Linear Algebra

一个短篇

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1 Vector

1.1 How to prove Two-vector collinear theorem

Definition 1.1 (scalar multiplication of vectors):

A real number λ times a vector $\boldsymbol{a}$ is equal to a vector $\lambda\boldsymbol{a}$:
- In the direction: when $\lambda > 0$, they are in the same direction, when $\lambda = 0$, $\lambda\boldsymbol{a} = \boldsymbol{0}$, when $\lambda < 0$, they are in the opposite direction.
- $|\lambda\boldsymbol{a}| = |\lambda||\boldsymbol{a}|$

Theorem 1.1 (Two-vector collinear theorem):

There are two non-zero vector $\boldsymbol{a}, \boldsymbol{b}$. If it exists a real number λ, such that:

$$\boldsymbol{a} \parallel \boldsymbol{b} \Longleftrightarrow \boldsymbol{b} = \lambda\boldsymbol{a}$$

1.1.1 Three-point collinear decision theorem

Theorem 1.2 (Three-point collinear decision theorem):

If A, B, C are collinear, then:

$$\overrightarrow{OB} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OC}$$

Proof: Since A, B, C are collinear, we have

$$\overrightarrow{CB} \parallel \overrightarrow{AC}$$
$$\overrightarrow{CB} = \lambda\overrightarrow{AC}$$

So

$$\overrightarrow{OB} - \overrightarrow{OC} = \lambda\left(\overrightarrow{OA} - \overrightarrow{OC}\right)$$

Hence

$$\overrightarrow{OB} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OC}$$

□

1.1.2 Expand to triangle

Theorem 1.3 (Three-point collinear decision theorem in triangle):

In $\triangleright_{ABC}$, D is the n equal component of $BC : nBD = kDC = BC$.

$$\overrightarrow{AD} = \frac{n}{n+k}\overrightarrow{AB} + \frac{k}{n+k}\overrightarrow{AC}$$

Proof: We have

$$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$$
$$\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$$

Since $nBD = kDC = BC$, We can have

$$\overrightarrow{BC} = n\overrightarrow{AD} - n\overrightarrow{AB}$$
$$\overrightarrow{BC} = k\overrightarrow{AC} - k\overrightarrow{AD}$$

So

$$k\overrightarrow{AC} - k\overrightarrow{AD} = n\overrightarrow{AD} - n\overrightarrow{AB}$$

Hence

$$\overrightarrow{AD} = \frac{n}{n+k}\overrightarrow{AB} + \frac{k}{n+k}\overrightarrow{AC}$$

□

1.2 In three dimensions space

Lemma （Fundamental theorem of space vectors）：

In three dimensions, we have three noncoplanar vectors $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$, for arbitrary vector $\boldsymbol{P}$, exist a unique tuple (x, y, z), hence:

$$\boldsymbol{P} = x\boldsymbol{e_1} + y\boldsymbol{e_2} + z\boldsymbol{e_3}$$

Hence, we can define:

Definition 1.2:

Let the three cross-perpendicular vector $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ as orthogonal bases in three dimensions space, and let (x, y, z) represent vector's coordinates.

Theorem 1.4 （Fundamental theorem of cooriented quantities）：

If threr are two non-collinear vector $\boldsymbol{x}, \boldsymbol{y}$, the sufficient and necessary condition that vector $\boldsymbol{P}$ is coplanar with $\boldsymbol{x}, \boldsymbol{y}$ is exists the unique real pair (λ, μ) such that:

$$\boldsymbol{P} = \lambda\boldsymbol{x} + \mu\boldsymbol{y}$$

1.3 Expand

1.3.1 Rotation of vector

Theorem 1.5:

In two-dimensions space, set the angle of vector $\boldsymbol{a}$ is θ, the cordinate of $\boldsymbol{a}$ is (x, y), the length of $\boldsymbol{a}$ is $l = \sqrt{x^2 + y^2}$, then the cordinate of $\boldsymbol{a}$ can be represent as $(l\cos\theta, l\sin\theta)$, set a vector $\boldsymbol{b}$ have angle α respect to $\boldsymbol{a}$, such that:

$$\boldsymbol{b} = (x\cos\alpha - y\sin\alpha, y\cos\alpha + x\sin\alpha)$$

Proof: We have angle α respect to $\boldsymbol{a}$, so that

$$\boldsymbol{b} = (l\cos(\theta + \alpha), l\sin(\theta + \alpha))$$

According to Triangle identity transformation

$$\boldsymbol{b} = (l\cos\theta\cos\alpha - l\sin\theta\sin\alpha, l\sin\theta\cos\alpha + l\cos\theta\sin\alpha)$$

According to $\boldsymbol{a} = (l\cos\theta, l\sin\theta)$, hence

$$\boldsymbol{b} = (x\cos\alpha - y\sin\alpha, y\cos\alpha + x\sin\alpha)$$

□

1.4 Inner product

Definition 1.3: **Inner product**

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = |\boldsymbol{\alpha}||\boldsymbol{\beta}| \cos\left(\widehat{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)$$

Lemma: **Cordinate representation**

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Proof: Set orthogonal bases $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we can have

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1$$
$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{i} \cdot \boldsymbol{k} = 0$$

Let $\boldsymbol{\alpha} = (x_1, y_1, z_1), \boldsymbol{\beta} = (x_2, y_2, z_2)$ since

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = (x_1 \boldsymbol{i} + y_1 \boldsymbol{j} + z_1 \boldsymbol{k}) \cdot (x_2 \boldsymbol{i} + y_2 \boldsymbol{j} + z_2 \boldsymbol{k})$$

Simplify it

$$\begin{aligned} \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = (x_1 x_2)\ \boldsymbol{i}^2 + (x_1 y_2)\ \boldsymbol{i} \cdot \boldsymbol{j} + (x_1 z_2)\ \boldsymbol{i} \cdot \boldsymbol{k} \\ +\ (y_1 y_2)\ \boldsymbol{j}^2 + (y_1 x_2)\ \boldsymbol{j} \cdot \boldsymbol{i} + (y_1 z_2)\ \boldsymbol{j} \cdot \boldsymbol{k} \\ +\ (z_1 z_2) \boldsymbol{k}^2 + (z_1 x_2)\ \boldsymbol{k} \cdot \boldsymbol{i} + (z_1 y_2)\ \boldsymbol{k} \cdot \boldsymbol{j} \\ = x_1 x_2 + y_1 y_2 + z_1 z_2 \end{aligned}$$

Hence

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

□

1.5 Outer product

Definition 1.4: **Outer product**

$\boldsymbol{\alpha} \times \boldsymbol{\beta}$ is a vector.
$\boldsymbol{\alpha} \times \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\beta}$ allows right hand rule.

$$|\boldsymbol{\alpha} \times \boldsymbol{\beta}| = |\boldsymbol{\alpha}||\boldsymbol{\beta}| \sin\left(\widehat{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)$$
$$\boldsymbol{\alpha} \times \boldsymbol{\beta} \perp \boldsymbol{\alpha} \perp \boldsymbol{\beta}$$

Lemma: **Reverse-exchange law**

$$\boldsymbol{\alpha} \times \boldsymbol{\beta} = -\boldsymbol{\beta} \times \boldsymbol{\alpha}$$

Outer product matrix representation in three dimensions

$$\boldsymbol{\alpha} \times \boldsymbol{\beta} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\ \boldsymbol{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\ \boldsymbol{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\ \boldsymbol{k}$$

2 Matrix

Normally, $\boldsymbol{I}$ represent the unit matrix: main diagonal all equal to 1, rest all equal to 0.

Definition 2.1:

Set the number of rows of a matrix X is r_X, column of it is c_X, X is arbitrary.

Homomorphic matrix : There are two matrix A, B, $r_A = r_B \wedge c_A = c_B$.

Square : There is a matrix A, $r_A = c_A$.

Main diagonal : There is a square A have n factorial, the set of the main diagonal element is

$$\{A_{ij} \mid i = j \in [1, n]\}$$

symmetric matrix : There is a square matrix of n factorials $A : A_{ij} = A_{ji} \ (i \in [1, n], j \in [1, n])$.

Diagonal matrix : A matrix have nonzero element only on the main diagonal, with the rest of the elements are being 0, It can represented as

$$\mathrm{diag}\{\lambda_1, \cdots, \lambda_n\} \ (\lambda_n = A_{nn})$$

Triangle matrix :