

Absence of the Spin-Glass Phase in Dense Associative Memories

1 Spin-Glass Phase in the Classical Hopfield Model

In the classical Hopfield model [?], the coupling matrix is

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^p \xi_i^\mu \xi_j^\mu, \quad (1)$$

and the local field acting on spin S_i in a state aligned with pattern ν is

$$h_i = \sum_{j \neq i} J_{ij} S_j = \xi_i^\nu (1 + \delta_i), \quad \delta_i = \frac{1}{N} \sum_{j \neq i} \sum_{\mu \neq \nu} \xi_i^\mu \xi_j^\mu \xi_j^\nu, \quad (2)$$

where δ_i is cross-talk noise with variance $\langle \delta_i^2 \rangle = (p-1)/N$. For $p = \alpha N$, this noise is $O(\sqrt{\alpha})$ and independent of N .

1.1 Order parameters

The mean-field theory uses two order parameters:

- Overlap (magnetization): $m^\mu = \frac{1}{N} \sum_i \langle S_i \rangle \xi_i^\mu$ — measures alignment with pattern μ .
- Edwards–Anderson parameter: $q = \langle N^{-1} \sum_i \langle S_i \rangle^2 \rangle$ — measures freezing (non-ergodicity).

1.2 Phase diagram (Amit, Gutfreund, Sompolinsky 1987)

The system exhibits three phases:

1. **Paramagnetic (P):** $m = 0, q = 0$. Above $T_g = 1 + \sqrt{\alpha}$, the system is ergodic.
2. **Spin-glass (SG):** $m = 0, q > 0$. Below T_g (for $\alpha > \alpha_c \approx 0.138$ at $T = 0$), the system freezes into a random state with no macroscopic overlap with any stored pattern. The SG order parameter satisfies $q \simeq T_g - T$ near the transition.
3. **Retrieval / Ferromagnetic (FM):** $m \neq 0, q > 0$. Below $T_M(\alpha)$, the system retrieves a stored pattern. For $\alpha < \alpha_c$, this is the ground state.

The SG phase arises because the coupling matrix J_{ij} is a rank- p perturbation of what is effectively a random symmetric matrix. From random matrix theory, J has p outlier eigenvalues (corresponding to the stored patterns) sitting on a *semicircular bulk* of width $\sim 2\sqrt{\alpha}$. When $\alpha > \alpha_c$, the outliers merge into the bulk, and the random-matrix structure dominates the energy landscape, creating an exponentially large number of metastable states — the spin-glass phase.

2 Dense Associative Memories: LSE and LSR

In dense (modern) associative memories, the energy function is qualitatively different. For the two models considered here:

2.1 LSE (Log-Sum-Exp)

$$E_{\text{LSE}}(\mathbf{S}) = -\log \sum_{\mu=1}^p \exp\left(\frac{1}{2} (\boldsymbol{\xi}^\mu \cdot \mathbf{S})^2 / N\right). \quad (3)$$

2.2 LSR (Log-Sum with $b = 2 + \sqrt{2}$)

$$E_{\text{LSR}}(\mathbf{S}) = -\frac{1}{b} \log \sum_{\mu=1}^p \exp\left(\frac{b}{2} (\boldsymbol{\xi}^\mu \cdot \mathbf{S})^2 / N\right), \quad b = 2 + \sqrt{2}. \quad (4)$$

The $\exp(\cdot)$ inside the log-sum acts as a **soft-max**: it exponentially amplifies the overlap with the best-matching pattern and suppresses all others.

3 Why No Spin-Glass Phase in Dense AM

3.1 Suppression of cross-talk

In the Hopfield model, the local field is *linear* in the overlaps:

$$h_i^{\text{Hopf}} = \frac{1}{N} \sum_{\mu} m^{\mu} \xi_i^{\mu} = \frac{m^{\nu}}{N} \xi_i^{\nu} + \underbrace{\frac{1}{N} \sum_{\mu \neq \nu} m^{\mu} \xi_i^{\mu}}_{\text{cross-talk noise}}. \quad (5)$$

All p patterns contribute equally to the local field, regardless of their overlap magnitude. The cross-talk noise scales as $\sqrt{\alpha}$ and generates the frustrated landscape responsible for the SG phase.

In dense AM, the effective local field is dominated by the **exponentially weighted** leading pattern:

$$h_i^{\text{dense}} \propto \frac{\sum_{\mu} (\boldsymbol{\xi}^{\mu} \cdot \mathbf{S}) \xi_i^{\mu} e^{\frac{b}{2} (\boldsymbol{\xi}^{\mu} \cdot \mathbf{S})^2 / N}}{\sum_{\mu} e^{\frac{b}{2} (\boldsymbol{\xi}^{\mu} \cdot \mathbf{S})^2 / N}}. \quad (6)$$

The softmax weights $\propto e^{bN(m^{\mu})^2/2}$ ensure that the pattern with the largest overlap exponentially dominates the sum. All other overlaps are suppressed by a factor $\sim e^{-bN[(m^{\text{max}})^2 - (m^{\mu})^2]/2}$, which vanishes exponentially in N .

3.2 No frustrated landscape at high α

The key consequence: when α exceeds the critical capacity α_c , the stored patterns become destabilized, but the cross-talk noise *does not create its own metastable states*.

- **Hopfield:** $\alpha > \alpha_c \Rightarrow$ patterns unstable, but the linear cross-talk creates $\sim e^{cN}$ random metastable states \Rightarrow SG phase ($m = 0$, $q > 0$).
- **Dense AM:** $\alpha > \alpha_c \Rightarrow$ patterns unstable, and the exponential suppression kills the diffuse noise that would form SG states. No metastable states survive \Rightarrow paramagnetic phase ($m = 0$, $q = 0$).

Physically, the $\log\text{-}\sum\text{-exp}$ energy landscape has deep, narrow wells around stored patterns (exponentially deep in N), but the regions between wells are *smooth* — no rugged random-matrix structure. When the wells disappear at high α , the landscape becomes flat, and the system is ergodic.

3.3 Direct $\mathbf{R} \rightarrow \mathbf{P}$ transition

The phase diagram of dense AM therefore contains only two phases:

Phase	$\varphi/\varphi_{\text{th}}(T)$	q_{EA}
Retrieval (R)	≈ 1	> 0
Paramagnetic (P)	≈ 0	≈ 0

with a direct $\mathbf{R} \rightarrow \mathbf{P}$ transition at $\alpha = \alpha_c(T)$, and no intermediate SG phase.

This is confirmed by our basin stability simulations (v7), which show $\text{SG} = 0$ for both LSE and LSR across the entire explored (α, T) range.

3.4 Summary

The absence of the spin-glass phase in dense AM is a direct consequence of the nonlinear (exponential) energy function:

1. The softmax mechanism concentrates the energy on the dominant overlap, suppressing cross-talk.
2. Without cross-talk, there is no Sherrington–Kirkpatrick-like frustrated coupling structure.
3. Without frustration, no exponentially many metastable states form.
4. Therefore, when retrieval fails ($\alpha > \alpha_c$), the system goes directly to the paramagnetic phase.

This is one of the key advantages of dense associative memories over the classical Hopfield model: a cleaner phase diagram with no spin-glass contamination.

References

- [1] D. J. Amit, H. Gutfreund, and H. Sompolinsky, “Statistical mechanics of neural networks near saturation,” *Ann. Phys.* **173**, 30–67 (1987).