2-SAT Resolution

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Definitions

Beforehand, we consider the following definitions :

- An atom (e.g. x_0) is a simple proposition ;
- A litteral is an atom or its negation (e.g. $x_0, \neg x_0$) ;
- X is a finite set of atoms and Y is the finite set of litterals defined from X (i.e. $\forall x \in X, x \in Y \wedge \neg x \in Y$) ;
- The lexicon of a formula Φ is the set of atoms $X_\Phi \subseteq X$ with at least one occurrence in Φ;
- A clause is a disjunction of a finite number of litterals ;

Definitions (cont'd)

- A formula in CNF (Conjunctive Normal Form) is a conjunction of a finite number of clauses ;
- A valuation $V : X \to \{0, 1\}$ is a function which assigns a truth value to each atom $x \in X$;
- A valuation defined on a lexicon naturally extends to all formulas defined on such a lexicon; if Φ is such a formula, then $v(\Phi)$ is the associated truth value ;
- A model of a formula is a valuation that satisfies the formula (i.e. the valuation of the formula is 1) ;
- A formula is consistent if it admits at least one model.

CNF-SAT : Problem statement

Goal : Given a CNF formula Φ defined on the lexicon X_Φ, give a model if Φ is consistent.

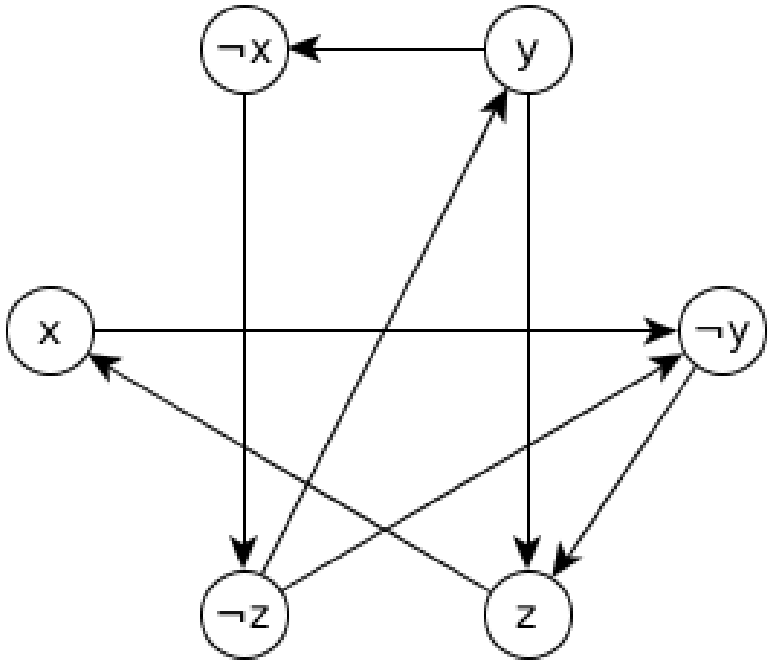
CNF-SAT : Properties and resolution

- NP Complete problem. No general polynomial-time solution is known.
- However, with at most 2 litterals per clause (2-SAT form), polynomial-time solving algorithms are known.
- Steps (exactly two litterals per clause) :
  - Build the conditional graph from the 2-SAT formula $(A \vee B \Leftrightarrow (\neg A \Rightarrow B) \vee (\neg B \Rightarrow A))$;
  - Fetch subgraphs that are strongly connected, i.e. each vertex is reachable from the others.
  - Decision rule : if a litteral and its negation are involved in the same component, formula is not consistent.
  - If the formula is consistent, assign 1 to litterals that are not negations and 0 to the others.

Implication graph example

Implication graph built from the following formula

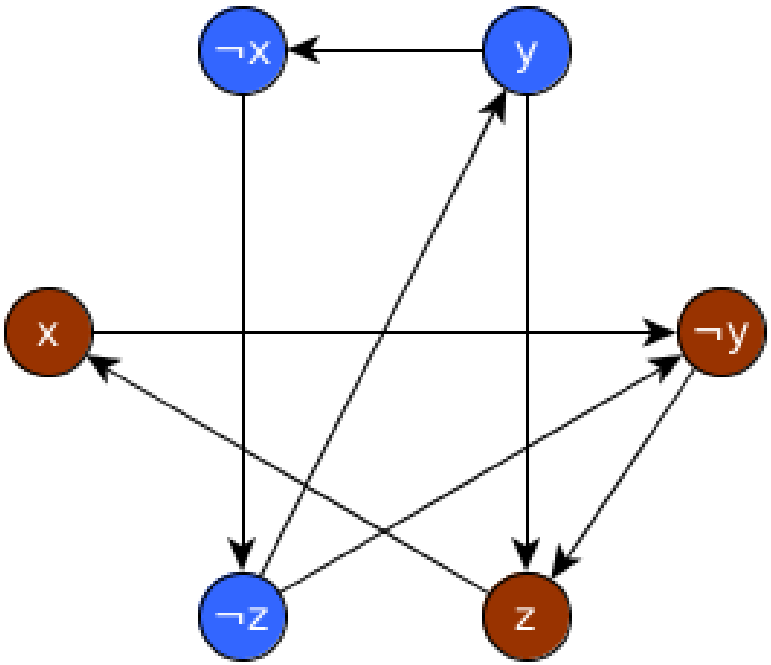
$$(\neg x \vee \neg y) \wedge (\neg y \vee z) \wedge (x \vee \neg z) \wedge (z \vee y)$$



Strongly connected component (SCC)

Implication graph built from the following formula

$$(\neg x \vee \neg y) \wedge (\neg y \vee z) \wedge (x \vee \neg z) \wedge (z \vee y)$$



Model : $x \to 1, y \to 0, z \to 1$

(Reverse list of SCC by topological order and assign True to litterals that do not have negation)

Known algorithms to fetch SCCs

- Kosaraju's algorithm. Two depths searches in respectively the graph and its transpose.
  - Transpose of a graph G is G with reverted edges.
- Tarjan's algorithm. One depth search with a stack. Use of indexes assigned to vertices to detect root nodes and then build SCCs.

A more Prolog-friendly algorithm ?

Path-based strong component algorithm

- Last (known) version of this algorithm by [Gabow,2000].
- Two stack are used. One to keep track of the current component and the second to keep track of the search path.
- List insertion/read in Prolog is done in a stack fashion.

Path-based strong component algorithm

Algorithm Strongly connected components with stacks and DFS

```
1:  C ← 0,S ← [],P ← []
2:  Prs ← {}, Sccs ← {}
3:  procedure PBSCA(v,{V, E})
4:      Prepend v to P and S
5:      Prs ← Prs ∪ {v, C}
6:      C ← C + 1
7:      for w in neighbors(v, E) do
8:          if {w, N} ∉ Prs then
9:              I ← I ∪ {i}
10:         else if ∄scc ∈ Sccs|w ∈ scc then
11:             M ← +∞
12:             while M > N do
13:                 z ← Pop(P)
14:                 Pick {z, m} ∈ Prs
15:                 M ← m
16:             end while
17:         end if
18:     end for
19:     if Top(P) = v then
20:         scc ← {}
21:         lastpop ← Pop(S)
22:         while lastpop ≠ v do
23:             scc ← scc ∪ lastpop
24:             lastpop ← Pop(S)
25:         end while
26:         Pop(P)
27:         Sccs ← Sccs ∪ scc
28:     end if
29: end procedure
30: for v in V do
31:     if {v, _} ∉ Prs then
32:         PBSCA(v, {V, E})
33:     end if
34: end for
```

Path-based strong component algorithm - Example

Execute the algorithm through the example shown in Slide 6.

2SAT - Scheduling problem

- We consider a classroom with n teachers and m cohorts of students.
- Each teacher have a set of working hours slots in which they are available in a week.
- Each teacher have a fixed number of hours to spend with each cohort of students.
- A teacher cannot be assigned to two cohorts of students at the same hour slot.
- A cohort of students cannot be assigned to two teachers at the same hour slot.
- The goal of the scheduling problem is to assign hours slots to teachers and cohorts in order to fulfill the time each teacher has to spend with cohorts of students.
  - NP-Complete. We consider here that each teacher has only one or two hours to spend with the students.

SAT - Homework

- Implement a 2SAT Solver using at least two algorithms from the literature (including the presented one).
  - Make sure to separate and explain clearly each algorithm you have implemented in your source code file.
- Benchmark them with a common set of formulas, providing examples and time execution.
- Provide a formalisation of the scheduling problem.
- Build instances of the scheduling problem and solve them using your 2SAT solver. Provide examples and time execution.

Homework - Input/Output (2SAT Solver)

- The input is a list of pairs of literals (an atom or a term $n(A)$ where A is an atom).
- The output is either :
  - The empty list if the formula Φ described in the input is not consistent.
  - A list of pairs in which each atom in the lexicon X_Φ is associated to a $\{0, 1\}$ valuation.

Homework - Input/Output (Scheduling Solver)

- The inputs are :
  - A list of list of natural numbers (only 1 or 2) R of size $n \times m$.
  - A list of list of list of binary numbers A of size $n \times 5 \times 10$.
- The output is either
  - An empty list if there is no solution,
  - A list of quadruplets that contains for each assignment the teacher i, the cohort j, the day d and the hour slot i.

That's all folks

Start with *https://en.wikipedia.org/wiki/2-satisfiability* to know more about 2SAT