SFT calculations and asymptotics

This document explores and implements the exact integration of the Fourier transform on the ARM sphere of a simple but suitable model of a molecular orbital which works well for simple di- and tri-atomic molecules. The form factor $R(\boldsymbol{p})$ of the ARM formalism can be expressed in terms of such a transform, which can be written as

$$\mathrm{SFT}\,(\boldsymbol{q}) = \int \frac{d\Omega}{2\pi}\, e^{-i\,\boldsymbol{q}\cdot\hat{\boldsymbol{r}}}\, F(\hat{\boldsymbol{r}}),$$

where $F(\hat{\boldsymbol{r}})$ is the angular part of the Dyson orbital, i.e. $\langle \boldsymbol{r} \mid n_D\rangle = C_\kappa\, \kappa^{3/2}\, e^{-\kappa r} (\kappa r)^{\frac{Q}{\kappa}-1}\, F(\hat{\boldsymbol{r}})$. This angular dependence can be modelled well, by all accounts, as

$$F(\hat{\boldsymbol{r}}) = \cosh(b\cos\theta)\left(1 + c\cos^2\theta\right)\cos^{N_z}\theta\,\sin^{N_x+N_y}\theta\,\cos^{N_x}\phi\,\sin^{N_y}\phi.$$

This model is taken from a paper by Ryan, Michael, Serguei and Misha (PRL **106** 173001), and they in turn cite A.A. Radzig and B. M. Smirnov, *Reference data on atoms, molecules and ions* (Springer-Verlag, Berlin, 1985), as a source for that form. I have not been able to get that reference, though I suspect there's a copy in MBI or nearby.

Anyway, as it turns out, this integral is exactly doable if you keep your analytical continuation wits about you, turn the cosh into exponentials, and you use a certain key formula (specifically, Gegenbauer's finite integral, in Watson's Bessel functions book, p. 50, or formulas 7.333.1 and .2 in Gradshteyn and Ryzhik). It comes out to the very nice form

$$\mathrm{SFT}(\boldsymbol{q}) = (-i)^{N_{\mathrm{xyz}}} \sum_\pm \frac{q_x^{N_x}\, q_y^{N_y} (q_z \pm i\,b)^{N_z}}{\left(q_x^2+q_y^2+(q_z\pm i\,b)^2\right)^{N_{\mathrm{xyz}}/2}} \left[\left(1 + c\,\frac{N_z+1/2}{N_{\mathrm{xyz}}+3/2}\right) j_{N_{\mathrm{xyz}}}\!\left(\sqrt{q_x^2+q_y^2+(q_z\pm i\,b)^2}\right) - c\left(\frac{(q_z\pm i\,b)^2}{q_x^2+q_y^2+(q_z\pm i\,b)^2} - \frac{N_z+1/2}{N_{\mathrm{xyz}}+3/2}\right) j_{N_{\mathrm{xyz}}+2}\!\left(\sqrt{q_x^2+q_y^2+(q_z\pm i\,b)^2}\right)\right],$$

where the js are spherical Bessel functions and $N_{\mathrm{xyz}} = N_x + N_y + N_z$.

One curious feature is that $\mathrm{SFT}(\boldsymbol{q})$ is only a function of the transverse momentum. This is because, by definition, $\boldsymbol{q} = a(\boldsymbol{p} + \boldsymbol{A}(t_s))$, and t_s itself is a function of the momentum, through the saddle-point equation $\frac{1}{2}(\boldsymbol{p} + \boldsymbol{A}(t_s)) + I_p = 0$, which can also be written as $p_\parallel + A(t_s) = -i\sqrt{\kappa^2 + p_\perp^2}$, and therefore $\boldsymbol{q} = a\left(\boldsymbol{p}_\perp - i\sqrt{\kappa^2 + p_\perp^2}\;\hat{\boldsymbol{n}}\right)$.

This document also contains some derived functions which help make sense of this solution.

- An approximation to SFT in the case of large a and small tunnelling angles. This can be further refined if necessary, but for now I left it in the form

$$\mathrm{SFT}(\boldsymbol{q}) \simeq (-i)^{N_{\mathrm{xyz}}} \frac{e^{\kappa a}}{\kappa a} \left(\frac{v_x}{\kappa}\right)^{\mathrm{nx}} \left(\frac{v_y}{\kappa}\right)^{\mathrm{ny}} e^{\frac{1}{2}\left(1-\frac{\kappa^2+p_\perp^2}{\kappa^2}\cos^2\theta\right)\frac{b^2}{\kappa a}} \frac{1}{2}$$
$$\sum_{\pm}\Bigg[\left(\frac{v_z}{\kappa} \pm i \frac{b}{\kappa a}\right)^{\mathrm{nz}} e^{\mp i b\left(-\frac{p_o}{\kappa}\sin\theta - i\frac{\sqrt{\kappa^2+p_\perp^2}}{\kappa}\cos\theta\right)} \Bigg(1 - \frac{N_{\mathrm{xyz}}+1}{2}\left(1-(N_{\mathrm{xyz}}+3)\frac{\kappa^2+p_\perp^2}{\kappa^2}\cos^2\theta\right)\frac{b^2}{\kappa^2 a^2} \pm$$
$$i\,(N_{\mathrm{xyz}}+1)\frac{b}{\kappa a}\left(-\frac{p_o}{\kappa}\sin\theta - i\frac{\sqrt{\kappa^2+p_\perp^2}}{\kappa}\cos\theta\right)\Bigg)\Bigg(\left(1+c\,\frac{N_z+1/2}{N_{\mathrm{xyz}}+3/2}\right) - c\Bigg(\left(\frac{v_z}{\kappa}\pm i\frac{b}{\kappa a}\right)^2$$
$$\left(1-\left(1-4\,\frac{\kappa^2+p_\perp^2}{\kappa^2}\cos^2\theta\right)\frac{b^2}{\kappa^2 a^2} \pm i\,2\,\frac{b}{\kappa a}\left(-\frac{\mathrm{po}}{\kappa}\sin\theta - i\frac{\sqrt{\kappa^2+p_\perp^2}}{\kappa}\cos\theta\right)\right) + \frac{N_z+1/2}{N_{\mathrm{xyz}}+3/2}\Bigg)\Bigg)\Bigg].$$

This is important because it displays the correct asymptotics with respect to a, as $\frac{e^{\kappa a}}{\kappa a}$, which is needed to counteract the factor of $\kappa\, a\, e^{-\kappa a}$ that comes from the spatial part of the orbital, as the complete form factor needs to be, at least approximately, independent of a. The exponential factor comes from the fact that the spherical Bessel functions are being taken at an argument which is close to $\sqrt{q^2} \simeq \sqrt{(-i\,\kappa\, a\,\hat{\boldsymbol{n}})^2} \simeq \sqrt{-\kappa^2 a^2}$. This is imaginary, and therefore the Bessel functions are in the modified Bessel function regime. The branch cut on that square root does not matter, as it gets exactly canceled out by a corresponding branch cut in the denominator $\left(q_x^2+q_y^2+(q_z \pm i\, b)^2\right)^{N_{xyz}/2}$, as long as both branch cuts are taken to be exactly the same. This is the case for the built-in functions in use in this implementation. Note here that the molecular axis is the z axis, and the laser is assumed to be in the (x, z) plane; p_o is the transverse momentum in that plane.

• Restricted versions of SFT for the case of on-axis ionization, or slight variations. Specifically, there are simpler expressions available for the case of $\mathrm{SFT}(p_\perp = 0)$, and for its derivatives with respect to p_o and $p_y$there. These are useful because they can be used to approximate it as $\mathrm{SFT}(q) = \mathrm{SFT}(p_\perp = 0) + q_o \frac{\partial}{\partial q_o}\mathrm{SFT}\,|_{p_\perp=0} + q_y \frac{\partial}{\partial q_y}\mathrm{SFT}\,|_{p_\perp=0}$, which is the corresponding version of the small-angle approximation that is given in the single-electron ARM paper (Torlina et al, PRA **86** 043409).

Definitions

Preliminaries

This assigns the value 1 to the formally undefined expression 0^0, which in this context is always a special case of r^n for real r and integer n. However, it is important to note that this is (in principle) dangerous, and it can potentially cause errors in other notebooks that are running on the same kernel.

```
Unprotect[Power];
Power[0, 0] = 1;
Power[0. + 0. i, 0] = 1;
Power[0., 0] = 1;
Protect[Power];
```

Some functions for support. The velocity vector **vel** at t_s as a function of alignment angle and transverse momentum and the corresponding $\boldsymbol{q}$ vector.

```
vel[θ_, po_, py_, κ_] :=
  (po {Cos[θ], 0, -Sin[θ]} + py {0, 1, 0} - ⅈ Sqrt[κ^2 + po^2 + py^2] {Sin[θ], 0, Cos[θ]});
qvec[a_, θ_, po_, py_, κ_] := a vel[θ, po, py, κ];
```

SFTnumeric

Uses parallelized memoization as per mm.se/q/1259

```
SFTnumeric[qx_, qy_, qz_, b_, c_, nx_, ny_, nz_] :=
  With[{result = SFTnumericParallelized[qx, qy, qz, b, c, nx, ny, nz]},
   (SFTnumeric[qx, qy, qz, b, c, nx, ny, nz] = result) /; result =!= Null];
SFTnumeric[qx_, qy_, qz_, b_, c_, nx_, ny_, nz_] :=
 SFTnumericParallelized[qx, qy, qz, b, c, nx, ny, nz] =
  SFTnumeric[qx, qy, qz, b, c, nx, ny, nz] = NIntegrate[
    1/(2 π) ⅇ^(-ⅈ (qx Sin[θ] Cos[ϕ] + qy Sin[θ] Sin[ϕ] + qz Cos[θ])) Cosh[b Cos[θ]]
     (1 + c Cos[θ]^2) Cos[θ]^nz Sin[θ]^(nx+ny+1) Cos[ϕ]^nx Sin[ϕ]^ny
    , {θ, 0, π}, {ϕ, 0, 2 π}
    , Method → "MultidimensionalRule"
   ]
SetSharedFunction[SFTnumericParallelized];
SFTnumeric[{qx_, qy_, qz_}, b_, c_, nx_, ny_, nz_] :=
 SFTnumeric[qx, qy, qz, b, c, nx, ny, nz]
```

SFTanalytic

```
SFTanalytic[qx_, qy_, qz_, b_, c_, nx_, ny_, nz_] := With[
  {ss = Function[s, Sqrt[qx^2 + qy^2 + (qz + s ⅈ b)^2]], j = SphericalBesselJ, n = nx + ny + nz},
  Sum[(-ⅈ)^(nx+ny+nz) qx^nx qy^ny (qz + s ⅈ b)^nz ((1 + c (nz + 1/2)/(n + 3/2)) j[n, ss[s]]/ss[s]^n -
      c ((qz + s ⅈ b)^2/(qx^2 + qy^2 + (qz + s ⅈ b)^2) - (nz + 1/2)/(n + 3/2)) j[n + 2, ss[s]]/ss[s]^n), {s, {1, -1}}]
 ]
SFTanalytic[{qx_, qy_, qz_}, b_, c_, nx_, ny_, nz_] :=
 SFTanalytic[qx, qy, qz, b, c, nx, ny, nz]
```

SFTasymptotic

AsymptoticBesselI

```
AsymptoticBesselI[n_, σ_, order_: 1] := Block[{n1, σ1},
  AsymptoticBesselI[n1_, σ1_, order] =
   Normal[Delete[Series[BesselI[n1, σ1], {σ1, ∞, order}], {2, 2}]];
  AsymptoticBesselI[n, σ, order]
 ]
```

This provides an appropriate asymptotic series for the spherical Bessel functions of the exact analytic SFT, which are in the modified-Bessel-function regime of the form $j_n(i\,\sigma)$.

It is important to note that the precise phrasing of this code is very important and it is overall very finick-y.This is because the naive command for the asymptotic series of the Bessel function gets the polynomial part correctly, but it returns subexponential terms which are not desired and in general not particularly correct:

```
Series[BesselI[n, σ], {σ, ∞, 1}]
```

$$e^{-\sigma}\left(e^{2\sigma}\left(\frac{\sqrt{\frac{1}{\sigma}}}{\sqrt{2\pi}}+O\left[\frac{1}{\sigma}\right]^{3/2}\right)+\left(\frac{i\,e^{i\,n\,\pi}\sqrt{\frac{1}{\sigma}}}{\sqrt{2\pi}}+O\left[\frac{1}{\sigma}\right]^{3/2}\right)\right)$$

Leaving aside the weird factorization, the $e^{-\sigma}e^{2\sigma}$ factor is correct but the pure exponential-decay factor in $e^{-\sigma}\times\text{poly}(\sigma)$ is not what we want for large σ. In some ways this is understandable as the half-integer modified Bessel functions come out in terms of hyperbolic sines and cosines,

```
BesselI[1 / 2, σ]
BesselI[11 / 2, σ]
```

$$\frac{\sqrt{\frac{2}{\pi}}\,\text{Sinh}[\sigma]}{\sqrt{\sigma}}$$

$$\frac{\left(2+\frac{1890}{\sigma^4}+\frac{210}{\sigma^2}\right)\text{Cosh}[\sigma]+\left(-\frac{1890}{\sigma^5}-\frac{840}{\sigma^3}-\frac{30}{\sigma}\right)\text{Sinh}[\sigma]}{\sqrt{2\pi}\,\sqrt{\sigma}}$$

but in the asymptotic regime these go away, and they are explicitly ignored in e.g. DLMF 10.40.1. Moreover, it is apparently impossible to get Mathematica to produce the asymptotic series without those terms, even by providing suitable **Assumptions**. To deal with this the code uses a **Delete** statement on the offending terms, but that relies on the to-be-deleted terms being in part ⟦2,2⟧ of the outputof **Series**, which is liable to break if the output is reordered through whatever reason.

So: the above code as stated works, just be very careful with these things when modifying it.

SFTasymptotic

```
ClearAll[SFTasymptotic];
SFTasymptotic[poo_, pyy_, θθ_, bb_, cc_, κκ_, aa_, nx_, ny_, nz_, order_] :=
 Block[{po, py, θ, b, c, κ, a},
  SFTasymptotic[po_, py_, θ_, b_, c_, κ_, a_, nx, ny, nz, order] = Block[
    {n = nx + ny + nz, qx, qy, qz, σ, s, κa},
    σ = κa Sqrt[(1 + b^2/κa^2 - 2 s b/κa Sqrt[1 + po^2/κ^2 + py^2/κ^2] Cos[θ] + 2 s ⅈ b/κa po/κ Sin[θ])];
    {qx, qy, qz} = {κa (po Cos[θ] - ⅈ Sqrt[po^2 + py^2 + κ^2] Sin[θ])/κ,
      κa py/κ, κa (-ⅈ Sqrt[po^2 + py^2 + κ^2] Cos[θ] - po Sin[θ])/κ};
    ⅇ^(κ a) ExpToTrig[Sum[
      Normal[Series[
         (-ⅈ)^n ⅇ^(-κa) qx^nx qy^ny (qz + s ⅈ b)^nz Sqrt[π/2] ((1 + c (nz + 1/2)/(n + 3/2)) 1/σ^(n+1/2)
            AsymptoticBesselI[n + 1/2, σ, order + 1] - c ((qz + s ⅈ b)^2 + (nz + 1/2)/(n + 3/2) σ^2)
            1/σ^(n+5/2) AsymptoticBesselI[n + 5/2, σ, order + 1])
         , {κa, ∞, order + 1}]] /. {κa → κ a}
      , {s, {1, -1}}]]
   ];
  SFTasymptotic[poo, pyy, θθ, bb, cc, κκ, aa, nx, ny, nz, order]
 ]
```

SFTrestricted

Benchmarking

SFTnumeric vs SFTanalytic

Pre-computation of SFTnumeric values

```
LaunchKernels[16];
```

```
Column@AbsoluteTiming@Block[{b = 2.5, c = 1, nx = 0, ny = 0, nz = 0, part = Re[e^i #] &},
    ParallelTable[
     part[SFTnumeric[qx, 0, qz, b, c, nx, ny, nz]]
     , {qx, -15, 15, 1}, {qz, -15, 15, 1}
    ]
   ];
```

NIntegrate::slwcon :
Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

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Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

About 300s=5min on Ramanujan.

```
AbsoluteTiming[Block[{b = 2.5, c = 1, nx = 0, ny = 0, nz = 0},
   ParallelTable[
    SFTnumeric[qx, 0, qz, b, c, nx, ny, nz]
    , {qx, -15, 15, 1}, {qz, -15, 15, 1}
   ]
  ][[1 ;; 5, 1 ;; 5]]]
```

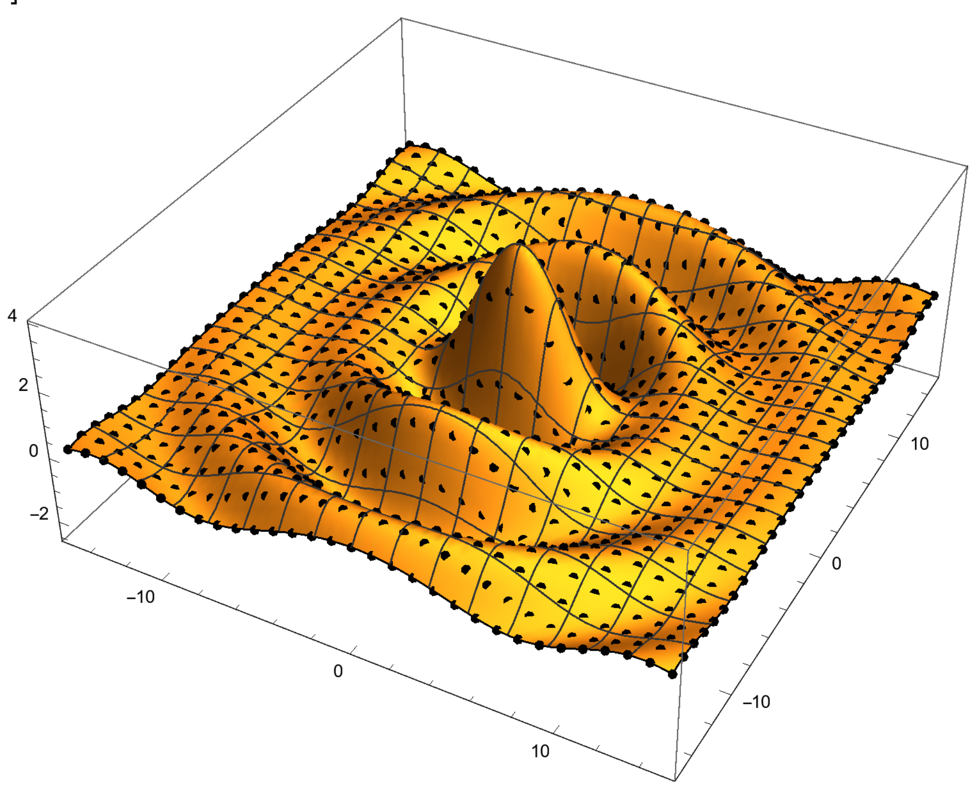
```
Save["~/Work/Scratch/data 17.03 numerical SFT integration.txt", SFTnumeric]
```

To recover

```
<< "~/Work/Scratch/data 17.03 numerical SFT integration.txt"
```

SFTnumeric vs SFTanalytic

```
Block[{b = 2.5, c = 1, nx = 0, ny = 0, nz = 0, part = Re[e^i #] &},
 Show[{
   Plot3D[
    part[SFTanalytic[qx, 0, qz, b, c, nx, ny, nz]]
    , {qx, -15, 15}, {qz, -15, 15}
    , PlotRange → Full
    , PlotPoints → 50
    , ImageSize → 500
   ],
   DiscretePlot3D[
    part[SFTnumeric[qx, 0, qz, b, c, nx, ny, nz]]
    , {qx, -15, 15, 1}, {qz, -15, 15, 1}
    , Filling → None
    , PlotStyle → Black
   ]
  }]
]
```



SFTanalytic vs SFTasymptotic

Precomputation ofasymptotics

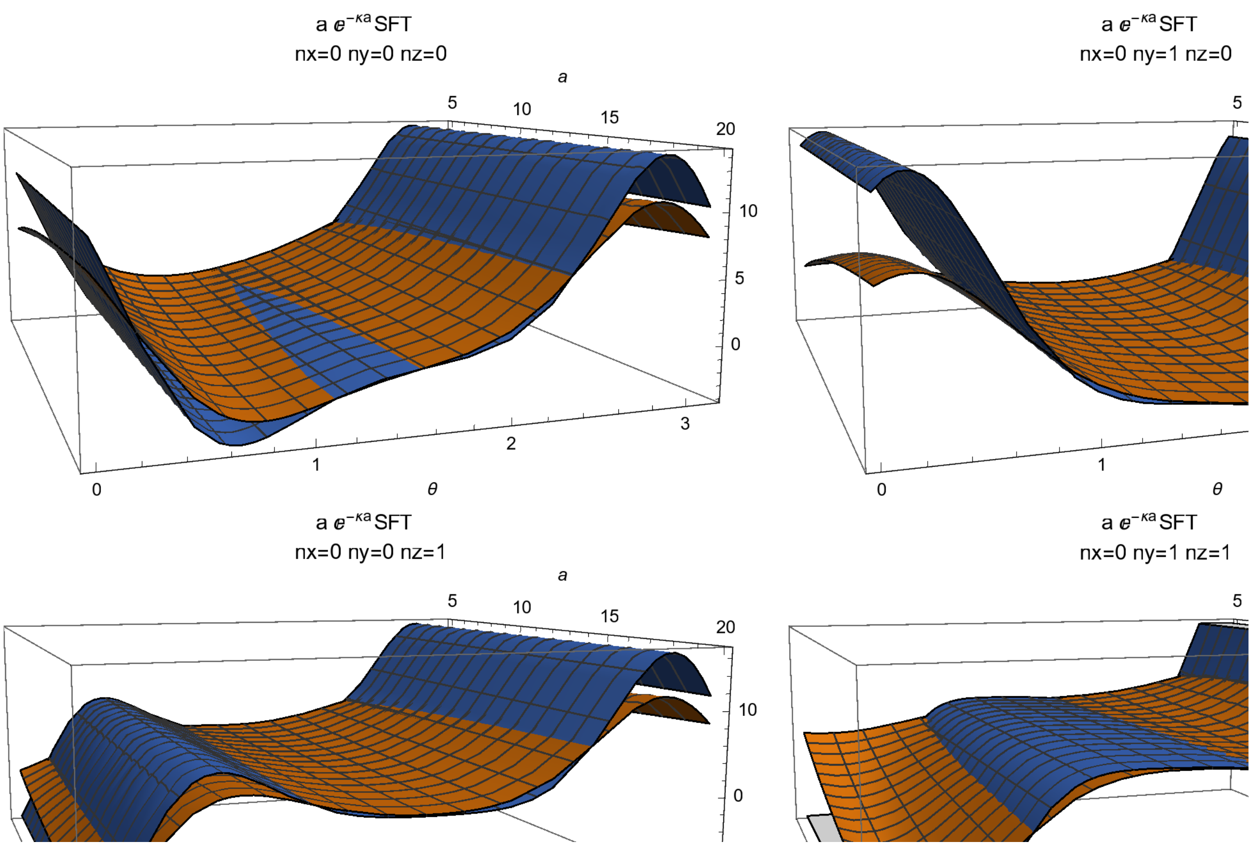
```
Table[ AbsoluteTiming[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order];
  {nx, ny, nz, order}] , {nx, 0, 1}, {ny, 0, 1}, {nz, 0, 1}, {order, 0, 2}]
(*re-run to check memoization*)
Table[ AbsoluteTiming[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order];
  {nx, ny, nz, order}] , {nx, 0, 1}, {ny, 0, 1}, {nz, 0, 1}, {order, 0, 2}]
```

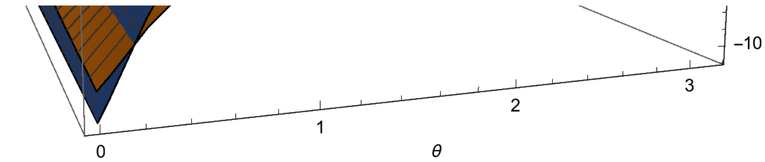
```
{{{{{2.33811, {0, 0, 0, 0}}, {3.52959, {0, 0, 0, 1}}, {5.94723, {0, 0, 0, 2}}},
   {{2.93044, {0, 0, 1, 0}}, {4.55546, {0, 0, 1, 1}}, {9.45347, {0, 0, 1, 2}}}},
  {{{2.88167, {0, 1, 0, 0}}, {4.34909, {0, 1, 0, 1}}, {8.28129, {0, 1, 0, 2}}},
   {{3.60642, {0, 1, 1, 0}}, {5.93972, {0, 1, 1, 1}}, {12.1271, {0, 1, 1, 2}}}}},
 {{{{2.94624, {1, 0, 0, 0}}, {4.48779, {1, 0, 0, 1}}, {8.93201, {1, 0, 0, 2}}},
   {{3.73377, {1, 0, 1, 0}}, {6.17458, {1, 0, 1, 1}}, {13.0961, {1, 0, 1, 2}}}},
  {{{3.66129, {1, 1, 0, 0}}, {5.85139, {1, 1, 0, 1}}, {10.5276, {1, 1, 0, 2}}},
   {{6.32087, {1, 1, 1, 0}}, {10.3875, {1, 1, 1, 1}}, {19.0367, {1, 1, 1, 2}}}}}}
```

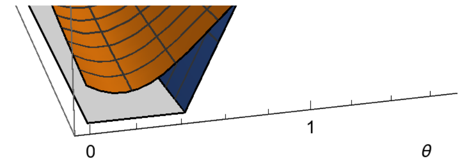
```
{{{{{0.000262, {0, 0, 0, 0}}, {0.001337, {0, 0, 0, 1}}, {0.003363, {0, 0, 0, 2}}},
   {{0.000214, {0, 0, 1, 0}}, {0.001584, {0, 0, 1, 1}}, {0.004476, {0, 0, 1, 2}}}},
  {{{0.004894, {0, 1, 0, 0}}, {0.002131, {0, 1, 0, 1}}, {0.005241, {0, 1, 0, 2}}},
   {{0.000535, {0, 1, 1, 0}}, {0.00148, {0, 1, 1, 1}}, {0.007515, {0, 1, 1, 2}}}}},
 {{{{0.000218, {1, 0, 0, 0}}, {0.002018, {1, 0, 0, 1}}, {0.005381, {1, 0, 0, 2}}},
   {{0.000277, {1, 0, 1, 0}}, {0.001667, {1, 0, 1, 1}}, {0.004562, {1, 0, 1, 2}}}},
  {{{0.000217, {1, 1, 0, 0}}, {0.001384, {1, 1, 0, 1}}, {0.004262, {1, 1, 0, 2}}},
   {{0.000279, {1, 1, 1, 0}}, {0.001877, {1, 1, 1, 1}}, {0.006671, {1, 1, 1, 2}}}}}}
```

Asymptotics with a for fixed momentum

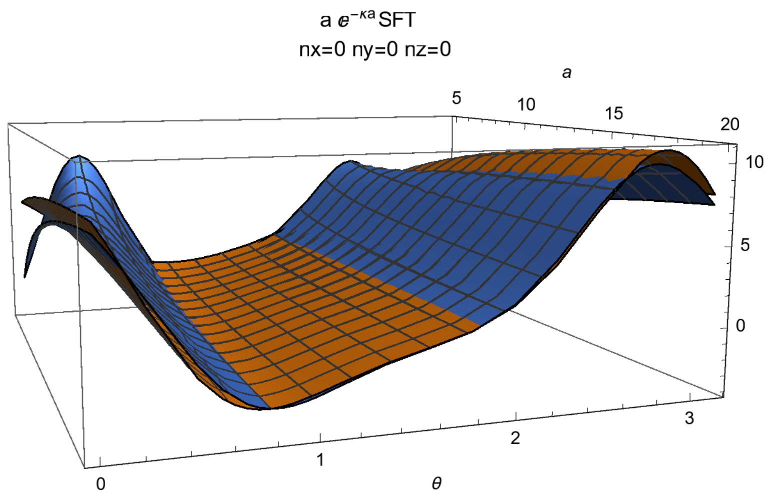
```
Column[Table[
  Column[
   AbsoluteTiming[Block[{κ = 1, c = 1, b = 2.5, po = -0.5, py = 0.15, part = Re[ⅇ^ⅈ #] &},
     Labeled[Grid[Transpose[Flatten[Table[
          Plot3D[
           {Tooltip[a ⅇ^(-κ a) part[SFTanalytic[
                qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"],
            Tooltip[a ⅇ^(-κ a) part[SFTasymptotic[po, py, θ, b, c, κ,
                a, nx, ny, nz, order]], "Asymptotic"]}
           , {a, 5, 20}, {θ, 0, π}
           , ImageSize → 400
           , AxesLabel → {"a", "θ"}
           , PlotLabel → "a ⅇ^(-κa)SFT \n nx=" <>
             ToString[nx] <> " ny=" <> ToString[ny] <> " nz=" <> ToString[nz]
           , ViewPoint → {2, -0.7, Scaled[0.0005]}
          ]
          , {nx, {0, 1}}, {ny, {0, 1}}, {nz, {0, 1}}], {1, 2}]]],
      Style["order=" <> ToString[order], 20], {Top, Left}]
    ]]]
  , {order, 0, 2}], Spacings → Scaled[0.1]]
11.9295
```

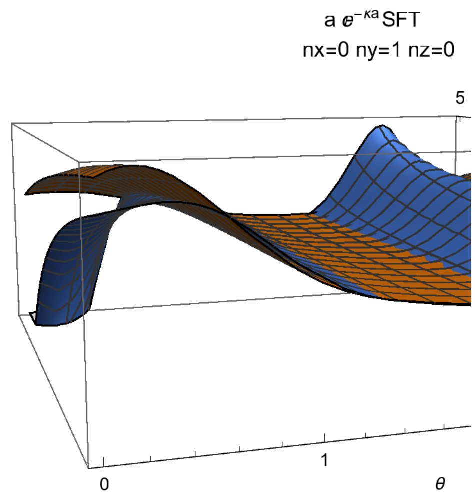


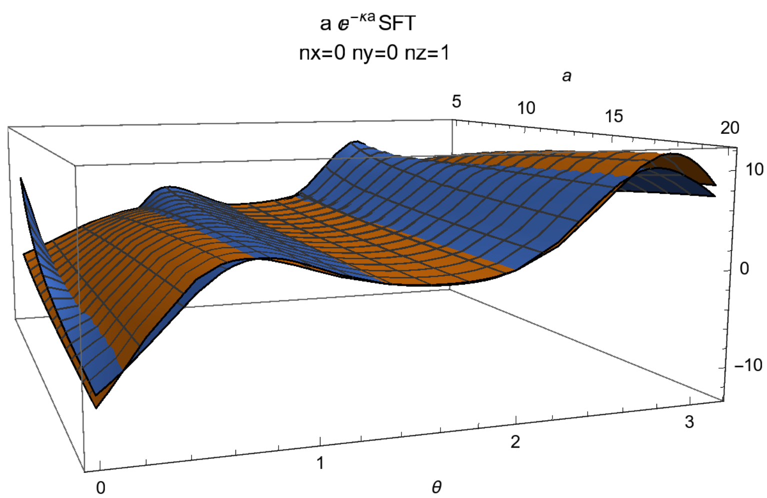


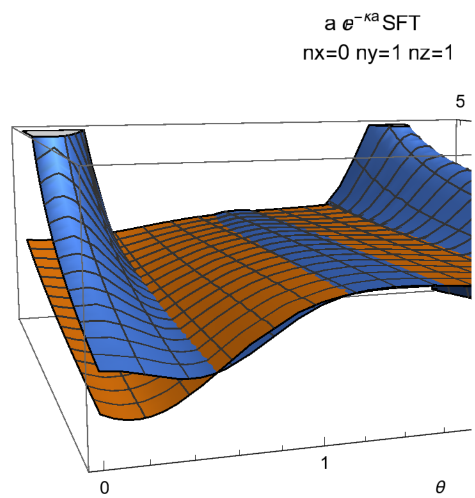


26.6124

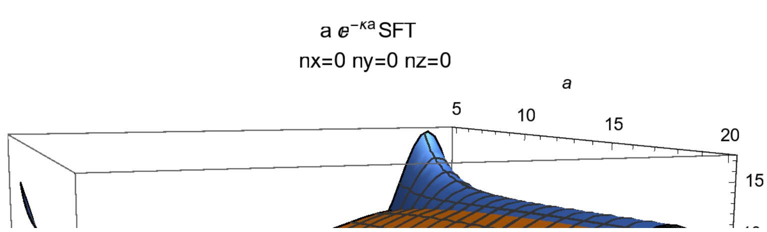


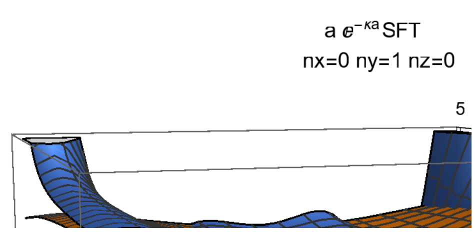


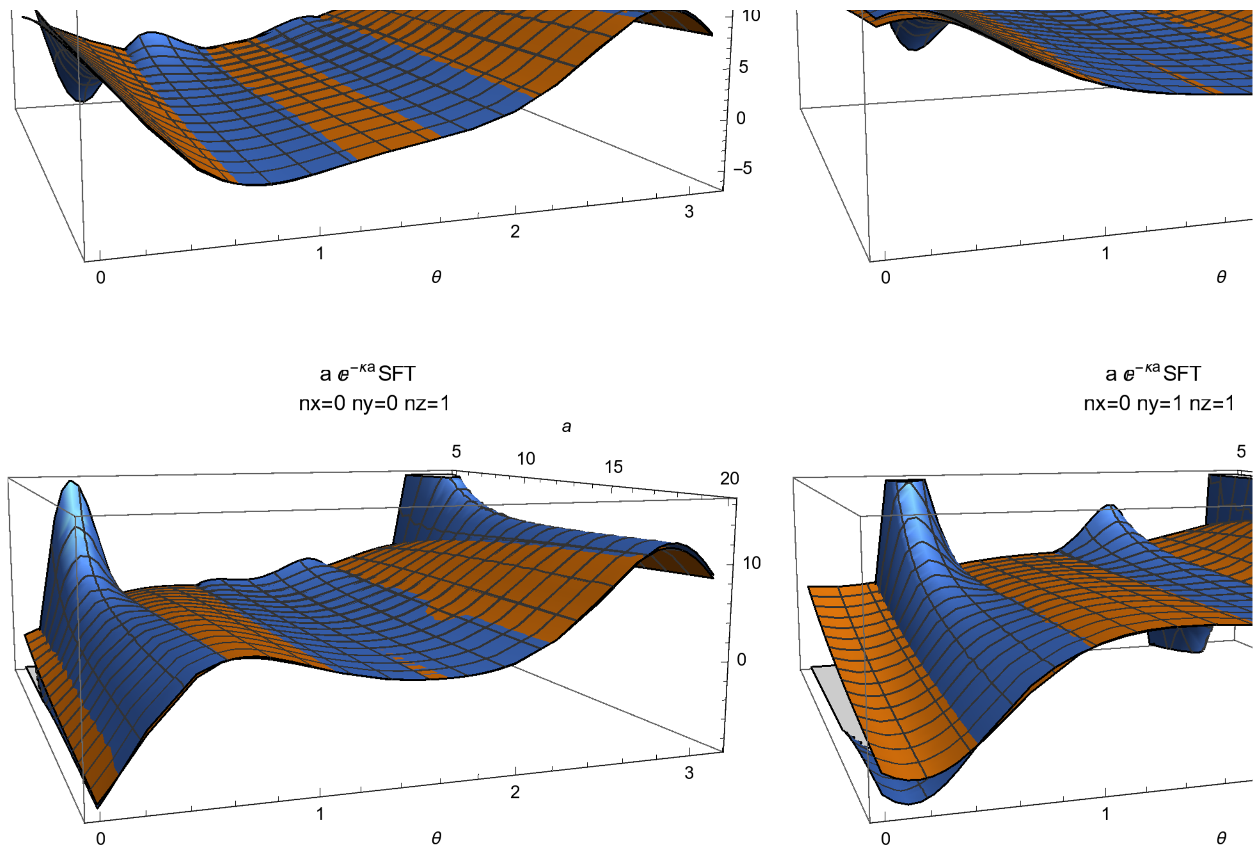




49.616

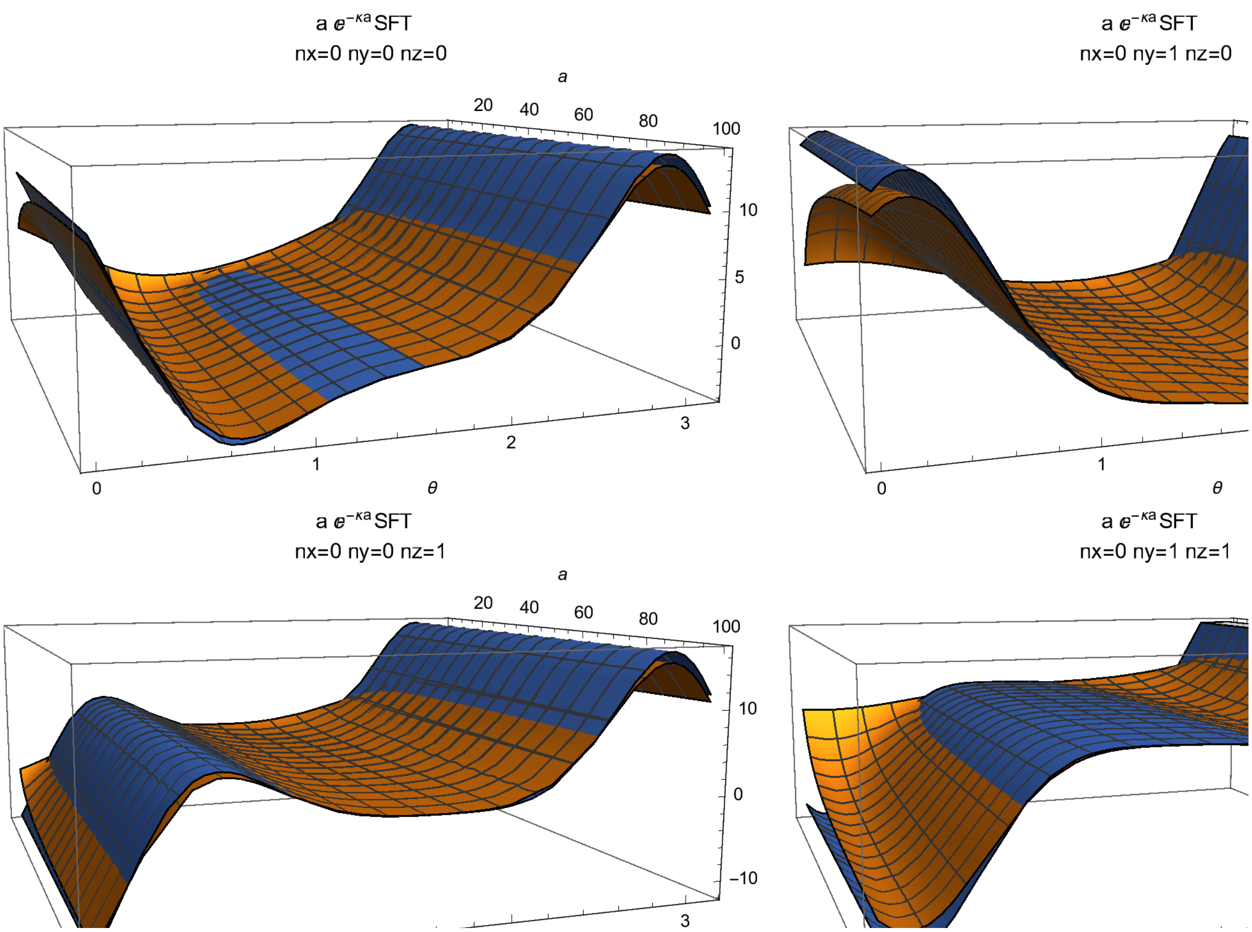


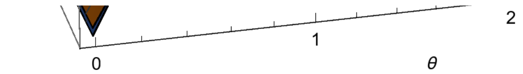


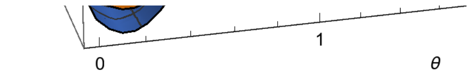


Long-range asymptotics for a fixed momentum.

```
Column[Table[
  Column[
   AbsoluteTiming[Block[{κ = 1, c = 1, b = 2.5, po = -0.5, py = 0.15, part = Re[ⅇ^ⅈ #] &},
     Labeled[Grid[Transpose[Flatten[Table[
          Plot3D[
           {Tooltip[a ⅇ^(-κ a) part[SFTanalytic[
                qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"],
            Tooltip[a ⅇ^(-κ a) part[SFTasymptotic[po, py, θ, b, c, κ,
                a, nx, ny, nz, order]], "Asymptotic"]}
           , {a, 5, 100}, {θ, 0, π}
           , ImageSize → 400
           , AxesLabel → {"a", "θ"}
           , PlotLabel → "a ⅇ^(-κa)SFT \n nx=" <>
             ToString[nx] <> " ny=" <> ToString[ny] <> " nz=" <> ToString[nz]
           , ViewPoint → {2, -0.7, Scaled[0.0005]}
          ]
          , {nx, {0, 1}}, {ny, {0, 1}}, {nz, {0, 1}}], {1, 2}]]],
      Style["order=" <> ToString[order], 20], {Top, Left}]
    ]]]
  , {order, 0, 2}], Spacings → Scaled[0.1]]
11.9678
```

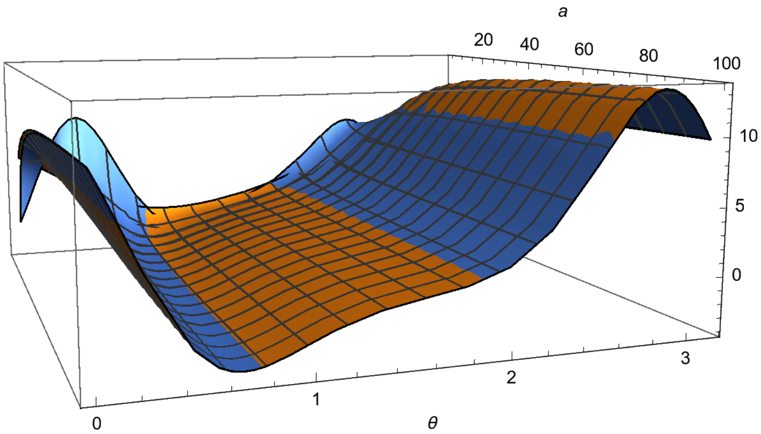




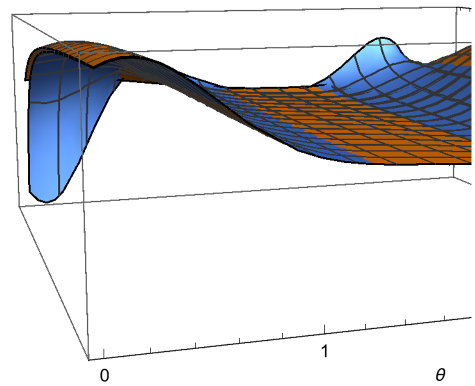


23.9256

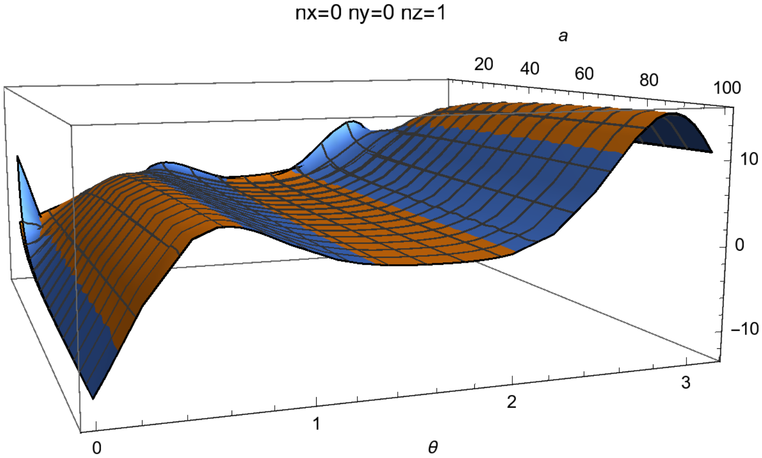
a $e^{-\kappa a}$ SFT
nx=0 ny=0 nz=0



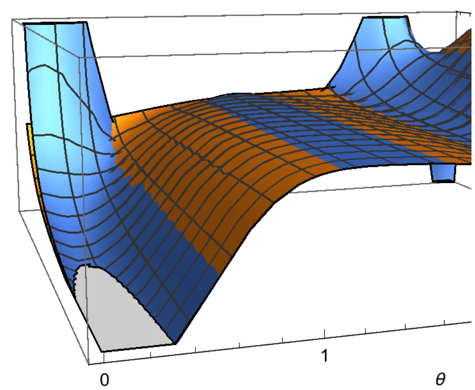
a $e^{-\kappa a}$ SFT
nx=0 ny=1 nz=0



a $e^{-\kappa a}$ SFT
nx=0 ny=0 nz=1

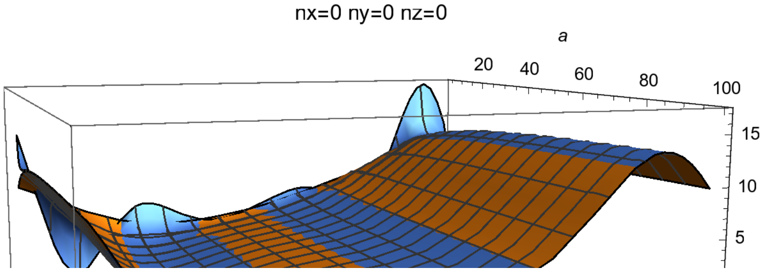


a $e^{-\kappa a}$ SFT
nx=0 ny=1 nz=1

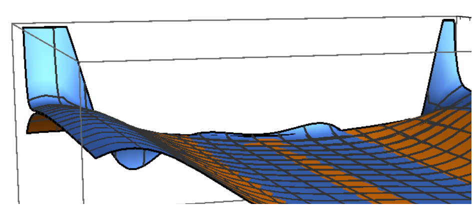


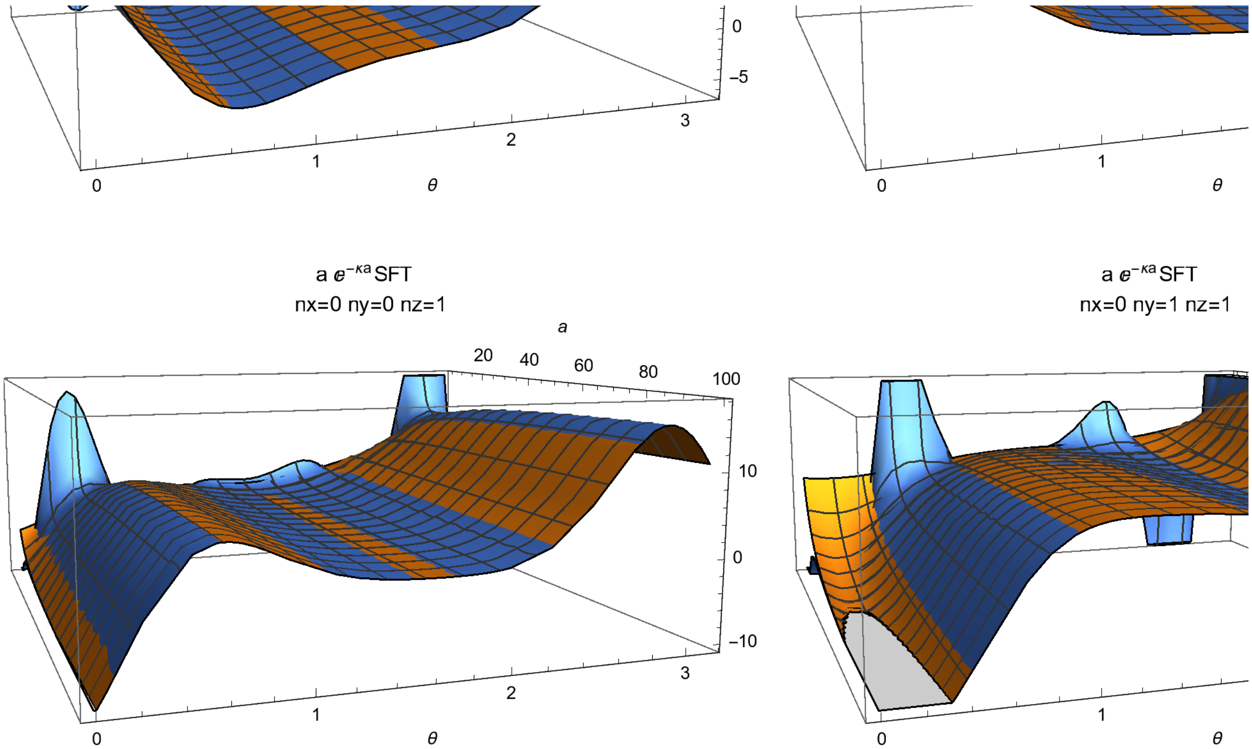
45.9839

a $e^{-\kappa a}$ SFT
nx=0 ny=0 nz=0



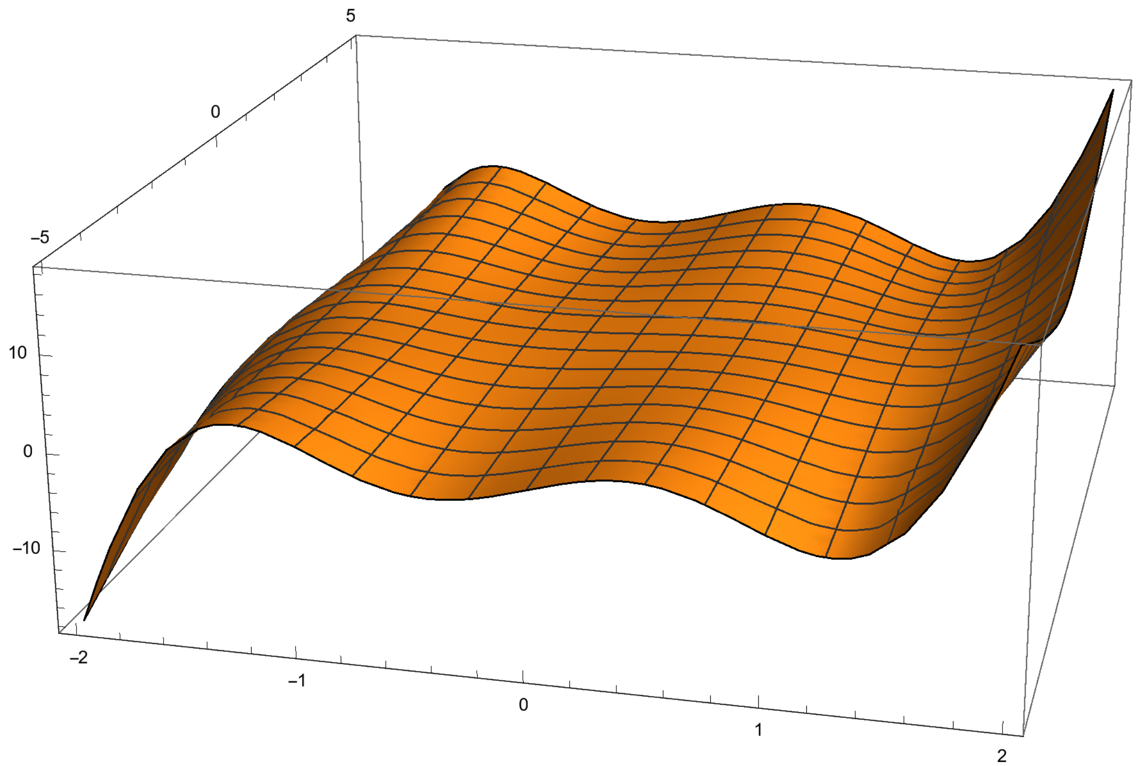
a $e^{-\kappa a}$ SFT
nx=0 ny=1 nz=0





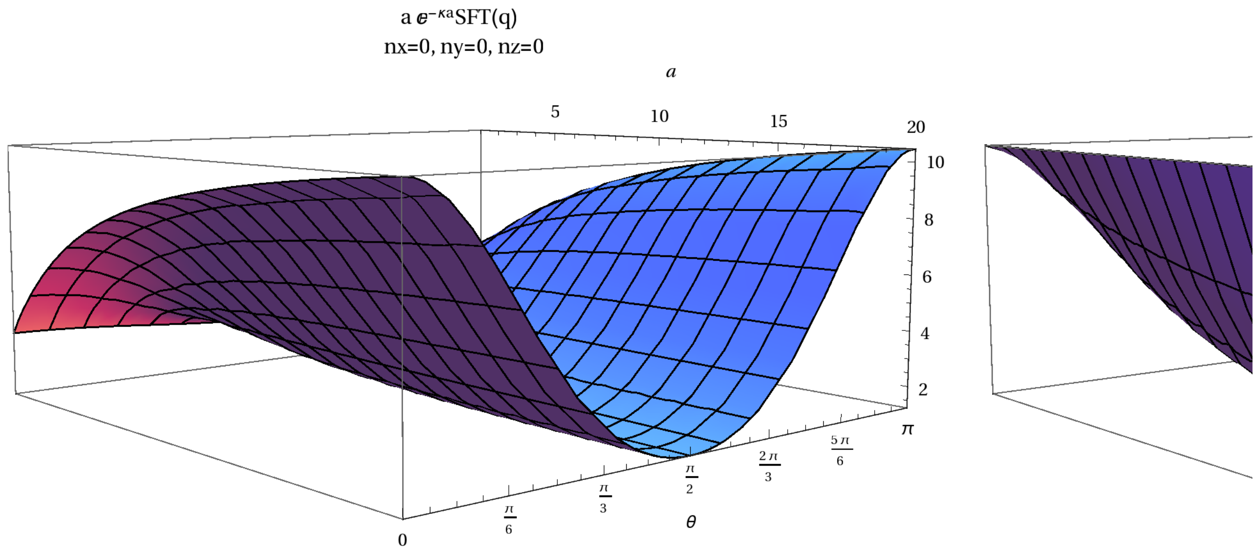
Other stuff

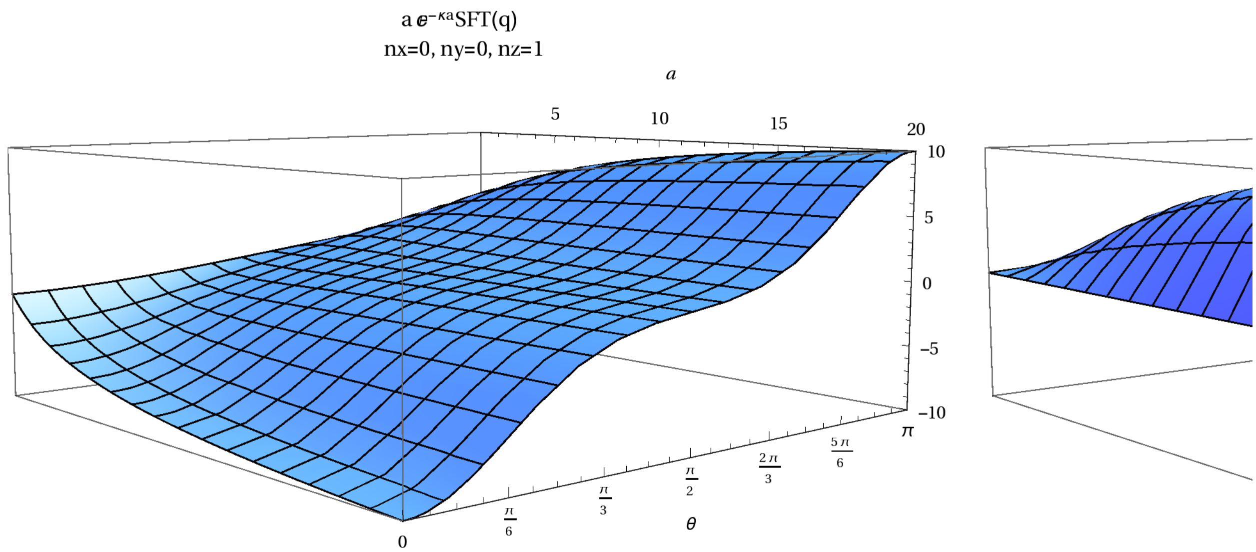
```
Block[{κ = 1, c = 1, b = 2.5, θ = 90 °, part = Re[e^ⅈ #] &, nx = 1, ny = 0, nz = 1, a = 15},
 Plot3D[
  Tooltip[a e^-κ a part[SFTanalytic[qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"]
  , {po, -2, 2}, {py, -5, 5}
  , PlotRange → Full
 ]
]
```




```
Column[AbsoluteTiming[Block[{κ = 1, c = 1, b = 2.5, po = 0, py = 0, part = Re[#] &},
   Grid[Transpose[Flatten[Table[
        Plot3D[
         Tooltip[
          a ⅇ^(-κ a) part[SFTanalytic[qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"]
         , {a, 1, 20}, {θ, 0, π}
         , ImageSize → 500
         , AxesLabel → {"a", "θ"}
         , PlotLabel → "a ⅇ^(-κa)SFT(q) \n nx=" <>
           ToString[nx] <> ", ny=" <> ToString[ny] <> ", nz=" <> ToString[nz]
         , ViewPoint → {2, -1.7, Scaled[0.0005]}
         , PlotRangePadding → None
         , PlotTheme → "Classic"
         , Ticks → {Automatic,
           Join[{#, #} & /@ Range[0, π, π / 6], {#, "", {0.01, 0}} & /@ Range[0, π, π / 24]]
           , Automatic}
        ]
        , {nx, {0, 1}}, {ny, {0}}, {nz, {0, 1}}], {1, 2}]]]
  ]]]
```

```
3.68851
```


a $e^{-\kappa a}$SFT(q)
nx=0, ny=0, nz=0
a
5
10
15
20
10
8
6
4
2
π
0
π/6
π/3
π/2
2π/3
5π/6
θ


a $e^{-\kappa a}$SFT(q)
nx=0, ny=0, nz=1
a
5
10
15
20
10
5
0
−5
−10
π
0
π/6
π/3
π/2
2π/3
5π/6
θ

Old stuff

SFTanalytic vs SFTasymptotic

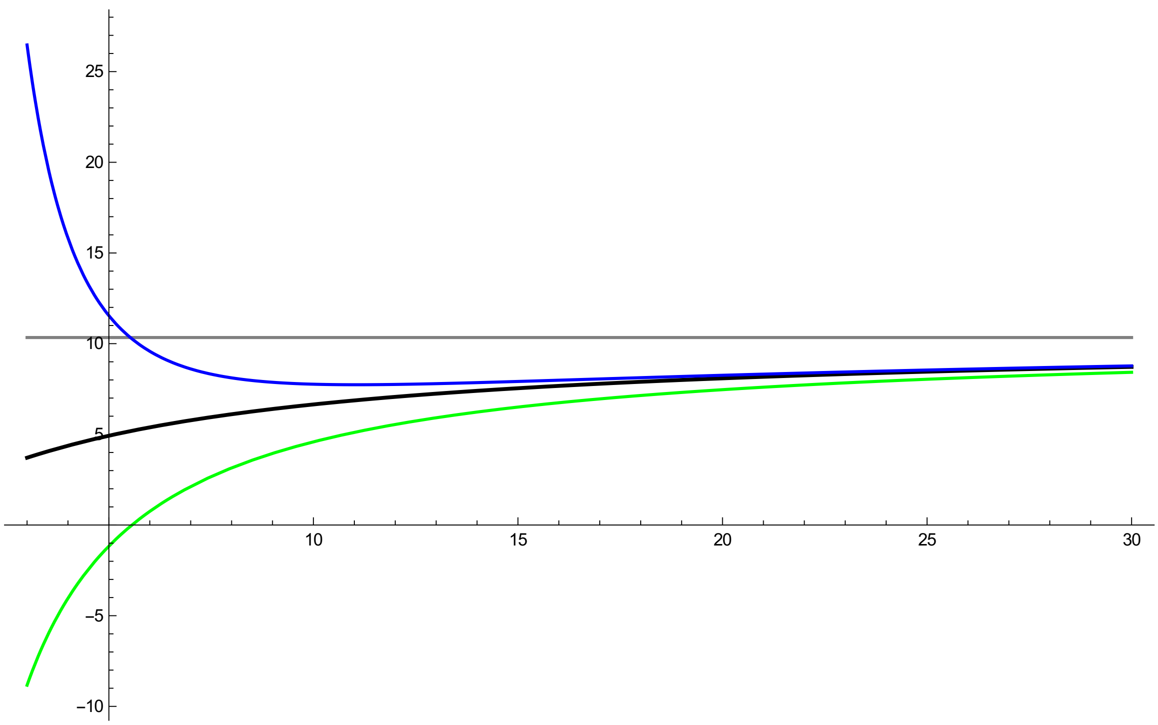
```
Quit
```

```
Table[ AbsoluteTiming[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order];
  {nx, ny, nz, order}] , {nx, 0, 1}, {ny, 0, 1}, {nz, 0, 1}, {order, 0, 2}]
(*re-run to check memoization*)
Table[ AbsoluteTiming[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order];
  {nx, ny, nz, order}] , {nx, 0, 1}, {ny, 0, 1}, {nz, 0, 1}, {order, 0, 2}]
```

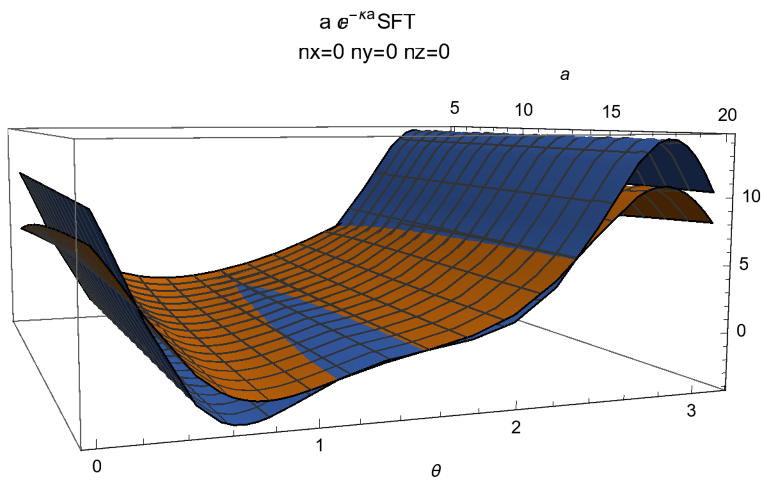
```
{{{{{2.33811, {0, 0, 0, 0}}, {3.52959, {0, 0, 0, 1}}, {5.94723, {0, 0, 0, 2}}},
   {{2.93044, {0, 0, 1, 0}}, {4.55546, {0, 0, 1, 1}}, {9.45347, {0, 0, 1, 2}}}},
  {{{2.88167, {0, 1, 0, 0}}, {4.34909, {0, 1, 0, 1}}, {8.28129, {0, 1, 0, 2}}},
   {{3.60642, {0, 1, 1, 0}}, {5.93972, {0, 1, 1, 1}}, {12.1271, {0, 1, 1, 2}}}}},
 {{{{2.94624, {1, 0, 0, 0}}, {4.48779, {1, 0, 0, 1}}, {8.93201, {1, 0, 0, 2}}},
   {{3.73377, {1, 0, 1, 0}}, {6.17458, {1, 0, 1, 1}}, {13.0961, {1, 0, 1, 2}}}},
  {{{3.66129, {1, 1, 0, 0}}, {5.85139, {1, 1, 0, 1}}, {10.5276, {1, 1, 0, 2}}},
   {{6.32087, {1, 1, 1, 0}}, {10.3875, {1, 1, 1, 1}}, {19.0367, {1, 1, 1, 2}}}}}}
```

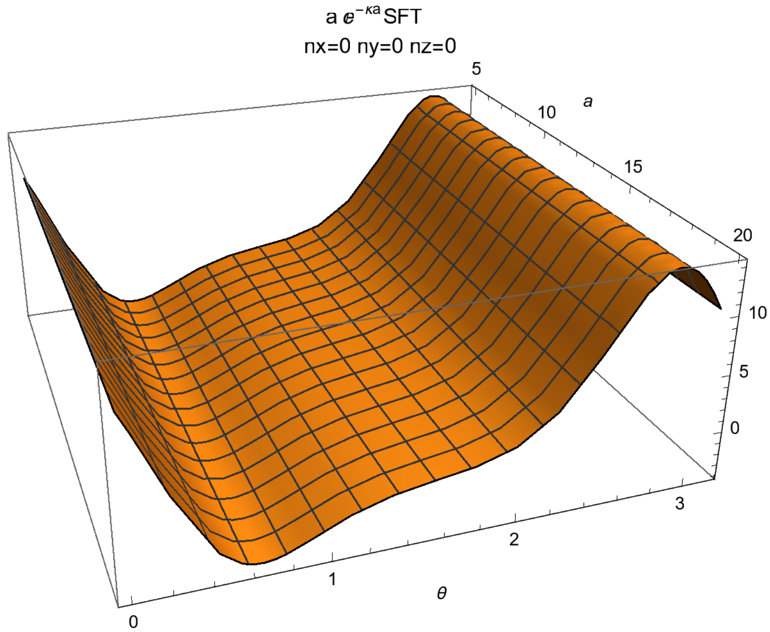
```
{{{{{0.000262, {0, 0, 0, 0}}, {0.001337, {0, 0, 0, 1}}, {0.003363, {0, 0, 0, 2}}},
   {{0.000214, {0, 0, 1, 0}}, {0.001584, {0, 0, 1, 1}}, {0.004476, {0, 0, 1, 2}}}},
  {{{0.004894, {0, 1, 0, 0}}, {0.002131, {0, 1, 0, 1}}, {0.005241, {0, 1, 0, 2}}},
   {{0.000535, {0, 1, 1, 0}}, {0.00148, {0, 1, 1, 1}}, {0.007515, {0, 1, 1, 2}}}}},
 {{{{0.000218, {1, 0, 0, 0}}, {0.002018, {1, 0, 0, 1}}, {0.005381, {1, 0, 0, 2}}},
   {{0.000277, {1, 0, 1, 0}}, {0.001667, {1, 0, 1, 1}}, {0.004562, {1, 0, 1, 2}}}},
  {{{0.000217, {1, 1, 0, 0}}, {0.001384, {1, 1, 0, 1}}, {0.004262, {1, 1, 0, 2}}},
   {{0.000279, {1, 1, 1, 0}}, {0.001877, {1, 1, 1, 1}}, {0.006671, {1, 1, 1, 2}}}}}}
```

```
Block[{nx = 0, ny = 0, nz = 0, order = 1, κ = 1,
  c = 1, b = 2.5, po = -0.5, py = 0.15, θ = 0 °, part = Re[ⅇ^ⅈ #] &},
 Plot[
  Evaluate[Join[
    {a ⅇ^(-κ a) part[SFTanalytic[qvec[a, θ, po, py, κ], b, c, nx, ny, nz]]},
    Table[
     a ⅇ^(-κ a) part[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order]], {order, 0, 2}]
   ]]
  , {a, 3, 30}
  , ImageSize → 600
  , PlotRange → Full
  , PlotStyle → {{Black, Thick}, Gray, Green, Blue}
 ]
]
```



```
Block[{nx = 0, ny = 0, nz = 0, order = 0, κ = 1,
  c = 1, b = 2.5, po = -0.5, py = 0.15, part = Re[ⅇ^ⅈ #] &},
 Plot3D[
  {Tooltip[
    a ⅇ^(-κ a) part[SFTanalytic[qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"],
   Tooltip[a ⅇ^(-κ a) part[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order]],
    "Asymptotic"]}
  , {a, 5, 20}, {θ, 0, π}
  , ImageSize → 400
  , AxesLabel → {"a", "θ"}
  , PlotLabel → "a ⅇ^(-κa)SFT \n nx=" <>
    ToString[nx] <> " ny=" <> ToString[ny] <> " nz=" <> ToString[nz]
  , ViewPoint → {2, -0.7, Scaled[0.0005]}
 ]
]
```

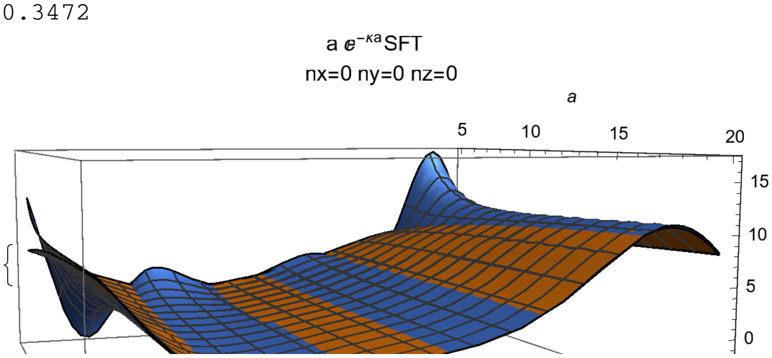




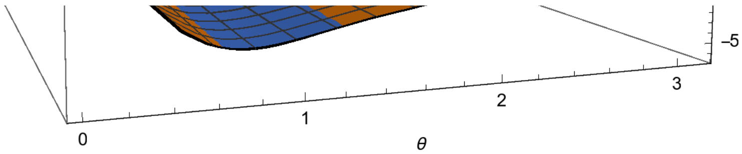

```
Column@AbsoluteTiming@
  Block[{order = 2, κ = 1, c = 1, b = 2.5, po = -0.5, py = 0.15, part = Re[e^(i #)] &},
   Transpose[Flatten[Table[
      Plot3D[
       {Tooltip[a e^(-κ a)
          part[SFTanalytic[qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"],
        Tooltip[a e^(-κ a) part[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order]],
          "Asymptotic"]}
       , {a, 5, 20}, {θ, 0, π}
       , ImageSize → 400
       , AxesLabel → {"a", "θ"}
       , PlotLabel → "a e^(-κa)SFT \n nx=" <>
         ToString[nx] <> " ny=" <> ToString[ny] <> " nz=" <> ToString[nz]
       , ViewPoint → {2, -0.7, Scaled[0.0005]}
      ]
      , {nx, {0, 1}}, {ny, {0, 1}}, {nz, {0, 1}}], {1, 2}]]
  ]
```

50.3472

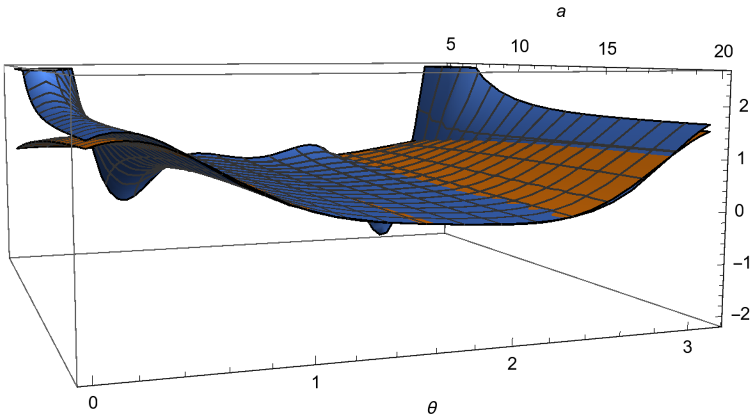
{{



,

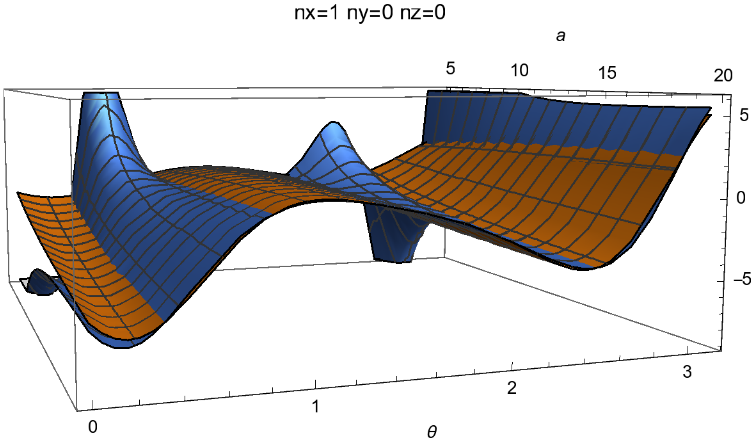


a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=0



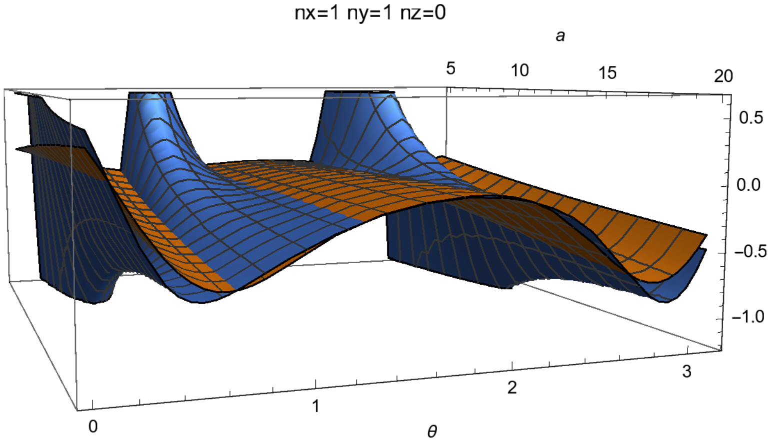
,

a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=0



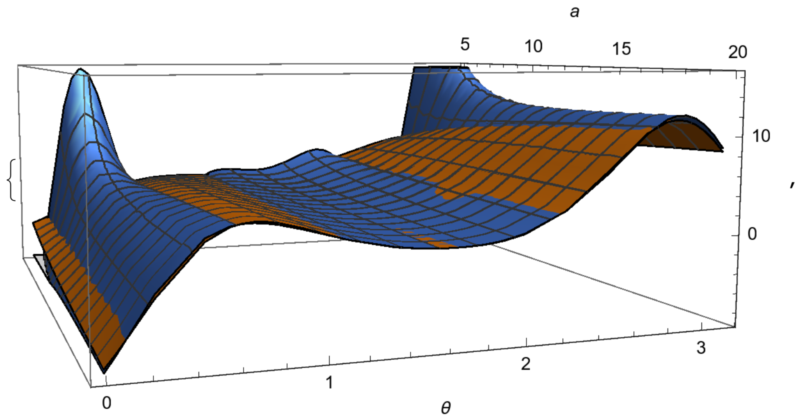
,

a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=0

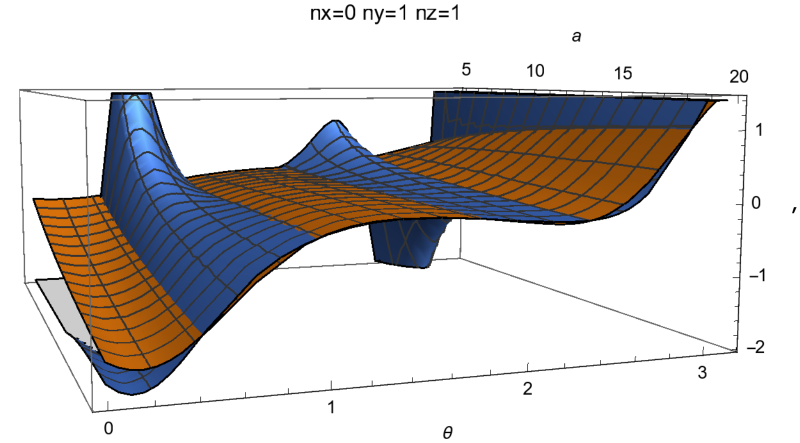


},

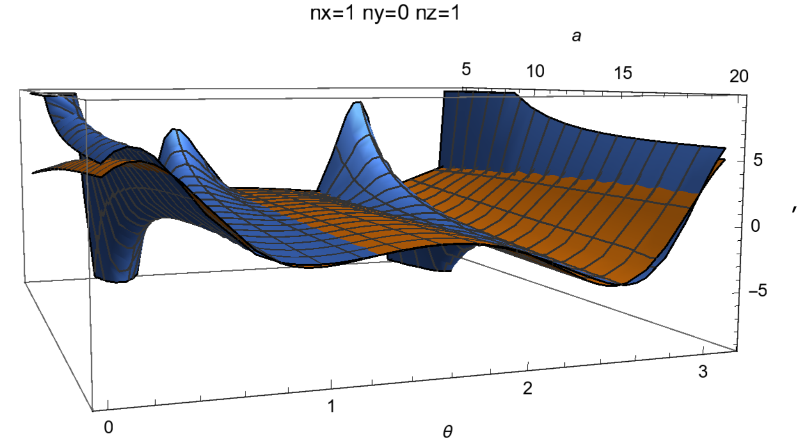
a $e^{-\kappa a}$SFT
nx=0 ny=0 nz=1



a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=1

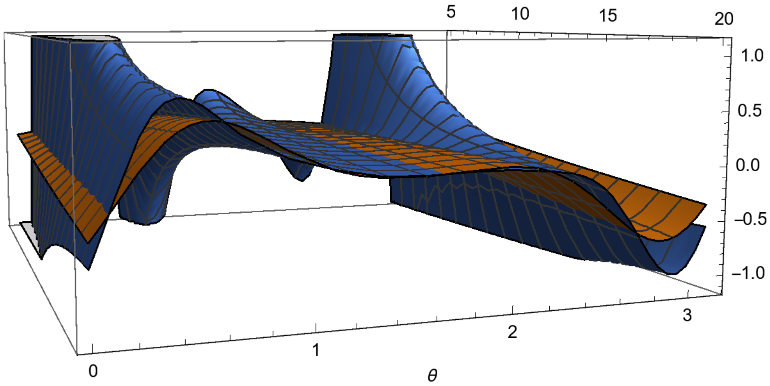


a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=1



a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=1

a



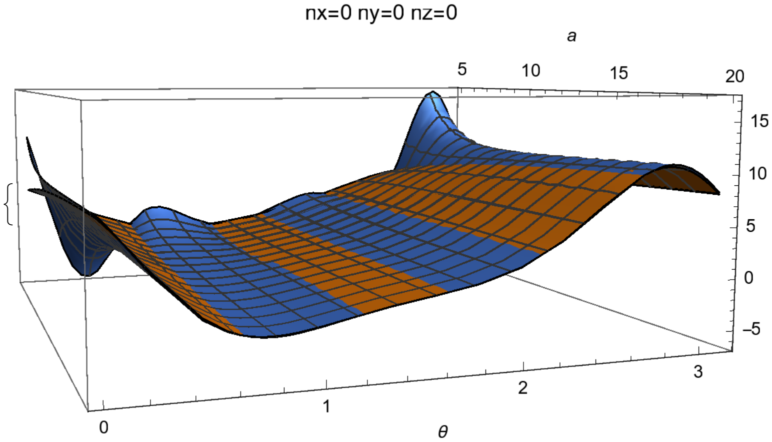
}}

264.853

{{

a $e^{-\kappa a}$SFT

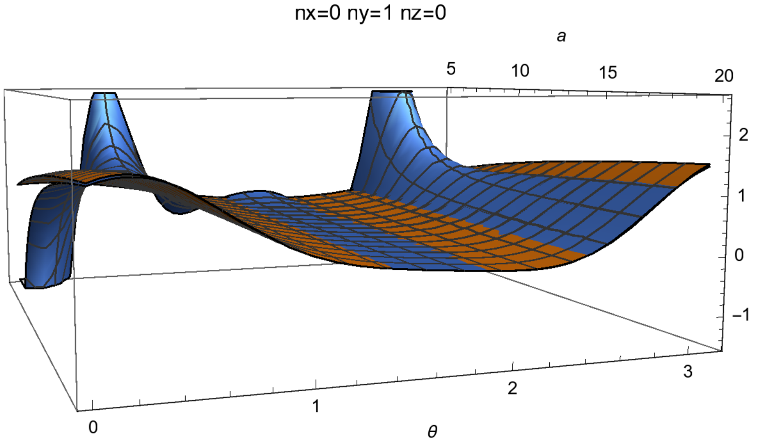
nx=0 ny=0 nz=0



,

a $e^{-\kappa a}$SFT

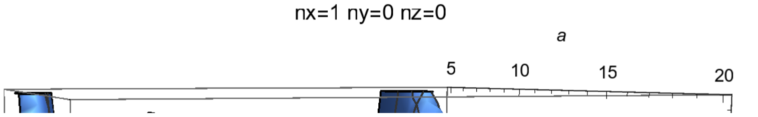
nx=0 ny=1 nz=0

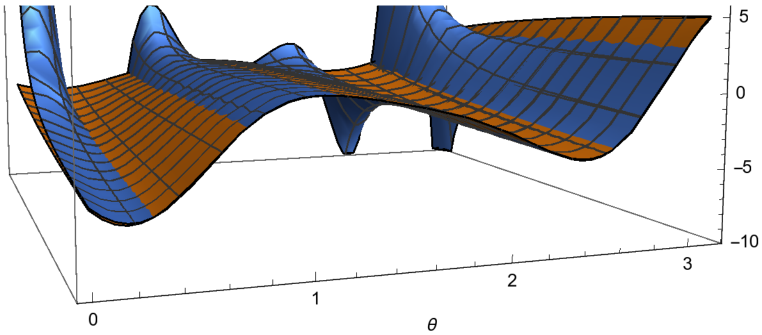


,

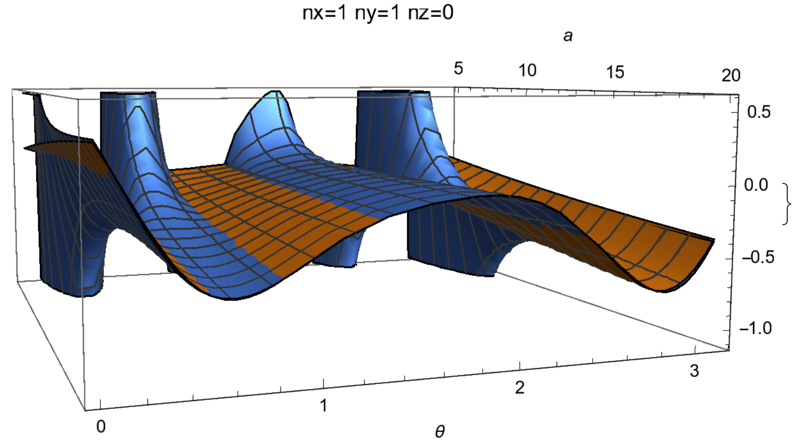
a $e^{-\kappa a}$SFT

nx=1 ny=0 nz=0

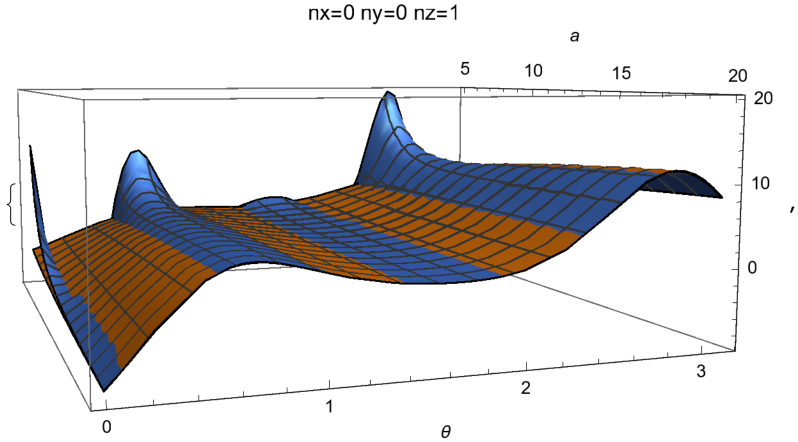




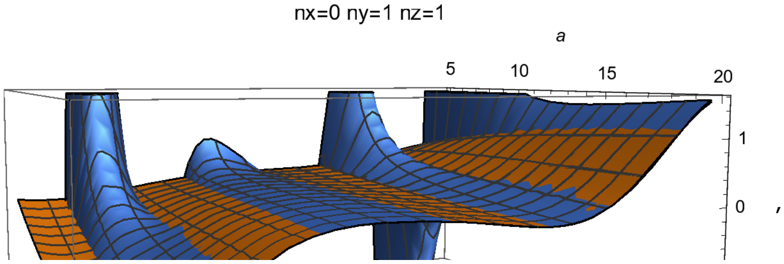
a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=0

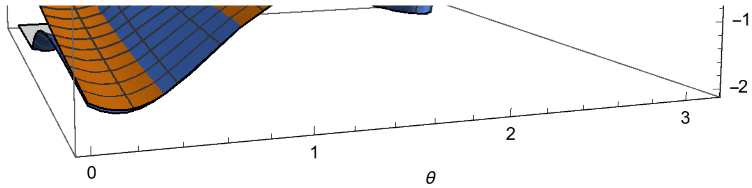


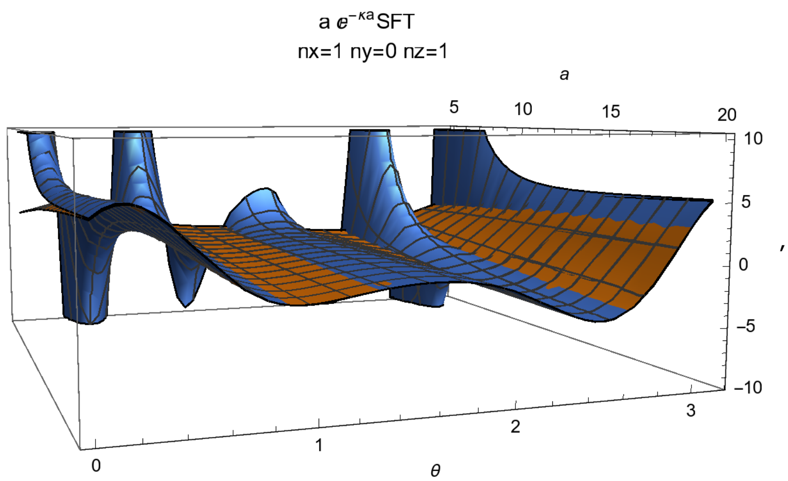
a $e^{-\kappa a}$SFT
nx=0 ny=0 nz=1

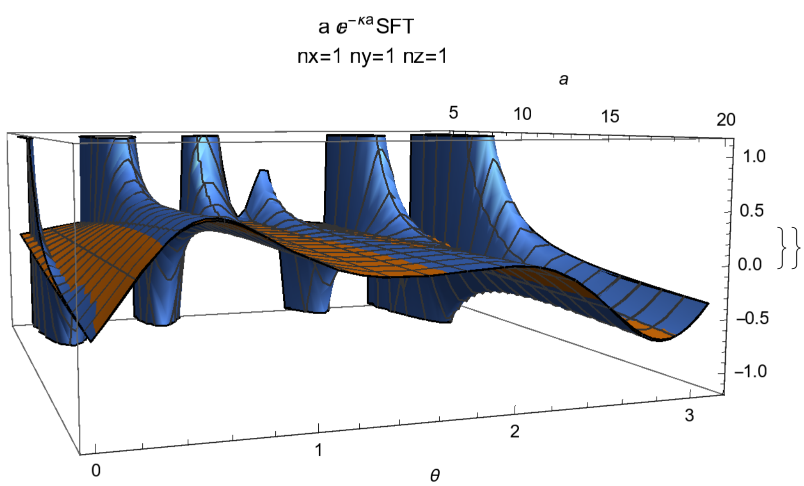


a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=1



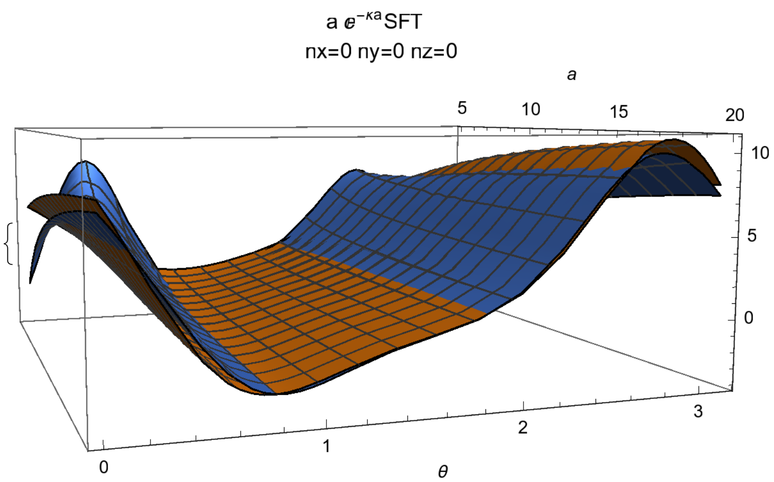

−1
−2
0
1
2
3
θ


a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=1
a
5
10
15
20
10
5
0
−5
−10
0
1
2
3
θ

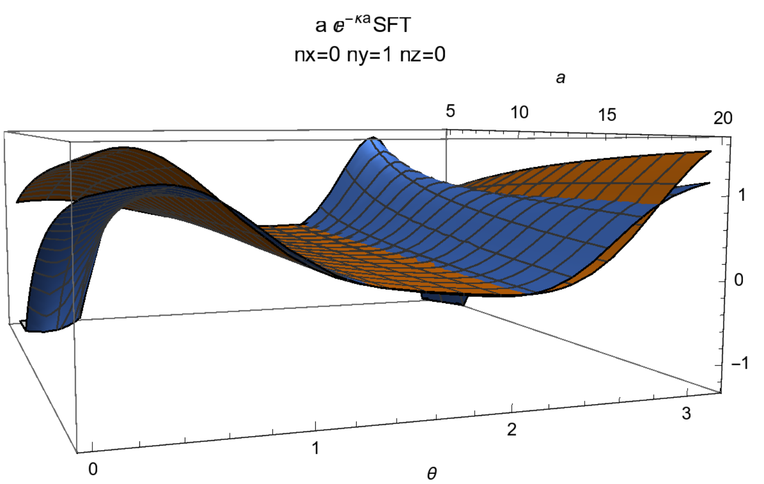

a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=1
a
5
10
15
20
1.0
0.5
0.0
−0.5
−1.0
0
1
2
3
θ

```
Column@AbsoluteTiming@
  Block[{order = 1, κ = 1, c = 1, b = 2.5, po = -0.5, py = 0.15, part = Re[ⅇ^ⅈ #] &},
   Transpose[Flatten[Table[
      Plot3D[
       {Tooltip[a ⅇ^(-κ a)
           part[SFTanalytic[qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"],
        Tooltip[a ⅇ^(-κ a) part[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order]],
         "Asymptotic"]}
       , {a, 5, 20}, {θ, 0, π}
       , ImageSize → 400
       , AxesLabel → {"a", "θ"}
       , PlotLabel → "a ⅇ^(-κa)SFT \n nx=" <>
         ToString[nx] <> " ny=" <> ToString[ny] <> " nz=" <> ToString[nz]
       , ViewPoint → {2, -0.7, Scaled[0.0005]}
      ]
      , {nx, {0, 1}}, {ny, {0, 1}}, {nz, {0, 1}}], {1, 2}]]
  ]
27.0004
```

{{

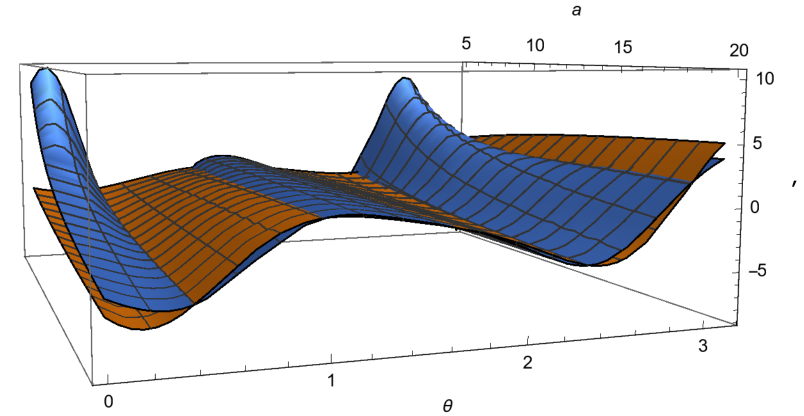


,

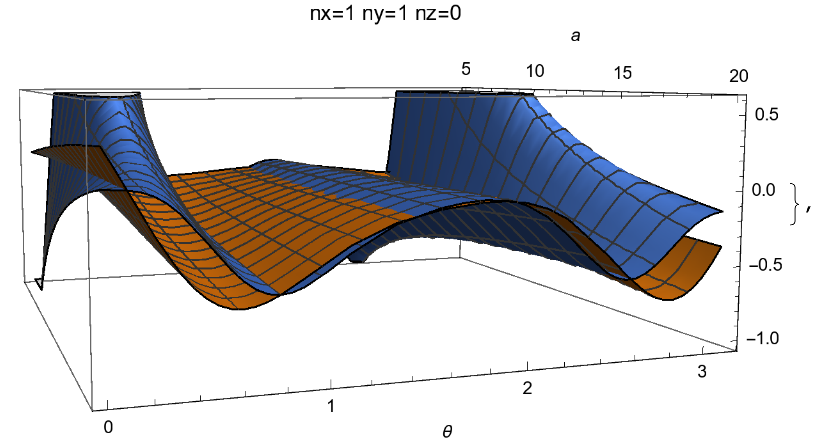


,

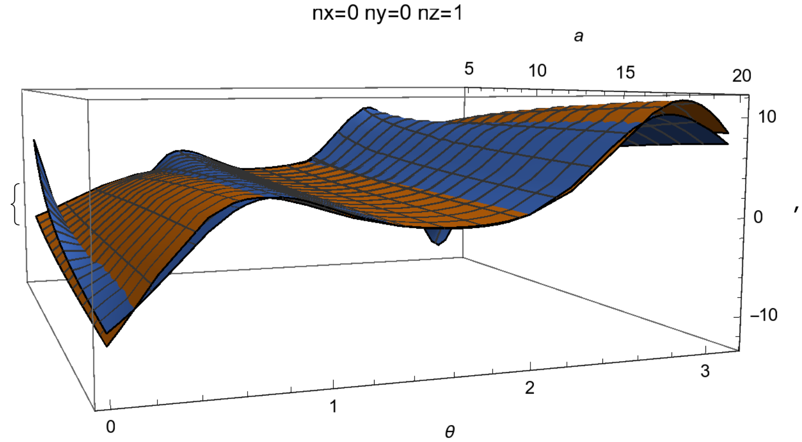
a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=0



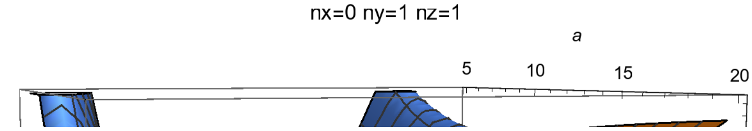
a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=0

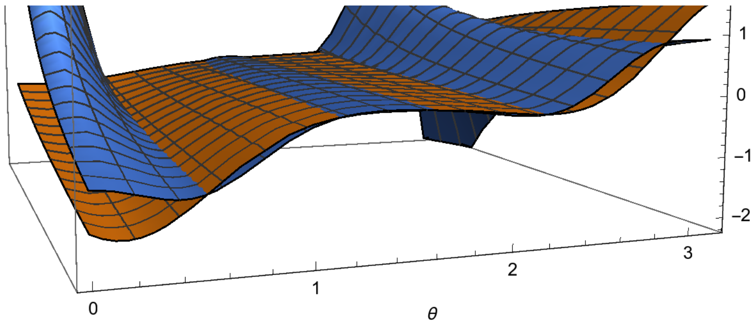


a $e^{-\kappa a}$SFT
nx=0 ny=0 nz=1



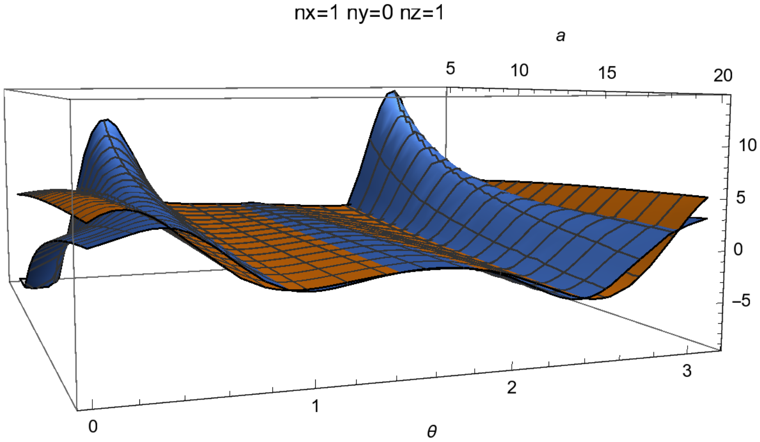
a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=1





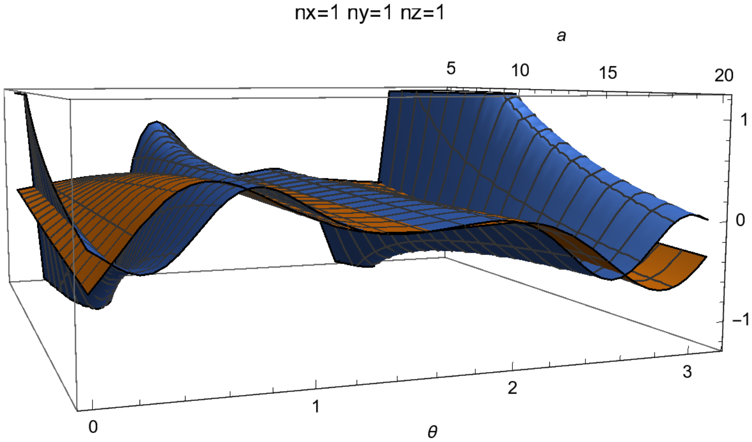
,

a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=1



,

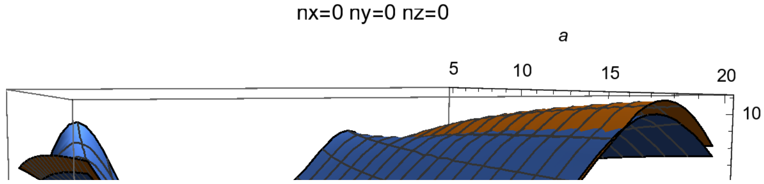
a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=1

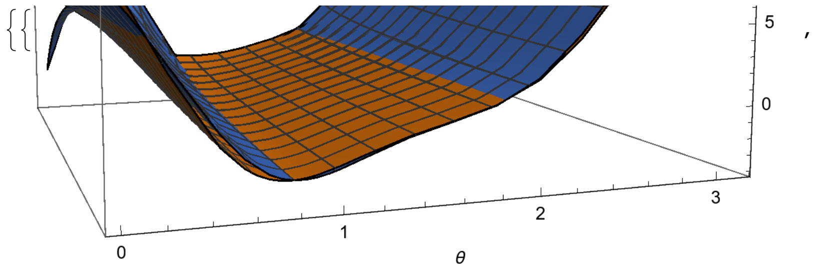


}}

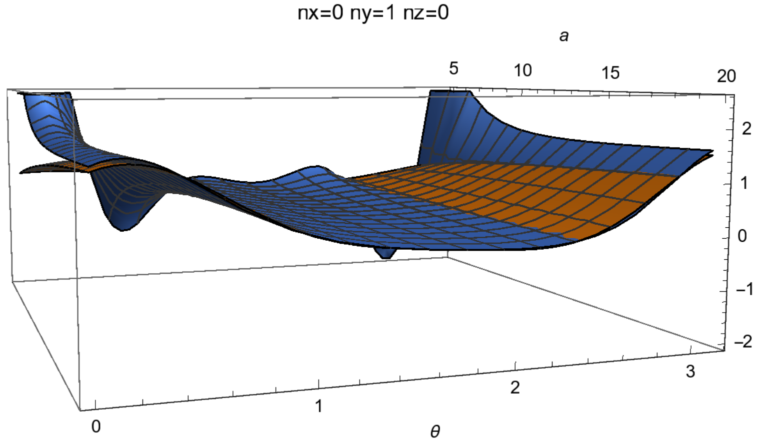
113.231

a $e^{-\kappa a}$SFT
nx=0 ny=0 nz=0

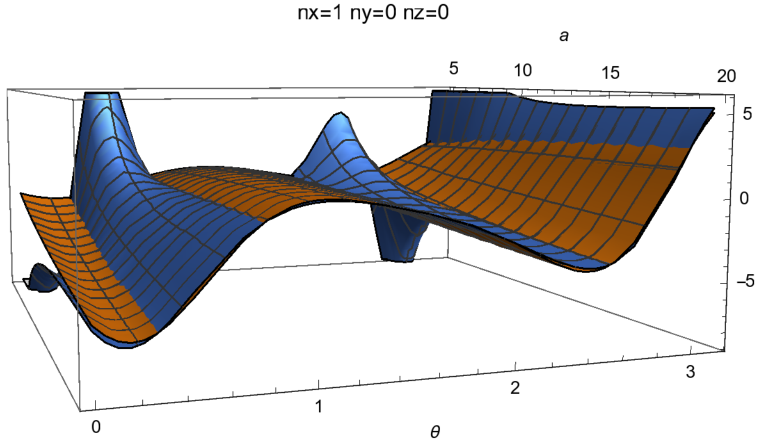




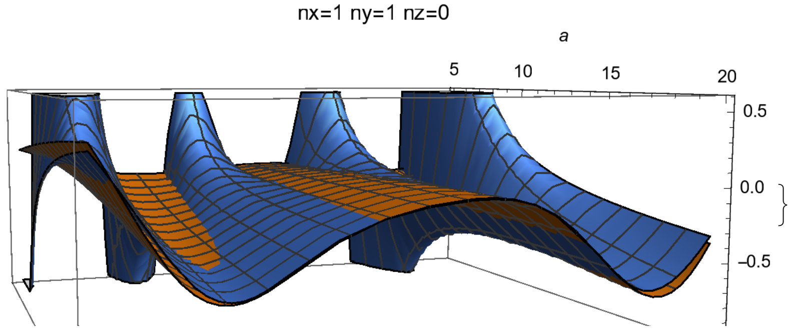
a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=0

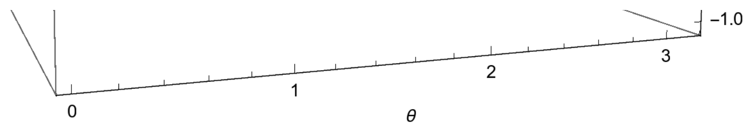


a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=0



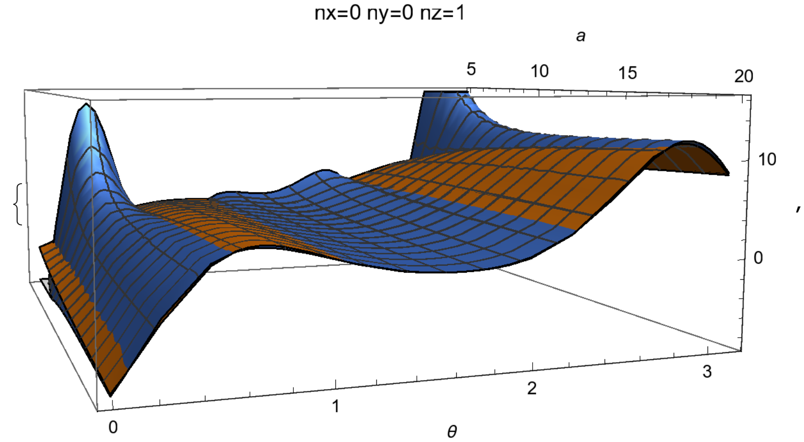
a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=0





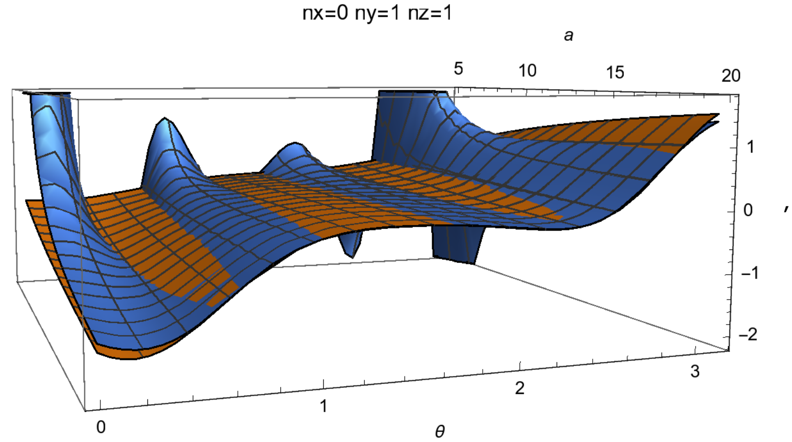
a $e^{-\kappa a}$SFT

nx=0 ny=0 nz=1



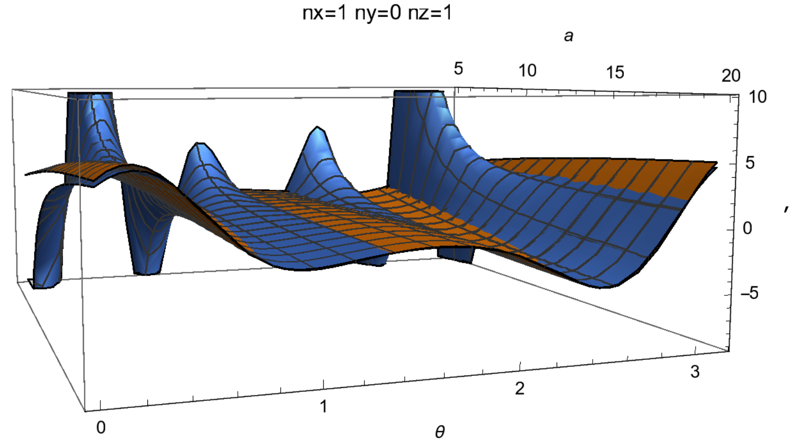
a $e^{-\kappa a}$SFT

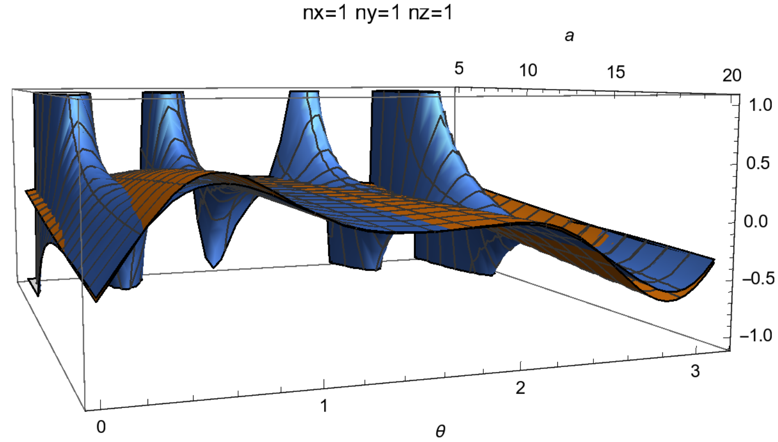
nx=0 ny=1 nz=1



a $e^{-\kappa a}$SFT

nx=1 ny=0 nz=1

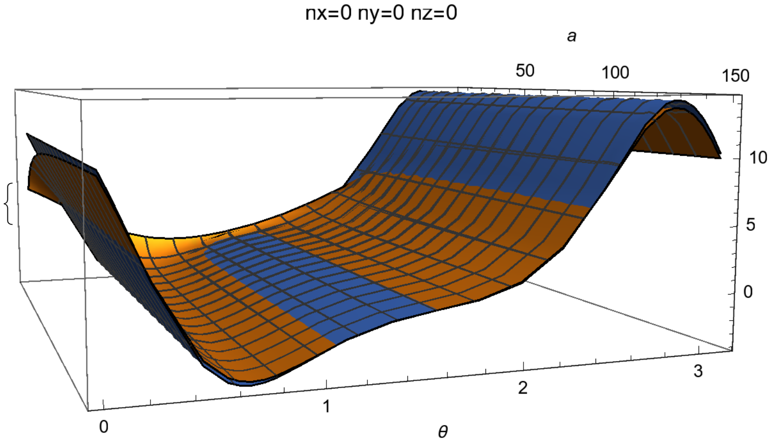




}}

```
Column@AbsoluteTiming@
  Block[{order = 0, κ = 1, c = 1, b = 2.5, po = -0.5, py = 0.15, part = Re[ⅇ^ⅈ #] &},
   Transpose[Flatten[Table[
      Plot3D[
       {Tooltip[a ⅇ^-κ a
          part[SFTanalytic[qvec[a, θ, po, py, κ], b, c, nx, ny, nz]], "Analytic"],
        Tooltip[a ⅇ^-κ a part[SFTasymptotic[po, py, θ, b, c, κ, a, nx, ny, nz, order]],
         "Asymptotic"]}
       , {a, 5, 150}, {θ, 0, π}
       , ImageSize → 400
       , AxesLabel → {"a", "θ"}
       , PlotLabel → "a ⅇ^-κa SFT \n nx=" <>
         ToString[nx] <> " ny=" <> ToString[ny] <> " nz=" <> ToString[nz]
       , ViewPoint → {2, -0.7, Scaled[0.0005]}
      ]
      , {nx, {0, 1}}, {ny, {0, 1}}, {nz, {0, 1}}], {1, 2}]]
  ]
12.1586
```

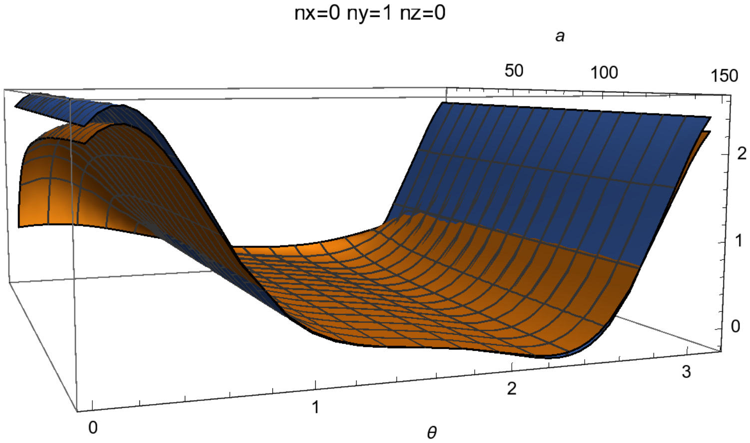
{{



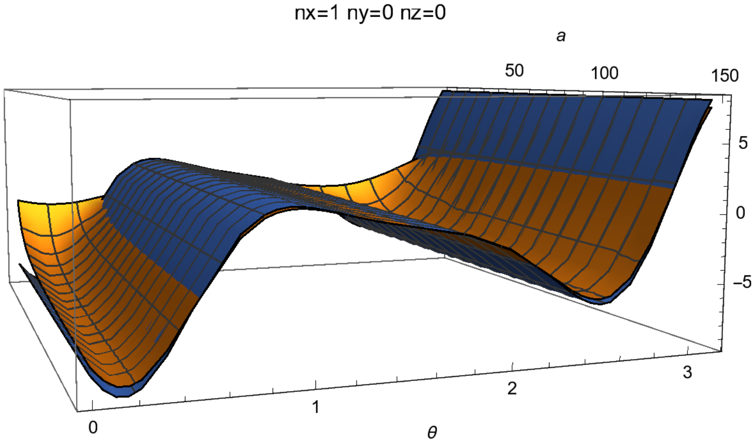
,

a ⅇ^-κa SFT

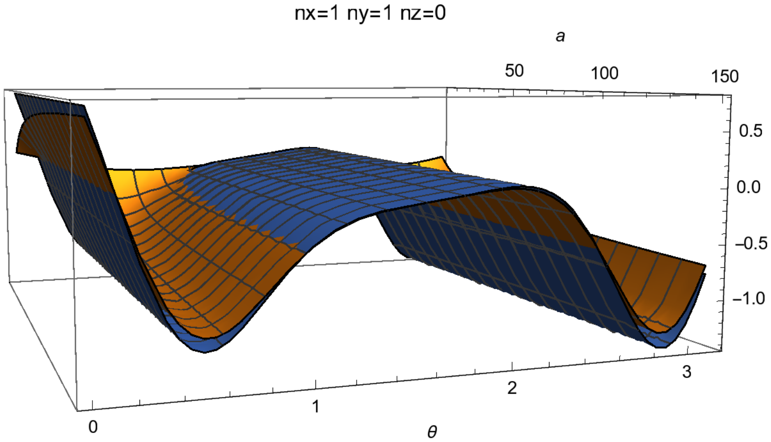
$a\,e^{-\kappa a}$SFT
nx=0 ny=1 nz=0



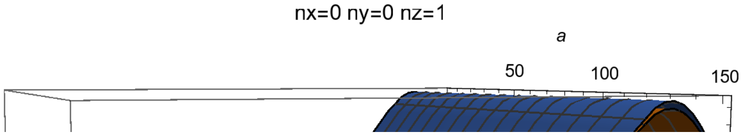
$a\,e^{-\kappa a}$SFT
nx=1 ny=0 nz=0



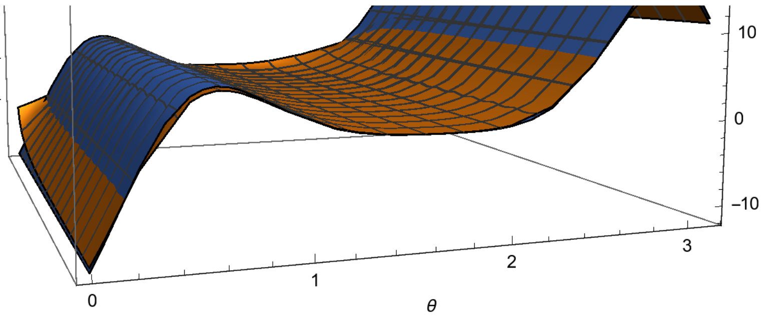
$a\,e^{-\kappa a}$SFT
nx=1 ny=1 nz=0



$a\,e^{-\kappa a}$SFT
nx=0 ny=0 nz=1



{



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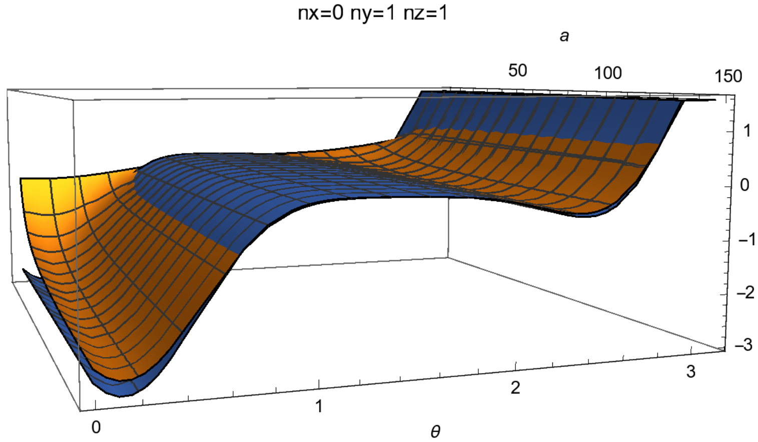
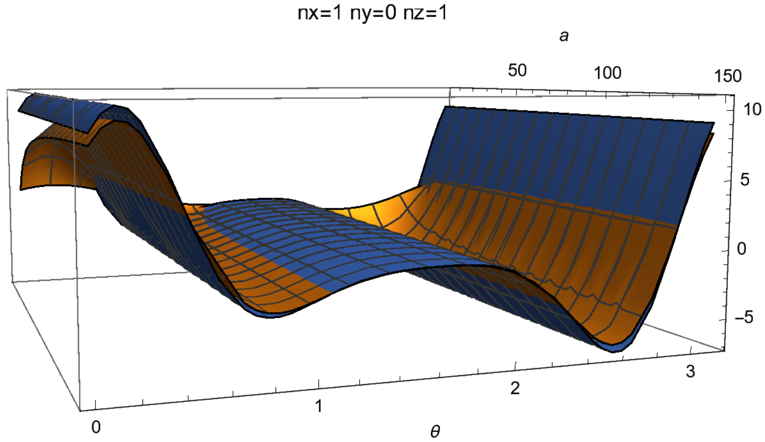
a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=1



,

a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=1



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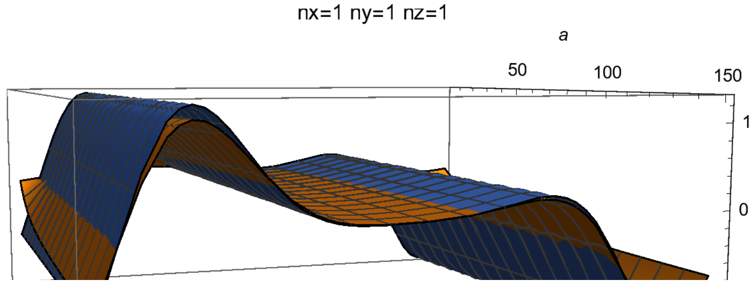
a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=1



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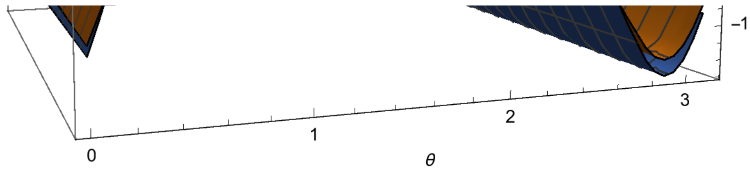
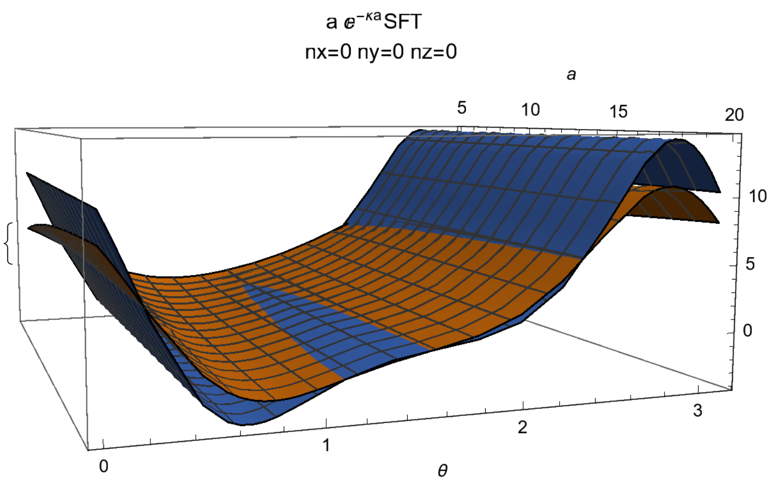
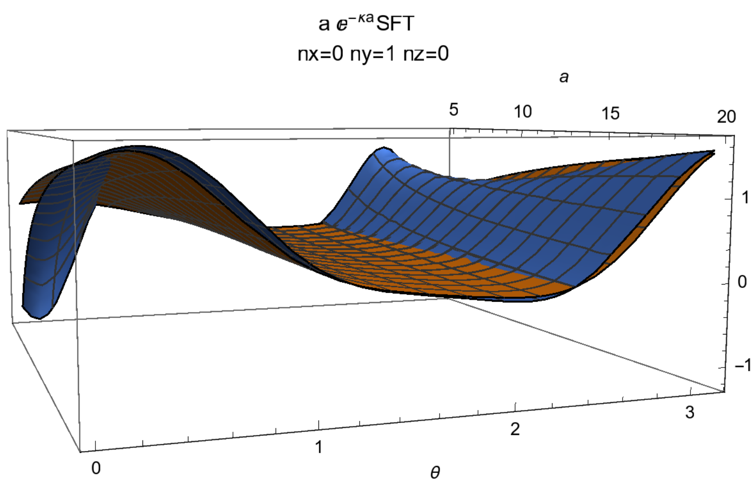

−1
0
1
2
3
θ

45.5745

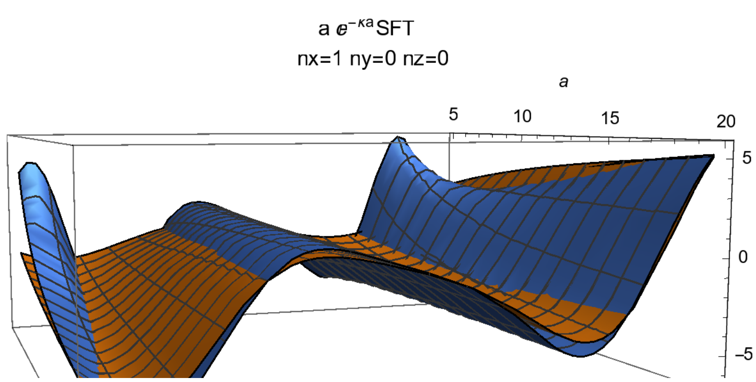
{{


a $e^{-\kappa a}$SFT
nx=0 ny=0 nz=0
a
5
10
15
20
10
5
0
0
1
2
3
θ

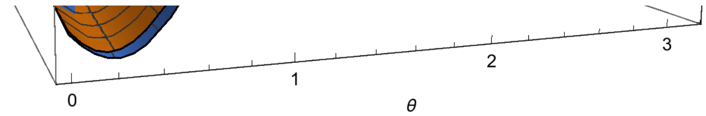
,


a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=0
a
5
10
15
20
1
0
−1
0
1
2
3
θ

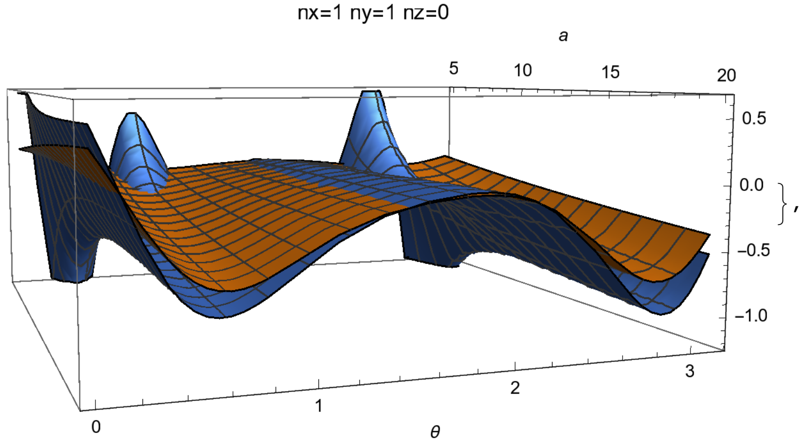
,


a $e^{-\kappa a}$SFT
nx=1 ny=0 nz=0
a
5
10
15
20
5
0
−5

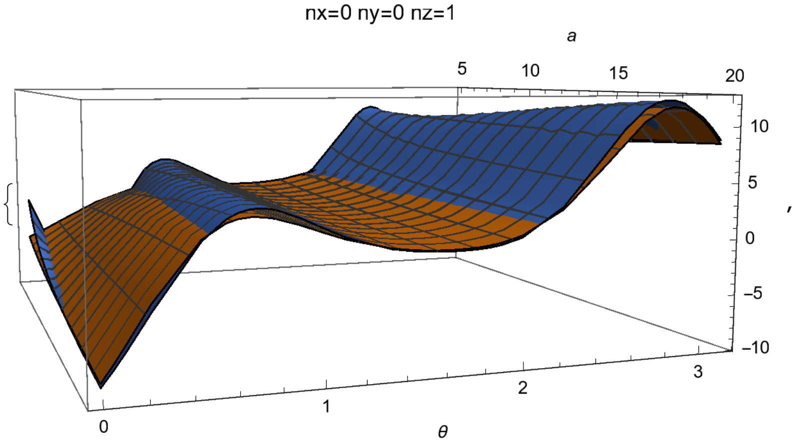
,



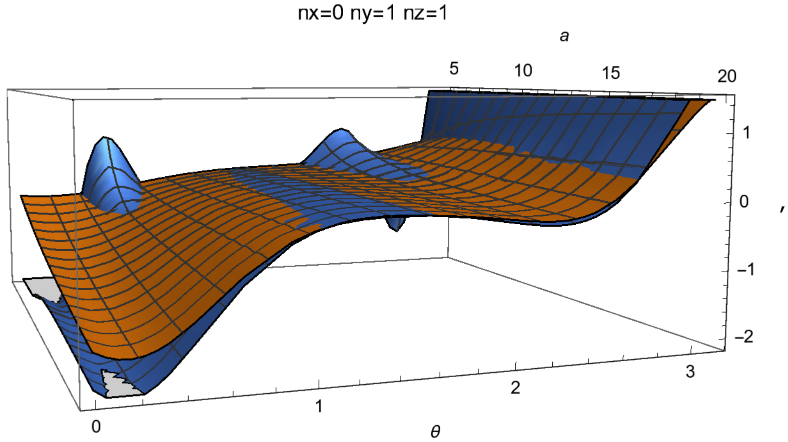
a $e^{-\kappa a}$SFT
nx=1 ny=1 nz=0

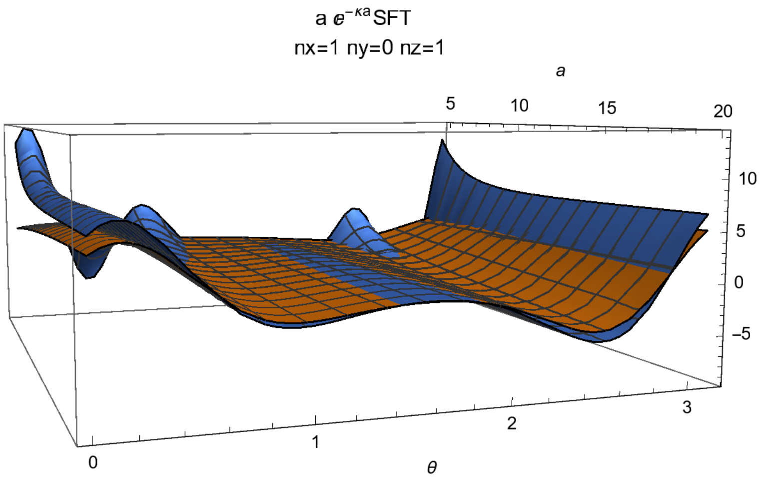


a $e^{-\kappa a}$SFT
nx=0 ny=0 nz=1



a $e^{-\kappa a}$SFT
nx=0 ny=1 nz=1





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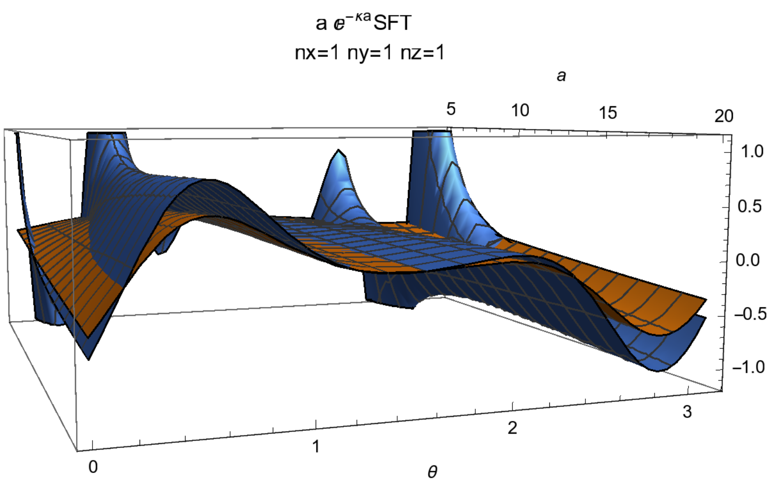

}}

```
SFTasymptotic[po, py, θ, b, c, κ, a, 1, 0, 0, 0] // Simplify // AbsoluteTiming
```

$$\left\{0.30229,\ -\frac{1}{a\,\kappa^4}\,\mathbb{i}\,e^{a\,\kappa}\,\mathrm{Cosh}\left[\frac{b\left(\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\mathbb{i}\,po\,\mathrm{Sin}[\theta]\right)}{\kappa}\right]\left(po\,\mathrm{Cos}[\theta]-\mathbb{i}\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[\theta]\right)\left(\kappa^2+c\left(po^2+py^2+\kappa^2\right)\mathrm{Cos}[\theta]^2-2\,\mathbb{i}\,c\,po\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-c\,po^2\,\mathrm{Sin}[\theta]^2\right)\right\}$$

$$
\Big\{29.643259\grave{}\,,\ \frac{1}{2\,a^2\,\kappa^8\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}}\,e^{a\,\kappa}\left(i\,po\,\mathrm{Cos}[\theta]+\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[\theta]\right)
$$

$$
\Bigg(\frac{1}{4}\,\mathrm{Cosh}\Bigg[b\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{i\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Bigg]
$$

$$
\left(c\,py^2+2\,\kappa^2+c\,\kappa^2+c\,\left(2\,po^2+py^2+\kappa^2\right)\,\mathrm{Cos}[2\,\theta]-2\,i\,c\,po\,\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[2\,\theta]\right)
$$

$$
\Bigg(\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \left(-4\,{\color{red}\mathbf{a}}\,\kappa^3+b^2\,\left(py^2-\kappa^2\right)\right)+b^2\,\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \left(2\,po^2+py^2+\kappa^2\right)
$$

$$
\mathrm{Cos}[2\,\theta]-2\,i\,b^2\,po\,\left(po^2+py^2+\kappa^2\right)\,\mathrm{Sin}[2\,\theta]\Bigg)+2\,b\,\kappa^2\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}
$$

$$
\Bigg(\Bigg(\Bigg(2\,\kappa^3\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ +c\,\Bigg(-2\,po^2\,\Bigg(\sqrt{po^2+py^2+\kappa^2}\ -\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\Bigg)+\kappa\,\Bigg(-2\,\kappa
$$

$$
\sqrt{po^2+py^2+\kappa^2}\ +3\,py^2\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ +3\,\kappa^2\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\Bigg)\Bigg)\Bigg)\ \mathrm{Cos}[\theta]\ +
$$

$$
c\,\Bigg(\kappa\,\left(py^2+\kappa^2\right)\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ +2\,po^2\,\Bigg(\sqrt{po^2+py^2+\kappa^2}\ +\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\Bigg)\Bigg)\Bigg)
$$

$$
\mathrm{Cos}[3\,\theta]-2\,i\,po\,\Bigg(c\,py^2+\kappa^2+2\,c\,\kappa\,\sqrt{po^2+py^2+\kappa^2}\ \sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ +
$$

$$
c\,\Bigg(2\,po^2+py^2+\kappa^2+2\,\kappa\,\sqrt{po^2+py^2+\kappa^2}\ \sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\Bigg)\,\mathrm{Cos}[2\,\theta]\Bigg)\,\mathrm{Sin}[\theta]\Bigg)
$$

$$
\mathrm{Sinh}\Bigg[b\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{i\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Bigg]\Bigg)\Big\}
$$

Note the big red *a* where there’s supposed to be no *a* dependence in an order = 0 expansion. This needs more work.

Re-testing the asymptotics

```
(*ClearAll[SFTasymptotic];*)
(*SFTasymptotic[poo_,pyy_,θθ_,bb_,cc_,κκ_,aa_,nx_,ny_,nz_,order_]:=*)
Block[{po, py, θ, b, c, κ, a},
 (*SFTasymptotic[po_,py_,θ_,b_,c_,κ_,a_,nx,ny,nz,order]=*)Block[
  {n = nx + ny + nz, qx, qy, qz, σ, s, κa,   nx = 1, ny = 0, nz = 0, order = 1},
```

$$\sigma = \kappa a \sqrt{\left(1 + \frac{b^2}{\kappa a^2} - 2\,s\,\frac{b}{\kappa a}\sqrt{1 + \frac{po^2}{\kappa^2} + \frac{py^2}{\kappa^2}}\,\mathrm{Cos}[\theta] + 2\,s\,\mathrm{i}\,\frac{b}{\kappa a}\,\frac{po}{\kappa}\,\mathrm{Sin}[\theta]\right)};$$

$$\{qx, qy, qz\} = \left\{\kappa a\,\frac{po\,\mathrm{Cos}[\theta] - \mathrm{i}\sqrt{po^2 + py^2 + \kappa^2}\,\mathrm{Sin}[\theta]}{\kappa},\ \kappa a\,\frac{py}{\kappa},\ \kappa a\,\frac{-\mathrm{i}\sqrt{po^2 + py^2 + \kappa^2}\,\mathrm{Cos}[\theta] - po\,\mathrm{Sin}[\theta]}{\kappa}\right\};$$

$$e^{\kappa\,a}\,\mathrm{ExpToTrig}\Big[\mathrm{Sum}\Big[\mathrm{Normal}\Big[\mathrm{Series}\Big[(-\mathrm{i})^n\,e^{-\kappa a}\,qx^{nx}\,qy^{ny}\,(qz + s\,\mathrm{i}\,b)^{nz}\sqrt{\frac{\pi}{2}}$$

$$\left(\left(1 + c\,\frac{nz + 1/2}{n + 3/2}\right)\frac{1}{\sigma^{n+1/2}}\mathrm{AsymptoticBesselI}[n + 1/2, \sigma, order + 1] - c\left((qz + s\,\mathrm{i}\,b)^2 + \frac{nz + 1/2}{n + 3/2}\,\sigma^2\right)\frac{1}{\sigma^{n+5/2}}\mathrm{AsymptoticBesselI}[n + 5/2, \sigma, order + 1]\right)$$

$$,\ \{\kappa a, \infty, order + 1\}\Big]\Big]\ /.\ \{\kappa a \to \kappa\,a\}$$

$$,\ \{s, \{1, -1\}\}\Big]\Big]$$

```
 ]
 (*SFTasymptotic[poo,pyy,θθ,bb,cc,κκ,aa,nx,ny,nz,order]*)
]
Simplify[%]
```

$$a\,\kappa\,e^{-\kappa\,a}\,\%$$

```
FreeQ[%, a]
```

$$e^{a\,\kappa}$$

$$\left(\frac{1}{a^2\,\kappa^2}\left(-\frac{1}{4\,\kappa^3}\,\mathrm{i}\left(po\,\mathrm{Cos}[\theta] - \mathrm{i}\sqrt{po^2 + py^2 + \kappa^2}\,\mathrm{Sin}[\theta]\right)\left(\kappa^2 + c\,po^2\,\mathrm{Cos}[\theta]^2 + c\,py^2\,\mathrm{Cos}[\theta]^2 + c\,\kappa^2\right.\right.\right.$$

$$\left.\mathrm{Cos}[\theta]^2 - 2\,\mathrm{i}\,c\,po\sqrt{po^2 + py^2 + \kappa^2}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] - c\,po^2\,\mathrm{Sin}[\theta]^2\right)$$

$$\left(b^2 - \frac{1}{4}\left(-2\,b\sqrt{1 + \frac{po^2}{\kappa^2} + \frac{py^2}{\kappa^2}}\,\mathrm{Cos}[\theta] + \frac{2\,\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\right)^2\right)$$

$$\left(\mathrm{Cosh}\left[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\, b\, po\, \mathrm{Sin}[\theta]}{\kappa}\right]-\right.$$

$$\left.\mathrm{Sinh}\left[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\, b\, po\, \mathrm{Sin}[\theta]}{\kappa}\right]\right)-$$

$$\frac{1}{\kappa}\,\mathrm{i}\sqrt{\frac{\pi}{2}}\left(po\,\mathrm{Cos}[\theta]-\mathrm{i}\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[\theta]\right)$$

$$\left(-\frac{1}{\sqrt{2\pi}}-\frac{c}{5\sqrt{2\pi}}+b\sqrt{\frac{2}{\pi}}\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]+\right.$$

$$\frac{1}{5}\,b\,c\sqrt{\frac{2}{\pi}}\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\, b\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{\kappa}-\frac{\mathrm{i}\, b\, c\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{5\,\kappa}+$$

$$c\left(\frac{1}{5}\,b\sqrt{\frac{2}{\pi}}\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{b\sqrt{\frac{2}{\pi}}\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]}{\kappa}+\right.$$

$$\frac{4\,\mathrm{i}\, b\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{5\,\kappa}-\frac{1}{5\,\kappa^2}\left(-3\sqrt{\frac{2}{\pi}}+2\,b\sqrt{\frac{2}{\pi}}\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\right.$$

$$\left.\frac{2\,\mathrm{i}\, b\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{\kappa}\right)\left(\kappa^2-5\,po^2\,\mathrm{Cos}[\theta]^2-5\,py^2\,\mathrm{Cos}[\theta]^2-\right.$$

$$\left.\left.\left.5\,\kappa^2\,\mathrm{Cos}[\theta]^2+10\,\mathrm{i}\,po\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]+5\,po^2\,\mathrm{Sin}[\theta]^2\right)\right)\right)$$

$$\left(\mathrm{Cosh}\left[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\, b\, po\, \mathrm{Sin}[\theta]}{\kappa}\right]-\mathrm{Sinh}\left[\right.\right.$$

$$b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Big]\Big)\Bigg)-$$

$$\frac{1}{2\,a\,\kappa^4}\,\mathrm{i}\left(po\,\mathrm{Cos}[\theta]-\mathrm{i}\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[\theta]\right)\Big(\kappa^2+c\,po^2\,\mathrm{Cos}[\theta]^2+$$
$$c\,py^2\,\mathrm{Cos}[\theta]^2+c\,\kappa^2\,\mathrm{Cos}[\theta]^2-$$
$$2\,\mathrm{i}\,c\,po\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-c\,po^2\,\mathrm{Sin}[\theta]^2\Big)$$
$$\Big(\mathrm{Cosh}\Big[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Big]-$$
$$\mathrm{Sinh}\Big[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Big]\Big)-$$

$$\frac{1}{2\,a\,\kappa^4}\,\mathrm{i}\left(po\,\mathrm{Cos}[\theta]-\mathrm{i}\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[\theta]\right)$$
$$\Big(\kappa^2+c\,po^2\,\mathrm{Cos}[\theta]^2+c\,py^2\,\mathrm{Cos}[\theta]^2+c\,\kappa^2\,\mathrm{Cos}[\theta]^2-$$
$$2\,\mathrm{i}\,c\,po\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-c\,po^2\,\mathrm{Sin}[\theta]^2\Big)$$
$$\Big(\mathrm{Cosh}\Big[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Big]+$$
$$\mathrm{Sinh}\Big[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Big]\Big)+$$

$$\frac{1}{a^2\,\kappa^2}\Bigg(-\frac{1}{4\,\kappa^3}\,\mathrm{i}\left(po\,\mathrm{Cos}[\theta]-\mathrm{i}\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[\theta]\right)\Big(\kappa^2+c\,po^2\,\mathrm{Cos}[\theta]^2+c\,py^2\,\mathrm{Cos}[\theta]^2+$$
$$c\,\kappa^2\,\mathrm{Cos}[\theta]^2-2\,\mathrm{i}\,c\,po\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-c\,po^2\,\mathrm{Sin}[\theta]^2\Big)$$
$$\left(b^2-\frac{1}{4}\left(2\,b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{2\,\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\right)^2\right)$$
$$\Big(\mathrm{Cosh}\Big[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\Big]+$$

$$\mathrm{Sinh}\left[b \sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\, b\, po\, \mathrm{Sin}[\theta]}{\kappa}\right]\Bigg)-$$

$$\frac{1}{\kappa}\,\mathrm{i}\sqrt{\frac{\pi}{2}}\ \left(po\,\mathrm{Cos}[\theta]-\mathrm{i}\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[\theta]\right)$$

$$\Bigg(-\frac{1}{\sqrt{2\pi}}-\frac{c}{5\sqrt{2\pi}}-b\sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-$$

$$\frac{1}{5}\,b\,c\sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]+\frac{\mathrm{i}\, b\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{\kappa}+\frac{\mathrm{i}\, b\, c\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{5\kappa}+$$

$$c\Bigg(-\frac{1}{5}\,b\sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]+\frac{b\sqrt{\frac{2}{\pi}}\ \sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]}{\kappa}-$$

$$\frac{4\,\mathrm{i}\, b\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{5\kappa}-\frac{1}{5\kappa^2}\Bigg(-3\sqrt{\frac{2}{\pi}}\ -2\,b\sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]+$$

$$\frac{2\,\mathrm{i}\, b\sqrt{\frac{2}{\pi}}\ po\,\mathrm{Sin}[\theta]}{\kappa}\Bigg)\Big(\kappa^2-5\,po^2\,\mathrm{Cos}[\theta]^2-5\,py^2\,\mathrm{Cos}[\theta]^2-$$

$$5\,\kappa^2\,\mathrm{Cos}[\theta]^2+10\,\mathrm{i}\,po\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]+5\,po^2\,\mathrm{Sin}[\theta]^2\Big)\Bigg)\Bigg)$$

$$\Bigg(\mathrm{Cosh}\left[b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\, b\, po\,\mathrm{Sin}[\theta]}{\kappa}\right]+\mathrm{Sinh}\Big[$$

$$b\sqrt{1+\frac{po^2}{\kappa^2}+\frac{py^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\, b\, po\,\mathrm{Sin}[\theta]}{\kappa}\Big]\Bigg)\Bigg)\Bigg)$$

$$
\begin{aligned}
&\frac{1}{2\,a^2\,\kappa^7}\,e^{a\,\kappa}\left(i\,po\,Cos[\theta]+\sqrt{po^2+py^2+\kappa^2}\,Sin[\theta]\right)\\
&\Bigg(Cosh\left[b\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,Cos[\theta]-\frac{i\,b\,po\,Sin[\theta]}{\kappa}\right]\Bigg(b^2\,c\,\left(po^2+py^2+\kappa^2\right)^2\,Cos[\theta]^4+\\
&\quad\left(po^2+py^2+\kappa^2\right)Cos[\theta]^2\left(-2\,c\,\kappa^2\,(-6+a\,\kappa)+b^2\left(\kappa^2-c\left(po^2+\kappa^2\right)\right)+b^2\,c\,po^2\,Cos[2\,\theta]\right)-\\
&\quad 2\,i\,b^2\,c\,po\left(po^2+py^2+\kappa^2\right)\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)Cos[\theta]^3\,Sin[\theta]+\\
&\quad po^2\,\kappa^2\left(b^2\,(-1+c)+2\,c\,(-6+a\,\kappa)\right)Sin[\theta]^2+\\
&\quad 2\,i\,b^2\,c\,po^3\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)Cos[\theta]\,Sin[\theta]^3+\\
&\quad b^2\,c\,po^4\,Sin[\theta]^4-\kappa\Bigg(\kappa^3\left(b^2+2\,(-1+c+a\,\kappa)\right)-\\
&\qquad i\,po\,\kappa\left(2\,c\,(-6+a\,\kappa)\sqrt{po^2+py^2+\kappa^2}+b^2\left(c\sqrt{po^2+py^2+\kappa^2}-\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\right)\\
&\qquad Sin[2\,\theta]+b^2\,c\,po^2\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,Sin[2\,\theta]^2\Bigg)\Bigg)+\\
&\quad 2\,b\,\kappa\Bigg(\Bigg(\Bigg(2\,\kappa^3\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+c\Bigg(-2\,po^2\left(\sqrt{po^2+py^2+\kappa^2}-\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)+\\
&\qquad\quad \kappa\left(-2\,\kappa\sqrt{po^2+py^2+\kappa^2}+3\,py^2\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+3\,\kappa^2\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\Bigg)\Bigg)\\
&\qquad Cos[\theta]+c\left(\kappa\left(py^2+\kappa^2\right)\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+2\,po^2\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\right)\\
&\qquad Cos[3\,\theta]-2\,i\,po\Bigg(c\,py^2+\kappa^2+2\,c\,\kappa\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+\\
&\qquad\quad c\left(2\,po^2+py^2+\kappa^2+2\,\kappa\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)Cos[2\,\theta]\Bigg)Sin[\theta]\Bigg)\\
&\quad Sinh\left[b\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,Cos[\theta]-\frac{i\,b\,po\,Sin[\theta]}{\kappa}\right]\Bigg)
\end{aligned}
$$

$$\frac{1}{2\,a\,\kappa^6}\left(\mathbb{i}\,po\,\mathrm{Cos}[\theta]+\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Sin}[\theta]\right)$$

$$\Bigg(\mathrm{Cosh}\left[b\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]-\frac{\mathbb{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\right]\Bigg(b^2\,c\,\left(po^2+py^2+\kappa^2\right)^2\mathrm{Cos}[\theta]^4+$$

$$\left(po^2+py^2+\kappa^2\right)\mathrm{Cos}[\theta]^2\left(-2\,c\,\kappa^2\,(-6+a\,\kappa)+b^2\left(\kappa^2-c\left(po^2+\kappa^2\right)\right)+b^2\,c\,po^2\,\mathrm{Cos}[2\,\theta]\right)-$$

$$2\,\mathbb{i}\,b^2\,c\,po\left(po^2+py^2+\kappa^2\right)\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\mathrm{Cos}[\theta]^3\,\mathrm{Sin}[\theta]+$$

$$po^2\,\kappa^2\left(b^2\,(-1+c)+2\,c\,(-6+a\,\kappa)\right)\mathrm{Sin}[\theta]^2+$$

$$2\,\mathbb{i}\,b^2\,c\,po^3\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^3+$$

$$b^2\,c\,po^4\,\mathrm{Sin}[\theta]^4-\kappa\Bigg(\kappa^3\left(b^2+2\,(-1+c+a\,\kappa)\right)-$$

$$\mathbb{i}\,po\,\kappa\left(2\,c\,(-6+a\,\kappa)\sqrt{po^2+py^2+\kappa^2}+b^2\left(c\sqrt{po^2+py^2+\kappa^2}-\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\right)$$

$$\mathrm{Sin}[2\,\theta]+b^2\,c\,po^2\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Sin}[2\,\theta]^2\Bigg)\Bigg)+$$

$$2\,b\,\kappa\Bigg(\Bigg(\Bigg(2\,\kappa^3\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+c\Bigg(-2\,po^2\left(\sqrt{po^2+py^2+\kappa^2}-\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)+$$

$$\kappa\left(-2\,\kappa\sqrt{po^2+py^2+\kappa^2}+3\,py^2\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+3\,\kappa^2\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\Bigg)\Bigg)$$

$$\mathrm{Cos}[\theta]+c\left(\kappa\left(py^2+\kappa^2\right)\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+2\,po^2\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\right)$$

$$\mathrm{Cos}[3\,\theta]-2\,\mathbb{i}\,po\Bigg(c\,py^2+\kappa^2+2\,c\,\kappa\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+$$

$$c\left(2\,po^2+py^2+\kappa^2+2\,\kappa\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\mathrm{Cos}[2\,\theta]\Bigg)\mathrm{Sin}[\theta]\Bigg)$$

$$\mathrm{Sinh}\left[b\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]-\frac{\mathbb{i}\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\right]\Bigg)$$

False

$$\mathrm{Series}\Bigg[\Bigg(\mathrm{Cosh}\Big[b\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,\mathrm{po}\,\mathrm{Sin}[\theta]}{\kappa}\Big]\ \Bigg(b^2\,c\,\left(\mathrm{po}^2+\mathrm{py}^2+\kappa^2\right)^2\mathrm{Cos}[\theta]^4+$$

$$\left(\mathrm{po}^2+\mathrm{py}^2+\kappa^2\right)\mathrm{Cos}[\theta]^2\left(-2\,c\,\kappa^2\,(-6+a\,\kappa)+b^2\left(\kappa^2-c\left(\mathrm{po}^2+\kappa^2\right)\right)+b^2\,c\,\mathrm{po}^2\,\mathrm{Cos}[2\,\theta]\right)-$$

$$2\,\mathrm{i}\,b^2\,c\,\mathrm{po}\left(\mathrm{po}^2+\mathrm{py}^2+\kappa^2\right)\left(\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}+\kappa\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\right)\mathrm{Cos}[\theta]^3\,\mathrm{Sin}[\theta]+$$

$$\mathrm{po}^2\,\kappa^2\left(b^2\,(-1+c)+2\,c\,(-6+a\,\kappa)\right)\mathrm{Sin}[\theta]^2+$$

$$2\,\mathrm{i}\,b^2\,c\,\mathrm{po}^3\left(\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}+\kappa\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\right)\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^3+$$

$$b^2\,c\,\mathrm{po}^4\,\mathrm{Sin}[\theta]^4-\kappa\Bigg(\kappa^3\left(b^2+2\,(-1+c+a\,\kappa)\right)-$$

$$\mathrm{i}\,\mathrm{po}\,\kappa\left(2\,c\,(-6+a\,\kappa)\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}+b^2\left(c\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}-\kappa\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\right)\right)$$

$$\mathrm{Sin}[2\,\theta]+b^2\,c\,\mathrm{po}^2\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}\ \sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\ \mathrm{Sin}[2\,\theta]^2\Bigg)\Bigg)+$$

$$2\,b\,\kappa\Bigg(\Bigg(2\,\kappa^3\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}+c\Bigg(-2\,\mathrm{po}^2\left(\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}-\kappa\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\right)+\kappa$$

$$\left(-2\,\kappa\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}+3\,\mathrm{py}^2\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}+3\,\kappa^2\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\right)\Bigg)\Bigg)\mathrm{Cos}[\theta]+$$

$$c\left(\kappa\left(\mathrm{py}^2+\kappa^2\right)\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}+2\,\mathrm{po}^2\left(\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}+\kappa\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\right)\right)\mathrm{Cos}[3\,\theta]-$$

$$2\,\mathrm{i}\,\mathrm{po}\Bigg(c\,\mathrm{py}^2+\kappa^2+2\,c\,\kappa\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}\ \sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}+$$

$$c\left(2\,\mathrm{po}^2+\mathrm{py}^2+\kappa^2+2\,\kappa\sqrt{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}\ \sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\right)\mathrm{Cos}[2\,\theta]\Bigg)\mathrm{Sin}[\theta]\Bigg)$$

$$\mathrm{Sinh}\Big[b\sqrt{\frac{\mathrm{po}^2+\mathrm{py}^2+\kappa^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{\mathrm{i}\,b\,\mathrm{po}\,\mathrm{Sin}[\theta]}{\kappa}\Big]\Bigg)$$

,

```
{a,
 0,
 4}]
```

$$\left(\text{Cosh}\left[b\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \text{Cos}[\theta]-\frac{i\,b\,po\,\text{Sin}[\theta]}{\kappa}\right]\right.$$

$$\left(2\kappa^4-b^2\kappa^4-2c\kappa^4+b^2c\left(po^2+py^2+\kappa^2\right)^2\text{Cos}[\theta]^4+\right.$$

$$\left(po^2+py^2+\kappa^2\right)\text{Cos}[\theta]^2\left(-b^2c\,po^2+b^2\kappa^2+12c\kappa^2-b^2c\kappa^2+b^2c\,po^2\,\text{Cos}[2\theta]\right)-$$

$$2\,i\,b^2c\,po\left(po^2+py^2+\kappa^2\right)\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\text{Cos}[\theta]^3\,\text{Sin}[\theta]+$$

$$\left(-b^2-12c+b^2c\right)po^2\kappa^2\,\text{Sin}[\theta]^2+2\,i\,b^2c\,po^3\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)$$

$$\text{Cos}[\theta]\,\text{Sin}[\theta]^3+b^2c\,po^4\,\text{Sin}[\theta]^4-12\,i\,c\,po\,\kappa^2\sqrt{po^2+py^2+\kappa^2}\ \text{Sin}[2\theta]+$$

$$i\,b^2c\,po\,\kappa^2\sqrt{po^2+py^2+\kappa^2}\ \text{Sin}[2\theta]-i\,b^2po\,\kappa^3\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \text{Sin}[2\theta]-$$

$$\left.b^2c\,po^2\kappa\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \text{Sin}[2\theta]^2\right)+$$

$$2b\kappa\left(\left(\left(2\kappa^3\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+c\left(-2po^2\left(\sqrt{po^2+py^2+\kappa^2}-\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)+\right.\right.\right.\right.$$

$$\left.\left.\left.\kappa\left(-2\kappa\sqrt{po^2+py^2+\kappa^2}+3py^2\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+3\kappa^2\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\right)\right)\text{Cos}[\theta]+$$

$$c\left(\kappa\left(py^2+\kappa^2\right)\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+2po^2\left(\sqrt{po^2+py^2+\kappa^2}+\kappa\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\right)\text{Cos}[3\theta]-$$

$$2\,i\,po\left(c\,py^2+\kappa^2+2c\kappa\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}+\right.$$

$$\left.\left.c\left(2po^2+py^2+\kappa^2+2\kappa\sqrt{po^2+py^2+\kappa^2}\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\right)\text{Cos}[2\theta]\right)\text{Sin}[\theta]\right)$$

$$\mathrm{Sinh}\left[b\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{i\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\right]\Bigg)+$$

$$\mathrm{Cosh}\left[b\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\ \mathrm{Cos}[\theta]-\frac{i\,b\,po\,\mathrm{Sin}[\theta]}{\kappa}\right]$$

$$\Big(-2\,\kappa^5-2\,c\,\kappa^3\,\left(po^2+py^2+\kappa^2\right)\mathrm{Cos}[\theta]^2+$$

$$2\,c\,po^2\,\kappa^3\,\mathrm{Sin}[\theta]^2+2\,i\,c\,po\,\kappa^3\sqrt{po^2+py^2+\kappa^2}\ \mathrm{Sin}[2\,\theta]\Big)\,a+O[a]^5$$

```
(*ClearAll[SFTasymptotic];*)
(*SFTasymptotic[poo_,pyy_,θθ_,bb_,cc_,κκ_,aa_,nx_,ny_,nz_,order_]:=*)
Block[{po, py, θ, b, c, κ, a},
  (*SFTasymptotic[po_,py_,θ_,b_,c_,κ_,a_,nx,ny,nz,order]=*)Block[
   {n = nx + ny + nz, qx, qy, qz, σ, s, κa,   nx = 1, ny = 0, nz = 0, order = 1},
   σ = κa Sqrt[(1 + b^2/κa^2 - 2 s b/κa Sqrt[1 + po^2/κ^2 + py^2/κ^2] Cos[θ] + 2 s I b/κa po/κ Sin[θ])];
   {qx, qy, qz} = {κa (po Cos[θ] - I Sqrt[po^2 + py^2 + κ^2] Sin[θ])/κ,
     κa py/κ, κa (-I Sqrt[po^2 + py^2 + κ^2] Cos[θ] - po Sin[θ])/κ};
   E^(κ a) ExpToTrig[Sum[
      Normal[Series[
         (-I)^n E^(-κa) qx^nx qy^ny (qz + s I b)^nz Sqrt[π/2] ((1 + c (nz + 1/2)/(n + 3/2)) 1/σ^(n+1/2)
             AsymptoticBesselI[n + 1/2, σ, order + 1] - c ((qz + s I b)^2 + (nz + 1/2)/(n + 3/2) σ^2)
             1/σ^(n+5/2) AsymptoticBesselI[n + 5/2, σ, order + 1])
         , {κa, ∞, order + 1}]] /. {κa → κ a}
      , {s, {1, -1}}]]
   ]
  (*SFTasymptotic[poo,pyy,θθ,bb,cc,κκ,aa,nx,ny,nz,order]*)
 ] // Simplify;
Simplify[Series[a κ E^(-κ a) %, {a, ∞, 5}]]
```

```
Block[{po, py, θ, b, c, κ, a,     n = nx + ny + nz,
   qx, qy, qz, σ, s, κa,    nx = 1, ny = 0, nz = 0, order = 0},
```

$$\sigma = \kappa a \sqrt{\left(1 + \frac{b^2}{\kappa a^2} - 2\,s\,\frac{b}{\kappa a}\sqrt{1 + \frac{po^2}{\kappa^2} + \frac{py^2}{\kappa^2}}\,\mathrm{Cos}[\theta] + 2\,s\,\mathrm{i}\,\frac{b}{\kappa a}\,\frac{po}{\kappa}\,\mathrm{Sin}[\theta]\right)};$$

$$\{qx, qy, qz\} = \left\{\kappa a\,\frac{po\,\mathrm{Cos}[\theta] - \mathrm{i}\sqrt{po^2 + py^2 + \kappa^2}\,\mathrm{Sin}[\theta]}{\kappa},\; \kappa a\,\frac{py}{\kappa},\; \kappa a\,\frac{-\mathrm{i}\sqrt{po^2 + py^2 + \kappa^2}\,\mathrm{Cos}[\theta] - po\,\mathrm{Sin}[\theta]}{\kappa}\right\};$$

```
Series[
```

$$(-\mathrm{i})^n\, e^{-\kappa a}\, qx^{nx}\, qy^{ny}\, (qz + s\,\mathrm{i}\,b)^{nz}\sqrt{\frac{\pi}{2}}$$

$$\left(\left(1 + c\,\frac{nz + 1/2}{n + 3/2}\right)\frac{\mathrm{AsymptoticBesselI}[n + 1/2, \sigma, order + 1]}{\sigma^{n+1/2}} - c\left((qz + s\,\mathrm{i}\,b)^2 + \frac{nz + 1/2}{n + 3/2}\,\sigma^2\right)\frac{\mathrm{AsymptoticBesselI}[n + 5/2, \sigma, order + 1]}{\sigma^{n+5/2}}\right)$$

```
 , {κa, ∞, n + order + 1}]

]
```

$$
\begin{aligned}
&-\frac{1}{2\,\kappa^3\,\kappa a} \\
&\quad \mathbb{i}\,\mathbb{e}^{-b\,s\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]+\frac{\mathbb{i}\,b\,po\,s\,\mathrm{Sin}[\theta]}{\kappa}}\left(po\,\mathrm{Cos}[\theta]-\mathbb{i}\,\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Sin}[\theta]\right)\left(\kappa^2+c\,po^2\,\mathrm{Cos}[\theta]^2+\right. \\
&\qquad \left.c\,py^2\,\mathrm{Cos}[\theta]^2+c\,\kappa^2\,\mathrm{Cos}[\theta]^2-2\,\mathbb{i}\,c\,po\,\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-c\,po^2\,\mathrm{Sin}[\theta]^2\right)+\frac{1}{\kappa a^2} \\
&\quad \left(-\frac{1}{4\,\kappa^3}\,\mathbb{i}\,\mathbb{e}^{\frac{-b\,s\,\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]+\mathbb{i}\,b\,po\,s\,\mathrm{Sin}[\theta]}{\kappa}}\left(po\,\mathrm{Cos}[\theta]-\mathbb{i}\,\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Sin}[\theta]\right)\left(\kappa^2+c\,po^2\,\mathrm{Cos}[\theta]^2+\right.\right. \\
&\qquad\quad \left.c\,py^2\,\mathrm{Cos}[\theta]^2+c\,\kappa^2\,\mathrm{Cos}[\theta]^2-2\,\mathbb{i}\,c\,po\,\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-c\,po^2\,\mathrm{Sin}[\theta]^2\right) \\
&\qquad \left(b^2-\frac{1}{4}\left(-2\,b\,s\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]+\frac{2\,\mathbb{i}\,b\,po\,s\,\mathrm{Sin}[\theta]}{\kappa}\right)^2\right)- \\
&\qquad \frac{1}{\kappa}\,\mathbb{i}\,\mathbb{e}^{\frac{-b\,s\,\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]+\mathbb{i}\,b\,po\,s\,\mathrm{Sin}[\theta]}{\kappa}}\,\sqrt{\frac{\pi}{2}}\,\left(po\,\mathrm{Cos}[\theta]-\mathbb{i}\,\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Sin}[\theta]\right) \\
&\qquad \left(\frac{1}{5\,\kappa}\left(5\,b\,\sqrt{\frac{2}{\pi}}\,s\,\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]+b\,c\,\sqrt{\frac{2}{\pi}}\,s\,\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]-\right.\right. \\
&\qquad\qquad \left.5\,\mathbb{i}\,b\,\sqrt{\frac{2}{\pi}}\,po\,s\,\mathrm{Sin}[\theta]-\mathbb{i}\,b\,c\,\sqrt{\frac{2}{\pi}}\,po\,s\,\mathrm{Sin}[\theta]\right)+\frac{1}{\sqrt{2\,\pi}} \\
&\qquad\quad c\left(\frac{1}{5\,\kappa}2\left(-5\,b\,s\,\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Cos}[\theta]+b\,s\,\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]+\right.\right. \\
&\qquad\qquad\quad \left.4\,\mathbb{i}\,b\,po\,s\,\mathrm{Sin}[\theta]\right)-\frac{1}{5\,\kappa^3}4\left(b\,s\,\kappa\,\sqrt{\frac{po^2+py^2+\kappa^2}{\kappa^2}}\,\mathrm{Cos}[\theta]-\mathbb{i}\,b\,po\,s\,\mathrm{Sin}[\theta]\right) \\
&\qquad\qquad \left(\kappa^2-5\,po^2\,\mathrm{Cos}[\theta]^2-5\,py^2\,\mathrm{Cos}[\theta]^2-5\,\kappa^2\,\mathrm{Cos}[\theta]^2+\right. \\
&\qquad\qquad\quad \left.\left.\left.\left.10\,\mathbb{i}\,po\,\sqrt{po^2+py^2+\kappa^2}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]+5\,po^2\,\mathrm{Sin}[\theta]^2\right)\right)\right)\right)+O\left[\frac{1}{\kappa a}\right]^3
\end{aligned}
$$