CS 2009
Design and Analysis of Algorithms

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HEAP SORT

BINARY TREE TERMINOLOGY

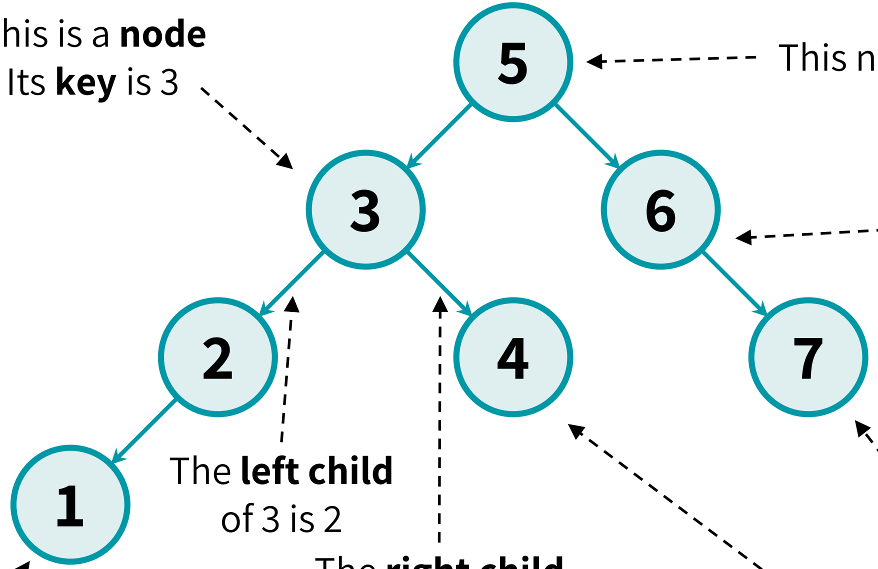
Each node has two children

Each node has a pointer to its left child, right child, and parent

The **left descendants** of 5 are 1, 2, 3, and 4

The **ancestors** of 1 are 2, 3, and 5

This is a **node**
Its **key** is 3



This node is the **root**

6 is the **parent** of 7

The **height** of this tree is 3
(max number of edges from root to a leaf)

The **left child** of 3 is 2

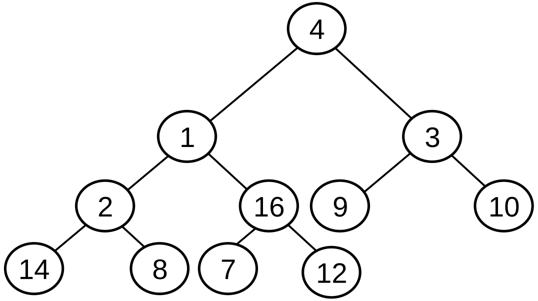
The **right child** of 3 is 4

Both children of a leaf node are **NIL**
(I usually won't draw them)

Nodes without children are **leaves** (aka **leaf nodes**)

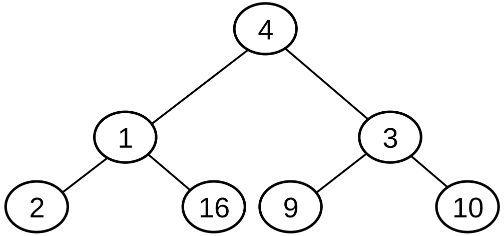
Special Types of Trees

- *Def:* **Full binary tree** = a binary tree in which each node is either a leaf or has degree exactly 2.



Full binary tree

- *Def:* **Complete binary tree** = a binary tree in which all leaves are on the same level and all internal nodes have degree 2.

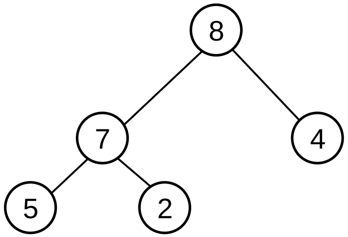


Complete binary tree

The Heap Data Structure

- *Def:* A **heap** is a nearly complete binary tree with the following two properties:
  - **Structural property:** all levels are full, except possibly the last one, which is filled from left to right
  - **Order (heap) property:** for any node x

Parent(x) ≥ x

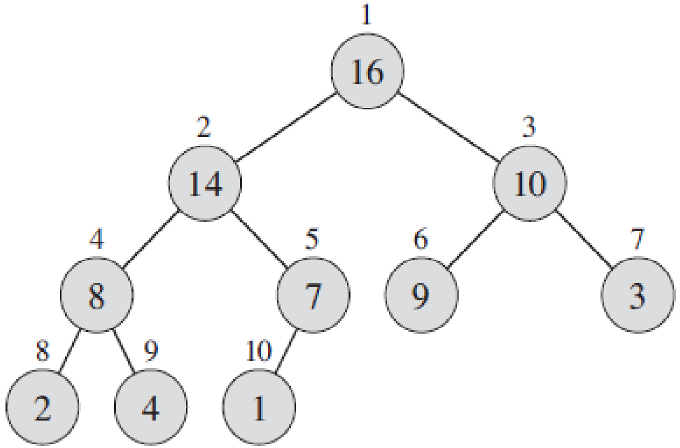


Heap

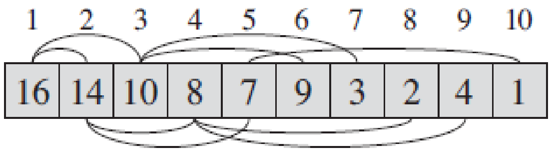
From the heap property, it follows that:
"The root is the maximum element of the heap!"

A heap is a binary tree that is filled in order

Array Representation of Heaps



(a)



(b)

Root of tree is $A[1]$

Left child of $A[i] = A[2i]$

Right child of $A[i] = A[2i + 1]$

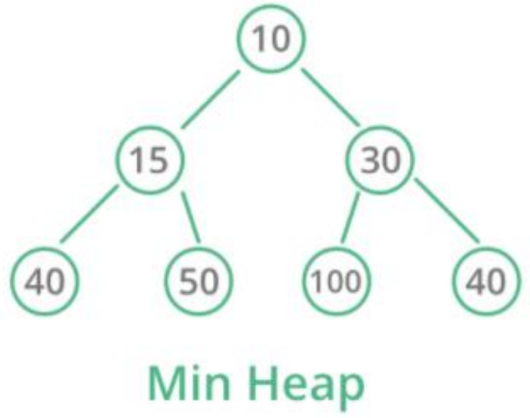
Parent of $A[i] = A[\lfloor i/2 \rfloor]$

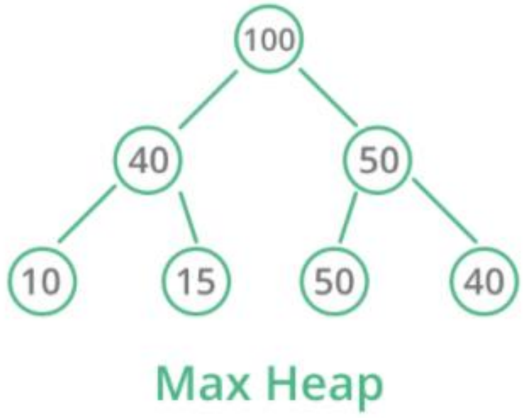
The elements in the subarray $A[(\lfloor n/2 \rfloor+1) \ .. \ n]$ are leaves

Figure 6.1 A max-heap viewed as **(a)** a binary tree and **(b)** an array. The number within the circle at each node in the tree is the value stored at that node. The number above a node is the corresponding index in the array. Above and below the array are lines showing parent-child relationships; parents are always to the left of their children.

Heap Types

- Max-heaps (largest element at root), have the max-heap property:
    - for all nodes i, excluding the root: $A[PARENT(i)] \geq A[i]$
- Min-heaps (smallest element at root), have the min-heap property:
    - for all nodes i, excluding the root: $A[PARENT(i)] \leq A[i]$
- For heap sort algorithm, we use max-heaps.
- Min-heaps are commonly used for implementing priority queues.





Adding/Deleting Nodes

- New nodes are always inserted at the bottom level (left to right)
- Nodes are removed from the bottom level (right to left)

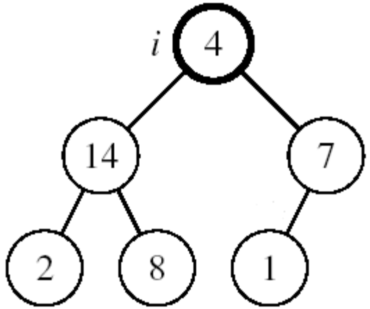
Operations on Heaps

- Maintain/Restore the max-heap property
  - MAX-HEAPIFY
- Create a max-heap from an unordered array
  - BUILD-MAX-HEAP
- Sort an array in place
  - HEAPSORT
- Priority queues

Maintaining the Heap Property

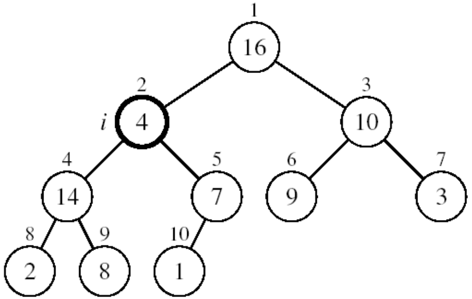
- Suppose a node is smaller than a child
  - Left and Right subtrees of *i* are max-heaps

- To eliminate the violation:
  - Exchange with larger child
  - Move down the tree
  - Continue until node is not smaller than children



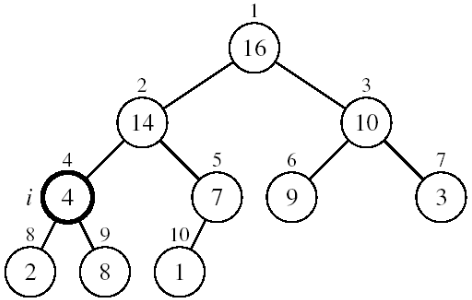
Example

MAX-HEAPIFY(A, 2, 10)



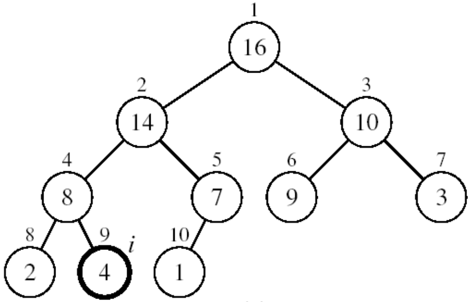
A[2] violates the heap property

A[2] ↔ A[4]



A[4] violates the heap property

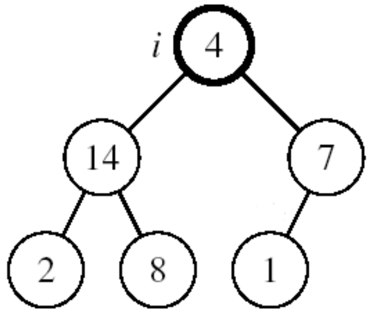
A[4] ↔ A[9]



Heap property restored

Maintaining the Heap Property

- Assumptions:
  - Left and Right subtrees of i are max-heaps
  - $A[i]$ may be smaller than its children

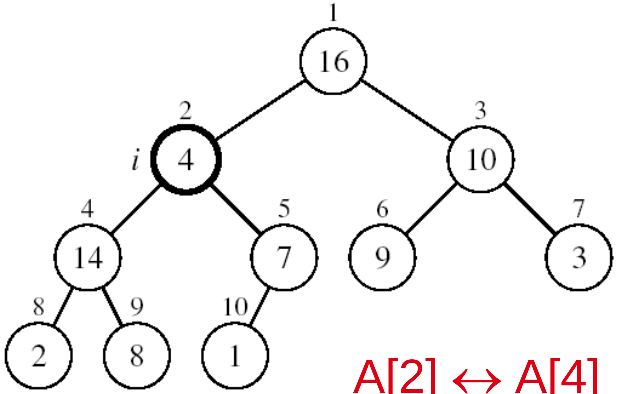


Alg: MAX-HEAPIFY(A, i, n)

1. l ← LEFT(i)
2. r ← RIGHT(i)
3. **if** l ≤ n and $A[l] > A[i]$
4. **then** *largest* ← l
5. **else** *largest* ← i
6. **if** r ≤ n and $A[r] > A[largest]$
7. **then** *largest* ← r
8. **if** *largest* ≠ i
9. **then** exchange $A[i] \leftrightarrow A[largest]$
10. **MAX-HEAPIFY(A, largest, n)**

Maintaining the Heap Property

- Assumptions:
  - Left and Right subtrees of *i* are max-heaps
  - $A[i]$ may be smaller than its children



A[2] ↔ A[4]

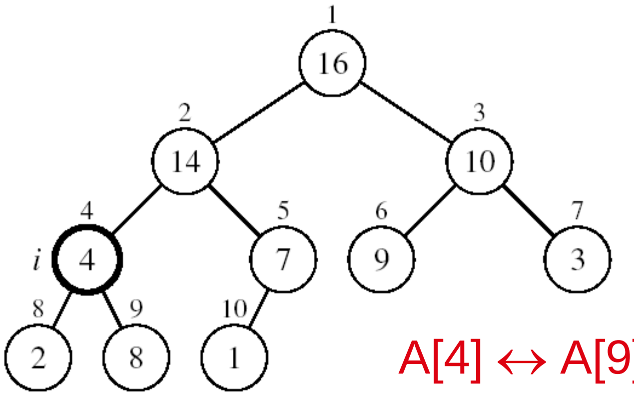
MAX-HEAPIFY(A, 2, 10)

Alg: MAX-HEAPIFY(*A, i, n*)

```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ n and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ n and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then exchange A[i] ↔ A[largest]
        MAX-HEAPIFY(A, largest, n)
```

Maintaining the Heap Property

- Assumptions:
    - Left and Right subtrees of *i* are max-heaps
    - $A[i]$ may be smaller than its children



A[4] ↔ A[9]

Alg: MAX-HEAPIFY(A, i, n)

```
l ← LEFT(i)
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if l ≤ n and A[l] > A[i]
    then largest ←l
    else largest ←i
if r ≤ n and A[r] > A[largest]
    then largest ←r
if largest ≠ i
    then exchange A[i] ↔ A[largest]
        MAX-HEAPIFY(A, largest, n)
```
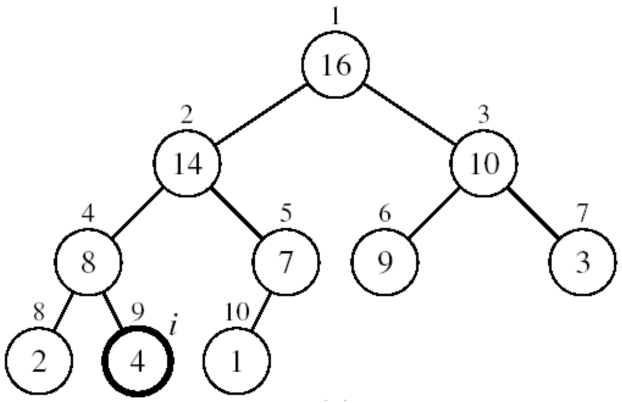

Maintaining the Heap Property

- Assumptions:
  - Left and Right subtrees of *i* are max-heaps
  - $A[i]$ may be smaller than its children



Alg: MAX-HEAPIFY(*A, i, n*)

1. $l \leftarrow$ LEFT(i)
2. $r \leftarrow$ RIGHT(i)
3. **if** $l \le n$ and $A[l] > A[i]$
4. **then** $largest \leftarrow l$
5. **else** $largest \leftarrow i$
6. **if** $r \le n$ and $A[r] > A[largest]$
7. **then** $largest \leftarrow r$
8. **if** $largest \neq i$
9. **then** exchange $A[i] \leftrightarrow A[largest]$
10. MAX-HEAPIFY(A, $largest$, n)

MAX-HEAPIFY Running Time

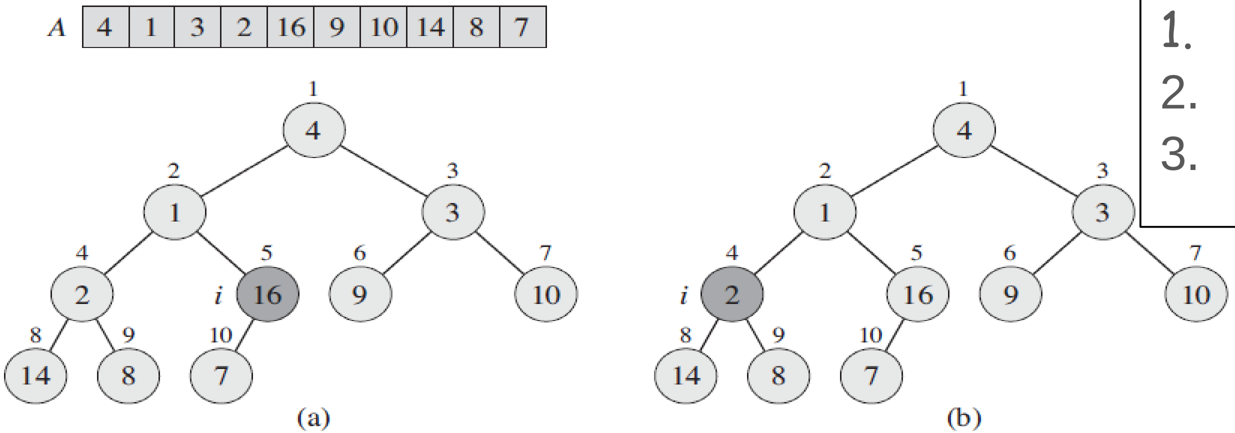
- Intuitively:
  - It traces a path from the root to a leaf (longest path length: h)
  - At each level, it makes exactly 2 comparisons
  - Total number of comparisons are 2h
  - Running time is O(h) or O(log n)

- Running time of MAX-HEAPIFY is $O(lgn)$
- Can be written in terms of the height of the heap, as being $O(h)$
  - Since the height of the heap is $\lfloor lgn \rfloor$

Build Max Heap Procedure

- Convert an array $A[1 \ldots n]$ into a max-heap ($n = length[A]$)
- The elements in the subarray $A[(\lfloor n/2 \rfloor+1) \ldots n]$ are leaves
- Apply MAX-HEAPIFY on elements between 1 and $\lfloor n/2 \rfloor$

Figure 6.3 :

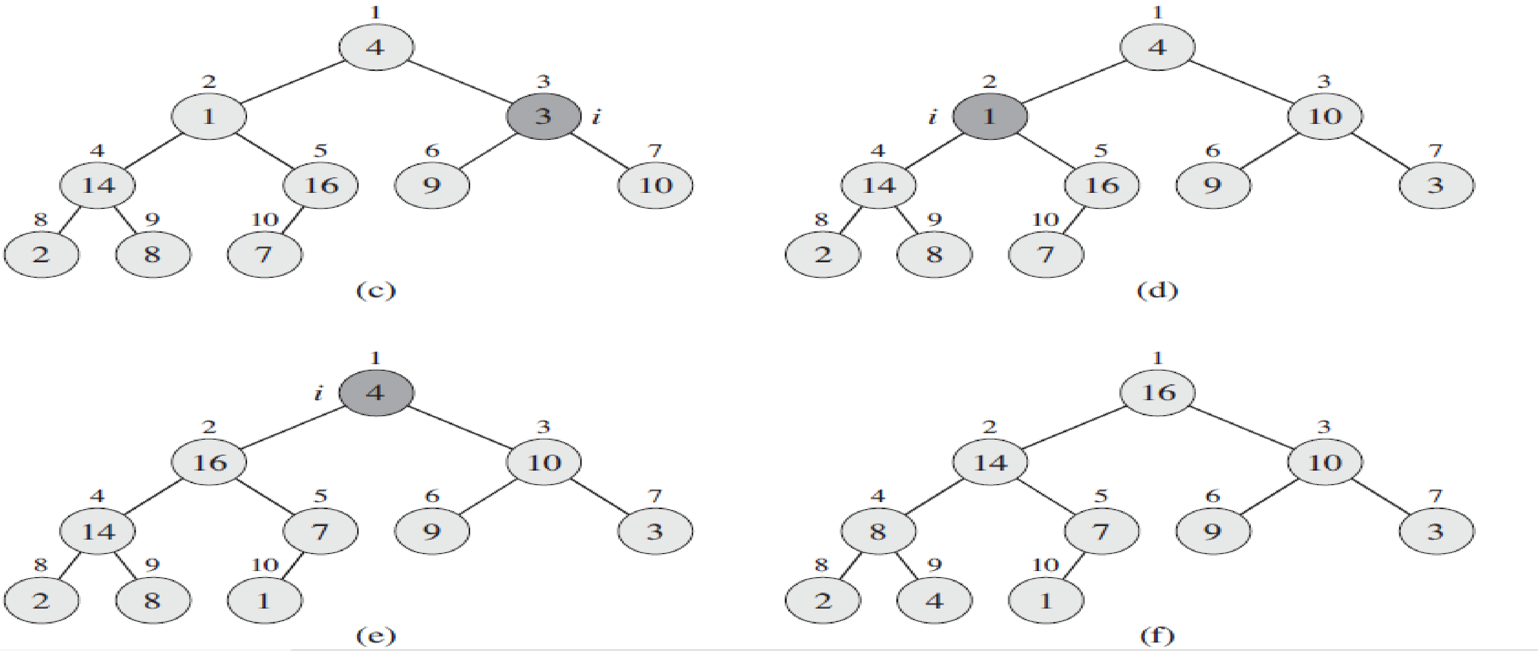


Alg: BUILD-MAX-HEAP(*A*)

```
n = length[A]
for i ← ⌊n/2⌋ downto 1
    do MAX-HEAPIFY(A, i, n)
```

Build Max Heap Procedure

- Figure 6.3 :



Heapsort

- It has time complexity of divide and conquer i-e “nlogn” but it does not behave like divide and conquer because it splits data into sorted and unsorted sections. It is not divide and conquer algorithm

- It is sorting in place algorithm i-e does not require extra space like merge sort.

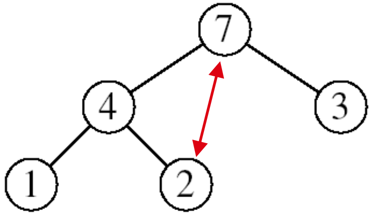
- Combines the better attributes of two sorting algorithms. Time complexity like mergeSort i-e “nlogn” and space complexity like insertion sort i-e “O(1)“

- Heap was used as “garbage collected storage” in languages like java, lisp etc.

- But here in heap sort, it will be used as a data structure and not the garbage collected storage.

Heapsort

- **Goal:**
  - Sort an array using heap representations
- **Idea:**
  - Build a **max-heap** from the array
  - Swap the root (the maximum element) with the last element in the array
  - “Discard” this last node by decreasing the heap size
  - Call MAX-HEAPIFY on the new root
  - Repeat this process until only one node remains

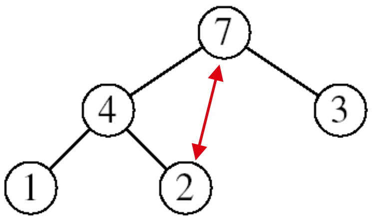


MAX-HEAPIFY(A, 1, 4)

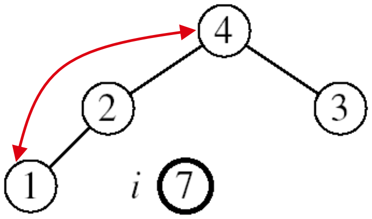
Extract Max

- Remove root
- Swap with last node
- Re-heapify

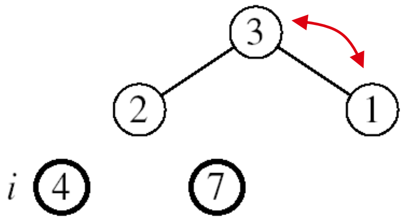
Example: A=[7, 4, 3, 1, 2]



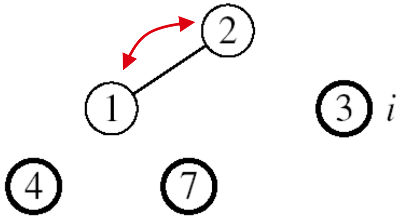
MAX-HEAPIFY(A, 1, 4)



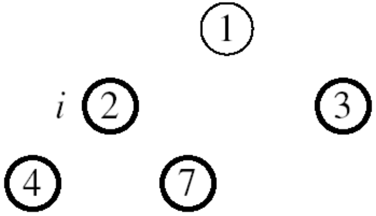
MAX-HEAPIFY(A, 1, 3)



MAX-HEAPIFY(A, 1, 2)



MAX-HEAPIFY(A, 1, 1)



A	1	2	3	4	7

Alg: HEAPSORT(*A*)

1. BUILD-MAX-HEAP(*A*)
2. **for** i ← *length*[*A*] **downto** 2
3. **do** exchange *A*[1] $\boxed{\leftrightarrow}$ *A*[i]
4. MAX-HEAPIFY(*A*, 1, i - 1)

$O(\lg n)$

n-1 times

- Running time: $O(n\lg n)$ --- *Can be shown to be* $\Theta(n\lg n)$

Priority Queue

- Queue – only access element in front
- Queue elements sorted by order of importance
- Implement as a heap where nodes store priority values

Why merge sort is preferred, in presence of Heap, although Heap does not require any extra space?

Comparison-based Sorting

- **You want to sort an array of items**
- **You can't access the items' values directly: you can only *compare* two items and find out which is bigger or smaller.**
- Examples: Insertion Sort, MergeSort, QuickSort

"Comparison-based sorting algorithms" are general-purpose.

The worst case complexity of comparison-based sorting can not be reduced more than "n.logn" (Proof in textbook)

Linear-time Sorting

Beyond comparison-based sorting algorithms!

A New Model Of Computation

The elements we're working with have meaningful values.

Before:

arbitrary elements whose values we could never directly access, process, or take advantage of (i.e. we could only interact with them via comparisons)

Now (examples):

9	18	27	4	9	18	27

not-too-large integers

Dec	Feb	Oct	May

months in a year

- The worst-case complexity can be reduced further from "n.logn" without making comparisons, called linear sorting. Counting, Radix and Bucket sort are three examples.
- However, it is possible only under restrictive circumstances, for example sorting small integers (exam score), characters etc.

Counting Sort

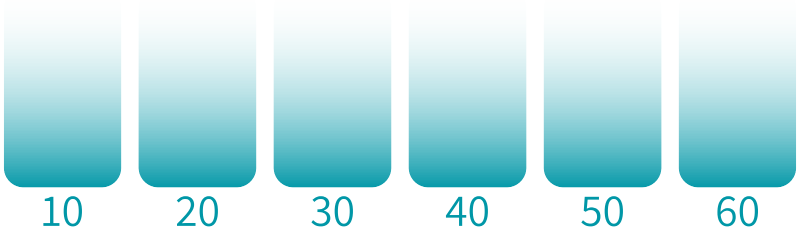
We assume that there are only k different possible values in the array (and we know these k values in advance)

For example: elements are integers in {10, 20, 30, 40, 50, 60}

Input:

30	50	20	30	10	60	50	20

Buckets:



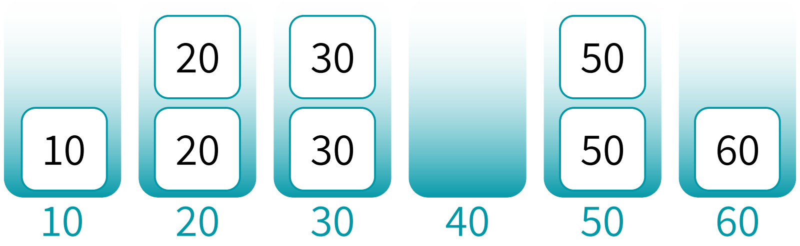
Counting Sort

We assume that there are only k different possible values in the array (and we know these k values in advance)

For example: elements are integers in {10, 20, 30, 40, 50, 60}

Input: 30 50 20 30 10 60 50 20

Buckets:



Counting Sort

We assume that there are only k different possible values in the array (and we know these k values in advance)

For example: elements are integers in {10, 20, 30, 40, 50, 60}

Input:

30	50	20	30	10	60	50	20

Buckets:

10	20	30	40	50	60
	20	30		50	
10	20	30		50	60

Sorted in time:

O(n)

Stability issue: Dr. Waheed

Output:

10	20	20	30	30	50	50	60

Because, no element is taken into consideration individually, instead only frequency of elements is counted, so order of elements can not be keep tracked.

Counting Sort

- Input: array A[1, … , n]; k (elements in A have values from 1 to k)
- Output: sorted array A

Algorithm:

1. Create a counter array C[1, …, k]
2. Create an auxiliary array B[1, …, n]
3. Scan A once, record element frequency in C
4. Calculate prefix sum in C
5. Scan A in the reverse order, copy each element to B at the correct position according to C.
6. Copy B to A

Counting Sort: Pseudocode

```
COUNTING-SORT(A, B, k):
    let C[1..k] be a new array
    for i = 1 to k
        C[i ] = 0
    for j = 1 to A.length
        C[A[ j ]] = C[A[ j ]] + 1

    for i = 2 to k
        C[i ] = C[i ] + C[i - 1]

    for j = A.length to 1
        B[C[A[ j ]]] = A[ j ]
        C[A[ j ]] = C[A[ j ]] - 1
```

1. Create a counter array C[1, …, k]
2. Create an auxiliary array B[1, …, n]
3. Scan A once, record element frequency in C
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5. Scan A in the reverse order, copy each element to B at the correct position according to C.
6. Copy B to A

Analysis of Counting Sort

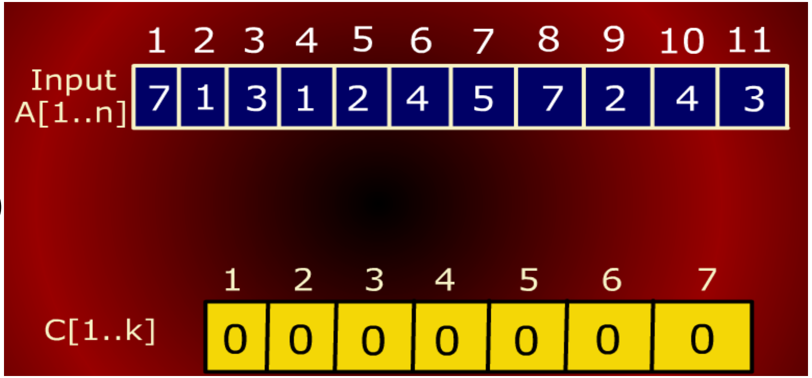
- Input: array A[1, … , n]; k (elements in A have values from 1 to k)
- Output: sorted array A

Algorithm:

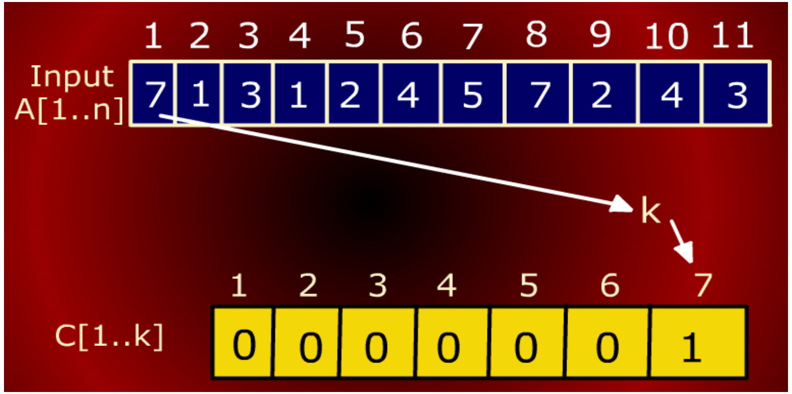
Step	Time	Space
1. Create a counter array C[1, …, k]		O(k)
2. Create an auxiliary array B[1, …, n]		O(n)
3. Scan A once, record element frequency in C	O(n)	
4. Calculate prefix sum in C	O(k)	
5. Scan A in the reverse order, copy each element to B at the correct position according to C.	O(n)	
6. Copy B to A	O(n)	
	O(n+k)=O(n) (if k=O(n))	O(n+k)=O(n) (if k=O(n))

Counting sort

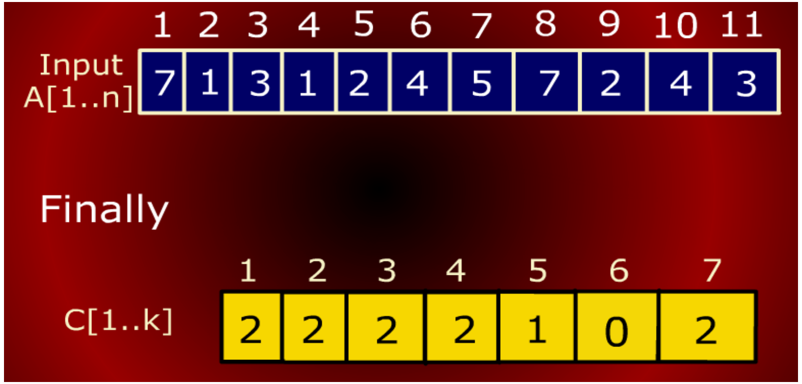
(1)



(2)



(3)

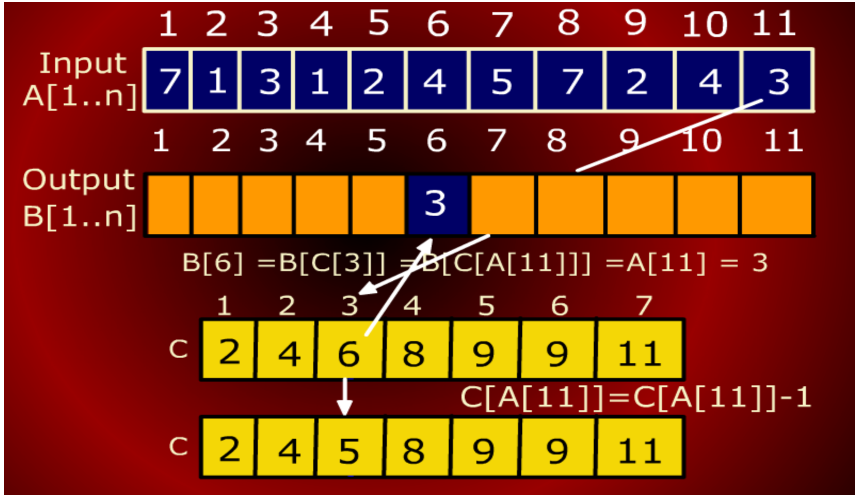


(4)

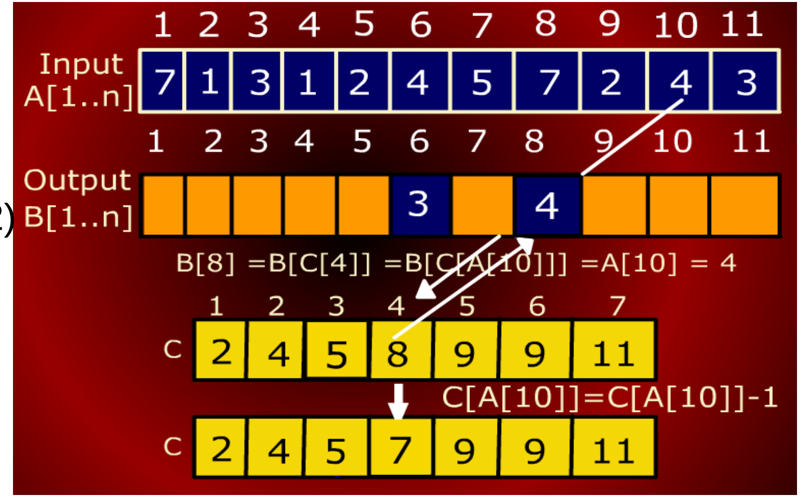
Counting sort

```
COUNTING-SORT(A, B, k):
    …
        for j = A.length to 1
            B[C[A[ j ]]] = A[ j ]
            C[A[ j ]] = C[A[ j ]] - 1
```

(1)



(2)



Counting sort

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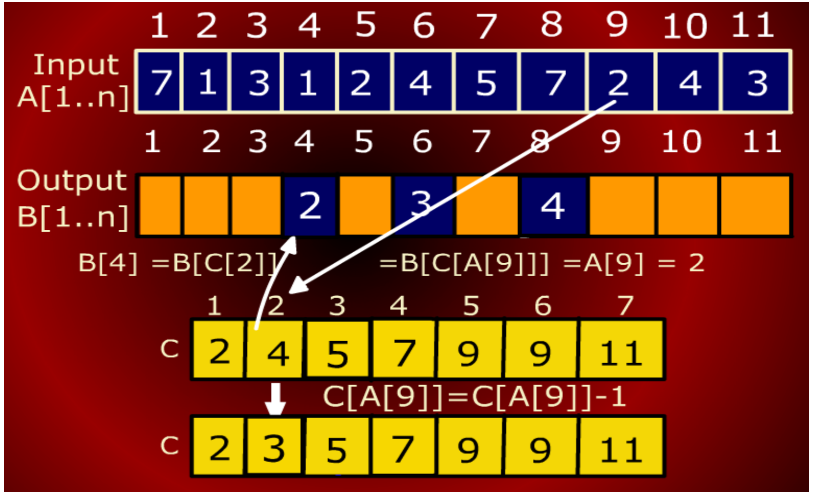
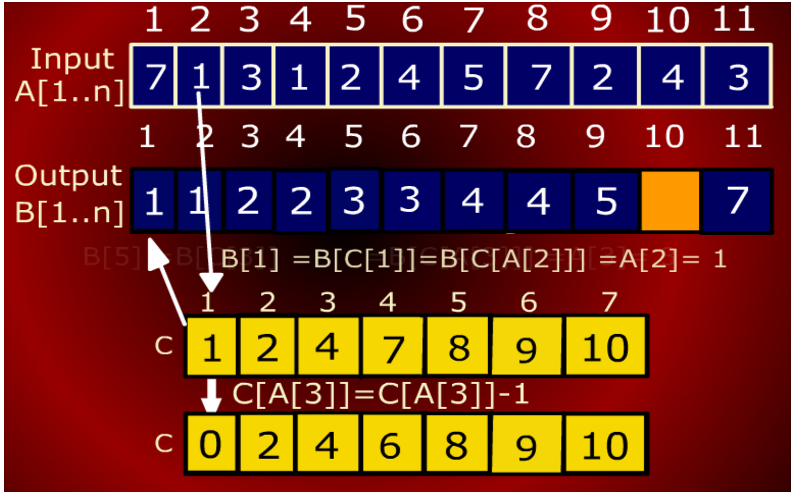


(4)



RADIX SORT

A sorting algorithm for integers up to size M

(or more generally, for sorting strings)

RADIX SORT

For sorting integers where the maximum value of any integer is M.
(This can be generalized to lexicographically sorting strings as well)

IDEA:

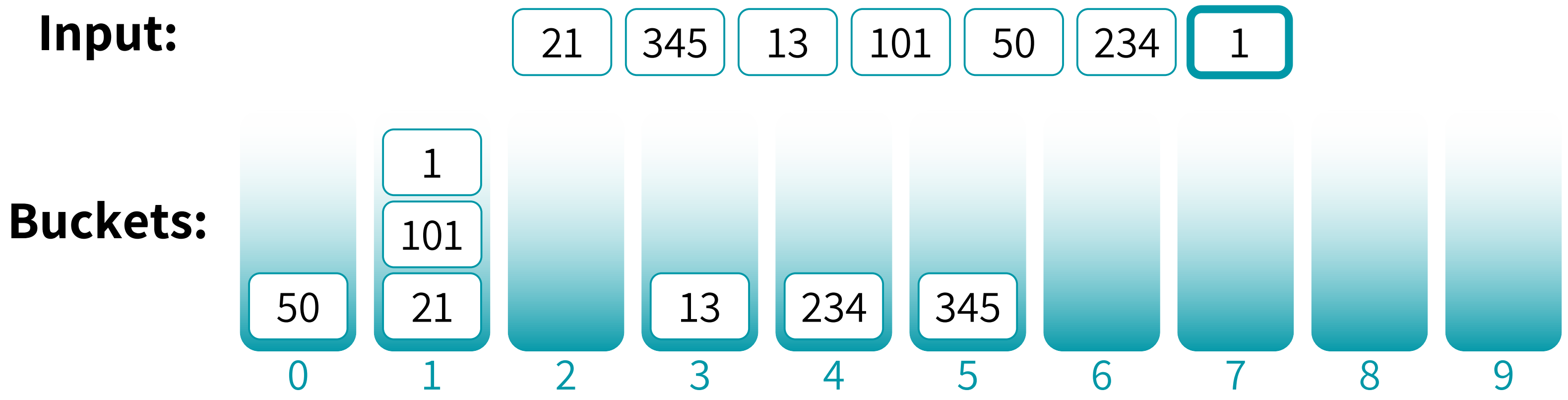
Perform CountingSort on the least-significant digit first,
then perform CountingSort on the next least-significant, and so on...

Instead of a bucket per possible value, **we just need to maintain a bucket per possible value that a single digit (or character) can take on!**

e.g. 10 buckets labeled 0, 1, …, 9

RADIX SORT

STEP 1: CountingSort on the least significant digit

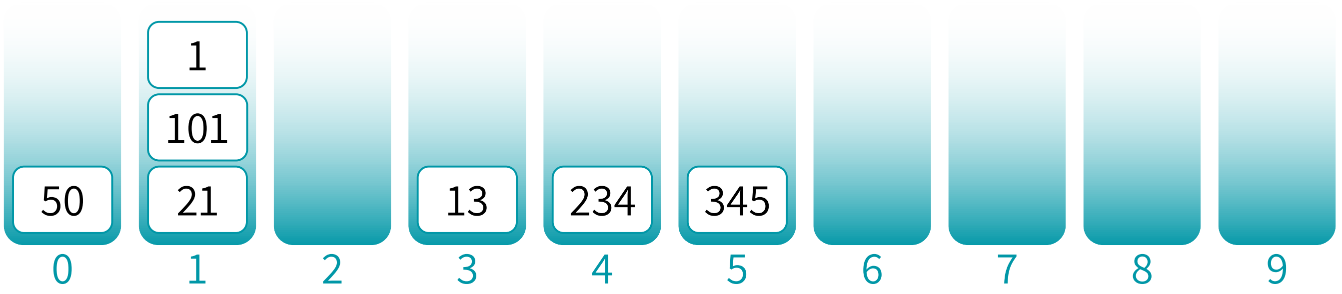


RADIX SORT

STEP 1: CountingSort on the least significant digit

Input: 21 | 345 | 13 | 101 | 50 | 234 | 1

Buckets:

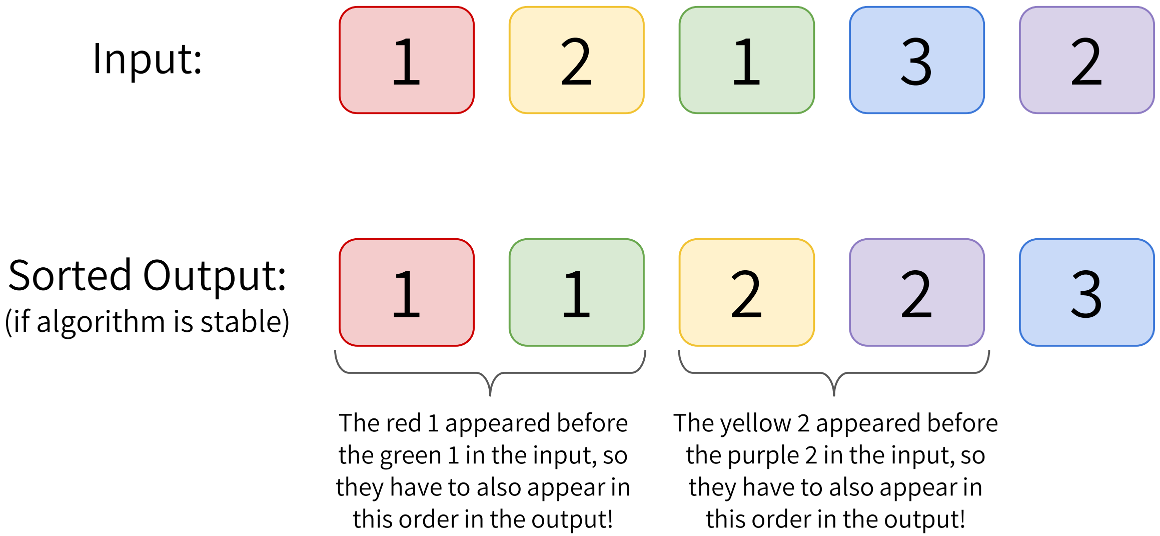


Output: 50 | 21 | 101 | 1 | 13 | 234 | 345

When creating the output list, make sure bucket items exit in FIFO order
(i.e. use a *stable* implementation of CountingSort, where buckets are FIFO queues)

QUICK ASIDE: STABLE SORTING

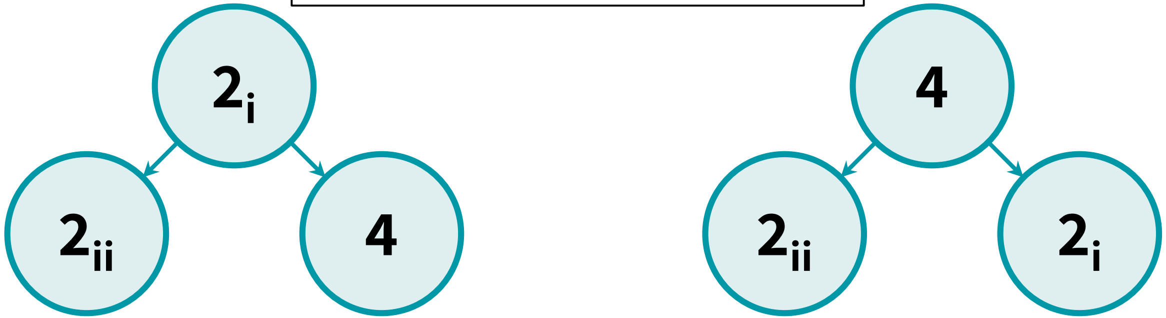
We say a sorting algorithm is STABLE if two objects with equal values appear in the same order in the sorted output as they appear in the input.



QUICK ASIDE: STABLE SORTING

We say a sorting algorithm is STABLE if two objects with equal values appear in the same order in the sorted output as they appear in the input.

Heap Sort: Not Stable



In-Place Sort: An sorting algorithm is one that uses no additional array for storage

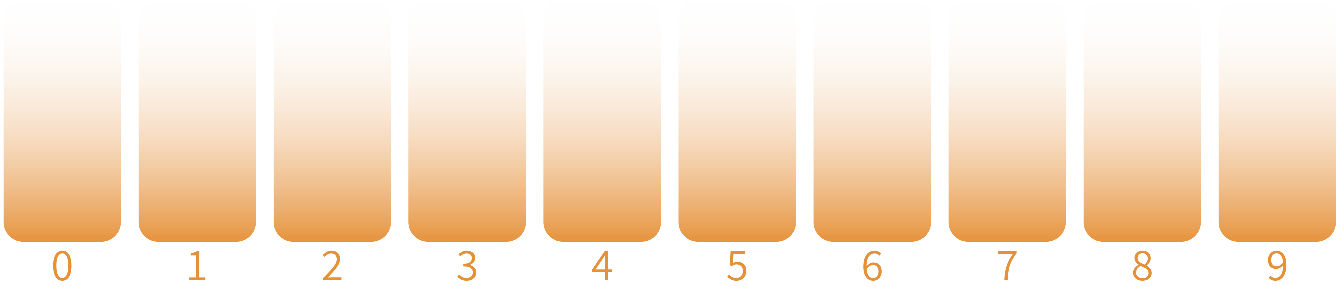
RADIX SORT

STEP 2: CountingSort on the 2nd least significant digit

Input:
(output from STEP 1)

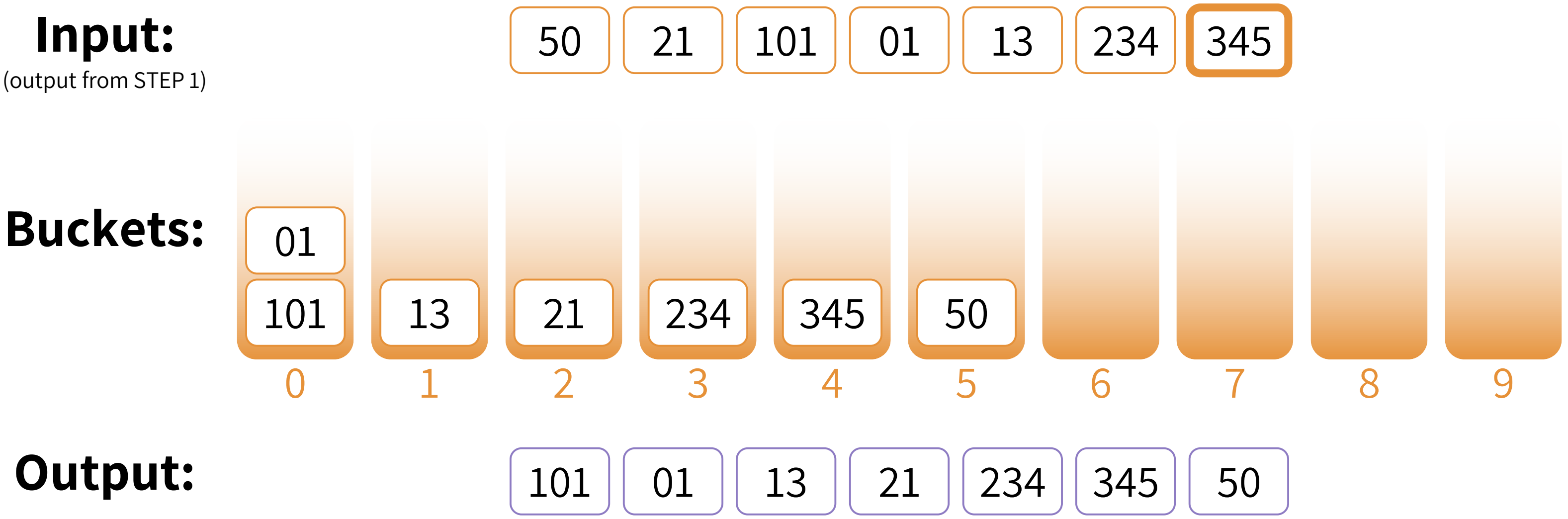
50	21	101	1	13	234	345

Buckets:



RADIX SORT

STEP 2: CountingSort on the 2nd least significant digit



When creating the output list, make sure bucket items exit in FIFO order
(i.e. use a *stable* implementation of CountingSort, where buckets are FIFO queues)

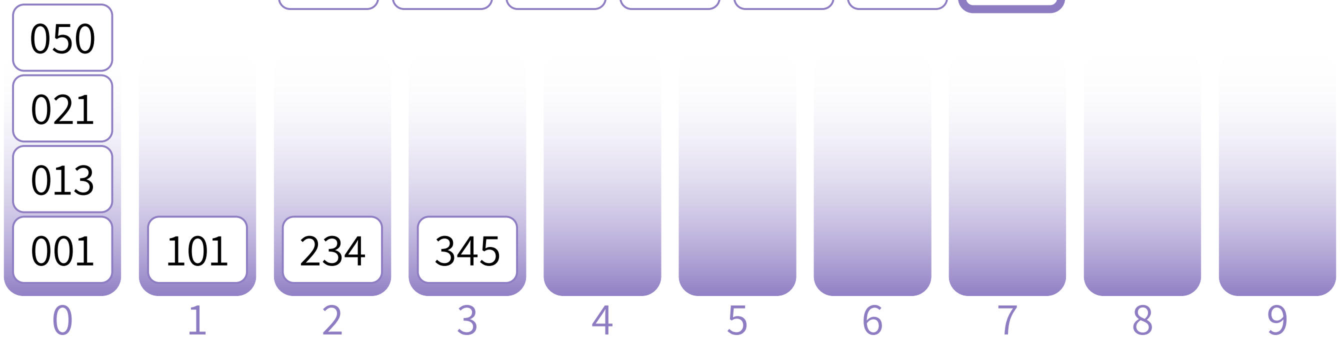
RADIX SORT

STEP 3: CountingSort on the 3rd least significant digit

Input:
(output from STEP 2)

101 001 013 021 234 345 050

Buckets:



Output:

001 013 021 050 101 234 345

It worked! But why does it work???

RADIX SORT RUNTIME

Suppose we are sorting **n** (up-to-)**d**-digit numbers in base 10 (e.g. n = 7, d = 3):

21	345	13	101	50	234	1

How many iterations are there?

d iterations

How long does each iteration take?

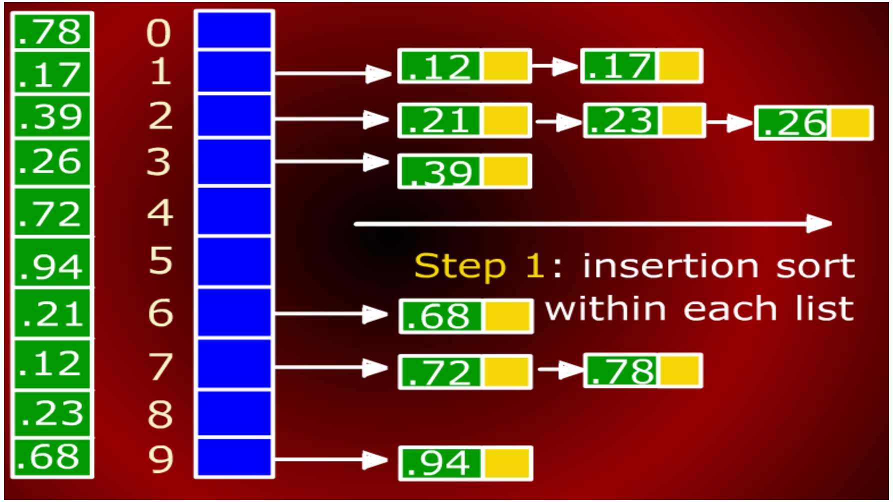
Initialize 10 buckets + put n numbers in 10 buckets ⇒ **O(n)**

What is the total running time?

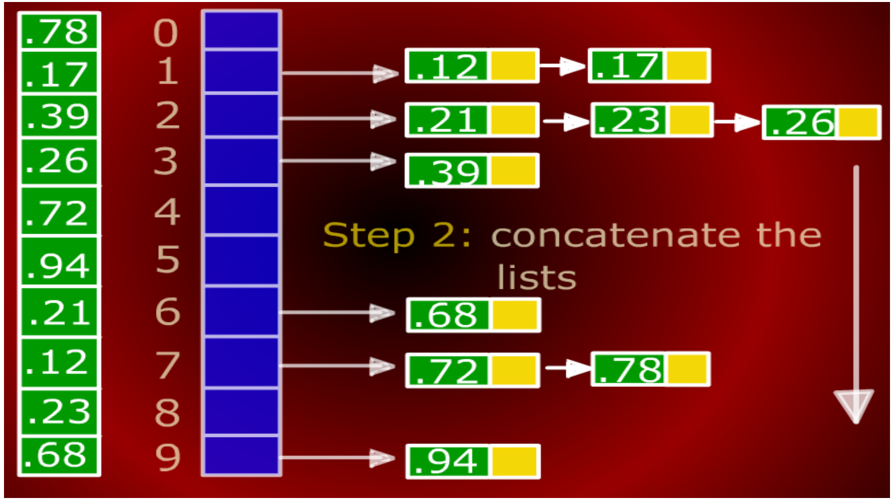
O(nd)

Bucket Sort

Assumption: Input elements are uniformly distributed over [0,1]



(a)



(b)

Bucket Sort

BUCKET-SORT(A)

let $B[0 \,.\,.\, n-1]$ be a new array
$n = A.length$
for $i = 0$ **to** $n - 1$
    make $B[i]$ an empty list
for $i = 1$ **to** n
    insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
for $i = 0$ **to** $n - 1$
    sort list $B[i]$ with insertion sort
concatenate the lists $B[0], B[1], \ldots, B[n-1]$ together in order

Comparison of Sorting Algorithms

Algorithm	Worst Time	Extra Memory	Stable
Insertion sort	$O(n^2)$	O(1) (in place)	Yes
Merge sort	$O(n\ lgn)$	O(n)	Yes
Quick sort	$O(n^2)$	O(1) (in place)	Yes
Heap sort	$O(n\ lgn)$	O(1) (in place)	No
Counting sort	$O(n+k)$	$O(n+k)$	Yes