2-SAT Resolution

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Definitions

Beforehand, we consider the following definitions :

- An atom (e.g. x_0) is a simple proposition ;
- A litteral is an atom or its negation (e.g. x_0,$\neg x_0$) ;
- X is a finite set of atoms and Y is the finite set of litterals defined from X (i.e. $\forall x \in X, x \in Y \wedge \neg x \in Y$) ;
- The lexicon of a formula Φ is the set of atoms $X_\Phi \subseteq X$ with at least one occurrence in Φ;
- A clause is a disjunction of a finite number of litterals ;

Definitions (cont'd)

- A formula in CNF (Conjunctive Normal Form) is a conjunction of a finite number of clauses ;
- A valuation $V : X \to \{0, 1\}$ is a function which assigns a truth value to each atom $x \in X$;
- A valuation defined on a lexicon naturally extends to all formulas defined on such a lexicon; if Φ is such a formula, then $v(\Phi)$ is the associated truth value ;
- A model of a formula is a valuation that satisfies the formula (i.e. the valuation of the formula is 1) ;
- A formula is consistent if it admits at least one model.

CNF-SAT : Problem statement

Goal : Given a CNF formula Φ defined on the lexicon X_Φ, give a model if Φ is consistent.

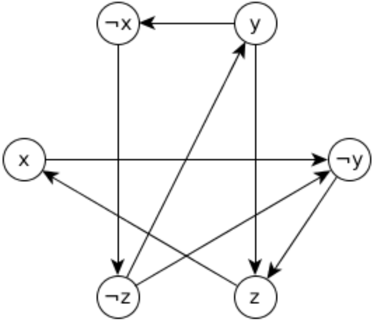
CNF-SAT : Properties and resolution

- NP Complete problem. No general polynomial-time solution is known.
- However, with at most 2 litterals per clause (2-SAT form), polynomial-time solving algorithms are known.
- Steps (exactly two litterals per clause) :
  - Build the conditional graph from the 2-SAT formula $(A \vee B \Leftrightarrow (\neg A \Rightarrow B) \vee (\neg B \Rightarrow A))$;
  - Fetch subgraphs that are strongly connected, i.e. each vertex is reachable from the others.
  - Decision rule : if a litteral and its negation are involved in the same component, formula is not consistent.
  - If the formula is consistent, assign 1 to litterals that are not negations and 0 to the others.

Implication graph example

Implication graph built from the following formula

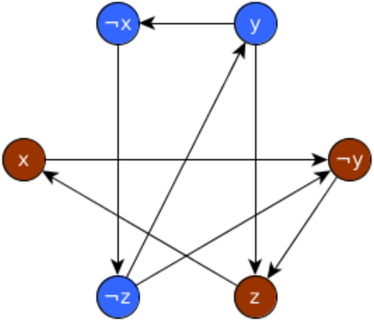
$(\neg x \vee \neg y) \wedge (\neg y \vee z) \wedge (x \vee \neg z) \wedge (z \vee y)$



Strongly connected component (SCC)

Implication graph built from the following formula

$$(\neg x \vee \neg y) \wedge (\neg y \vee z) \wedge (x \vee \neg z) \wedge (z \vee y)$$



Model : $x \rightarrow 1, y \rightarrow 0, z \rightarrow 1$

Known algorithms to fetch SCCs

- Kosaraju's algorithm. Two depths searches in respectively the graph and its transpose.
  - Transpose of a graph G is G with reverted edges.
- Tarjan's algorithm. One depth search with a stack. Use of indexes assigned to vertices to detect root nodes and then build SCCs.

A more Prolog-friendly algorithm ?

Path-based strong component algorithm

- Last (known) version of this algorithm by [Gabow,2000].
- Two stack are used. One to keep track of the current component and the second to keep track of the search path.
- List insertion/read in Prolog is done in a stack fashion.

Path-based strong component algorithm

Algorithm Strongly connected components with stacks and DFS

```
C ← 0, S ← [], P ← []
Prs ← {}, Sccs ← {}
procedure PBCSA(v,E)
    Prepend v to P and S
    Prs ← Prs ∪ {v, C}
    C ← C + 1
    for w in neighbors(v, E) do
        if {w, N} ∉ Prs then
            PBCSA(w, E)
        else if ∄scc ∈ Sccs|w ∈ scc then
            M ← +∞
            while M > N do
                z ← Pop(P)
                Pick {z, m} ∈ Prs
                M ← m
            end while
        end if
    end for
    if Top(P) = v then
        scc ← {}
        lastpop ← Pop(S)
        while lastpop ≠ v do
            scc ← scc ∪ lastpop
            lastpop ← Pop(S)
        end while
        Pop(P)
    end if
    return I
end procedure
Given a graph {V, E} :
for v in V do
    if {v, _} ∉ Prs then
        PBCSA(v, E)
    end if
end for
```

Path-based strong component algorithm - Example

Execute the algorithm through the example shown in Slide 6.

2SAT - Scheduling problem

- We consider a classroom with n teachers and m cohorts of students.
- Each teacher have a set of working hours slots in which they are available in a week.
- Each teacher have a fixed number of hours to spend with each cohort of students.
- A teacher cannot be assigned to two cohorts of students at the same hour slot.
- A cohort of students cannot be assigned to two teachers at the same hour slot.
- The goal of the scheduling problem is to assign hours slots to teachers and cohorts in order to fulfill the time each teacher has to spend with cohorts of students.
  - NP-Complete. We consider here that each teacher has only one or two hours to spend with the students.

SAT - Homework

- Implement a 2SAT Solver using at least two algorithms from the literature (including the presented one).
  - Make sure to separate and explain clearly each algorithm you have implemented in your source code file.
- Benchmark them with a common set of formulas, providing examples and time execution.
- Provide a formalisation of the scheduling problem.
- Build instances of the scheduling problem and solve them using your 2SAT solver. Provide examples and time execution.

Homework - Input/Output (2SAT Solver)

- The input is a list of pairs of literals (an atom or a term $n(A)$ where A is an atom).
- The output is either :
  - The empty list if the formula Φ described in the input is not consistent.
  - A list of pairs in which each atom in the lexicon X_Φ is associated to a $\{0, 1\}$ valuation.

Homework - Input/Output (Scheduling Solver)

- The inputs are :
  - A list of list of natural numbers (only 1 or 2) R of size $n \times m$.
  - A list of list of list of binary numbers A of size $n \times 5 \times 10$.
- The output is either
  - An empty list if there is no solution,
  - The number of days + a list of quadruplets that contains for each assignment the teacher i, the cohort j, the day d and the hour slot i.

That's all folks

Start with *https://en.wikipedia.org/wiki/2-satisfiability* to know more about 2SAT