Vaccination optimisation on an SIR model using JuMP.jl

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Introduction

This example examines the optimal control of an SIR model through vaccination, which reduces the number of susceptible individuals according to the following set of equations:

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI - \upsilon(t)S, \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta SI - \gamma I, \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= \beta SI
\end{aligned}
$$

Similar to our previous examples, the population is divided into three categories: susceptible individuals (`S`), infected individuals (`I`), and the total number of cases (`C`). However in this case, Susceptible individuals are vaccinated at a per-capita rate ` (t)`.

The optimal control problem is defined as the policy that minimises the total number of cases (i.e., the final size of the epidemic) while adhering to the following constraints: (a) the vaccination rate, , cannot exceed a maximum value, indicating a limit on the rate of vaccination, and (b) there is a cost associated with the vaccination process, measured as the integral of ` (t)*S(t)` over time, which cannot exceed a predetermined level. Again, we determine the optimal policy numerically using a simple Euler discretisation and then JuMP.jl with IPOPT to optimise.

Libraries

```
using OrdinaryDiffEq
using DiffEqCallbacks
using JuMP
using Ipopt
using Plots
using DataInterpolations
using NonlinearSolve;
```

Functions

ODE system

```
function sir_ode!(du,u,p,t)
    (S, I, C) = u
    ( ,  ,  ) = p
    @inbounds begin
        du[1] = - *S*I -  *S
        du[2] =  *S*I -  *I
        du[3] =  *S*I
    end
    nothing
end;
```

Running the model without intervention

Parameters

```
u0 = [0.99, 0.01, 0.0]; #S, I, C (cumulative incidence)
p = [0.5, 0.25, 0]; #  ,  ,
```

```
t0 = 0.0
tf = 100
dt = 0.1
ts = collect(t0:dt:tf)
alg = Tsit5();
```

Using ODEProblem

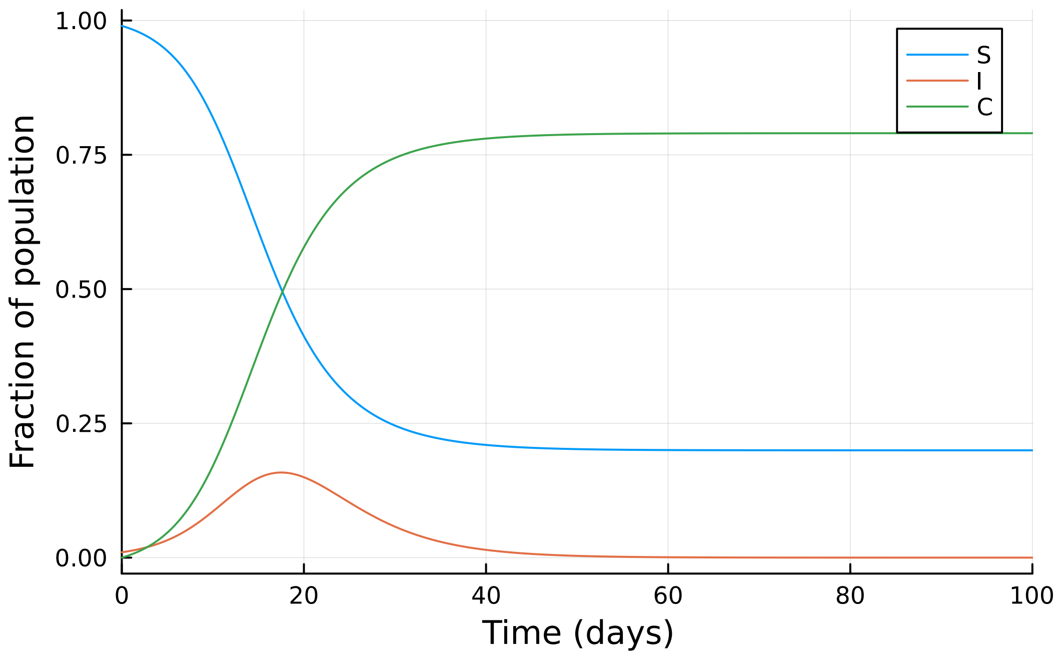
```
prob1 = ODEProblem(sir_ode!, u0, (t0, tf), p)
sol1 = solve(prob1, alg, saveat=ts);
```

Without control the peak fraction of infected individuals is $\sim 16\%$

```julia
peak_value, peak_index = findmax(sol1[2, :])
println("The maximum fraction of infected at a `dt` time is: ", peak_value)
```

```
The maximum fraction of infected at a `dt` time is: 0.15845528864997238
```

```julia
plot(sol1,
     xlim=(0, 100),
     labels=["S" "I" "C"],
     xlabel="Time (days)",
     ylabel="Fraction of population")
```



Searching for the optimal intervention constrained by maximum vaccination rate _max, and cost

Parameters

```julia
p2 = copy(p)
p2[3] = 0.05;   #Set maximum vaccination rate to 0.05
  = p2[1]
  = p2[2]
 _max = p2[3]
 _total = 1      # Maximum cost
```

```
S0 = u0[1]
I0 = u0[2]
C0 = u0[3]

T = Int(tf/dt)

silent = true;
```

Model setup

```
model = Model(Ipopt.Optimizer)
set_optimizer_attribute(model, "max_iter", 1000)
if !silent
    set_optimizer_attribute(model, "output_file", "JuMP_ftc.txt")
    set_optimizer_attribute(model, "print_timing_statistics", "yes")
end;
```

Variables:

From their definition, the variables `S`, `I` and `C` are constrained to values between 0 and 1. We constrain our vaccination policy, `(t)` to lie between 0 and `_max`.

```
@variable(model, 0 <= S[1:(T+1)] <= 1)
@variable(model, 0 <= I[1:(T+1)] <= 1)
@variable(model, 0 <= C[1:(T+1)] <= 1)
@variable(model, 0 <=  [1:(T+1)] <=  _max);
```

We discretise the SIR model using a simple Euler discretisation:

```
@expressions(model, begin
        infection[t in 1:T],   * I[t] * dt * S[t]
        recovery[t in 1:T],   * dt * I[t]
        vaccination[t in 1:T],  [t]* dt * S[t]
    end);
```

We constrain the integral of the intervention to be less than or equal to `_total`, assuming that the intervention is piecewise constant during each time step.

```
@constraints(model, begin
    S[1]==S0
    I[1]==I0
    C[1]==C0
    [t=1:T], S[t+1] == S[t] - infection[t] - vaccination[t]
    [t=1:T], I[t+1] == I[t] + infection[t] - recovery[t]
    [t=1:T], C[t+1] == C[t] + infection[t]
    dt * sum( [t]*S[t] for t in 1:T+1) <=  _total
end);
```

This scenario's objective is to minimise the total number of cases:

```
@objective(model, Min, C[T+1]);
```

```
if silent
    set_silent(model)
end
optimize!(model)
```

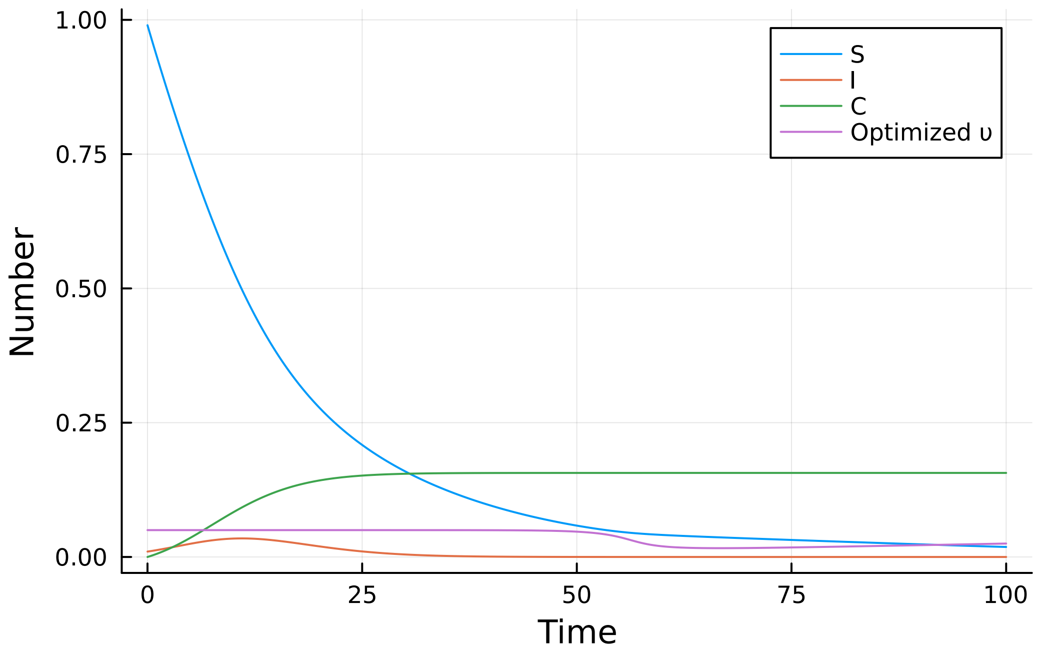
```
******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit https://github.com/coin-or/Ipopt
******************************************************************************
```

```
termination_status(model)
```

```
LOCALLY_SOLVED::TerminationStatusCode = 4
```

```
S_opt = value.(S)
I_opt = value.(I)
C_opt = value.(C)
 _opt = value.( );
```

```
plot(ts, S_opt, label="S", xlabel="Time", ylabel="Number")
plot!(ts, I_opt, label="I")
plot!(ts, C_opt, label="C")
plot!(ts,  _opt, label="Optimized  ")
```



With the optimised vaccine intervention, we can observe that the maximum number of fraction of infected is $\sim 3\%$

```
peak_value_opt, peak_index_opt = findmax(I_opt)
println("The maximum fraction of infected at a `dt` time is: ", peak_value_opt)
```

```
The maximum fraction of infected at a `dt` time is: 0.034552884095835804
```

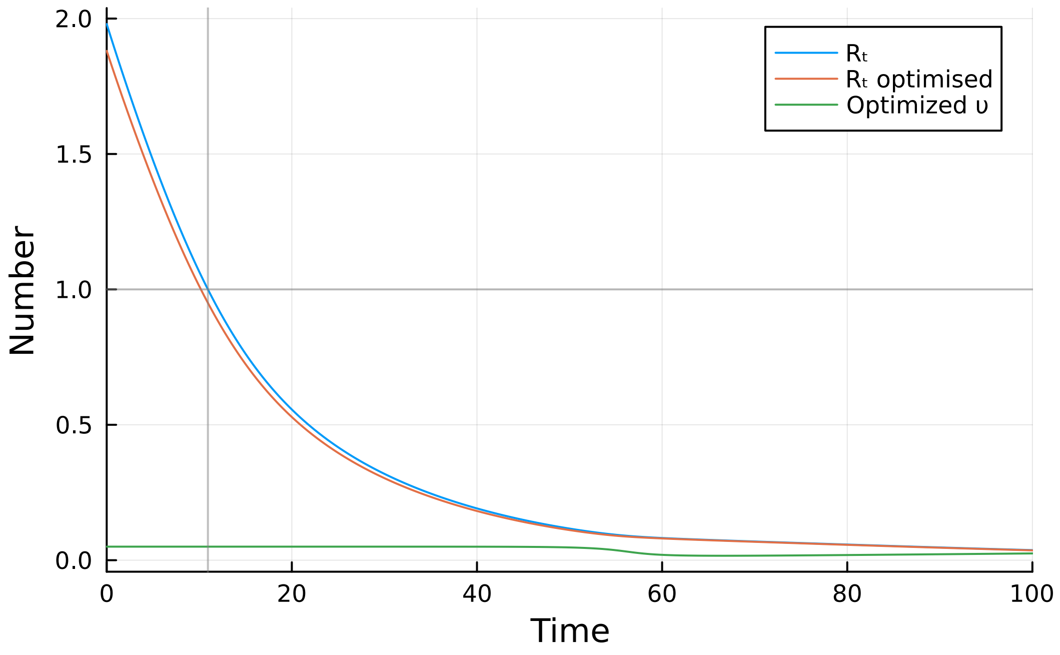
Again, we can calculate the effective reproductive number, `R` in the presence of the intervention:

```
R _opt =  .* S_opt ./    #Not taking into account the intervention
R _opt = R _opt .* (1 .-  _opt);  #Taking into account the intervention
```

And the time at which `R ==1` using a root-finding approach:

```
R _interp = CubicSpline(R _opt,ts)
f(u, p) = [R _interp(u[1]) - 1.0]
u0 = [(tf-t0)/5]
Rtprob = NonlinearProblem(f, u0)
Rtsol = solve(Rtprob, NewtonRaphson(), abstol = 1e-9).u[1];
```

```
plot(ts, R _opt, label="R ", xlabel="Time", ylabel="Number", legend=:topright, xlim=(0,100)
plot!(ts, R _opt, label="R  optimised")
plot!(ts,  _opt, label="Optimized  ")
vline!([Rtsol], color=:gray, alpha=0.5, label=false)
hline!([1.0], color=:gray, alpha=0.5, label=false)
```



Discussion

Assuming that a vaccine is available at the start of an epidemic, the results in this example suggest that the optimal policy is to vaccinate early and at the maximum level available until the vaccine supply is exhausted.

The plot of `R ` over time shows that, in this scenario as well as in the lockdown scenario, the optimal policy does not aim to keep `R ` at or below 1 to prevent an increase in the infected population. Instead, it focuses on using the available vaccine supply to achieve the lowest possible total number of cases.