Proof. From the definition of virtual queue in (8), we can get $Q_k(t+1)^2 = Q_k(t)^2 + 2Q_k(t)(\beta_k - \mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t)) + (\beta_k - \mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t))$, which can be combined with (11).

$$
\begin{aligned}
\triangle(Q(t)) &= \mathbb{E}[L(Q(t+1)) - L(Q(t))|Q(t)] \\
&= \mathbb{E}[\frac{1}{2}\sum_{k\in\mathcal{K}} Q_k(t+1)^2 - \frac{1}{2}\sum_{k\in\mathcal{K}} Q_k(t)^2|Q(t)] \\
&\le \mathbb{E}[\frac{1}{2}\sum_{k\in\mathcal{K}}(\beta_k - \mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t))^2 + \sum_{k\in\mathcal{K}} Q_k(t)(\beta_k - \mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t))|Q(t)] \\
&\le B + \mathbb{E}[\sum_{k\in\mathcal{K}} Q_k(t)(\beta_k - \mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t))|Q(t)]
\end{aligned}
$$

Here, let $B(t) = \frac{1}{2}\sum_{k\in\mathcal{K}}(\beta_k - \mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t))^2$, and assume that the second time of arrival and service in each queue is bounded, so there is a finite constant $B > 0$ for all t and all possible queue vectors $Q(t)$ such that $\mathbb{E}[B(t)|Q(t)] \le B$ holds. So we have:

$$
\begin{aligned}
\sum_{t\in T} VU(t) - \triangle(Q(t)) &= \sum_{t\in T}\sum_{k\in\mathcal{K}}\{V\mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t)u_k(t) - \mathbb{E}[L(Q(t+1)) - L(Q(t))|Q(t)]\} \\
&\ge \sum_{t\in T}\sum_{k\in\mathcal{K}}\{V\mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t)u_k(t) - Q_k(t)(\beta_k - \mathbb{I}_{\{\tau_k(t)\le\tau_{max}^t\}}x_k(t)) - B\}
\end{aligned}
$$