Applying the linear chain trick using AlgebraicPetri.jl

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Introduction

This example extends the basic SIR model using the linear chain trick (see this paper for more background), in which we chain together multiple infected stages in order to change the infectious period distribution.

Libraries

```
using AlgebraicPetri,AlgebraicPetri.TypedPetri
using Catlab, Catlab.CategoricalAlgebra, Catlab.Programs
using Catlab.WiringDiagrams
using AlgebraicDynamics.UWDDynam
using LabelledArrays
using OrdinaryDiffEq
using Plots
```

Transitions

For convenience, we define `sub`, a function that generate subscripted variables as strings.

```
nstages = 4
sub(i::Int) = i<0 ? error("$i is negative") : join(' '+d for d in reverse(digits(i)))
sub(x::String,i::Int) = x*sub(i);
```

We define a labelled Petri net that has the different types of transition in our models. The first argument is an array of state names as symbols (here, a generic `:Pop`), followed by the transitions in the model. Transitions are given as `transition_name=>((input_states)=>(output_states))`.

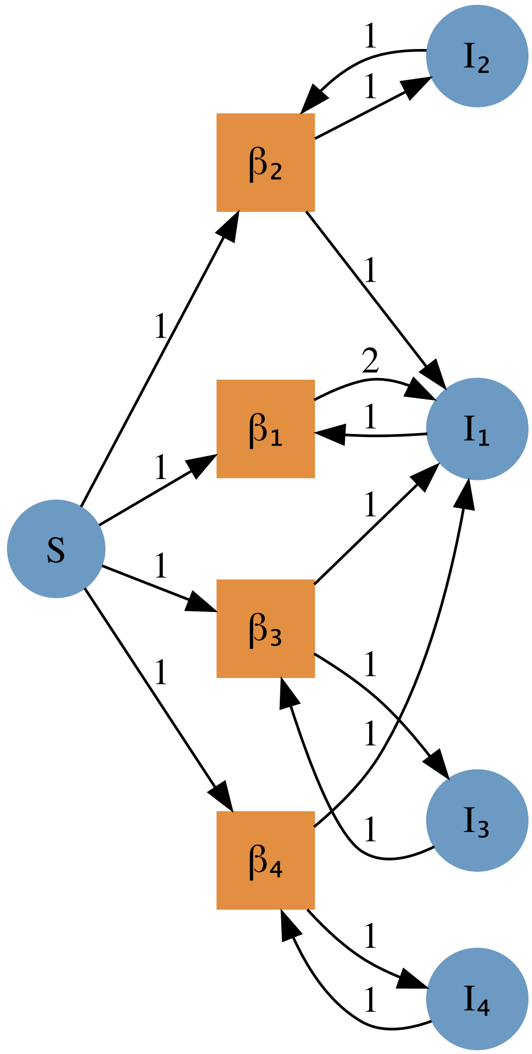
```
epi_lpn = LabelledPetriNet(
  [:Pop],
  :infection=>((:Pop, :Pop)=>(:Pop, :Pop)),
  :progression=>(:Pop=>:Pop),
  :recovery=>(:Pop=>:Pop)
);
```

Next, we define the transmission model as an undirected wiring diagram using the `@relation` macro, as in the basic SIR example. The steps are as follows:

1. Define the undirected wiring diagram.
2. Convert to an `ACSetTransformation` by composing the wiring diagram with the labelled Petri net of transitions.
3. Extract the composed Petri net from the `ACSetTransformation`.

We use four terms that capture the generation of `I` from the four infectious stages, with a separate parameter for each stage.

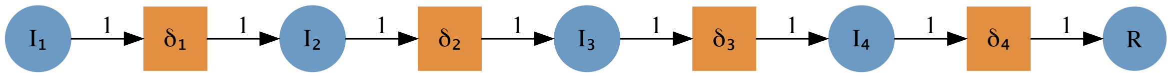
```
si_uwd = @relation (S, I , I , I , I ) where (S::Pop, I ::Pop, I ::Pop, I ::Pop, I ::Pop) begi
    infection(S,I ,I ,I )
    infection(S,I ,I ,I )
    infection(S,I ,I ,I )
    infection(S,I ,I ,I )
end
betas = Symbol.([sub(" ",i) for i=1:nstages])
si_acst = oapply_typed(epi_lpn, si_uwd, betas)
si_lpn = dom(si_acst)
Graph(si_lpn)
```



We repeat for the progression (`I` to `I`, etc.) and recovery (`I` to `R`) transitions; we will use an indexed parameter for both the progression and recovery rates.

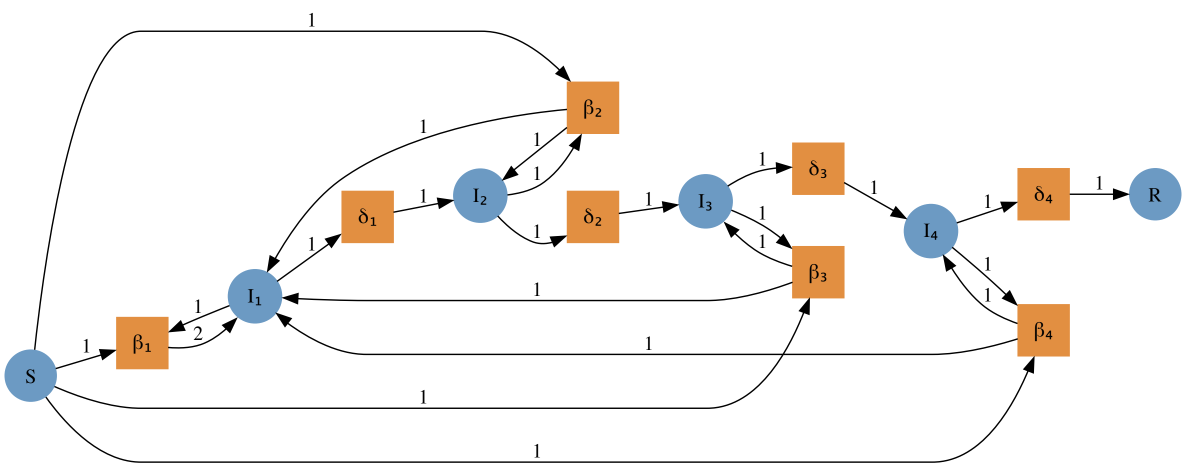
```
ir_uwd = @relation (I , I , I , I , R) where (I ::Pop, I ::Pop, I ::Pop, I ::Pop, R::Pop) begi
    progression(I ,I )
    progression(I ,I )
    progression(I ,I )
    recovery(I ,R)
end
deltas = Symbol.([sub(" ",i) for i=1:nstages])
ir_acst = oapply_typed(epi_lpn, ir_uwd, deltas)
```

```
ir_lpn = dom(ir_acst)
Graph(ir_lpn)
```



To glue the SI and IR models together to make an SIR model, we perform the following: 1. We define an undirected wiring diagram which contains all our states, and two transitions. 2. We then create a `StructuredMulticospan` using this wiring diagram and a dictionary that maps the objects in the wiring diagram with the transmission and recovery Petri nets generated previously. 3. We extract the composed labelled Petri net.

```
sir_uwd = @relation (S, I , I , I , I , R) where (S::Pop, I ::Pop, I ::Pop, I ::Pop, I ::Pop,
    si(S, I , I , I , I )
    ir(I , I , I , I , R)
end
sir_smc = oapply(sir_uwd, Dict(
    :si => Open(si_lpn),
    :ir => Open(ir_lpn),
))
sir_lpn = apex(sir_smc)
Graph(sir_lpn)
```



Running the model

To run an ODE model from the labelled Petri net, we define a vector field, the initial conditions, the parameter array, and the time span.

```
sir_vf = vectorfield(sir_lpn);
u0 = @LArray [990.0, 10.0, 0.0, 0.0, 0.0, 0.0] Tuple(snames(sir_lpn))
p = @LArray vec([repeat([0.5/1000],4); repeat([0.25/4],4)]) Tuple(tnames(sir_lpn))
tspan = (0.0, 40.0);
```

We can now solve the system.

```
sir_prob = ODEProblem(sir_vf, u0, tspan, p)
sir_sol = solve(sir_prob, Rosenbrock32())
plot(sir_sol)
```