Stochastic heat equation

We solve numerically the stochastic heat equation and represents the solution for some values of its parameters in the space-time plane. We also investigate the effect on a nonlinear term in the equation.

a) The stochastic heat equation is a partial differential equation showing a space-time white noise $W(x,t)$ and can be written in 1 dimension as

$$\partial_t u = \Delta u + \sigma \dot{W}(x,t), \qquad t > 0, x \in (0,1), \tag{1}$$

where $\partial_t u$ is the time derivative of u with respect to t, Δu is the Laplacian of u, and σ is a constant with real values. We impose Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ and the initial conditions $u(x,0) = x(1-x)$.

After discretization in space with finite differences, equation (1) has the form

$$du_i(t) = \Big(\frac{u_{i-1} - 2u_i + u_{i+1}^m}{dx^2}\Big)dt + \sigma\frac{dW_i}{\sqrt{dx}}, \qquad i = 2, \dots, N_x - 1 \tag{2}$$

with Dirichlet boundary conditions $u_0(t) = u_{N_x}(t) = 0$ and independent Wiener processes $W_i(t)$.

To discretize the equation in time, we divide the global interval $[0,1]$ into several intervals dt. We could approximate the temporal derivative $du_i(t)$ with the difference $du_i = u_i^{m+1} - u_i^m$, where m indicates the m-th sub-interval of $[0,1]$. This way of expressing the temporal derivative makes it easy to express u_i^{m+1} as a function of terms at time m. However, this explicit Euler method has problems of stability, which can be avoided only by multiplying the Laplacian by a very small constant.

We therefore discretise the equation (2) in time by using a "backward" difference, consisting in approximating the time derivative $du_i(t)$ with the difference $du_i = u_i^m - u_i^{m-1}$. This approach is implicit and the term u_i^m cannot be expressed easily, but can be found only by solving a system of equations. By discretizing the time in this way, we have the system of equations for $i = 2, \dots, N_x - 1$

$$u_i^m - u_i^{m-1} = \Big(\frac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{dx^2}\Big)dt + \sigma\frac{dW_i^m}{\sqrt{dx}},$$

which can be rewritten in the following way :

$$-\frac{u_{i-1}^m}{dx^2}dt + u_i^m\Big(1 + \frac{2dt}{dx^2}\Big) - \frac{u_{i+1}^m dt}{dx^2} = u_i^{m-1} + \frac{\sigma dW_i^m}{\sqrt{dx}}, \qquad i = 2, \dots, N_x - 1. \tag{3}$$

In other terms, the system of equations to solve at every time t is

$$au_{i-1}^m + bu_i^m - cu_{i+1}^m = d_i,$$

with the coefficients

$$a = -\frac{1}{dx^2}dt, \qquad b = 1 + \frac{2dt}{dx^2}, \qquad c = -\frac{dt}{dx^2}, \qquad d_i = u_i^{m-1} + \frac{\sigma dW_i^m}{\sqrt{dx}}.$$

The system of equations in matrix form is

$$Au = d, \tag{4}$$

and more precisely

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \dots & 0 \\ a_2 & b_2 & c_2 & 0 & \dots & 0 \\ 0 & a_3 & b_3 & c_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ 0 & 0 & \dots & 0 & a_{N_x} & b_{N_x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N_x-1} \\ u_{N_x} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{N_x-1} \\ d_{N_x} \end{bmatrix}.$$

In order to satisfy the Dirichlet boundary conditions, we set the equalities

$$b_1 = 1, \qquad c_1 = 0, \qquad d_1 = 0,$$

$$a_{N_z} = 0, \qquad b_{N_x} = 1, \qquad d_{N_x} = 0,$$

with coefficients d_1 and d_{N_x} put to 0 at every time.
Being tridiagonal, the matrix A can be factorised into two matrices L and U, such that $A = LU$ and that can be written as

$$L = \begin{bmatrix} e_1 & & & & & 0 \\ a_2 & e_2 & & & & \\ & & \ddots & \ddots & & \\ & & & a_{N_x-1} & e_{N_x-1} & \\ 0 & & & & a_{N_x} & e_{N_x} \end{bmatrix} U = \begin{bmatrix} 1 & f_1 & & & & 0 \\ & 1 & f_2 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & f_{N_x-1} & \\ 0 & & & & & 1 \end{bmatrix}.$$

The matrices L and U are thus defined by the vectors a, e and f. The coefficients of the vectors e and f can be expressed as

$$e_1 = b_1, \qquad f_1 = \frac{c_1}{e_1} = \frac{c_1}{b_1}$$

$$e_i = b_i - a_1 f_{i-1}, \qquad f_i = \frac{c_i}{e_i}.$$

The factorization of A in L and U is carried out in the function `LU_tridiag.m`. The system of equations (4) can be written as

$$L(Uu) = d.$$

Consequently, we first solve the equation in the matrix form $Ls = d$, and then $Uu = s$. These operations are performed in the `solve_Aud.m` function, which uses the fact that L is a lower triangular matrix and U a higher triangular one.

The program `StocHeatEq_implicitEuler.m` implements the steps described above to solve the stochastic heat equation. Figure 1 shows the solution of the deterministic heat equation with parameters $\alpha = 1$, $\sigma = 0$ (no stochastic noise) and $\beta = 0$ (no nonlinear terms). The solution $u(x,t)$, from its initial condition $u(x,0)$, decreases progressively in the considered time interval. This damping effect is controlled by the constant α.

In these conditions, some stochastic noise is added. Figure 2 shows the space-time solution of the stochastic heat equation for two values of σ. At $\sigma = 0.1$ the solution $u(x,t)$ begins to be significantly perturbed.

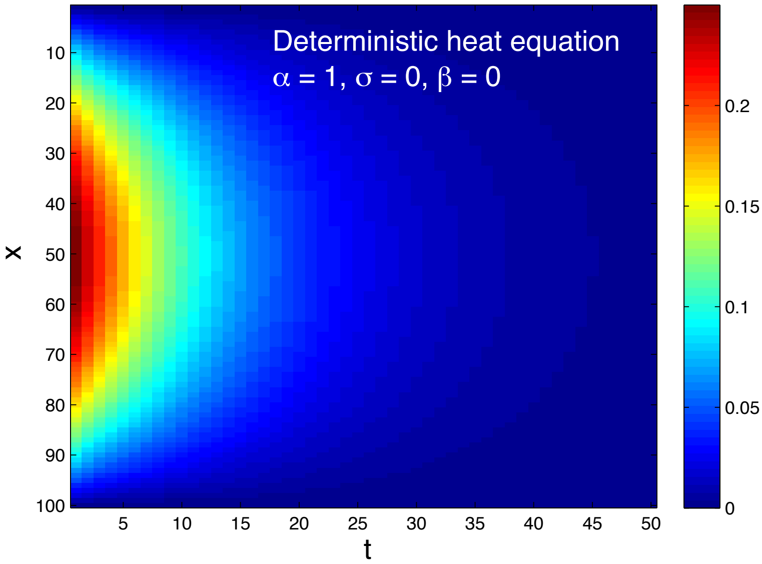

FIGURE 1 – Representation in the space-time plane of the solution of the deterministic heat equation ($\alpha = 1$).

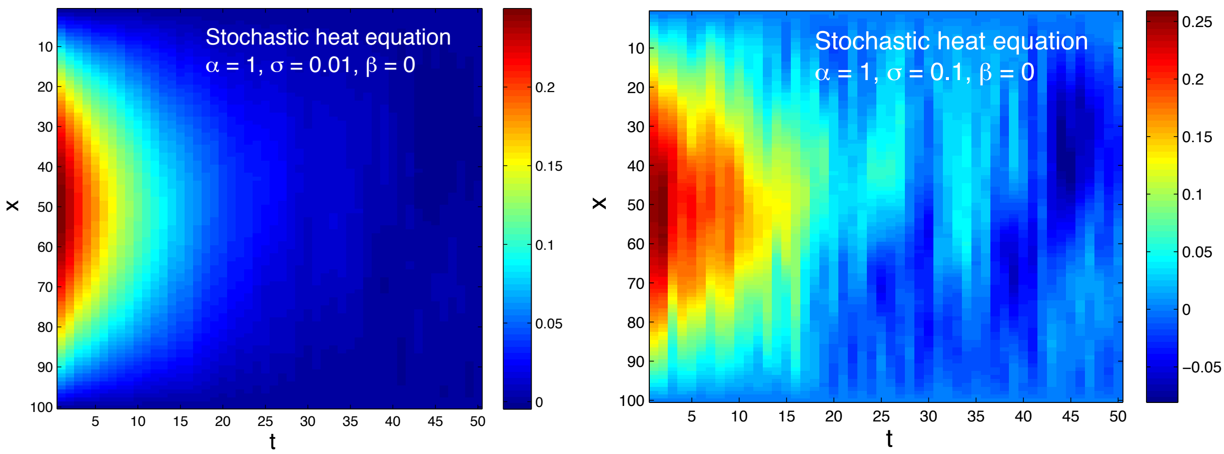

FIGURE 2 – Representation in the space-time plane of the solution of the stochastic heat equation for a stochastic noise $\sigma = 0.01$ (left) and $\sigma = 0.1$ (right).

b) We take into account a nonlinear term in the stochastic heat equation

$$\partial_t u = (\Delta u + \sin u) + \sigma \dot{W}(x,t), \qquad t > 0, x \in (0,1).$$

The Laplacian Δu is treated implicitly and the nonlinear term $\sin u$ explicitly. In this way, we add to the right-hand side of the equations (3) a term $\sin u_i^{m-1} dt$, which is in this way included in the coefficients d_i

$$d_i = u_i^{m-1} + \beta \sin u_i^{m-1} dt + \frac{\sigma dW_i^m}{\sqrt{dx}}.$$

We have added a constant β which multiplies $\sin u$ to be able to vary the contribution of this term. The method that has been illustrated in the previous point can therefore be easily generalized without a significant increase of the computational time. On the other hand, in order to treat the $\sin u$ term in an implicit way, it would be necessary to add $\sin u_i^m dt$. As it contains the variable at time m, this quantity could not be included in the coefficients d_i, but should be treated as an additional term in the system of equations to be solved, which would significantly increase the complexity of the problem and the computational time.

Figure 3 (left) shows the solution in the space-time plane of the stochastic heat equation for $\sigma = 0.1$ and $\beta = 1$. In these conditions, the introduction of the nonlinear term does not seem to provide a significant modification of the solution $u(x,t)$. On the other hand, when one increases the value of β to 10 (figure 3 (right)), the trajectory is strongly perturbed. The solutions diverge over time for values of β greater than 10.

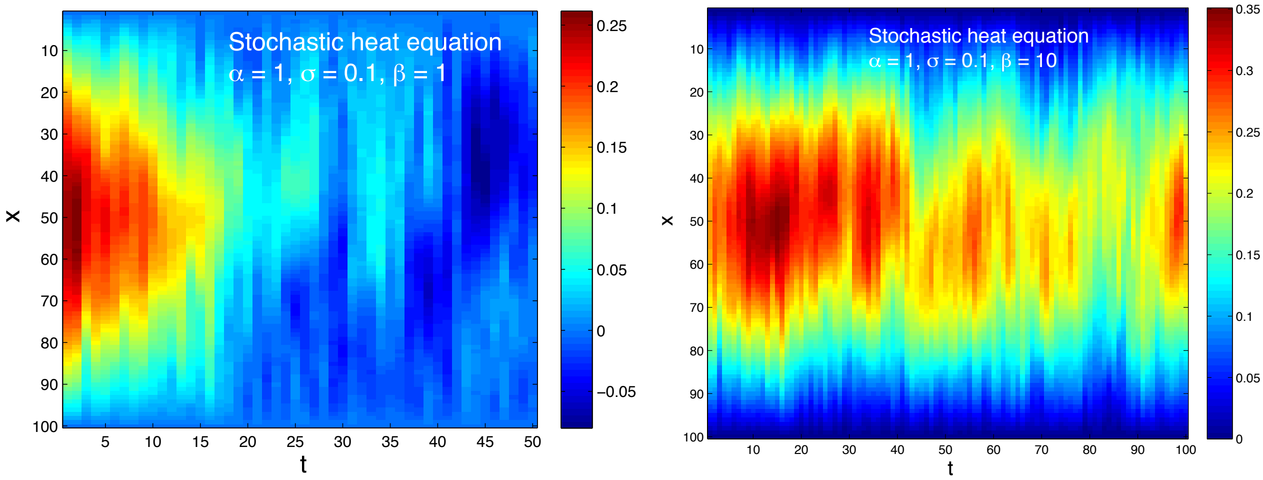

FIGURE 3 – Representation in the space-time plane of the solution of the stochastic heat equation for a stochastic noise $\sigma = 0.1$ and a constant $\beta = 1$ (left) and $\beta = 10$ (right).