Convergence of numerical methods for the integration of stochastic differential equations

We study the strong and weak convergence of some numerical methods (Euler-Maruyama, Milstein, Heun) for solving linear and nonlinear stochastic differential equations (SDE).

a) We first study the strong convergence of the Euler-Maruyama method in the calculation of the solution of the linear SDE the linear SDE

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \tag{1}$$

with initial conditions $X(0) = X_0$, real parameters λ and μ, and exact solution

$$X(t) = X_0 e^{\left(\lambda - \frac{\mu^2}{2}\right)t + \mu W(t)}.$$

To study the strong convergence, we calculate in the program `StrongConv_eqL.m` the error between the numerical solution X_N and the exact one at time T, $X(T)$, through the quantity

$$\mathbb{E}\left(\left|X_N - X(T)\right|\right).$$

The figure 3 a) shows the errors calculated for $M = 10^3$ trajectories as a function of the discretisation step Δt. The Euler-Maruyama method shows a strong convergence of order $\frac{1}{2}$. The slope of the line fitting the obtained values is 0.5952.

b) We study the weak convergence of the Euler-Maruyama method in the calculation of the solution of the linear SDE (1).
In this case we calculate the error between the numerical solution X_N (after N integration steps) and the exact solution at time T, $X(T)$, through the quantity

$$\left|\,\mathbb{E}\left(\phi(X_N) - \phi(X(T))\right)\right|,$$

where we take into account two test functions $\phi(x) = x$ and $\phi(x) = x^2$. We calculate the two error terms

$$\left|\,\mathbb{E}(X_N) - \mathbb{E}(X(T))\right| \qquad \text{et} \qquad \left|\,\mathbb{E}(X_N^2) - \mathbb{E}(X^2(T))\right|. \tag{2}$$

The program `WeakConv_eqL.m` is written with two `for` cycles to take into account a rather high number of Brownian trajectories M.
The interval $[0, T]$ is divided into 5 discretisation steps $\Delta t = 2^{p-10}$ with $1 \leq p \leq 5$. For each step Δt, we calculate a column vector $\mathtt{X}_N$ of size $M \times 1$ corresponding to the value of X at time T. This calculation is performed by adding at each time step the quantity `dt * lambda * Xt + mu * Xt .* dW` to the previous value of `Xt`. `dW` is a column vector with Brownian increments of size $M \times 1$. For each discretisation step, we calculate the mean of the values of $\mathtt{X}_N$ and the mean of their square.
The moments of the exact solution $\mathbb{E}(X(T))$ and $\mathbb{E}(X^2(T))$ in the equation (2) can be estimated analytically. In fact,

$$\mathbb{E}[X(T)] = X_0 e^{\left(\lambda - \frac{\mu^2}{2}\right)T}\,\mathbb{E}[e^{\mu W(T)}] \qquad \text{et} \qquad \mathbb{E}[X^2(T)] = X_0^2 e^{2\left(\lambda - \frac{\mu^2}{2}\right)T}\,\mathbb{E}[e^{2\mu W(T)}].$$

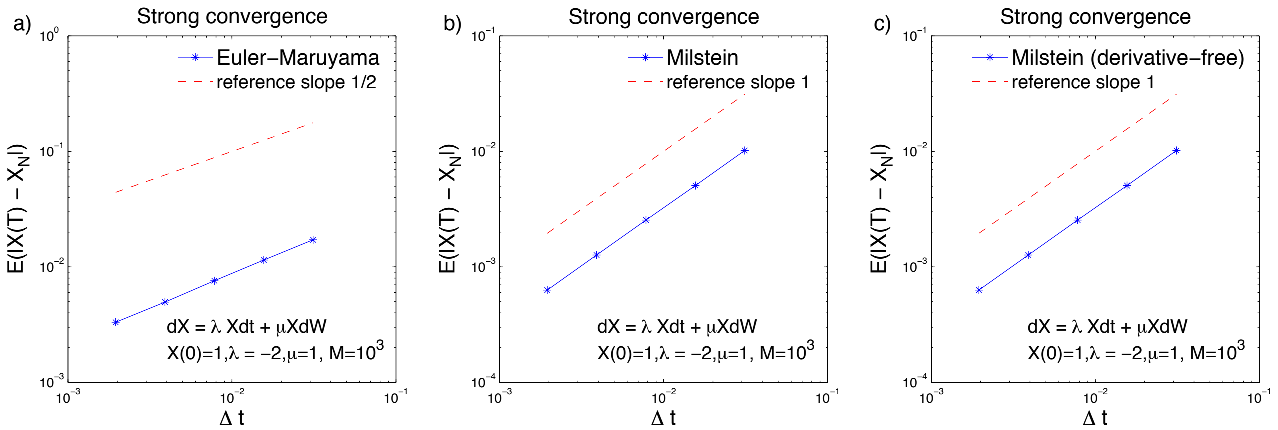

FIGURE 1 – Strong convergence of the methods Euler-Maruyama (a), Milstein (b) and Milstein without direct calculation of the derivative (c) applied to the linear equation $dX = \lambda X dt + \mu X dW$. The errors $\mathbb{E}(|X_N - X(T)|)$ of the numerical solution of the equation with respect to the exact solution at time T are shown as a function of the discretisation step Δt. The results are obtained for $\lambda = -2$, $\mu = 1$, $M = 10^3$ trajectories and brownian increments dW with gaussian probability distribution.

The expected value of a function $f(x) = e^{\alpha x}$, where x is a random variable with gaussian distribution of mean m and variance σ^2, can be expressed as

$$\mathbb{E}\left[e^{\alpha x}\right] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\alpha x} e^{-\frac{(x-m)^2}{2\sigma^2}} = e^{-\frac{m^2}{2\sigma^2}} e^{\frac{\sigma^2}{2}\left(\alpha+\frac{m}{\sigma^2}\right)^2}, \tag{3}$$

where we use the result $\int_{-\infty}^{\infty} e^{-Ax^2+B} = \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}}$.
Therefore, as $W(t)$ follows a normal distribution with $m = 0$ and $\sigma^2 = T$, we obtain

$$\mathbb{E}[X(T)] = X_0 e^{\lambda T} \qquad \text{et} \qquad \mathbb{E}[X^2(T)] = X_0^2 e^{2\lambda T + \mu^2 T}.$$

The figures 2 a) and b) show the errors of the numerical solution as a function of the discretisation step Δt. The errors are calculated according to the equation (2) with a number of trajectories $M = 10^6$. In the case of the two test functions, the error goes to 0 for decreasing discretisation steps.
The slopes of the lines fitting the obtained results are 1.0280 and 0.9955 for the two considered test functions. A dashed reference line with slope equal to 1 confirms that the weak convergence of the Euler-Maruyama method has order 1, while the strong one has order $\frac{1}{2}$ (figure 1 a).

We replace the random variables with normal probability distribution $dW \sim \mathcal{N}(0, h)$ with independent discrete random variables such that $P(dW = \sqrt{h}) = P(dW = -\sqrt{h}) = \frac{1}{2}$, by using the command `dW = sqrt(dt)*sign(randn(m,1))`. The slopes of the lines fitting the obtained results are 1.0282 and 0.9855 with $M = 10^6$ for the two considered test functions. Thus, the Euler-Maruyama method keeps an order of weak convergence of 1, even if we consider discrete random variables.

c) We study the strong and weak convergence of the Euler-Maruyama method in the calculation of the solution of the nonlinear SDE

$$dX(t) = \left(\frac{X(t)}{2} + \sqrt{X^2(t)+1}\right)dt + \sqrt{X^2(t)+1}\,dW(t), \tag{4}$$

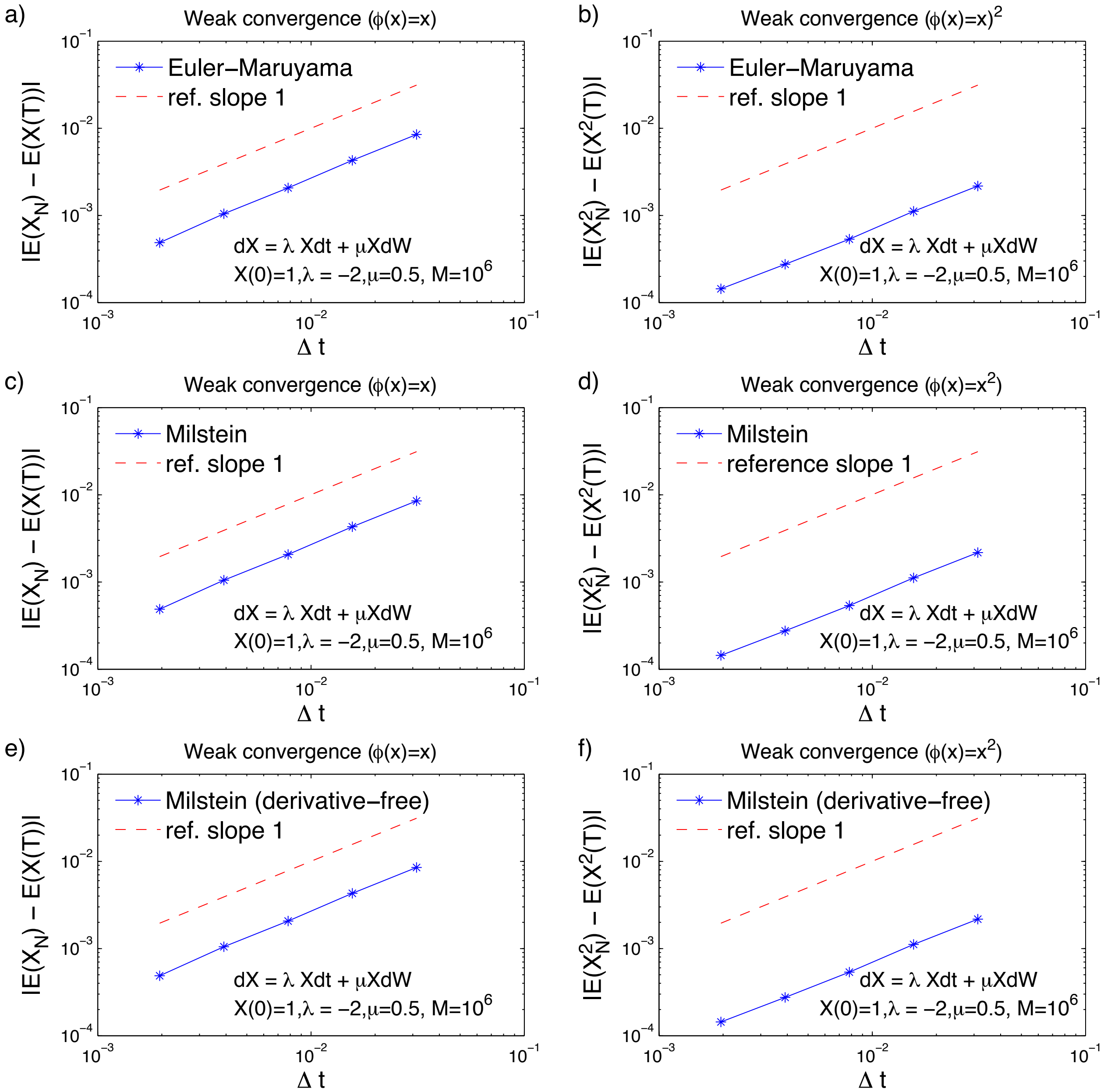

FIGURE 2 – Weak convergence of the methods Euler-Maruyama (a), Milstein (b) and Milstein without direct calculation of the derivative (c) applied to the linear equation $dX = \lambda X dt + \mu X dW$. The error $|\,\mathbb{E}(X_N) - \mathbb{E}(X(T))|$ (a,c,e) and $|\,\mathbb{E}(X_N^2) - \mathbb{E}(X^2(T))|$ (b,d,f) of the numerical solution of the equation with respect to the exact solution at time T are shown as a function of the discretisation step Δt. The results are obtained for $\lambda = -2$, $\mu = 0.5$, $M = 10^6$ trajectories and for Brownian motions dW with gaussian probability distribution.

with initial conditions $X(0) = X_0$ and exact solution given by

$$X(t) = sinh\big(t + W(t) + arcsinh(X_0)\big).$$

Also in this case, to study the strong convergence, we calculate the error between the numerical solution X_N and the exact one at time T, $X(T)$, through the quantity

$$\mathbb{E}\left(|X_N - X(T)|\right).$$

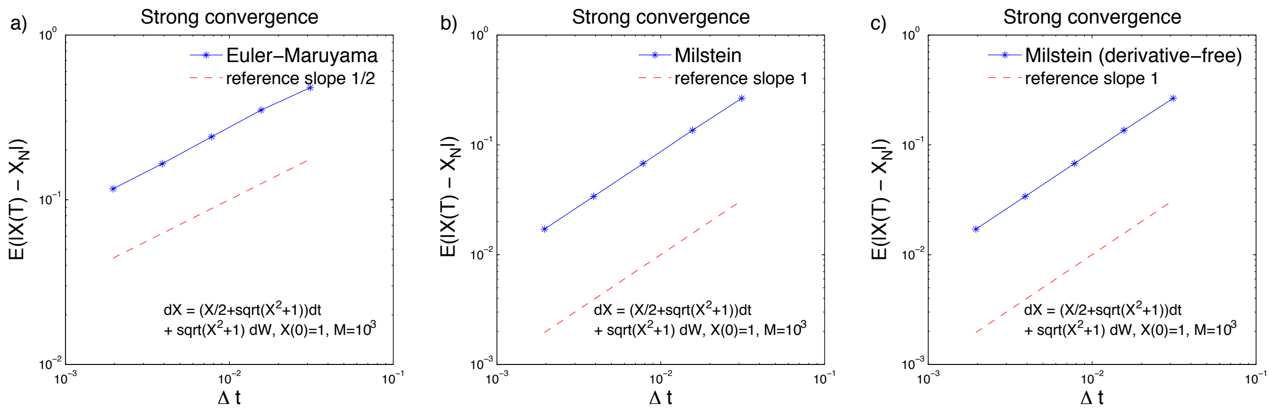

FIGURE 3 – Strong convergence of the methods Euler-Maruyama (a), Milstein (b) and Milstein without direct calculation of the derivative (c) applied to the nonlinear equation $dX = \Big(\frac{X}{2} + \sqrt{X^2+1}\Big)dt + \sqrt{X^2+1}dW$. The errors $\mathbb{E}(|X_N - X(T)|)$ of the numerical solution of the equation with respect to the exact solution at time T are shown as a function of the discretisation step Δt. The results are obtained for $M = 10^3$ trajectories and brownian increments dW with gaussian probability distribution.

The figure 3 a) shows the errors calculated for $M = 10^3$ trajectories as a function of the discretisation step Δt. The Euler-Maruyama method shows a strong convergence of order $\frac{1}{2}$ even for the nonlinear problem. The slope of the line fitting the obtained values is 0.5161.

To study the weak convergence, we calculate the errors (2), for which we need to estimate the expectation values of the exact solution $\mathbb{E}(X(T))$ and $\mathbb{E}(X^2(T))$. We use the result (3), which yields

$$\mathbb{E}[e^{\pm x}] = e^{\pm m + \frac{\sigma^2}{2}}, \qquad \text{et}$$
$$\mathbb{E}[e^{\pm 2x}] = e^{2(\pm m + \sigma^2)},$$

where m and σ^2 are the mean and the variance of the gaussian variable x respectively. Therefore, as $sinh(x) = \frac{1}{2}\big(e^x - e^{-x}\big)$, we have

$$\mathbb{E}[X(T)] = \frac{1}{2}\Big(e^{T+arcsinh(X_0)}\,\mathbb{E}[e^{W(t)}] - e^{-T-arcsinh(X_0)}\,\mathbb{E}[e^{-W(t)}]\Big) = e^{\frac{T}{2}} sinh(T + arcsinh(X_0))$$

and similarly

$$\mathbb{E}[X^2(T)] = -\frac{1}{2} + \frac{1}{4}\Big(e^{4T+2arcsinh(X_0)} + e^{-2arcsinh(X_0)}\Big).$$

Figure 4 a, b) shows the errors in the cases of weak convergence (2) calculated with $M = 10^6$ trajectories as a function of Δt. The Euler-Maruyama method shows a weak convergence of order 1 even in the nonlinear case. The slopes of the lines fitting the points represented in figure 4 a, b) have values 0.9732 and 0.8377.

d) We analyze the strong and weak convergence of the Milstein method applied to the linear (1) and nonlinear (4) SDE.

The strong convergence in the linear equation is studied in the program `StrongConv_eqL.m`. As shown in the figure 1, the strong convergence of the Milstein method has order 1 (slope

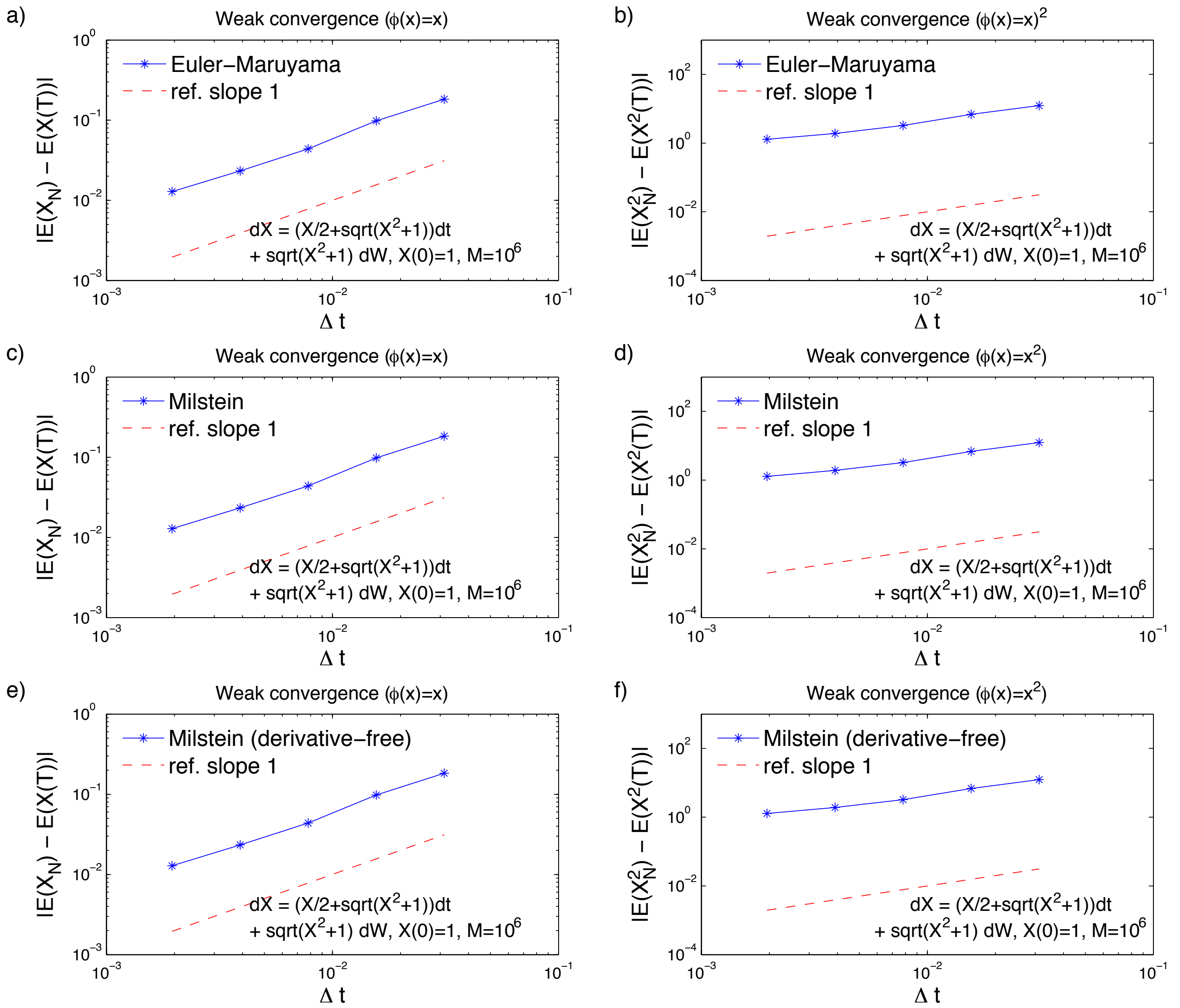

FIGURE 4 – Weak convergence of the methods Euler-Maruyama (a), Milstein (b) and Milstein without direct calculation of the derivative (c) applied to the nonlinear equation $dX = \left(\frac{X}{2} + \sqrt{X^2+1}\right)dt + \sqrt{X^2+1}dW$. The error $|\mathbb{E}(X_N) - \mathbb{E}(X(T))|$ (a,c,e) and $|\mathbb{E}(X_N^2) - \mathbb{E}(X^2(T))|$ (b,d,f) of the numerical solution of the equation with respect to the exact solution at time T are shown as a function of the discretisation step Δt. The results are obtained for $M = 10^6$ trajectories and for Brownian motions dW with gaussian probability distribution.

of the fitting line :1.0030), whereas the Euler-Murayama method has order $\frac{1}{2}$ (slope of the regression line : 0.5952).

The weak convergence in the linear equation is studied in the program `WeakConv_eqL.m`. As shown in figure 2 c) and d), the weak convergence of the Milstein method has order 1 (slope of the fitting line : 1.0278 for the case $\phi = x$ and 0.9951 for $\phi = x^2$). We use $M = 10^6$ trajectories.

The strong convergence in the nonlinear equation is studied with the code `StrongConv_eqNL.m` and the results are shown in the figure 3. We find similar results with respect to the linear case (slopes of the fitting lines : 0.5161 for the Euler-Maruyama method and 0.9917 for the

Milstein one).

The weak convergence in the nonlinear equation is analyzed in the program `WeakConv_eqNL.m`. As shown in figure 4 c) and d), the weak convergence of the Milstein method has order 1 (slope of the regression line : 0.9732 for $\phi = x$ and 0.8377 for $\phi = x^2$). we use $M = 10^6$ trajectories. In all cases, we obtain similar results with the discrete random variables dW.

e) In the 4 programs adopted for the previous points, we also study the convergence of the variant of the Milstein method which uses for the derivative of g an approximation of finite differences

$$g'(Xn)dx \approx \frac{1}{2}\big(g(Xn+dX) - g(Xn-dX)\big).$$

We find a strong convergence of order 1 in the case of the linear (figure 1 c, slope of the fitting line : 1.0030) and nonlinear equation (figure 3 c, slope of the fitting line : 0.9917). We get a similar result for the weak convergence of the linear equation (figure 2 e,f, slopes of the regression lines : 1.0278 (for $\phi = x$) and 0.9951 (for $\phi = x^2$)) and the nonlinear one (figure 4 e,f, slopes of the fitting lines : 0.9732 (for $\phi = x$) and 0.8377 (for $\phi = x^2$)) .

f) We consider a SDE with a constant additive noise

$$dX = f(X)dt + \sigma dW. \tag{5}$$

We study the weak convergence of the Heun method applied to this SDE, which is based on two steps

$$K = X_n + f(X_n)h + \sqrt{h}\sigma\Delta Wn,$$

$$X_{n+1} = X_n + \frac{h}{2}f(X_n) + \frac{h}{2}f(K) + \sqrt{h}\sigma\Delta W_n.$$

We calculate the exact solution of (5) with a linear function $f(X) = \alpha X$, where α is constant. The Itô formula for a generic function $u(x,t)$ is

$$du(x,t) = \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}a(x,t) + \frac{1}{2}b^2(x,t)\frac{\partial^2 u}{\partial x^2}\right]dt + b(x,t)\frac{\partial u}{\partial x}dW.$$

By applying this formula to the function $u(X,t) = Xe^{-\alpha t}$, we have :

$$du = \big[X(-\alpha)e^{-\alpha t} + \alpha X e^{-\alpha t} + 0\big]dt + \sigma e^{-\alpha t}dW,$$

$$u(t) - u(0) = \sigma\int_0^t e^{-\alpha s}dW_s,$$

$$X(t) = X_0 e^{\alpha t} + \sigma e^{\alpha t}\int_0^t e^{-\alpha s}dW_s.$$

By using the properties of the stochastic integral, we can express the expectation value of $X(t)$ and $X^2(t)$ as

$$\mathbb{E}[X(t)] = X_0 e^{\alpha t} + \sigma e^{\alpha t}\,\mathbb{E}\left[\int_0^t e^{-\alpha s}dW_s\right] = X_0 e^{\alpha t}.$$

$$\mathbb{E}[X^2(t)] = X_0^2 e^{2\alpha t} + \sigma^2 e^{2\alpha t}\,\mathbb{E}\left[\left(\int_0^t e^{-\alpha s}dW_s\right)^2\right] + 2X_0\sigma e^{2\alpha t}\,\mathbb{E}\left[\int_0^t e^{-\alpha s}dW_s\right] =$$

$$= X_0^2 e^{2\alpha t} + \sigma^2 e^{2\alpha t} \int_0^t \mathbb{E}[e^{-\alpha s}] ds = X_0^2 e^{2\alpha t} + \sigma^2 e^{2\alpha t} \int_0^t e^{-\alpha s} ds =$$

$$= e^{2\alpha t} \left[X_0 + \frac{\sigma^2}{2\alpha} \right] - e^{\alpha t} \frac{\sigma^2}{2\alpha}.$$

The code `WeakConvH.m` allows one to analyze the weak convergence of the Heun method for the solution of the equation (5) with a linear function $f(X) = \alpha X$.
The figure 5 shows a weak convergence of order 2 for the Heun method applied to the equation (5) with $\alpha = 1$, $\sigma = 0.001$ and $M = 10^6$. The noise is kept very low because this method, given its high order of convergence, is very costly in terms of computational time. The slopes of two lines fitting the data are 2.0138 and 1.9377 for the two test functions ϕ.

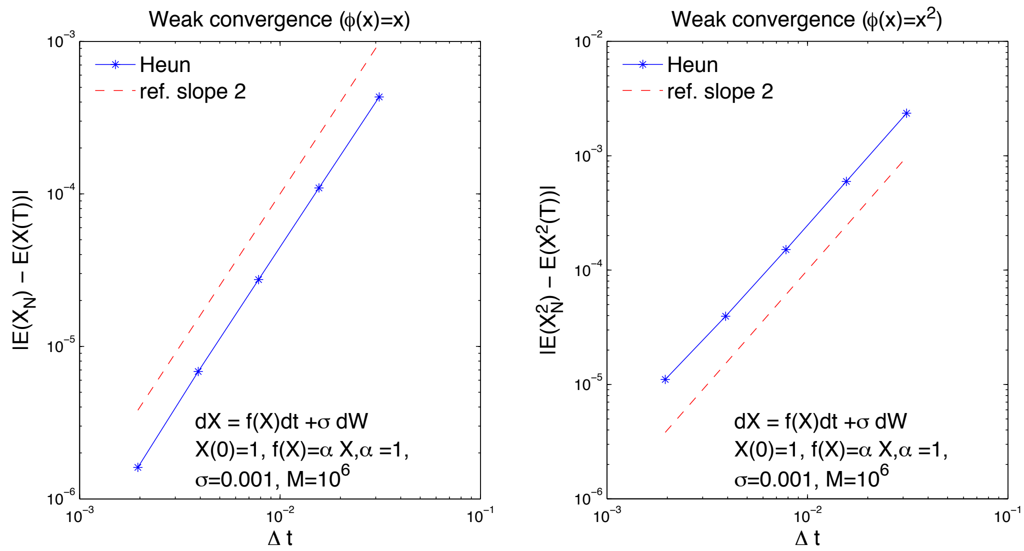

FIGURE 5 – Weak convergence of the Heun method applied to the linear equation $dX = f(X)dt + \sigma dW$ avec $f(X) = \alpha X$. The errors $|\mathbb{E}(X_N) - \mathbb{E}(X(T))|$ (left) and $|\mathbb{E}(X_N^2) - \mathbb{E}(X^2(T))|$ (right) of the numerical solution of the equation with respect to the exact solution at time T are shown as a function of the discretisation step Δt. The results are obtained for $\alpha = 0.001$, $M = 10^6$ trajectories and Brownian increments dW with gaussian probability distribution.