Note on Calculus

一个短篇

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目录

1 重积分

1.1 二重积分

Definition 1.1：

$$\iint\limits_D f(x,y)\,\mathrm{d}\sigma = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\sigma_i$$

1.1.1 直角坐标系

$$\mathrm{d}\sigma = \mathrm{d}x\,\mathrm{d}y$$

Formula 1.1： 先横切再竖切

$$D \coloneqq \{(x,y) \mid y\in[c,d], x\in[\varphi_1(y),\varphi_2(y)]\}$$
$$\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,\mathrm{d}x\right]\mathrm{d}y$$

Also represented as

$$\int_c^d \mathrm{d}y \int_{\varphi_1(y)}^{\varphi_1(y)} f(x,y)\,\mathrm{d}x$$

Formula 1.2： 先竖切再横切

$$D \coloneqq \{(x,y) \mid x\in[a,b], y\in[\varphi_1(x),\varphi_2(x)]\}$$
$$\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)\,\mathrm{d}y$$

1.1.2 极坐标系

Formula 1.3：

$$\iint\limits_D f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_\alpha^\beta \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho$$

1.2 三重积分

Definition 1.2： 三重积分

$$\iiint\limits_\Omega f(x,y,z)\,\mathrm{d}v = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i$$

1.2.1 直角坐标系

$$\iiint\limits_\Omega f(x,y,z)\,\mathrm{d}v = \iiint\limits_\Omega f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Formula 1.4：投影穿线法：（先对 z 积分，再对 x, y 做二重积分的处理）

将 封闭区域 Ω 投影至 xOy 面上得到封闭面 D_{xy}，因此：

$$\Omega := \left\{(x, y, z) \mid z_1(x, y) \le z \le z_2(x, y), (x, y) \in D_{xy}\right\}$$
$$D_{xy} := \{(x, y) \mid y_1(x) \le y \le y_2(x), a \le x \le b\}$$

则

$$\iiint\limits_{\Omega} f(x, y, z)\,\mathrm{d}v = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,\mathrm{d}z$$

Formula 1.5：投影切面法：

记 l 为 Ω 在 z 轴上的投影，D_z 为 Ω 在 $z = z$ 的截面：

$$\Omega := \{(x, y, z) \mid x, y \in D_z, a \le z \le b\}$$
$$\iiint\limits_{\Omega} f(x, y, z)\,\mathrm{d}v = \int_a^b \mathrm{d}z \iint\limits_{D_x} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y$$

1.2.2 柱坐标系

圆柱，圆锥，旋转体

Formula 1.6：

$$\iiint\limits_{\Omega} f(\rho\cos\theta, \rho\sin\theta, z)\rho\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}z = \int_\alpha^\beta \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} \rho\,\mathrm{d}\rho \int_{z_1}^{z_2} f(x, y, z)\,\mathrm{d}z$$

1.2.3 球坐标系

积分区域与球有关

Definition 1.3：

在球面坐标系中，球半径设为 r，r 与 z 轴的夹角设为 φ，r 在 xoy 面上的投影距离 x 轴的夹角设为 θ，有：

$$\begin{cases} z = r\cos\varphi \\ x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \end{cases}$$

体积元 $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta$

Formula 1.7：

$$\begin{aligned} I &= \iiint\limits_{\Omega} f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\, r^2\sin\varphi\ \mathrm{d}r\ \mathrm{d}\varphi\ \mathrm{d}\theta \\ &= \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \int_{\varphi_1}^{\varphi_2} \mathrm{d}\varphi \int_{r_1}^{r_2} F(r, \varphi, \theta)\, r^2\sin\varphi\ \mathrm{d}r \end{aligned}$$

2 曲线积分和曲面积分

2.1 曲线积分

2.1.1 对弧长的曲线积分

Definition 2.1:

$$\int_L f(x,y)\,\mathrm{d}s = \lim_{\lambda\to 0}\sum f(\xi_i,\eta_i)\Delta s_i$$

Formula 2.1:

有参数方程:

$$L := \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \quad (\alpha \le t \le \beta)$$

所以:

$$\int_L f(x,y)\,\mathrm{d}s = \int_\alpha^\beta f[\varphi(t),\psi(t)]\sqrt{\varphi'^2(t)+\psi'^2(t)}\,\mathrm{d}t \quad (\alpha < \beta)$$

2.1.2 对坐标的曲面积分

Definition 2.2:

$$\begin{aligned}\int_L F(x,y)\cdot\mathrm{d}\boldsymbol{r} &= \int_L P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y \\ &= \lim_{\lambda\to 0}\sum_{i=1}^{n}[Q(\xi_i,\eta_i)\Delta y_i + P(\xi_i,\eta_i)\Delta x_i]\end{aligned}$$

Formula 2.2: 有参数方程

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

则

$$\begin{aligned} I &= \int_L P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y \\ &= \int_\alpha^\beta \{P[\varphi(t),\psi(t)]\varphi'(t) + Q[\varphi(t),\psi(t)]\psi'(t)\}\,\mathrm{d}t \end{aligned}$$

2.1.3 格林公式

Formula 2.3:

$$\oint_L P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \pm\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

2.2 曲面积分

2.2.1 对面积的曲面积分

Formula 2.4：

$$
\begin{aligned}
&\iint\limits_{\Sigma} f(x,y,z)\,\mathrm{d}S \\
=&\iint\limits_{D_{xy}} f[x,y,z(x,y)]\sqrt{1+z_x^2+z_y^2}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

2.2.2 对坐标的曲面积分

Formula 2.5：

$$
\begin{aligned}
&\iint\limits_{\Sigma} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y \\
=&\pm\iint\limits_{D_{xy}} f[x,y,z(x,y)]\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

2.2.3 高斯公式

Formula 2.6：

$$
\oiint\limits_{\Sigma} P\,\mathrm{d}y\,\mathrm{d}z+Q\,\mathrm{d}z\,\mathrm{d}x+R\,\mathrm{d}x\,\mathrm{d}y=\iiint\limits_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}v
$$