Proof. From the definition of virtual queue in (8), we can get
$Q_k(t+1)^2 \leq (Q_k(t) + \beta_k - \mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t))^2$, which can be combined with (11).

$$
\begin{aligned}
\triangle(Q(t)) &= \mathbb{E}[L(Q(t+1)) - L(Q(t))|Q(t)] \\
&= \mathbb{E}[\frac{1}{2}\sum_{k\in\mathcal{K}} Q_k(t+1)^2 - \frac{1}{2}\sum_{k\in\mathcal{K}} Q_k(t)^2|Q(t)] \\
&\leq \mathbb{E}[\frac{1}{2}\sum_{k\in\mathcal{K}} (\beta_k - \mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t))^2 + \sum_{k\in\mathcal{K}} Q_k(t)(\beta_k - \mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t))|Q(t)] \\
&\leq B + \mathbb{E}[\sum_{k\in\mathcal{K}} Q_k(t)(\beta_k - \mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t))|Q(t)]
\end{aligned}
$$

Here, we have $B = \mathrm{argmax}\{\frac{1}{2}(\beta_k - \mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t))\}$, which shows that:

$$
\begin{aligned}
\sum_{t\in T} VU(t) - \triangle(Q(t)) &= \sum_{t\in T}\sum_{k\in\mathcal{K}} \{V\mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t) u_k(t) - \mathbb{E}[L(Q(t+1)) - L(Q(t))|Q(t)]\} \\
&\geq \sum_{t\in T}\sum_{k\in\mathcal{K}} \{V\mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t) u_k(t) - Q_k(t)(\beta_k - \mathbb{I}_{\{\tau_k(t) \leq \tau^t_{max}\}} x_k(t)) - B\}
\end{aligned}
$$