Modes of an arbitrary free scalar field φ have the equation of motion

$$\varphi'' + 2\left(\frac{a'}{a}\right)\varphi' - \vec{\nabla}^2\varphi = 0 \qquad (122)$$

with primes derivatives with respect to conformal time τ.
Gravitational waves obey this evolution equation.
For comoving wavevectors $\vec{k}$, mode equation is

$$\varphi_k'' + 2\left(\frac{a'}{a}\right)\varphi_k' + k^2\varphi_k = 0 \qquad (126)$$

or changing to $u_k \equiv a(\tau)\,\varphi_k(\tau)$,

$$u_k'' + \left(k^2 - \frac{a''}{a}\right)u_k = 0. \qquad (128)$$

$$\frac{a''}{a} = \frac{4\pi G}{3}(\rho - 3P)$$

Now change dependent variables to dimensionless $x = k\tau$.

$$dx = k\,d\tau$$

$$\frac{d}{d\tau} = k\frac{d}{dx}$$

$$u_k'' + \left(1 - \frac{a''}{a}\right)u_k = 0$$

where now primes are $\frac{d}{dx}$.

$$\frac{d^2 u_k(x)}{dx^2} + \left(1 - \frac{1}{a}\frac{d^2 a}{dx}\right) u_k(x) = 0$$

This is a Schrodinger-type equation, with a turning point at $\frac{d^2 a}{dx^2} = a$.

$$\frac{d^2 u_k}{dx^2} + \underbrace{\left(1 - \frac{H_0^2}{2k^2}\left(\Omega_{mat}\, a^{-1} + 4\Omega_\Lambda a^2\right)\right)}_{-Q_k(x)} u_k = 0.$$

Turning point at x_* which satisfies

$$\frac{2k^2}{H_0^2} = \Omega_{mat}\, a(x_*)^{-1} + 4\Omega_\Lambda a(x_*)^2$$

Right side has a minimum at

$$-\Omega_{mat}\, a^{-2} + 8\Omega_\Lambda a = 0$$

$$a^3 = \frac{\Omega_{mat}}{8\Omega_\Lambda}$$

$$a_{min} = \left(\frac{\Omega_{mat}}{8\Omega_\Lambda}\right)^{1/3} = 0.38 \text{ for } \Omega_\Lambda = 0.7$$

When $a = 1$, right side is $\Omega_{mat} + 4\Omega_\Lambda$.

If $k^2 > \frac{1}{2} H_0^2 (\Omega_{mat} + 4\Omega_\Lambda)$ then only one turning point in the interval $0 < a < 1$.

So only consider modes with

$$\frac{k}{H_0} > \frac{1}{2}(\Omega_{mat} + 4\Omega_\Lambda) \approx 1.55$$

The turning point satisfies: $a_* \equiv a(x_*)$

$$4\Omega_\Lambda a_*^3 - 2\frac{k^2}{H_0^2} a_* + \Omega_{mat} = 0$$

$$a_*^3 - \frac{1}{2\Omega_\Lambda}\frac{k^2}{H_0^2} a_* + \frac{\Omega_{mat}}{4\Omega_\Lambda} = 0,$$

This should have a total of 3 real roots with 1 between

$0 < a_* < a_{min}$. Vieta's formula gives

$$a_* = 2\frac{k}{H_0}\frac{1}{\sqrt{6\Omega_\Lambda}} \cos\left[\frac{1}{3}\arccos\left(-\frac{3\sqrt{6}}{4}\frac{H_0^3}{k^3}\Omega_{mat}\Omega_\Lambda^{1/2}\right) - \frac{2\pi}{3} j\right],$$

$j = 0, 1, 2$

$$\frac{3\sqrt{6}}{4}\Omega_{mat}\Omega_\Lambda^{1/2} = 0.46$$

$$\frac{2}{\sqrt{6\Omega_\Lambda}} = 0.98$$

$$a_* = 0.98\frac{k}{H_0}\cos\left[\frac{1}{3}\arccos\left(-0.46\frac{H_0^3}{k^3}\right) - \frac{2\pi}{3} j\right], \quad j = 0, 1, 2$$

For $\frac{k}{H_0} > 1.55$, which j gives a_* between 0 and a_{min}? $\boxed{j=1}$

For $k/H_0 = 1.55$, $a_* = 0.063$

$\epsilon = 0.46\frac{H_0^3}{k^3}$ $\quad \cos\left[\frac{1}{3}\arccos(-\epsilon) - \frac{2\pi}{3}\right] \sim \frac{\epsilon}{3}$ ✓

So the turning point is at $x = x_*$ such that

$$a(x_*) = \frac{2}{\sqrt{6\Omega_\Lambda}} \frac{k}{H_0} \cos\left[\frac{1}{3}\arccos\left(-\frac{3\sqrt{6}}{4}\Omega_{mat}\Omega_\Lambda^{1/2}\frac{H_0^3}{k^3}\right) - \frac{2\pi}{3}\right]$$

If we only consider modes with one turning point for $0 < a < 1$, $\frac{k}{H_0} > 1.55$. Arg of $\arccos(-\epsilon)$ is

$$\epsilon < \frac{3\sqrt{6}}{4}\Omega_{mat}\Omega_\Lambda^{1/2}\frac{8}{(\Omega_{mat}+4\Omega_\Lambda)^3}$$

$$= 0.12$$

$$\arccos(-\epsilon) \sim \frac{\pi}{2} + \epsilon + \frac{\epsilon^3}{6} \quad , \quad \epsilon \to 0$$

so approximating as $\frac{\pi}{2} + \epsilon$ is good to 1% in our range.

$$a(x_*) \approx \frac{2}{\sqrt{6\Omega_\Lambda}} \frac{k}{H_0} \cos\left[\frac{\sqrt{6}}{4}\Omega_{mat}\Omega_\Lambda^{1/2}\frac{H_0^3}{k^3} - \frac{\pi}{2}\right]$$

$$\sim \frac{2}{\sqrt{6\Omega_\Lambda}} \frac{k}{H_0} \frac{\sqrt{6}}{4}\Omega_{mat}\Omega_\Lambda^{1/2}\frac{H_0^3}{k^3}$$

$$\boxed{a(x_*) \simeq \frac{1}{2}\Omega_{mat}\frac{H_0^2}{k^2}}$$

Given the single turning point $0 < x_* < 1$, we can then approximate the solution by the first-order uniform asymptotic approximation. Let $y = x_* - x$

so that $Q_k(y) > 0$ for $y > 0$.

$$\frac{d^2 u_k}{dy^2} = Q_k(y)\, u_k$$

$$u_k(y) \simeq 2\sqrt{\pi}\, C \left(\frac{3S_0(y)}{2}\right)^{1/6} (Q(y))^{-1/4} Ai\left(\left(\frac{3S_0(y)}{2}\right)^{2/3}\right),$$

$$S_0(y) \equiv \int_0^y |Q(y)|^{1/2}\, dy$$

This uniform solution matches the asymptotic solutions at $y=0$, $y \to \pm\infty$. (Need to characterize the size of the error.)

y will range up to $+x_*$ (corresponding to $x=0$, $\tau=0$ which we choose to correspond to the end of inflation) and down to $x_* - x_0$ corresponding to $x = x_0$, $\tau = \tau_0$ today.

$\tau_0 = 3.27\, H_0$ (for $\Omega_\Lambda = 0.69$, $\Omega_{mat} = 0.31$)

$x_0 = 3.27\left(\frac{H_0}{k}\right)$.

Since $Q_k(x) = \frac{1}{a}\frac{d^2a}{dx^2} - b$

In the $k \to 0$ limit, $Q_k(x) \to \frac{1}{a}\frac{d^2a}{dx^2}$

and

$$\frac{d^2u_k}{dx^2} \sim \frac{1}{a}\frac{d^2a}{dx^2}\,u_k$$

Thus $u_k = Ca$ is a solution, $\varphi_k = \frac{u_k}{a} = C$.

So while a mode is outside the horizon, its amplitude will be approx. constant.

We can normalize the WKB solution this way. If φ_{ki} is the post-inflation field amplitude,

$$\lim_{y \to x_r} \frac{u_k(y)}{a(y)} = \varphi_{ki}.$$

or $$\lim_{a \to 0} \frac{u_k(a)}{a} = \varphi_{ki}$$

$$S_0(y) \equiv \int_0^y |Q(y')|^{1/2}\, dy'$$

$$= \int_0^y \left| \frac{H_0^2}{2k^2}\left(\frac{\Omega_{mat}}{a} + 4\Omega_\Lambda a^2 \right) - 1 \right|^{1/2} dy$$

$$= \int_{a_*}^{a(y)} \left| \frac{H_0^2}{2k^2}\left(\Omega_{mat}\, a^{-1} + 4\Omega_\Lambda a^2 \right) - 1 \right|^{1/2} \frac{dy}{da}\, da$$

$$= -k \int_{a_*}^{a(y)} \left| \frac{H_0^2}{2k^2}\left(\Omega_{mat}\, a^{-1} + 4\Omega_\Lambda a^2 \right) - 1 \right|^{1/2} \frac{d\tau}{da}\, da$$

$$= -\frac{1}{\sqrt{2}} \int_{a_*}^{a(y)} da \left| \frac{\Omega_{mat}\, a^{-1} + 4\Omega_\Lambda a^2 - 2k^2/H_0^2}{\Omega_r + \Omega_m a + \Omega_\Lambda a^4} \right|^{1/2}$$

As $a \to 0$, integrand $\sim \left(\frac{\Omega_{mat}}{\Omega_{rad}} \right)^{1/2} a^{-1/2}$ so convergent

$$S_0(x_*) = -\int_{a_*}^{0} \frac{1}{\sqrt{2}}\, da \left| \frac{\Omega_{mat}\, a^{-1} + 4\Omega_\Lambda a^2 - 2k^2/H_0^2}{\Omega_r + \Omega_m a + \Omega_\Lambda a^4} \right|^{1/2}$$

$$= \frac{1}{\sqrt{2}} \int_0^{a_*} da \left(\frac{\Omega_{mat}\, a^{-1} + 4\Omega_\Lambda a^2 - 2k^2/H_0^2}{\Omega_r + \Omega_m a + \Omega_\Lambda a^4} \right)^{1/2}$$

$$\rightarrow \quad = \frac{1}{\sqrt{2}} \int_0^{\epsilon} \cdot + \int_{\epsilon}^{a_*} \left(\qquad \right)^{1/2}$$

$$\int_0^{\epsilon} da \left(\frac{\Omega_{mat}\, a^{-1} + 4_\Lambda a^2 - \frac{2k^2}{H_0^2}}{\Omega_r + \Omega_m a + \Omega_\Lambda a^4} \right)^{1/2} \sim \int_0^{\epsilon} da \left(\frac{\Omega_{mat}\, a^{-1} - \frac{2k^2}{H_0^2}}{\Omega_r + \Omega_m a} \right)^{1/2}$$

$$\left(\frac{\Omega_{mat} a^{-1} - \frac{2k^2}{H_0^2}}{\Omega_r + \Omega_m a} \right)^{1/2}$$

$$\sim \left(\frac{\Omega_{mat}}{\Omega_r} \right)^{1/2} a^{-1/2} \left(\frac{1 - \frac{2k^2}{H_0^2} \frac{a}{\Omega_m}}{1 + \frac{\Omega_m}{\Omega_r} a} \right)^{1/2}$$