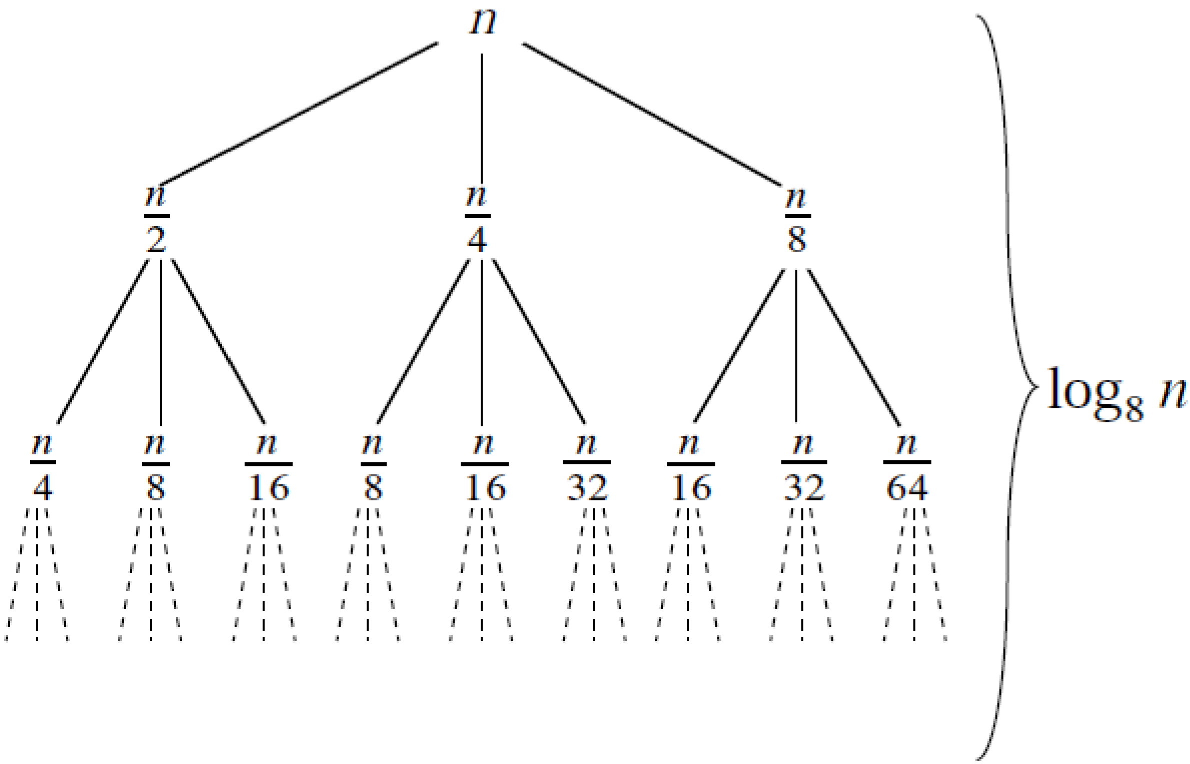
If $f(n) \in \Theta(n^{\log_b a} \log^k n)$ *for some* $k \geq 0$ *then*

$$T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$$

Recursion Tree

Recursion Tree (Example # 2)

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$



$$n$$

$$n\left(\frac{4+2+1}{8}\right) = \frac{7}{8}n$$

$$n\left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{2}{32} + \frac{1}{64}\right)$$
$$= n\,\frac{16+16+12+4+1}{64}$$
$$= n\,\frac{49}{64} = \frac{7}{8}^{2} n$$

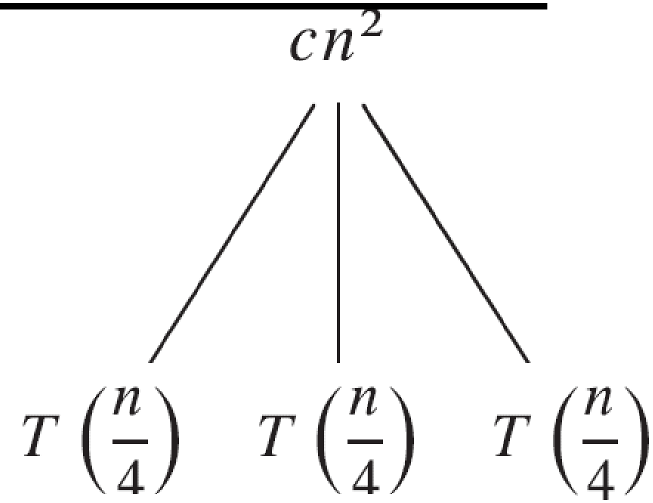
$$\vdots$$

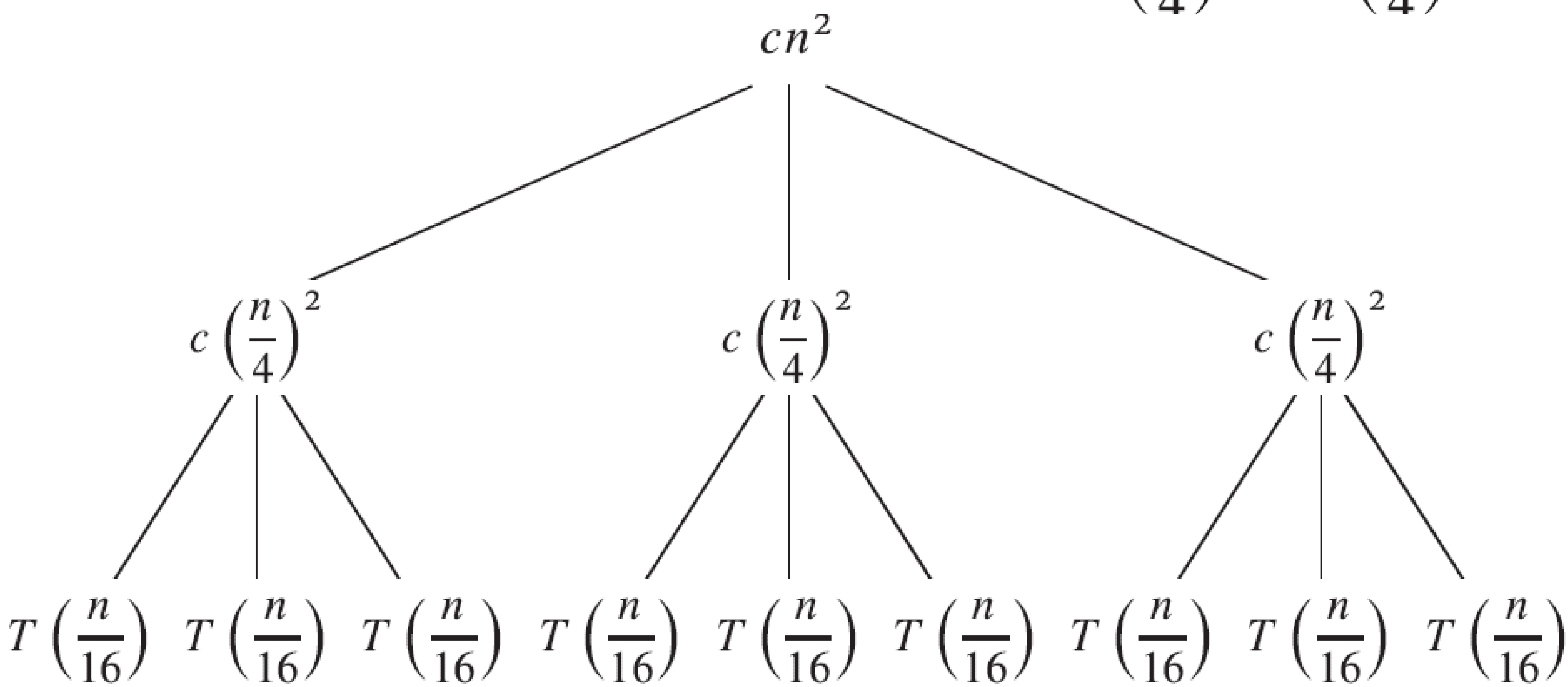
$$\sum_{i=1}^{\log n} \left(\frac{7}{8}\right)^{i} n = \Theta(n)$$

Recursion Tree (Example # 3i)

$T(n)$

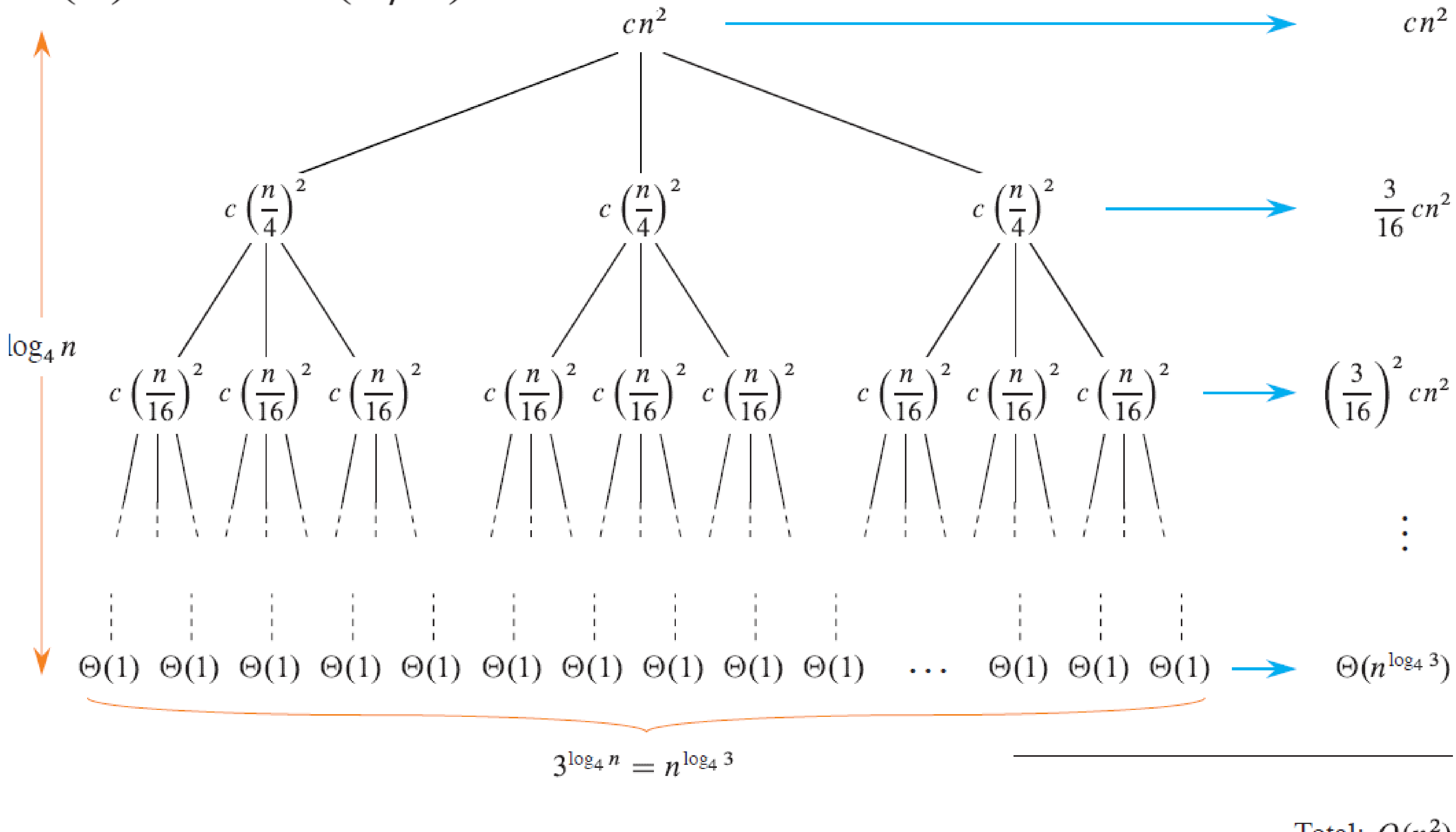
$$T(n) = 3T(n/4) + cn^2$$





Recursion Tree (Example # 3ii)

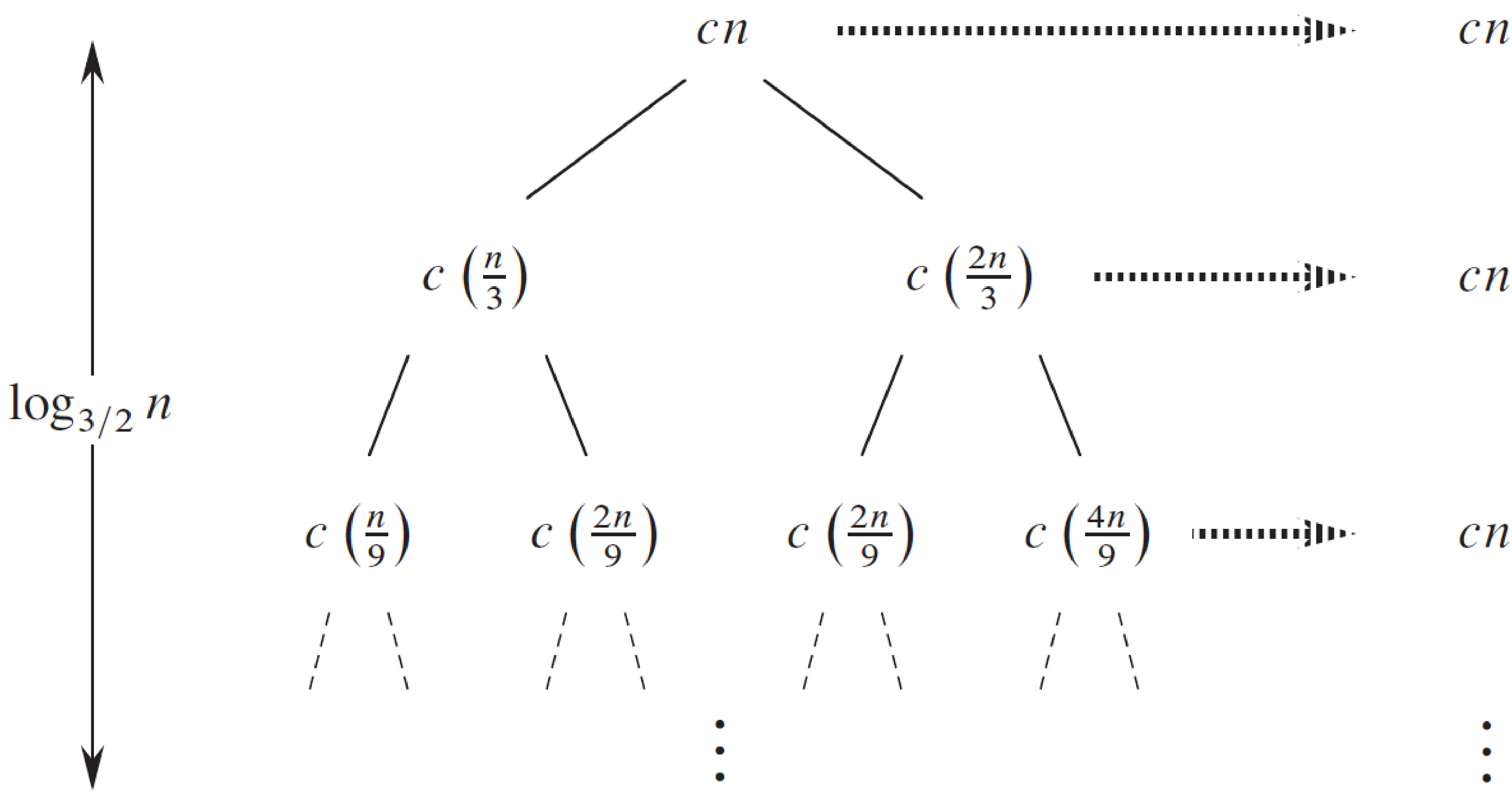
$$T(n) = 3T(n/4) + cn^2$$



(d)

Recursion Tree (Example # 4)

$$T(n) = T(n/3) + T(2n/3) + cn$$

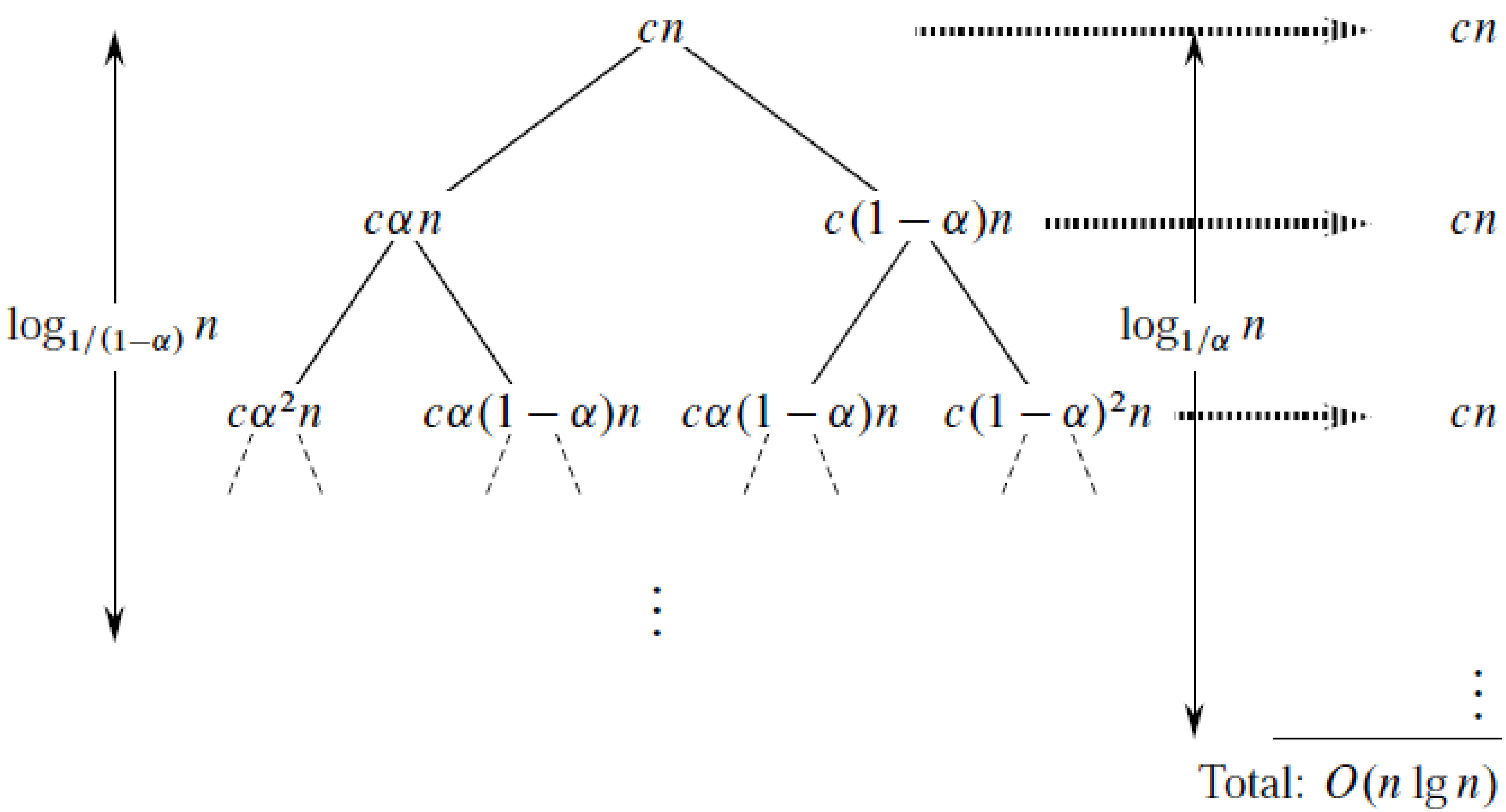


Total: $O(n \lg n)$

Recursion Tree (Example # 5)

$$T(n) = T(\alpha n) + T((1-\alpha)n) + cn$$

Without loss of generality, let $\alpha \geq 1-\alpha$, so that $0 < 1-\alpha \leq 1/2$ and $1/2 \leq \alpha < 1$.



Substitution Method (Guess and Test)

1. Guess the form of the solution or Guess what the answer is
   (iterative substitution: iteratively apply the recurrence equation to itself to find a possible pattern)

2. Prove your guess is correct, using mathematical induction
   (Guess and Test method).

Solving Recurrences by Substitution: Guess-and-Test

$$\mathbf{T(n) = 2T(n/2) + n}$$

Guess (#1)	$\mathbf{T(n) = O(n)}$	
Inductive Hypothesis	$\mathbf{T(n) <= cn}$	for some constant c>0
Inductive Step	$T(n/2) <= cn/2$	

$$T(n) = 2T(n/2) + n$$

$$T(n) \leq 2 \cdot c(n/2) + n$$

$$T(n) \leq cn + n$$

$$T(n) \leq (c+1)\, n$$

no choice of c could ever make (c + 1) n ≤ cn!

Our guess was wrong!!

Solving Recurrences by Substitution: G #2

$T(n) = 2T(n/2) + n$

Guess (#2) **$T(n) = O(n^2)$**

IH **$T(n) \le cn^2$** for some constant $c>0$

Inductive Step $T(n/2) \le c.\frac{n^2}{4}$

$$T(n) = 2T(n/2) + n$$

$$T(n) \le 2 \cdot \left(\frac{cn^2}{4}\right) + n$$

$$T(n) \le \frac{cn^2}{2} + n$$

$$T(n) \le \frac{cn2}{2} + n \le cn^2$$

Works for all n as long as c>=2 !!

Solving Recurrences by Substitution: G #3

T(n) = 2T(n/2) + n

Guess (#3) **T(n) = O(nlogn)**

IH **T(n) <= cnlogn** for some constant c>0

Inductive Step $T\left(\frac{n}{2}\right) \leq c\,\frac{n}{2}\log(\frac{n}{2})$

$$T(n) = 2T(n/2) + n$$

$$T(n) \leq 2 \cdot c\,\frac{n}{2}\log(\frac{n}{2}) + n$$

$$T(n) \leq cn\,(logn - \log 2) + n$$

$$T(n) \leq cn \log n - cn + n$$

Thus $T(n) \leq cn \log n - cn + n$ <= cnlogn

Works for all n as long as c>=1 !!

Guess and Test Method by Substitution: Ex #2, G # 1

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

Guess (# 1) $\quad$ T(n) = O(n log n)

(Inductive Hypothesis): $\quad \mathrm{T(n)} \leq \mathrm{c\,n\,log\,n} \quad$ for $\mathrm{c} > 0$

Inductive step, Assume $\quad \mathrm{T}\left(\frac{\mathrm{n}}{2}\right) \leq \mathrm{c}\,\frac{\mathrm{n}}{2}\,\log\left(\frac{\mathrm{n}}{2}\right)$

$$\mathrm{T(n)} = 2\mathrm{T}\left(\frac{\mathrm{n}}{2}\right) + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq 2 \cdot \mathrm{c}\,\frac{\mathrm{n}}{2}\,\log\left(\frac{\mathrm{n}}{2}\right) + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq cn\,(logn - \log 2) + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq cn\log n - cn + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq (c+b)n\log n - cn$$

Wrong: we cannot make this last line be less than cn log n

Guess and Test Method by Substitution: Ex #2, G # 2

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

Guess (# 1) $\quad T(n) = O(n \log^2 n)$

(Inductive Hypothesis): $\quad \mathrm{T(n)} \leq \mathrm{c\,n\,log^2\,n} \quad$ for $c > 0$

Inductive step, Assume $\quad \mathrm{T}\left(\frac{\mathrm{n}}{2}\right) \leq \mathrm{c}\,\frac{\mathrm{n}}{2}\,\log^2(\frac{\mathrm{n}}{2})$

$$\mathrm{T(n)} = 2\mathrm{T}\left(\frac{\mathrm{n}}{2}\right) + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq 2 \cdot \mathrm{c}\,\frac{\mathrm{n}}{2}\,\log^2(\frac{\mathrm{n}}{2}) + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq \mathrm{cn}\,\log^2(\frac{\mathrm{n}}{2}) + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq cn\,(logn - \log 2)^2 + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq cn\log^2 n - 2cnlogn + cn + \mathrm{bnlogn}$$

$$\mathrm{T(n)} \leq cn\log^2 n + (b - 2c)nlogn + cn$$

if c > b.
So, T(n) is $O(n \log^2 n)$.
In general, to use this method, you need to have a good guess and you need to be good at induction proofs.

Backup Slides

Understanding the Master Theorem

- Let $a \geq 1, b > 1,$ and d be constants.
- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

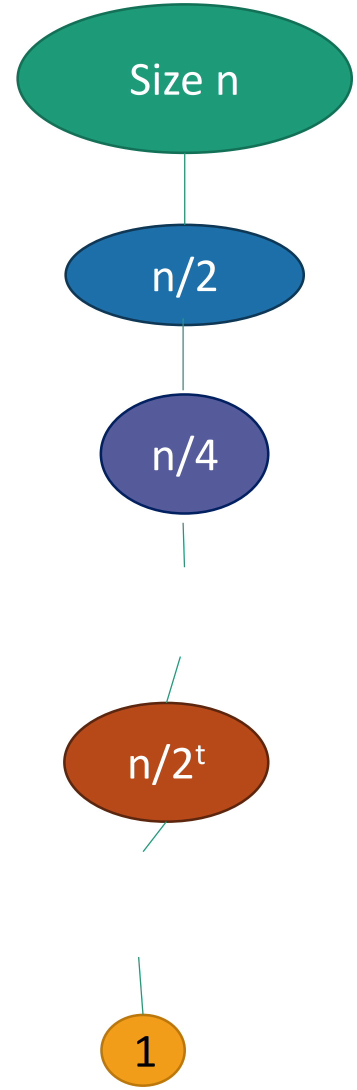
Consider the three examples

1. $T(n) = T\left(\frac{n}{2}\right) + n$
2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$
3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n$

First example

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

$$1.\ T(n) = T\left(\frac{n}{2}\right) + n, \qquad \left(a < b^d\right)$$



- The amount of work done at the top (the biggest problem) is higher than the amount of work done anywhere else.

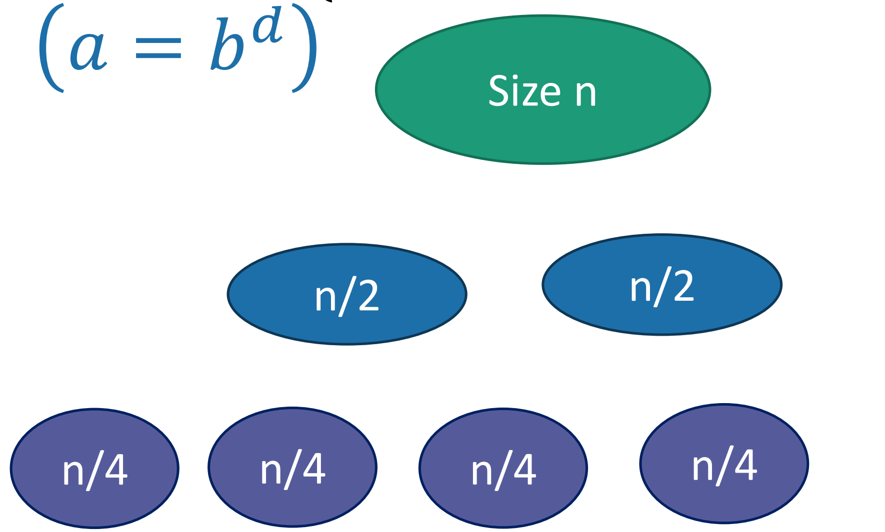
- T(n) = O(work at top) = O(n)

Most work at the top of the tree!

Second example

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n, \qquad \left(a = b^d\right)$

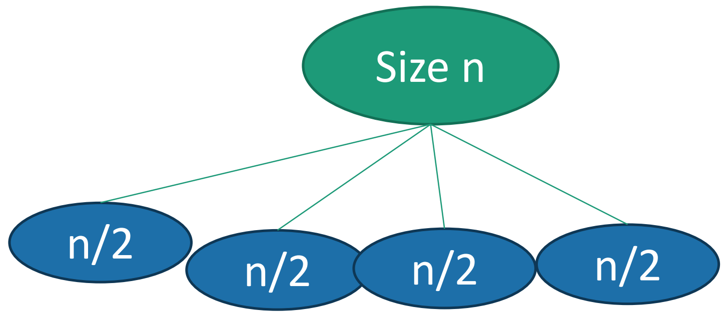


- The branching **just** balances out the amount of work.
- The same amount of work is done at every level.
- T(n) = (number of levels) * (work per level)
- = log(n) * O(n) = O(nlog(n))

Third example

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

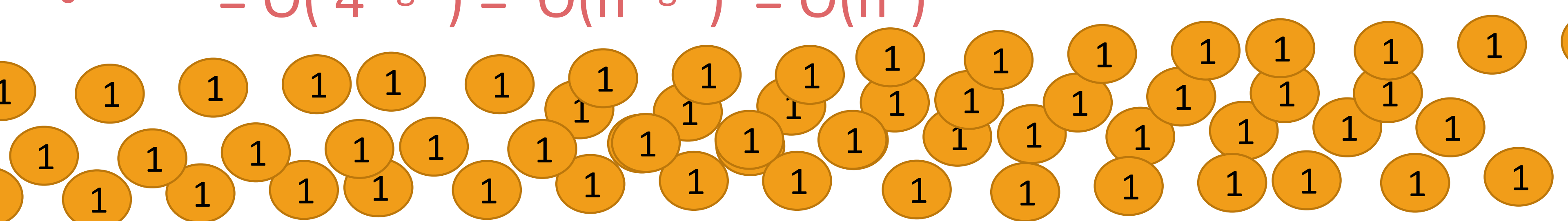
3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n, \qquad \left(a > b^d\right)$



Most work at the bottom of the tree!

- There are a HUGE number of leaves, and the total work is dominated by the time to do work at these leaves.

- T(n) = O(work at bottom) = O($4^{\text{depth of tree}}$)
- = O($4^{\log n}$) = O($n^{\log 4}$) = O(n^2)



Solve the Recurrence (Example # 2)

Just for backup.
Could confuse

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n > 1 \end{cases}$$

For $2^k = n \quad \Rightarrow \quad k = \log n$

For $2^{k-1} = 2^k\, 2^{-1} = 2^k / 2^1 \Rightarrow n/2$

$T(n) = n + T(n/2)$

$T(n) = n + (n/2 + T(n/4))$

$T(n) = n + n/2 + T(n/4)$

$T(n) = n + n/2 + n/4 + T(n/8))$

$T(n) = n + n/2 + n/4 + T(n/8).$

$T(n) = n\ (1 + 1/2 + 1/\ 2^{3-1}\ ) + T(n/2^3)$

.....

$T(n) = n(1 + ½ + ¼ \ldots\ldots. + 1/\ 2^{k-1}) + T(n/2^k)$

$T(n) = n(1 + ½ + ¼ + \ldots. + 1/n) + 1/n + T(1)$

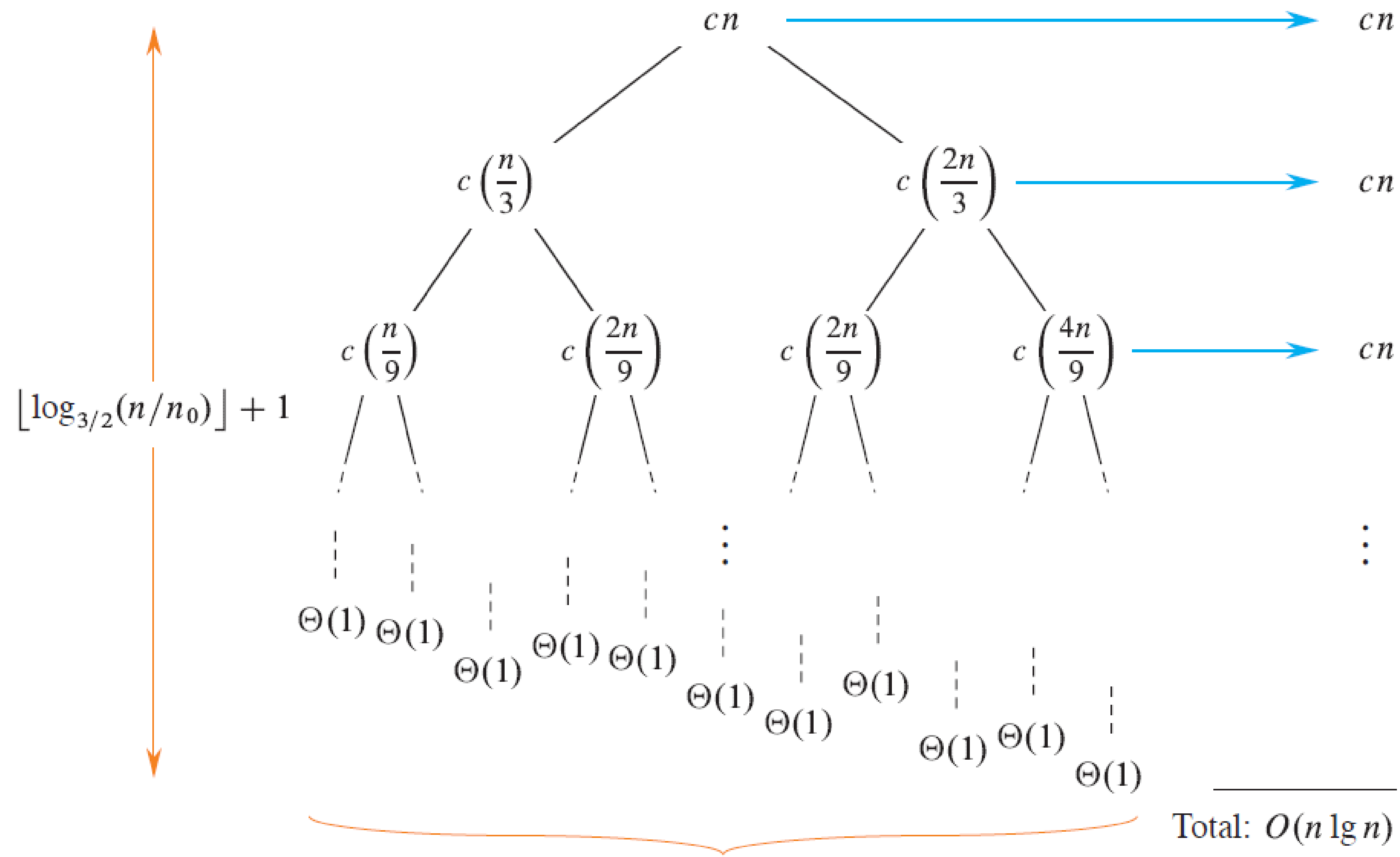
$T(n) = n.(1+1) + 1/n + 1$

$T(n) = \text{Theta}(n)$

Recursion Tree (Example # 3)

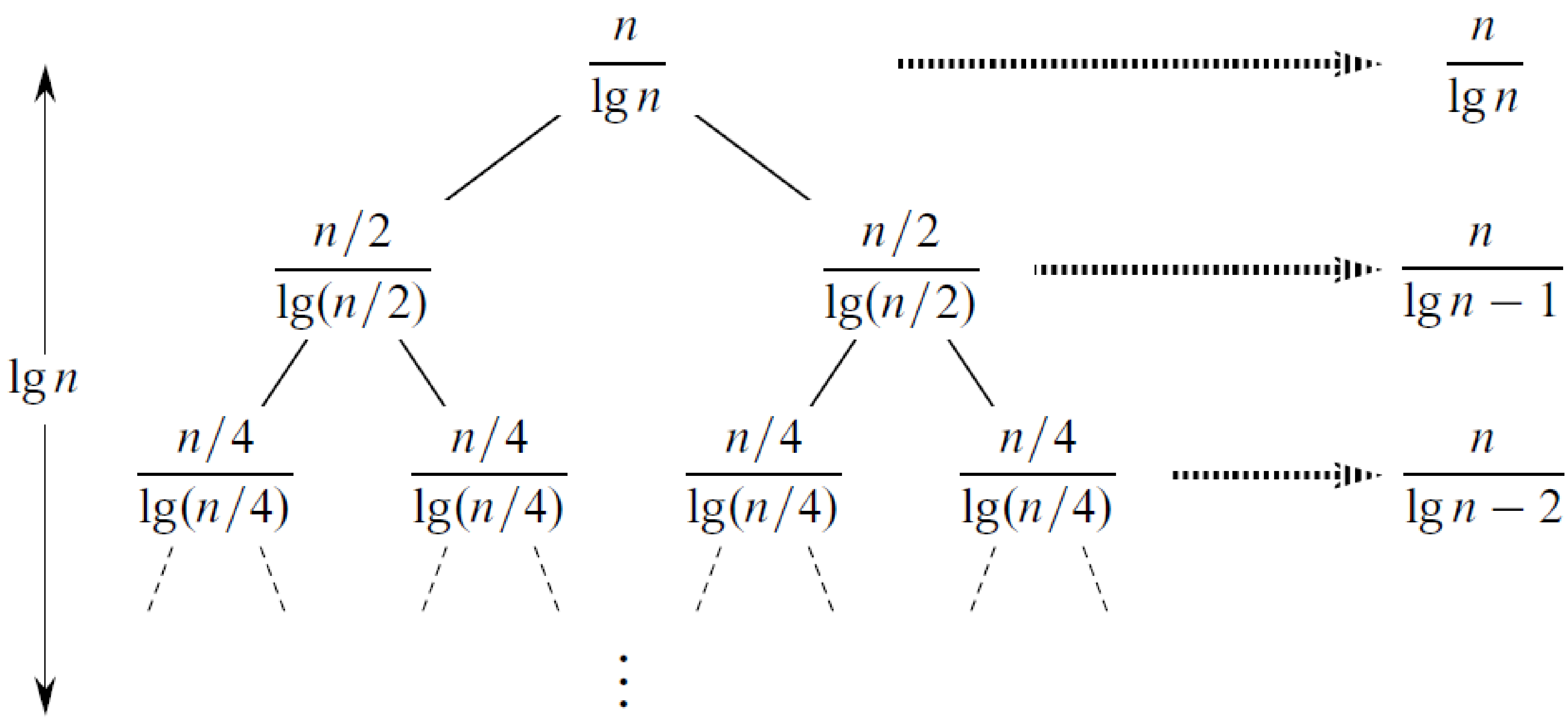
Just for backup.
Could confuse

$$T(n) = T(n/3) + T(2n/3) + cn$$



Recursion Tree (Example # 6)

$$T(n) = 2T(n/2) + n/\lg n$$



- # leaves = n, as reducing by 2 times n/2
- n/2 minimum decrement, so $\log_2 n$ levels

$$\sum_{i=0}^{\lg n - 1} \frac{n}{\lg n - i} = \Theta(n \lg \lg n)$$

Now let's check all the cases

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

Case 1: $a = b^d$

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

- $T(n) = c \cdot n^d \cdot \sum_{t=0}^{log_b(n)} \left(\frac{a}{b^d}\right)^t$

$$= c \cdot n^d \cdot \sum_{t=0}^{log_b(n)} 1$$

Equal to 1!

$$= c \cdot n^d \cdot (\log_b(n) + 1)$$

$$= c \cdot n^d \cdot \left(\frac{\log(n)}{\log(b)} + 1\right)$$

$$= \Theta\left(n^d \log(n)\right)$$

Case 2: $a < b^d$

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

- $T(n) = c \cdot n^d \cdot \sum_{t=0}^{log_b(n)} \left(\frac{a}{b^d}\right)^t$

$= c \cdot n^d \cdot [\text{some constant}]$

$= \Theta\left(n^d\right)$

Less than 1!

Case 3: $a > b^d$

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- $T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$ Larger than 1!

$$= c \cdot n^d \left(\frac{\left(\frac{a}{b^d}\right)^{\log_b(n)+1} - 1}{\frac{a}{b^d} - 1} \right)$$

$$= \Theta\left(n^d \left(\frac{a}{b^d}\right)^{\log_b(n)}\right) = \Theta\left(n^d \left(\frac{a^{\log_b(n)}}{{b^d}^{\log_b(n)}}\right)\right)$$

$$= \Theta\left(n^d \left(\frac{n^{\log_b(a)}}{n^{\log_b(b^d)}}\right)\right) = \Theta\left(n^d \left(\frac{n^{\log_b(a)}}{n^d}\right)\right)$$

$$= \Theta\left(n^{\log_b(a)}\right)$$

Recursion Tree method

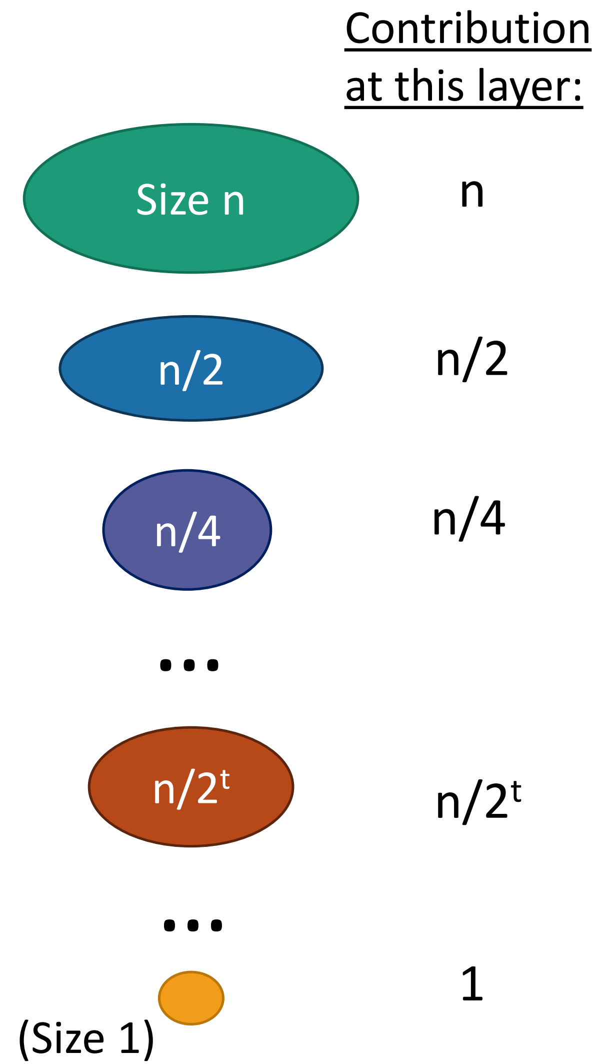
Understanding Master Theorem
Detailed solution, First Example

- $T_1(n) = T_1\left(\frac{n}{2}\right) + n, \quad T_1(1) = 1.$
- Adding up over all layers:

$$\sum_{i=0}^{\log(n)} \frac{n}{2^i}$$

$$= n.\sum_{i=0}^{\log(n)} \frac{1}{2^i} = n.2$$

- So $T_1(n) = O(n)$.



Recursion Tree method

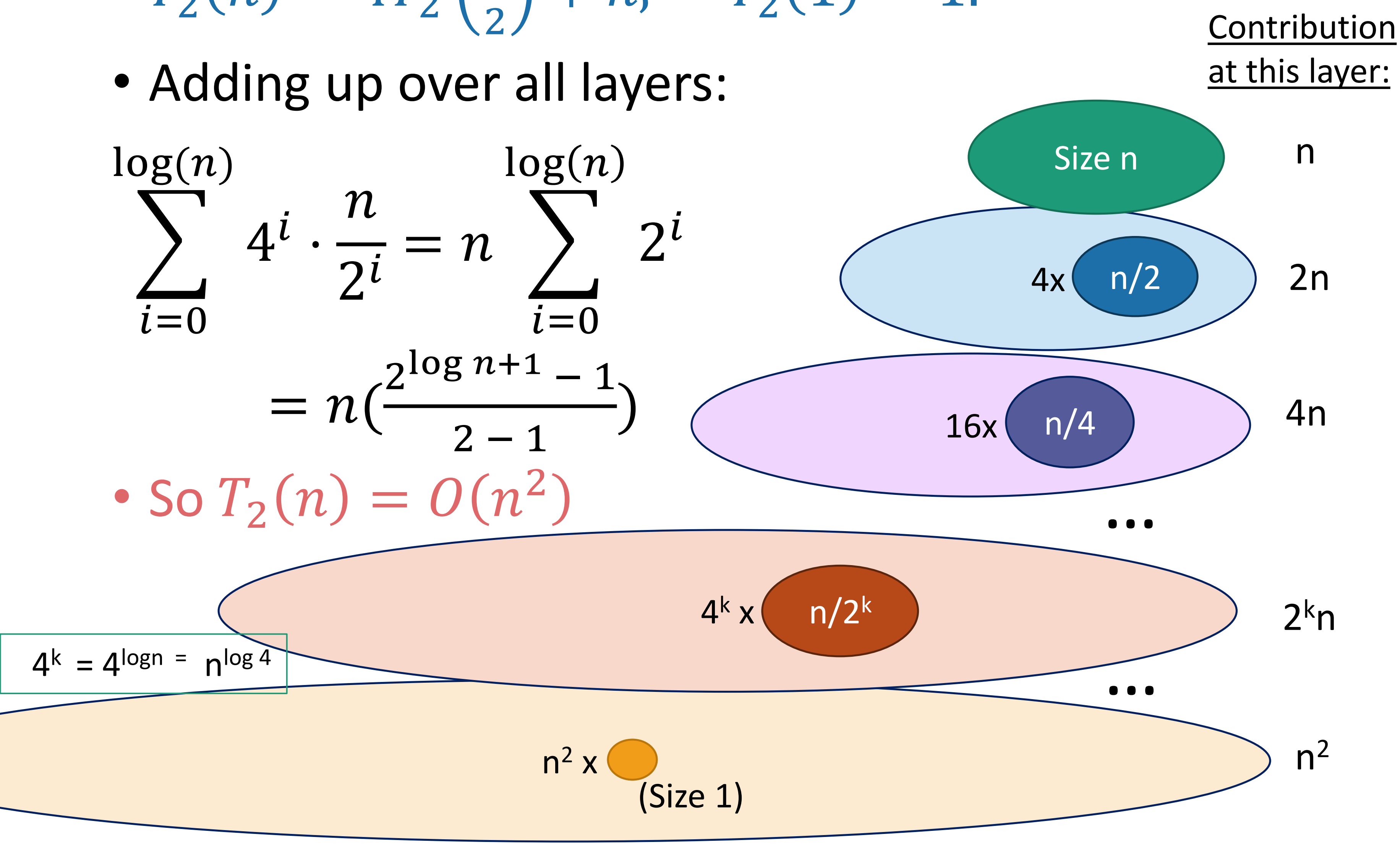
Understanding Master Theorem
Detailed solution, Third Example

- $T_2(n) = 4T_2\left(\frac{n}{2}\right) + n, \quad T_2(1) = 1.$
- Adding up over all layers:

$$\sum_{i=0}^{\log(n)} 4^i \cdot \frac{n}{2^i} = n \sum_{i=0}^{\log(n)} 2^i$$

$$= n\left(\frac{2^{\log n + 1} - 1}{2 - 1}\right)$$

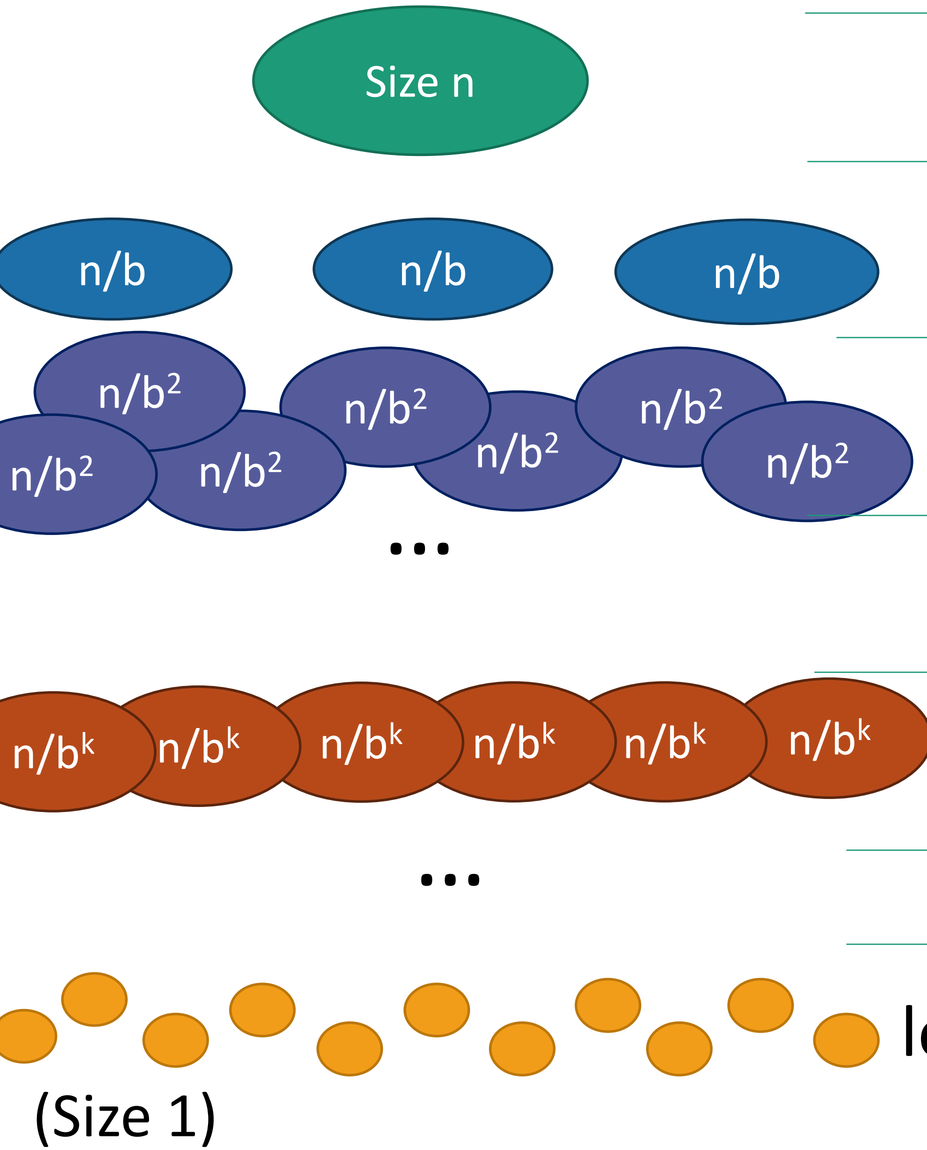
- So $T_2(n) = O(n^2)$



$4^k = 4^{\log n} = n^{\log 4}$

Recursion tree

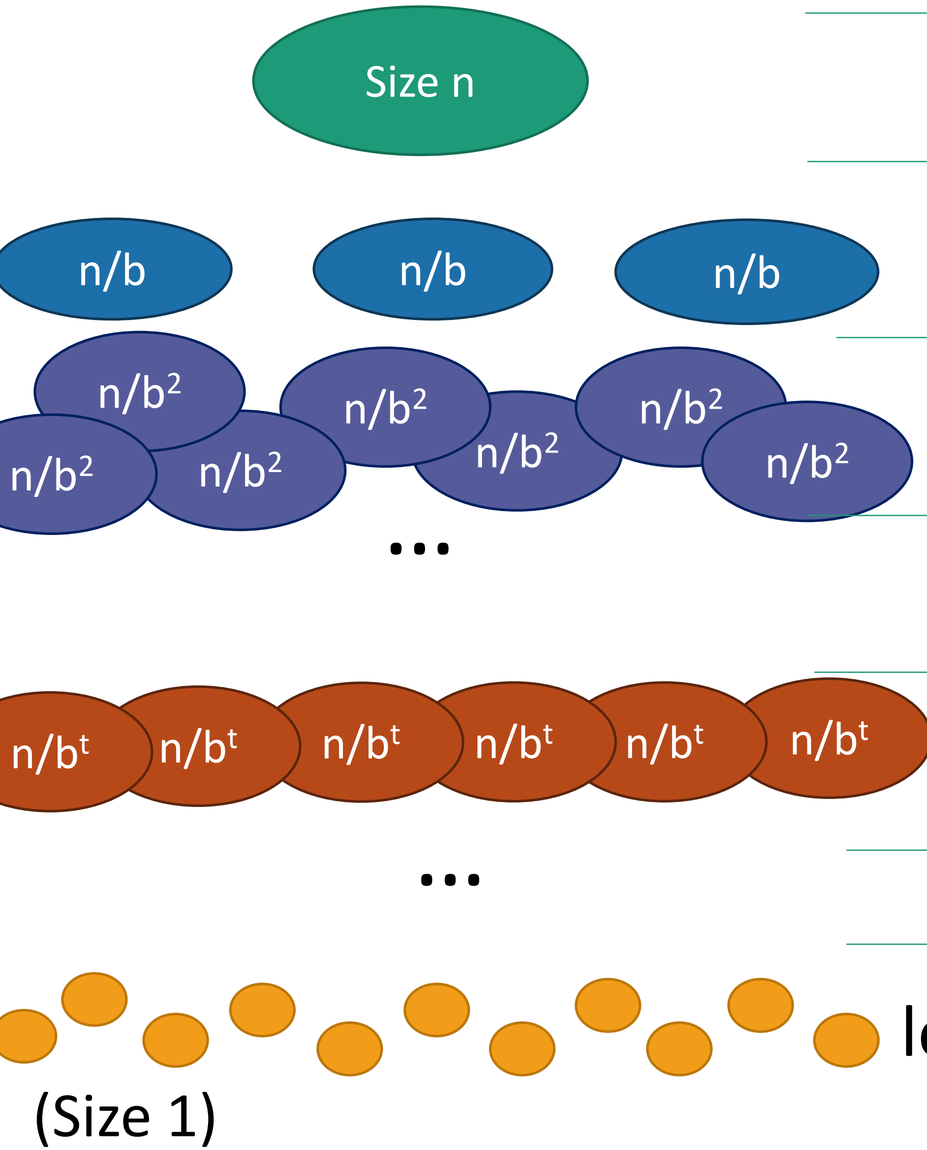
$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	
1	a	n/b	
2	a^2	n/b^2	
k	a^k	n/b^k	
…			
$\log_b(n)$	$a^{\log_b(n)}$	1	

Recursion tree

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
2	a^2	n/b^2	$a^2c\left(\frac{n}{b^2}\right)^d$
t	a^t	n/b^t	$a^tc\left(\frac{n}{b^t}\right)^d$
…			
$\log_b(n)$	$a^{\log_b(n)}$	1	$a^{\log_b(n)}c$

(Let's pretend that the base case is T(1) = c for convenience).

Recursion tree

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

Size n

n/b n/b n/b

Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
			$a^2 c\left(\frac{n}{b^2}\right)^d$
			$a^t c\left(\frac{n}{b^t}\right)^d$
…			
$\log_b(n)$	$a^{\log_b(n)}$	1	$a^{\log_b(n)} c$

…

(Size 1)

(Let's pretend that the base case isT(1) = c for convenience).

Total work (derivation on board) is at most:

$$c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$$