A Bitmapper's Companion

epilys November 28, 2021

an introduction
to basic bitmap
mathematics
and algorithms
with code
samples in **Rust**

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All non-screenshot figures were generated by hand in Inkscape unless otherwise stated.

The skull in the cover is a transformed bitmap of the skull in the 1533 oil painting by Hans Holbein the Younger, *The Ambassadors*, which features a floating distorted skull rendered in anamorphic perspective.

A Bitmapper's Companion, 2021

Special Topics ► Computer Graphics ► Programming

006.6'6-dc20

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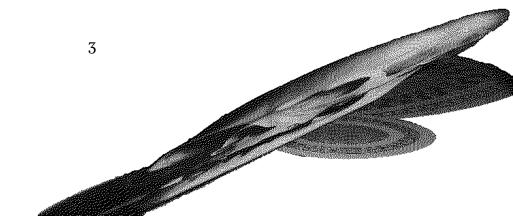
The source code for this work is available under the GNU GENERAL PUBLIC LICENSE version 3 or later. You can view it, study it, modify it for your purposes as long as you respect the license if you choose to distribute your modifications.

The source code is available here

https://github.com/epilys/bitmappers-companion

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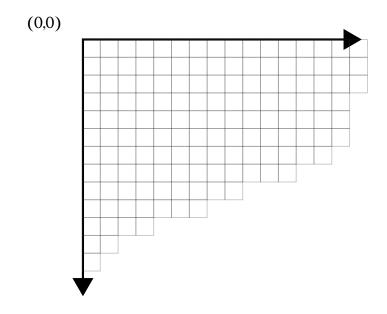


Part I Introduction

Data representation

The data structures we're going to use is *Point* and *Image*. *Image* represents a bitmap, although we will use full RGB colors for our points therefore the size of a pixel in memory will be u8 instead of 1 bit.

We will work on the cartesian grid representing the framebuffer that will show us the pixels. The *origin* of this grid (i.e. the center) is at (0,0).



We will represent points as pairs of signed integers. When actually drawing them though, negative values and values outside the window's geometry will be ignored (clipped).

src/lib.rs:



Displaying pixels to your screen

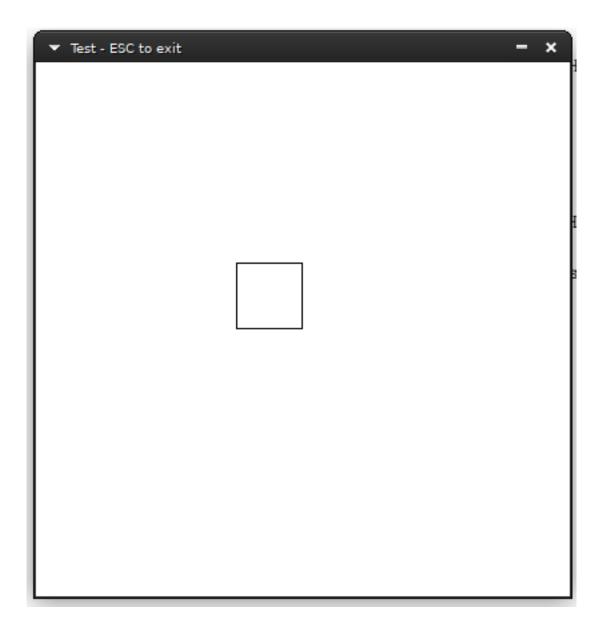
A way to display an *Image* is to use the minifb crate which allows you to create a window and draw pixels directly on it. Here's how you could set it up:

src/bin/introduction.rs:



This code file is a PDF attachment

Running this will show you something like this:



Bits to byte pixels

Let's define a way to convert bit information to a byte vector:

```
pub fn bits_to_bytes(bits: &[u8], width: usize) -> Vec<u32> {
    let mut ret = Vec::with_capacity(bits.len() * 8);
    let mut current_row_count = 0;
    for byte in bits {
        for n in 0..8 {
            if byte.rotate_right(n) & 0x01 > 0 {
                ret.push(BLACK);
            } else {
                ret.push(WHITE);
            }
            current_row_count += 1;
            if current_row_count == width {
                     current_row_count = 0;
                break;
            }
        }
    }
    ret
}
```

Real pixels to byte pixels



Loading xbm files in Rust

xbm files are C source code files that contain the pixel information for an image as macro definitions for the dimensions and a static char array for the pixels, with each bit column representing a pixel. If the width dimension doesn't have 8 as a factor, the remaining bit columns are left blank/ignored.

They used to be a popular way to share user avatars in the old internet and are also good material for us to work with, since they are small and numerous. The following is such an image:



Then, we can convert the xbm file from C to **Rust** with the following transformations:

```
| #define news_width 48
| #define news_height 48
| static char news_bits[] = {
```

to

```
const NEWS_WIDTH: usize = 48;
const NEWS_HEIGHT: usize = 48;
const NEWS_BITS: &[u8] = &[
```

And replace the closing } with].

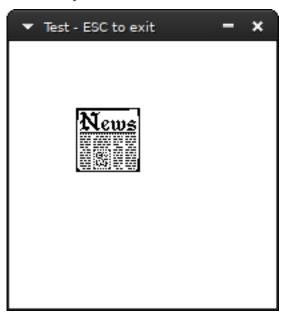
We can then include the new file in our source code:

```
| include!("news.xbm.rs");
```

load the image:

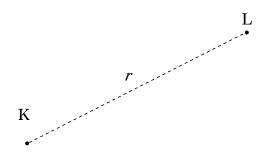
```
let mut image = Image::new(NEWS_WIDTH, NEWS_HEIGHT, 25, 25);
image.bytes = bits_to_bytes(NEWS_BITS, NEWS_WIDTH);
```

and finally run it:



Part II Points and Lines

Distance between two points



Given two points, K and L, an elementary application of Pythagoras' Theorem gives the distance between them as

$$r = \sqrt{(x_L - x_K)^2 + (y_L - y_K)^2}$$
 (6.1)

which is simply coded:

```
pub fn distance_between_two_points(p_k: Point, p_1: Point) -> f64 {
    let (x_k, y_k) = p_k;
    let (x_1, y_1) = p_1;
    let xlk = x_1 - x_k;
    let ylk = y_1 - y_k;
    f64::sqrt((xlk*xlk + ylk*ylk) as f64)
}
```

Equations of a line

There are several ways to describe a line mathematically. We'll list the convenient ones for drawing pixels.

The equation that describes every possible line on a two dimensional grid is the *implicit* form ax + by = c, $(a, b) \neq (0, 0)$. We can generate equivalent equations by adding the equation to itself, i.e. $ax + by = c \equiv 2ax + 2by = 2c \equiv a'x + b'y = c'$, a' = 2a, b' = 2b, c' = 2c as many times as we want. To "minimize" the constants a, b, c we want to satisfy the relationship $a^2 + b^2 = 1$, and thus can convert the equivalent equations into one representative equation by multiplying the two sides with $\frac{1}{\sqrt{a^2+b^2}}$; this is called the normalized equation.

The *slope intercept form* describes any line that intercepts the y axis at $b \in \mathbb{R}$ with a specific slope a:

$$y = ax + b$$

The *parametric* form...

7.1 Line through a point $P = (x_p, y_p)$ and a slope m

$$y - y_p = m(x - x_p)$$

7.2 Line through two points

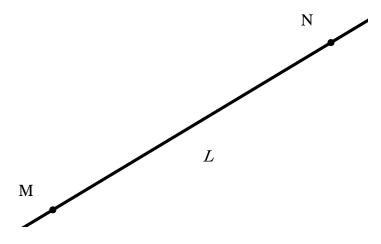


Figure 7.1:

It seems sufficient, given the coordinates of two points M, N, to calculate a, b and c to form a line equation:

$$ax + by + c = 0$$

If the two points are not the same, they necessarily form such a line. To get there, we start from expressing the line as parametric over t: at t=0 it's at point M and at t=1 it's at point N:

$$c = c_M + (c_N - c_M)t, t \in R, c \in \{x, y\}$$
$$c = c_M, t \in R, c \in \{x, y\}$$

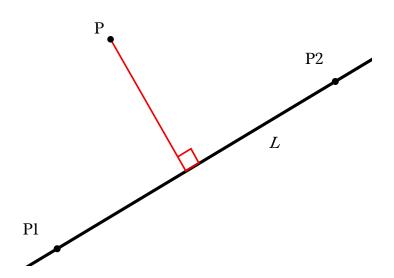
Substituting *t* in one of the equations we get:

$$(y_M - y_N)x + (x_N - x_M)y + (x_My_N - x_Ny_M) = 0$$

Which is what we were after. We finish by normalising what we found with $\frac{1}{\sqrt{a^2+b^2}}$:

lines

Distance from a point to a line



8.1 Using the implicit equation form

Let's find the distance from a given point P and a given line L. Let d be the distance between them. Bring L to the implicit form ax + by = c.

$$d = \frac{|ax_p + by_p + c|}{\sqrt{a^2 + b^2}}$$

8.2 Using an L defined by two points P_1, P_2

With $P = (x_0, y_0)$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$.

$$d = \frac{|(x_2 - x_1)(y_1 - y_0) - (x_1 - x_0)(y_2 - y_1)|}{\sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

8.3 Using an L defined by a point P_l and angle θ

$$d = |cos(\theta)(P_{ly} - y_p) - sin(\theta)(P_{lx} - P_x)|$$

Angle between two lines

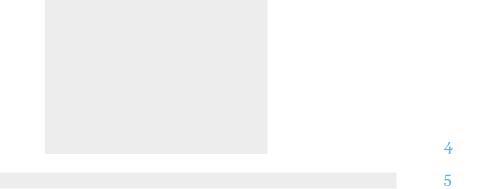


Intersection of two lines



Figure 11.1:

Line equidistant from two points



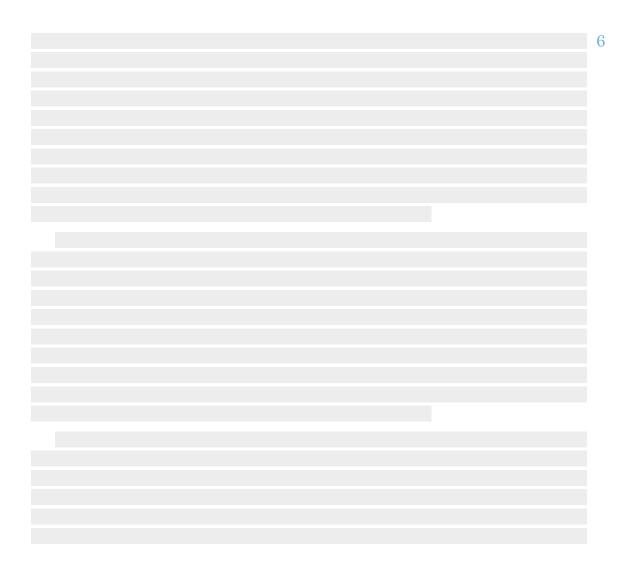
Let's name this line L. From the previous chapter we know how to get the line that's created by the two points M and N. If only we knew how to get a perpendicular line over the midpoint of a line segment!

Thankfully that midpoint also satisfies *L*'s equation, ax + by + c. The midpoint's coordinates are intuitively:

$$(\frac{x_M + x_N}{2}, \frac{y_M + y_N}{2})$$

Putting them into the equation we can generate a triple of (a',b',c') and then normalize it to get L.

Normal to a line through a point



Part III Points, Lines and Circles

circles

Chapter 13

Equations of a circle



circles

Chapter 14

Bounding circle

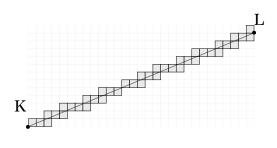


Part IV

Points, Line Segments and Arcs

Drawing a line segment from its two endpoints

For any line segment with any slope, pixels must be matched with the infinite amount of points contained in the segment. As shown in the following figure, a segment *touches* some pixels; we could fill them using an algorithm and get a bitmap of the line segment.



The algorithm presented here was first derived by Bresenham. In the *Image* implementation, it is used in the plot_line_width method.

```
pub fn plot_line_width(&mut self, (x1, y1): (i64, i64), (x2, y2): (i64, i64)) {
    /* Bresenham's line algorithm */
    let mut d;
    let mut x: i64;
    let mut y: i64;
    let ax: i64;
    let sx: i64;
    let sx: i64;
    let sy: i64;
    let dx: i64;
    let dx: i64;
    let dx: i64;
    let x: i64;
    le
```

Drawing line segments with width

```
pub fn plot_line_width(&mut self, (x1, y1): (i64, i64), (x2, y2): (i64, i64), _wd: f64) {
    /* Bresenham's line algorithm */
    let mut x; i64;
    let mut x; i64;
    let mut x; i64;
    let ax: i64;
    let ax: i64;
    let xy: i64;
    let xy: i64;
    let xy: i64;
    let dx: i64;
    let bx: i64;
```

Intersection of two line segments

Let points $\mathbf{l} = (x_1, y_1)$, $\mathbf{2} = (x_2, y_2)$, $\mathbf{3} = (x_3, y_3)$ and $\mathbf{4} = (x_4, y_4)$ and $\mathbf{l}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ two line segments they form. We wish to find their intersection:

First, get the equation of line L_{12} and line L_{34} from chapter *Equations of a line*.

Substitute points **3** and **4** in equation L_{12} to compute $r_3 = L_{12}(\mathbf{3})$ and $r_4 = L_{12}(\mathbf{4})$ respectively.

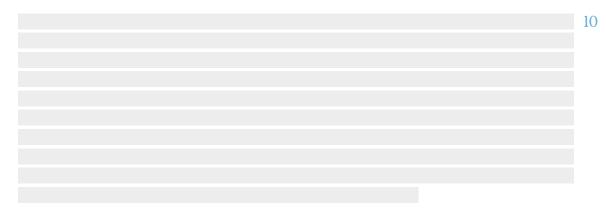
If $r_3 \neq 0$, $r_4 \neq 0$ and $sgn(r_3) == sign(r_4)$ the line segments don't intersect, so stop.

In L_{34} substitute point 1 to compute r_1 , and do the same for point 2.

If $r_1 \neq 0$, $r_2 \neq 0$ and $sgn(r_1) == sign(r_2)$ the line segments don't intersect, so stop.

At this point, L_{12} and L_{34} either intersect or are equivalent. Find their intersection point. (Refer to *Intersection of two lines*.)

17.1 Fast intersection of two line segments



curves

Part V

Curves other than circles

curves

Chapter 18

Parametric ellipictal arcs

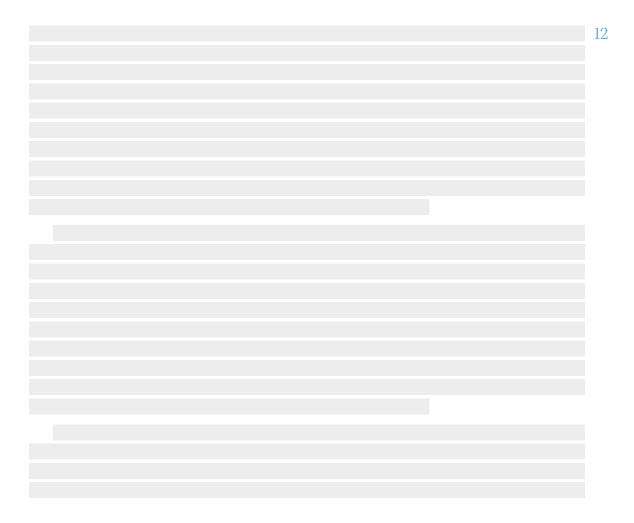


Part VI Points, Lines and Shapes

shapes

Chapter 19

Union, intersection and difference of polygons



shapes

Chapter 20

Centroid of polygon



Part VII

Vectors, matrices and transformations

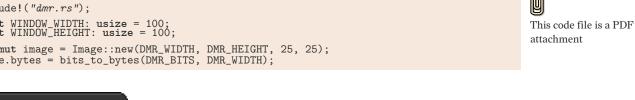
Rotation of a bitmap

$$p' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$\begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

$$c=cos\theta, s=sin\theta, x_{p'}=x_pc-y_ps, y_{p'}=x_ps+y_pc.$$

Let's load an xface. We will use bits_to_bytes (See Introduction).

```
include!("dmr.rs");
const WINDOW_WIDTH: usize = 100;
const WINDOW_HEIGHT: usize = 100;
let mut image = Image::new(DMR_WIDTH, DMR_HEIGHT, 25, 25);
image.bytes = bits_to_bytes(DMR_BITS, DMR_WIDTH);
```



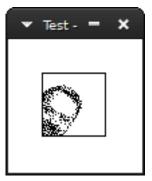


This is the xface of dmr. Instead of displaying the bitmap, this time we will rotate it 0.5 radians. Setup our image first:

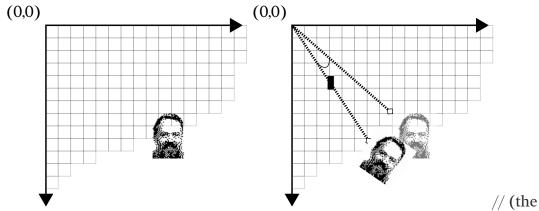
```
let mut image = Image::new(DMR_WIDTH, DMR_HEIGHT, 25, 25);
image.draw_outline();
let dmr = bits_to_bytes(DMR_BITS, DMR_WIDTH);
```

And then, loop for each byte in dmr's face and apply the rotation transformation.

The result:



We didn't mention in the beginning that the rotation has to be relative to a *point* and the given transformation is relative to the *origin*, in this case the upper left corner (0,0). So dmr was rotated relative to the origin:



distance to the origin (actually 0 pixels) has been exaggerated for the sake of the example)

Usually, we want to rotate something relative to itself. The right point to choose is the *centroid* of the object.

If we have a list of n points, the centroid is calculated as:

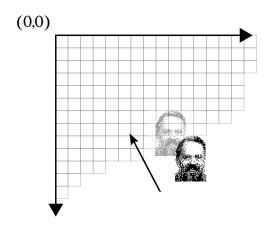
$$x_c = \frac{1}{n} \sum_{i=0}^{n} x_i$$

$$y_c = \frac{1}{n} \sum_{i=0}^n y_i$$

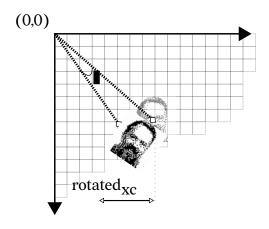
Since in this case we have a rectangle, the centroid has coordinates of half the width and half the height.

By subtracting the centroid from each point before we apply the transformation and then adding it back after we get what we want:

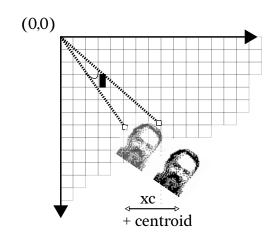
Here's it visually: First subtract the center point.



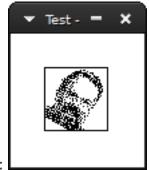
Then, rotate.



And subtract back to the original position.

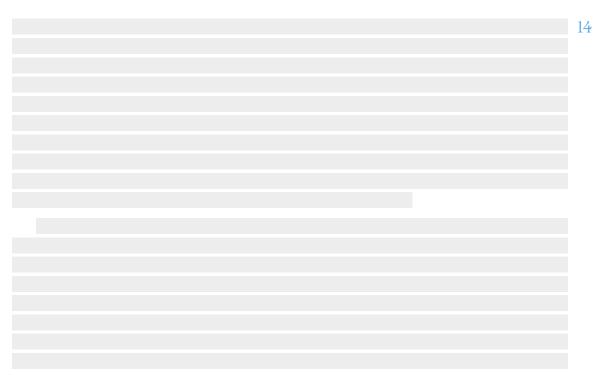


In code:



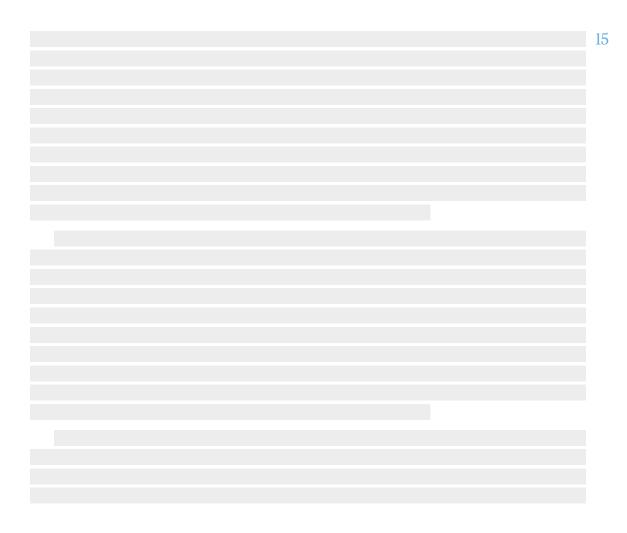
The result:

21.1 Fast 2D Rotation



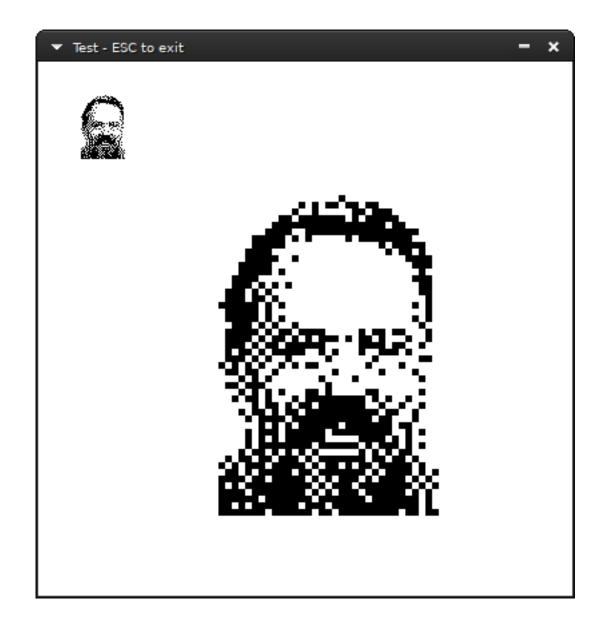
Chapter 22

90° Rotation of a bitmap by parallel recursive subdivision



Chapter 23

Magnification/Scaling



```
let mut original = Image::new(DMR_WIDTH, DMR_HEIGHT, 25, 25);
original.bytes = bits_to_bytes(DMR_BITS, DMR_WIDTH);
original.draw(&mut buffer, BLACK, None, WINDOW_WIDTH);
let mut scaled = Image::new(DMR_WIDTH * 5, DMR_HEIGHT * 5, 100, 100);
let mut sx: i64; //source
let mut sy: i64; //source
let mut dx: i64; //destination
let mut dy: i64 = 0; //destination
let og_height = original.height as i64;
let og_width = original.width as i64;
let scaled_height = scaled.height as i64;
let scaled_width = scaled.width as i64;
while dy < scaled_height {
    sy = (dy * og_height) / scaled_height;
    dx = 0;
    while dx < scaled_width {
        sx = (dx * og_width) / scaled_width;
        if original.get(sx, sy) == Some(BLACK) {
            scaled.plot(dx, dy);
        }
        dx += 1;
    }
    dy += 1;
}
scaled.draw(&mut buffer, BLACK, None, WINDOW_WIDTH);</pre>
```

src/bin/scale.rs:



This code file is a PDF attachment

23.1 Smoothing enlarged bitmaps

16

23.2 Stretching lines of bitmaps

trans-
forma-
tions

Chapter 24

Mirroring



Chapter 25

Shearing



Chapter 26

Projections



areas

Part VIII

Areas

21

areas

areas

Chapter 27

Flood filling



23

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Part IX

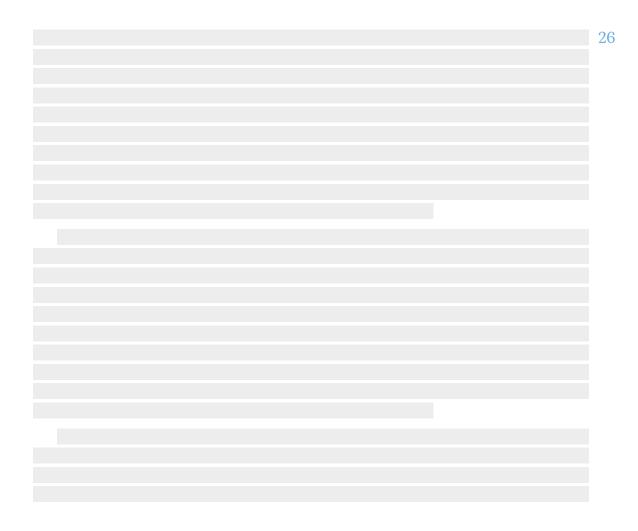
Advanced

27.1 Faster Drawing a line segment from its two endpoints using Symmetry



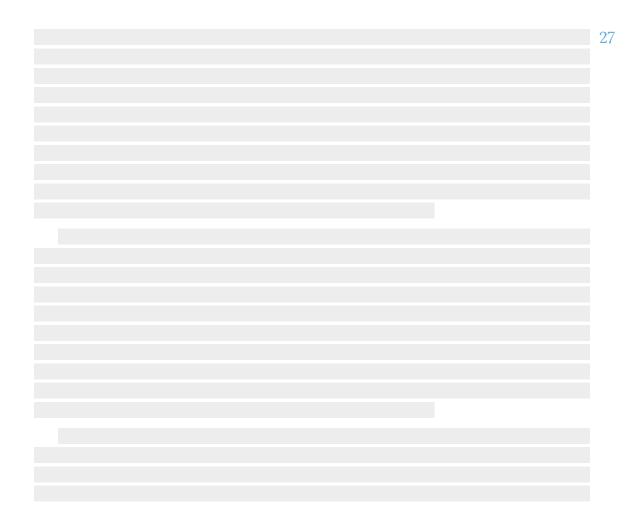
Chapter 28

Joining the ends of two wide line segments together



Chapter 29

Composing monochrome bitmaps with separate alpha channel data



Chapter 30

Orthogonal connection of two points



Chapter 31

Join segments with round corners



Chapter 32

Faster line clipping



Chapter 33

Space-filling Curves



About this text

The text has been typeset in $X_{\overline{A}} \text{Le} T_{\overline{E}} X$ using the book class and:

- **Redaction** for the main text.
- $\boldsymbol{\mathsf{Fira}}$ $\boldsymbol{\mathsf{Sans}}$ for referring to the programming language $\boldsymbol{\mathsf{Rust}}$.
- **Redaction20** for referring to the words bitmap and pixels as a concept.