Research Statement

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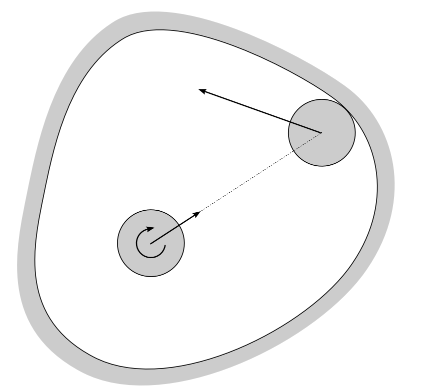
1 A brief introduction to no-slip billiards

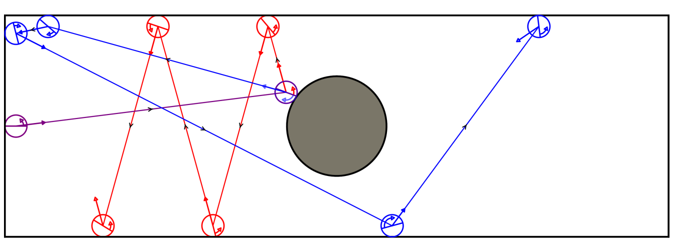
My research seeks to broadly understand *no-slip billiards*, a model in which angular and linear momentum may be exchanged at particle collisions subject to certain conservation laws. Specifically, I am interested in the geometry of the physically motivated models for no-slip collisions, in the intriguing dynamics of no-slip billiard systems, and in incorporating rotational models into statistical mechanics, leading to new insights into diffusion and thermodynamics, and also directly applying to macroscopic models such as non-holonomic systems.



Angular and linear momentum are exchanged at no-slip collisions. (Angle in ≠ angle out!)

Over the last century the study of mathematical billiards, based on the *specular* (sometimes called optical) model in which the angle in equals the angle out, has proven immensely valuable. Long-studied mechanical billiard models such as Lorentz gases continue to inspire active research, while in recent years entirely new fields such as quantum billiards have arisen, as well as as specialized applications as diverse as microwave oven design or robot motion planning. In contrast, the idea of no-slip or "rough" collisions has been considered at least since Richard Garwin's 1969 paper on Super Balls, arising as a second ideal collision model on equal footing with the standard specular model. Yet, very little work has been done on understanding no-slip billiard dynamical systems, incorporating them in statistical mechanical applications, or unraveling the connection to non-holonomic systems. My goal is to fill in many of these gaps.

At many points of entry there are opportunities either to strengthen the analytic foundation or to use computational techniques to shed light on aspects not yet tractable by other methods. These questions encompass both applied and pure mathematics, both subtle foundational and more approachable experimental questions. In the latter case, my priority is to continue to get students involved according to their interests and talents. To the left, a figure created by the Tarleton State University 2019 Billiards Summer Research Group, in which Scott Cook and I worked with four talented undergraduate students.



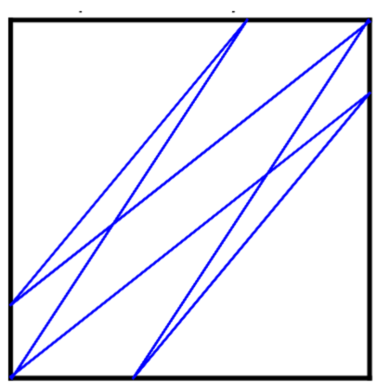
The trajectories with specular collisions (red) and no-slip collisions (blue) in a rectangular billiard with a scatterer.

2 Some areas of inquiry

This section gives a more detailed description of specific open questions and areas of interest, with a brief bibliography of outside work, our finished results, and ongoing work in each case.

2.1 The foundations of the no-slip model

The collision model for two dimensional disks and a special case of three dimensional spheres were previously known [1], but in [2] Renato Feres and I show that they may be viewed as special cases of a more general class of linear maps on the tangent bundle of a Riemannian manifold. In this framework we may adapt the model to a variety of physical assumptions and consider a broader picture (in the spirit of Klein's Erlangen program) including a hierarchy of manifolds corresponding to collision conditions imposed on the boundary map.



All orbits, regardless of initial velocity or spin, are periodic.

Using the versatility of this model, in work with graduate student Bishwas Ghimire, varying the mass distribution of the colliding particle led to many new examples of *persistently periodic* no-slip billiards, intriguing dynamical systems in which all orbits are periodic regardless of initial conditions, including the persistently periodic square in the figure to the left and persistently periodic pentagons.

References

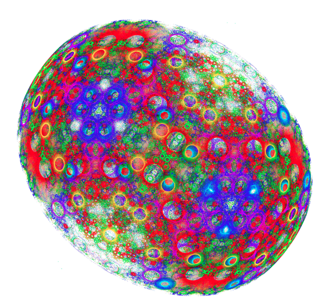
[1] R. L. Garwin, *Kinematics of an ultraelastic rough ball.* American Journal of Physics, (1) 37 (1969), 88-92.

[2] C. Cox, R. Feres, *Differential geometry of rigid bodies collisions and non-standard billiards.* Discrete and Continuous Dynamical Systems-A, 33 (2016) no. 11, 6065–6099.

[3] C. Cox, B. Ghimire, *No-slip billiards with varying mass distribution*, (in preparation).

2.2 No-slip dynamics

The two forays into no-slip dynamics prior to our recent work were by Broomhead and Gutkin [4], showing that the "no-slip strip" is bounded, and by Wojtkowski [5], who considered the linear stability of certain no-slip periodic orbits. Whether any no-slip billiard can be ergodic–completely mixing in a sense that allows the application of probabilistic techniques of statistical mechanics–is an open question. Computationally, phase portraits (like the one shown to the right) suggest elliptic islands precluding ergodicity are ubiquitous. In [6] we show that the invariance found in [4] generalizes to no-slip polygons, which in contrast to standard polygonal billiards are never ergodic. To the right is a phase portrait projection of a no-slip pentagon, with orbits coded by color to show the numerous elliptic islands where quasiperiodic orbits are trapped



Elliptic islands abound, defying ergodicity.

in the neighborhood of (sometimes very high period) periodic points. We also extend the linear stability result from [5].

References

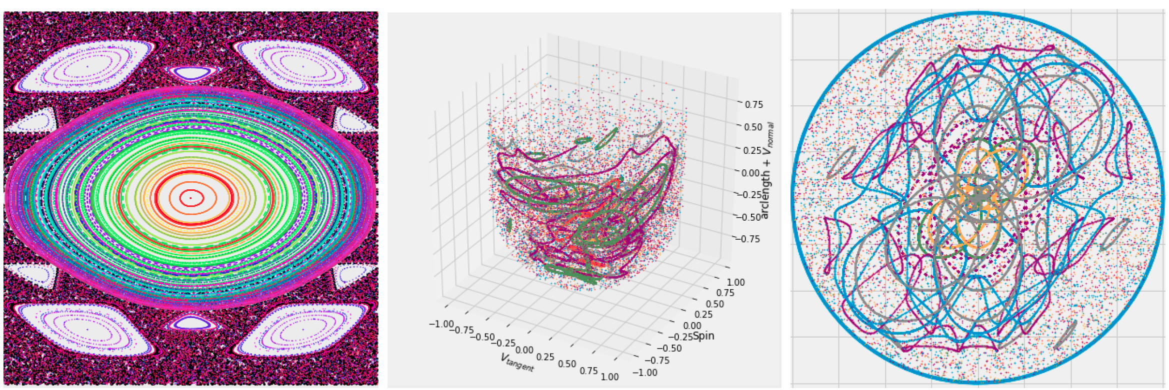
[4] D.S. Broomhead, E. Gutkin. *The dynamics of billiards with no-slip collisions.* Physica D 67 (1993) 188-197.

[5] M. Wojtkowski, *The system of two spinning disks in the torus.* Physica D 71 (1994) 430-439.

[6] C. Cox, R. Feres, H.-K. Zhang, *Stability of periodic orbits of no-slip billiards*, Nonlinearity, 31 (10), 2018, 4433-4471.

2.3 Computational mathematics

Billiard simulations have proven useful both in suggesting directions for analytic results (for example, in [6] above) and in producing experimental results of interest. In [7], Maria Correia, Hongkun Zhang, and I used SageMath billiard simulations of billiards tables motivated by quantum billiards. To search for ergodicity, we calculated Lyapunov exponents and produced phase portraits, including the one to the left below with stable elliptic islands in an ergodic sea. Clayton Boone and Ed Smith, two students in my graduate modeling class with strong Python skills, handled the tricky task of simulating billiards having walls without constant curvature, yielding the new results in [8]. In a summer REU project, we used python billiard simulations to compare known statistical results of billiard scatterers such as the Galton board and Lorentz gases to the analogous no-slip cases, creating the phase portraits in the middle and right below.



A phase portrait for a standard lemon billiard (left) versus the no-slip Lorentz gas (middle and right).

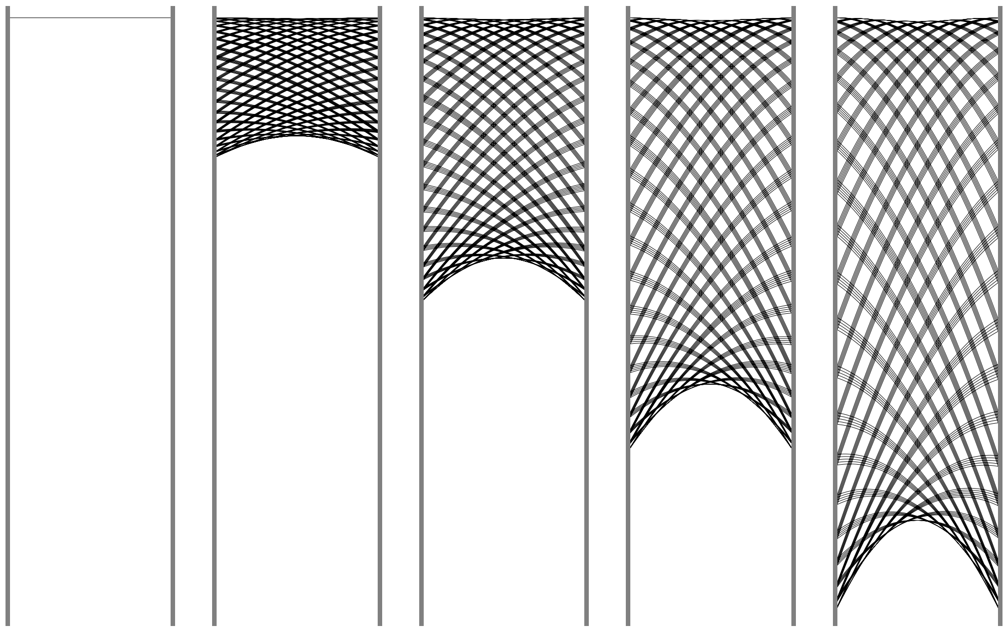
References

[7] M. Correia, C. Cox, H.-K. Zhang, *Ergodicity in umbrella billiards*, New Horizons in Mathematical Physics, Vol. 1, No. 2, September 2017, pp. 56-67.

[8] C. Boone, C. Cox, E. Smith, *Specular and no-slip billiards with cusps*, Proceedings of the ICTCM, to appear.

[9] T. Chumley, S. Cook, C. Cox, H. Grant, N. Petela, B. Rothrock, R. Xhafaj, *The no-slip Galton board*, (ongoing).

2.4 No-slip and non-holonomic billiards



No-slip trajectories in the strip remain bounded under an increasing external force.

Physical models imposing non-holonomic constraints, like the rolling coin or the Chaplygin sleigh, model curious systems sometimes demonstrating counter-intuitive behavior. For example, most basketball fans have seen rolling balls seeming to come out of the basket while golfers know the phenomenon described in [10]. In [11] we show that the invariance of the no-slip strip (more generally, hyperplanes in any dimension) persists under gravity, as in the figure above. Additionally, we show that the non-holonomic model of a continuously rolling ball in a cylinder can be seen as the limit of small no-slip collisions under conditions characterized by a certain *rolling defect*. The connection between non-holonomic mechanics and no-slip billiards appears to be much broader, as in [12] where the authors' "non-holonomic billiards" correspond, for certain cases, to no-slip billiards. An important direction of our current work it to better understand this connection.

References

[10] M. Gualtieri, T. Tokieda, L. Advis-Gaete, B. Carry, E. Reffet and C. Guthmann, *The Golfer's dilemma*, American Journal of Physics, **74** (2006), 497–501.

[11] T. Chumley, S. Cook, C. Cox, R. Feres, *Rolling and no-slip bouncing in cylinders*, Journal of Geometric Mechanics, 12 (1) 2020. (arxiv:1808.08448).

[12] A. V. Borisov, I. S. Mamaev and A. A. Kilin, *On the model of non-holonomic billiard*, Regul. Chaotic Dyn., **16** (2011), 653–662.