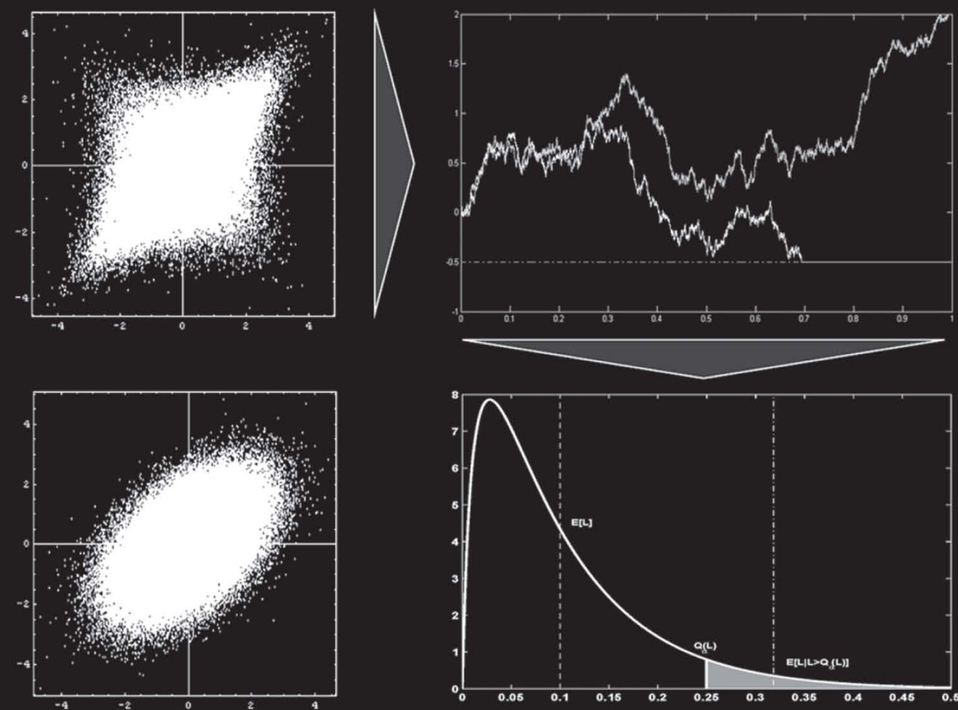
Christian Bluhm, Ludger Overbeck,
and Christoph Wagner

Introduction to Credit Risk Modeling

Second Edition



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Preface to Second Edition

The first edition of this book appeared eight years ago. Since then the banking industry experienced a lot of change and challenges. The most recent financial crisis which started around May 2007 and lasted in its core period until early 2009 gave rise to a lot of scepticism about whether credit risk models are appropriate to capture the true nature of risks inherent in credit portfolios in general and structured credit products in particular. In a recent article we discuss common credit risk modeling approaches in light of the most recent crisis and invite readers to participate in the discussion; see [26].

A key observation in a discussion like the one in [26] is that the universe of available models and tools is sufficiently rich for doing a good job even in a severe crisis scenario as banks recently experienced it. What seems to be more critical is an appropriate *model choice*, *parameterization of models*, dealing with *uncertainties*, e.g., based on insufficient data, and *communication* of model outcomes to decision makers and executive senior management. These are the four main areas of challenge where we think that a lot of work and rethinking needs to be done in a “post-crisis” reflection of credit risk models.

In the first edition of this book we focused on the description of common mathematical approaches to model credit portfolios. We did not change this philosophy for the second edition. Therefore, we left large parts of the book unchanged in its core message but supplemented the exposition with new model developments and with details we omitted in the first edition. The aforementioned four areas of challenge in a “post-crisis” reflection of credit risk models would justify another separate exposition in book form and it should be clear that a second edition of an existing book cannot be an appropriate substitute for that. However, we included a few comments in the text where appropriate.

A brief outline of the track record of changes and updates is as follows.

Chapter 1 is updated in line with developments over the last eight years. We also included more details in the case of some topics. For instance, the sections on the probability of default (PD), exposure-at-default (EAD) and loss-given-default (LGD) are still brief compared to what could be said but have been extended in comparison to the first edition of this book. The brief section on regulatory capital has also been changed and updated.

Chapter 2 has been updated where necessary but is left unchanged in large parts. What is new is a longer section on techniques for the generation of loss distributions. Because such techniques rely on a bunch of tools from probability theory we included for the convenience of the reader an introductory section on "prerequisites from probability theory" in order to keep the survey on calculation and simulation techniques self-contained as much as possible. Readers will also benefit from the probability toolkit in Chapter 4 on CreditRisk$^+$ where generating functions play a major role.

Chapter 3 is left unchanged except for the correction of typos.

Chapter 4 is enhanced by a new section on technical details regarding the calculation of the loss distribution in CreditRisk$^+$.

Chapter 5 has been updated and some new developments are included now. For instance, spectral risk measures and an axiomatic approach to capital allocation are introduced.

Chapter 6 is left unchanged except for a brief section on term structures of default probabilities based on non-homogeneous Markov chains. This new approach has been included because it fits models far better than the time-homogeneous Markov chain approach.

Chapter 7 is left unchanged except for the correction of typos.

Chapter 8 is updated to some extent as well as enhanced. We kept the presentation of cash flow structures because the basic principles and structures remain unchanged. During the most recent crisis, structured products came under pressure and markets dried up. However, we are convinced that securitization as well as portfolio structuring remain a core competence and major tool of banks' financial engineering departments. Therefore, we extended Chapter 8 by a section on multi-period models and a brief section on recent developments. However, we keep the presentation short because we have dedicated a separate book [24] to the topic of CDO modeling and it would not have made much sense to carry sections from [24] over to this book.

We need to make a disclaimer regarding data in examples. In all cases we used data for illustrative purposes only. Therefore, we decided not to re-run all examples with more recent data. Interested readers can find in [24] many more examples with up-to-date data.

Altogether we can say that doing math in the context of credit risk modeling still means a lot of personal satisfaction to us. Credit risk in particular and finance in general are great fields to apply mathematical concepts to real life situations.

However, when doing this one should never forget that senior management, regulators, investors, etc. rely in their decision making on models and valuation outcomes. The most recent crisis showed that it is important to appropriately communicate model outcomes and to make sure that the variation in results is made transparent to decision makers in ways that they can understand.

A last remark we want to make concerns model choice and model risk. It is not recommended to use one and only one model for a particular problem. Instead, we recommend using various models to shed some light on different aspects of the true nature of a credit risk problem. In this way, the problem is viewed from different angles. The most recent crisis showed that more modeling and more analysis are superior to just one model relying on various simplifying assumptions. Regulators, for instance, talk a lot about stress testing. From our perspective, stress testing should already be part of the model. What people consider as stress, for example a market scenario where banks lose several billion euros, is historically seen not as stress but as a 10-year (give or take) regular event. It should be treated as such.

Munich and Giessen, April 2010

Christian Bluhm, Ludger Overbeck, Christoph Wagner

Acknowledgements

Christian Bluhm would like to thank his wife Tabea, and his children Sarah and Noa for their continuous support. Various book projects in the last years consumed a significant amount of time and it is just great to have a patient and understanding family. Ludger Overbeck is grateful to his wife Bettina and his children Leonard, Daniel, Clara, and Benjamin for their ongoing support.

We had great feedback, support, and comments on the first edition of this book by many colleagues, friends and readers from all over the world. We are grateful they let us know of typos, mistakes, and errors and we are happy about input on how the exposition can be improved. We hope that readers will continue to let us know if they find errors or unclear passages in the book and we apologize for still undiscovered shortfalls of the manuscript. Feedback and input can be sent to the contact email addresses at

http://www.christian-bluhm.net
http://www.uni-giessen.de/∼gc1156.

Disclaimer

This book reflects the view of the authors and not the opinion of their current or former employers. The content of the book has been written for educational purposes only. The authors are by no means liable for any damage arising from any application of the theory, examples or data presented in this book.

Preface

In banking, especially in risk management, portfolio management, and structured finance, solid quantitative know-how becomes more and more important. We had a two-fold intention when writing this book.

First, this book is designed to help mathematicians and physicists leaving the academic world and starting a profession as risk or portfolio managers to get quick access to the world of credit risk management. Second, our book is aimed at being helpful to risk managers looking for a more quantitative approach to credit risk.

Following this intention on one side, our book is written in a *lecture notes* style very much reflecting the keyword "introduction" already used in the title of the book. We consequently avoid elaborating on technical details not really necessary for understanding the underlying idea. On the other side, we kept the presentation mathematically precise and included some proofs as well as many references for readers interested in diving deeper into the mathematical theory of credit risk management.

The main focus of the text is on *portfolio* rather than *single obligor* risk. Consequently, correlations and factors play a major role. Moreover, most of the theory in many aspects is based on probability theory. We, therefore, recommend that the reader consult some standard text on this topic before going through the material presented in this book. Nevertheless, we tried to keep it as self-contained as possible.

Summarizing our motivation for writing an introductory text on credit risk management one could say that *we tried to write the book we would have liked to read before starting a profession in risk management* some years ago.

Munich and Frankfurt, August 2002

Christian Bluhm, Ludger Overbeck, Christoph Wagner

About the Authors

Christian Bluhm worked for Deutsche Bank in their risk methodology department, for McKinsey as an associate in their risk management practice, for HypoVereinsbank's Group Credit Portfolio Management in Munich where he headed the Structured Finance Analytics team which was responsible for the evaluation of structured credit assets (short and long side) and for Credit Suisse where he worked as a Managing Director heading the Credit Portfolio Management unit in Zurich. At the end of 2009 Christian left Credit Suisse to enjoy a sabbatical break with much time for his family.

Christian holds a Ph.D. degree in mathematics from the University of Erlangen-Nürnberg and was a one-year post-doctoral member of the mathematics department of Cornell University, Ithaca, New York. He has authored several papers and research articles on harmonic and fractal analysis of random measures, stochastic processes, and random fields. Since he started to work in risk management more than 10 years ago, he continuously publishes in this field and regularly speaks at risk management conferences and workshops. He also lectures at universities.

Ludger Overbeck worked for Deutsche Bundesbank in their supervision department and headed the Research and Development team in the Risk Analytics and Instruments department of Deutsche Bank's credit risk management unit until 2003. Since then he has been Professor for Probability Theory and Quantitative Finance and Risk Management at the Institute of Mathematics of the University of Giessen. Besides this he is serving in different positions in credit portfolio management and securitizaton units for HypoVereinsbank/UniCredit, DZ-Bank, and recently for Commerzbank. His main responsibilities include quantitative credit risk models, bank portfolio steering, valuation of structured credit products, economic capital and capital allocation.

Ludger holds a Ph.D. in probability theory from the University of Bonn. After two post-doctoral years in Paris and Berkeley, from 1995 to 1996, he finished his Habilitation in applied mathematics during his affiliation with the Bundesbank. From the University of Frankfurt, Ludger received a Habilitation in Business and Economics in 2001. He has published papers in mathematical and statistical journals as well as in journals on finance and economics. Ludger frequently speaks at academic and practitioner conferences.

Christoph Wagner worked for Deutsche Bank in their risk methodology department, the risk methodology team of Allianz Group Center in Munich, UniCredit/HypoVereinsbank in Munich, and Allianz Risk Transfer in Zurich. His main responsibilities in his positions were (and continue to be) credit risk and insurance-linked securities, securitizations and alternative risk transfer as well as the valuation of all kinds of structured credit products.

Christoph holds a Ph.D. in statistical physics from the Technical University of Munich. Before joining Deutsche Bank, which was his first position in risk management, he spent several years in postdoctoral positions, both at the Center of Nonlinear Dynamics and Complex Systems, Brussels, and at the Siemens Research Department in Munich. He has published several articles on nonlinear dynamics and stochastic processes, as well as on risk modeling. He regularly speaks at conferences on quantitative topics in risk management.

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Chapter 1

The Basics of Credit Risk Management

Why is credit risk management an important issue in banking? To answer this question let us construct an example which is, although simplified, nevertheless not too unrealistic: Assume a major building company is asking its house bank for a loan in the size of 100 million Euro. Somewhere in the bank's credit department a senior analyst has the difficult job of deciding if the loan will be given to the customer or if the credit request will be rejected. Let us further assume that the analyst knows that the bank's chief credit officer has known the chief executive officer of the building company for many years, and to make things even worse, the credit analyst knows from recent default studies that the building industry is under hard pressure and that the *bank-internal rating*[1] of this particular building company is just on the way down to a low *subinvestment grade* (low credit quality).

What should the analyst do? Well, the most natural answer would be that the analyst should reject the deal based on the information she or he has about the company and the current market situation. An alternative would be to grant the loan to the customer but to *insure* the loss potentially arising from the engagement by means of some credit risk management instrument (e.g., a so-called *credit derivative*).

Admittedly, we intentionally exaggerated in our description, but situations like the one just constructed happen from time to time and it is never easy for a credit officer to make a decision under such difficult circumstances. A brief look at any typical banking portfolio will be sufficient to convince people that defaulting obligors belong to the daily business of banking the same way as credit applications or ATM machines. Banks therefore started to think about ways of *loan insurance*

[1] A rating is an indication of creditworthiness; see Section 1.1.1.1.

many years ago, and the insurance paradigm will now directly lead us to the first central building block of credit risk management.

1.1 Expected Loss

Situations as the one described in the introduction suggest the need of a *loss protection* in terms of an *insurance*, as one knows it from car or health insurances. Moreover, history shows that even good customers have a potential to default on their financial obligations, such that an insurance for not only the critical but all loans in the bank's credit portfolio makes much sense.

The basic idea behind insurance is always the same. For example, in health insurance the costs of a few sick customers are covered by the total sum of revenues from the fees paid to the insurance company by all customers. Therefore, the fee that a man at the age of thirty has to pay for health insurance protection somehow reflects the insurance company's experience regarding *expected costs* arising from this particular group of clients.

For bank loans one can argue exactly the same way: Charging an appropriate *risk premium* for every loan and collecting these risk premiums in an internal bank account called *expected loss reserve* will create a capital cushion for covering losses arising from defaulted loans.

1.1.1 Remark Note that for many banks the paradigm of an expected loss reserve in the sense of *saving money in good times for spending it in bad times* is just a theoretical concept. For instance, US-GAAP[2] banks like Deutsche Bank or Credit Suisse who are both exchange-listed at Wall Street need to build loss reserves like, for instance, the so-called FAS-5 reserve, in a *period-conform* manner which means they can not be used in the afore-mentioned sense of an expected loss reserve.

But the paradigm of a reserve for expected losses is still used as a theoretical concept even in US-GAAP banks and, as it will be explained in a moment, expected loss is then applied as part of the risk premium charged to the borrower.

[2]US-GAAP stands for United States Generally Accepted Accounting Principles.

In probability theory the attribute *expected* always refers to an *expectation* or *mean value*, and this is also the case in risk management. The basic idea is as follows: The bank assigns to every customer a *probability of default* (PD), a loss fraction called the *loss given default* (LGD), describing the fraction of the loan's exposure expected to be lost in case of default, and the *exposure at default* (EAD) subject to be lost in the considered time period. The loss of any obligor is then defined by a *loss variable*

$$\tilde{L} \;=\; \text{EAD} \times \text{LGD} \times L \qquad \text{with} \qquad L = \mathbf{1}_D, \quad \mathbb{P}(D) = \text{PD}, \tag{1.1}$$

where D denotes the *event* that the obligor defaults in a certain period of time (most often one year), and $\mathbb{P}(D)$ denotes the probability of D.

The constituents of formula (1.1) are random variables. Although we will not go too much into technical details, we should mention here that underlying our model is some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consisting of a *sample space* Ω, a *σ-Algebra* $\mathcal{F}$, and a probability measure $\mathbb{P}$. The elements of $\mathcal{F}$ are the *measurable events* of the model, and intuitively it makes sense to claim that the event of default should be measurable. Moreover, it is common to identify $\mathcal{F}$ with the *information* available, and the information if an obligor defaults or survives should be included in the set of measurable events.

Note that the quantities PD, LGD, EAD and all quantities derived from those three are measured w.r.t. a specified time horizon. We drop the time aspect for now but will come back to it later in the text.

In the setup we just described it is very natural to define the *expected loss* (EL) of any customer or, more general, credit-risky asset as follows.

1.1.2 Definition *Given a loss variable $\tilde{L}$ as in (1.1), its expectation*

$$\text{EL} \;=\; \mathbb{E}[\tilde{L}]$$

is called the expected loss of the underlying credit-risky asset.

A common well-known formula for the EL appears in the following special situation.

1.1.3 Proposition *If the constituents of $\tilde{L}$ in (1.1) are independent, the expected loss can be written as*

$$\mathrm{EL} \ = \ \mathbb{E}[\mathrm{EAD}] \times \mathbb{E}[\mathrm{LGD}] \times \mathrm{PD}. \tag{1.2}$$

Moreover, if EAD and LGD are constant values the formula reads as

$$\mathrm{EL} \ = \ \mathrm{EAD} \times \mathrm{LGD} \times \mathrm{PD}. \tag{1.3}$$

Proof. The expectation of any *Bernoulli* random variable like $\mathbf{1}_D$ is its event probability. If the three factors in (1.1) are independent the expectation of their product is the product of their expectation. □

Note that making the assumption that EAD and LGD are constant values can be a good starting point for a back-of-the-envelope calculation to assign fixed values to EAD and LGD. However, in realistic situations EAD has to be modeled as a random variable due to uncertainties in payment profiles like, for instance, amortization, usage, and other drivers of EAD up to the chosen planning horizon.

In Section 1.1.4 we will briefly touch on the question of independence of PD, LGD and EAD. In fact, the independence assumption indeed is rather questionable and very much simplifying. Altogether one can say that (1.3) is the most simple representation formula for the expected loss one can have. The more simplifying assumptions are dropped the more one moves away from closed formulas like (1.3).

Although our focus in the book is pretty much on *portfolio risk* rather than on *single obligor risk* we briefly describe the three constituents of Formula (1.3) in the following paragraphs.

1.1.1 Probability of Default (PD)

The derivation of default probabilities is the "bread and butter" of a credit risk analytics team. There are situations where the assignment of default probabilities is straightforward and there are situations where it seems almost impossible to come up with a reasonable approach.

First of all we want to mention that later in Chapter 6 we will find that it is not sufficient to have default probabilities w.r.t. one particular time horizon. For instance, it is not enough to know for each credit-risky asset in the portfolio what its likelihood is to default within one year. Instead, one needs a whole *term structure*

$$(p_t)_{t\geq 0}$$

of default probabilities where t denotes time and for each point t in time the likelihood p_t is the default probability of the considered asset or client w.r.t. the time interval $[0, t]$. Note that in the literature PD term structures are often called *credit curves*. We dedicated the whole Chapter 6 in this book to this topic. In this section we focus on some basic remarks regarding PDs w.r.t. a fixed time horizon, say, one year. We also speak of 'clients' mostly in this section but what we essentially mean is any kind of credit-risky asset.

1.1.1.1 Ratings

Let us start with an upfront remark. Originally ratings were not developed for the derivation of PDs but only for the discrimination of credit quality on an ordinal scale. And in case of rating agencies which we will introduce later it still is the case that they do not assign PDs directly to rated clients but assign ratings in the sense of Table 1.1. So one has to be careful to put ratings and PDs in one bucket without keeping in mind that they are in fact different objects, as we will point out in a moment. However, because PDs are assigned to ratings and PDs are a main driver of the portfolio loss as well as all kinds of important ratios in banking, including regulatory capital related quantities, it is a common pattern that ratings and PDs are associated. Having said that, we continue our presentation from the viewpoint of the practitioner who uses ratings in the sense explained in the sequel.

The assignment of default probabilities to clients typically functions via so-called *rating systems*. A rating system can be thought of as a discretization of PDs on an *ordinal scale* which is called the *rating scale*. Discretization of a continuous metric quantity like a PD to an ordinal scale makes life in large organizations easier although one could argue that discretization seems a bit artificial and in the context of pricing introduces unnecessary jumps in pricing grids.

Well-known discretizations of PDs are the rating scales by the rating agencies *Moody's*, *Standard & Poor's*, and *Fitch*. Readers unfamiliar with the term "rating agency" can access background information on rating agencies, their work and their publications via their websites.

- For Moody's Investors Service go to: www.moodys.com;
- for Standard & Poor's go to: www.standardandpoors.com;
- for Fitch Ratings go to: www.fitchratings.com.

Rating scales of rating agencies look as follows. Standard & Poor's and Fitch use AAA, AA, A, BBB, BB, B, CCC, CC, C as a rating scale for rating best credit quality (AAA), 2nd-best credit quality (AA), and so on, until worst credit quality (C). The default state indicating that a company already failed in some payment obligation is denoted by D. Moody's uses Aaa, Aa, A, Baa, Ba, B, Caa, etc. to denote a comparable rating scale, again in decreasing order of credit quality. Each of the rating agencies has a finer rating scale in place to allow for a finer distinction of credit quality among obligors. Standard & Poor's and Fitch, for instance, refine AA in AA+, AA and AA− where AA+ and AA− have lower respectively higher PDs than AA. Later in Section 1.1.1.2 we will work with the fine rating scale from Moody's. As an example and to underline what we just explained, Table 1.1 shows a definition of rating grades as it is used by Standard & Poor's. The wording in the table makes explicit that a rating grade and its assigned default probability address the *creditworthiness* of a client. The table in the upper half of Figure 1.1 shows a discretization of PDs to rating grades, this time w.r.t. Moody's ratings and their data history. The procedure of discretization of PDs, namely the assignment of a PD to every rating grade in the given rating scale is called a *rating calibration*; see Section 1.1.1.2. But before we come to that we want to briefly discuss rating systems in general.

One can divide the universe of rating systems into four broad categories which we will briefly describe in the sequel. It is important to note *that the rating type categories as we introduce them are not fully disjoint. In many cases a rating system has a main flavor but combines it with technology from some other rating model type. For instance, a rating model could be causal in principal but also use elements from scoring theory and regression.*

TABLE 1.1: S&P Rating Categories [172].

AAA	*best credit quality* *extremely reliable with regard to financial obligations*
AA	*very good credit quality* *very reliable*
A	*more susceptible to economic conditions* *still good credit quality*
BBB	*lowest rating in investment grade*
BB	*caution is necessary* *best sub-investment credit quality*
B	*vulnerable to changes in economic conditions* *currently showing the ability to meet its financial obligations*
CCC	*currently vulnerable to non-payment* *dependent on favorable economic conditions*
CC	*highly vulnerable to a payment default*
C	*close to or already bankrupt* *payments on the obligation currently continued*
D	*payment default on some financial obligation has actually occured*

Causal Rating Systems

We consider this type of rating system as superior to all other approaches. Whenever possible, this should be the way to proceed. As the name indicates, *causal rating systems* rely in their mechanism on a causal relationship between underlying credit risk drivers and the default event of an asset or borrower. To mention an example, ratings assigned to tranches in collateralized debt obligations (CDO; see [24]) typically are of causal type because the CDO model derives scenarios where the considered tranche is hit by a loss as well as the loss severity of the tranche as a direct consequence of "turbulences" in the underlying reference portfolio of credit-risky instruments. The model-derived hitting probability, for instance, can then be mapped onto a rating scale such that, for instance, a tranche with low hitting probability might have a letter combination like AAA or AA whereas a tranche with a low capital cushion below might get a rating letter combination of B or even in the C-range.

Why do we think that causal rating models are the best way to think about ratings? The reason is that a causal model approach forces the modeler to extensively analyze, understand and model the "true" mechanism of default. This is under all circumstances the best a modeler can do. For instance, a CDO model requires a fully-fledged model for both the cash flow structure of the CDO as well as the credit risk of the underlying reference portfolio. Causal models force the modeling team to really understand how defaults can happen and how losses will accumulate under certain circumstances.

As another important example let us briefly touch on causal ratings for public companies. The most famous representative of this type of rating systems is the concept of *Expected Default Frequencies* (EDF) from Moody's KMV[3]. An example of how Moody's KMV proceeds in their model is summarized in Section 1.2.3 and in Chapter 3. Note that Moody's KMV is continuously updating and improving their model framework and, therefore, our outline of their approach is indicative and illustrative only.

As a last example for default probabilities with a causal background based on market data we mention spread-implied default probabilities for companies with public debt outstanding. Spread-implied means

[3]See: www.moodyskmv.com; see also www.creditedge.com.

that default probabilities are derived from credit spreads of traded products like corporate bonds and credit derivatives; see Chapter 7.

Balance Sheet Scorings

In some situations a causal approach is rather difficult to follow or maybe it is even impossible to directly model the default mechanism. In such cases one can switch to *scoring systems* which are a good choice and well-established in rating units in banks across the globe. For instance, whereas for stock exchange-listed corporate clients a causal modeling of PDs is market standard as mentioned before, it is hardly thinkable to follow a causal approach for private corporates. Moreover, there are many companies which do not have a so-called external rating, which is a rating assigned by the afore-mentioned rating agencies. In such cases, a balance sheet scoring model is the usual approach to assign a bank-internal[4] rating to such companies. This is typically done by the credit analysts of the bank based on the rating tools developed by the rating quant team. For rating assignment the credit analysts consider various different quantitative and qualitative *drivers* of the considered firm's economic future like, for instance,

- Future *earnings* and *cashflows*,
- *debt*, *short- and long-term liabilities*, and *financial obligations*,
- *capital structure* (e.g., *leverage*),
- *liquidity* of the firm's assets,
- situation (e.g., political, social, etc.) of the firm's home *country*,
- situation of the *market* (e.g., *industry*), in which the company has its main activities,
- *management quality*, company *structure*, etc.

[4]Without going into details we would like to add that banks always should base the decision about creditworthiness on their bank-internal rating systems. As a main reason one could argue that banks know their customers best. Moreover, it is well known that external ratings do not react quickly enough to changes in the economic health of a company. Banks should be able to do it better based on their long-term relationship with their customers.

From this by no means exhaustive list it can be read-off that *rating drivers* can be *quantitative* as well as *qualitative*. To mention another important example, succession planning can be important for smaller firms but can not be captured as a solid quantitative figure like, for instance, a debt-equity relation.

It is best practice in banking that ratings as an outcome of a statistical tool are always re-evaluated by the credit analyst who makes the credit decision which leads to "approved" or "rejected". Credit analysts typically have, in line with their credit decision competence, the right to *overrule* or *override* the calculated rating. In most of the cases this will be an override to a better or worse rating grade by not more than one or two notches. The *overruling quote* which measures the relation of overruled ratings compared to overall assigned ratings is a good measure of the acceptance of a rating system by the practitioners in the credit unit, namely, the credit analysts who distinguish themselves from the rating quants who developed the rating system. An example for a "no-concern" value of an overruling quote is $5-10\%$ give or take, depending on the considered client segment. Overruling competence is crucial because especially for smaller firms one can expect that certain aspects driving the ability to pay of the client might not be captured by a standardized statistical tool.

The afore-mentioned *quantitative drivers* of the rating are taken from the balance sheet and annual report of the borrowing company. These sources of information are important for the lending credit institute because it is pretty much all one can get if a company is not listed at an exchange and has no public debt outstanding. Because the balance sheet is the primary source of information the name of the approach, *balance sheet scoring*, does not come much as a surprise. The afore-mentioned rating drivers are then grouped and set in relation to form a list of *balance sheet ratios* which are mapped into so-called *scores* as a metric-scale measure of default remoteness. The total score of a client is then based on a weighted sum of ratio transformation functions, often involving a lot of *regression analysis* in the development process. The score of a client is then *calibrated* to a PD based on the history of default frequencies; see Section 1.1.1.2. A typical calibration function in this context could look as follows,

$$\mathrm{PD}_{client} \;=\; \frac{1}{1+\exp(-\mathrm{SCORE}_{client})}\,, \tag{1.4}$$

where SCORE_{client} represents the final score based on the afore-mentioned sum of transformed ratios. Readers interested in a deeper dive into internal rating systems should read the the article from Fritz, Luxenburger and Miehe [72]. Equation (1.4) is a representative of the class of so-called *logit* calibration functions which is a common transformation approach to get PDs out of scores in balance sheet scorings.

An industry example for an off-the-shelf model to obtain ratings for private companies is the RiskCalc[5] model by Moody's KMV.

Private Client Scorings

In the same way as one can build scoring systems for private companies one can derive scoring systems for private individuals, for instance, for clients which borrow money in the context of a residential mortgage. The basic mechanism is exactly the same but the rating score drivers are different. For instance, personal wealth, income situation, family context, etc. are typical drivers for a private client scoring. Moreover, practitioners know that the main drivers for default in a residential mortgage lending context and, more general, in any lending to private individuals are unemployment, divorce, and poor health.

Expert Rating Systems

There are portfolios where over many years hardly any default occurred. For example, municipalities in Switzerland can be grouped into a portfolio where almost no defaults occurred. In such situations it is difficult to work with balance sheet scorings because the number of defaults in the portfolio is too low for deriving statistically sound conclusions. This deficiency gave such portfolios a name. They are called *low default portfolios*; see Wilde and Lee [188] for more information and for ideas about how one could treat such portfolios. When objective data is missing, expert opinion is requested to come up with at least something. A common approach then is to overcome the problem of missing defaults by involving groups of experts in the considered segment in the bank to assign manual ratings to test cases. The modeling team then can apply techniques like *ordinal response methods* to establish a ranking of default remoteness (in the same way as rating grades) and later also a calibration to PDs for the client segment. Because of

[5] See: www.moodyskmv.com

the expert involvement, such rating approaches are called *expert rating systems*.

In general, and for all four approaches which we briefly outlined, one can measure the quality of a rating system by means of its so-called *discriminatory power*, a concept which goes beyond the purpose of this exposition. A beginner's crash course can be found in [16], pages 36 to 41. Readers interested in the topic should ask some web search engine with the keyword "discriminatory power" which will give them a large list of papers on the predictive power of rating systems, how predictive power is measured, how it can be improved and even what it means for a pricing system if discriminatory power can be increased due to a rating revision.

1.1.1.2 Calibration of Default Probabilities to Ratings

As mentioned several times before, the process of assigning a default probability to a rating grade (say, a letter combination in the sense of the rating agencies) is called a *calibration*. In this paragraph we will demonstrate how such a calibration works in principal. In a true-life situation the process is a bit more complex but for illustrative purposes our outline will be sufficient.

The end product of a calibration of default probabilities to ratings is a mapping of letter combinations (ratings) to default probabilities,

$$R \mapsto \mathrm{PD}(R),$$

such that to every rating R a certain default probability $\mathrm{PD}(R)$ is assigned. For example, the domain of such a mapping in case of Standard & Poor's letter ratings would be $\{AAA, AA, ..., C\}$ and, because we are dealing with probabilities, the range of the calibration function is in all cases the unit interval $[0, 1]$.

In the following, we explain by means of Moody's data how a calibration of default probabilities to external ratings can be done. From Moody's website or from other resources it is easy to get access to their study [141] of *historic corporate bond defaults*. There one can find a table like the one shown in Table 1.2 (see [141] Exhibit 40) showing historic default frequencies for the years 1983 up to 2000. The same exercise we are doing now can be done with Moody's most recent default study [167] or with comparable data from Standard & Poor's [171].

TABLE 1.2: Moody's Historic Corporate Bond Default Frequencies.

Rating	1983	1984	1985	1986	1987	1988
Aaa	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa3	0.00%	1.06%	0.00%	4.82%	0.00%	0.00%
Ba1	0.00%	1.16%	0.00%	0.88%	3.73%	0.00%
Ba2	0.00%	1.61%	1.63%	1.20%	0.95%	0.00%
Ba3	2.61%	0.00%	3.77%	3.44%	2.95%	2.59%
B1	0.00%	5.84%	4.38%	7.61%	4.93%	4.34%
B2	10.00%	18.75%	7.41%	16.67%	4.30%	6.90%
B3	17.91%	2.90%	13.86%	16.07%	10.37%	9.72%

Rating	1989	1990	1991	1992	1993	1994
Aaa	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa3	1.40%	0.00%	0.00%	0.00%	0.00%	0.00%
A1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa1	0.00%	0.00%	0.76%	0.00%	0.00%	0.00%
Baa2	0.80%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa3	1.07%	0.00%	0.00%	0.00%	0.00%	0.00%
Ba1	0.79%	2.67%	1.06%	0.00%	0.81%	0.00%
Ba2	1.82%	2.82%	0.00%	0.00%	0.00%	0.00%
Ba3	4.71%	3.92%	9.89%	0.74%	0.75%	0.59%
B1	6.24%	8.59%	6.04%	1.03%	3.32%	1.90%
B2	8.28%	22.09%	12.74%	1.54%	4.96%	3.66%
B3	19.55%	28.93%	28.42%	24.54%	11.48%	8.05%

Rating	1995	1996	1997	1998	1999	2000
Aaa	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.29%
Baa2	0.00%	0.00%	0.00%	0.32%	0.00%	0.00%
Baa3	0.00%	0.00%	0.00%	0.00%	0.34%	0.98%
Ba1	0.00%	0.00%	0.00%	0.00%	0.47%	0.91%
Ba2	0.00%	0.00%	0.00%	0.61%	0.00%	0.66%
Ba3	1.72%	0.00%	0.47%	1.09%	2.27%	1.51%
B1	4.35%	1.17%	0.00%	2.13%	3.08%	3.25%
B2	6.36%	0.00%	1.50%	7.57%	6.68%	3.89%
B3	4.10%	3.36%	7.41%	5.61%	9.90%	9.92%

As said earlier, note that in our illustrative example we choose the *fine ratings scale* of Moody's, making finer differences regarding the creditworthiness of obligors.

Now, an important observation is that for best ratings no defaults at all have been observed. This is not as surprising as it looks at first sight: For example rating class Aaa is often calibrated with a default probability of 2 bps ("bp" stands for *"basispoint"* and means 0.01%), essentially meaning that one expects a Aaa-default on average twice in $10,000$ years. This is a long time to go; so, one should not be surprised that quite often best rating grades are lacking any default history. Nevertheless we believe that it would not be correct to take the historical zero-balance as an indication that these rating classes are risk-free opportunities for credit investment. Therefore, we have to find a way to assign small but positive default probabilities to those ratings.

Figure 1.1 shows our "quick-and-dirty working solution" of the problem, where we use the attribute "quick-and-dirty" because, as mentioned before, in a true life situation one would try to do the calibration a little more sophisticatedly[6].

Summarized in a recipe-like style, the calibration has three steps:

1. Denote by $h_i(R)$ the historic default frequency of rating class R for year i, where i ranges from 1983 to 2000. For example, $h_{1993}(Ba1) = 0.81\%$. Then compute the mean value and the standard deviation of these frequencies over the years, where the rating is fixed, namely

$$m(R) = \frac{1}{18}\sum_{i=1983}^{2000} h_i(R) \qquad \text{and}$$

$$s(R) = \sqrt{\frac{1}{17}\sum_{i=1983}^{2000} \big(h_i(R) - m(R)\big)^2} \; .$$

   The mean value $m(R)$ for rating R is our first guess of the potential default probability assigned to rating R. The standard deviation $s(R)$ gives us some insight about the volatility and therefore about the error we eventually make when believing that $m(R)$

[6] For example, one could look at investment and sub-investment grades separately.

is a good estimate of the default probability of R-rated obligors. Figure 1.1 shows the values $m(R)$ and $s(R)$ for the considered rating classes. Because even best rated obligors are not free of default risk, we write "not observed" in the cells corresponding to $m(R)$ and $s(R)$ for ratings R=Aaa,Aa1,Aa2,A1,A2,A3 (ratings where no defaults have been observed) in Figure 1.1.

2. Next, we plot the mean values $m(R)$ into a coordinate system, where the x-axis refers to the rating classes (here numbered from 1 (Aaa) to 16 ($B3$)). One can see in the chart in Figure 1.1 that on a *logarithmic scale* the mean default frequencies $m(R)$ can be fitted by a regression line. Here we should add a comment that there is strong evidence from various empirical default studies that default frequencies grow exponentially with decreasing creditworthiness. For this reason we have chosen an exponential fit (linear on logarithmic scale). Using standard regression theory, see, e.g., [155] Chapter 4, or by simply using any software providing basic statistical functionality, one can easily obtain the following exponential function fitting our data:

$$\mathrm{PD}(x) \;=\; 3\times 10^{-5}\, e^{\,0.5075\,x} \qquad (x = 1, ..., 16).$$

3. As a last step, we use our regression equation for the estimation of default probabilities $\mathrm{PD}(x)$ assigned to rating classes x ranging from 1 to 16. Figure 1.1 shows our result, which we now call *a calibration of default probabilities to Moody's ratings*. Note that based on our regression even the best rating Aaa has a small but positive default probability. Moreover, we can hope that our regression analysis has smoothed out sampling errors from the historically observed data.

Although there would be much more to say about default probabilities, we stop the discussion for now and turn our attention to EAD and LGD from Formula (1.3).

1.1.2 The Exposure at Default

EAD is the quantity in Equation (1.3) specifying the exposure the bank does have to its borrower. In practice, banks grant to obligors so-called *credit lines* which function like a credit limit for the single-obligor exposure.

Rating	Mean	Standard-Deviation	Default Probability
Aaa	not observed	not observed	0.005%
Aa1	not observed	not observed	0.008%
Aa2	not observed	not observed	0.014%
Aa3	0.08%	0.33%	0.023%
A1	not observed	not observed	0.038%
A2	not observed	not observed	0.063%
A3	not observed	not observed	0.105%
Baa1	0.06%	0.19%	0.174%
Baa2	0.06%	0.20%	0.289%
Baa3	0.46%	1.16%	0.480%
Ba1	0.69%	1.03%	0.797%
Ba2	0.63%	0.86%	1.324%
Ba3	2.39%	2.35%	2.200%
B1	3.79%	2.49%	3.654%
B2	7.96%	6.08%	6.070%
B3	12.89%	8.14%	10.083%

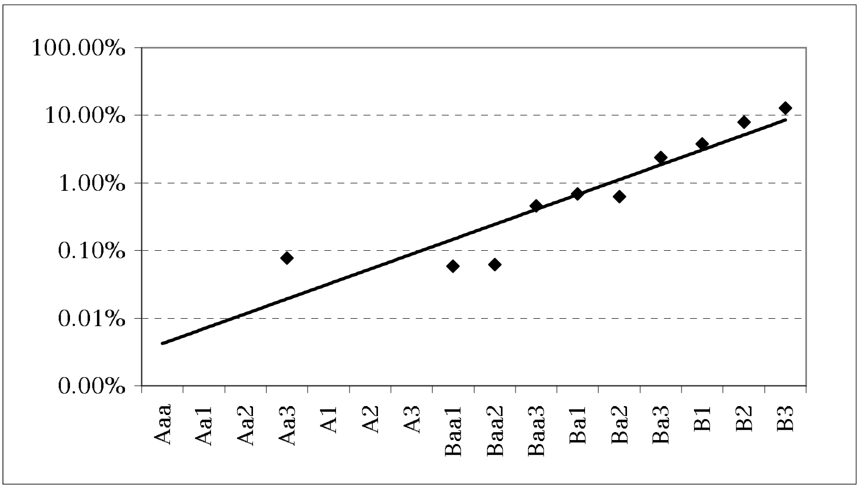

FIGURE 1.1: Calibration of ratings to default probabilities.

For the sake of a better understanding let us introduce a working example which will accompany us through this whole section on EAD. Let us assume that a credit analyst assigns to a borrower, say, a medium-sized firm, a credit line with a total limit of EUR 20m. Let us assume that the credit line is structured in the following way:

- Total credit line is EUR 20m.
- The borrower can draw EUR 12m as cash and can use the remaining EUR 8m of the credit line for so-called *contingent liabilities*, e.g., guarantees or comparable credit constructs but not for cash.

Now let us assume the borrower has drawn EUR 10m already. This part of the credit line is then called the *outstandings* of the client's exposure. The remaining open EUR 10m of the credit line are called *commitments*. In other words, the outstandings refer to the portion of the overall client exposure the obligor is already using. There is no randomness involved, drawn is drawn, and if the obligor defaults then the outstandings are subject to recovery and in a worst case situation could potentially be lost in total.

Of course, there is some time dynamics involved in outstandings. For instance, if the obligor pays back borrowed amounts over time then it makes a big difference whether an obligor defaults today or sometime in the future. Especially in mortgages where one often finds pre-determined amortization schemes the timing of default has a direct impact on the EAD. In our example one would need to accurately evaluate incoming cash from repayments versus newly opened parts of the credit line of the obligor which are subject to be drawn again, depending on the lending contract framework the bank and the obligor agreed to and signed.

The commitments, i.e., the remaining open EUR 10m of the borrower's credit line, are rather tricky to take into account. There is no other way than considering the exposure arising from the open part of the credit line as a random variable. So in our particular example we have EUR 10m open in the credit line but only EUR 2m can be drawn as cash. The other 8m can only be used for contingent liabilities. The two parts of the open line address different random effects:

- The EUR 2m which can be drawn as cash are driven by the likelihood that the borrower draws on them as well as by the fraction quantifying how much of the 2m she or he draws. Describing the situation by a simple equation we could write

$$\mathrm{EAD}_{cash} \; = \; \mathbf{1}_D \times X \times [2\mathrm{m}] \; (\mathrm{EUR}) \tag{1.5}$$

for the random exposure adding to current outstandings. Here, D describes the event (in the σ-field $\mathcal{F}$) that the obligor draws on the open cash credit line and X is a random variable defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $X(\omega) \in [0, 1]$ for each $\omega \in \Omega$ quantifying the random fraction describing how much of the open 2m line is drawn. Altogether we are dealing with two random variables here. The equation could be made significantly more complex if one wants to take a stepwise drawing behavior into account, say, the obligor draws a partial amount in the future and another amount even later, and so on.

- The remaining EUR 8m which can be used for contingent liabilities are also subject to various random effects. First of all, there are again one or more indicator variables reflecting the optionality of usage of free parts of the credit line. Second, there is randomness in the fact that contingent liabilities not necessarily lead to cash exposure. A guarantee has no real exposure as of today but might converge into exposure in the future. Such random effects are typically treated by so-called *conversion factors*.

Let us put the pieces together for EAD calculation. We assume that the bank has a huge loss database useful for the calibration of exposure parameters. One common exposure parameter is the so-called *draw-down factor* (DDF). In our example it could be the case that the bank is able to say that the given type of obligor tends to draw on the free part of the credit line (EUR 2m) in 80% of the cases and on average uses 60% of the available cash. In other words, based on historic experience the bank obtains parameters in (1.5) like

$$\mathbb{P}(D) \; = \; 80\% \quad \text{and} \quad \mathbb{E}[X] \; = \; 60\%.$$

Assuming independence of $\mathbf{1}_D$ and X, this leads to an expected cash exposure for the unused part of the cash credit line of

$$\mathbb{E}[\mathrm{EAD}_{cash}] \; = \; \mathbb{P}(D) \times \mathbb{E}[X] \times [2\mathrm{m}] \; (\mathrm{EUR}) \; = \; 48\% \times [2\mathrm{m}] \; (\mathrm{EUR}).$$

The 48% would then be used as the DDF for this particular situation. Note that the DDF is one particular common example for conversion factors. For the contingent liability part of the credit line we assume again the existence of a rich database which allows for the calibration of a DDF of, say, 40% for the contingent liability part and a so-called *cash equivalent exposure factor* (CEEF) of 80% which is another conversion factor quantifying the conversion of the specific contingent liability, say, a guarantee, into a cash exposure. Altogether we obtain (assuming independence) the following representation for the EAD in our example:

$$
\begin{aligned}
\mathbb{E}[\mathrm{EAD}] &= [10m] + 48\% \times [2\mathrm{m}] + 32\% \times [8\mathrm{m}] \text{ (EUR)} \qquad (1.6)\\
&= [10m + 0.96m + 2.56m] \text{ (EUR)}\\
&= [13.52\mathrm{m}] \text{ (EUR)}
\end{aligned}
$$

where $32\% = 40\% \times 80\%$. So altogether our (expected) EAD is between the already utilized 10m and the overall committed 20m but higher than the committed cash line of 12m.

Our example provided some flavor on how complicated EAD calculations can be and in real life it actually is even more complex. For example, commitments of banks to clients often include various so-called *covenants*, which are *embedded options* which, for example, may force an obligor in times of financial distress to provide more collateral[7] or to renegotiate the terms of the loan.

A problem is that often the obligor has some informational advantage in that the bank recognizes financial distress of its borrowers with some delay. In case of covenants allowing the bank to close committed lines triggered by some early default indication, it really is a matter of timing whether the bank picks up such indications early enough to react before the customer has drawn on her or his committed lines. Bankers here often speak of a *race to default* which addresses the problem that distressed clients tend to exhaust their lines just before they default as much as possible.

The *Basel Committee on Banking Supervision*[8] provides conversion factors for banks who are unable or not allowed by their regulator to

[7]Collateral means assets securing a loan, e.g., mortgages, bonds, guarantees, etc. In case a loan defaults, the value of the collateral reduces the realized loss.

[8]The Basel Commitee coordinates the rules and guidelines for banking supervision. Its members are central banks and other national offices or government agencies responsible for banking supervision.

calibrate their own internal conversion factors like DDFs and CEEFs; see [149].

We stop here and come to the last of the three EL-relevant quantities, namely, the *loss-given-default* (LGD).

1.1.3 The Loss Given Default

A first distinction we need to make when it comes to LGDs is that of LGD as an amount of money and LGD as a percentage quote. The first mentioned is often denoted as \$LGD which means loss given default in monetary units. The concept of LGD is best demonstrated by means of an example in the same way as we proceeded for EAD.

Let us assume that a client has m credit products with the bank and pledged n collateral securities to the bank which can in case of default be used for recovery purposes in order to mitigate the realized loss arising from the client's default. Each credit product gets assigned an EAD such that for m credit products we get EAD_1, ..., EAD_m as well as expected recovery proceeds from the n collateral securities. We denote such recovery proceeds by $\$\text{REC}_1$, ..., $\$\text{REC}_n$. Such a constellation, having m credit products and n collateral securities is called an m-to-n situation. It can be difficult to get the interdependence and relation between products and collateral right, especially in cases where we have to deal with *dedicated collateral* which can be used for certain purposes under certain circumstances only. Here we assume that we can simply collect "good cash" (recovery proceeds) and "bad cash" (loss exposure) together in two separate buckets which we then compare to obtain our net balance with the defaulted client. What we get from that approach is the following:

$$\begin{aligned} \$\text{LGD} \;=\; \max\Big(0, (\text{EAD}_1 + \cdots + \text{EAD}_m) \\ -\;(\$\text{REC}_1 + \cdots + \$\text{REC}_n)\Big) \end{aligned} \tag{1.7}$$

which leads to a percentage LGD of

$$\text{LGD} \;=\; \frac{\$\text{LGD}}{\text{EAD}_1 + \cdots + \text{EAD}_m}. \tag{1.8}$$

Note that we easily wrote down the quantities $\$\text{REC}_i$ but, in fact, their derivation can be quite complex and needs a rich database storing historic proceeds from collateral security categories, collected with sufficient granularity. A typical discussion point in such calculations is, for instance, the time value of money. Recovery proceeds coming in later in time should be discounted in order to reflect the time value of money. The determination of an appropriate discount rate is just one out of many questions one has to solve in this context.

Summarizing one can say that LGD calibration is a long story and far from being trivial. The current regulatory framework forces banks with approval to use their internal PD, EAD and LGD calibrations to come up with good ideas on LGD calibration but we believe there is still a lot of ground to cover.

1.1.4 A Remark on the Relation between PD, EAD, LGD

Proposition 1.1.3 is based on the assumption that PD, EAD and LGD are independent. In real life this assumption is not realistic. When making such an assumption one should keep awareness that every calculation based on such an assumption creates a laboratory-like model environment in the same way as physicists make simplifying assumptions and create simplified environments in their laboratory.

Why is the independence assumption questionable? Let us focus on LGD and PD first. A fundamental principle in any market is the principle of *supply and demand*. Put in the context of defaults and losses this means the following. In a recession scenario one can expect that default rates increase. As a consequence, banks will be forced to sell collateral securities related to defaulted loans or assets to the market. This will increase supply for certain goods, for instance, in a residential mortgage crisis the market will be swampled with private homes offered for sale. A situation like this occurred in the subprime mortgage crisis in the US two years ago. Now, the principle of *supply and demand* leads to a price drop of such collateral securities which are now over-supplied in the market. But this in turn reduces recovery proceeds achievable by selling collateral to the market. Formula (1.7) shows that this leads to higher LGDs. So altogether we found that higher PDs in a recession can lead to higher LGDs which, neglecting time lagging, means that default rates and realized losses are positively related. One could also reformulate this statement and simply say that *defaults and recoveries to some extent are influenced by the same underlying systematic risk drivers* so that they can not be independent. A study by Altman et al. [3] shows empirical evidence for the comments we just made. What about EAD and PD? We mentioned before that in times of financial distress firms tend to draw on their open credit lines. This increases EADs in times where default rates are going high systematically. So even EAD can not safely be considered as independent from default rates and default rates are the basis for PD estimation.

For a nice approach to dependent LGD modeling we refer to the paper by Hillebrand [92].

1.2 Unexpected Loss

At the beginning of this chapter we introduced the EL of a transaction and imagined it as an insurance or loss reserve in order to cover losses the bank expects from historical default experience. But a focus on *expected* losses is not enough. In fact, the bank should in addition to the expected loss also make sure that thay have a good understanding on how much money would be necessary for covering *unexpected* losses where the attribute 'unexpected' addresses losses exceeding the historic average observed in the past. As a measure for the magnitude of the deviation of losses from the EL, the standard deviation of the loss variable $\tilde{L}$ as defined in (1.1) is a natural first choice.

1.2.1 Definition *The standard deviation*

$$\mathrm{UL} \;=\sqrt{\mathbb{V}[\tilde{L}]} \;=\; \sqrt{\mathbb{V}\big[\mathrm{EAD}\times\mathrm{LGD}\times L\big]}$$

of the loss variable $\tilde{L}$ from (1.1) is called the unexpected loss of the underlying loan or asset.

In analogy to Proposition 1.1.3 one can prove the following representation formula for the UL of a loan.

1.2.2 Proposition *Under the assumption that EAD is deterministic and that LGD and the default event D are independent, the unexpected* *loss of a loan is given by*

$$\mathrm{UL} \;=\; \mathrm{EAD} \times \sqrt{\mathbb{V}[\mathrm{LGD}] \times \mathrm{PD} + \mathbb{E}[\mathrm{LGD}]^2 \times \mathrm{PD}(1 - \mathrm{PD})}\;.$$

Proof. First, we square UL and get

$$\begin{aligned}\mathrm{UL}^2 &= \mathbb{V}\big[\mathrm{EAD} \times \mathrm{LGD} \times L\big] \\ &= \mathrm{EAD}^2 \times \mathbb{V}\big[\mathrm{LGD} \times L\big].\end{aligned}$$

From the identity $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ we get

$$\begin{aligned}\mathbb{V}\big[\mathrm{LGD} \times L\big] &= \mathbb{E}\big[\mathrm{LGD}^2 \times L^2\big] - \mathbb{E}\big[\mathrm{LGD} \times L\big]^2 \\ &= \mathbb{E}\big[\mathrm{LGD}^2\big] \times \mathbb{E}\big[L^2\big] - \mathbb{E}[\mathrm{LGD}]^2 \times \mathbb{E}[L]^2\end{aligned}$$

because LGD and L are independent by assumption. Because $L = \mathbf{1}_D$ is a Bernoulli variable we have $\mathbb{E}[L^2] = \mathbb{E}[L] = \mathbb{P}[D] = \mathrm{PD}$ such that

$$\mathbb{V}\big[\mathrm{LGD} \times L\big] \;=\; \mathbb{E}\big[\mathrm{LGD}^2\big] \times \mathrm{PD} - \mathbb{E}[\mathrm{LGD}]^2 \times \mathrm{PD}^2.$$

Now we add $0 = \mathrm{PD} \times \mathbb{E}[\mathrm{LGD}]^2 - \mathrm{PD} \times \mathbb{E}[\mathrm{LGD}]^2$ and find

$$\mathbb{V}\big[\mathrm{LGD} \times L\big] \;=\; \mathrm{PD} \times \mathbb{V}[\mathrm{LGD}] + \mathbb{E}[\mathrm{LGD}]^2 \times \mathrm{PD}(1 - \mathrm{PD}).$$

Collecting the pieces together we have our proof. □

We are now ready for a major step forward. So far we always looked at the credit risk of a single facility although banks have to manage *large portfolios* consisting of many different products with different risk characteristics. We therefore will now indicate how one can model the total loss of a credit portfolio.

For this purpose we consider a family of m loans

$$\begin{aligned}\tilde{L}_i &= \mathrm{EAD}_i \times \mathrm{LGD}_i \times L_i\,, \qquad (1.9)\\ \text{with} \quad L_i &= \mathbf{1}_{D_i}\,, \quad \mathbb{P}(D_i) = \mathrm{PD}_i\;.\end{aligned}$$

which we call a *portfolio* from now on.

1.2.3 Definition *A portfolio is a collection of loss variables $\tilde{L}_i$ as in (1.9). The portfolio loss is then defined as the random variable*

$$\tilde{L}_{PF} \;=\; \sum_{i=1}^{m} \tilde{L}_i \;=\; \sum_{i=1}^{m} \mathrm{EAD}_i \times \mathrm{LGD}_i \times L_i\;. \qquad (1.10)$$

Similar to the "standalone" quantities EL and UL we now obtain portfolio quantities EL_{PF} and UL_{PF}, defined as follows.

1.2.4 Definition *Given a portfolio of m loss variables as in (1.9), the expected and unexpected loss of the portfolio are given by*

$$\mathrm{EL}_{PF} \;=\; \mathbb{E}[\tilde{L}_{PF}] \quad \textit{and} \quad \mathrm{UL}_{PF} \;=\; \sqrt{\mathbb{V}[\tilde{L}_{PF}]}.$$

We briefly call them portfolio EL and portfolio UL in the sequel.

For the portfolio EL one has the following representation.

1.2.5 Proposition *Given a portfolio of m loss variables as in (1.9), the portfolio EL is always given by*

$$\mathrm{EL}_{PF} \;=\; \sum_{i=1}^{m} \mathrm{EL}_i \;=\; \sum_{i=1}^{m} \mathrm{EAD}_i \times \mathrm{LGD}_i \times \mathrm{PD}_i \tag{1.11}$$

where EL_i denotes the EL of the single loss $\tilde{L}_i$.

Proof. The assertion follows directly from the linearity of $\mathbb{E}[\,\cdot\,]$. □

In case of the portfolio UL, linearity in general holds only if the loss variables $\tilde{L}_i$ are pairwise uncorrelated (see BIENAYMÉ's Theorem in [12] Chapter 8). If the loss variables are correlated we can no longer expect that variance behaves linearly. Unfortunately, correlated loss variables are the standard case and the modeling of correlated variables is what this book is all about. So the portfolio UL is the first risk quantity we meet where correlations (say, covariances) between single-name risks play a fundamental role.

1.2.6 Proposition *Given a portfolio of m loss variables as in (1.9) with deterministic* EAD*'s, the portfolio UL is given by*

$$\mathrm{UL}_{PF} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} \mathrm{EAD}_i \times \mathrm{EAD}_j \times \mathrm{Cov}[\mathrm{LGD}_i \times L_i\,,\, \mathrm{LGD}_j \times L_j]}\;. \tag{1.12}$$

Proof. The proposition is a direct consequence of the formula

$$\mathbb{V}\Big[\sum_{i=1}^{m} c_i X_i\Big] \;=\; \sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j \mathrm{Cov}[X_i, X_j]$$

for square-integrable random variables $X_1, ..., X_m$ and arbitrary constants $c_1, ..., c_m$. □

1.2.7 Proposition *Given a portfolio of m loss variables as in (1.9) with deterministic* EADs *and deterministic* LGDs *we have*

$$\mathrm{UL}_{PF}^2 \;=\; \sum_{i,j=1}^{m} \mathrm{EAD}_i \times \mathrm{EAD}_j \times \mathrm{LGD}_i \times \mathrm{LGD}_j \times$$

$$\times \sqrt{\mathrm{PD}_i(1-\mathrm{PD}_i)\mathrm{PD}_j(1-\mathrm{PD}_j)}\,\rho_{ij}$$

where $\rho_{ij} = \mathrm{Corr}[L_i, L_j] = \mathrm{Corr}[\mathbf{1}_{D_i}, \mathbf{1}_{D_j}]$ denote the so-called default correlation between counterparties (or assets) i and j.

Proof. The proof follows from the representation of the portfolio UL according to Equation 1.12, from

$$\mathrm{Cov}[L_i, L_j] \;=\; \sqrt{\mathbb{V}[L_i]\,\mathbb{V}[L_j]}\,\mathrm{Corr}[L_i, L_j]$$

and from $\mathbb{V}[L_i] = \mathrm{PD}_i(1-\mathrm{PD}_i)$ for each $i = 1, ..., m$. □

Before continuing we want to spend a moment with thinking about the meaning and interpretation of correlation. For simplicity let us consider a portfolio consisting of two loans with LGD= 100% and EAD= 1. We then only deal with L_i for $i = 1, 2$, and we set $\rho = \mathrm{Corr}[L_1, L_2]$ and $p_i = \mathrm{PD}_i$. Then, the squared UL of our portfolio is given by

$$\mathrm{UL}_{PF}^2 \;=\; p_1(1-p_1)+p_2(1-p_2)+2\rho\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}\;. \quad (1.13)$$

We consider three possible cases regarding the default correlation ρ:

- $\rho = 0$. In this case, the third term in (1.13) vanishes. Although unusual in this context, one could say that $\rho = 0$ stands for *optimal diversification.* The concept of diversification is easily explained. Investing in many different assets generally reduces the overall portfolio risk, because usually it is very unlikely to see a large number of loans defaulting all at once. The less the loans in the portfolio have in common, the higher the chance that default of one obligor does not mean a lot to the economic future of other loans in the portfolio. The case $\rho = 0$ refers to a situation where the loans in the portfolio are completely unrelated. Interpreting the UL as a substitute[9] for portfolio risk, we see that this case minimizes the risk of joint defaults.

- $\rho > 0$. In this case our two counterparties are interrelated in that default of one counterparty increases the likelihood that the other counterparty will also default. We can make this precise by looking at the conditional default probability of counterparty 2 under the condition that obligor 1 already defaulted:

$$\mathbb{P}[L_2 = 1 \mid L_1 = 1] = \frac{\mathbb{P}[L_1 = 1, L_2 = 1]}{\mathbb{P}[L_1 = 1]} = \frac{\mathbb{E}[L_1 L_2]}{p_1} \tag{1.14}$$

$$= \frac{p_1 p_2 + \mathrm{Cov}[L_1, L_2]}{p_1} = p_2 + \frac{\mathrm{Cov}[L_1 L_2]}{p_1}.$$

So we see that positive correlation respectively covariance leads to a conditional default probability higher (because $\mathrm{Cov}[L_1, L_2] > 0$) than the unconditional default probability p_2 of obligor 2. In other words, in case of positive correlation any default in the portfolio has an important implication on other facilities in the portfolio, namely that there might be more losses to be encountered. The extreme case in this scenario is the case of *perfect correlation* ($\rho = 1$). In the case of $p = p_1 = p_2$, Equation (1.13) shows that in the case of perfect correlation we have $\mathrm{UL}_{PF} = 2\sqrt{p(1-p)}$, essentially meaning that our portfolio contains the risk of only one obligor but with double intensity (*concentration risk*). In this situation it follows immediately from (1.14) that default of one obligor makes the other obligor defaulting almost surely.

- $\rho < 0$. This is the mirrored situation of the case $\rho > 0$. We therefore only discuss the extreme case of perfect anti-correlation ($\rho = -1$). One then can view an investment in asset 2 as an

[9]Note that in contrast to the EL, the UL is the "true" uncertainty the bank faces when investing in a portfolio because it captures the deviation from the expectation.

almost *perfect hedge* against an investment in asset 1, if (additionally to $\rho = -1$) the characteristics (exposure, rating, etc.) of the two loans match. Admittedly, this terminology makes much more sense when following a *marked-to-market*[10] approach to loan valuation, where an increase in market value of one of the loans immediately (under the assumption $\rho = -1$) would imply a decrease in market value of the other loan. However, from (1.13) it follows that in the case of a perfect hedge the portfolio's UL completely vanishes ($\mathrm{UL}_{PF} = 0$). This means that our perfect hedge (investing in asset 2 with correlation -1 w.r.t. a comparable and already owned asset 1) completely eliminates (neutralizes) the risk of asset 1.

We now turn to the important notion of economic capital.

1.2.1 Economic Capital

We have learned so far that banks should hold some capital cushion against unexpected losses. However, defining the UL of a portfolio as the *risk capital* saved for cases of financial distress is not the best choice, because there might be a significant likelihood that losses will exceed the portfolio's EL by more than one standard deviation of the portfolio loss. Therefore one seeks other ways to quantify risk capital, hereby taking a *target level* of *statistical confidence* into account.

The most common way to quantify risk capital is the concept of *economic capital*[11]

1.2.8 Definition *Let a portfolio $(\tilde{L}_i)_{i=1,\ldots,m}$ be given. The economic capital (EC) w.r.t. a prescribed level of confidence α is defined as the α-quantile of the portfolio loss $\tilde{L}_{PF}$ minus the EL of the portfolio:*

$$\mathrm{EC}_\alpha \;=\; q_\alpha - \mathrm{EL}_{PF} \tag{1.15}$$

where q_α is the α-quantile of $\tilde{L}_{PF}$ given as

$$q_\alpha \;=\; \inf\{q > 0 \mid \mathbb{P}[\tilde{L}_{PF} \le q] \ge \alpha\}\ . \tag{1.16}$$

[10]In a marked-to-market framework loans do not live in a two-state world (default or survival) but rather are evaluated w.r.t. their *market value.*

[11]Synonymously called *Capital at Risk* (CaR) in the literature; the quantile q_α from Definition 1.2.8 sometimes is called the (credit) *Value-at-Risk* (VaR).

For example, if the level of confidence is set to $\alpha = 99.98\%$, then the risk capital EC_α will (on average) be sufficient to cover unexpected losses in 9,998 out of 10,000 years, hereby assuming a planning horizon of one year. Unfortunately, under such a calibration one can on the other side expect that in 2 out of 10,000 years the economic capital $EC_{99.98\%}$ will not be sufficient to protect the bank from insolvency. This is the downside when calibrating risk capital by means of quantiles. However, today most major banks use an EC framework for their internal credit risk model.

The reason for reducing the quantile q_α by the EL is due to the "best practice" of decomposing the total risk capital (i.e., the quantile) into a first part covering expected losses and a second part meant as a cushion against unexpected losses. Altogether the pricing of a loan typically takes several cost components into account. First of all, the price of the loan should include the costs of administrating the loan and maybe some kind of upfront fees. Second, expected losses are charged to the customer, hereby taking the creditworthiness captured by the customer's rating into account. More risky customers have to pay a higher risk premium than customers showing high credit quality. Third, the bank will also ask for some compensation for taking the risk of unexpected losses coming with the new loan into the bank's credit portfolio. The charge for unexpected losses is often calculated as the *contributory* EC of the loan in reference to the lending bank's portfolio; see Chapter 5. In contrast to the EL which is priced in completely, the EC often is only partially charged in form of

$$\text{EC-charge} \;=\; EC_{contributory} \times \text{HR}\ [\%]$$

where HR denotes some *hurdle rate*, e.g., 25%.

Note that there is an important difference between the EL and the EC charges: The EL charge is independent from the composition of the reference portfolio, whereas the EC charge strongly depends on the current composition of the portfolio in which the new loan will be included. For example, if the portfolio is already well diversified, then the EC charge as a price for taking unexpected risk does not have to be as high as it would be in the case for a portfolio in which, for example, the new loan would induce or increase some concentration risk. Summarizing one can say the EL charges are *portfolio independent*, but EC charges are *portfolio dependent*. This makes the calculation

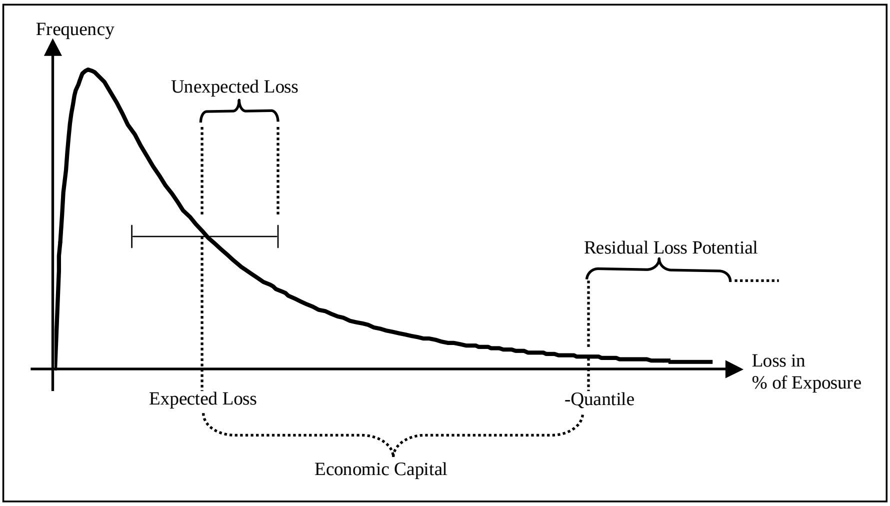

FIGURE 1.2: The portfolio loss distribution.

of the contributory EC in pricing tools more complicated, because one always has to take the complete reference portfolio into account. *Risk contributions* will be discussed in Chapter 5.

An alternative to EC is a risk capital based on *Expected Shortfall* (ES). A capital definition according to ES very much reflects an insurance point of view of the credit risk business. Today it is known that ES is superior to EC as a risk capital measure for various reasons. We will come back to ES and its properties in Chapter 5.

1.2.2 The Loss Distribution

All risk quantities on a portfolio level are based on the portfolio loss variable $\tilde{L}_{PF}$. Therefore it does not come much as a surprise that the distribution of $\tilde{L}_{PF}$, the so-called *loss distribution* of the portfolio, plays a central role in credit risk management. In Figure 1.2 it is illustrated that all risk quantities of the credit portfolio can be identified by means of the loss distribution of the portfolio. This is an important observation, because it shows that in cases where the distribution of the portfolio loss can only be determined in an empirical way one can use empirical statistical quantities as a proxy for the respective "true" risk quantities.

In practice there are various ways to generate a loss distribution; see Section 2.8. The first method is based on *Monte Carlo simulation*; the second is based on a so-called *analytical approximation*. We describe both methods in short in the following section but come back to the topic of generating loss distributions in greater detail in Section 2.8.

1.2.2.1 Monte Carlo Simulation of Losses

In a Monte Carlo simulation, losses are simulated and tabulated in form of a *histogram* in order to obtain an *empirical loss distribution* of the underlying portfolio. The *empirical distribution function* can be determined as follows:

Assume we have simulated n potential portfolio losses $\tilde{L}_{PF}^{(1)}, ..., \tilde{L}_{PF}^{(n)}$, hereby taking the driving distributions of the single loss variables and their correlations[12] into account. Then the empirical loss distribution function is given by

$$F(x) \;=\; \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{[0,x]}(\tilde{L}_{PF}^{(j)}) \; . \tag{1.17}$$

Figure 1.3 shows the shape of the density (histogram of the randomly generated numbers $(\tilde{L}_{PF}^{(1)}, ..., \tilde{L}_{PF}^{(n)})$) of the empirical loss distribution of some test portfolio.

From the empirical loss distribution we can derive all of the portfolio risk quantities introduced in the previous paragraphs. For example, the α-quantile of the loss distribution can directly be obtained from our simulation results $\tilde{L}_{PF}^{(1)}, ..., \tilde{L}_{PF}^{(n)}$ as follows:

Starting with the order statistics of $\tilde{L}_{PF}^{(1)}, ..., \tilde{L}_{PF}^{(n)}$, say

$$\tilde{L}_{PF}^{(i_1)} \leq \tilde{L}_{PF}^{(i_2)} \leq \cdots \leq \tilde{L}_{PF}^{(i_n)} \; ,$$

the α-quantile $\hat{q}_\alpha$ of the empirical loss distribution for any confidence level α is given by

$$\hat{q}_\alpha \;=\; \begin{cases} \alpha\tilde{L}_{PF}^{(i_{[n\alpha]})} + (1-\alpha)\tilde{L}_{PF}^{(i_{[n\alpha]+1})} & \text{if} \quad n\alpha \in \mathbb{N} \\ \tilde{L}_{PF}^{(i_{[n\alpha]})} & \text{if} \quad n\alpha \notin \mathbb{N} \end{cases} \tag{1.18}$$

[12] We will later see that correlations are incorporated by means of a *factor model*.

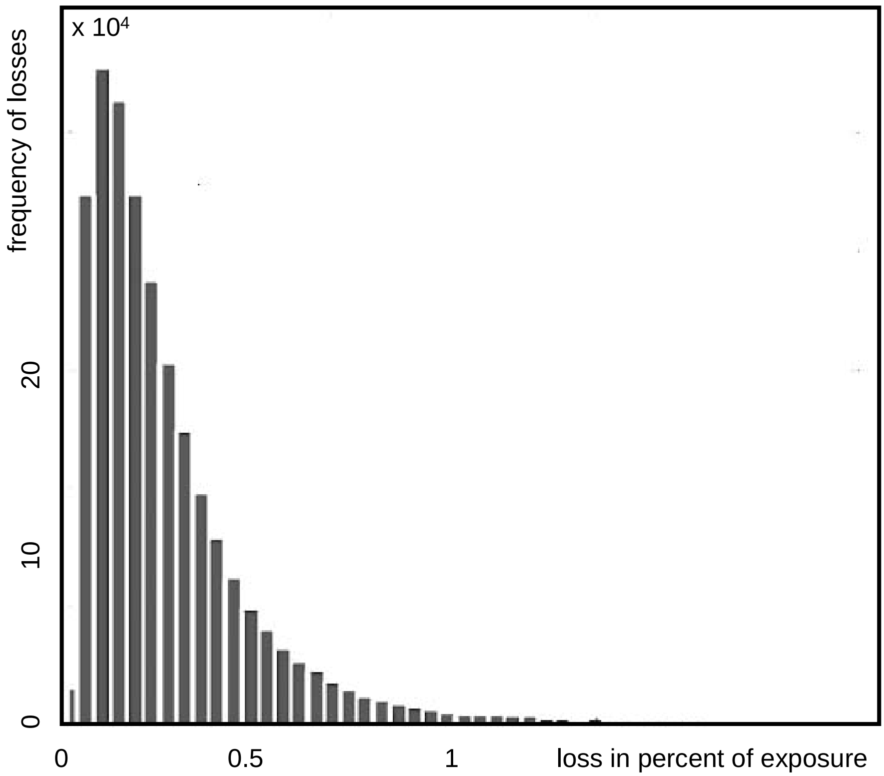

FIGURE 1.3: An empirical portfolio loss distribution obtained by Monte Carlo simulation. The histogram is based on a portfolio of 2.000 middle-size corporate loans.

$$\text{where} \qquad [n\alpha] \;=\; \min\left\{k \in \{1,...,n\} \mid n\alpha \le k\right\} .$$

The economic capital can then be estimated by

$$\widehat{\mathrm{EC}}_\alpha \;=\; \hat{q}_\alpha - \frac{1}{n}\sum_{j=1}^{n} \tilde{L}_{PF}^{(j)} . \tag{1.19}$$

In an analogous manner, any other risk quantity can be obtained by calculating the corresponding empirical statistics.

Approaching the loss distribution of a large portfolio by Monte Carlo simulation always requires a sound *factor model*; see Section 1.2.3. The classical statistical reason for the existence of factor models is the wish to explain the variance of a variable in terms of underlying factors. Despite the fact that in credit risk we also wish to explain the variability of a firm's economic success in terms of global underlying influences, the necessity for factor models comes from two major reasons.

First of all, the correlation between single loss variables should be made interpretable in terms of *economic variables*, such that large losses can be explained in a sound manner. For example, a large portfolio loss might be due to the *downturn* of an industry common to many counterparties in the portfolio. Along this line, a factor model can also be used as a tool for *scenario analysis*. For example, by setting an industry factor to a particular fixed value and then starting the Monte Carlo simulation again, one can study the impact of a down- or upturn of the respective industry.

The second reason for the need of factor models is a reduction of the computational effort. For example, for a portfolio of 100,000 transactions, $\frac{1}{2} \times 100,000 \times 99,999$ correlations have to be calculated. In contrast, modeling the correlations in the portfolio by means of a factor model with 100 indices reduces the number of involved correlations by a factor of 1,000,000. We will come back to factor models in 1.2.3 and also in later chapters.

1.2.2.2 Analytical Approximation

Another approach to the portfolio loss distribution is by analytical approximation. Roughly speaking, the analytical approximation maps an actual portfolio with unknown loss distribution to an equivalent portfolio with known loss distribution. The loss distribution of the equivalent portfolio is then taken as a substitute for the "true" loss distribution of the original portfolio.

In practice this is often done as follows. Choose a family of distributions characterized by its first and second moment, showing the typical shape (i.e., right-skewed with fat tails[13]) of loss distributions as illustrated in Figure 1.2.

From the known characteristics of the original portfolio (e.g., rating distribution, exposure distribution, maturities, etc.) calculate the first moment (EL) and estimate the second (centered) moment (UL^2).

Note that the EL of the original portfolio usually can be calculated based on the information from the rating, exposure, and LGD distributions of the portfolio.

Unfortunately the second moment can not be calculated without any assumptions regarding the default correlations in the portfolio; see Equation (1.13). Therefore, one now has to make an assumption regarding an *average default correlation* ρ. Note that in case one thinks in terms of *asset value models*, see Section 2.4.1, one would rather guess an average *asset correlation* instead of a default correlation and then calculate the corresponding default correlation by means of applying Proposition 2.5.1 to the definition of the default correlation. However, applying Equation (1.13) by setting all default correlations ρ_{ij} equal to ρ will provide an estimated value for the original portfolio's UL.

Now one can choose from the parametrized family of loss distribution the distribution best matching the original portfolio w.r.t. first and second moments. This distribution is then interpreted as the loss distribution of an equivalent portfolio which was selected by a *moment matching* procedure.

Obviously the most critical part of an analytical approximation is the determination of the average asset correlation. Here one has to rely on practical experience with portfolios where the average asset correlation is known. For example, one could compare the original portfolio with a set of typical bank portfolios for which the average asset correlations are known. In some cases there is empirical evidence regarding a reasonable range in which one would expect the unknown correlation to be

[13]In our terminology, a distribution has *fat tails*, if its quantiles at high confidence are higher than those of a normal distribution with matching first and second moments.

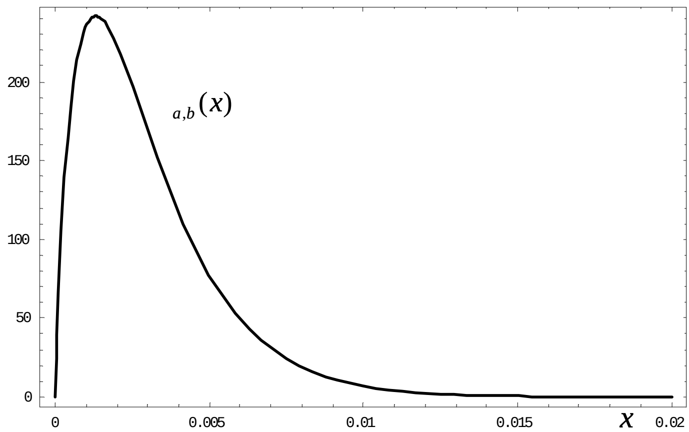

FIGURE 1.4: Analytical approximation by some beta distribution.

located. For example, if the original portfolio is a retail portfolio, then one would expect the average asset correlation of the portfolio to be a small number, maybe contained in the interval $[1\%, 5\%]$. If the original portfolio contains loans given to large firms, then one would expect the portfolio to have a high average asset correlation, maybe somewhere between 40% and 60%. Just to give another example, the new *Basel Capital Accord* (see Section 1.3) assumes an average asset correlation of 20% for corporate loans; see [148]. In Section 2.7 we estimate the average asset correlation in Moody's universe of rated corporate bonds to be around 25%. Summarizing, we can say that calibrating[14] an average correlation is on one hand a typical source of *model risk*, but on the other hand nevertheless often supported by some practical experience.

As an illustration of how the moment matching in an analytical approximation works, assume that we are given a portfolio with an EL of 30 bps and an UL of 22.5 bps, estimated from the information we have about some credit portfolio combined with some assumed average correlation.

Now, in Section 2.5 we will introduce a typical family of two-parameter loss distributions used for analytical approximation. Here, we want to approximate the loss distribution of the original portfolio by a beta

[14]The calibration might be more honestly called a "guestimate", a mixture of a guess and an estimate.

distribution, matching the first and second moments of the original portfolio. In other words, we are looking for a random variable

$$X \sim \beta(a, b) \ ,$$

representing the percentage portfolio loss, such that the parameters a and b solve the following equations:

$$0.003 = \mathbb{E}[X] = \frac{a}{a+b} \qquad \text{and} \tag{1.20}$$

$$0.00225^2 = \mathbb{V}[X] = \frac{ab}{(a+b)^2(a+b+1)} \ .$$

Hereby recall that the probability density φ_X of X is given by

$$\varphi_X(x) \ = \ \beta_{a,b}(x) \ = \ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1} \tag{1.21}$$

($x \in [0,1]$) with first moment and second (centered) moment

$$\mathbb{E}[X] = \frac{a}{a+b} \qquad \text{and} \qquad \mathbb{V}[X] = \frac{ab}{(a+b)^2(a+b+1)} \ .$$

Equations (1.20) represent the moment matching addressing the "correct" beta distribution matching the first and second moments of our original portfolio. It turns out that $a = 1.76944$ and $b = 588.045$ solve equations (1.20). Figure 1.4 shows the probability density of the so calibrated random variable X.

The analytical approximation takes the random variable X as a proxy for the unknown loss distribution of the portfolio we started with. Following this assumption, the risk quantities of the original portfolio can be approximated by the respective quantities of the random variable X. For example, quantiles of the loss distribution of the portfolio are calculated as quantiles of the beta distribution. Because the "true" loss distribution is substituted by a closed-form, analytical, and well-known distribution, all necessary calculations can be done in fractions of a second. The price we have to pay for such convenience is that all calculations are subject to significant *model risk*. Admittedly, the beta distribution as shown in Figure 1.4 has the shape of a loss distribution, but there are various two-parameter families of probability densities having the typical shape of a loss distribution. For example, some gamma distributions, the F-distribution, and also the distributions introduced in Section 2.5 have such a shape. Unfortunately they all have different tails, such that in case one of them would approximate really well the unknown loss distribution of the portfolio, the others automatically would be the wrong choice. Therefore, the selection of an appropriate family of distributions for an analytical approximation is a remarkable source of model risk. Nevertheless there are some families of distributions that are established as *best practice* choices for particular cases. For example, the distributions in Section 2.5 are a very natural choice for analytical approximations, because they are limit distributions of a well understood model.

In practice, analytical approximation techniques can be applied quite successfully to so-called *homogeneous portfolios*. These are portfolios where all transactions in the portfolio have comparable risk characteristics, for example, no exposure concentrations, default probabilities in a band with moderate bandwidth, only a few (better: one single!) industries and countries, and so on. There are many portfolios satisfying such constraints. For example, many retail banking portfolios and also many portfolios of smaller banks can be evaluated by analytical approximations with sufficient precision.

In contrast, a full Monte Carlo simulation of a large portfolio can last several hours, depending on the number of counterparties and the number of scenarios necessary to obtain sufficiently rich tail statistics for the chosen level of confidence.

The main advantage of a Monte Carlo simulation is that it accurately captures the dependencies inherent in the portfolio instead of relying on a whole bunch of assumptions. Moreover, a Monte Carlo simulation takes into account all the different risk characteristics of the loans in the portfolio. However, in Section 2.8 we also touch on valuable alternatives to the Monte Carlo approach.

1.2.3 Modeling Correlations by Means of Factor Models

Factor models are a well established technique from multivariate statistics, applied in credit risk models, for identifying underlying drivers of correlated defaults and for reducing the computational effort regarding the calculation of correlated losses. We start by discussing the basic meaning of a *factor*.

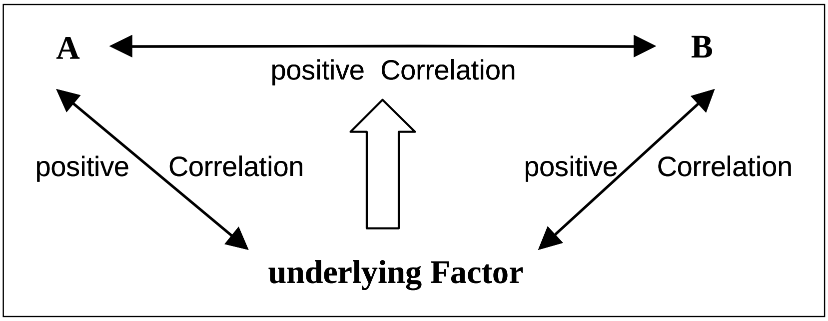

FIGURE 1.5: Correlation induced by an underlying factor.

Assume we have two firms A and B which are positively correlated. For example, let A be DaimlerChrysler and B stand for BMW. Then, it is quite natural to explain the positive correlation between A and B by the correlation of A and B with an *underlying factor*; see Figure 1.5. In our example we could think of the *automotive industry* as an underlying factor having significant impact on the economic future of the companies A and B. Of course there are probably some more underlying factors driving the riskiness of A and B. For example, DaimlerChrysler is to a certain extent also influenced by a factor for *Germany*, the *United States*, and eventually by some factors incorporating *Aero Space* and *Financial Companies*. BMW is certainly correlated with a country factor for *Germany* and probably also with some other factors. However, the crucial point is that factor models provide a way to express the correlation between A and B exclusively by means of their correlation with common factors. As already mentioned in the previous section, we additionally wish underlying factors to be *interpretable* in order to identify the reasons why two companies experience a down- or upturn at about the same time. For example, assume that the automotive industry gets under pressure. Then we can expect that companies A and B also get under pressure, because their fortune is related to the automotive industry. The part of the *volatility* of a company's financial success (e.g., incorporated by its asset value process) related to systematic factors like industries or countries is called the *systematic risk* of the firm. The part of the firm's asset volatility that can not be explained by systematic influences is called the *specific* or *idiosyncratic risk* of the firm. We will make both notions precise later on in this section.

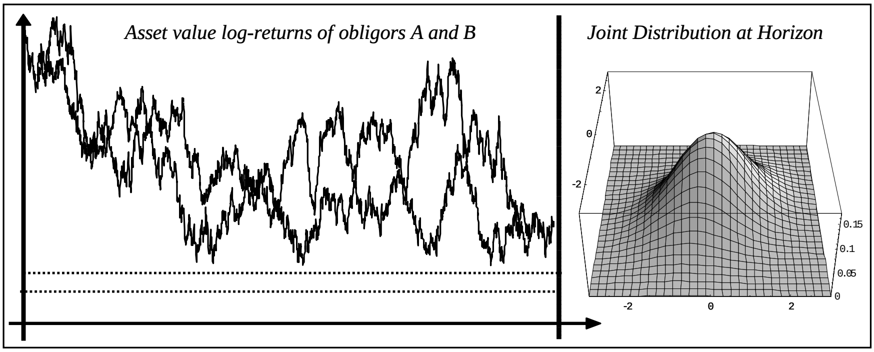

FIGURE 1.6: Correlated processes of obligor's asset value log-returns.

In the sequel we introduce an example of a typical factor model as it has been used by two industry leaders, namely, Moody's KMV and the RiskMetrics Group in their models for years. The companies behind the models continue to contribute in the area of credit risk research and modeling. Readers interested in information about the two firms can consult their websites

- www.moodyskmv.com
- www.riskmetrics.com

and will find a lot of information including research articles. Both firms continuously develop and improve their models so that one has to read through their most recent documentation in order to get a fresh taste on the current state of their models. However, our exposition of a typical factor model is meant as an *illustrative example only* which has the sole purpose to demonstrate how such a model works in principal.

Both models incorporate the idea that every firm admits a process of *asset values*, such that default or survival of the firm depends on the state of the asset values at a certain planning horizon. If the process has fallen below a certain critical threshold, called the *default point* of the firm, then the company has defaulted. If the asset value process is above the critical threshold, the firm survives. Asset value models have their roots in Merton's seminal paper [137] and will be explained in detail in Chapter 3 and also to some extent in Section 2.4.1.

Figure 1.6 illustrates the asset value model for two counterparties. Two correlated processes describing two obligor's asset values are shown. The correlation between the processes is called the *asset correlation.* In case the asset values are modeled by *geometric Brownian motions* (see Chapter 3), the asset correlation is just the correlation of the driving Brownian motions. At the planning horizon, the processes induce a bivariate asset value distribution. In the classical Merton model, where asset value processes are correlated geometric Brownian motions, the log-returns of asset values are normally distributed, so that the joint distribution of two asset value log-returns at the considered horizon is bivariate normal with a correlation equal to the asset correlation of the processes, see also Proposition 2.5.1. The dotted lines in Figure 1.6 indicate the critical thresholds or default points for each of the processes. Regarding the calibration of these default points we refer to CROSBIE [36] for an introduction.

Now let us start with the model used for years by Moody's KMV. They named it the *Global Correlation Model^{TM}*. A highly readable summary of the model can be found in CROUHY, GALAI, and MARK [38]. Our approach to describe the model is slightly different than other presentations because we want to have the relevant formulas in a way supporting a convenient algorithm for the calculation of asset correlations.

Following Merton's model[15], the *Global Correlation Model^{TM}* focuses on the asset value log-returns r_i of counterparties $(i = 1, ..., m)$ at a certain planning horizon (typically 1 year), admitting a representation

$$r_i = \beta_i \Phi_i + \varepsilon_i \qquad (i = 1, ..., m). \tag{1.22}$$

Here, Φ_i is called the *composite factor* of firm i, because in multi-factor models Φ_i typically is a weighted sum of several factors. Equation (1.22) is nothing but a standard *linear regression* equation, where the sensitivity coefficient, β_i, captures the linear correlation of r_i and Φ_i. In analogy to the *capital asset pricing model* (CAPM) (see, e.g., [38]) β is called the *beta* of counterparty i. The variable ε_i represents the *residual* part of r_i, essentially meaning that ε_i is the error one makes

[15]Actually, although the *Global Correlation Model^{TM}* in principal follows Merton's model, it does not really work with Gaussian distributions but rather relies on an empirically calibrated framework; see CROSBIE [36] and also Chapter 3.

when substituting r_i by $\beta_i\Phi_i$. Merton's model lives in a log-normal world, so that $\boldsymbol{r} = (r_1, ..., r_m) \sim N(\boldsymbol{\mu}, \Gamma)$ is multivariate Gaussian with a correlation matrix Γ. The composite factors Φ_i and ε_i are accordingly also normally distributed. Another basic assumption is that ε_i is independent of the Φ_i's for every i. Additionally the residuals ε_i are assumed to be uncorrelated[16]. Therefore, the returns r_i are exclusively correlated by means of their composite factors. This is the reason why Φ_i is thought of as the *systematic* part of r_i, whereas ε_i due to its independence from all other involved variables can be seen as a random effect just relevant for counterparty i. Now, in regression theory one usually decomposes the *variance* of a variable in a systematic and a specific part. Taking variances on both sides of Equation (1.22) yields

$$\mathbb{V}[r_i] \;=\; \underbrace{\beta_i^2\mathbb{V}[\Phi_i]}_{systematic} \;+\; \underbrace{\mathbb{V}[\varepsilon_i]}_{specific} \qquad (i = 1, ..., m). \tag{1.23}$$

Because the variance of r_i captures the risk of unexpected movements of the asset value of counterparty i, the decomposition (1.23) can be seen as a splitting of total risk of firm i in a *systematic* and a *specific* risk. The former captures the variability of r_i coming from the variability of the composite factor, which is $\beta_i^2\mathbb{V}[\Phi_i]$; the latter arises from the variability of the residual variable, $\mathbb{V}[\varepsilon_i]$. Note that some people say *idiosyncratic* instead of *specific*.

Alternatively to the beta of a firm one could also look at the *coefficient of determination* of the regression Equation (1.22). The coefficient of determination quantifies how much of the variability of r_i can be explained by Φ_i. This quantity is usually called the *R-squared*, R^2, of counterparty i and constitutes an important input parameter in all credit risk models based on asset values. It is usually defined as the systematic part of the variance of the standardized[17] returns $\tilde{r}_i = (r_i - \mathbb{E}[r_i])/\sqrt{\mathbb{V}[r_i]}$, namely

$$R_i^2 \;=\; \frac{\beta_i^2\mathbb{V}[\Phi_i]}{\mathbb{V}[r_i]} \qquad (i = 1, ..., m). \tag{1.24}$$

The residual part of the total variance of the standardized returns $\tilde{r}_i$ is then given by $1 - R_i^2$, thereby quantifying the percentage value of the specific risk of counterparty i.

[16]Recall that in the Gaussian case *uncorrelated* is equivalent to *independent*.

[17]That is, normalized in order to have mean zero and variance one.

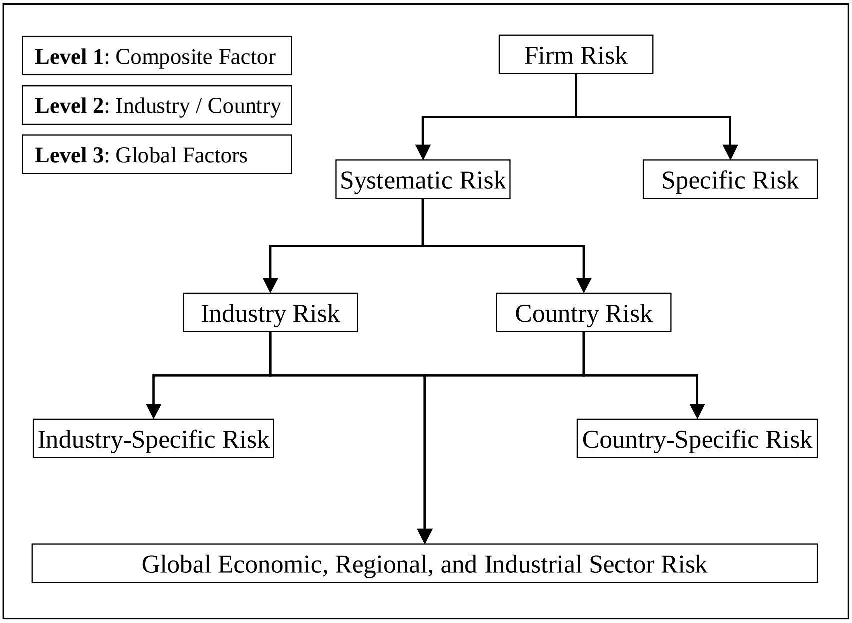

FIGURE 1.7: Three-level factor structure in the *Global Correlation Model*^{TM}; see also comparable presentations in the literature, e.g., Figure 9.9. in [38] and Figure 1.7 in [24].

Now we will look more carefully at the composite factors. The decomposition of a firm's variance in a systematic and a specific part is the first out of three levels in the *Global Correlation Model*^{TM}; see Figure 1.7. The subsequent level is the decomposition of the firm's composite factor Φ in industry and country indices.

Before writing down the level-2 decomposition, let us rewrite Equation (1.22) in vector notation[18] which is more convenient for further calculations. For this purpose denote by $\boldsymbol{\beta} = (\beta_{ij})_{1 \leq i,j \leq m}$ the diagonal matrix in $\mathbb{R}^{m \times m}$ with $\beta_{ij} = \beta_i$ if $i = j$ and $\beta_{ij} = 0$ if $i \neq j$. Equation (1.22) then can be rewritten in vector notation as follows:

$$\boldsymbol{r} \;=\; \boldsymbol{\beta\Phi} \;+\; \boldsymbol{\varepsilon} \;, \tag{1.25}$$

$$\boldsymbol{\Phi}^T = (\Phi_1, ..., \Phi_m) \;, \qquad \boldsymbol{\varepsilon}^T = (\varepsilon_1, ..., \varepsilon_m) \;.$$

[18] Note that in the sequel we write vectors as column vectors.

For the second level, the *Global Correlation Model*^{TM} decomposes every Φ_i w.r.t. an *industry* and *country* breakdown,

$$\Phi_i = \sum_{k=1}^{K} w_{i,k}\Psi_k \qquad (i = 1, ..., m), \tag{1.26}$$

where $\Psi_1, ..., \Psi_{K_0}$ are industry indices and $\Psi_{K_0+1}, ..., \Psi_K$ are country indices. The coefficients $w_{i,1}, ..., w_{i,K_0}$ are called the *industry weights* and the coefficients $w_{i,K_0+1}, ..., w_{i,K}$ are called the *country weights* of counterparty i. It is assumed that $w_{i,k} \geq 0$ for all i and k, and that

$$\sum_{k=1}^{K_0} w_{i,k} \;=\; \sum_{k=K_0+1}^{K} w_{i,k} \;=\; 1 \qquad (i = 1, ..., m).$$

In vector notation, (1.25) combined with (1.26) can be written as

$$\boldsymbol{r} \;=\; \boldsymbol{\beta W \Psi} \;+\; \boldsymbol{\varepsilon}\;, \tag{1.27}$$

where $\boldsymbol{W}=(w_{i,k})_{i=1,...,m;\; k=1,...,K}$ denotes the matrix of industry and country weights for the counterparties in the portfolio, and $\boldsymbol{\Psi}^T = (\Psi_1, ..., \Psi_K)$ means the vector of industry and country indices. This constitutes the second level of the *Global Correlation Model*^{TM}.

At the third and last level, a representation by a weighted sum of *independent global factors* is constructed for representing industry and country indices,

$$\Psi_k \;=\; \sum_{n=1}^{N} b_{k,n}\Gamma_n \;+\; \delta_k \qquad (k = 1, ..., K), \tag{1.28}$$

where δ_k denotes the Ψ_k-specific residual. Such a decomposition is typically done by a *principal components analysis* (PCA) of the industry and country indices. In vector notation, (1.28) becomes

$$\boldsymbol{\Psi} \;=\; \boldsymbol{B\Gamma} \;+\; \boldsymbol{\delta} \tag{1.29}$$

where $\boldsymbol{B}=(b_{k,n})_{k=1,...,K;\; n=1,...,N}$ denotes the matrix of *industry* and *country betas*, $\boldsymbol{\Gamma}^T = (\Gamma_1, ..., \Gamma_N)$ is the global factor vector, and $\boldsymbol{\delta}^T = (\delta_1, ..., \delta_K)$ is the vector of industry and country residuals. Combining (1.27) with (1.29), we finally obtain

$$\boldsymbol{r} \;=\; \boldsymbol{\beta W}(\boldsymbol{B\Gamma} + \boldsymbol{\delta}) \;+\; \boldsymbol{\varepsilon}\;. \tag{1.30}$$

So in the *Global Correlation Model™* the vector of the portfolio's returns $\boldsymbol{r}^T = (r_1, ..., r_m)$ can conveniently be written by means of underlying factors. Note that for computational purposes Equation (1.30) is the most convenient one, because the underlying factors are independent. In contrast, for an economic interpretation and for scenario analysis one would rather prefer Equation (1.27), because the industry and country indices are easier to interpret than the global factors constructed by PCA. In fact, the industry and country indices have a clear economic meaning, whereas the global factors arising from a PCA are of synthetic type. Although they admit some vague interpretation as shown in Figure 1.7, their meaning is not as clear as is the case for the industry and country indices.

As already promised, the calculation of asset returns in the model as introduced above is straightforward now. First of all, we standardize the asset value log-returns,

$$\tilde{r}_i \;=\; \frac{r_i - \mathbb{E}[r_i]}{\sigma_i} \qquad (i = 1, ..., m)$$

where σ_i denotes the volatility of the asset value log-return of counterparty i. From Equation (1.30) we then obtain a representation of standardized log-returns,

$$\tilde{r}_i \;=\; \frac{\beta_i}{\sigma_i}\,\tilde{\Phi}_i \;+\; \frac{\tilde{\varepsilon}_i}{\sigma_i} \qquad \text{where} \qquad \mathbb{E}[\tilde{\Phi}_i] = \mathbb{E}[\tilde{\varepsilon}_i] = 0\;. \tag{1.31}$$

Now, the asset correlation between two counterparties is given by

$$\mathrm{Corr}[\tilde{r}_i, \tilde{r}_j] \;=\; \mathbb{E}\big[\tilde{r}_i \tilde{r}_j\big] \;=\; \frac{\beta_i}{\sigma_i}\frac{\beta_j}{\sigma_j}\,\mathbb{E}\big[\tilde{\Phi}_i \tilde{\Phi}_j\big] \tag{1.32}$$

because the *Global Correlation Model™* assumes the residuals $\tilde{\varepsilon}_i$ to be uncorrelated and independent of the composite factors. For calculation purposes it is convenient to get rid of the volatilities σ_i and the betas β_i in Equation (1.32). This can be achieved by replacing the betas by the R-squared parameters of the involved firms. From Equation (1.24) we know that

$$R_i^2 \;=\; \frac{\beta_i^2}{\sigma_i^2}\,\mathbb{V}[\Phi_i] \qquad (i = 1, ..., m). \tag{1.33}$$

Therefore, Equation (1.32) combined with (1.33) yields

$$\mathrm{Corr}[\tilde{r}_i, \tilde{r}_j] \;=\; \frac{R_i}{\sqrt{\mathbb{V}[\Phi_i]}}\frac{R_j}{\sqrt{\mathbb{V}[\Phi_j]}}\,\mathbb{E}\big[\tilde{\Phi}_i \tilde{\Phi}_j\big] \tag{1.34}$$

$$= \frac{R_i}{\sqrt{\mathbb{V}[\tilde{\Phi}_i]}} \frac{R_j}{\sqrt{\mathbb{V}[\tilde{\Phi}_j]}} \mathbb{E}\big[\tilde{\Phi}_i \tilde{\Phi}_j\big]$$

because by construction we have $\mathbb{V}[\Phi_i] = \mathbb{V}[\tilde{\Phi}_i]$.

Based on Equation (1.30) we can now easily compute asset correlations according to (1.34). After standardization, (1.30) changes to

$$\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{\beta}} \boldsymbol{W}(\boldsymbol{B}\tilde{\boldsymbol{\Gamma}} + \tilde{\boldsymbol{\delta}}) + \tilde{\boldsymbol{\varepsilon}} \;, \tag{1.35}$$

where $\tilde{\boldsymbol{\beta}} \in \mathbb{R}^{m \times m}$ denotes the matrix obtained by scaling every diagonal element in $\boldsymbol{\beta}$ by $1/\sigma_i$, and

$$\mathbb{E}\big[\tilde{\boldsymbol{\Gamma}}\big] = 0, \quad \mathbb{E}\big[\tilde{\boldsymbol{\varepsilon}}\big] = 0, \quad \mathbb{E}\big[\tilde{\boldsymbol{\delta}}\big] = 0 \;.$$

Additionally, the residuals $\tilde{\boldsymbol{\delta}}$ and $\tilde{\boldsymbol{\varepsilon}}$ are assumed to be uncorrelated and independent of $\tilde{\boldsymbol{\Gamma}}$. We can now calculate asset correlations according to (1.34) just by computing the matrix

$$\mathbb{E}\big[\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^T\big] \;=\; \boldsymbol{W}\left[\boldsymbol{B}\mathbb{E}\big[\tilde{\boldsymbol{\Gamma}}\tilde{\boldsymbol{\Gamma}}^T\big]\boldsymbol{B}^T + \mathbb{E}\big[\tilde{\boldsymbol{\delta}}\tilde{\boldsymbol{\delta}}^T\big]\right]\boldsymbol{W}^T \tag{1.36}$$

because the matrix of standardized composite factors is given by $\tilde{\boldsymbol{\Phi}} = \boldsymbol{W}(\boldsymbol{B}\tilde{\boldsymbol{\Gamma}} + \tilde{\boldsymbol{\delta}})$. Let us quickly prove that (1.36) is true. By definition, we have

$$\begin{aligned}
\mathbb{E}\big[\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^T\big] \;&=\; \mathbb{E}\left[\boldsymbol{W}(\boldsymbol{B}\tilde{\boldsymbol{\Gamma}} + \tilde{\boldsymbol{\delta}})\Big(\boldsymbol{W}(\boldsymbol{B}\tilde{\boldsymbol{\Gamma}} + \tilde{\boldsymbol{\delta}})\Big)^T\right] \\
&=\; \boldsymbol{W}\,\mathbb{E}\left[(\boldsymbol{B}\tilde{\boldsymbol{\Gamma}} + \tilde{\boldsymbol{\delta}})(\boldsymbol{B}\tilde{\boldsymbol{\Gamma}} + \tilde{\boldsymbol{\delta}})^T\right]\boldsymbol{W}^T \\
&=\; \boldsymbol{W}\Big(\boldsymbol{B}\mathbb{E}\big[\tilde{\boldsymbol{\Gamma}}\tilde{\boldsymbol{\Gamma}}^T\big]\boldsymbol{B}^T + \underbrace{\boldsymbol{B}\mathbb{E}\big[\tilde{\boldsymbol{\Gamma}}\tilde{\boldsymbol{\delta}}^T\big]}_{=\,0} + \underbrace{\mathbb{E}\big[\tilde{\boldsymbol{\delta}}(\boldsymbol{B}\tilde{\boldsymbol{\Gamma}})^T\big]}_{=\,0} + \mathbb{E}\big[\tilde{\boldsymbol{\delta}}\tilde{\boldsymbol{\delta}}^T\big]\Big)\boldsymbol{W}^T \;.
\end{aligned}$$

The two expectations above vanish due to our orthogonality assumptions. This proves (1.36). Note that in equation (1.36), $\mathbb{E}\big[\tilde{\boldsymbol{\Gamma}}\tilde{\boldsymbol{\Gamma}}^T\big]$ is a diagonal matrix (because we are dealing with orthogonal global factors) with diagonal elements $\mathbb{V}[\Gamma_n]$ $(n = 1, ..., N)$, and $\mathbb{E}\big[\tilde{\boldsymbol{\delta}}\tilde{\boldsymbol{\delta}}^T\big]$ is a diagonal matrix with diagonal elements $\mathbb{V}[\delta_k]$ $(k = 1, ..., K)$. Therefore, the calculation of asset correlations according to (1.36) can conveniently be implemented in case one knows the variances of global factors, the variances of industry and country residuals, and the beta of the industry and country indices w.r.t. the global factors.

The factor model used by the RiskMetrics Group is quite similar to the *Global Correlation Model*^{TM} just described. So there is no need to start all over again, and we refer to the documentation which, in case of RiskMetrics, is called the *CreditMetrics*^{TM} Technical Document [88]. However, there are two fundamental differences between the models which are worthwhile and important to be mentioned.

First, the *Global Correlation Model*^{TM} is calibrated w.r.t. *asset* value processes, whereas the factor model of *CreditMetrics*^{TM} uses *equity* processes instead of asset value processes, thereby taking *equity correlations as a proxy for asset correlations*; see [88], page 93. We consider this difference to be fundamental, because a very important feature of the model world owned by Moody's KMV is that it really manages the admittedly difficult process of translating equity and market information into asset values; see Chapter 3.

Second, the framework *CreditMetrics*^{TM} uses indices[19] referring to a *combination* of some industry in some particular country, whereas the *Global Correlation Model*^{TM} considers industries and countries separately. So a German automotive company in the *CreditMetrics*^{TM} factor model would get a 100%-weight w.r.t. an index describing the *German automotive* industry, whereas in the *Global Correlation Model*^{TM} this company would have industry *and* country weights equal to 100% w.r.t. an *automotive index* and a *country index* representing Germany. Both approaches are quite different and have their own advantages and disadvantages.

1.3 Regulatory Capital and the Basel Initiative

It is worthwhile to mention that in the first edition of this book we started with the remark that the regulatory capital approach currently is under review. Today, eight years later, this statement is true again. The most recent crisis gave rise to uncountably many discussions on the current regulatory approach. Because regulation is an ongoing issue and frameworks are subject to change it does not make sense in a

[19]MSCI indices; see www.msci.com.

book like this to spend too much time with a topic like regulatory capital. However, it does make sense to provide at least some remarks and some flavor on how regulatory capital is calculated. We concentrate on the calculation aspects of regulatory capital. Topics like disclosure or reporting are left out intentionally. The currently valid regulatory framework can be found in the document [149] and its supplementary papers on www.bis.org. So let us start with a bit of history and then let us briefly present examples of capital formulas in the current framework.

In 1983 the banking supervision authorities of the main industrialized countries (G7) agreed on rules for banking regulation, which should be incorporated into national regulation laws. Since the national regulators discussed these issues, hosted and promoted by the *Bank of International Settlement* (www.bis.org) located in *Basel* in Switzerland, these rules were called *The Basel Capital Accord*.

The best known rule therein is the *8-percent rule*. Under this rule, banks have to prove that the capital they hold is larger than 8% of their so-called *risk-weighted assets* (RWA), calculated for all balance sheet positions. This rule implied that the capital basis for banks was mainly driven by the exposure of the loans to their customers. The RWA were calculated by a simple *weighting scheme*. Roughly speaking, for loans to any government institution the *risk weight* was set to 0%, reflecting the broad opinion that the governments of the world's industrial nations are likely to meet their financial obligations. The risk weight for loans to OECD banks was fixed at 20%. Regarding corporate loans, the committee agreed on a *standard risk weight* of 100%, no matter if the borrowing firm is a more or less risky obligor. The RWA were then calculated by adding up all of the bank's weighted credit exposures, yielding a *regulatory capital* of $8\% \times$ RWA.

The main weakness of this capital accord was that it made no distinction between obligors with different creditworthiness. In 1988 an amendment to this Basel Accord opened the door for the use of internal models to calculate the regulatory capital for off-balance sheet positions in the *trading book*. The trading book was mostly seen as containing deals bearing *market risk*, and therefore the corresponding internal models captured solely the market risk in the trading business. Still, corporate bonds and derivatives contributed to the RWA, since the default risk was not captured by the market risk models.

In 1997 the *Basel Committee on Banking Supervision* allowed the banks to use so-called *specific risk models*, and the eligible instruments no longer fell under the 8%-rule. Around that time regulators recognized that banks already internally used sophisticated models to handle the credit risk for their balance sheet positions with an emphasis on default risk. These models were quite different from the standard specific risk models. In particular, they produced a loss distribution of the entire portfolio and did not so much focus on the volatility of the spreads as in most of the specific risk models.

At the end of the 20th century, the Basel Committee started to look intensively at the models presented in this book. However, in the finally agreed regulatory framework [149], shortly called *Basel II*, they do not allow the use of internal credit risk models for the calculation of regulatory capital. Instead, they use a more or less complicated risk-weighting scheme for bank's credit risk positions. The Basel II approach was switched live in most banks worldwide on January 1st in 2007. In the sequel, we briefly outline the currently used approach.

A major improvement of Basel II compared to the former approach (shortly called Basel I) is that the new capital rules are much more risk sensitive. As already mentioned, the standard risk weight under Basel 1 was 100%, which led to a regulatory capital of

$$[\text{risk weight}] \times [\text{solvability coefficient}] \;=\; 100\% \times 8\% \;=\; 8\%$$

for various assets originated (and later often securitized) by banks. In the Basel II framework, often called the *new capital accord*, risk weights are working in the way they are supposed to work, namely, by weighting positions w.r.t. their credit risk. Depending on the level of sophistication a bank operates, the Basel II accord offers different approaches to regulatory capital. The most sophisticated approach a bank can implement is the so-called *internal ratings-based approach* (IRB). In this approach, banks calculate the risk weight of an asset in the following way[20] (see [149], §271-272):

$$\begin{aligned} \text{RWA} \;&=\; 12.5 \times \text{EAD} \times \text{LGD} \times K(\text{PD}) \times M(\text{PD}, \text{MAT}) \\ K(\text{PD}) \;&=\; N\left[\frac{N^{-1}[\text{PD}] + \sqrt{\varrho(\text{PD})}\, q_{99.9\%}(Y)}{\sqrt{1-\varrho(\text{PD})}}\right] - \text{PD} \end{aligned}$$

[20]Note that the function K is the quantile function (here, with respect to a confidence level of 99.9%) of the limit distribution in Formula (2.54).

$$\varrho(\mathrm{PD}) \;=\; 0.12 \times \frac{1-e^{-50\times\mathrm{PD}}}{1-e^{-50}} \;+\; 0.24 \times \left(1 - \frac{1-e^{-50\times\mathrm{PD}}}{1-e^{-50}}\right).$$

The meaning of the parameters in the formula is as follows:

- $M(\mathrm{PD}, \mathrm{MAT})$ is an adjustment factor depending on the effective maturity MAT of the asset and its PD.
- $N[\,\cdot\,]$ is the standard normal distribution function and $N^{-1}[\,\cdot\,]$ is its inverse.
- The quantity $q_{99.9\%}(Y)$ is the 99.9%-quantile of a standard normal random variable Y.
- The quantity ϱ has the meaning of a correlation parameter; see Formula (2.54).
- The formula for the correlation parameter ϱ is an interpolation between 12% and 24% quantifying the systematic risk ("R-squared"; see Formula 1.23 and the discussion thereafter) of an asset as a function of its PD.

Different asset classes get different parameterizations in the Basel II world. For instance, for SMEs[21] some firm size adjustment is applied ([149], §273-274); for retail exposures, e.g., residential mortgages or revolving retail exposures, other correlation parameterizations are prescribed ([149], §327-330), and so on. The RWA formula as presented by us refers to a *standard corporate loan.*

A good question one could ask is why the correlation parameter $\varrho = \varrho(\mathrm{PD})$ is chosen in dependence on the PD. A mathematical answer to this question is that there really are no good reasons for introducing such a functional relation between the parameters. A practitioner would probably argue that one expects on average to have better ratings for larger firms (e.g., multi-nationals like Nestle, Novartis, Deutsche Bank, IBM, etc.) and "large" often is associated with higher systematic risk so that lower PD corresponds to higher systematic risk and, therefore, a higher correlation parameter. However, reducing a two-parameter distribution model (see Formula (2.54)) to a one-parameter model by making one of the parameters a function of the other parameter is a questionable approach, like it or dislike it.

[21] Small- and medium-sized enterprises.

The Basel II approach has its strengths and weaknesses. Clearly a strength is that IRB-banks can use their internal ratings, LGDs and EADs as an input into the RWA-function. This is a huge progress in Basel II compared to Basel I. It basically means that *regulatory and economic approaches exhibit high convergence for single-name risks.*

A true weakness is that Basel II does neither penalize concentration nor award diversification. The approach is based on a simple but fully-fledged portfolio model but does not use the model itself and instead relies on a risk-weighting scheme which is *not portfolio context sensitive.* As an illustration imagine a loan in a first scenario in the credit portfolio of Deutsche Bank and in a second scenario the same loan in the credit portfolio of Credit Suisse. Although both credit books are fundamentally different the RWA-formula does not reflect those differences, the capital calculation according to Basel II is not sensitive to the surrounding credit portfolio.

This disadvantage becomes even more dramatic when one considers the way Basel II treats securitizations. Here one can safely say that Basel II simply fails to capture effects and risks of such so-called *correlation products.* We will briefly touch on the topic of *regulatory arbitrage* in structured credit products in Chapter 8.

What comes next from Basel? Nobody really knows yet. But it is certain that the accord will be revised again although the timing of a new accord and its final content are still open. However, there are many smaller changes and addenda which are published on continuous base on the website www.bis.org. Visiting this website and scanning available documentation and press releases is a must for every credit risk professional. We can only recommend to visit it.

Further Reading

As a general guide to quantitative risk management we recommend the book by McNeil, Frey and Embrechts [136]. Their book is a rich source for quantitative model techniques, not only for credit but also for other risks. Another book on risk management which contains also non-quantitative aspects is the book by Crouhy, Galai and Mark [38]. A book dealing with the integration of credit and interest rate risk is [45] by van Deventer, Imai and Mesler [45]. There are many other books on credit risk modeling available where each book has its own focus and flavor. To mention a few examples we refer to Ammann [5], Duffie and Singleton [51], Lando [120] and Schmid [161]

As a reference to Section 1.3 we refer to the Basel capital accord in its original form [149]. There are some books in the market where the Basel II standards are not only explained but also illustrated by means of examples. For this we refer to the book by Ong [151]. Another book which provides guidance regarding all quantitative aspects of the Basel II capital accord is the book by Engelmann and Rauhmeier [57].

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