Thévenin's theorem

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Thévenin's Theorem

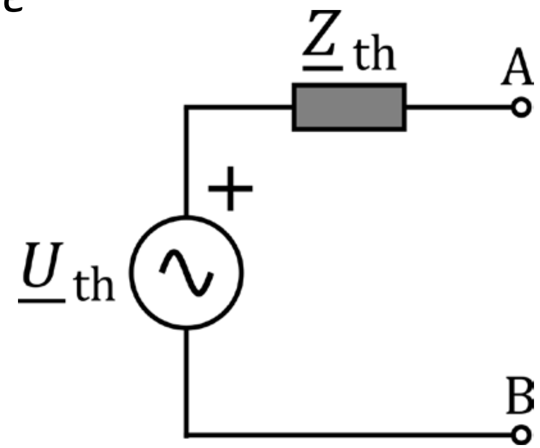
Basics

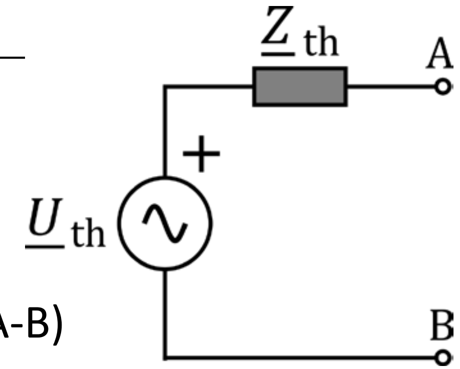
- Thévenin's theorem is one of the most important theorems of electric circuits

- The theorem was independently derived in 1853 by Hermann von Helmholtz and in 1883 by Léon Charles Thévenin

- *Any linear electrical network with voltage and current sources and impedances can be reduced to a two-terminal circuit consisting of a single voltage source in series with a single impedance*

- Thévenin's theorem allows to reduce complicated circuits to a single voltage source $\underline{U}_{\mathrm{th}}$ and a single impedance $\underline{Z}_{\mathrm{th}}$





Thévenin's Theorem

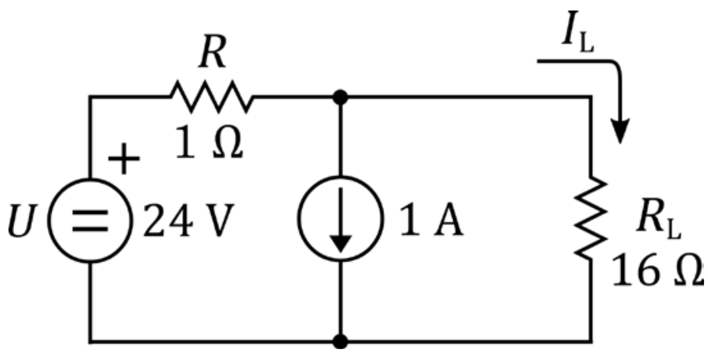
Procedure to obtain a circuit Thévenin equivalent

1. **Remove the load** from the circuit and label the two terminals (A-B)
2. **Determine the open circuit voltage** between the terminals A-B. The resulting open circuit voltage will be the value of the Thévenin voltage, $\underline{U}_{\mathrm{th}}$
3. **Determine the Thévenin equivalent impedance** $\underline{Z}_{\mathrm{th}}$ by calculating the impedance "seen" between terminals A-B. Two different methods:
    1. Direct calculation of the impedance (no dependent sources and magnetic couplings in the circuit) using series-parallel association imposing:
        - Voltage sources are set to zero by replacing them with short circuits
        - Current sources are set to zero by replacing them with open circuits
    2. Short circuit current method
        - Short circuit terminals A-B, measure the short circuit current and calculate the Thévenin equivalent impedance $\underline{Z}_{\mathrm{th}}$ using the Thévenin voltage $\underline{U}_{\mathrm{th}}$
4. **Draw and solve the equivalent circuit** using the calculated Thévenin voltage (Step 2) and the Thévenin impedance (Step 3), connecting the load

Thévenin's Theorem

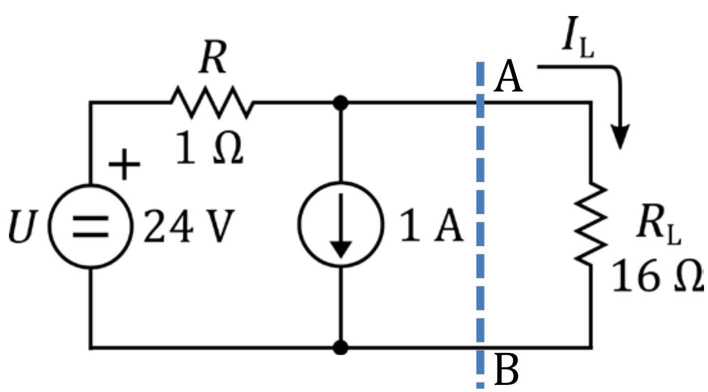
Application example 1 – DC circuit

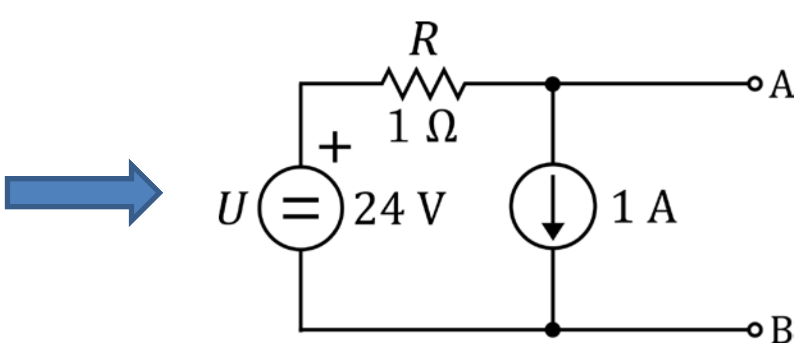
- The same procedure is applied considering that impedances are resistances
- Objectives:
  - Obtain the Thévenin equivalent seen from the load R_L
  - Solve the circuit (obtain the load current I_L)



Procedure to obtain a circuit Thévenin equivalent (Example in DC current)

1. **Remove the load** from the circuit and label the two resulting terminals

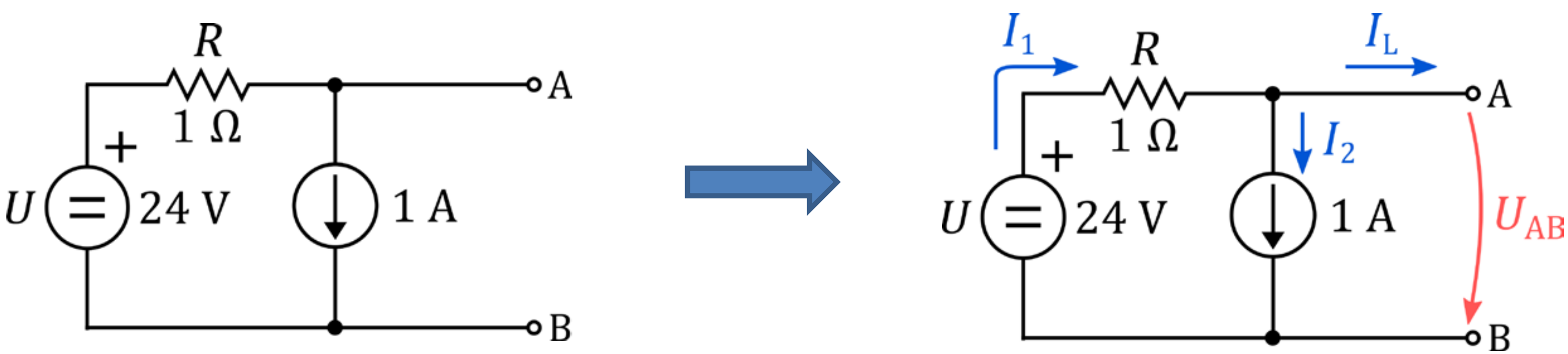




Thévenin's Theorem

Procedure to obtain a circuit Thévenin equivalent (Example in DC current)

2. **Determine the open circuit voltage** between the A-B terminals. The resulting open circuit voltage will be the value of the Thévenin voltage, U_{th}.



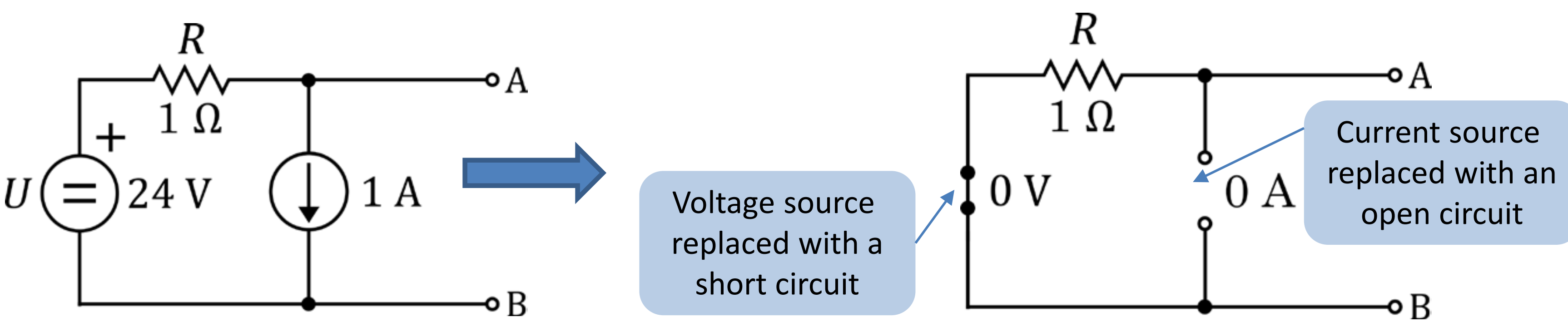
Circuit equations:

$$\left.\begin{aligned} I_1 &= I_2 + \cancel{I_L} = I \\ U - U_{AB} &= R\,I \end{aligned}\right\} \quad U_{\mathrm{th}} = U_{AB} = U - R\,I = 24 - 1 \cdot 1 = 23\ \mathrm{V}$$

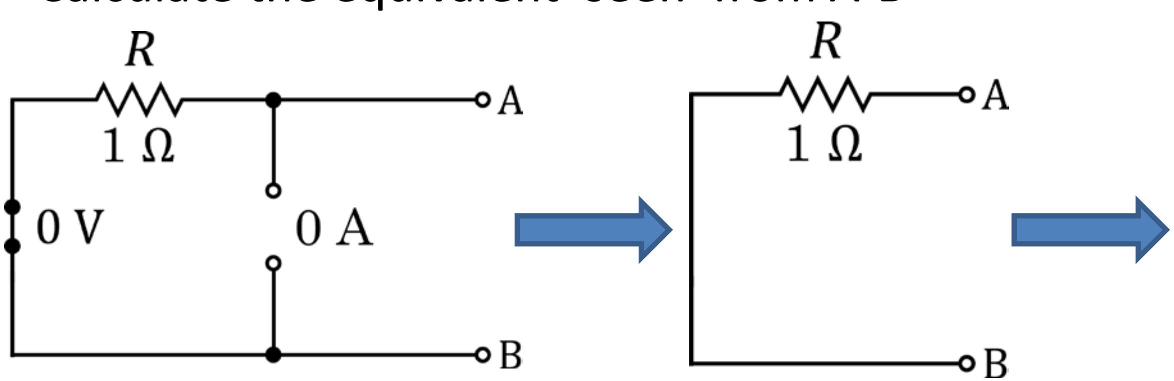
Thévenin's Theorem

Procedure to obtain a circuit Thévenin equivalent (Example in DC current)

3. **Determine the Thévenin equivalent resistance** R_{th} by calculating the resistance "seen" between terminals A-B
   'Direct calculation method' is used for this example (simple DC circuit)
   - Set the voltage and current sources to zero



   - Calculate the equivalent 'seen' from A-B

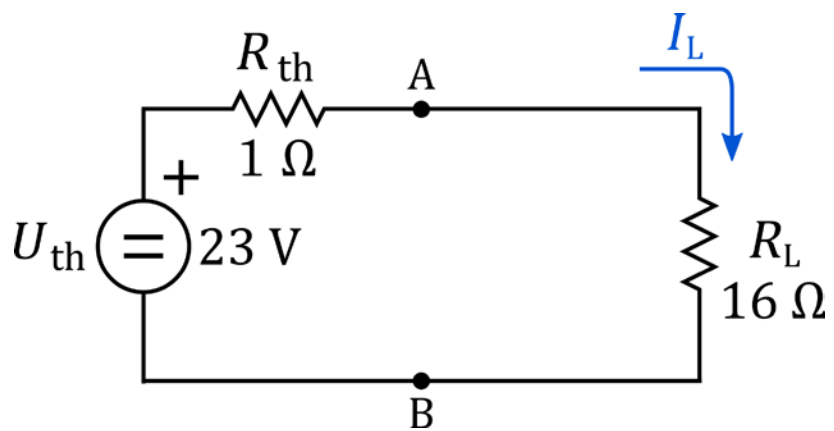


$$R_{\mathrm{th}} = R = 1\ \Omega$$

Thévenin's Theorem

Procedure to obtain a circuit Thévenin equivalent (Example in DC current)

4. **Draw the equivalent circuit** using the calculated Thévenin voltage (Step 2) and the Thévenin resistance (Step 3), connecting again the load



Then, the **circuit can be solved**

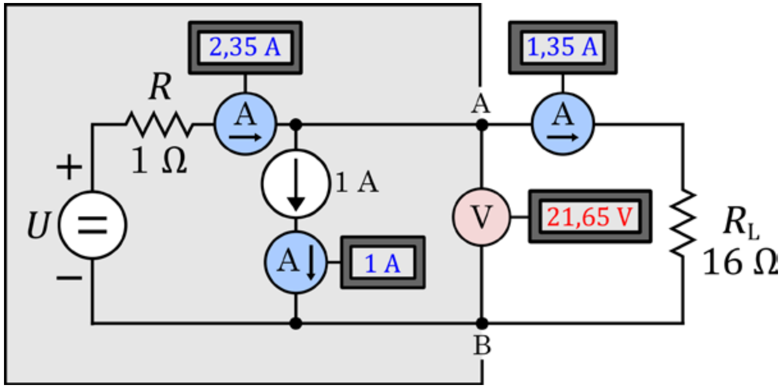
$$U_{th} = (R_{th}+R_L)\, I_L \longrightarrow I_L = \frac{U_{th}}{R_{th}+R_L} \longrightarrow I_L = \frac{23}{1+16} = 1{,}35\ \text{A}$$

The positive sign of the current means that it is following in the same direction as we have assumed

Thévenin's Theorem

Understanding the results

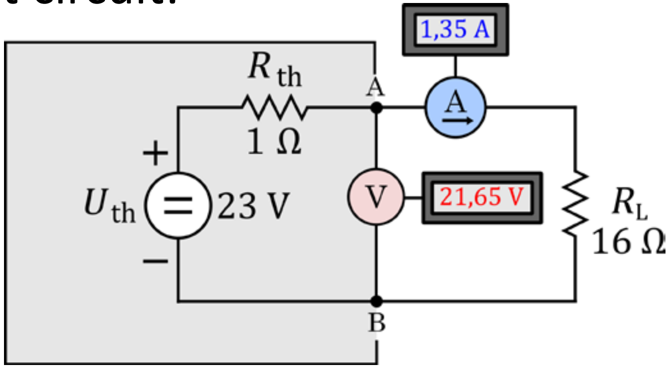
The original circuit is:



And its Thévenin equivalent circuit:

$$U_{th} = 23\ \text{V}$$

$$R_{th} = 1\ \Omega$$

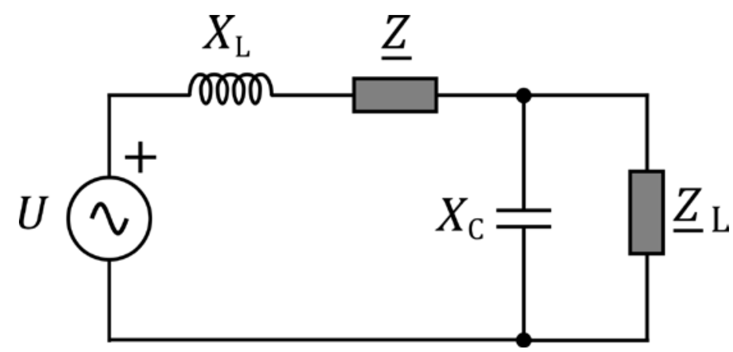


In both circuits the same voltage is applied to the load and therefore the current is identical

Thévenin's Theorem

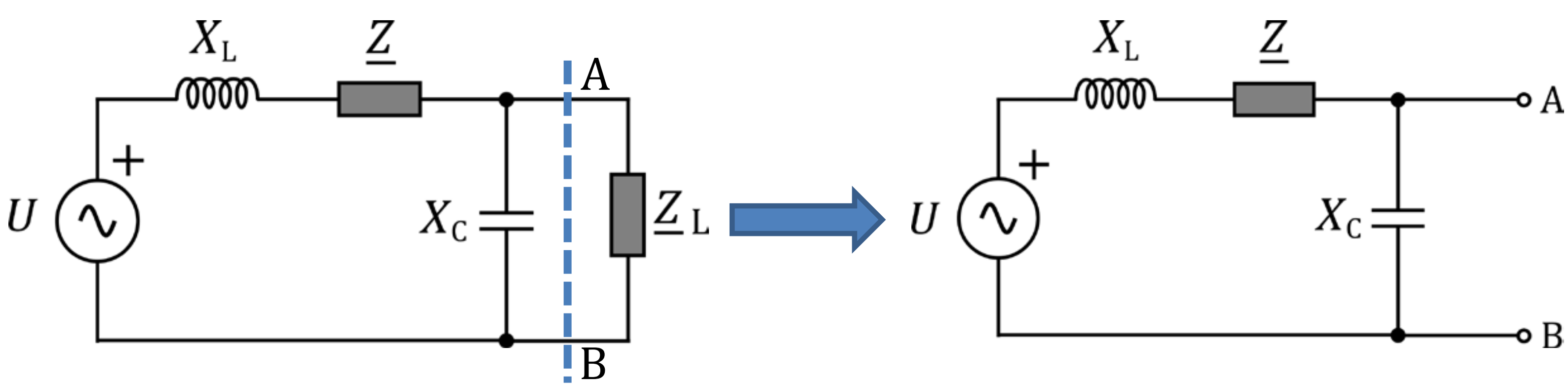
Application example 2 – AC circuit

- Objective: obtain the Thévenin equivalent seen from the load $\underline{Z}_{\mathrm{L}}$ and solve the circuit
- Circuit data:
  - Voltage: $230\ \mathrm{V}$
  - Frequency: $50\ \mathrm{Hz}$
  - Load: $\underline{Z}_{\mathrm{L}} = 50 + j2\ \Omega$
  - $X_{\mathrm{L}} = 0{,}5\ \Omega$
  - $X_{\mathrm{C}} = 200\ \Omega$
  - $\underline{Z} = 2 + j2\ \Omega$



Procedure to obtain a circuit Thévenin equivalent (Example in AC current)

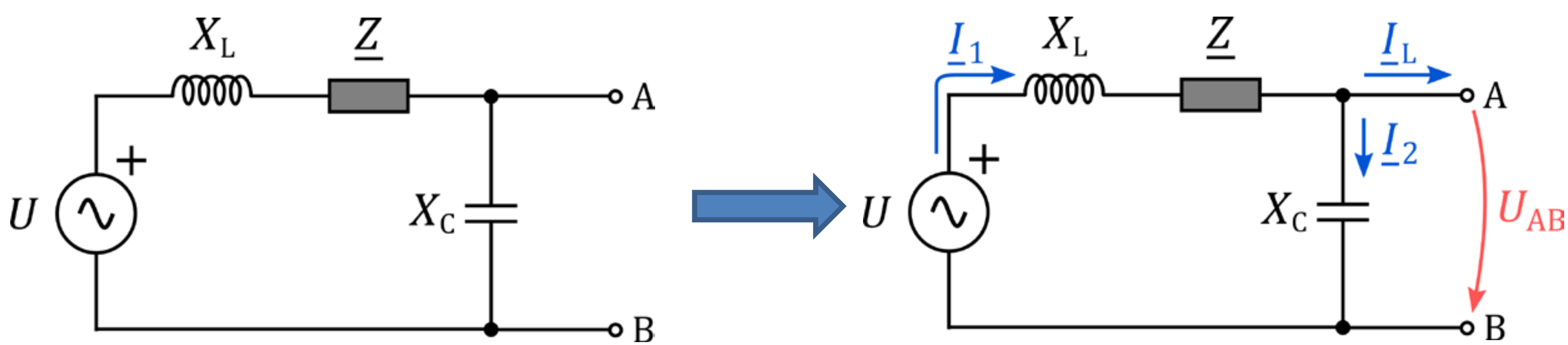
1. **Remove the load** from the circuit and label the two resulting terminals



Thévenin's Theorem

Procedure to obtain a circuit Thévenin equivalent (Example in AC current)

2. **Determine the open circuit voltage** between the A-B terminals. The resulting open circuit voltage will be the value of the Thévenin voltage, $\underline{U}_{\text{th}}$.



Circuit equations:

$$\left.\begin{aligned} \underline{I}_1 &= \underline{I}_2 + \cancel{\underline{I}_L} = \underline{I} \\ \underline{U} &= \underline{I}\,(jX_L + \underline{Z} - jX_C) \\ \underline{U}_{AB} &= -jX_C\,\underline{I} \\ \underline{U} &= U \end{aligned}\right\}$$

$$\underline{I} = \frac{U}{jX_L + \underline{Z} - jX_C} = \frac{230}{j0{,}5 + (2 + j2) - j200} = 0{,}012 + j1{,}16\ \text{A}$$

$$\underline{U}_{\text{th}} = \underline{U}_{AB} = -jX_C\,\underline{I} = 232{,}89 - j2{,}35\ \text{V}$$

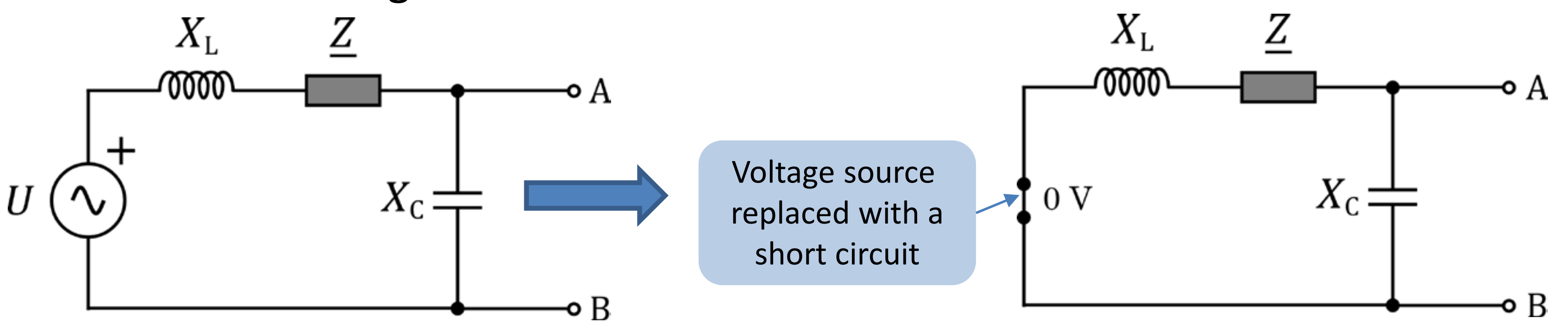
$$U_{\text{th}} = 232{,}90\ \text{V}$$

Thévenin's Theorem

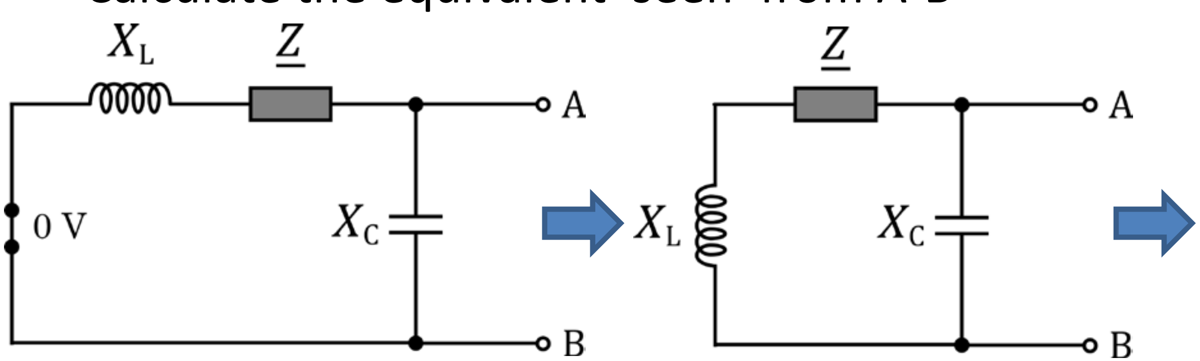
Procedure to obtain a circuit Thévenin equivalent (Example in AC current)

3. **Determine the Thévenin equivalent impedance** $\underline{Z}_{\text{th}}$ by calculating the impedance "seen" between terminals A-B.
   First, the **'Direct calculation method'** is used.

- Set the voltage source to zero



- Calculate the equivalent 'seen' from A-B



// → parallel association of impedances

$$\underline{Z}_{\text{th}} = \left(\underline{Z} + jX_L\right) // -jX_C$$

Thévenin's Theorem

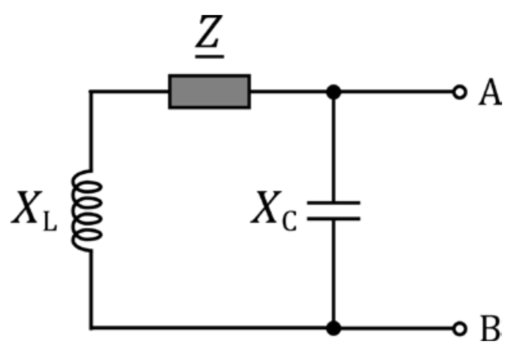
Procedure to obtain a circuit Thévenin equivalent (Example in AC current)

3. Based on the **'Direct calculation method'**, the impedance can be calculated using the following data:

$$\underline{Z} = 2 + j2\ \Omega$$

$$X_{\mathrm{L}} = 0{,}5\ \Omega$$

$$X_{\mathrm{C}} = 200\ \Omega$$



$$\underline{Z}_{\mathrm{th}} = (\underline{Z} + jX_{\mathrm{L}}) // -jX_{\mathrm{C}}$$

$$\underline{Z}_{\mathrm{th}} = \frac{1}{\frac{1}{jX_{\mathrm{L}} + \underline{Z}} + \frac{1}{-jX_{\mathrm{C}}}} = \frac{1}{\frac{1}{j0{,}5 + (2 + j2)} + \frac{1}{-j200}} = 2{,}05 + j2{,}51\ \Omega$$

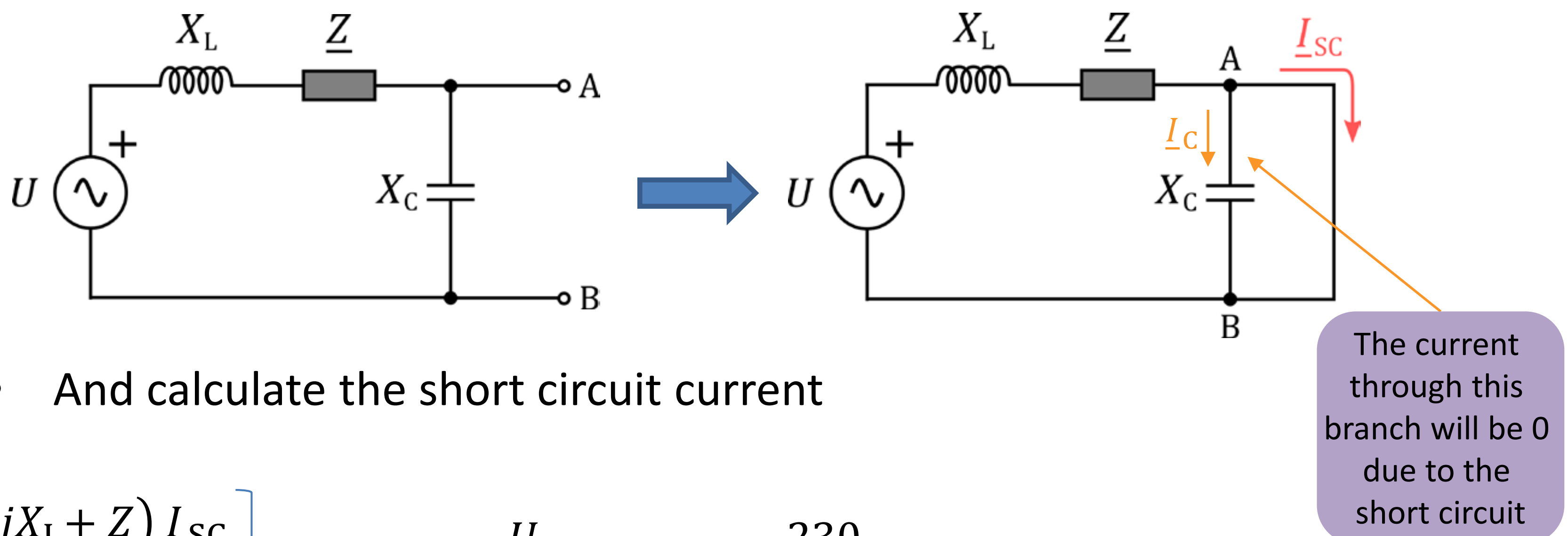
This method is limited to simple circuits that can be reduced by series-parallel associations. It is not applicable if magnetic couplings are present in the circuit.

Thévenin's Theorem

Procedure to obtain a circuit Thévenin equivalent (Example in AC current)

3. An alternative method to find the Thévenin impedance is the **'Short-circuit current method'** which does not have limitations as the 'Direct impedance method'. It can be applied as follows

- Create a short circuit between A and B in the initial circuit



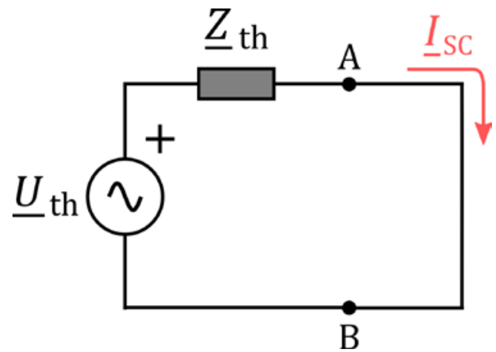
- And calculate the short circuit current

$$\left.\begin{aligned} \underline{U} &= (jX_L + \underline{Z})\,\underline{I}_{SC} \\ \underline{U} &= U \end{aligned}\right\} \quad \underline{I}_{SC} = \frac{U}{(jX_L + \underline{Z})} = \frac{230}{(j0{,}5 + 2 + j2)} = 44{,}88 - j56{,}10\ \text{A}$$

Thévenin's Theorem

Procedure to obtain a circuit Thévenin equivalent (Example in AC current)

3. Similarly, applying a short-circuit to the Thévenin equivalent



- The Thévenin impedance can be expressed as

$$\underline{U}_{\text{th}} = \underline{Z}_{\text{th}}\,\underline{I}_{\text{SC}} \longrightarrow \underline{Z}_{\text{th}} = \frac{\underline{U}_{\text{th}}}{\underline{I}_{\text{SC}}}$$

- Then, based on the obtained Thévenin voltage and short circuit current, the Thévenin impedance is:

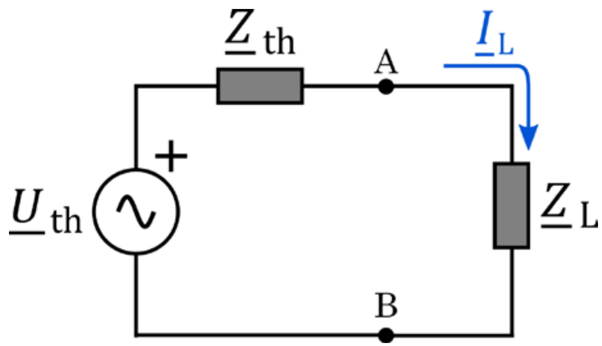
$$\underline{Z}_{\text{th}} = \frac{\underline{U}_{\text{th}}}{\underline{I}_{\text{SC}}} = \frac{232{,}89 - j2{,}35}{44{,}88 - j56{,}09} = 2{,}05 + j2{,}51\ \Omega$$

- Impedance comparison between methods
  - Direct method: $\underline{Z}_{\text{th}} = 2{,}05 + j2{,}51\ \Omega$
  - Short circuit method: $\underline{Z}_{\text{th}} = 2{,}05 + j2{,}51\ \Omega$

Thévenin's Theorem

Procedure to obtain a circuit Thévenin equivalent (Example in AC current)

4. **Draw the equivalent circuit** using the calculated Thévenin voltage (step 2) and the Thévenin impedance (Step 3), connecting again the load.



Then, the **circuit can be solved**

$$\underline{U}_{\text{th}} = (\underline{Z}_{\text{th}} + \underline{Z}_{\text{L}})\,\underline{I}_{\text{L}} \longrightarrow \underline{I}_{\text{L}} = \frac{\underline{U}_{\text{th}}}{(\underline{Z}_{\text{th}} + \underline{Z}_{\text{L}})} = 4{,}44 - j0{,}43\text{ A}$$

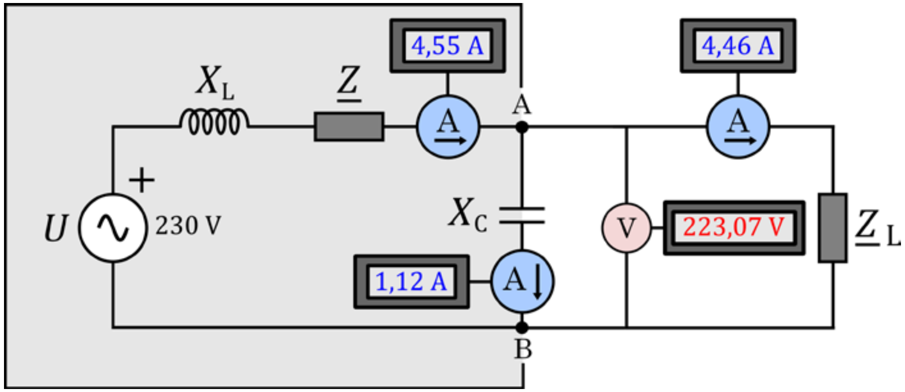
Then, the current flowing through the circuit is

$$I_{\text{L}} = 4{,}46\text{ A}$$

Thévenin's Theorem

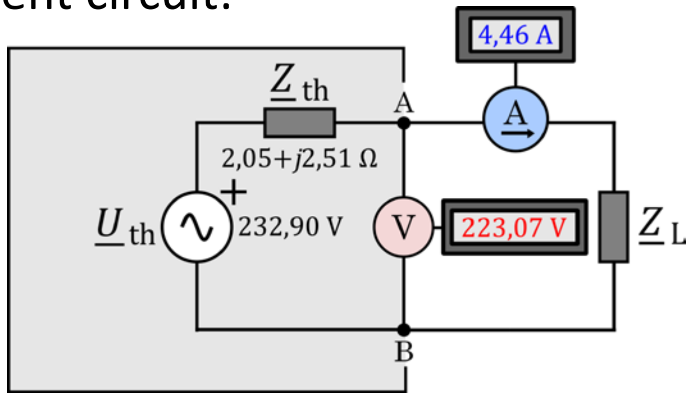
Understanding the results

The original circuit is:

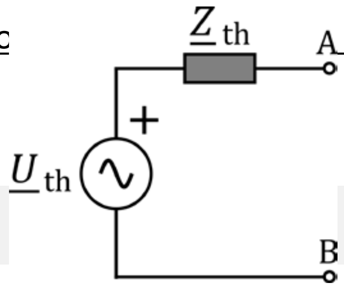


And its Thévenin equivalent circuit:

$$\underline{U}_{th} = 232{,}89 - j2{,}35 \text{ V}$$
$$\underline{Z}_{th} = 2{,}05 + j2{,}51 \ \Omega$$



In both circuits the same voltage is applied to the load and therefore the current is identical.



Grid Thévenin equivalent

Short circuit power

- Usually, electric companies use the Thévenin equivalent to characterize a certain point of the network. Two concepts should be previously defined (according to IEC*):
  - The **Short circuit power** S_{SC} is the **product between the current in the short circuit** at a point of a system, and a **conventional voltage**, generally the operating voltage.
  - The **grid operating voltage** U_{g} in a system is the value of the **voltage under normal conditions**, at a given instant and a given point of the system (this value may be expected, estimated or measured)

- Then, the grid Thévenin equivalent can be calculated as

$$U_{\mathrm{th}} = U_{\mathrm{g}} \qquad Z_{\mathrm{th}} = \frac{U_{\mathrm{g}}^2}{S_{\mathrm{SC}}}$$

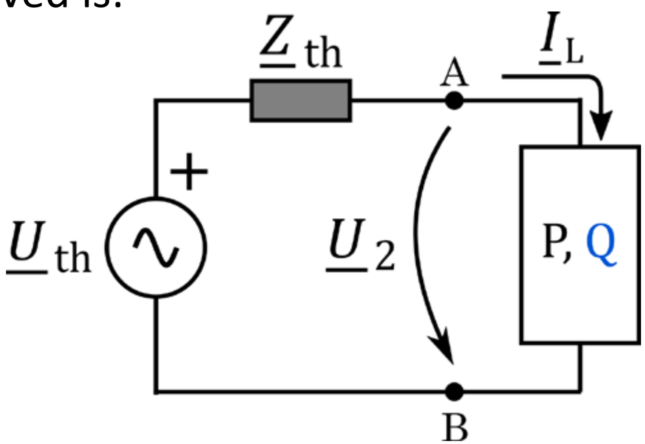
- The ratio between R and X of the grid impedance Z_{th} depends on the type of grid (transmission or distribution)

*IEC (International Electrotechnical Commission)

Relevant Thévenin application case

Thévenin circuit equivalent connected to a load

- The problem to be solved is:

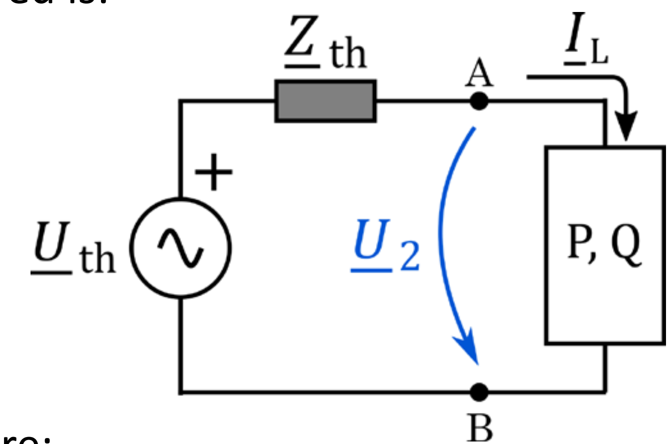


- It is a simple 'Power flow analysis' case, considering only two nodes.
- Two different cases can be differentiated based on the known data
  - It is assumed that the Grid Thévenin is known ($\underline{U}_{\mathrm{th}}$ and $\underline{Z}_{\mathrm{th}}$)
  - Case 1: P and Q known – Find voltage U_2
  - Case 2: P and U_2 known – Find reactive power Q

Relevant Thévenin application case

Case 1: P and Q known, U_2 unknown

- The problem to be solved is:



- The circuit equations are:

Equation 1: $\underline{U}_{\text{th}} = \underline{Z}_{\text{th}}\,\underline{I}_{\text{L}} + \underline{U}_2 = (R_{\text{th}} + jX_{\text{th}})\,\underline{I}_{\text{L}} + \underline{U}_2$

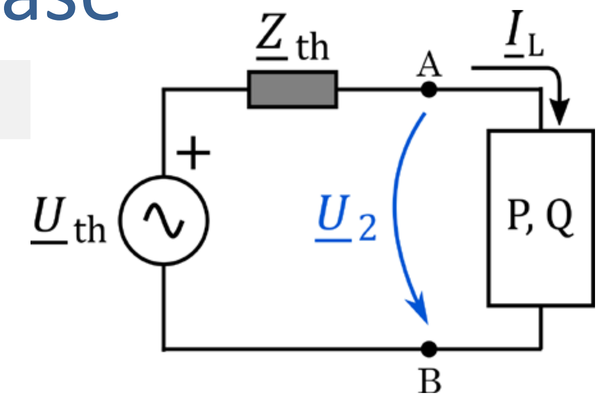
Equation 2: $\underline{S} = \underline{U}_2\,\underline{I}_{\text{L}}^* = P + jQ \longrightarrow \underline{I}_{\text{L}}^* = \dfrac{\underline{S}}{\underline{U}_2} \longrightarrow \underline{I}_{\text{L}} = \dfrac{\underline{S}^*}{\underline{U}_2^*}$

- Substituting the expression of the current $\underline{I}_{\text{L}}$ in equation 1

$$\underline{U}_{\text{th}} = (R_{\text{th}} + jX_{\text{th}})\frac{\underline{S}^*}{\underline{U}_2^*} + \underline{U}_2 \xrightarrow{\times\ \underline{U}_2^*} \underline{U}_{\text{th}}\,\underline{U}_2^* = (R_{\text{th}} + jX_{\text{th}})\,\underline{S}^* + \overbrace{\underline{U}_2\,\underline{U}_2^*}^{U_2^2}$$

Relevant Thévenin application case

Case 1: P and Q known, U_2 unknown



- Then:

$$\underline{U}_{\text{th}}\ \underline{U}_2^* = (R_{\text{th}} + jX_{\text{th}})\underline{S}^* + U_2^2$$

$$\underline{U}_{\text{th}}\ \underline{U}_2^* = (R_{\text{th}} + jX_{\text{th}})(P - jQ) + U_2^2$$

- Operating the previous equation:

$$j^2 = -1$$

$$\underline{U}_{\text{th}}\ \underline{U}_2^* = R_{\text{th}}P - jR_{\text{th}}Q + jX_{\text{th}}P - j^2X_{\text{th}}Q + U_2^2$$

$$\underline{U}_{\text{th}}\ \underline{U}_2^* = \underbrace{R_{\text{th}}P + X_{\text{th}}Q + U_2^2}_{\text{Real part}} + j\underbrace{(X_{\text{th}}P - R_{\text{th}}Q)}_{\text{Imaginary part}}$$

- Then, calculating the square of the absolute value of the previous expression

$$U_{\text{th}}^2\ U_2^2 = \underbrace{(R_{\text{th}}P + X_{\text{th}}Q + U_2^2)^2}_{\text{Real part}} + \underbrace{(X_{\text{th}}P - R_{\text{th}}Q)^2}_{\text{Imaginary part}}$$

$$U_{\text{th}}^2\ U_2^2 = R_{\text{th}}^2P^2 + X_{\text{th}}^2Q^2 + U_2^4 + \cancel{2R_{\text{th}}PX_{\text{th}}Q} + 2R_{\text{th}}P\ U_2^2 + 2X_{\text{th}}Q\ U_2^2 +$$
$$+X_{\text{th}}^2P^2 + R_{\text{th}}^2Q^2 - \cancel{2R_{\text{th}}PX_{\text{th}}Q}$$

Relevant Thévenin application case

Case 1: P and Q known, U_2 unknown

- Reordering the previous expression

$$U_2^4 - U_{\text{th}}^2\ U_2^2 + R_{\text{th}}^2 P^2 + X_{\text{th}}^2 Q^2 + 2R_{\text{th}} P\ U_2^2 + 2X_{\text{th}} Q\ U_2^2 + X_{\text{th}}^2 P^2 + R_{\text{th}}^2 Q^2 = 0$$

- Knowing that:

$$R_{\text{th}}^2 P^2 + X_{\text{th}}^2 Q^2 + X_{\text{th}}^2 P^2 + R_{\text{th}}^2 Q^2 = \overbrace{\left(R_{\text{th}}{}^2 + X_{\text{th}}^2\right)}^{Z_{\text{th}}^2}\overbrace{(P^2 + Q^2)}^{S^2} = Z_{\text{th}}^2\ S^2$$

- The expression can be simplified as

$$U_2^4 - U_{\text{th}}^2\ U_2^2 + 2R_{\text{th}} P\ U_2^2 + 2X_{\text{th}} Q\ U_2^2 + Z_{th}^2 S^2 = 0$$

- Simplifying (U_2^2 common factor):

$$U_2^4 - U_2^2(U_{\text{th}}^2 - 2(R_{\text{th}} P + X_{\text{th}} Q)) + Z_{\text{th}}^2 S^2 = 0$$

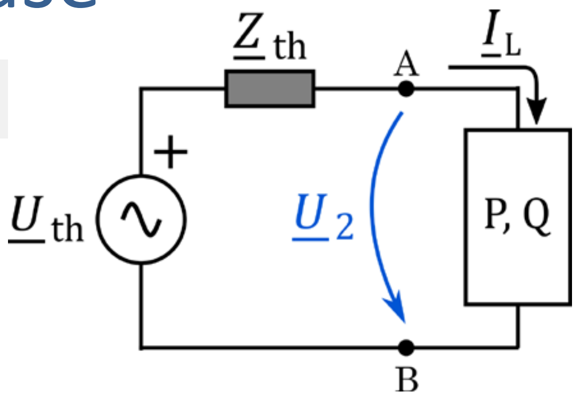
- Provided that

$$\text{Re}\left(\underline{Z}_{\text{th}}\ \underline{S}^*\right) = \text{Re}\left((R_{\text{th}} + jX_{\text{th}})\ (P - jQ)\right) = \overbrace{\text{Re}(R_{\text{th}} P - j^2 X_{\text{th}} Q - jR_{\text{th}} Q + jX_{\text{th}} P)}^{R_{\text{th}} P + X_{\text{th}} Q}$$

- Then, the final expression to obtain the load voltage U_2 is:

$$U_2^4 - U_2^2\left(U_{\text{th}}^2 - 2\ \text{Re}\left(\underline{Z}_{\text{th}}\ \underline{S}^*\right)\right) + Z_{\text{th}}^2\ S^2 = 0$$

Relevant Thévenin application case



Case 1: P and Q known, U_2 unknown

- Then, the equation must be solved

$$U_2^4 - U_2^2(U_{\mathrm{th}}^2 - 2\,\mathrm{Re}(\underline{Z}_{\mathrm{th}}\,\underline{S}^*)) + Z_{\mathrm{th}}^2\,S^2 = 0$$

- As an example:
  - Voltage: $\underline{U}_{\mathrm{th}} = 230\ \mathrm{V}$
  - Frequency: $50\ \mathrm{Hz}$
  - Impedance: $\underline{Z}_{\mathrm{th}} = 0{,}5 + j1{,}2\ \Omega$
  - Load:
    - $P = 1\ \mathrm{kW}$
    - $\cos\varphi = 0{,}9$
  - $S = \dfrac{P}{\cos\varphi}$

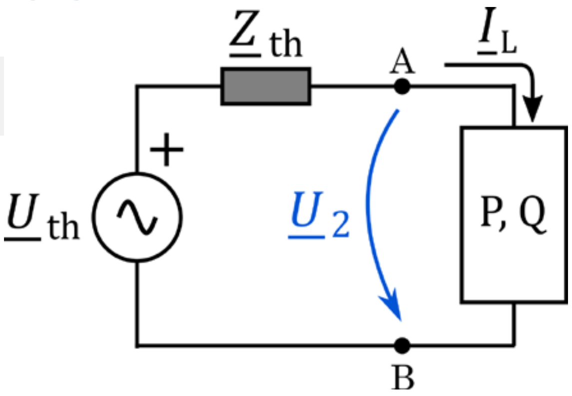
- The numeric solutions of the biquadratic equation are $\pm\,225{,}16\ \mathrm{V}$ and $\pm 6{,}42\ \mathrm{V}$

- Considering the positive values: $225{,}16\ \mathrm{V}$ and $6{,}42\ \mathrm{V}$

- Finally, the adequate solution should be selected, considering the system operational voltage and its constraints. The most suitable solution is $U_2 = 225{,}16\ \mathrm{V}$ as the voltage magnitude is close to the grid 230 V value

Relevant Thévenin application case

Case 1: P and Q known, U_2 unknown



- In order to confirm this selection, the current flowing through the circuit can be obtained (taking U_2 as reference)

$$\underline{S} = U_2\, \underline{I}_{\text{L}}^* = P + jQ \longrightarrow \underline{I}_{\text{L}} = \frac{\underline{S}^*}{U_2}$$

- The current for both solutions are
  - For $U_2 = 225{,}16\ \text{V} \rightarrow I_{\text{L}} = 4{,}93\ \text{A}$ (logical current)
  - For $U_2 = 6{,}42\ \text{V} \rightarrow I_{\text{L}} = 173{,}20\ \text{A}$ (high current)

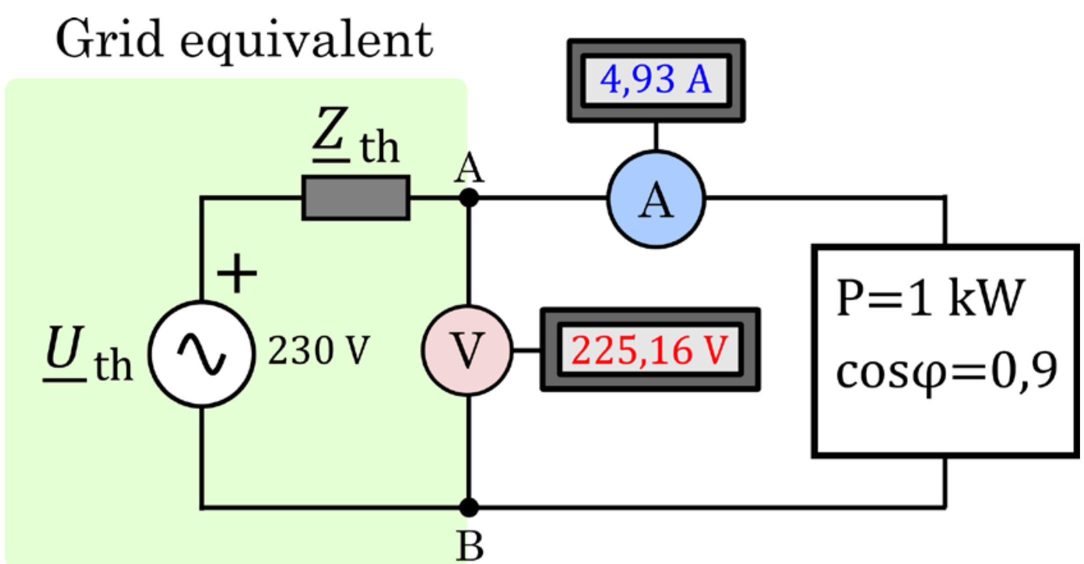
- Also, the equivalent impedance of the load can be obtained:

$$\left.\begin{aligned}\underline{S} &= U_2\, \underline{I}_{\text{L}}^* \\ \underline{U}_2 &= \underline{Z}_{\text{L}}\, \underline{I}_{\text{L}}\end{aligned}\right\} \quad \underline{Z}_{\text{L}} = \frac{U_2^2}{\underline{S}^*} \longrightarrow \underline{Z}_{\text{L}} = 41{,}06 + j19{,}89\ \Omega$$

Relevant Thévenin application case

Understanding the results

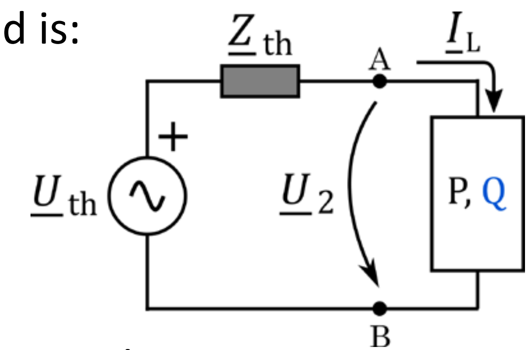
The obtained results can be depicted as



Relevant Thévenin application cases

Case 2: P and U_2 known, Q unknown

- The problem to be solved is:



- Recalling the obtained expression:

$$U_2^4 - U_2^2\left(U_{\text{th}}^2 - 2\text{Re}\left(\underline{Z}_{\text{th}}\,\underline{S}^*\right)\right) + Z_{\text{th}}^2\,S^2 = 0$$

- Expanding terms

$$U_2^4 - U_{\text{th}}^2\;U_2^2 + 2R_{\text{th}}P\;U_2^2 + 2X_{\text{th}}Q\;U_2^2 + Z_{th}^2\;(P^2 + Q^2) = 0$$

$$U_2^4 - U_{\text{th}}^2\;U_2^2 + 2R_{\text{th}}P\;U_2^2 + 2X_{\text{th}}\,Q\,U_2^2 + Z_{\text{th}}^2P^2 + Z_{\text{th}}^2\,Q^2 = 0$$

- Then, the reactive power can be obtained:

$$Q^2\,Z_{\text{th}}^2 + 2X_{\text{th}}\;U_2^2\;Q + (U_2^4 - U_{\text{th}}^2\;U_2^2 + 2R_{\text{th}}P\;U_2^2 + Z_{\text{th}}^2P^2) = 0$$