Explanation of Code for Deyan

In the paper (that I will reference in the email) there is a statement that modes with long wavelengths (typically longer than the time taken for the transition to take place) ought to obey perturbations that take a form like spherical Bessel functions. In the case where we have shorter wavelengths we can use a WKB like approach. In my code I call these two functions ⇒

h_instantaneous

h_wkb_uniform

I will include a brief breakdown of each function.

h_ instantaneous appears like so

for $\tau < \tau_{eq}$ — conformal time

spherical bessel

$$h\_\text{instantaneous} = j_0(k\tau)$$

for $\tau > \tau_{eq}$

$$h\_\text{instantaneous} = \frac{\tau_{eq}}{\tau}\left[A j_1(k\tau) + B y_1(k\tau)\right]$$

A, B are expressions dependent on k, τ_{eq}

$$A = \frac{\frac{3}{2} k\tau_{eq} - \frac{1}{2} k\tau_{eq}\cos(2k\tau_{eq}) + \sin(2k\tau_{eq})}{k^2 \tau_{eq}^2}$$

$$B = \frac{2 - 2k^2\tau_{eq}^2 - 2\cos(2k\tau_{eq}) - k\tau_{eq}\sin(2k\tau_{eq})}{2k^2\tau_{eq}^2}$$

Which is exactly in the ref. paper page 12.

u_wkb_uniform

~~u_wkb_uniform~~ is informed by Arthur's computation for the ~~ston~~ hyperspherical bessel function. This calculation is based on a couple things:

$\epsilon = 1$ in our calc, different from slow-roll

we are effectively solving $\epsilon^2 y'' = Q(x) y$

In our case,

$$Q(x) = -1 + \frac{H_0^2}{2k^2}\left(\Omega_{mat} a^{-1} + 4\Omega_\Lambda a^2\right)$$

Our turning point must occur when $x=0$ such that x_* satisfies

$$\frac{2k^2}{H_0^2} = \Omega_{mat}\, a(x_*)^{-1} + 4\Omega_\Lambda\, a(x_*)^2$$

and our scale factor at the turning point appears as,

$$a_* = a(x_*) \cong \frac{1}{2}\Omega_{mat}\frac{H_0^2}{k^2}$$

We can then use a single function that uniformly approximates u_K ~~and~~ where $S_0(x) = \int_0^x \sqrt{Q(t)}\, dt$ in,

$$u_K(y) \cong 2\sqrt{\pi}\, C\left(\frac{3 S_0(y)}{2}\right)^{1/6}(Q(y))^{-1/4} Ai\left(\frac{3 S_0(y)}{2}\right)^{2/3}$$

I did attempt a third function that in theory would produce the expected h^2 vs k for both long and short wavelengths. This is under the assumption that the condition that marks regimes occurs when $\frac{k}{H_0} = 1.55$.

I am not 100% yet convinced of that and I realize I need to divide u_wkb_uniform by scale factor a. This function is called combined_h