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# An alternative linear matrix method for normal modes in collisionless stellar disks

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ABSTRACT

We present an alternative matrix method

## 1. INTRODUCTION

## 2. THE MATRIX METHOD

$$V(u,\theta)\mathrm{e}^{-u/2} = \Phi(R,\theta) = \sum_m \Phi_m(R)\mathrm{e}^{\mathrm{i}m\theta} \tag{1}$$

$$S(u,\theta)\mathrm{e}^{-3u/2} = \Sigma(R,\theta) = \sum_m \Sigma_m(R)\mathrm{e}^{\mathrm{i}m\theta} \tag{2}$$

**I. Surface density**

$$A_m(\alpha) = \int\limits_{-\infty}^{\infty} \mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\alpha u} S_m(u) = \int\limits_{-\infty}^{\infty} \mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\alpha u + 3u/2}\Sigma_m(R) \tag{3}$$

$$R\Sigma(R,\theta) = R\sum_m \Sigma_m(R)\mathrm{e}^{\mathrm{i}m\theta} = \sum_{lm}\int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{w}\mathcal{F}_{lm}(\mathbf{J})\mathrm{e}^{\mathrm{i}(lw_1+mw_2)}\delta(R-r(\mathbf{J},w))\delta(\theta-\vartheta(\mathbf{J},\mathbf{w})) \tag{4}$$

Recall that ($w_1$ and $w$ is used interchangeably):

$$w_2 = \theta + \phi(\mathbf{J},w_1), \quad \phi(\mathbf{J},w) = \frac{\Omega_2(\mathbf{J})}{\Omega_1(\mathbf{J})}w - \frac{L}{\Omega_1(\mathbf{J})}\int\limits_0^w \frac{\mathrm{d}w'}{r^2(\mathbf{J},w')} \tag{5}$$

and

$$\vartheta(\mathbf{J},\mathbf{w}) = w_2 - \phi(\mathbf{J},w_1) \tag{6}$$

Thus one has:

$$R\Sigma_m(R)\mathrm{e}^{\mathrm{i}m\theta} = \sum_l \mathrm{e}^{\mathrm{i}m\theta}\int \mathrm{d}\mathbf{J}\mathcal{F}_{lm}(\mathbf{J})\int\limits_{-\pi}^{\pi}\mathrm{d}w\,\mathrm{e}^{\mathrm{i}(lw+m\phi(\mathbf{J},w))}\delta[R-r(\mathbf{J},w)] \tag{7}$$

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or (index $m$ for $\mathcal{F}_{lm}$ is dropped):

$$R\Sigma_m(R) = \sum_l \int \mathrm{d}\mathbf{J}\mathcal{F}_l(\mathbf{J}) \int_{-\pi}^{\pi} \mathrm{d}w\, \mathrm{e}^{i(lw+m\phi(\mathbf{J},w))}\delta[R - r(\mathbf{J},w)] \tag{8}$$

$$\begin{aligned} A_m(\alpha) = \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\alpha u + u/2} \sum_l \int \mathrm{d}\mathbf{J}\mathcal{F}_l(\mathbf{J}) \int_{-\pi}^{\pi} \mathrm{d}w\, \mathrm{e}^{\mathrm{i}[lw+m\phi(\mathbf{J},w)]}\delta(R - r(\mathbf{J},w)) \\ = \sum_l \int \mathrm{d}\mathbf{J}\mathcal{F}_l(\mathbf{J}) \int_{-\pi}^{\pi} \frac{\mathrm{d}w}{r(\mathbf{J},w)^{1/2}}\, \mathrm{e}^{\mathrm{i}[lw+m\phi(\mathbf{J},w)-\alpha\log\, r(\mathbf{J},w)]} = 2\pi \sum_l \int \mathrm{d}\mathbf{J}\mathcal{F}_l(\mathbf{J}) W_l(\mathbf{J},\alpha) \end{aligned} \tag{9}$$

where

$$W_l(\mathbf{J},\alpha) = \frac{1}{\pi}\int_0^{\pi} \frac{\mathrm{d}w}{r(\mathbf{J},w)^{1/2}} \cos\Big[lw + m\phi(\mathbf{J},w)\Big]\mathrm{e}^{-\mathrm{i}\alpha\log r(\mathbf{J},w)}\,. \tag{10}$$

Note that $w$ and $\phi(\mathbf{J},w)$ are odd functions of $w$, while $r(\mathbf{J},w)$ is even, so $W_l$ is complex.

## II. Potential

According to BT08, page 518, Eq. (5.80),

$$\begin{aligned} \Phi_{lm}(\mathbf{J}) = \frac{1}{(2\pi)^2}\int \mathrm{d}\mathbf{w}\, \Phi(r(\mathbf{J},w),\vartheta)\mathrm{e}^{-\mathrm{i}(lw_1+mw_2)} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} \mathrm{d}w\mathrm{e}^{-\mathrm{i}lw}\sum_{m'}\Phi_{m'}(r(\mathbf{J},w_1))\int_{-\pi}^{\pi}\mathrm{d}w_2\mathrm{e}^{\mathrm{i}m'\vartheta}\mathrm{e}^{-\mathrm{i}m(\vartheta+\phi(\mathbf{J},w)} \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}w\,\mathrm{e}^{-\mathrm{i}lw-\mathrm{i}m\phi(\mathbf{J},w)}\Phi_m(r(\mathbf{J},w)) \end{aligned} \tag{11}$$

or (index $m$ for $\Phi_{lm}$ is dropped):

$$\begin{aligned} \Phi_l(\mathbf{J}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}w\,\mathrm{e}^{-\mathrm{i}(lw+m\phi(\mathbf{J},w))}\Phi_m(r(\mathbf{J},w)) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}w\,\mathrm{e}^{-\mathrm{i}(lw+m\phi(\mathbf{J},w))-v/2}V_m(v) \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}w\,\mathrm{e}^{-\mathrm{i}(lw+m\phi(\mathbf{J},w))-v/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}V_m(\alpha)\mathrm{e}^{\mathrm{i}\alpha v} = -\frac{G}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}w\,\mathrm{e}^{-\mathrm{i}(lw+m\phi(\mathbf{J},w))-v/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}N(\alpha,m)A_m(\alpha)\mathrm{e}^{\mathrm{i}\alpha v} \end{aligned} \tag{12}$$

where $v = v(\mathbf{J},w) = \log r(\mathbf{J},w)$,

$$N(\alpha,m) \equiv \pi\frac{\Gamma([m+1/2+\mathrm{i}\alpha]/2)\Gamma([m+1/2-\mathrm{i}\alpha]/2)}{\Gamma([m+3/2+\mathrm{i}\alpha]/2)\Gamma([m+3/2-\mathrm{i}\alpha]/2)} \tag{13}$$

(see BT08, p. 108, Eq. 2.198 – it is $2\pi$ times larger than in Zang 1976).

Substituting $A_m = 2\pi\sum_{l'}\int\mathrm{d}\mathbf{J}'\mathcal{F}_{l'}W_{l'}$:

$$\Phi_l(\mathbf{J}) = -G\int_{-\infty}^{\infty}\mathrm{d}\alpha N(\alpha,m)W_l^*(\mathbf{J},\alpha)\sum_{l'}\int\mathrm{d}\mathbf{J}'\mathcal{F}_{l'}(\mathbf{J}')W_{l'}(\mathbf{J}',\alpha) \tag{14}$$

where $W_l^*$ denotes the complex conjugate of $W_l$.

Now, let's insert all into the linearized Collisionless Boltzmann Equation:

$$[\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})]\mathcal{F}_l(\mathbf{J}) = -\mathcal{F}_{0,l}(\mathbf{J})\,\Phi_l(\mathbf{J}), \quad \mathcal{F}_{0,l} \equiv \left[l\frac{\partial \mathcal{F}_0}{\partial J} + m\frac{\partial \mathcal{F}_0}{\partial L}\right] \tag{15}$$

or

$$[\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})]\mathcal{F}_l(\mathbf{J}) = G\mathcal{F}_{0,l}(\mathbf{J}) \int_{-\infty}^{\infty} \mathrm{d}\alpha N(\alpha, m) W_l^*(\mathbf{J}, \alpha) \sum_{l'} \int \mathrm{d}\mathbf{J}' \mathcal{F}_{l'}(\mathbf{J}') W_{l'}(\mathbf{J}', \alpha) \tag{16}$$

or

$$[\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})]\mathcal{F}_l(\mathbf{J}) = G\mathcal{F}_{0,l}(\mathbf{J}) \sum_{l'} \int \mathrm{d}\mathbf{J}' \Pi_{l,l'}(\mathbf{J}, \mathbf{J}') \mathcal{F}_{l'}(\mathbf{J}') \tag{17}$$

where

$$\Pi_{l,l'}(\mathbf{J}, \mathbf{J}') \equiv \int_{-\infty}^{\infty} \mathrm{d}\alpha\, N(\alpha, m) W_l^*(\mathbf{J}, \alpha) W_{l'}(\mathbf{J}', \alpha) \tag{18}$$

## Derivation of (9)

$$R\,\Sigma_m(R) = \sum_l \int \mathcal{F}_l(\mathbf{J})\, e^{i\,[lw + m\phi(w,\mathbf{J})]} dv_R\, dL = 2\sum_l \int \mathcal{F}_l(\mathbf{J})\, e^{i\,[lw + m\phi(w,\mathbf{J})]}\, \frac{\Omega_1(\mathbf{J})\, d\mathbf{J}}{|v_R(\mathbf{J}, R)|}.$$

Hence,

$$A_m(\alpha) = 2\int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\alpha u + u/2} \sum_l \int \mathrm{d}\mathbf{J} \mathcal{F}_l(\mathbf{J})\, \frac{\Omega_1(\mathbf{J})}{|v_R|}\, e^{\mathrm{i}\,[lw + m\phi(w,\mathbf{J})]}.$$

Since

$$du = \frac{dR}{R} \quad \text{and} \quad \frac{\Omega_1 dR}{v_R} = dw,$$

we have

$$du\,\frac{\Omega_1}{v_R} = \frac{1}{R}\frac{\Omega_1\, dR}{v_R} = e^{-u}\, dw. \tag{19}$$

and finally

$$A_m(\alpha) = \sum_l \int \mathrm{d}\mathbf{J} \mathcal{F}_l(\mathbf{J}) \int_{-\pi}^{\pi} \frac{\mathrm{d}w}{R(\mathbf{J}, w)^{1/2}}\, \mathrm{e}^{\mathrm{i}\,[lw + m\phi(\mathbf{J},w) - \alpha \ln\, R(\mathbf{J},w)]} = 2\pi \sum_l \int \mathrm{d}\mathbf{J} \mathcal{F}_l(\mathbf{J}) W_l(\mathbf{J}, \alpha).$$

or

$$A_m(\alpha) = 2\pi \sum_l \int \mathrm{d}\mathbf{J} \mathcal{F}_l(\mathbf{J}) W_l(\mathbf{J}, \alpha). \tag{20}$$

### 2.1. *The linear form*

The kernel $\Pi_{l,l'}(\mathbf{J}, \mathbf{J}')$ has a separable (degenerate) structure: it is represented as an integral over $\alpha$ of products of functions depending separately on $\mathbf{J}$ and $\mathbf{J}'$. This structure can be exploited to reduce the integral equation to a finite-dimensional matrix eigenvalue problem.

Since the right-hand side of equation (16) depends on $\mathcal{F}_{l'}(\mathbf{J}')$ only through the integral with the separable kernel, we seek the eigenfunctions in the form of an expansion in the kernel basis functions:

$$\mathcal{F}_l(\mathbf{J}) = \int_{-\infty}^{\infty} \mathrm{d}\alpha\, C_l(\alpha) W_l^*(\mathbf{J}, \alpha) \tag{21}$$

where $C_l(\alpha)$ are unknown coefficient functions. Substituting into equation (16) and using the separable form of $\Pi_{l,l'}$:

$$[\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})] \int_{-\infty}^{\infty} \mathrm{d}\alpha\, C_l(\alpha) W_l^*(\mathbf{J},\alpha)$$

$$= G\mathcal{F}_{0,l}(\mathbf{J}) \sum_{l'} \int \mathrm{d}\mathbf{J}' \int_{-\infty}^{\infty} \mathrm{d}\alpha\, N(\alpha,m) W_l^*(\mathbf{J},\alpha) W_{l'}(\mathbf{J}',\alpha) \int_{-\infty}^{\infty} \mathrm{d}\alpha'\, C_{l'}(\alpha') W_{l'}^*(\mathbf{J}',\alpha') \tag{22}$$

Multiplying both sides by $W_l(\mathbf{J},\beta)$ and integrating over $\mathrm{d}\mathbf{J}$:

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha\, C_l(\alpha) \int \mathrm{d}\mathbf{J}\, [\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})] W_l(\mathbf{J},\beta) W_l^*(\mathbf{J},\alpha)$$

$$= G \sum_{l'} \int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} \mathrm{d}\alpha'\, C_{l'}(\alpha') \int \mathrm{d}\mathbf{J} \int \mathrm{d}\mathbf{J}'\, \mathcal{F}_{0,l}(\mathbf{J}) N(\alpha,m) W_l(\mathbf{J},\beta) W_l^*(\mathbf{J},\alpha) W_{l'}(\mathbf{J}',\alpha) W_{l'}^*(\mathbf{J}',\alpha') \tag{23}$$

Separating the $\omega$-dependent and $\omega$-independent parts, we define:

$$D_l(\beta,\alpha) \equiv \int \mathrm{d}\mathbf{J}\, W_l(\mathbf{J},\beta) W_l^*(\mathbf{J},\alpha) \tag{24}$$

$$E_l(\beta,\alpha) \equiv \int \mathrm{d}\mathbf{J}\, [l\Omega_1(\mathbf{J}) + m\Omega_2(\mathbf{J})] W_l(\mathbf{J},\beta) W_l^*(\mathbf{J},\alpha) \tag{25}$$

$$B_{l,l'}(\beta,\alpha') \equiv G \int \mathrm{d}\mathbf{J} \int \mathrm{d}\mathbf{J}' \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathcal{F}_{0,l}(\mathbf{J}) N(\alpha,m) W_l(\mathbf{J},\beta) W_l^*(\mathbf{J},\alpha) W_{l'}(\mathbf{J}',\alpha) W_{l'}^*(\mathbf{J}',\alpha') \tag{26}$$

Note that matrices $D_l$ and $E_l$ are Hermitian: $D_l(\beta,\alpha) = D_l^*(\alpha,\beta)$, $E_l(\beta,\alpha) = E_l^*(\alpha,\beta)$. The matrix $B_{l,l'}$ can be factored by defining:

$$F_l(\beta,\alpha) \equiv \int \mathrm{d}\mathbf{J}\, \mathcal{F}_{0,l}(\mathbf{J}) W_l(\mathbf{J},\beta) W_l^*(\mathbf{J},\alpha) \tag{27}$$

so that

$$B_{l,l'}(\beta,\alpha') = G \int_{-\infty}^{\infty} \mathrm{d}\alpha\, N(\alpha,m)\, F_l(\beta,\alpha)\, D_{l'}(\alpha,\alpha') \tag{28}$$

This yields the linear eigenvalue equation for $\omega$:

$$\omega \int_{-\infty}^{\infty} \mathrm{d}\alpha\, D_l(\beta,\alpha) C_l(\alpha) = \int_{-\infty}^{\infty} \mathrm{d}\alpha\, E_l(\beta,\alpha) C_l(\alpha) + \sum_{l'} \int_{-\infty}^{\infty} \mathrm{d}\alpha'\, B_{l,l'}(\beta,\alpha') C_{l'}(\alpha') \tag{29}$$

Discretizing $\alpha \to \alpha_i$ $(i = 1, \ldots, N)$, we have for each $l$:

$$\omega \sum_{j=1}^{N} D_l^{ij} C_l^j = \sum_{j=1}^{N} E_l^{ij} C_l^j + \sum_{l'=l_{\min}}^{l_{\max}} \sum_{j=1}^{N} B_{l,l'}^{ij} C_{l'}^j \tag{30}$$

where $D_l^{ij} = D_l(\alpha_i,\alpha_j)$, $E_l^{ij} = E_l(\alpha_i,\alpha_j)$, and $B_{l,l'}^{ij} = B_{l,l'}(\alpha_i,\alpha_j)$. Combining all $l$ indices and $\alpha$ grid points $(l,i) \to$ single composite index, this becomes a standard generalized eigenvalue problem:

$$\omega\, \mathbf{DC} = (\mathbf{E} + \mathbf{B})\mathbf{C} \tag{31}$$

where $\mathbf{D}$ is block-diagonal in $l$ (each block is $D_l^{ij}$), $\mathbf{E}$ is also block-diagonal (each block is $E_l^{ij}$), and $\mathbf{B}$ couples different $l$ blocks (with off-diagonal blocks $B_{l,l'}^{ij}$). The eigenvalue $\omega$ and eigenvector $\mathbf{C}$ can be found using standard linear algebra routines (e.g., `scipy.linalg.eig`).

### 2.2. *Zang form*

We seek a solution in the form:

$$\mathcal{F}_l(\mathbf{J}) = \frac{1}{\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})} \int_{-\infty}^{\infty} \mathrm{d}\alpha\, C_l(\alpha) W_l(\mathbf{J}, \alpha) \tag{32}$$

where $C_l(\alpha)$ are unknown coefficient functions to be determined. Substituting this ansatz into equation (16):

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha\, C_l(\alpha) W_l(\mathbf{J}, \alpha) =$$
$$G\mathcal{F}_{0,l}(\mathbf{J}) \sum_{l'} \int \mathrm{d}\mathbf{J}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} N(\alpha, m) W_l^*(\mathbf{J}, \alpha) W_{l'}(\mathbf{J}', \alpha) \frac{1}{\omega - l'\Omega_1(\mathbf{J}') - m\Omega_2(\mathbf{J}')} \int_{-\infty}^{\infty} \mathrm{d}\alpha'\, C_{l'}(\alpha') W_{l'}(\mathbf{J}', \alpha') \tag{33}$$

Multiplying both sides by $W_l(\mathbf{J}, \beta)$ and integrating over $\mathrm{d}\mathbf{J}$:

$$\int \mathrm{d}\mathbf{J}\, W_l(\mathbf{J}, \beta) \int_{-\infty}^{\infty} \mathrm{d}\alpha\, C_l(\alpha) W_l(\mathbf{J}, \alpha) =$$
$$G \sum_{l'} \int_{-\infty}^{\infty} \mathrm{d}\alpha'\, C_{l'}(\alpha') \int \mathrm{d}\mathbf{J} \int \mathrm{d}\mathbf{J}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} \frac{\mathcal{F}_{0,l}(\mathbf{J}) N(\alpha, m)}{\omega - l'\Omega_1(\mathbf{J}') - m\Omega_2(\mathbf{J}')} W_l(\mathbf{J}, \beta) W_l^*(\mathbf{J}, \alpha) W_{l'}(\mathbf{J}', \alpha) W_{l'}(\mathbf{J}', \alpha') \tag{34}$$

Defining the matrix elements:

$$M_{l,l'}(\beta, \alpha'; \omega) \equiv G \int \mathrm{d}\mathbf{J} \int \mathrm{d}\mathbf{J}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} \frac{\mathcal{F}_{0,l}(\mathbf{J}) N(\alpha, m)}{\omega - l'\Omega_1(\mathbf{J}') - m\Omega_2(\mathbf{J}')} W_l(\mathbf{J}, \beta) W_l^*(\mathbf{J}, \alpha) W_{l'}(\mathbf{J}', \alpha) W_{l'}(\mathbf{J}', \alpha') \tag{35}$$

and

$$D_l(\beta, \alpha) \equiv \int \mathrm{d}\mathbf{J}\, W_l(\mathbf{J}, \beta) W_l(\mathbf{J}, \alpha) \tag{36}$$

we obtain the matrix eigenvalue equation:

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha\, D_l(\beta, \alpha) C_l(\alpha) = \sum_{l'} \int_{-\infty}^{\infty} \mathrm{d}\alpha'\, M_{l,l'}(\beta, \alpha') C_{l'}(\alpha') \tag{37}$$

In practice, we discretize the continuous variable $\alpha$ into $N$ points: $\alpha_i$ $(i = 1, \ldots, N)$, transforming the integral equation into a finite-dimensional generalized matrix eigenvalue problem:

$$\sum_{j=1}^{N} D_l^{ij} C_l^j = \sum_{l'} \sum_{j=1}^{N} M_{l,l'}^{ij}(\omega) C_{l'}^j \tag{38}$$

where $D_l^{ij} \equiv D_l(\alpha_i, \alpha_j)$ and $M_{l,l'}^{ij}(\omega) \equiv M_{l,l'}(\alpha_i, \alpha_j; \omega)$. The eigenvalue $\omega$ enters the matrix $M_{l,l'}^{ij}$ through the denominator in equation (18), making this a nonlinear eigenvalue problem that can be solved iteratively using Newton-Raphson method.

## Another approach to non-linear eigenvalue problem

We have for the $l$-harmonics of DF, $\mathcal{F}_l(\mathbf{J})$, from linearized Collisionless Boltzmann Equation (CBE) in action-angle presentation

$$\mathcal{F}_l(\mathbf{J}) = -\frac{\mathcal{F}_{0,l}(\mathbf{J})\,\Phi_l(\mathbf{J})}{\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})}, \quad \mathcal{F}_{0,l} \equiv l\,\frac{\partial \mathcal{F}_0}{\partial J} + m\frac{\partial \mathcal{F}_0}{\partial L}, \tag{39}$$

where Fourier harmonics $\Phi_l(\mathbf{J})$ of perturbed potential are (see (12))

$$\Phi_l(\mathbf{J}) = -\frac{G}{2\pi}\int d\alpha\, N_m(\alpha)\, A(\alpha)\, W_l^*(\mathbf{J},\alpha), \tag{40}$$

where

$$W_l(\mathbf{J},\alpha) = \frac{1}{\pi}\int_0^{\pi} \frac{\mathrm{d}w}{\sqrt{R(\mathbf{J},w)}}\cos\Big[lw + m\phi(\mathbf{J},w)\Big]\mathrm{e}^{-\mathrm{i}\alpha\ln R(\mathbf{J},w)}\,. \tag{41}$$

The Fourier expansion coefficients $A(\alpha)$ of the radial part of perturbed surface density $\Sigma(R)$ multiplied by factor $R^{3/2}$, over logarithmic spirals $\exp(\mathrm{i}\alpha u) \equiv \exp(\mathrm{i}\alpha\ln R)$,[1] are expressed through $\mathcal{F}_l(\mathbf{J})$ as follows from (20):

$$A(\beta) = 2\pi\sum_l \int d\mathbf{J}\mathcal{F}_l(\mathbf{J})\, W_l(\mathbf{J},\beta). \tag{42}$$

Substitution $\mathcal{F}_l(\mathbf{J})$ from (39) with $\Phi_l(\mathbf{J})$ from (40) to the r.h.s. of (42) yields

$$A(\beta) = G\sum_l \int d\mathbf{J}\,\frac{\mathcal{F}_{0,l}(\mathbf{J})\,W_l(\mathbf{J},\beta)}{\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})}\int_{-\infty}^{\infty} W_l^*(\mathbf{J},\alpha)N_m(\alpha)\,A(\alpha)\,\mathrm{d}\alpha, \tag{43}$$

or

$$A(\beta) = \int_{-\infty}^{\infty} \mathcal{M}(\beta,\alpha;\omega)\,A(\alpha)\,d\alpha, \tag{44}$$

where

$$\mathcal{M}(\beta,\alpha;\omega) = G\,N_m(\alpha)\sum_l\int d\mathbf{J}\,\frac{\mathcal{F}_{0,l}(\mathbf{J})\,W_l(\mathbf{J},\beta)\,W_l^*(\mathbf{J},\alpha)}{\omega - l\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})}\,. \tag{45}$$

Here

$$N_m(\alpha) \equiv \pi\,\frac{\Gamma\Big(\frac{1}{2}\left[m+\frac{1}{2}+\mathrm{i}\,\alpha\right]\Big)\,\Gamma\Big(\frac{1}{2}\left[m+\frac{1}{2}-\mathrm{i}\,\alpha\right]\Big)}{\Gamma\Big(\frac{1}{2}\left[m+\frac{3}{2}+\mathrm{i}\,\alpha\right]\Big)\,\Gamma\Big(\frac{1}{2}\left[m+\frac{3}{2}-\mathrm{i}\,\alpha\right]\Big)}. \tag{46}$$

The integral equation (44) with the kernel (45) is a complete analogue of the Kalnajs matrix equation. However, instead of expanding the radial parts of the potential and surface density into discrete set of biorthogonal pairs of basic potential-density functions, a continuum Fourier expansion into logarithmic spirals $\exp(\mathrm{i}\,\alpha\ln r)$ is used.

[1]

$$R^{3/2}\,\Sigma(R) = \int_{-\infty}^{\infty}\frac{d\alpha}{2\pi}\,A(\alpha)\,\exp(\mathrm{i}\alpha u), \quad u = \ln R, \quad A(\alpha) = \int_{-\infty}^{\infty}\Sigma\big(R(u)\big)\,\exp\Big(\tfrac{3}{2}\,u - \mathrm{i}\alpha u\Big)\,du,$$

$$R^{1/2}\,\Phi(R) = \int_{-\infty}^{\infty}\frac{d\alpha}{2\pi}\,A(\alpha)\,N_m(\alpha)\exp(\mathrm{i}\alpha u)$$

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## 3. RESULTS AND DISCUSSION

## REFERENCES