# Graph Coloring

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# Graph Coloring

*Given a bidirected graph* $G = (V, E)$, *find the lowest integer* $k$ *such that each vertex is assigned a label* $1 \leqslant k' \leqslant k$, *i.e. that* $c(v) = k'$ *and* $\forall (v, v') \in E,\ c(v) \neq c(v')$.

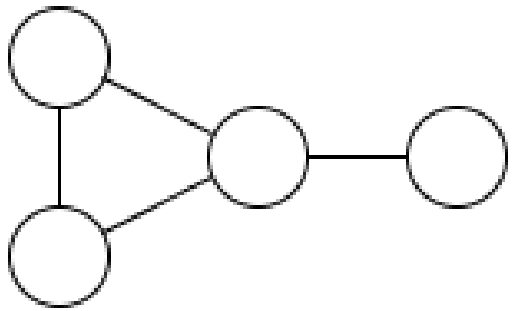
# (Many) Applications

- Bandwidth allocation to radio stations.
  - Unit disk graphs. Vertices (disks) are connected if they lie within a unit distance of each other.
- Aircraft allocation to flights.
  - Interval graph. Vertices (intervals) are connected if they overlap with each other.
- Register allocation to compiled programs (interference graph).
  - Interference graph. Vertices (variables) are connected if they are used at the same time.
- Sudoku can be solved through graph coloring (and you will).

# Definitions

- The degree of a vertex $v$ is the number of edges that involves $v$.
- The saturation of a vertex $v$ is the number of different colors of its neighbours.
- A coloring of a graph $G = (V, E)$ is an application $C : V \to \mathbb{N}$ such that $\forall (v, v') \in E$, $c \in C$ is a coloration iff $c(v) \neq c(v')$.
- The chromatic number of a graph $G$ is the minimal coloring of a graph G.
- The $k$-chromatic polynomial of a graph $G$ is the number of possible colorings with $k$ colors of G.

# Coloring a Graph



Minimum $k$-coloring ? Chromatic polynomial ?

# Remarks

- The minimum $k - coloring$ problem is NP-Hard.
- If $k$ is provided, the problem can be modeled as a Constraint Satisfaction Problem (CSP).
- Exact chromatic number : Contraction/addition with Zykov Theorem.
  - Zykov Theorem says that $\mathcal{X}(G) = min\{\mathcal{X}(G + uv), \mathcal{X}(G/uv)\}$.
  - Running time depends on the order where $u$ and $v$ are chosen. Still close to exponential in the worst case : $O(1.6180^{n+m})$
- Heuristic search : Welsh-Powell, DSATUR.

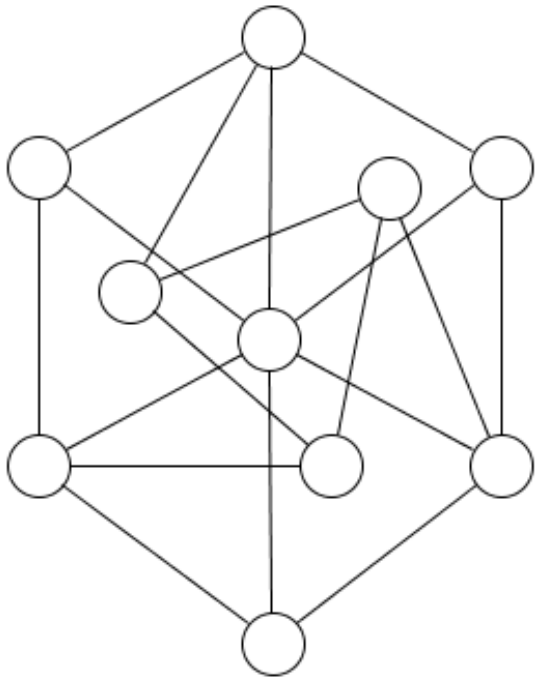
# Welsh-Powell algorithm

- Steps :
  - Sort all vertices in a decreasing order of their degrees.
  - Repeat until all vertices are colored :
    - Give the smallest color possible to the first non colored vertex.
    - Allocate the same color to the vertices that are not adjacent to a vertex with this color.
- In some cases, this algorithm can give the worst coloration possible. DSATUR is a better known heuristic.

# DSATUR

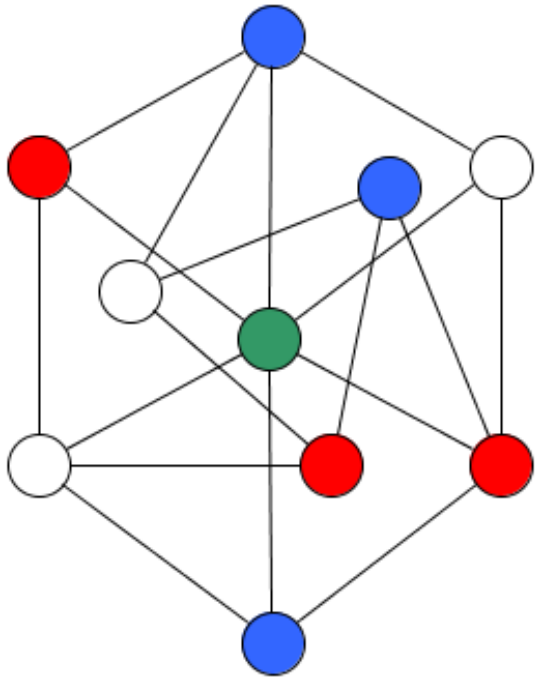
- Sort all vertices in a decreasing order of their degrees.
- Pick the first vertex and give it the smallest color.
- Repeat until all vertices are colored :
  - Choose a vertex that have the maximal saturation, or in case of a tie, with the maximal degree.
  - Assign the smallest possible color to that vertex

# Apply !



Apply DSATUR and Welsh-Powell algorithm to this graph.

# Chromatic number



The chromatic number is 4. Which colors have been yield from both algorithms ?

# Sudoku - Rules

- We consider $N \times M$ grid, a given set of partitions of the whole set of cells and a set of symbols.
  - The classic $9 \times 9$ Sudoku is partitionned into 9 contigous squares, for example.
- The goal is to fill the grid with the set of symbols such that there is no repetition of a symbol in (i) a column, (ii) a line or (iii) inside a partition.

# Graph Coloring and Sudoku - Homework

- Build the best possible approximation of a Graph Coloring exact solver.
- Build a Sudoku solver on top on your solver. The following assignments are not mandatory for implementation but needs at least a short discussion on possible approachs.
  - We say that a Sudoku input grid is valid if it admits an unique solution. How to determine if a Sudoku input is a valid Sudoku Grid ?
  - A Killer Sudoku is a configuration when an empty Sudoku input grid is provided and admits an unique solution. How to solve such a grid, with or without the set of symbols ?
- Deliverable is expected on the same fashion as previous projects and the oral presentation is scheluded to 17/04/2018.

# Graph Coloring - Inputs/Output

- The inputs are :
  - A list of pairs of atoms,
  - An integer $k$. if $k = 0$, then the algorithm is trying to compute the chromatic number. Otherwise, it will color the graph with $k$ colors if it is possible.
- The output is an empty list if there is no possible coloration, otherwise, it is an integer and a list of pair of atom and integer.

# Sudoku - Inputs/Output

- The inputs are :
  - A list of list of integers (the input Sudoku grid).
    - 0 is an empty cell.
  - A list of list of integers (the partitionning of the sudoku grid).
- The output is a list of list of integers (the solution of the Sudoku grid).
- If you address the validity problem, an additional binary output is required to tell if the Sudoku is valid or not. If it is not valid, the other output is an empty list.

# That's all folks

Have a good lunch !