A Bitmapper's Geometry

an introduction to basic bitmap mathematics and algorithms with code samples in **Rust**

epilys

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Manos Pitsidianakis (epilys)
`https://nessuent.xyz`
`https://github.com/epilys`
`epilys@nessuent.xyz`

All non-screenshot figures were generated by hand in Inkscape unless otherwise stated.

The skull in the cover is a transformed bitmap of the skull in the 1533 oil painting by Hans Holbein the Younger, *The Ambassadors*, which features a floating distorted skull rendered in anamorphic perspective.

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The source code is available here

`https://github.com/epilys/bitmappers-geometry`

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Part I

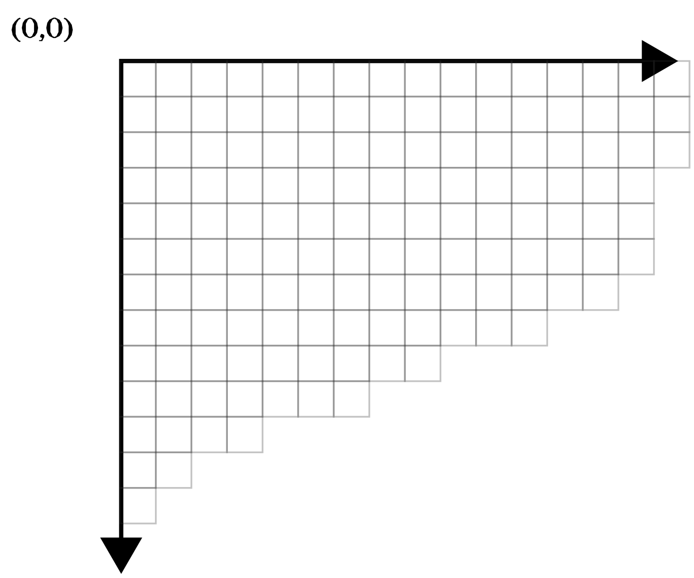
Introduction

Chapter 1

Data representation

The data structures we're going to use is *Point* and *Image*. *Image* represents a bitmap, although we will use full RGB colors for our points therefore the size of a pixel in memory will be u8 instead of 1 bit.

We will work on the cartesian grid representing the framebuffer that will show us the pixels. The *origin* of this grid (i.e. the center) is at $(0, 0)$.



We will represent points as pairs of signed integers. When actually drawing them though, negative values and values outside the window's geometry will be ignored (clipped).

```
pub type Point = (i64, i64);

pub const fn from_u8_rgb(r: u8, g: u8, b: u8) -> u32 {
    let (r, g, b) = (r as u32, g as u32, b as u32);
    (r << 16) | (g << 8) | b
}
pub const AZURE_BLUE: u32 = from_u8_rgb(0, 127, 255);
pub const RED: u32 = from_u8_rgb(157, 37, 10);
pub const WHITE: u32 = from_u8_rgb(255, 255, 255);
pub const BLACK: u32 = 0;

pub struct Image {
    pub bytes: Vec<u32>,
    pub width: usize,
    pub height: usize,
    pub x_offset: usize,
    pub y_offset: usize,
}

impl Image {
    pub fn new(width: usize,
        height: usize,
        x_offset: usize,
        y_offset: usize) -> Self;
    pub fn draw(&self,
        buffer: &mut Vec<u32>,
        fg: u32,
        bg: Option<u32>,
        window_width: usize);
    pub fn draw_outline(&mut self);
    pub fn clear(&mut self);
```

```
    pub fn plot(&mut self, x: i64, y: i64);
    pub fn get(&mut self, x: i64, y: i64) -> u32;
    pub fn plot_ellipse(
        &mut self,
        (xm, ym): (i64, i64),
        (a, b): (i64, i64),
        quadrants: [bool; 4],
        _wd: f64,
    );
    pub fn plot_line_width(&mut self,
              point_a: Point,
              point_b: Point,
              wd: f64);
    pub fn flood_fill(&mut self, mut x: i64, y: i64);
}
```

Chapter 2

Displaying pixels to your screen

A way to display an *Image* is to use the `minifb` crate which allows you to create a window and draw pixels directly on it. Here's how you could set it up:

```
use bitmappers_geometry::*;
use minifb::{Key, Window, WindowOptions};

const WINDOW_WIDTH: usize = 400;
const WINDOW_HEIGHT: usize = 400;

fn main() {
    let mut buffer: Vec<u32> = vec![WHITE; WINDOW_WIDTH * WINDOW_HEIGHT];
    let mut window = Window::new(
        "Test - ESC to exit",
        WINDOW_WIDTH,
        WINDOW_HEIGHT,
        WindowOptions {
            title: true,
            //borderless: true,
            //resize: false,
            //transparency: true,
            ..WindowOptions::default()
        },
```

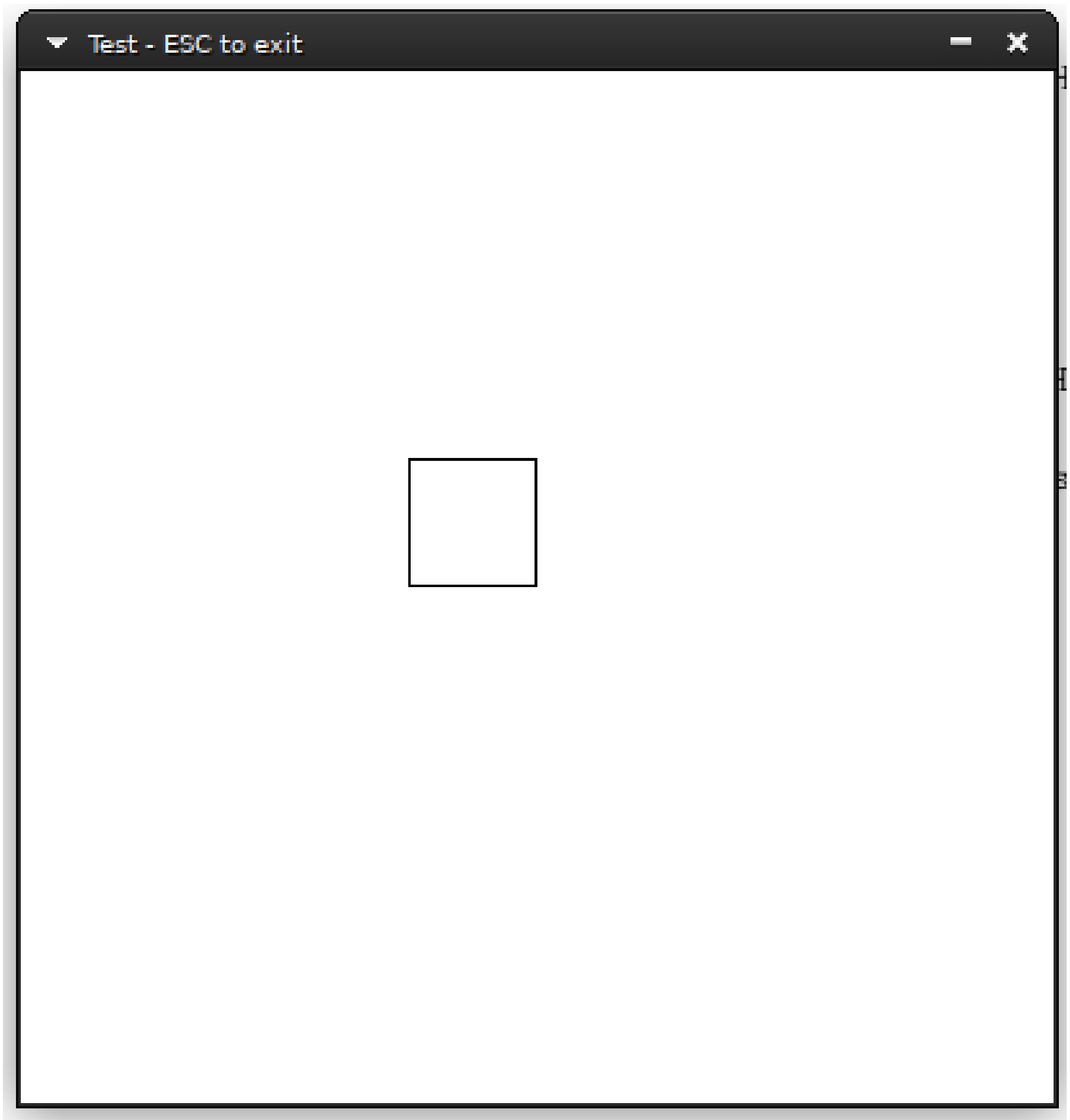
```
    )
    .unwrap();

    // Limit to max ~60 fps update rate
    window.limit_update_rate(Some(std::time::Duration::from_micros(16600)));

    let mut image = Image::new(50, 50, 150, 150);
    image.draw_outline();
    image.draw(&mut buffer, BLACK, None, WINDOW_WIDTH);

    while window.is_open()
         && !window.is_key_down(Key::Escape)
         && !window.is_key_down(Key::Q) {
        window
            .update_with_buffer(&buffer, WINDOW_WIDTH, WINDOW_HEIGHT)
            .unwrap();
        let millis = std::time::Duration::from_millis(100);
        std::thread::sleep(millis);
    }
}
```

Running this will show you something like this:


Test - ESC to exit

Chapter 3

Bits to byte pixels

Let's define a way to convert bit information to a byte vector:

```
pub fn bits_to_bytes(bits: &[u8], width: usize) -> Vec<u32> {
    let mut ret = Vec::with_capacity(bits.len() * 8);
    let mut current_row_count = 0;
    for byte in bits {
        for n in 0..8 {
            if byte.rotate_right(n) & 0x01 > 0 {
                ret.push(BLACK);
            } else {
                ret.push(WHITE);
            }
            current_row_count += 1;
            if current_row_count == width {
                current_row_count = 0;
                break;
            }
        }
    }
    ret
}
```

Chapter 4

Real pixels to byte pixels

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Chapter 5

Loading `xbm` files in **Rust**

`xbm` files are C source code files that contain the pixel information for an image as macro definitions for the dimensions and a static `char` array for the pixels, with each bit column representing a pixel. If the width dimension doesn't have 8 as a factor, the remaining bit columns are left blank/ignored.

They used to be a popular way to share user avatars in the old internet and are also good material for us to work with, since they are small and numerous. The following is such an image:

Then, we can convert the `xbm` file from C to **Rust** with the following transformations:

```
#define news_width 48
#define news_height 48
static char news_bits[] = {
```

to

```
const NEWS_WIDTH: usize = 48;
const NEWS_HEIGHT: usize = 48;
const NEWS_BITS: &[u8] = &[
```

And replace the closing } with].

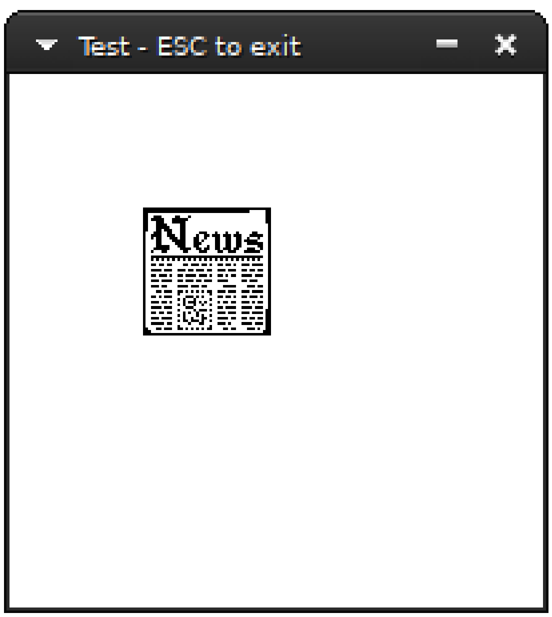
We can then include the new file in our source code:

```
include!("news.xbm.rs");
```

load the image:

```
let mut image = Image::new(NEWS_WIDTH, NEWS_HEIGHT, 25, 25);
image.bytes = bits_to_bytes(NEWS_BITS, NEWS_WIDTH);
```

and finally run it:

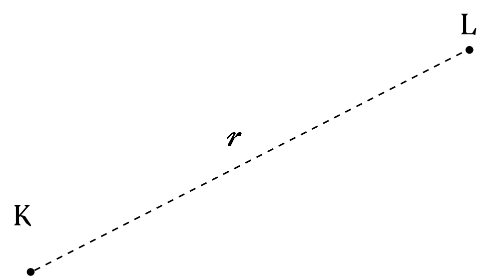


Part II

Points and Lines

Chapter 6

Distance between two points



Given two points, K and L, an elementary application of Pythagoras' Theorem gives the distance between them as

$$r = \sqrt{(x_L - x_K)^2 + (y_L - y_K)^2} \tag{6.1}$$

which is simply coded:

```
pub fn distance_between_two_points(p_k: Point, p_l: Point) -> f64 {
    let (x_k, y_k) = p_k;
    let (x_l, y_l) = p_l;
    let xlk = x_l - x_k;
    let ylk = y_l - y_k;
    f64::sqrt((xlk*xlk + ylk*ylk) as f64)
}
```

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Equations of a line

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Figure 11.1: 8

It seems sufficient, given the coordinates of two points M, N, to calculate a, b and c to form a line equation:

$$ax + by + c = 0$$

If the two points are not the same, they necessarily form such a line. To get there, we start from expressing the line as parametric over t: at $t = 0$ it's at point M and at $t = 1$ it's at point N:

$$c = c_M + (c_N - c_M)t, t \in R, c \in \{x, y\}$$

Substituting t in one of the equations we get:

$$(y_M - y_N)x + (x_N - x_M)y + (x_M y_N - x_N y_M) = 0$$

Which is what we were after. We finish by normalising what we found with $\frac{1}{\sqrt{a^2+b^2}}$:

Chapter 12

Line equidistant from two points

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Figure 12.1: 10

Let's name this line L. From the previous chapter we know how to get the line that's created by the two points M and N. If only we knew how to get a perpendicular line over the midpoint of a line segment!

Thankfully that midpoint also satisfies L's equation, $ax + by + c$. The midpoint's coordinates are intuitively:

$$\left(\frac{x_M + x_N}{2}, \frac{y_M + y_N}{2}\right)$$

Putting them into the equation we can generate a triple of (a', b', c') and then normalize it to get L.

Chapter 13

Normal to a line through a point

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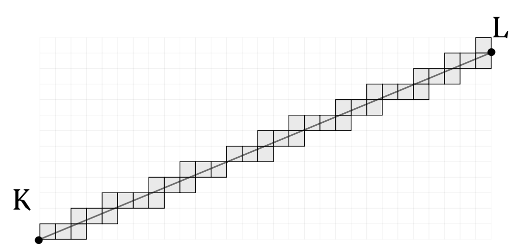
Part IV

Points, Line Segments and Arcs

Chapter 16

Drawing a line segment from its two endpoints

For any line segment with any slope, pixels must be matched with the infinite amount of points contained in the segment. As shown in the following figure, a segment *touches* some pixels; we could fill them using an algorithm and get a bitmap of the line segment.



The algorithm presented here was first derived by Bresenham. In the *Image* implementation, it is used in the `plot_line_width` method.

```
pub fn plot_line_width(&mut self, (x1, y1): (i64, i64), (x2, y2): (i64, i64)) {
    /* Bresenham's line algorithm */
    let mut d;
```

```
let mut x: i64;
let mut y: i64;
let ax: i64;
let ay: i64;
let sx: i64;
let sy: i64;
let dx: i64;
let dy: i64;

dx = x2 - x1;
ax = (dx * 2).abs();
sx = if dx > 0 { 1 } else { -1 };

dy = y2 - y1;
ay = (dy * 2).abs();
sy = if dy > 0 { 1 } else { -1 };

x = x1;
y = y1;

let b = dx / dy;
let a = 1;
let double_d = (_wd * f64::sqrt((a * a + b * b) as f64)) as i64;
let delta = double_d / 2;

if ax > ay {
    d = ay - ax / 2;
    loop {
        self.plot(x, y);
        if x == x2 {
            return;
        }
        if d >= 0 {
            y = y + sy;
```

```
                d = d - ax;
            }
            x = x + sx;
            d = d + ay;
        }
    } else {
        d = ax - ay / 2;
        let delta = double_d / 3;
        loop {
            self.plot(x, y);
            if y == y2 {
                return;
            }
            if d >= 0 {
                x = x + sx;
                d = d - ay;
            }
            y = y + sy;
            d = d + ax;
        }
    }
}
```

Chapter 17

Drawing Line segments With Width

```
pub fn plot_line_width(&mut self, (x1, y1): (i64, i64),
                       (x2, y2): (i64, i64), _wd: f64) {
    /* Bresenham's line algorithm */
    let mut d;
    let mut x: i64;
    let mut y: i64;
    let ax: i64;
    let ay: i64;
    let sx: i64;
    let sy: i64;
    let dx: i64;
    let dy: i64;

    dx = x2 - x1;
    ax = (dx * 2).abs();
    sx = if dx > 0 { 1 } else { -1 };

    dy = y2 - y1;
    ay = (dy * 2).abs();
    sy = if dy > 0 { 1 } else { -1 };

    x = x1;
```

```
y = y1;

let b = dx / dy;
let a = 1;
let double_d = (_wd * f64::sqrt((a * a + b * b) as f64)) as i64;
let delta = double_d / 2;

if ax > ay {
    d = ay - ax / 2;
    loop {
        self.plot(x, y);
        {
            let total = |_x| _x - (y * dx) / dy + (y1 * dx) / dy - x1;
            let mut _x = x;
            loop {
                let t = total(_x);
                if t < -1 * delta || t > delta {
                    break;
                }
                _x += 1;
                self.plot(_x, y);
            }
            let mut _x = x;
            loop {
                let t = total(_x);
                if t < -1 * delta || t > delta {
                    break;
                }
                _x -= 1;
                self.plot(_x, y);
            }
        }
        if x == x2 {
            return;
```

```
            }
            if d >= 0 {
                y = y + sy;
                d = d - ax;
            }
            x = x + sx;
            d = d + ay;
        }
    } else {
        d = ax - ay / 2;
        let delta = double_d / 3;
        loop {
            self.plot(x, y);
            {
                let total = |_x| _x - (y * dx) / dy + (y1 * dx) / dy - x1;
                let mut _x = x;
                loop {
                    let t = total(_x);
                    if t < -1 * delta || t > delta {
                        break;
                    }
                    _x += 1;
                    self.plot(_x, y);
                }
                let mut _x = x;
                loop {
                    let t = total(_x);
                    if t < -1 * delta || t > delta {
                        break;
                    }
                    _x -= 1;
                    self.plot(_x, y);
                }
            }
```

```
            if y == y2 {
                return;
            }
            if d >= 0 {
                x = x + sx;
                d = d - ay;
            }
            y = y + sy;
            d = d + ax;
        }
    }
}
```

Chapter 18

Intersection of two line segments

Let points $\mathbf{1} = (x_1, y_1)$, $\mathbf{2} = (x_2, y_2)$, $\mathbf{3} = (x_3, y_3)$ and $\mathbf{4} = (x_4, y_4)$ and **1,2**, **3,4** two line segments they form. We wish to find their intersection:

First, get the equation of line L_{12} and line L_{34} from chapter FIXME.

Substitute points **3** and **4** in equation L_{12} to compute $r_3 = L_{12}(\mathbf{3})$ and $r_4 = L_{12}(\mathbf{4})$ respectively.

If $r_3 \neq 0$, $r_4 \neq 0$ and $sgn(r_3) == sign(r_4)$ the line segments don't intersect, so stop.

In L_{34} substitute point **1** to compute r_1, and do the same for point **2**.

If $r_1 \neq 0$, $r_2 \neq 0$ and $sgn(r_1) == sign(r_2)$ the line segments don't intersect, so stop.

At this point, L_{12} and L_{34} either intersect or are equivalent. Find their intersection point. (Refer to FIXME.)

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Points, Lines and Planes

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Vectors, matrices and transformations

Chapter 22

Rotation of a bitmap

$$p' = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

$$c = cos\theta, s = sin\theta, x_{p'} = x_p c - y_p s, y_{p'} = x_p s + y_p c.$$

Let's load an `xface`. We will use `bits_to_bytes` (See Introduction).

```
include!("dmr.rs");

const WINDOW_WIDTH: usize = 100;
const WINDOW_HEIGHT: usize = 100;

let mut image = Image::new(DMR_WIDTH, DMR_HEIGHT, 25, 25);
image.bytes = bits_to_bytes(DMR_BITS, DMR_WIDTH);
```

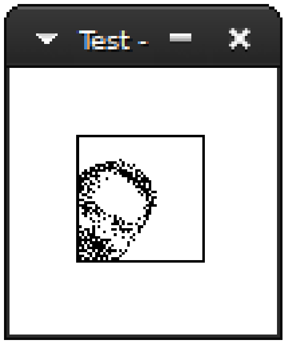
This is the xface of dmr. Instead of displaying the bitmap, this time we will rotate it 0.5 radians. Setup our image first:

```
let mut image = Image::new(DMR_WIDTH, DMR_HEIGHT, 25, 25);
image.draw_outline();
let dmr = bits_to_bytes(DMR_BITS, DMR_WIDTH);
```

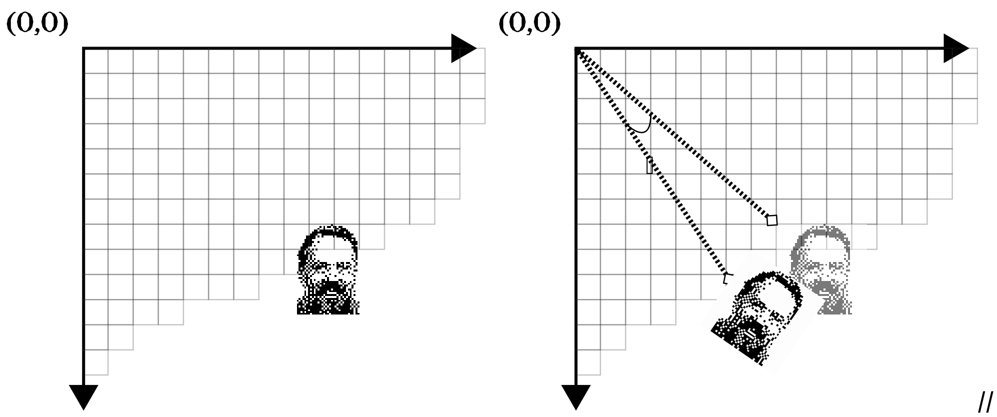
And then, loop for each byte in dmr's face and apply the rotation transformation.

```
let angle = 0.5;

let c = f64::cos(angle);
let s = f64::sin(angle);

for y in 0..DMR_HEIGHT {
    for x in 0..DMR_WIDTH {
        if dmr[y * DMR_WIDTH + x] == BLACK {
            let x = x as f64;
            let y = y as f64;
            let xr = x * c - y * s;
            let yr = x * s + y * c;
            image.plot(xr as i64, yr as i64);
        }
    }
}
```

The result:



We didn't mention in the beginning that the rotation has to be relative to a *point* and the given transformation is relative to the *origin*, in this case the upper left corner $(0, 0)$. So `dmr` was rotated relative to the origin:



(the distance to the origin (actually 0 pixels) has been exaggerated for the sake of the example)

Usually, we want to rotate something relative to itself. The right point to choose is the *centroid* of the object.

If we have a list of n points, the centroid is calculated as:

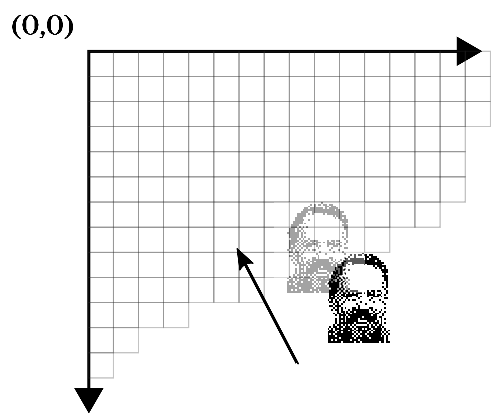
$$x_c = \frac{1}{n}\sum_{i=0}^{n} x_i$$

$$y_c = \frac{1}{n}\sum_{i=0}^{n} y_i$$

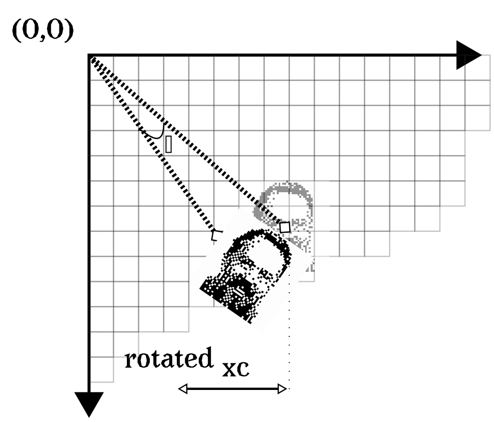
Since in this case we have a rectangle, the centroid has coordinates of half the width and half the height.

By subtracting the centroid from each point before we apply the transformation and then adding it back after we get what we want:

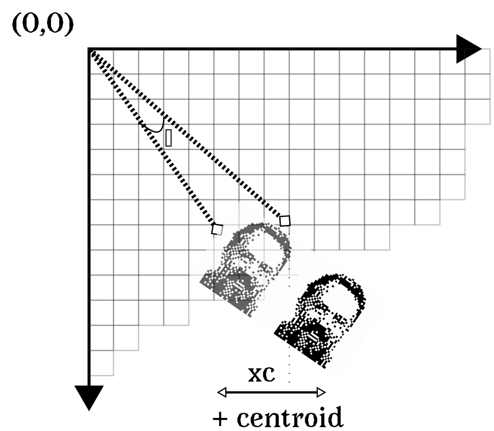
Here's it visually: First subtract the center point.



Then, rotate.



And subtract back to the original position.



In code:

```
let center_point = ((DMR_WIDTH/2) as i64, (DMR_HEIGHT/2) as i64);
for y in 0..DMR_HEIGHT {
    for x in 0..DMR_WIDTH {
        if dmr[y * DMR_WIDTH + x] == BLACK {
            let x = (x as i64 -center_point.0) as f64;
            let y = (y as i64 -center_point.1) as f64;
            let xr = x * c - y * s;
            let yr = x * s + y * c;
            image.plot(xr as i64+center_point.0,
                       yr as i64 + center_point.1);
        }
    }
}
```

The result: 

22.1 Fast 2D Rotation

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90° Rotation of a bitmap by parallel recursive subdivision

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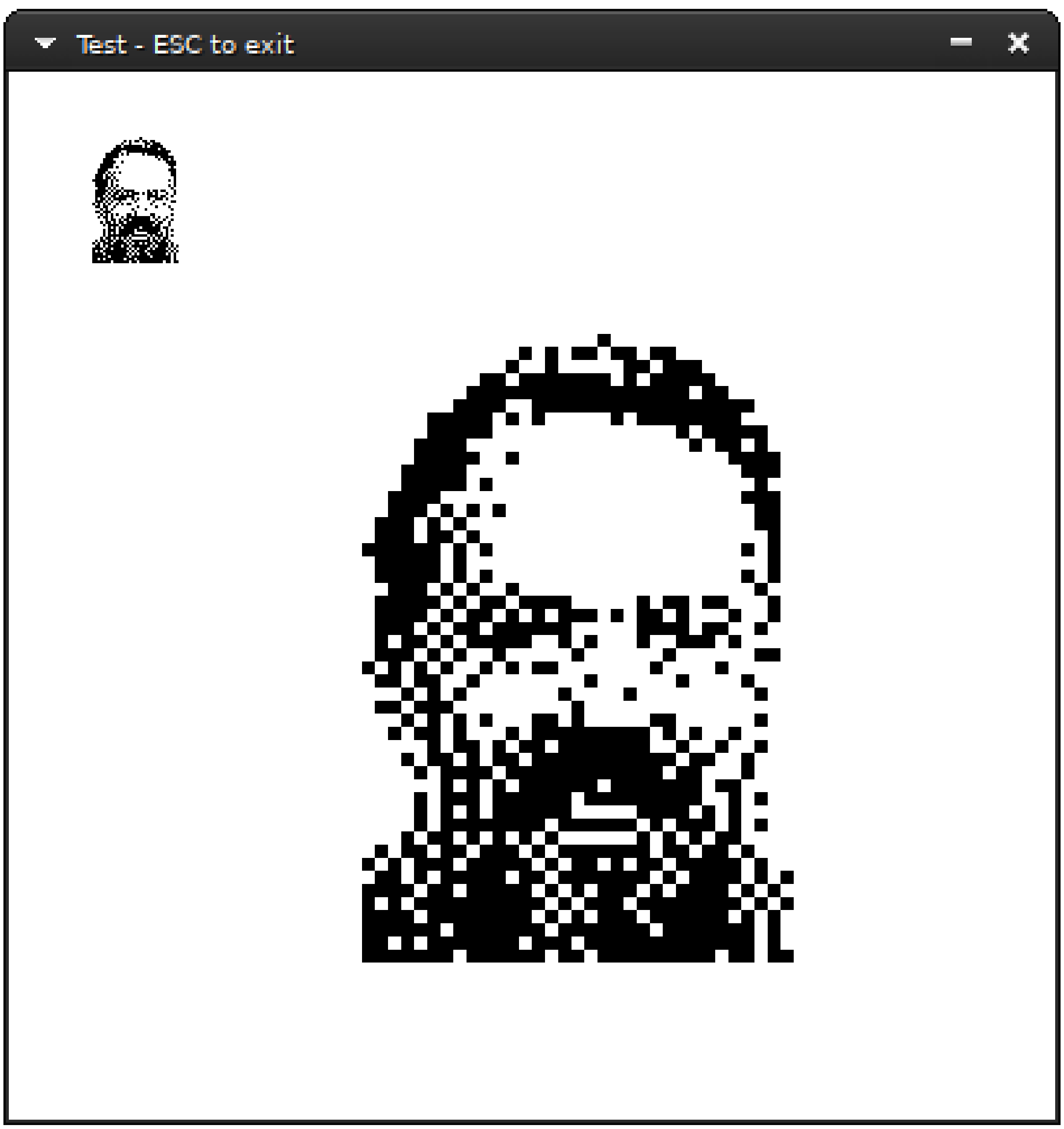
Chapter 24

Magnification/Scaling



```
let mut original = Image::new(DMR_WIDTH, DMR_HEIGHT, 25, 25);
original.bytes = bits_to_bytes(DMR_BITS, DMR_WIDTH);
original.draw(&mut buffer, BLACK, None, WINDOW_WIDTH);

let mut scaled = Image::new(DMR_WIDTH * 5, DMR_HEIGHT * 5, 100, 100);
let mut sx: i64; //source
let mut sy: i64; //source
let mut dx: i64; //destination
let mut dy: i64 = 0; //destination

let og_height = original.height as i64;
let og_width = original.width as i64;
let scaled_height = scaled.height as i64;
let scaled_width = scaled.width as i64;

while dy < scaled_height {
    sy = (dy * og_height) / scaled_height;
    dx = 0;
    while dx < scaled_width {
        sx = (dx * og_width) / scaled_width;
        if original.get(sx, sy) == Some(BLACK) {
            scaled.plot(dx, dy);
        }
        dx += 1;
    }
    dy += 1;
}
scaled.draw(&mut buffer, BLACK, None, WINDOW_WIDTH);
```

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About this text

The text has been typeset in XƎLaTeX using the `book` class and:

- **Avara** for the main text.
- *Fira Sans* for referring to the programming language **Rust**.
- Terminal Grotesque for referring to the word bitmap as a concept.