Timestep sensitivity analysis: Exponential vs Euler discretisation

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Introduction

This script performs a timestep sensitivity analysis for Case Scenario 1 (lockdown) comparing two discretisation methods:

Method 1: Exponential discretisation

- `infection[t] == (1 - exp(-(1 - υ[t]) * β * I[t] * dt)) * S[t]`
- `recovery[t] == (1 - exp(-γ * dt)) * I[t]`

Method 2: Simple Euler discretisation

- `infection[t] == (1 - υ[t]) * β * I[t] * dt * S[t]`
- `recovery[t] == γ * dt * I[t]`

```
using JuMP
using Ipopt
using Plots
Plots.default(fmt = :png)   # or :svg
using DataFrames
using Statistics
```

We add a warmup step to trigger JIT compilation and eliminate compilation overhead time

```
function warmup_jit()
    warmup_model = Model(Ipopt.Optimizer)
    set_optimizer_attribute(warmup_model, "print_level", 0)  # Silent

    # Small test problem
    @variable(warmup_model, 0 <= x[1:5] <= 1)
    @variable(warmup_model, y[1:4] >= 0)
    @constraint(warmup_model, [t=1:4], y[t] == (1 - exp(-0.5 * x[t])) * x[t])
    @constraint(warmup_model, [t=1:4], x[t+1] == x[t] - y[t])
    @objective(warmup_model, Min, x[5])

    optimize!(warmup_model)
end
warmup_jit()
```

Model parameters

```
β = 0.5          # infection rate
γ = 0.25         # recovery rate
υ_max = 0.5      # maximum intervention
υ_total = 10.0   # maximum cost

t0 = 0.0
tf = 100.0

S0 = 0.99
I0 = 0.01
C0 = 0.00;
```

```
# Timesteps to test
dt_values = [2.0, 1.0, 0.5, 0.25, 0.1, 0.05]

# Number of repetitions per dt cases (to calculate mean and std)
n_repetitions = 50

# Initialise results storage
results_exponential = []
results_euler = [];
```

Running optimisation for each timestep on both methods:

Method 1: Exponential discretisation

```
for dt in dt_values

    # Number of time points
    T = Int(tf/dt)

    # Storage results
    solve_times_exp = []
    objectives_exp = []
    statuses_exp = []

    # Trajectories
    local υ_opt_exp, C_opt_exp, S_opt_exp, I_opt_exp

    # Run n_repetitions times
    for rep in 1:n_repetitions
        model_exp = Model(Ipopt.Optimizer)
        set_optimizer_attribute(model_exp, "print_level", 0)

        # Variables
        @variable(model_exp, 0 <= S[1:(T+1)] <= 1)
        @variable(model_exp, 0 <= I[1:(T+1)] <= 1)
        @variable(model_exp, 0 <= C[1:(T+1)] <= 1)
        @variable(model_exp, 0 <= υ[1:(T+1)] <= υ_max)

        # Exponential discretisation
        @expressions(model_exp, begin
            infection[t in 1:T], (1 - exp(-(1 - υ[t]) * β * I[t] * dt)) * S[t]
            recovery[t in 1:T], (1 - exp(-γ * dt)) * I[t]
        end)

        @constraints(model_exp, begin
            S[1]==S0
            I[1]==I0
            C[1]==C0
            [t=1:T], S[t+1] == S[t] - infection[t]
            [t=1:T], I[t+1] == I[t] + infection[t] - recovery[t]
            [t=1:T], C[t+1] == C[t] + infection[t]
            dt * sum(υ[t] for t in 1:T+1) <= υ_total
        end)

        # Objective
        @objective(model_exp, Min, C[T+1])

        # Solve and time
        solve_start = time()
        optimize!(model_exp)
        solve_time = time() - solve_start

        # Extract results
        status = termination_status(model_exp)
        obj_value = objective_value(model_exp)

        # Store results
        push!(solve_times_exp, solve_time)
        push!(objectives_exp, obj_value)
        push!(statuses_exp, status)

        # Save trajectories from first run
        if rep == 1
            υ_opt_exp = value.(υ)
            C_opt_exp = value.(C)
            S_opt_exp = value.(S)
            I_opt_exp = value.(I)
        end
    end

    # Results
    mean_time_exp = mean(solve_times_exp)
    std_time_exp = std(solve_times_exp)

    # Verify ALL runs converged successfully
    if !all(s == MOI.LOCALLY_SOLVED for s in statuses_exp)
        error("Not all Exponential runs converged! Statuses: $statuses_exp")
    end
    status_exp = statuses_exp[1]  # All LOCALLY_SOLVED

    # Verify that all objectives are identical
    if !all(abs(obj - objectives_exp[1]) < 1e-10 for obj in objectives_exp)
        error("Objectives varied across repetitions:\n" *
              "  Values: $objectives_exp\n" *
              "  Max difference: $(maximum(objectives_exp) - minimum(objectives_
    end
    obj_value_exp = objectives_exp[1]
    # Control
    threshold = 0.01
    control_start_idx = findfirst(x -> x > threshold, υ_opt_exp)
    control_start_exp = control_start_idx !== nothing ? (control_start_idx - 1)
    control_active = [υ_opt_exp[i] > threshold for i in 1:length(υ_opt_exp)]
    control_duration_exp = sum(control_active) * dt

    # Exponential results
    result_exp = Dict(
        :dt => dt,
        :method => "Exponential",
        :time_points => T+1,
        :objective => obj_value_exp,
        :solve_time => mean_time_exp,
        :solve_time_std => std_time_exp,
        :solve_times_all => solve_times_exp,
        :convergence => string(status_exp),
        :control_start => control_start_exp,
        :control_duration => control_duration_exp,
        :final_infections => C_opt_exp[end],
        :υ_trajectory => υ_opt_exp,
        :C_trajectory => C_opt_exp,
        :S_trajectory => S_opt_exp,
        :I_trajectory => I_opt_exp
    )
    push!(results_exponential, result_exp)
end
```

Method 2: Simple Euler discretisation (Linear approximation)

```
for dt in dt_values

    # Number of time points
    T = Int(tf/dt)

    # Storage results
    solve_times_euler = []
    objectives_euler = []
    statuses_euler = []
```

```
# Trajectories
local υ_opt_euler, C_opt_euler, S_opt_euler, I_opt_euler

# Run n_repetitions times
for rep in 1:n_repetitions
    model_euler = Model(Ipopt.Optimizer)
    set_optimizer_attribute(model_euler, "print_level", 0)

    # Variables
    @variable(model_euler, 0 <= S[1:(T+1)] <= 1)
    @variable(model_euler, 0 <= I[1:(T+1)] <= 1)
    @variable(model_euler, 0 <= C[1:(T+1)] <= 1)
    @variable(model_euler, 0 <= υ[1:(T+1)] <= υ_max)

    # Simple Euler discretisation
    @expressions(model_euler, begin
        infection[t in 1:T], (1 - υ[t]) * β * I[t] * dt * S[t]
        recovery[t in 1:T], γ * dt * I[t]
    end)

    @constraints(model_euler, begin
        S[1]==S0
        I[1]==I0
        C[1]==C0
        [t=1:T], S[t+1] == S[t] - infection[t]
        [t=1:T], I[t+1] == I[t] + infection[t] - recovery[t]
        [t=1:T], C[t+1] == C[t] + infection[t]
        dt * sum(υ[t] for t in 1:T+1) <= υ_total
    end)

    # Objective
    @objective(model_euler, Min, C[T+1])

    # Solve and time
    solve_start = time()
    optimize!(model_euler)
    solve_time = time() - solve_start

    # Extract results
    status = termination_status(model_euler)
    obj_value = objective_value(model_euler)
```

```
        # Store results
        push!(solve_times_euler, solve_time)
        push!(objectives_euler, obj_value)
        push!(statuses_euler, status)

        # Save trajectories from first run
        if rep == 1
            υ_opt_euler = value.(υ)
            C_opt_euler = value.(C)
            S_opt_euler = value.(S)
            I_opt_euler = value.(I)
        end
    end

    # Calculate statistics
    mean_time_euler = mean(solve_times_euler)
    std_time_euler = std(solve_times_euler)

    # Verify ALL runs converged successfully
    if !all(s == MOI.LOCALLY_SOLVED for s in statuses_euler)
        error("Not all Euler runs converged! Statuses: $statuses_euler")
    end
    status_euler = statuses_euler[1]

    # Verify all objectives are identical
    if !all(abs(obj - objectives_euler[1]) < 1e-10 for obj in objectives_euler)
        error("Objectives varied across repetitions:\n" *
              "  Values: $objectives_euler\n" *
              "  Max difference: $(maximum(objectives_euler) - minimum(objectiv
    end
    obj_value_euler = objectives_euler[1]

    # Control
    threshold = 0.01
    control_start_idx = findfirst(x -> x > threshold, υ_opt_euler)
    control_start_euler = control_start_idx !== nothing ? (control_start_idx -
    control_active = [υ_opt_euler[i] > threshold for i in 1:length(υ_opt_euler)
    control_duration_euler = sum(control_active) * dt

    # Verify all objectives are identical
    if !all(abs(obj - objectives_euler[1]) < 1e-10 for obj in objectives_euler)
        error("Euler: Not deterministic! Objectives varied across repetitions:\
               "  Values: $objectives_euler\n" *
               "  Max difference: $(maximum(objectives_euler) - minimum(objectiv
    end

    # Euler results
    result_euler = Dict(
        :dt => dt,
        :method => "Euler",
        :time_points => T+1,
        :objective => obj_value_euler,
        :solve_time => mean_time_euler,
        :solve_time_std => std_time_euler,
        :solve_times_all => solve_times_euler,
        :convergence => string(status_euler),
        :control_start => control_start_euler,
        :control_duration => control_duration_euler,
        :final_infections => C_opt_euler[end],
        :υ_trajectory => υ_opt_euler,
        :C_trajectory => C_opt_euler,
        :S_trajectory => S_opt_euler,
        :I_trajectory => I_opt_euler
    )
    push!(results_euler, result_euler)
end
```

Exponential discretisation results:

```
6×9 DataFrame
 Row │ dt       time_points  objective  solve_time_mean  solve_time_std  conver
     │ Float64  Float64      Float64    Float64          Float64         String
─────┼──────────────────────────────────────────────────────────────────────────
   1 │    2.0          51.0     0.731            0.0228          0.0053  LOCALL
   2 │    1.0         101.0     0.6686           0.05            0.0033  LOCALL
   3 │    0.5         201.0     0.6332           0.2312          0.0053  LOCALL
   4 │    0.25        401.0     0.6145           0.2603          0.0123  LOCALL
   5 │    0.1        1001.0     0.6028           1.273           0.1185  LOCALL
   6 │    0.05       2001.0     0.5989           1.8665          0.1421  LOCALL
```

Simple Euler discretisation results:

```
6×9 DataFrame
```

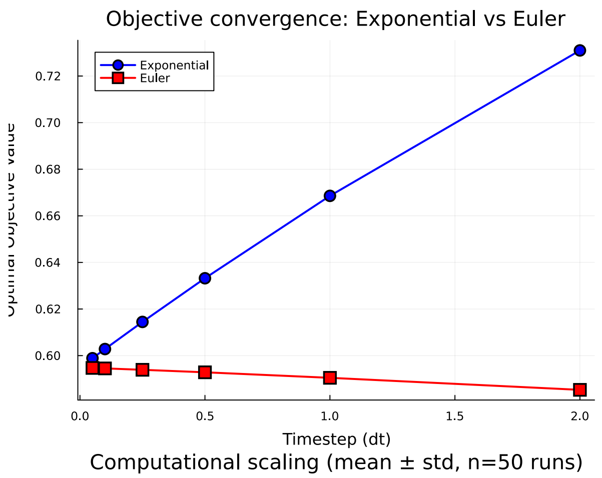
```
 Row │ dt         time_points  objective  solve_time_mean  solve_time_std  conver
     │ Float64    Float64      Float64    Float64          Float64         String
─────┼
   1 │    2.0            51.0     0.5853           0.0312          0.0078  LOCALL
   2 │    1.0           101.0     0.5905           0.0466          0.0244  LOCALL
   3 │    0.5           201.0     0.5928           0.2013          0.008   LOCALL
   4 │    0.25          401.0     0.5939           0.3019          0.0319  LOCALL
   5 │    0.1          1001.0     0.5945           1.4404          0.0862  LOCALL
   6 │    0.05         2001.0     0.5947           1.3914          0.0918  LOCALL
```

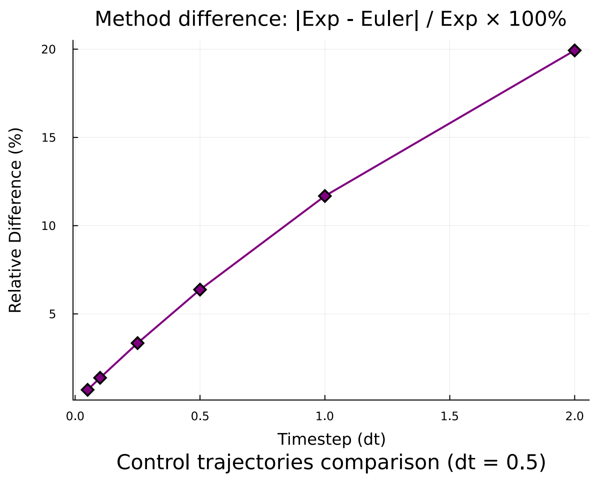
Relative errors of each method using ‘dt=0.05’ as reference

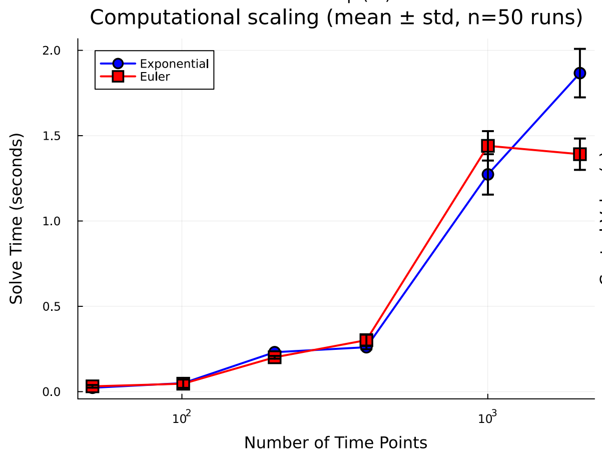
```
ref_idx_exp = findfirst(r -> r[:dt] == 0.05, results_exponential)
ref_idx_euler = findfirst(r -> r[:dt] == 0.05, results_euler)

if ref_idx_exp !== nothing && ref_idx_euler !== nothing
    ref_obj_exp = results_exponential[ref_idx_exp][:objective]
    ref_obj_euler = results_euler[ref_idx_euler][:objective]

    # Exponential method
    rel_errors_exp = []
    for r in results_exponential
        obj_error = abs(r[:objective] - ref_obj_exp) / ref_obj_exp * 100
        push!(rel_errors_exp, Dict(
            :dt => r[:dt],
            :obj_rel_error_pct => obj_error,
            :method => "Exponential"
        ))
    end

    # Euler method
    rel_errors_euler = []
    for r in results_euler
        obj_error = abs(r[:objective] - ref_obj_euler) / ref_obj_euler * 100
        push!(rel_errors_euler, Dict(
            :dt => r[:dt],
            :obj_rel_error_pct => obj_error,
            :method => "Euler"
        ))
    end

    errors_exp_df = DataFrame(rel_errors_exp)
    errors_euler_df = DataFrame(rel_errors_euler)

    # Convergence table
    convergence_comparison_df = DataFrame(
        dt = [r[:dt] for r in results_exponential],
        exp_error_pct = [r[:obj_rel_error_pct] for r in rel_errors_exp],
        euler_error_pct = [r[:obj_rel_error_pct] for r in rel_errors_euler]
    )

    show(transform(convergence_comparison_df, names(convergence_comparison_df, 

end
```

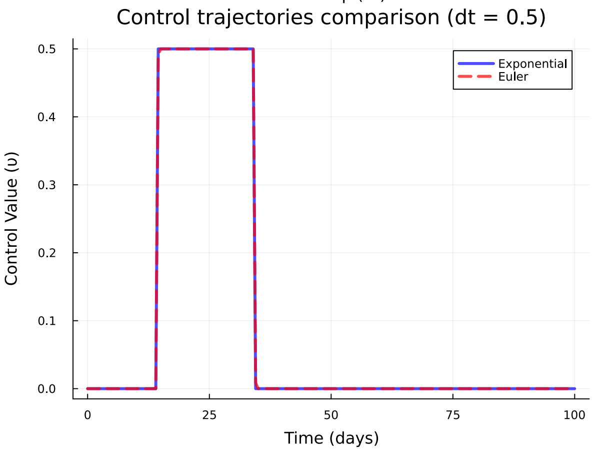
```
6×3 DataFrame
 Row │ dt       exp_error_pct  euler_error_pct
     │ Float64  Float64        Float64
─────┼─────────────────────────────────────────
   1 │    2.0         22.0589           1.5812
   2 │    1.0         11.6362           0.7131
   3 │    0.5          5.7315           0.3145
   4 │    0.25         2.6017           0.1343
   5 │    0.1          0.6581           0.0335
   6 │    0.05         0.0              0.0
```

Plots


Objective convergence: Exponential vs Euler
Exponential
Euler
Optimal Objective Value
Timestep (dt)
0.60
0.62
0.64
0.66
0.68
0.70
0.72
0.0
0.5
1.0
1.5
2.0


Method difference: |Exp - Euler| / Exp × 100%
Relative Difference (%)
Timestep (dt)
5
10
15
20
0.0
0.5
1.0
1.5
2.0


Computational scaling (mean ± std, n=50 runs)
Exponential
Euler
Solve Time (seconds)
Number of Time Points
0.0
0.5
1.0
1.5
2.0
10^2
10^3


Control trajectories comparison (dt = 0.5)
Exponential
Euler
Control Value (υ)
Time (days)
0.0
0.1
0.2
0.3
0.4
0.5
0
25
50
75
100