PETR 5313: CRN 38950, Fall 2017
Numerical Application in Petroleum Engineering,
Lesson 09: Integration

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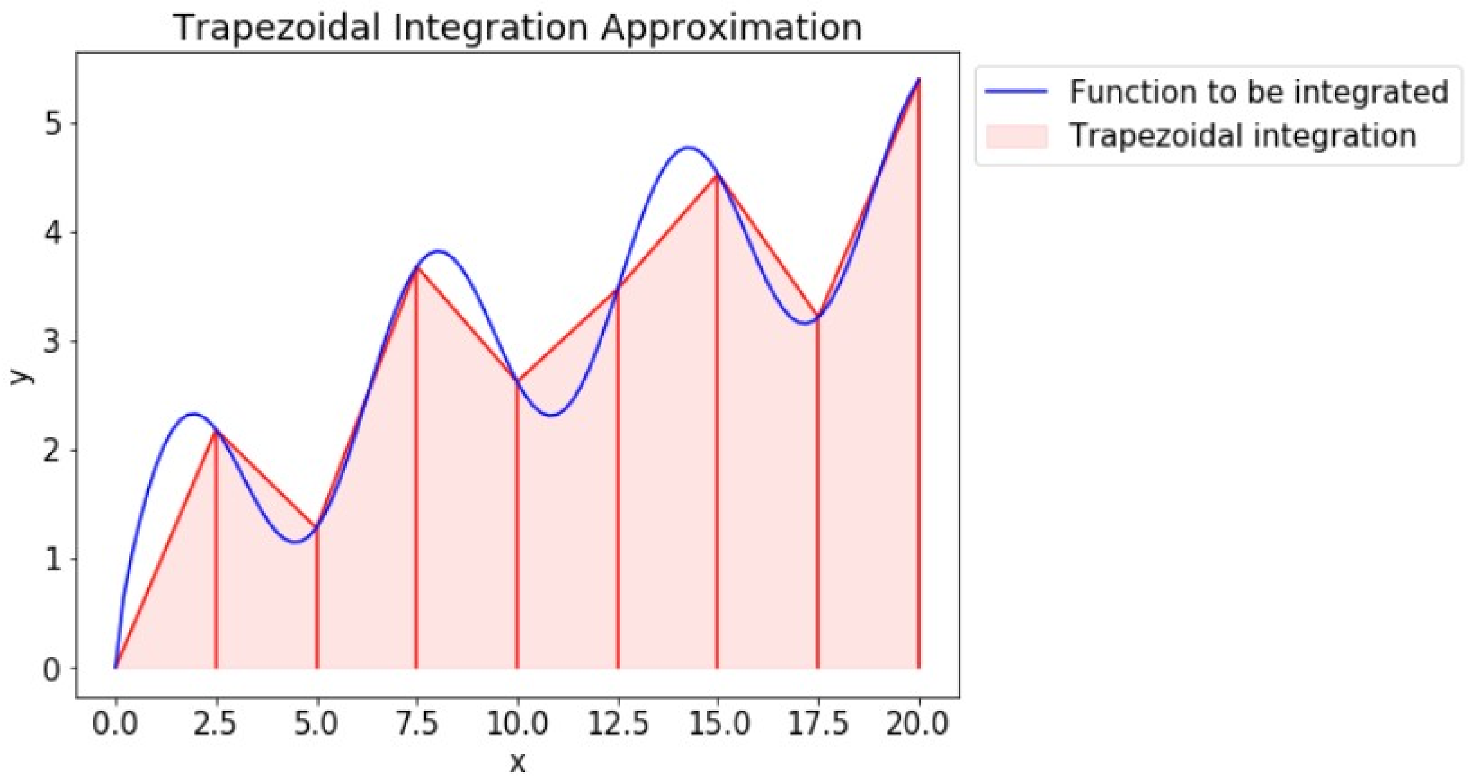
Outline

- Trapezoidal
  - Uniform discretization
  - Non-uniform discretization
- Simpson Method
  - Simpson 1/3
  - Simpson 3/8
- Double integration
  - Sympy
  - Scipy dblquad
  - Simpson twice!

Trapezoidal Method

$$\int_a^b f(x)dx \approx 0.5 * (f(a) + f(b)) * (a - b)$$

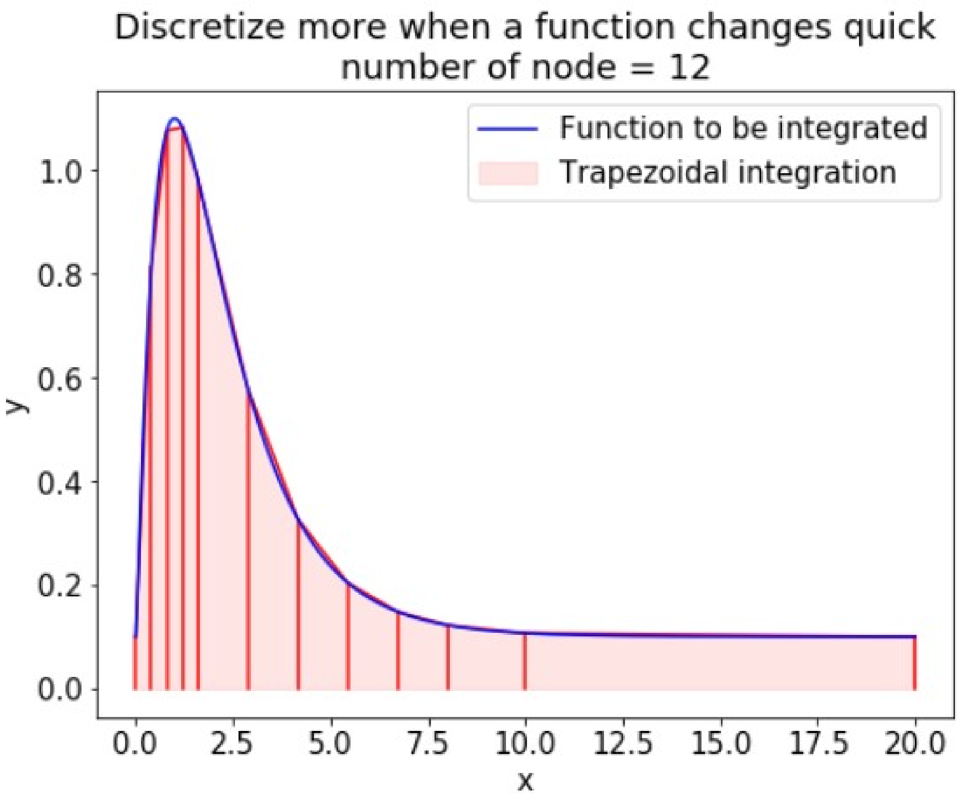
Approximate areas with several trapezoidal shapes and add them together

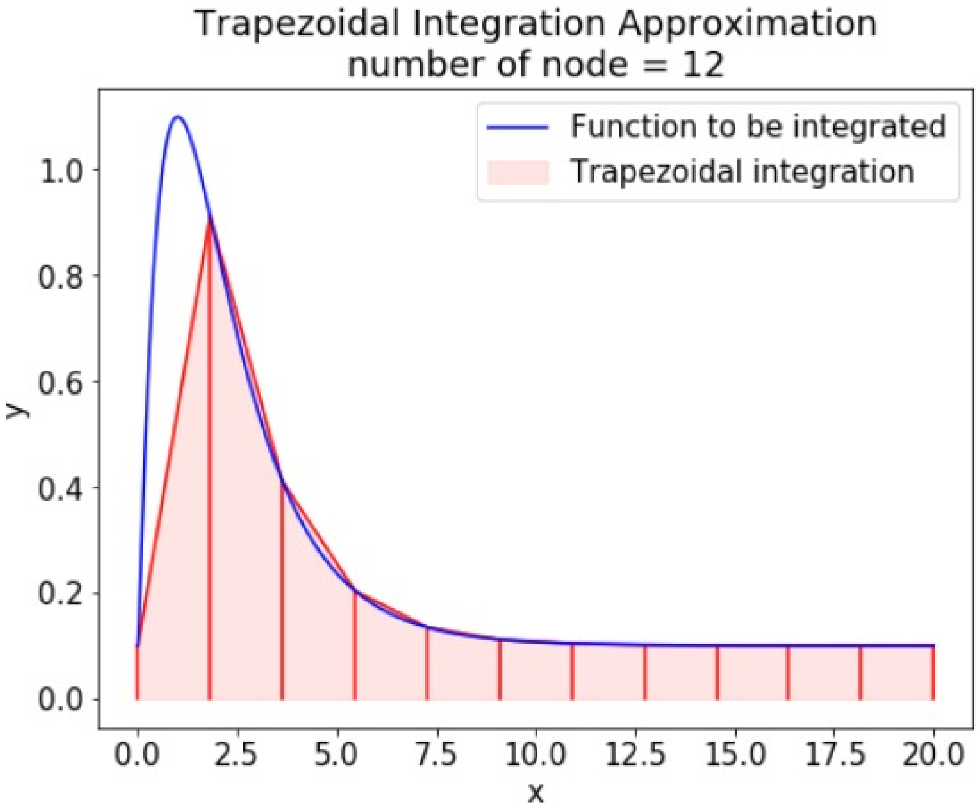


Trapezoidal Method: Non-uniform discretization

Discretize more where it is needed give better accuracy

- More point where function changes quick





Benefit of finer grids where it is needed

- Achieve a better accuracy with the same number of grid point
- Faster calculation with the same accuracy as compared to the uniform finer grids cases
- Less numerical error from adding small number together

Error from adding float together

- Float has about 15 – 16 significant digit
- More discretization is generally good, but too much can cause error

Error from adding numbers

It may seem that using trapezoidal with many sections just slow down the process, but give a more accurate answer.

- This is true, to a certain point

Consider $\sum_i x_i = 5$

- Where x_i is 1, 0.1, ..., 10^{-7}

Summation result in the format of (a,b) where
a is the negative value of the exponent of xi
b is the summation result

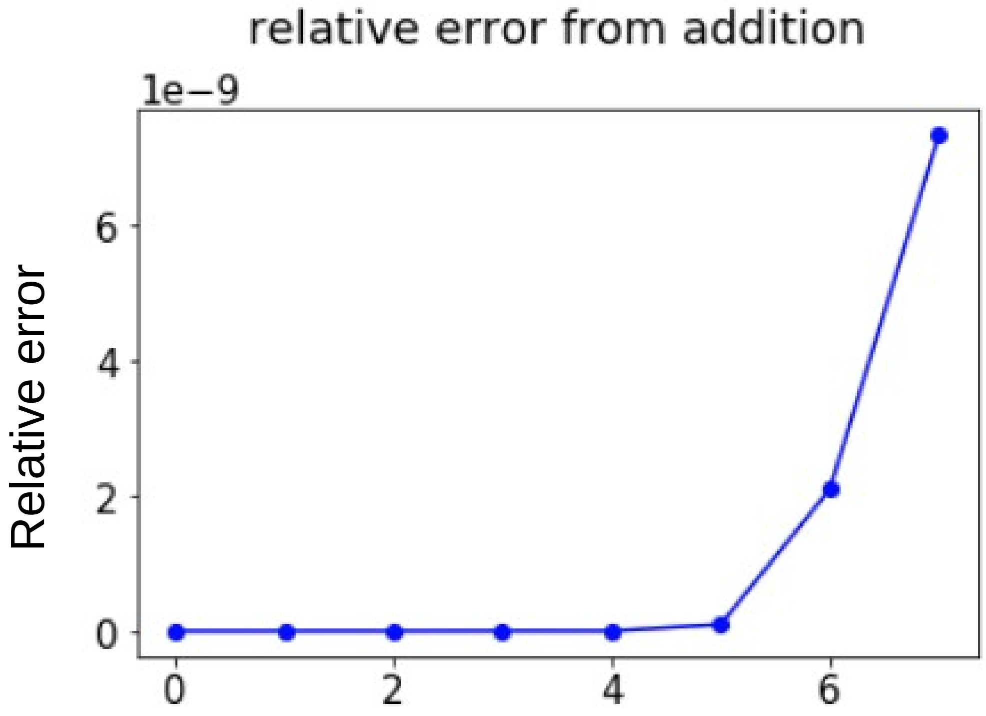
[(0, 5.0),
 (1, 5.0),
 (2, 4.999999999999999),
 (3, 4.999999999999916),
 (4, 4.9999999999999485),
 (5, 4.999999999995016),
 (6, 4.999999999895295),
 (7, 4.99999999633759)]

- When x is too small, error increases

Error from adding numbers

If we can have 1e200 sections for trapezoidal calculation, will we get exact or very accurate solution? No, if we use float!

Error from adding small areas together increases quickly as each interval become smaller



Negative value of the exponent
x = 10^(-0) to x = 10^(-7)

Simpson 1/3: Derivation Summary

- Quadratic polynomial is used to connect dot (instead of a straight line used in trapezoidal method)
- Lagrange polynomial interpolation is used to get the function approximation for that quadratic polynomial
- Then the function is integrated to get Simpson 1/3 rule

$$\int_a^b P(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

- This method requires function value at the middle point
- Cannot be used with non-uniform grids (unless re-deriving the formula)

Composite Simpson 1/3

- Single section has coefficient of 1-4-1
- Multiple sections has coefficient of 1-4-2-4-2-4-2-...-2-4-1
- Both end has coefficient of 1, in the middle is 4-2-4

$$\int_a^b f(x)\,dx \approx$$

$$\frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)\right]$$

Simpson 3/8

Simpson 3/8 is similar to Simpson 1/3, but the number of interval must be the multiplication of 3. m is just an integer. h is the interval size. Coefficient is 1-3-3-1 for a **single set of 4 points**

$$\int_a^b f(x)\,dx \approx$$

$$\frac{3h}{8}\sum_{k=1}^{m}(f(x_{3k-3} + 3f(x_{3k-2}) + 3f(x_{3k-1}) + f(x_{3k}))$$

$$h = \frac{b-a}{3m}$$

Numerical Error from Trapezoidal & Simpson method

Error of trapezoidal method is proportional to h^2

$$E = \mathcal{O}(h^2)$$

Error of Simpson method is proportional to h^4

$$E = \mathcal{O}(h^4)$$

Roughly speaking, h^4 means as the interval decrease 10 times, the error decrease 10,000 times (or 100 for h^2 case).

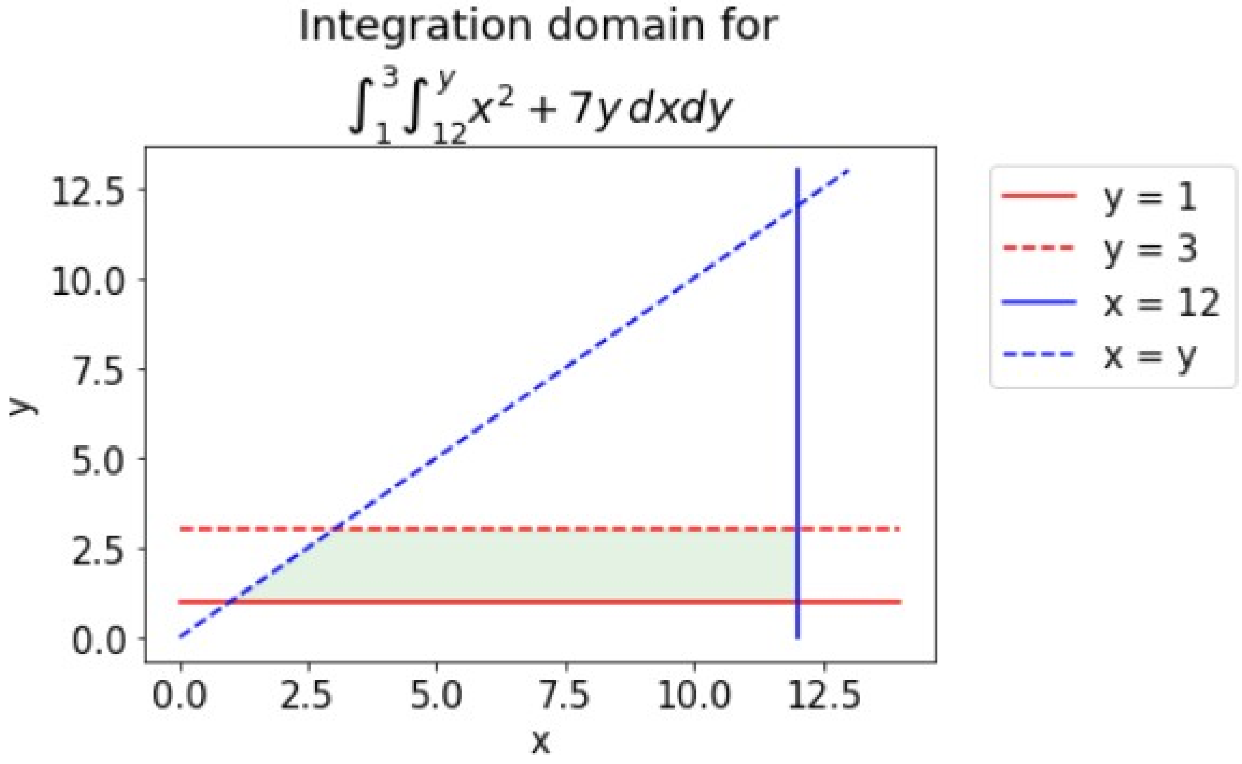
Ability to give exact solution

- Simpson: 3rd degree (or less) polynomial
- Trapezoidal: Straight line equation

Double Integration

$$\int_1^3 \int_{12}^y x^2 + 7y \, dx dy$$

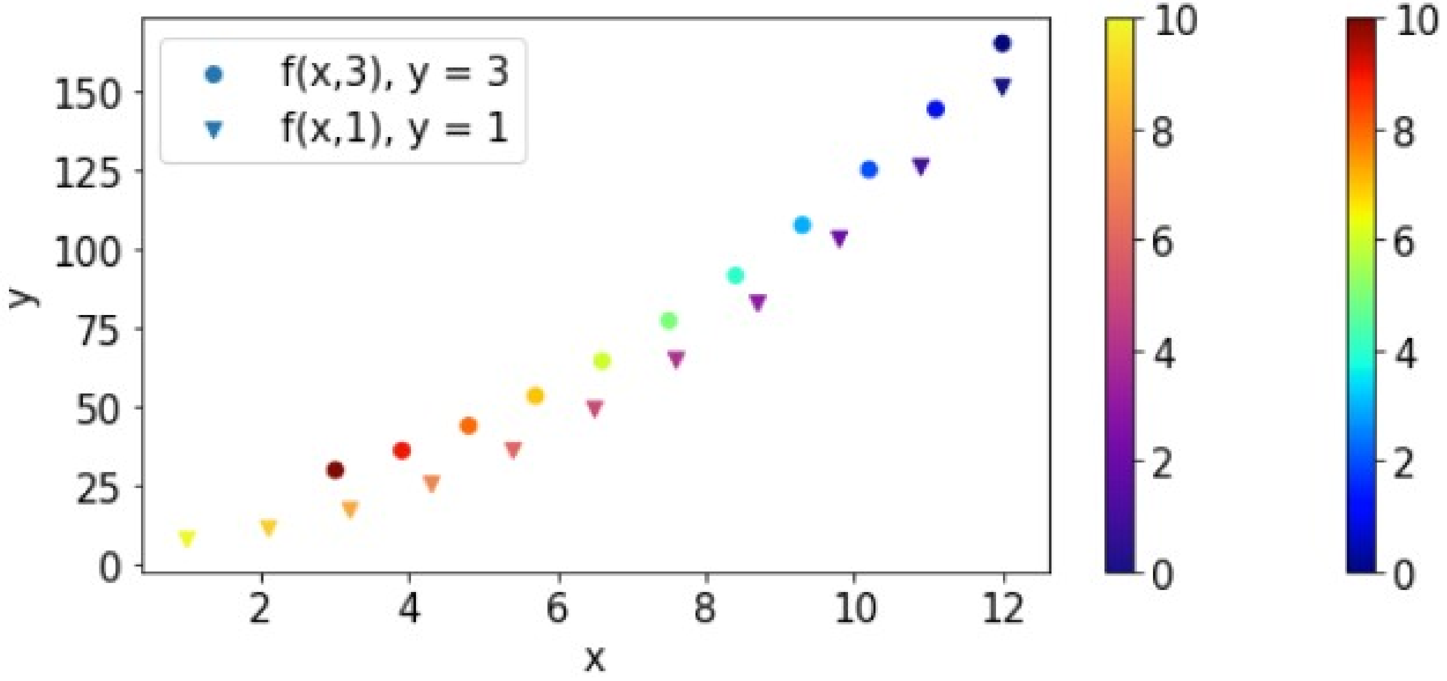
Integration Domain



Double Integration

f(x,y) for y = 1 and y = 3.

Notice that when y = 3, we integrate from x = 12 to x = 3 (**backward**)



Double Integration...

Method 1: Use scipy.integrate.dblquad

```
fxy3 = lambda x,y: x**2+7*y
sp.integrate.dblquad(fxy3,1,3,lambda y:12, lambda y:y)
```

- Quick and easy

Method 2: Use Simpson Method twice

- Should know to fully understand the double integration idea

Double Integration: Use Simpson method twice!

Step 1: Create $\int_a^y f(x,y)dx$ for every point in y domain

This work as one equation one unknown (which is y),

- so we have

$$\mathcal{G}(y) = \int_a^y f(x,y)dx$$

- With the input of y, integration from a to y is done by Simpson method. Note that y in f(x,y) and the integration limit **MUST** be the same value

Double Integration: Use Simpson method twice...

Use Simpson method second time to integrate G(y) dy

$$\int_1^3 \int_{12}^{y} (x^2 + 7y)\, dx dy = \int_1^3 \mathcal{G}(y) dy$$

```
y_fxy3 = np.linspace(1,3,11)
fxy3 = lambda x,y: x**2+7*y
# fy3 is the integration for x = 12 to y at a certain y
fy3 = lambda y: simps(fxy3(np.linspace(12,y,11),y),
                      np.linspace(12,y,11))
simps([fy3(i) for i in y_fxy3],y_fxy3)
```