An Empirical Bayes Approach to Nonparametric Regression with Correlated Errors

Research Presentation
Cal Poly, San Luis Obispo, CA

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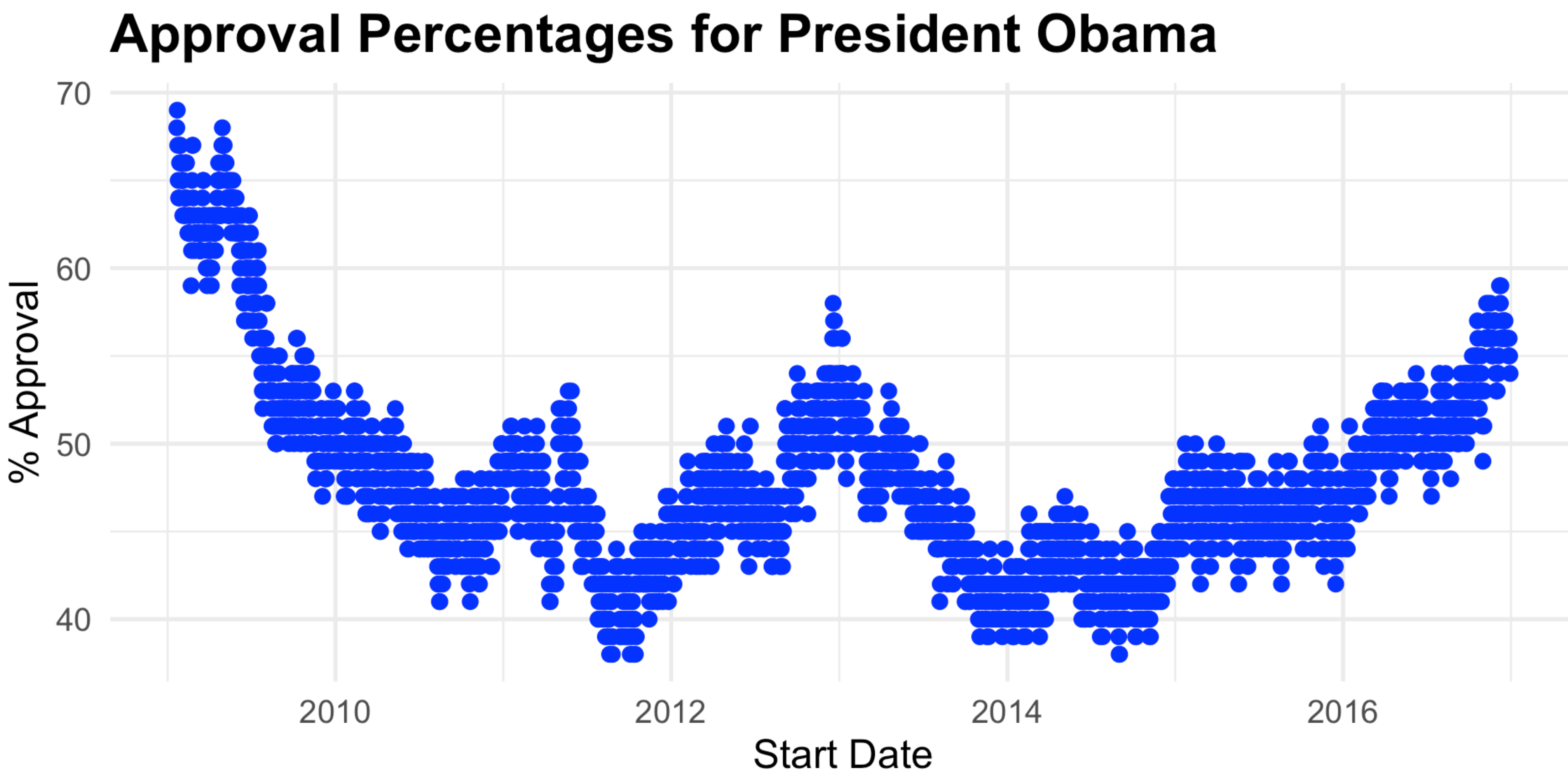
January 12, 2026

Presidential Approval Ratings

Visualization



Approval Ratings for President Obama across his two terms in office. Each point represents a three-day average.

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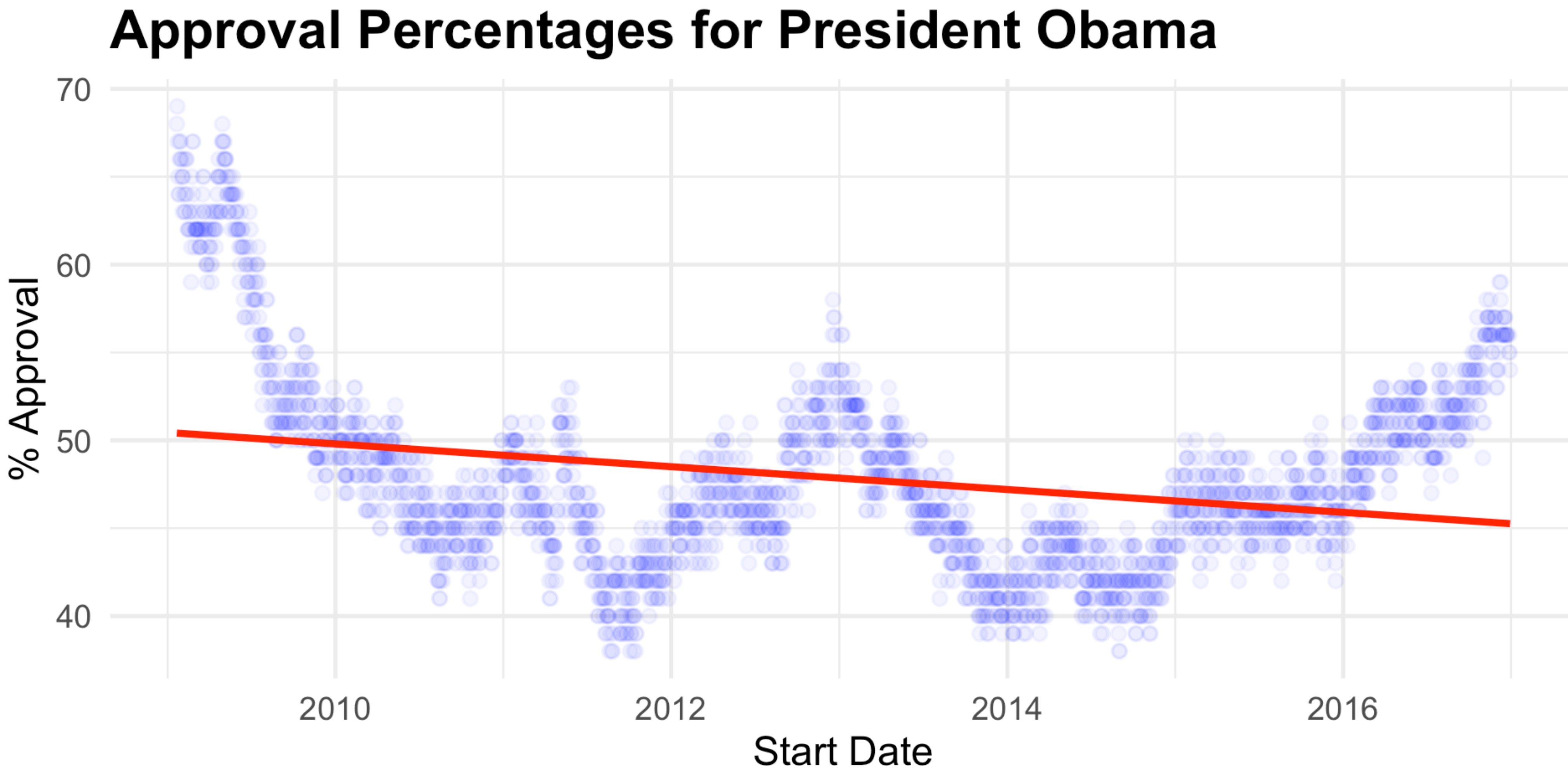
Linear Fit



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Nonparametric Regression

- A **nonparametric regression** problem
  - → I.e. can we estimate the true underlying **signal** function with as few assumptions as possible?
- Equally-spaced univariate data: $\mathcal{D} = \{(t, y_t)\}_{t=1}^{n}$
  - → $y_t =$ approval rating on day *t*
- **Model:** $y_t = f(t) + \omega_t$
  - → **Signal Function** $f(\cdot)$ is to be estimated
- Many pre-existing nonparametric regression techniques assume i.i.d. Gaussian **Noise:** ω_t; for this dataset, however, this is a *poor* assumption!
  - → Approval ratings have been reported as *three-day averages*

Main Ideas

An Overview of Our Procedure

- As such, we seek to develop an estimation procedure under the assumption of *correlated* noise.
- We leverage two main ideas in our approach:
    1. Utilizing a **Bayesian** approach (technically, an **Empirical Bayes**)
    2. Utilizing the **spectral domain** (**Fourier Transforms**)
- With these two ideas, we are able to reduce the estimation problem to a **Generalized Linear Model** (GLM).
- To ensure we're all on roughly equal footing, let's briefly discuss these topics in generality first

Bayesian Estimation

A High-Level Overview

- In a typical (**frequentist**) paradigm, parameters are treated as *deterministic* (i.e. nonrandom)
- Example: $X_1, X_2, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with σ^2 known; then μ, the unknown mean, is treated as fixed
  - → To account for variability across samples, we develop **estimators** (e.g. $\bar{X}$) to estimate the "true value" of μ.
- The Bayesian framework acknowledges that we are coming into our problem with certain *prior beliefs*, and, consequently, treats μ as a *random variable in itself*, following some prespecified **prior distribution**.
  - → Rather than looking at point estimators (like $\bar{X}$), we instead estimate μ with a **posterior distribution** $\pi(\mu \mid X_1, X_2, \cdots, X_n)$.

Bayesian Estimation

An Example

- **Example:** the so-called **normal-normal problem**:

$$\begin{aligned}(X_1, X_2, \cdots, X_n \mid \mu) &\overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2) \\ \mu &\sim \mathcal{N}(\nu, \tau^2)\end{aligned}$$

- With some work, the posterior distribution can be shown to be:

$$(\mu \mid X_1, X_2, \cdots, X_n) \sim \mathcal{N}(\mu_{\text{post}},\ \sigma^2_{\text{post}})$$

$$\begin{aligned}\mu_{\text{post}} &:= \left(\frac{\frac{1}{\tau^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} \right) \nu + \left(\frac{\frac{n}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} \right) \bar{X} \\ \sigma^2_{\text{post}} &:= \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\end{aligned}$$

Bayesian Estimation

An Example

$$\mu_{\text{post}} := \left(\frac{\frac{\sigma^2}{n}}{\tau^2 + \frac{\sigma^2}{n}} \right) \nu + \left(\frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}} \right) \bar{X}$$

- What we see is that the posterior mean is a weighted average of the *prior estimate* (ν) and the *frequentist estimator* ($\bar{X}$), with the weights proportional to the sums of the prior and model variances.
- A smaller prior variance (indicating more *certainty* in our choice of prior) leads to a higher weighting of ν over $\bar{X}$, and vice-versa.
- This assumes σ^2 to be known; if it were not, we could either assign a **hyperprior** (pure Bayesian) or estimate it using data (**Empirical Bayes**)

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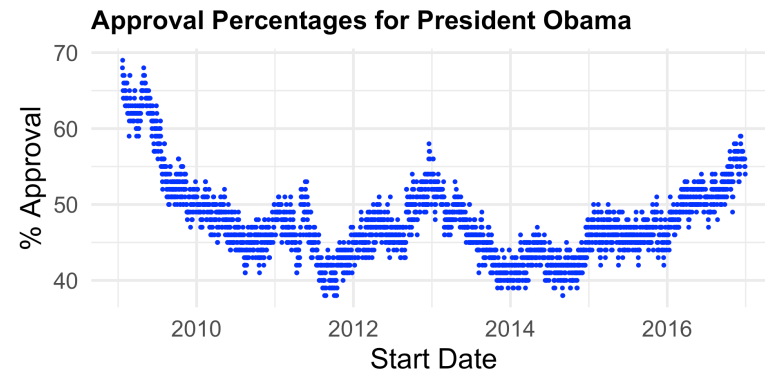
The Model

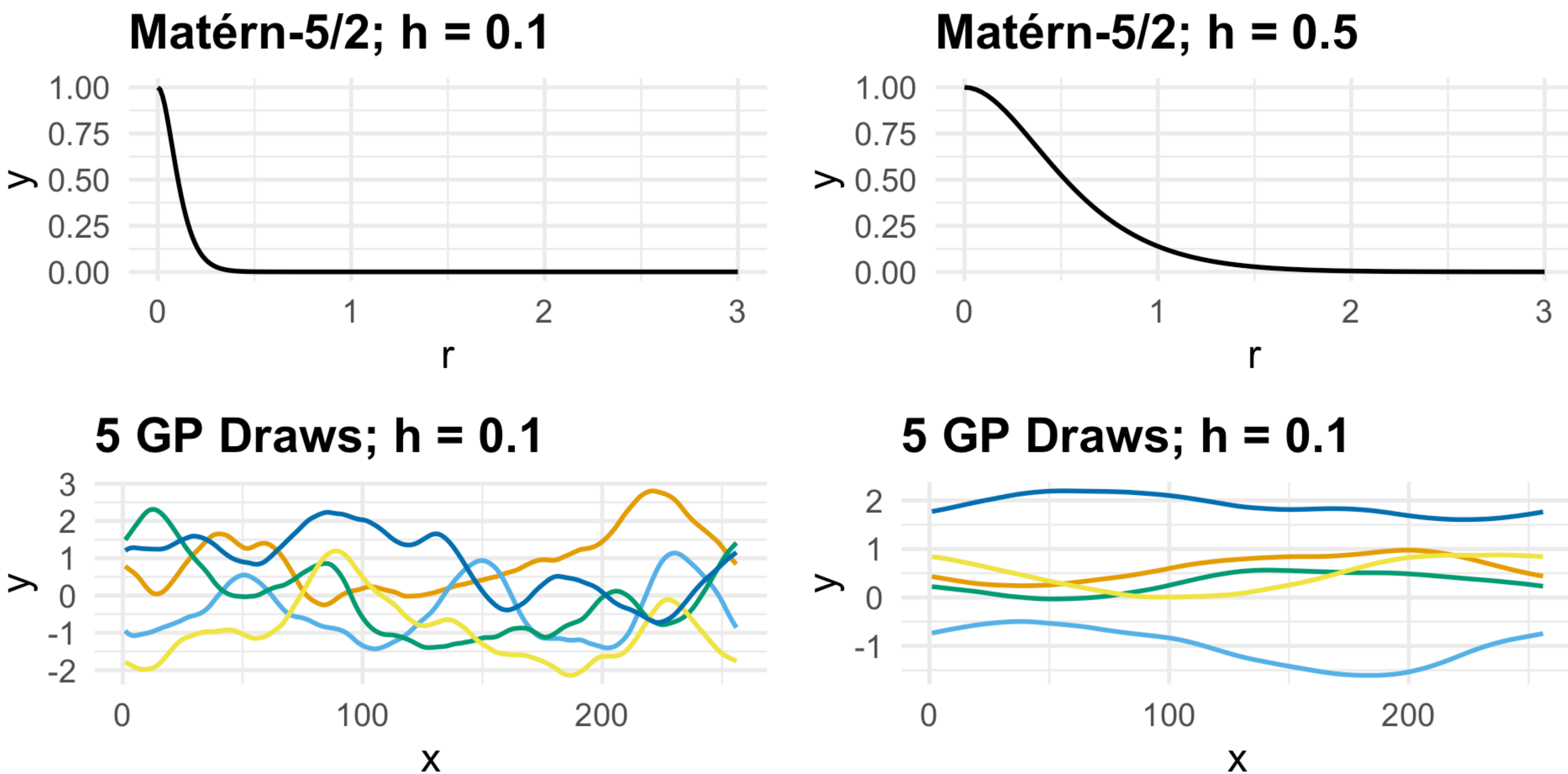
- **Model:** $y_t = f(t) + \omega_t$
  - → $y_t =$ approval rating on day *t*
  - → Signal function $f(\cdot)$ is to be estimated



- Adopting a Bayesian perspective, we assign a prior to the *signal function*
  - → The prior we use is called a **Gaussian Process**; think of it as a distribution such that draws from this distributions are *functions* (as opposed to *random vectors*).
    - ⤞ Prescribed by a mean *function* and a covariance *kernel* (akin to μ and σ^2, respectively)

Gaussian Process

Example Draws



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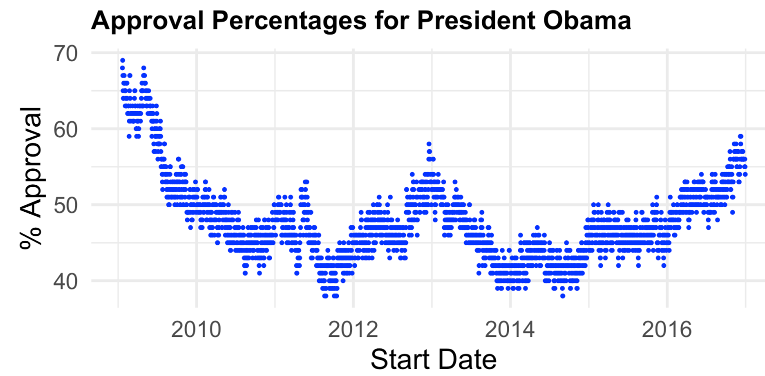

- **Prior:** $f \sim \mathcal{GP}(0,\ \tau^2 C(r))$ where $C(r)$ denotes the **Matérn-5/2 Kernel**

$$C(r) = \left(1 + \frac{r\sqrt{5}}{h} + \frac{5r^2}{3h^2} \right) \exp\left\{ -\frac{r\sqrt{5}}{h} \right\}$$

- **Noise:** **Moving Average** process of order *q*: $\omega_t = \varepsilon_t + \sum_{j=1}^{q} \psi_j \varepsilon_{t-j}$ where $\varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,\ \sigma^2)$

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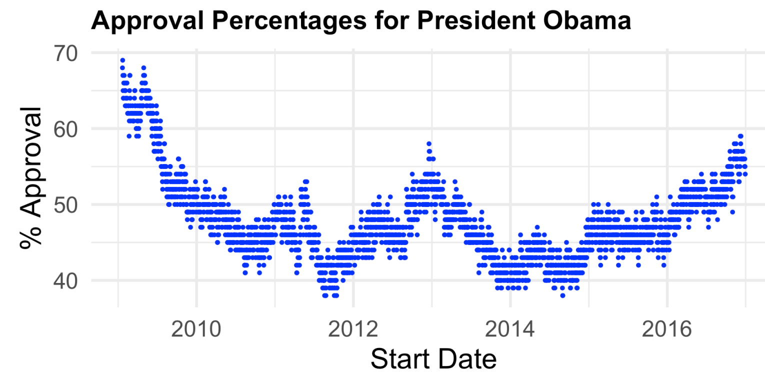

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The Problem

- Note that this model contains three parameters (often called **hyperparameters** in the Bayesian framework, since they are parameters not of particular interest to us in themselves beyond their link to the signal function).
- We adopt an Empirical Bayes framework, opting to estimate these parameters using *data* (as opposed to imposing further *hyperpriors*).
- A key proposition we make is to convert the estimation problem into the **spectral domain**.
  - → This is related to the notion of a **Fourier Transform**

Spectral Domain

- Recall that a vector space can be described using a set of **basis vectors**.
  - → These basis vectors allow any element of the vector space to be expressed as a series of **coordinates**, with respect to the selected basis vectors.
- A **function space** is essentially an extension of a vector space, where elements are now *functions*.
  - → Function spaces can be expressed in terms of **basis functions**, which then allow us to describe functions in term of coordinates wrt. the selected basis *functions*.
- It turns out that most functions can be expressed as some (infinite) combinations of sines and cosines of different periods.
- So, we can consider using a collection of sinusoids to form a basis, with coordinates expressed with respect to these sines and cosines.

Fourier Transforms

Intuition

- This is, in essence, what the **Fourier Transform** represents: a decomposition of a function into a superposition of sines and cosines of differing periods.
- For example, the **Sawtooth Wave**

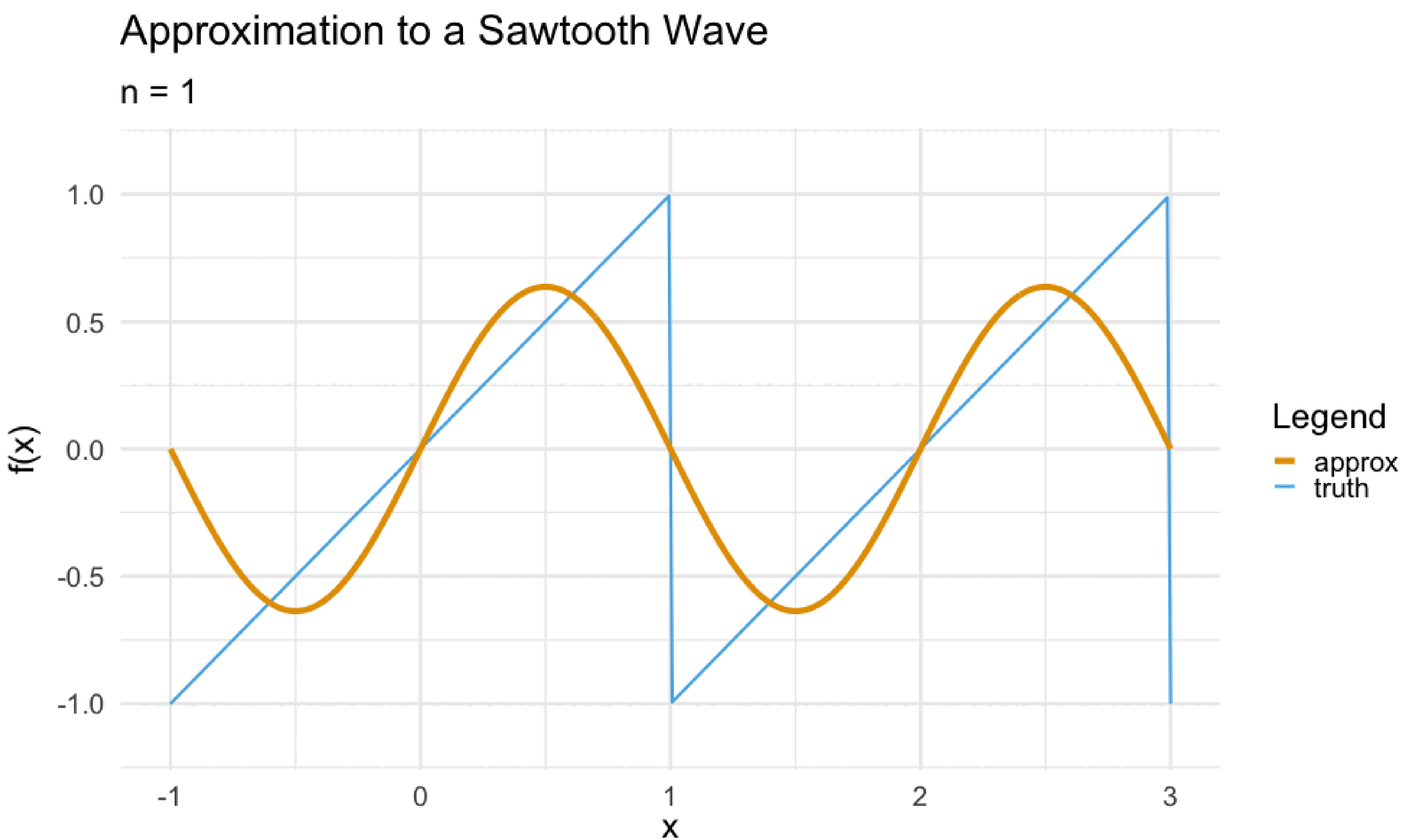
$$f(x) = x - 2 \cdot \mathbf{round}(x/2)$$

admits the following Fourier decomposition:

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x)$$

Fourier Transforms

Example: Sawtooth Wave



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The Model

- **Model:** $y_t = f(t) + \omega_t$
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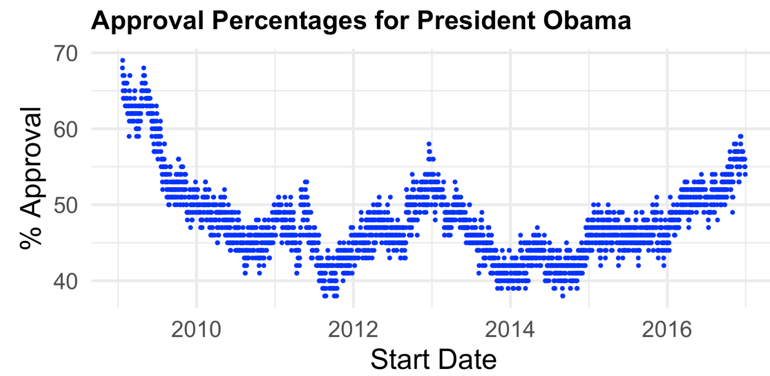
  - → Signal function $f(\cdot)$ is to be estimated



- Let $\{\eta_k\}_{k \geq 0}$ denote the Fourier coefficients of the signal function and let $\{\xi_k\}_{k \geq 0}$ denote the Fourier coefficients of the noise process. By standard results, these will be a collection of independent *complex*-normal random variables:

$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ v_k \mathbf{I}_2\right)$$
$$\xi_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right)$$

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The Model

- **Model:** $y_t = f(t) + \omega_t$
  - → $y_t =$ approval rating on day t
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$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ v_k \mathbf{I}_2\right)$$
$$\xi_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right)$$

- Given appropriate estimators $\hat{\eta}_k$ for η_k, we can construct an estimator for the signal function as

$$\widehat{f}(t) = \sum_{k=-\infty}^{\infty} \hat{\eta}_k e^{-2\pi i k t / n}$$

- Therefore, we're pretty much back to a normal-normal problem!

The Model

Time- and Frequency-Domain Specifications

- **Time-Domain Model:**

  $y_t = f(t) + \omega_t$

  → $f \sim \mathcal{GP}(0,\ \sigma^2 C(\cdot))$

  → $\omega_t = \varepsilon_t + \sum_{j=1}^{q} \psi_j \varepsilon_{t-j}$ where $\varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,\ \sigma_0^2)$

- **Frequency-Domain Model:**

$$(\eta_k + \xi_k) \mid \eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(\eta_k,\ v_k \mathbf{I}_2\right)$$
$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right)$$

- The "best" Bayesian point estimator is the posterior mean:

$$\mathbb{E}[\eta_k \mid \eta_k + \xi_k] = \frac{v_k}{v_k + u_k}(\eta_k + \xi_k)$$

- Standard results tell us that the variances v_k and u_k are related to the **spectral densities** of the signal and noise processes, respectively.

The Model

Frequency-Domain

$$(\eta_k + \xi_k) \mid \eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(\eta_k,\ v_k \mathbf{I}_2\right) \qquad \mathbb{E}[\eta_k \mid \eta_k + \xi_k] = \frac{v_k}{v_k + u_k}(\eta_k + \xi_k)$$
$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right)$$

- For a Matérn-5/2 kernel, $\boldsymbol{v_k \propto \tau^2 h\left(1 + k^2 h^2\right)^{-3}}$
- For an MA(q) noise process, $\boldsymbol{u_k = \frac{\sigma^2}{n}\left[b_0 + \sum_{j=1}^{q} b_j \cos\left(j \cdot \frac{2\pi k}{n}\right)\right]}$

Problems:

- η_k and ξ_k are **unknown** (they are like "population Fourier transforms")
- $\boldsymbol{v_k}$ and $\boldsymbol{u_k}$ depend on the unknown hyperparameters $\boldsymbol{\tau^2}$ (signal variance), $\boldsymbol{h}$ (signal bandwidth), and $\boldsymbol{\sigma^2}$ (noise variance).

The Model

Frequency-Domain

$$(\eta_k + \xi_k) \mid \eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(\eta_k,\ v_k \mathbf{I}_2\right) \qquad \mathbb{E}[\eta_k \mid \eta_k + \xi_k] = \frac{v_k}{v_k + u_k}(\eta_k + \xi_k)$$

$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right)$$

- For a Matérn-5/2 kernel, $v_k \propto \tau^2 h\left(1 + k^2 h^2\right)^{-3}$
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Resolution: Unknown η_k and ξ_k

- Since η_k and ξ_k appear in $\hat{\eta}_k$ only through their *sum*, we can replace their sum with $\check{y}_k$, the **Discrete Fourier Transform** (DFT) of the observed y_t values.

The Model

Frequency-Domain

$$(\eta_k + \xi_k) \mid \eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(\eta_k,\ v_k \mathbf{I}_2\right)$$
$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right)$$

$$\hat{\eta}_k := \left(\frac{v_k}{v_k + u_k} \right) \check{y}_k =: w_k \cdot \check{y}_k$$

- For a Matérn-5/2 kernel, $v_k \propto \tau^2 h \left(1 + k^2 h^2\right)^{-3}$
- For an MA(q) noise process, $u_k = \frac{\sigma^2}{n} \left[b_0 + \sum_{j=1}^{q} b_j \cos\left(j \cdot \frac{2\pi k}{n}\right) \right]$

- We still have the “ultimate” problem of estimating the three hyperparameters τ^2, h, and σ^2.
  → This is a highly *nonlinear* estimation problem, making it somewhat challenging.

The Model

Estimating the Hyperparameters

- Recall that η_k and ξ_k are *complex* random variables, meaning they each have two components (real and imaginary).

$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right) \qquad v_k \propto \tau^2 h\left(1 + k^2 h^2\right)^{-3}$$

$$\xi_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ v_k \mathbf{I}_2\right) \qquad u_k = \tfrac{\sigma^2}{n}\left[b_0 + \textstyle\sum_{j=1}^{q} b_j \cos\left(j \cdot \tfrac{2\pi k}{n}\right)\right]$$

$$\implies (\eta_k + \xi_k) \sim \mathcal{CN}\Big(0,\ (u_k + v_k)\mathbf{I}_2\Big)$$

$$\implies \frac{\|\eta_k + \xi_k\|^2}{2} \sim \text{Exponential}\left(u_k + v_k\right)$$

The Model

Estimating the Hyperparameters

- Recall that η_k and ξ_k are *complex* random variables, meaning they each have two components (real and imaginary).

$$\eta_k \overset{\text{ind}}{\sim} \mathcal{CN}\left(0,\ u_k \mathbf{I}_2\right) \qquad v_k \propto \tau^2 h\left(1 + k^2 h^2\right)^{-3}$$

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$$\underbrace{\frac{\|\eta_k + \xi_k\|^2}{2}}_{:=\theta_k} \sim \text{Exponential}\left[\tau^2 h\left(1 + k^2 h^2\right)^{-3} + \frac{\sigma^2}{n} b_0 + \sum_{j=1}^{q} \frac{\sigma^2}{n} b_j \cos\left(j \cdot \frac{2\pi k}{n}\right)\right]$$

The Model

Estimating the Hyperparameters

$$\underbrace{\frac{\|\eta_k + \xi_k\|^2}{2}}_{:=\theta_k} \sim \mathrm{Exponential}\left[\tau^2 h\left(1 + k^2 h^2\right)^{-3} + \frac{\sigma^2}{n} b_0 + \sum_{j=1}^{q} \frac{\sigma^2}{n} b_j \cos\left(j \cdot \frac{2\pi k}{n}\right)\right]$$

- Again, η_k and ξ_k appear only through their sum, which can be replaced with (estimated by) the DFT of the y_k values, $\check{y}_k$!
- **Proposition:** Fit a **Gamma Generalized Linear Model** (GLM) of the form

$$\frac{\|\check{y}_k\|^2}{2} \sim \mathrm{Exponential}\left[\delta_1\left(1 + k^2 h^2\right)^{-3} + \beta_0 + \sum_{j=1}^{q} \beta_k \cos\left(j \cdot \frac{2\pi k}{n}\right)\right]$$

The Model

Estimating the Hyperparameters

- **Proposition:** Fit a **Gamma Generalized Linear Model** (GLM) of the form

$$\frac{\|\check{y}_k\|^2}{2} \sim \mathrm{Exponential}\left[\delta_1\left(1 + k^2 h^2\right)^{-3} + \beta_0 + \sum_{j=1}^{q} \beta_k \cos\left(j \cdot \frac{2\pi k}{n}\right)\right]$$

- This is still nonlinear in the bandwidth; as such, we propose fitting the GLM for a fixed bandwidth and running a grid search over candidate bandwidth values to find the one that minimizes the model deviance.
  - → We are exploring more sophisticated techniques, like Newton-Raphson; research is ongoing (and perhaps one of you can help with this in the future!)

The Algorithm

1. Obtain the coefficients $\check{y}_k$ of the Discrete Fourier Transform of the y_t values, and use this to compute $\tilde{\theta}_k := (1/2)\|\check{y}_k\|^2$ for $k = 0, \cdots, B$ where $B = \lfloor (n-1)/2 \rfloor$ if n is odd and $B = n/2$ if n is even.
2. Identify the bandwidth $\widehat{h}$ that minimizes the deviance of the following Gamma Generalized Linear Model:

$$\tilde{\theta}_k \sim \mathbf{Exponential}\left[\delta_1 \left(1 + k^2 h^2\right)^{-3} + \beta_0 + \sum_{j=1}^{q} \beta_j \cos\left(j \cdot \frac{2\pi k}{n}\right)\right]$$

3. Utilizing the optimal bandwidth from step (2) above, refit the Gamma GLM and obtain fitted values $\hat{\theta}_k$, $k = 1, \cdots, B$ for θ_k. Estimates for v_k are computed as

$$\hat{v}_k := \hat{\theta}_k - \sum_{j=0}^{q} \widehat{\beta_j} \cos(2\pi jk/n)$$

The Algorithm (cont'd)

4. Multiply the $\check{\boldsymbol{y}}_k$ terms from step 1 by weights

$$\widehat{w}_k := \left(\frac{\hat{v}_k}{\hat{\theta}_k} \right)$$

taking care to not downweight the zeroth frequency $\check{\boldsymbol{y}}_{\boldsymbol{0}}$.

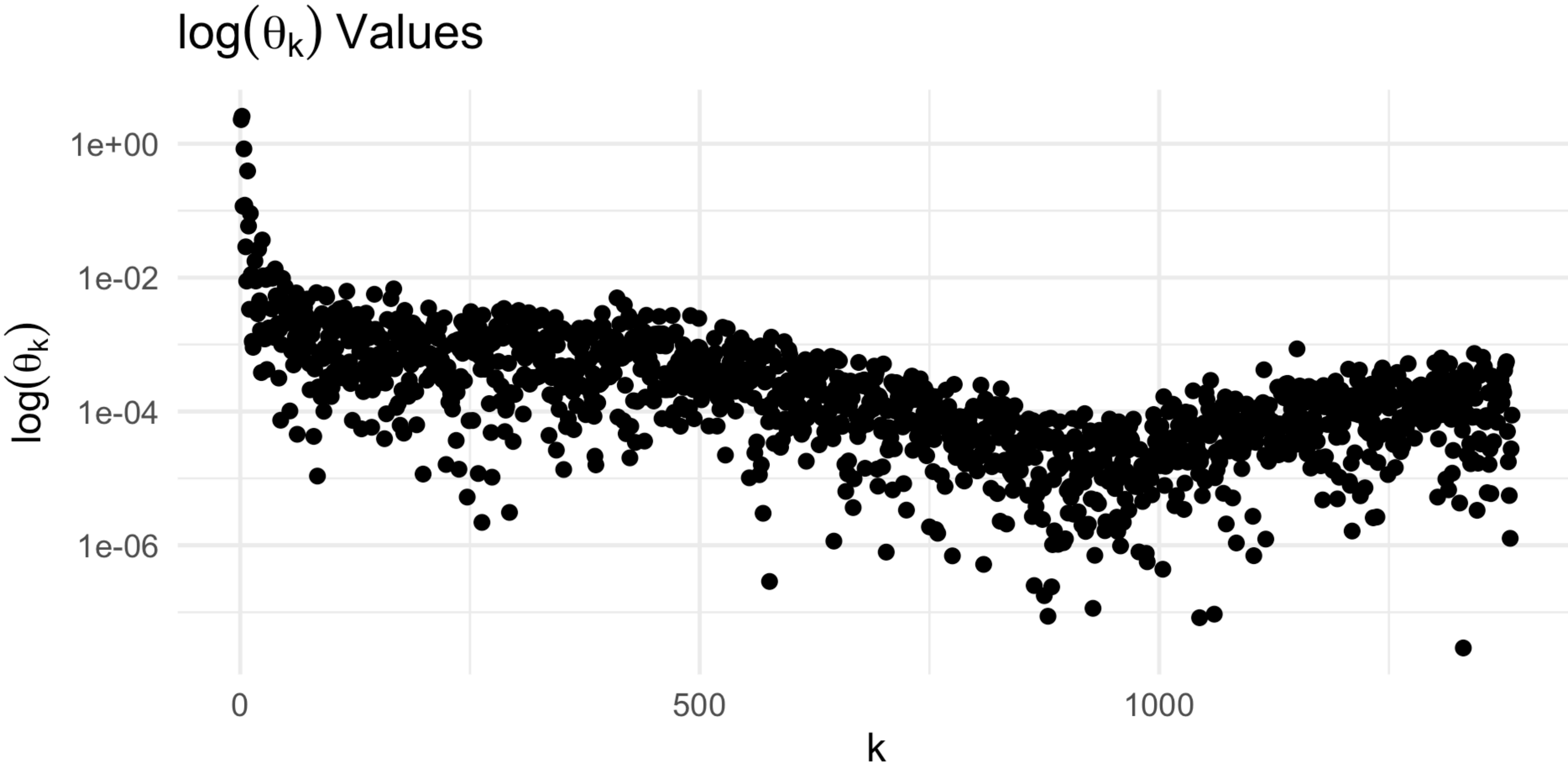
5. Apply the inverse Fourier transform to the weighted coefficients from step (4) above to obtain fitted values $\widehat{\boldsymbol{y}}$.

- Let us explore the application of this algorithm to the Presidential Approval Ratings dataset from the beginning of this presentation.

Presidential Approval Ratings

Step 1

▶ Code



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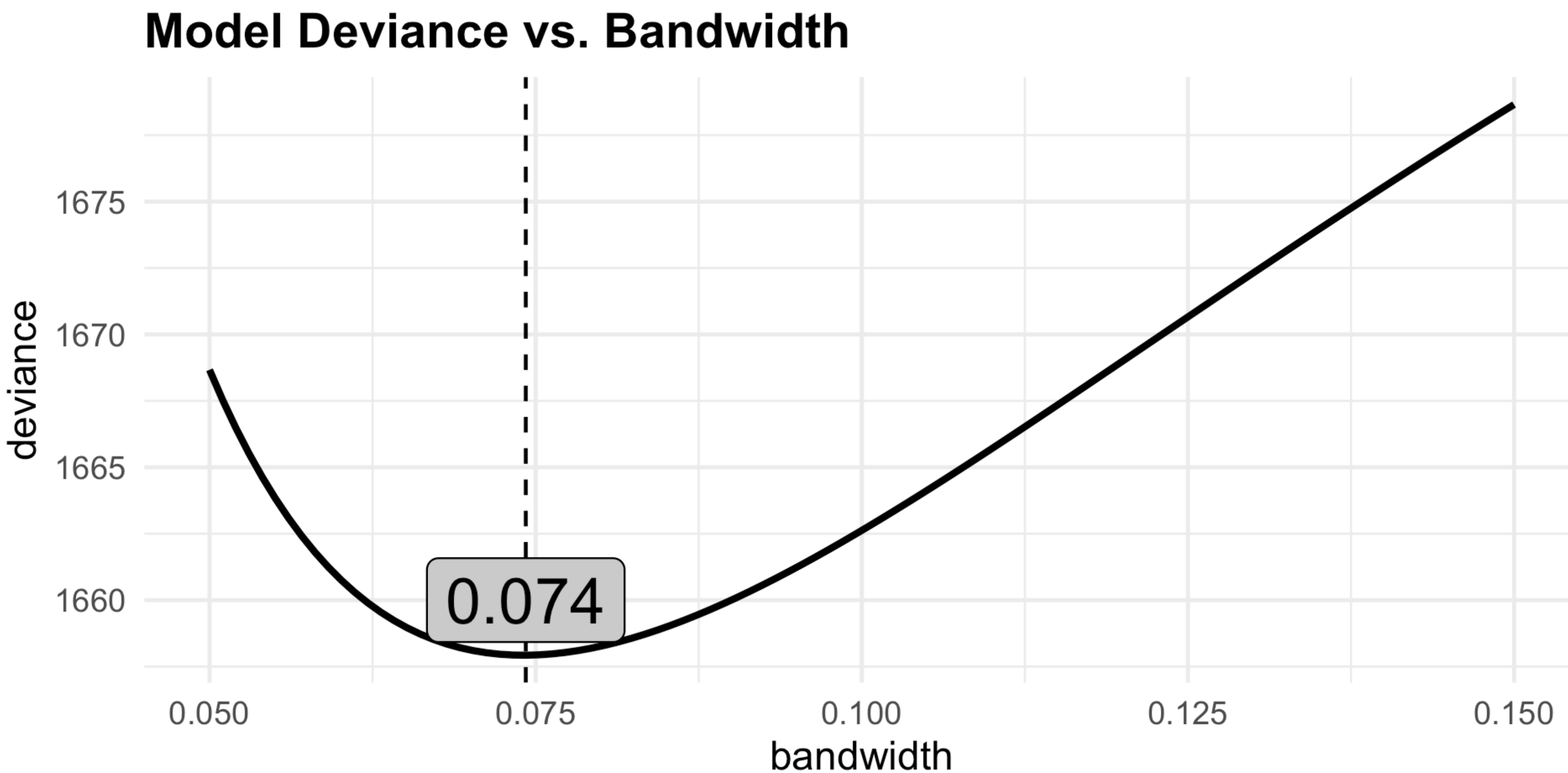
Step 2

▶ Code



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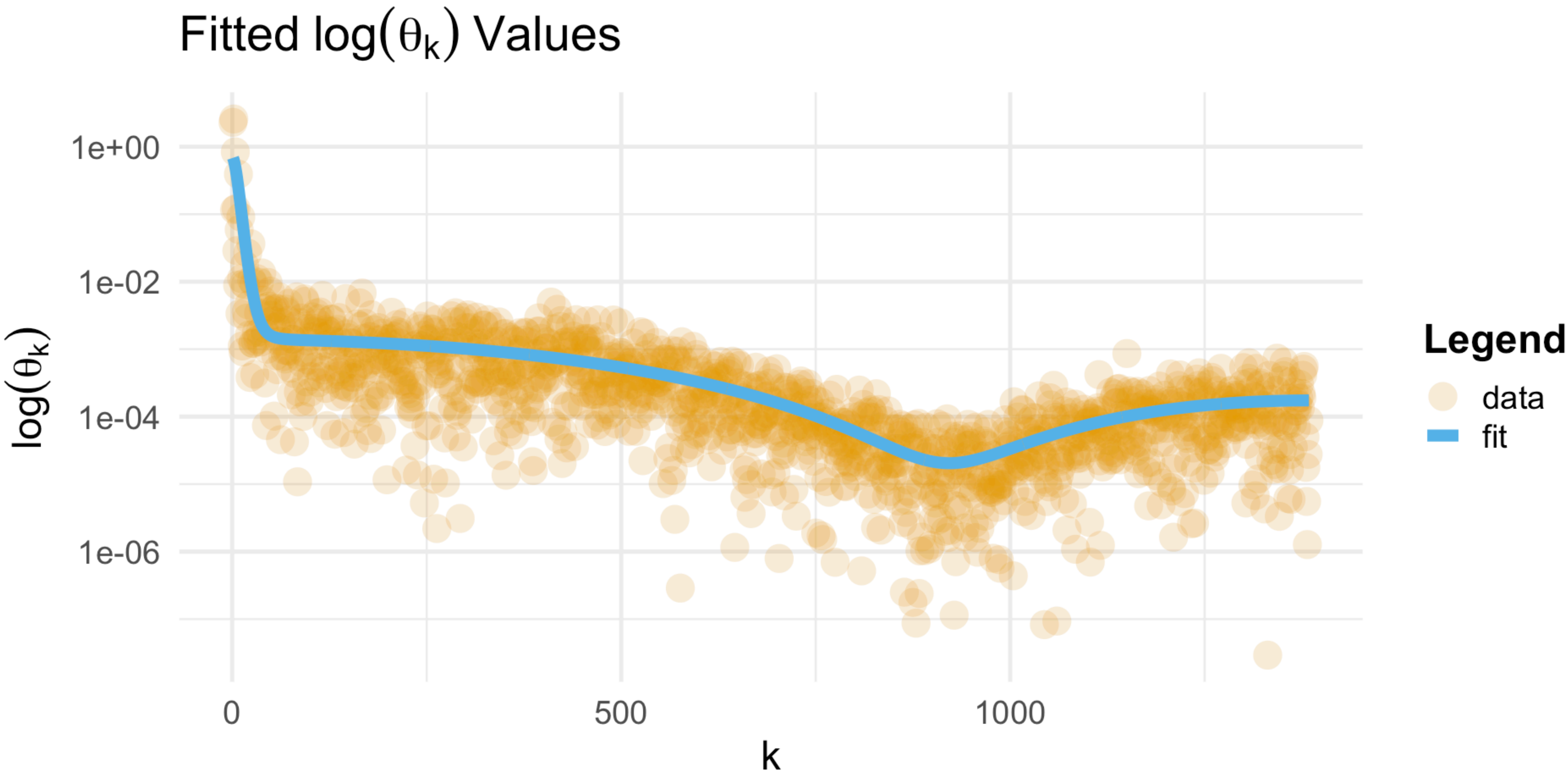
Step 3



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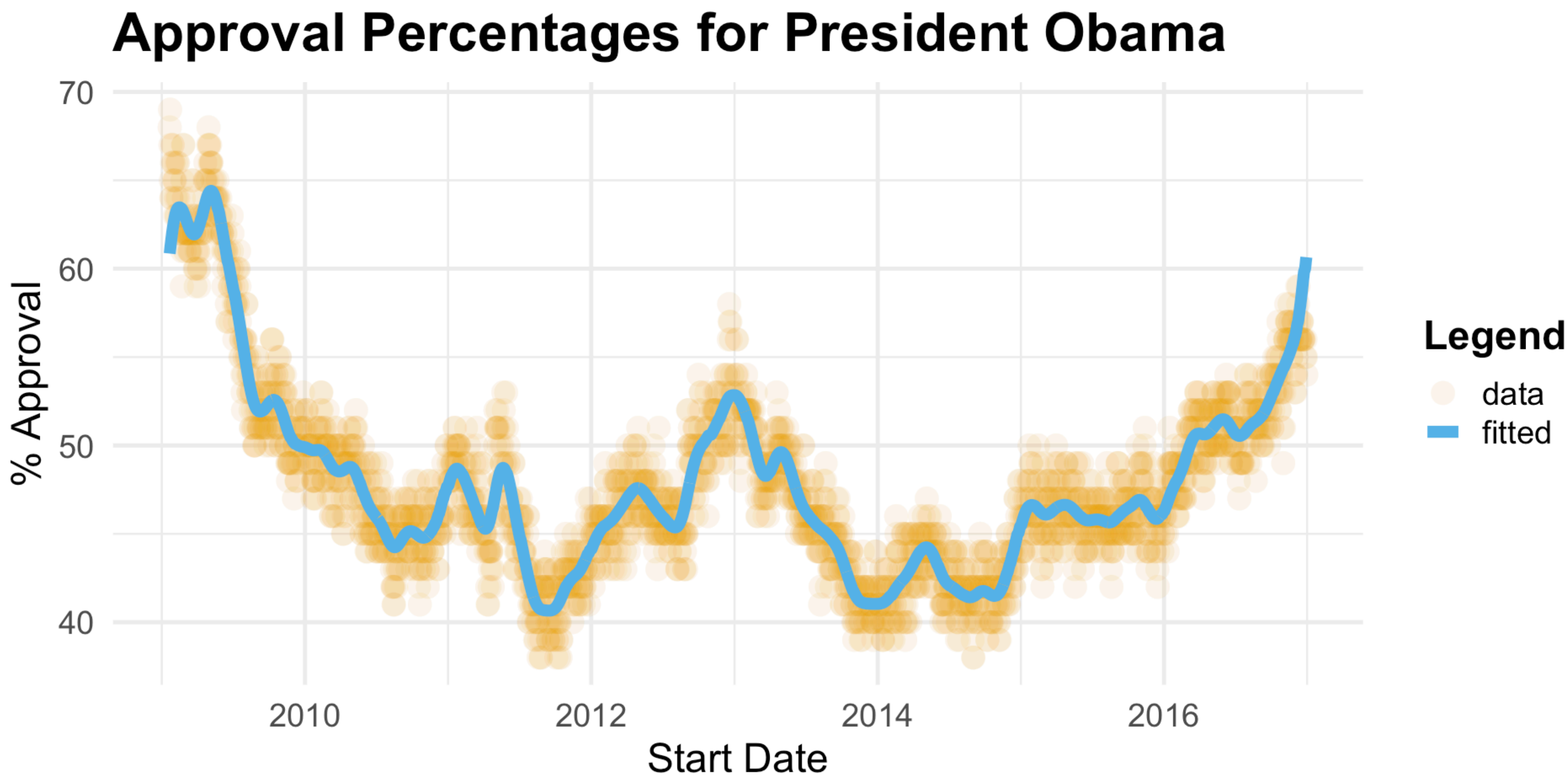
Steps 4 and 5

▶ Code



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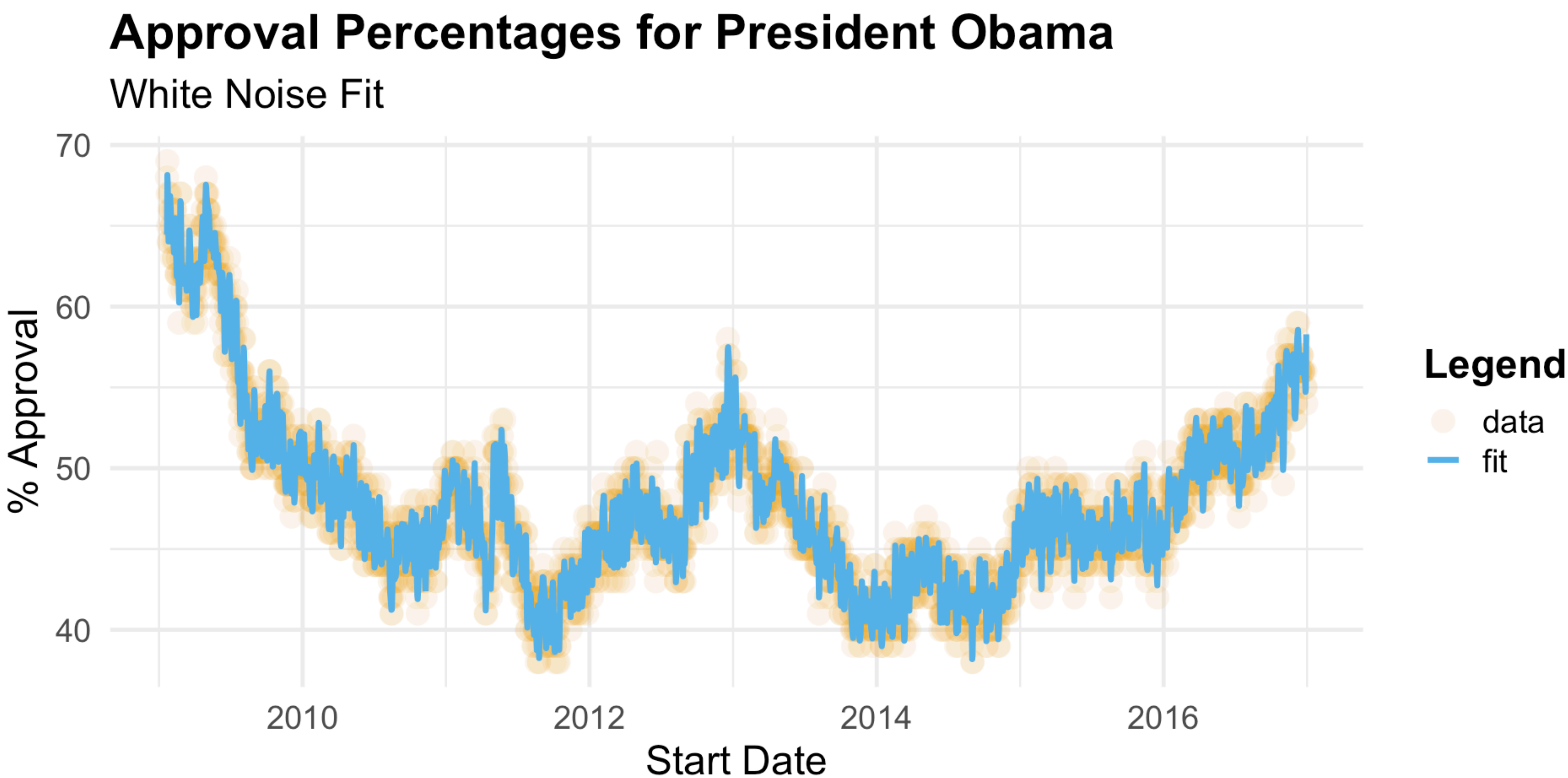
White Noise Fit



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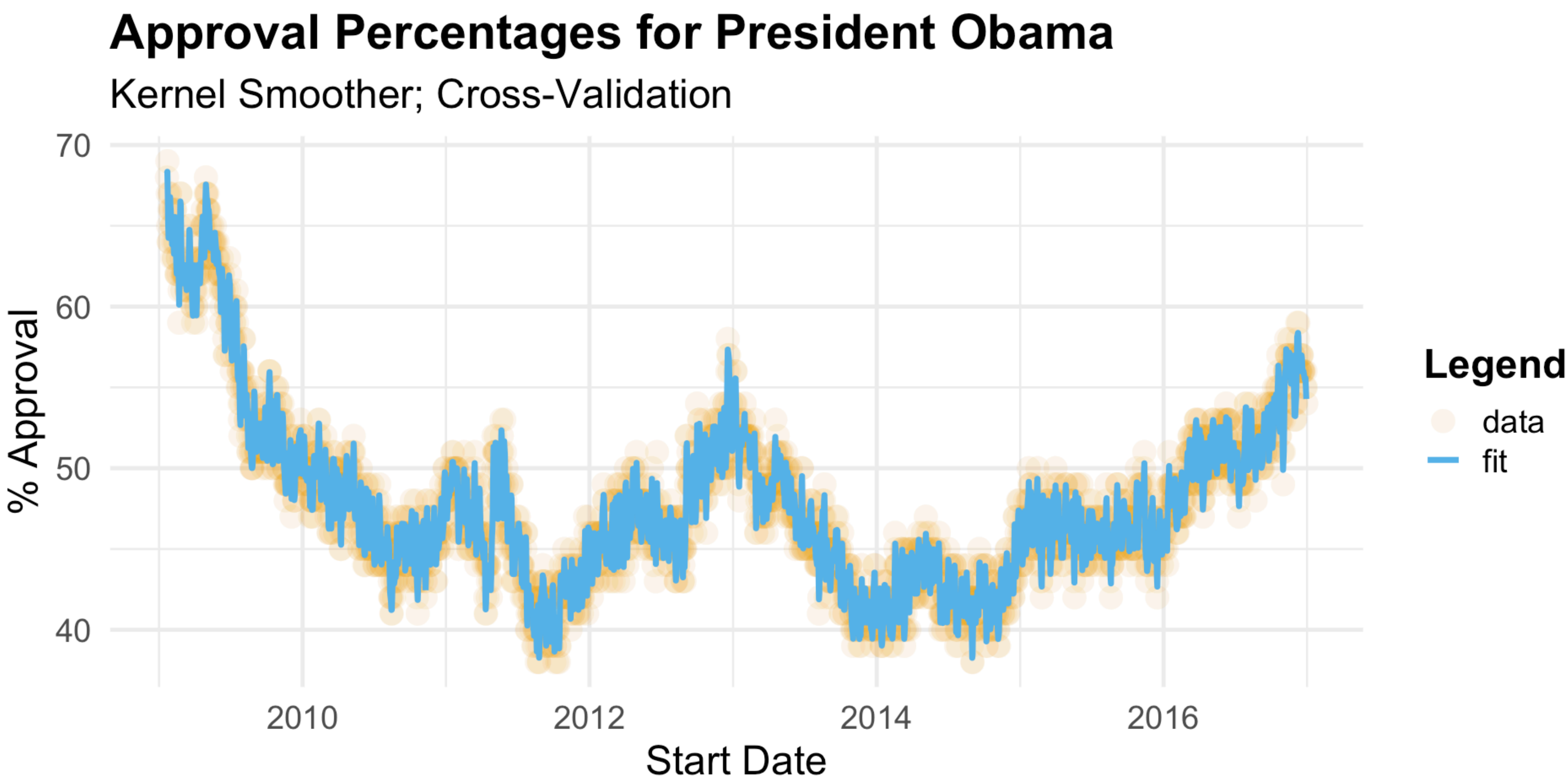
Kernel Smoothing Fit



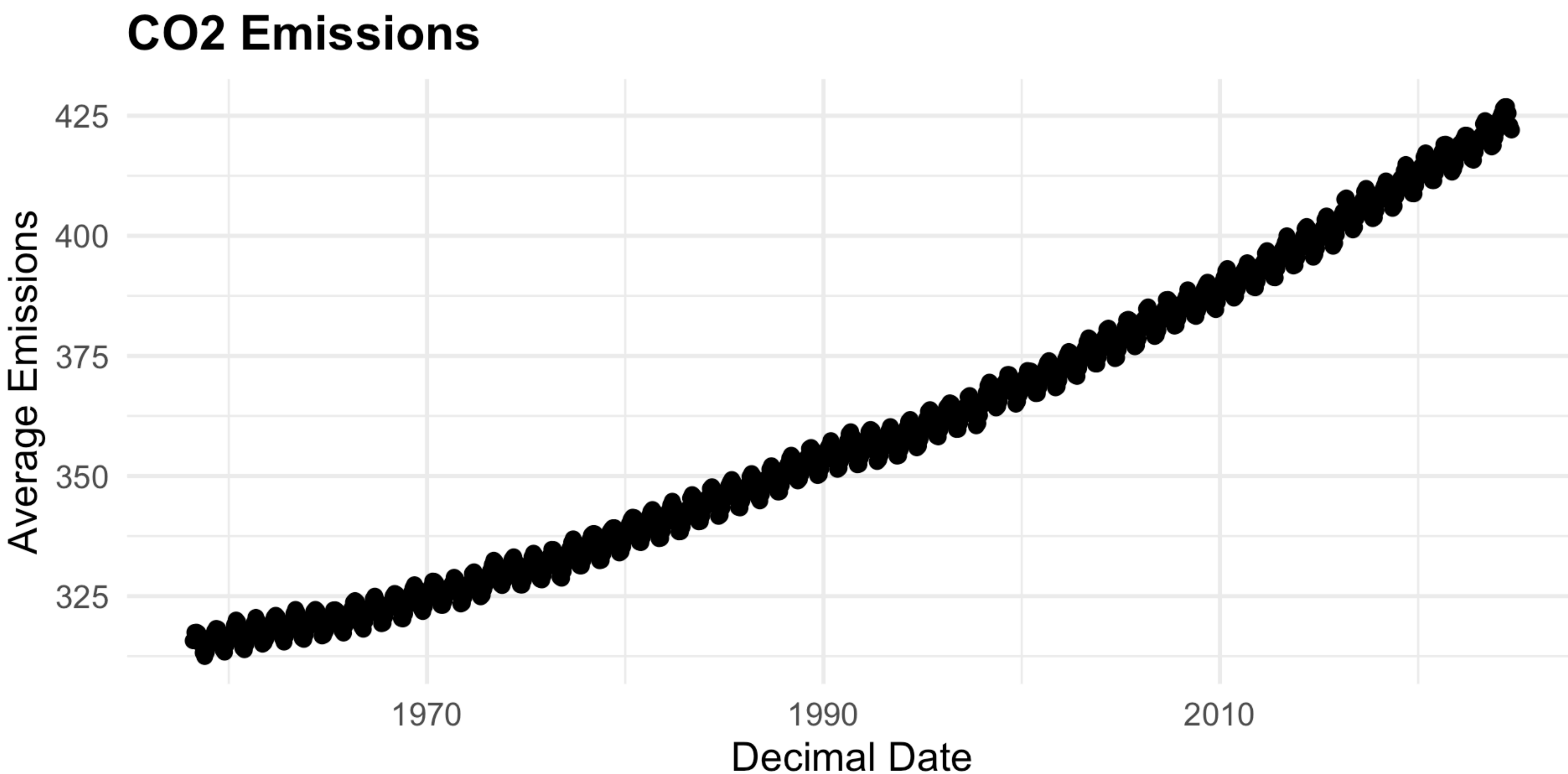
Trends

An Overview

- Fourier-based methods assume an inherent amount of *periodicity* in the signal function.
  - → As such, most Fourier-based estimators suffer from "edge effects" where the starting and ending fitted values are constrained to be equal, even if this is not desired.
- Our algorithm, admittedly, is no exception, which poses an additional challenge for datasets with a noticeable trend.
  - → As a somewhat "famous" example, we consider the CO_2 Emissions recorded at the Mauna Loa observatory in Hawai'i.

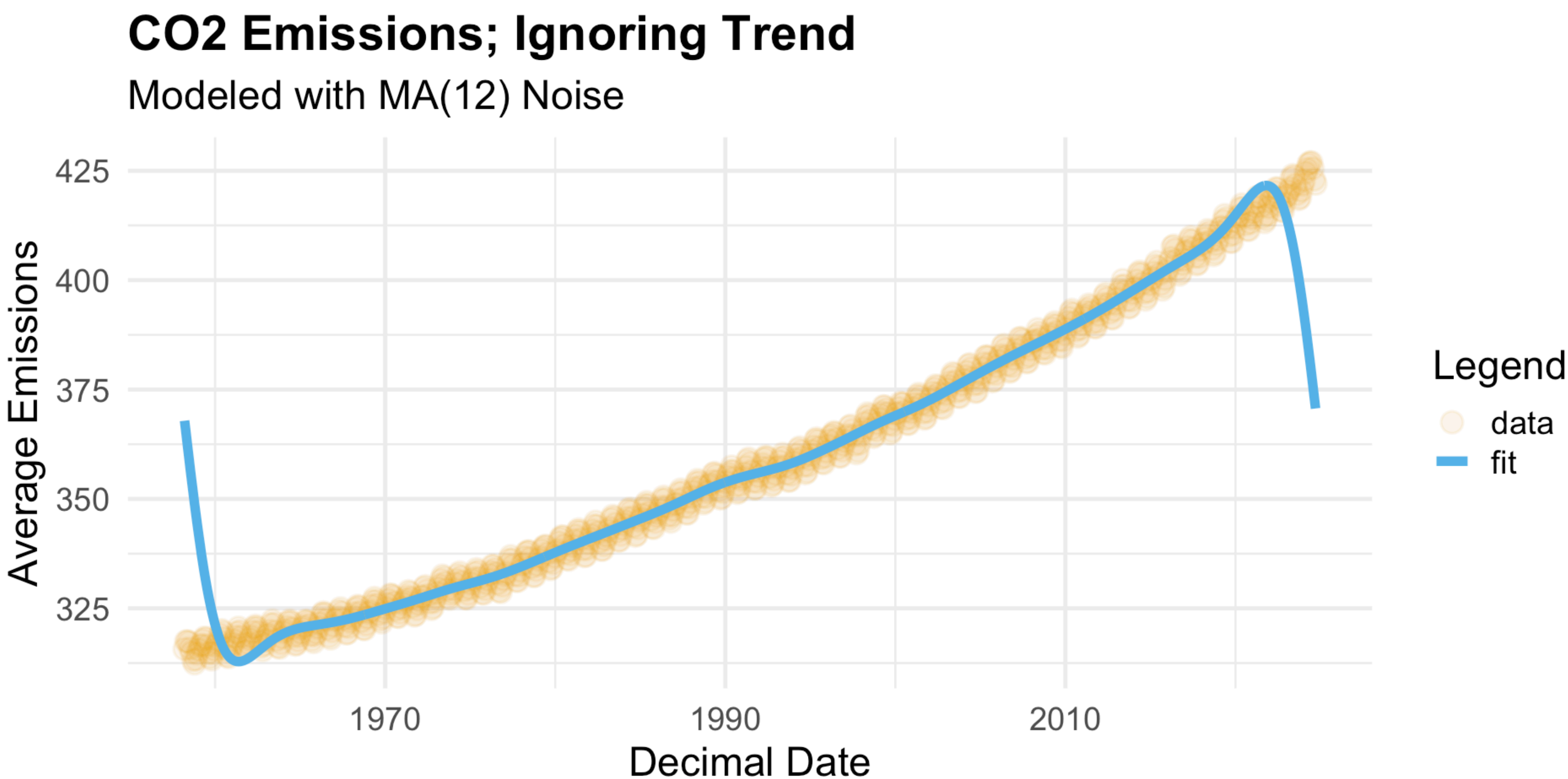
CO_2 Dataset

Visualization



CO_2 Dataset

Initial Fit



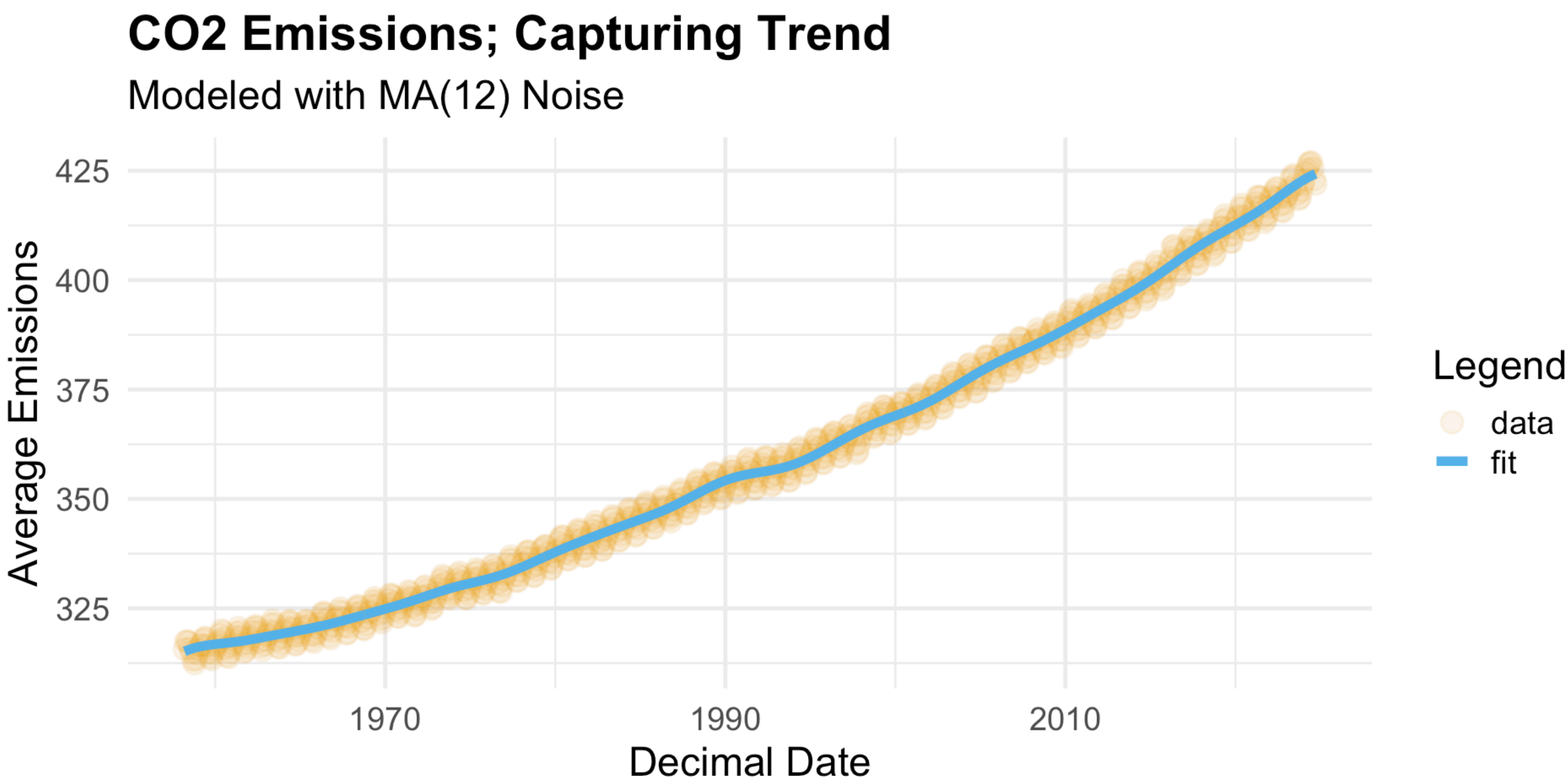
Trends

Our Approach

- Our current approach is to first remove the linear portion of the trend by way of a **simple linear regression**, and then apply our algorithm to the resulting residuals.
- Admittedly, this induces some heteroskedasticity which must be accounted for.
    - → Preliminary results indicate that the magnitude of this heteroskedasticity is relatively small, though, and can be ignored.
    - → A future direction of research that is of interest to us is finding a more systematic way to handle this heteroskedasticity.

CO_2 Dataset

Optimal Fit



Credible Intervals

A Work In Progress

- Finally, as a statistician, I would be remiss to not at least *mention* credible intervals.
- We are currently in the process of finalizing our treatment of credible intervals for our predicted signal values $f(t)$.
  - → We anticipate our treatment to be relatively sparse, with the intention of pursuing a more rigorous framework in future work.
- A preprint (containing further details of our algorithm and also our work on credible intervals) is expected to be completed in the coming months.

Conclusion

Summary

- In conclusion, we propose an Empirical Bayes estimation procedure for use in models with correlated data, with a Gaussian Process prior.
- A key aspect of our estimation procedure is the utilization of the spectral domain to carry out the estimation of necessary hyperparameters.
  - → This allows the hyperparameter estimation to be carried out by way of an Exponential Generalized Linear Regression, which is fast and efficient.
  - → Final fitted values are found by weighting the original spectrum, and inverting the Fourier transform.
- In cases where a noticeable trend is present, our algorithm can be applied to the residuals resulting from a Simple Linear Regression.

Conclusion

Future Work

- Three main avenues of future work immediately present themselves.
- From a computational standpoint, we would like to adopt a slightly more sophisticated procedure for estimating the bandwidth (beyond a simple grid-search).
- We would also like to explore the notion of credible intervals as they pertain to our final fitted values; work on this is ongoing.
  - → A quick note: we are also planning on exploring the asymptotics of our algorithm in the coming months.
- I would also like to explore whether it is feasible to extend our algorithm to *bivariate* data (though it is unclear whether this is feasible or not).

Acknowledgments

- I am deeply grateful for the tutelage of my advisor, Dr. Andrew Carter, and the other members of my doctoral committee (Dr. Yuedong Wang and Dr. Mengyang Gu)
- The Presidential Approval Ratings dataset has been accessed through the American Presidency Project at the University of California, Santa Barbara
- The CO_2 dataset has been accessed through the Global Monitoring Laboratory, a part of the National Oceanic and Atmospheric Administration
- This slide deck has been created using Quarto, in the RStudio IDE. Graphics have been produced with the `ggplot2` package, with colorblind-friendly Okabe-Ito Palette functionality added through the `ggokabeito` package.
  - → Slide icons have been sourced from `fontawesome`, and have been integrated into this slide deck using the `fontawesome` Quarto Extension.
  - → The base theme for this slide deck is the Quarto Clean Theme