Composing an SIR model from individual transitions using AlgebraicPetri.jl

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Introduction

This example serves as a 'Hello World' to composing Petri net models, where models are 'glued' together using the shared states of the individual submodels.

Libraries

```
using AlgebraicPetri,AlgebraicPetri.TypedPetri
using Catlab, Catlab.CategoricalAlgebra, Catlab.Programs
using Catlab.WiringDiagrams
using AlgebraicDynamics.UWDDynam
using LabelledArrays
using OrdinaryDiffEq
using Plots
```

Transitions

We first define a labelled Petri net that has the different types of transition in our models. The first argument is an array of state names as symbols (here, a generic `:Pop`), followed by the transitions in the model. Transitions are given as `transition_name=>((input_states)=>(output_states))`.

```
epi_lpn = LabelledPetriNet(
  [:Pop],
  :infection=>((:Pop, :Pop)=>(:Pop, :Pop)),
```

```
  :recovery=>(:Pop=>:Pop)
);
```

Labelled Petri nets contain four types of fields; `S`, states or species; `T`, transitions; `I`, inputs; and `O`, outputs.

Next, we define the transmission model (from susceptibles, `S` to infecteds, `I`) as an undirected wiring diagram using the `@relation` macro, referring to the transitions in our labelled Petri net above (`infection`). We include a reference to `Pop` in the definition of the state variables to allow us to do this.

```
si_uwd = @relation (S, I) where (S::Pop, I::Pop) begin
    infection(S, I, I, I)
end;
```

We then use `oapply_typed`, which takes in a labelled Petri net (here, `epi_lpn`) and an undirected wiring diagram (`si_uwd`), where each of the boxes is labeled by a symbol that matches the label of a transition in the Petri net, in addition to an array of symbols for each of the rates in the wiring diagram. This produces a Petri net given by colimiting the transitions together, and returns the `ACSetTransformation` from that Petri net to the type system.

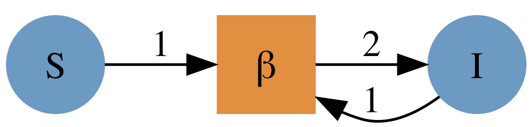
```
si_acst = oapply_typed(epi_lpn, si_uwd, [:β]);
```

To obtain the labelled Petri net, we extract the domain of the `ACSetTransformation` using `dom`.

```
si_lpn = dom(si_acst);
```

We can obtain a GraphViz representation of the labelled Petri net using `Graph`.

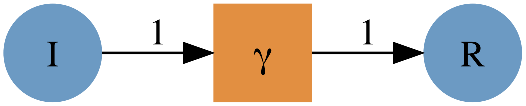
```
Graph(si_lpn)
```



We repeat for the recovery (`I` to `R`) transition.

```
ir_uwd = @relation (I, R) where (I::Pop, R::Pop) begin
    recovery(I, R)
```

```
end
ir_acst = oapply_typed(epi_lpn, ir_uwd, [: ])
ir_lpn = dom(ir_acst)
Graph(ir_lpn)
```



To glue the SI and IR models together to make an SIR model, we first define an undirected wiring diagram which contains all our states, and two transitions.

```
sir_uwd = @relation (S, I, R) where (S::Pop, I::Pop, R::Pop) begin
    si(S, I)
    ir(I, R)
end;
```

We then create a `StructuredMulticospan` using this wiring diagram, telling `oapply` that `si` in the wiring diagram corresponds to the `si_lpn` labelled Petri net, etc.. `Open` converts a PetriNet to an OpenPetriNet where each state is exposed as a leg of the cospan, allowing it to be composed over an undirected wiring diagram.

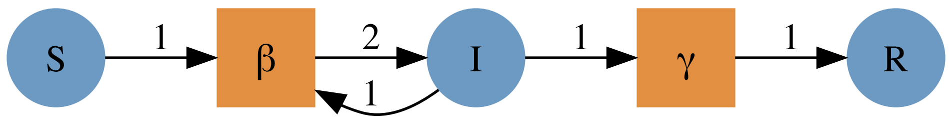
```
sir_smc = oapply(sir_uwd, Dict(
    :si => Open(si_lpn),
    :ir => Open(ir_lpn),
))
```

```
StructuredMulticospan{Catlab.CategoricalAlgebra.StructuredCospans.DiscreteACSet{AnonACSet{Ty
  T = 1:2
  S = 1:3
  I = 1:3
  O = 1:3
  it : I → T = [1, 1, 2]
  is : I → S = [1, 2, 2]
  ot : O → T = [1, 1, 2]
  os : O → S = [2, 2, 3]
  tname : T → Name = [: , : ]
  sname : S → Name = [:S, :I, :R], StructTightACSetTransformation{TypeLevelBasicSchema{Symbol
  S = 1:1
  sname : S → Name = [:S], AnonACSet{TypeLevelBasicSchema{Symbol, Tuple{:S}, Tuple{}, Tuple{
```

```
S = 1:1
sname : S → Name = [:I], AnonACSet{TypeLevelBasicSchema{Symbol, Tuple{:S}, Tuple{}, Tuple{
S = 1:1
sname : S → Name = [:R]])
```

We extract the labelled Petri net by extracting the object that is the codomain of all the legs, using the `apex` function.

```
sir_lpn = apex(sir_smc)
Graph(sir_lpn)
```



Running the model

To run an ODE model from the labelled Petri net, we generate a function that can be passed to SciML's `ODEProblem` using `vectorfield`.

```
sir_vf = vectorfield(sir_lpn);
```

The initial conditions and parameter values are written as labelled arrays.

```
u0 = @LArray [990.0, 10.0, 0.0] Tuple(snames(sir_lpn))
```

```
3-element LArray{Float64, 1, Vector{Float64}, (:S, :I, :R)}:
 :S => 990.0
 :I => 10.0
 :R => 0.0
```

```
p = @LArray [0.5/1000, 0.25] Tuple(tnames(sir_lpn))
```

```
2-element LArray{Float64, 1, Vector{Float64}, (: , : )}:
 :  => 0.0005
 :  => 0.25
```

```julia
tspan = (0.0, 40.0);
```

We can now use the initial conditions, the time span, and the parameter values to simulate the system.

```julia
sir_prob = ODEProblem(sir_vf, u0, tspan, p)
sir_sol = solve(sir_prob, Rosenbrock32())
plot(sir_sol)
```