Proof. From the definition of virtual queue in (8), we can get $Q_k(t+1)^2 \leq (Q_k(t)+\beta_k-x_k(t))^2$, which can be combined with (11).

$$
\begin{aligned}
\triangle(Q(t)) &= \mathbb{E}[L(Q(t+1))-L(Q(t))|Q(t)] &(1)\\
&= \mathbb{E}[\frac{1}{2}\sum_{k\in\mathcal{K}} Q_k(t+1)^2 - \frac{1}{2}\sum_{k\in\mathcal{K}} Q_k(t)^2|Q(t)] &(2)\\
&\leq \mathbb{E}[\frac{1}{2}\sum_{k\in\mathcal{K}}(\beta_k-x_k(t))^2 + \sum_{k\in\mathcal{K}} Q_k(t)(\beta_k-x_k(t))|Q(t)] &(3)\\
&\leq B+\mathbb{E}[\sum_{k\in\mathcal{K}} Q_k(t)(\beta_k-x_k(t))|Q(t)] &(4)
\end{aligned}
$$

Here, we have $B=\text{argmax}\{\frac{1}{2}(\beta_k-x_k(t))\}$, which shows that:

$$
\begin{aligned}
\sum_{t\in T} VU_k^t - \triangle(Q(t)) &= \sum_{t\in T}\sum_{k\in\mathcal{K}}\{V\mathbb{1}_{\{\tau_k^t\leq\tau_{max}^t\}}u_k^t - \mathbb{E}[L(Q(t+1))-L(Q(t))|Q(t)]\} &(5)\\
&\geq \sum_{t\in T}\sum_{k\in\mathcal{K}}\{V\mathbb{1}_{\{\tau_k^t\leq\tau_{max}^t\}}u_k^t - Q_k(t)(\beta_k-x_k(t)) - B\} &(6)
\end{aligned}
$$