# Basin Percolation in the LSR Associative Memory

## 1 Setup: LSR energy and the hard wall

The Log-Sum-ReLU (LSR) associative memory with $M$ patterns $\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^M$ on the sphere $S^{N-1}(\sqrt{N})$ has energy

$$E(\mathbf{S}) = -\frac{N}{b} \ln \sum_{\mu=1}^{M} \big[\, 1 - b + b\, \varphi_\mu(\mathbf{S}) \,\big]_+, \qquad b = 2 + \sqrt{2}, \tag{1}$$

where $\varphi_\mu(\mathbf{S}) = (\boldsymbol{\xi}^\mu \cdot \mathbf{S})/N$ is the overlap and $[\cdot]_+ = \max(0, \cdot)$.

The hard wall activates at $\varphi_c = (b-1)/b = 1/\sqrt{2} \approx 0.707$: pattern $\mu$ contributes to the energy only when $\varphi_\mu(\mathbf{S}) > \varphi_c$. When no pattern is active, the argument of the logarithm vanishes and $E = +\infty$.

**Basin of a single pattern.** For an isolated target $\boldsymbol{\xi}^1$, the accessible region is the spherical cap

$$X_0 = \big\{\, \mathbf{S} \in S^{N-1}(\sqrt{N}) \;:\; \varphi_1(\mathbf{S}) > \varphi_c \,\big\}, \tag{2}$$

with an infinite energy barrier at the boundary $\varphi_1 = \varphi_c$. The state is permanently confined to $X_0$.

## 2 Joint basins and continuous connectivity

With $M$ patterns, the accessible region is the set where $\sum_\mu [\cdots]_+ > 0$, i.e. where at least one pattern is active. Two individual basins $X_\mu$ and $X_\nu$ form a *joint basin* if their intersection $X_\mu \cap X_\nu \neq \varnothing$.

Along any continuous path through $X_\mu \cap X_\nu$, both terms are simultaneously positive, so the energy is always finite—the state never encounters $\ln 0$. The energy does increase at the "bottleneck" (narrowest part of the overlap corridor), creating a finite barrier

$$\Delta E \;\sim\; \frac{N}{2b} \ln \frac{1}{q}, \tag{3}$$

where $q = (\boldsymbol{\xi}^\mu \cdot \boldsymbol{\xi}^\nu)/N$ is the mutual overlap, but this barrier is finite for any $q > 0$.

**Geometric overlap criterion.** Two spherical caps of angular radius $\theta_c = \arccos \varphi_c = \pi/4$ overlap when the angle between their centers satisfies $\theta < 2\theta_c = \pi/2$, i.e. when

$$q = \frac{\boldsymbol{\xi}^\mu \cdot \boldsymbol{\xi}^\nu}{N} > \cos\!\big(\tfrac{\pi}{2}\big) = 0. \tag{4}$$

Since random patterns have $q \sim \mathcal{N}(0, 1/N)$, approximately $M/2$ patterns satisfy $q > 0$ geometrically. However, the energy barrier (**??**) scales as $\sim (N/4b) \ln N$ for $q \sim 1/\sqrt{N}$, making such corridors thermally inaccessible.

# 3 Percolation framework

## 3.1 Thermal connectivity threshold

We define two basins as *thermally connected* if the energy barrier along the optimal path through their overlap is comparable to the thermal energy scale $T$. This requires a minimum mutual overlap $q > q_{\text{eff}}(T)$.

At zero temperature, only basins with a wide, low-barrier corridor are connected. The natural hard-wall threshold is $q_{\text{eff}} = \varphi_c$, since patterns with $q > \varphi_c$ are already active at the target center (the term $[1 - b + b\,q]_+ > 0$).

## 3.2 Poisson model for the number of neighbors

For random patterns on $S^{N-1}(\sqrt{N})$, the overlap $q = (\boldsymbol{\xi}^\mu \cdot \boldsymbol{\xi}^1)/N$ with a fixed target is approximately $\mathcal{N}(0, 1/N)$ for large $N$. The number of patterns with $q > \varphi_c$ follows a Poisson distribution with rate

$$\lambda = M \cdot \Pr(q > \varphi_c) \;\approx\; \frac{\exp\bigl(N(\alpha - \varphi_c^2/2)\bigr)}{\varphi_c\sqrt{2\pi N}}, \tag{5}$$

where $M = e^{N\alpha}$ and we used the Mill's ratio approximation for the Gaussian tail.

**Finite-$N$ correction.** The exact overlap distribution on $S^{N-1}$ has density $f(q) \propto (1-q^2)^{(N-3)/2}$, which has a lighter tail than the Gaussian for moderate $N$. The Mill's ratio therefore *overestimates* $\lambda$; the true percolation threshold is shifted to higher $\alpha$ for finite $N$. This correction vanishes as $N \to \infty$.

## 3.3 Percolation transition

The basins of $M$ random patterns form a random geometric graph on $S^{N-1}$: each basin is a node, and two nodes are connected if their mutual overlap exceeds $q_{\text{eff}}$. The connected component containing the target undergoes a **percolation transition** at $\lambda = 1$.

Setting $\lambda = 1$ in (**??**) with $q_{\text{eff}} = \varphi_c$:

$$\boxed{\alpha_c = \frac{\varphi_c^2}{2} + \frac{\ln(\varphi_c\sqrt{2\pi N})}{N} \xrightarrow{N\to\infty} \frac{\varphi_c^2}{2} = \frac{1}{4} = 0.25\,.} \tag{6}$$

This coincides exactly with the zero-temperature critical capacity $\alpha_{\text{th}} = \frac{1}{2}(1 - 1/b)^2 = \frac{1}{2}\varphi_c^2$ from the mean-field theory.

**Physical picture.**

- **Subcritical** ($\alpha < \alpha_c$): $\lambda < 1$; the target's connected cluster has $O(1)$ patterns. The state is confined near the target $\to$ retrieval.
- **Critical** ($\alpha \approx \alpha_c$): $\lambda \approx 1$; cluster size diverges; critical fluctuations.
- **Supercritical** ($\alpha > \alpha_c$): $\lambda > 1$; a giant connected component forms. The state delocalizes among exponentially many basins $\to$ paramagnetic phase.

### 3.4 Temperature dependence

At temperature $T > 0$, thermal fluctuations lower the effective barrier, so narrower corridors become traversable: $q_{\text{eff}}(T) < \varphi_c$. The percolation threshold shifts to

$$\alpha_c(T) \approx \frac{q_{\text{eff}}(T)^2}{2}, \tag{7}$$

which decreases with $T$—the retrieval region shrinks, consistent with the phase diagram.

From the mean-field free energy calculation: $\alpha_c(T) = \frac{1}{2}[1 - f_{\text{ret}}(T)]^2$, where $f_{\text{ret}}(T) = u(\varphi(T)) - T\,s(\varphi(T))$ is the retrieval free energy. Identifying $q_{\text{eff}}(T) = 1 - f_{\text{ret}}(T)$ connects the percolation threshold to the thermodynamic phase boundary.

## 4 BFS depth: chain connections

Starting from the target, we explore the basin connectivity graph via breadth-first search:

1. **Depth 0**: target pattern $\boldsymbol{\xi}^1$ with basin $X_0$.
2. **Depth 1**: all patterns $\mu$ with $(\boldsymbol{\xi}^\mu \cdot \boldsymbol{\xi}^1)/N > \varphi_c$ ($K_1 \sim \text{Poisson}(\lambda)$ patterns).
3. **Depth 2**: for each depth-1 pattern $\mu$, find patterns $\nu$ with $(\boldsymbol{\xi}^\mu \cdot \boldsymbol{\xi}^\nu)/N > \varphi_c$ that are *not* already at depth 1.

**Depth-2 patterns are absent for large $N$.** For a depth-1 pattern $\mu$ (overlap $\varphi_\mu > \varphi_c$ with target) and a candidate depth-2 pattern $\nu$ (overlap $\varphi_\nu < \varphi_c$ with target), their mutual overlap is

$$\frac{\boldsymbol{\xi}^\mu \cdot \boldsymbol{\xi}^\nu}{N} \;\approx\; \varphi_\mu \cdot \varphi_\nu + \sqrt{(1-\varphi_\mu^2)(1-\varphi_\nu^2)}\,\frac{u_\mu \cdot u_\nu}{\sqrt{N-1}}, \tag{8}$$

where $u_\mu, u_\nu$ are random unit vectors in the $(N-1)$-dimensional subspace perpendicular to the target.

The deterministic part satisfies $\varphi_\mu \cdot \varphi_\nu < \varphi_c^2 = 1/2 < \varphi_c = 1/\sqrt{2}$, and the random fluctuation $\sim \mathcal{O}(1/\sqrt{N})$ cannot bridge the gap $\varphi_c - \varphi_c^2 \approx 0.207$ for large $N$.

At finite $N$, the fluctuation has standard deviation $\sim 0.5/\sqrt{N-1}$, requiring a $\sim 0.207\sqrt{N}$ sigma event for a depth-2 connection. For $N = 25$: $\sim 1\,\sigma$ (rare but possible); for $N = 50$: $\sim 1.5\,\sigma$ (very rare); for $N \geq 100$: negligible.

## 5 Simulation design

The accompanying script `percolation_LSR.jl` tests four aspects of the percolation hypothesis:

1. **Panel 1: Analytical** $\lambda(\alpha)$ for $N = 30, 50, 75, 150$. All curves cross $\lambda = 1$ near $\alpha \approx 0.25$, with finite-$N$ corrections $\sim \ln N/N$ shifting the threshold to higher $\alpha$.
2. **Panel 2: Direct neighbor counting** ($N = 25$, 3000 realizations). Generate $M = e^{N\alpha}$ random patterns on $S^{N-1}(\sqrt{N})$, count neighbors $K$ with overlap $> \varphi_c$. Compare histogram with Poisson($\lambda$). The Mill's ratio approximation overestimates $\lambda$ at moderate $N$ because the exact spherical tail is lighter than the Gaussian.

3. **Panel 3: Branching process survival** ($N = 50$). Simulate a Galton–Watson process with Poisson($\lambda$) offspring. The survival probability transitions from 0 to 1 at $\alpha_c$, sharpening with BFS depth (analogous to increasing system size in percolation).

4. **Panel 4: BFS depth analysis** ($N = 25$, 200 realizations). Count depth-1 neighbors $K_1$ and *new* depth-2 neighbors $K_2$. Confirms that $\langle K_2 \rangle \ll \langle K_1 \rangle$: the BFS effectively terminates at depth 1.

# 6 Implications for the simulation protocol

The percolation picture justifies the Poisson pattern reduction scheme (v5 approach) with an important correction: **the background energy $S_{\mathrm{bg}}$ should be removed**.

1. The energy computed from just the $K + 1$ retained patterns (target + $K$ active neighbors) is *exact* for all states reachable from the target basin. Patterns outside the connected component are separated by infinite barriers and cannot be reached.

2. The $S_{\mathrm{bg}}$ term in v5 artificially smooths the energy landscape, removing the infinite barriers that confine the state. This distorts the dynamics near the phase boundary.

3. With the Poisson scheme (no $S_{\mathrm{bg}}$), memory scales as $\mathcal{O}(N \cdot K_{\max})$ instead of $\mathcal{O}(N \cdot M)$, enabling much larger $N$ for sharper phase boundaries.

4. For $\alpha > \alpha_c$ (supercritical), $K$ grows exponentially but can be capped at some $K_{\mathrm{cap}}$ without affecting the measured overlap $\varphi$ at the target—the state delocalizes among the $K_{\mathrm{cap}}$ nearest basins, which is sufficient to show non-retrieval.

**Connection to the Hopfield model.** In the classical Hopfield model, the coupling structure creates an exponentially large number of spin-glass metastable states via random-matrix frustration. In the LSR model, the log-$\sum$-exp energy function confines the state to discrete basins separated by infinite barriers. The phase transition is purely *geometric*—basin percolation—rather than arising from frustrated interactions. This explains the absence of the spin-glass phase in dense associative memories.

# References


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