Structural Dynamics

Introduction

- To discuss the dynamics of a single-degree-of freedom spring-mass system.
- To derive the finite element equations for the time-dependent stress analysis of the one-dimensional bar, including derivation of the lumped and consistent mass matrices.
- To introduce procedures for numerical integration in time, including the central difference method, Newmark's method, and Wilson's method.
- To describe how to determine the natural frequencies of bars by the finite element method.
- To illustrate the finite element solution of a time-dependent bar problem.

Structural Dynamics

Introduction

- To develop the beam element lumped and consistent mass matrices.
- To illustrate the determination of natural frequencies for beams by the finite element method.
- To develop the mass matrices for truss, plane frame, plane stress, plane strain, axisymmetric, and solid elements.
- To report some results of structural dynamics problems solved using a computer program, including a fixed-fixed beam for natural frequencies, a bar, a fixed-fixed beam, a rigid frame, and a gantry crane-all subjected to time-dependent forcing functions.

Structural Dynamics

Introduction

This chapter provides an elementary introduction to time-dependent problems.

We will introduce the basic concepts using the single-degree-of-freedom spring-mass system.

We will include discussion of the stress analysis of the one-dimensional bar, beam, truss, and plane frame.

Structural Dynamics

Introduction

We will provide the basic equations necessary for structural dynamic analysis and develop both the lumped- and the consistent-mass matrices involved in the analyses of a bar, beam, truss, and plane frame.

We will describe the assembly of the global mass matrix for truss and plane frame analysis and then present numerical integration methods for handling the time derivative.

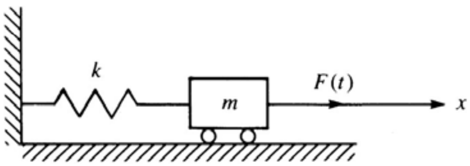
We will provide longhand solutions for the determination of the natural frequencies for bars and beams, and then illustrate the time-step integration process involved with the stress analysis of a bar subjected to a time dependent forcing function.

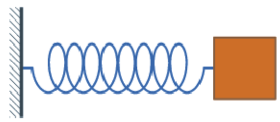
Structural Dynamics

Dynamics of a Spring-Mass System

In this section, we will discuss the motion of a single-degree-of-freedom spring-mass system as an introduction to the dynamic behavior of bars, trusses, and frames.

Consider the single-degree-of-freedom spring-mass system subjected to a time-dependent force $F(t)$ as shown in the figure below.



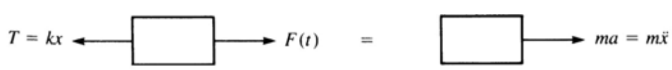


The term k is the stiffness of the spring and m is the mass of the system.

Structural Dynamics

Dynamics of a Spring-Mass System

The free-body diagram of the mass is shown below.



The spring force $T = kx$ and the applied force $F(t)$ act on the mass, and the mass-times-acceleration term is shown separately.

Applying Newton's second law of motion, $f = ma$, to the mass, we obtain the equation of motion in the x direction:

$$F(t) - kx = m\ddot{x}$$

where a dot over a variable indicates differentiation with respect to time.

Structural Dynamics

Dynamics of a Spring-Mass System

The standard form of the equation is: $m\ddot{x} + kx = F(t)$

The above equation is a second-order linear differential equation whose solution for the displacement consists of a homogeneous solution and a particular solution.

The homogeneous solution is the solution obtained when the right-hand-side is set equal to zero.

A number of useful concepts regarding vibrations are available when considering the free vibration of a mass; that is when $F(t) = 0$.

Structural Dynamics

Dynamics of a Spring-Mass System

Let's define the following term: $\omega^2 = \dfrac{k}{m}$

The equation of motion becomes: $\ddot{x} + \omega^2 x = 0$

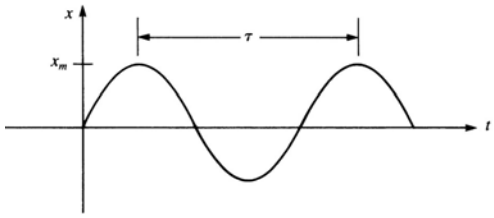
where ω is called the ***natural circular frequency*** of the free vibration of the mass (radians per second).

Note that the natural frequency depends on the spring stiffness k and the mass m of the body.

Structural Dynamics

Dynamics of a Spring-Mass System

The motion described by the homogeneous equation of motion is called ***simple harmonic motion***. A typical displacement/time curve is shown below.



where x_m denotes the maximum displacement (or ***amplitude*** of the vibration).

Structural Dynamics

Dynamics of a Spring-Mass System

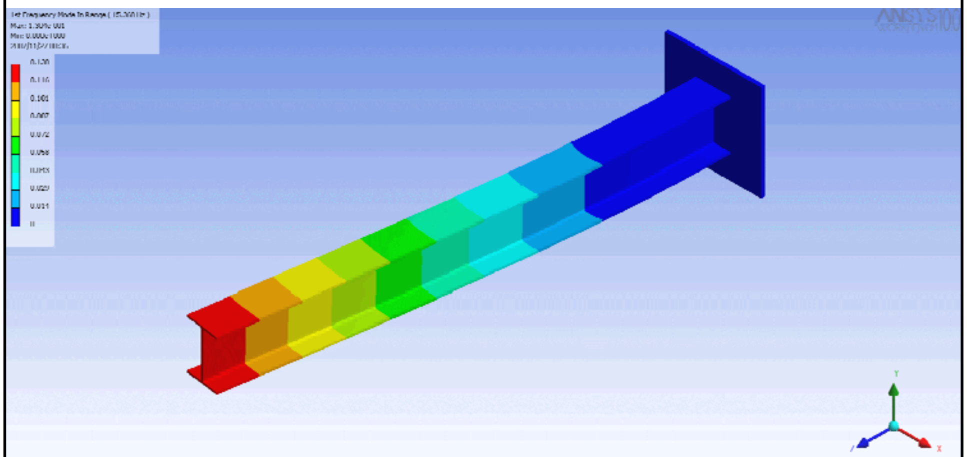
The time interval required for the mass to complete one full cycle of motion is called the ***period*** of the vibration τ (in seconds) and is defined as:

$$\tau = \frac{2\pi}{\omega}$$

The frequency in hertz (Hz = 1/s) is $f = 1/\tau = \omega/(2\pi)$.

Structural Dynamics

Dynamics of a Spring-Mass System



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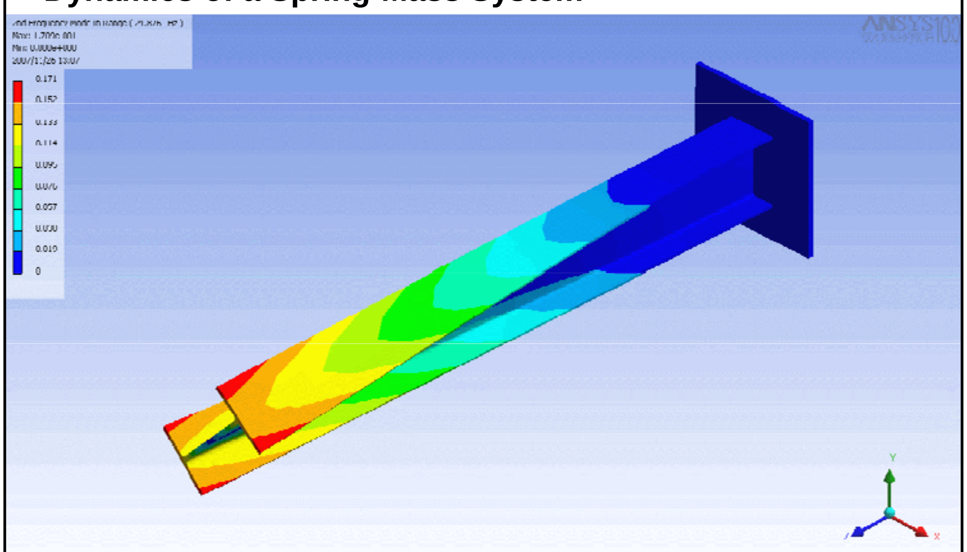
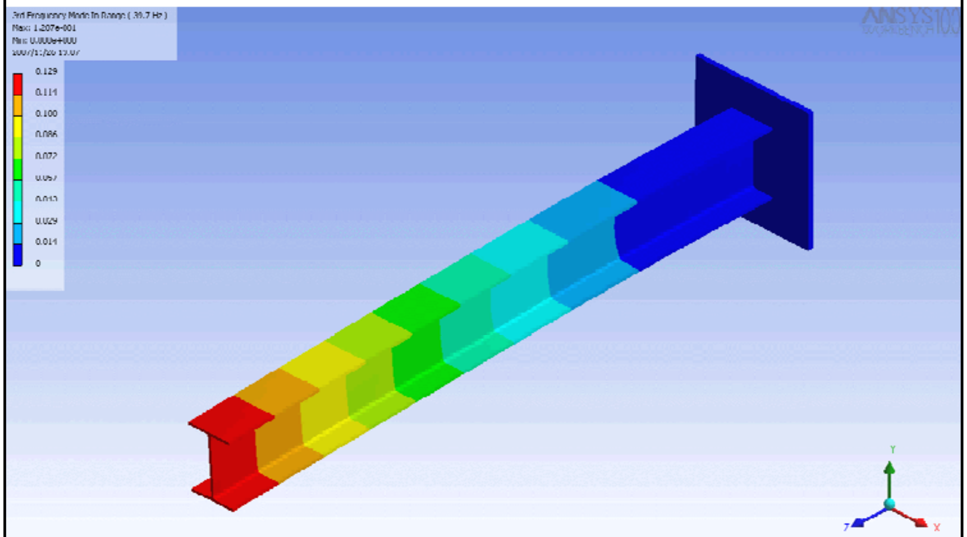
Dynamics of a Spring-Mass System



Structural Dynamics

Dynamics of a Spring-Mass System



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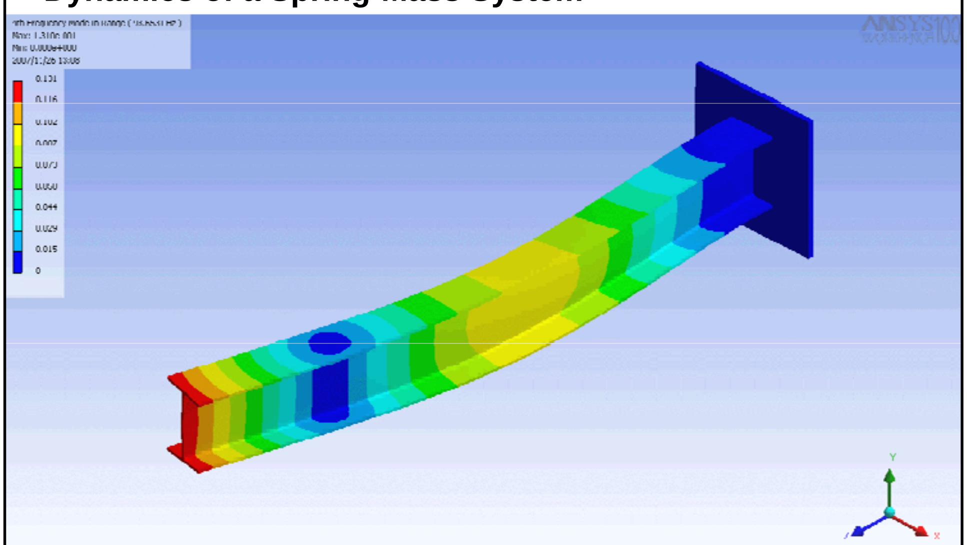
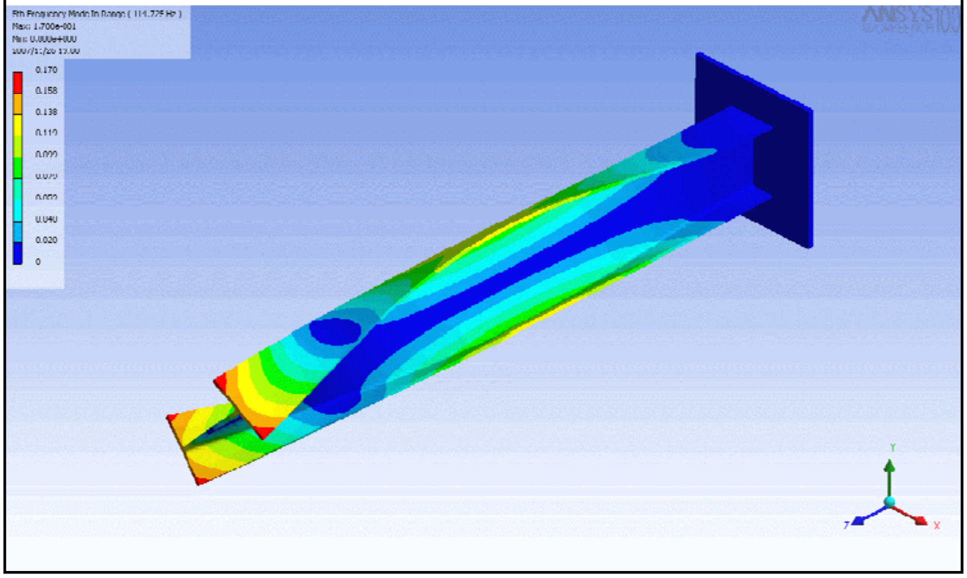
Dynamics of a Spring-Mass System



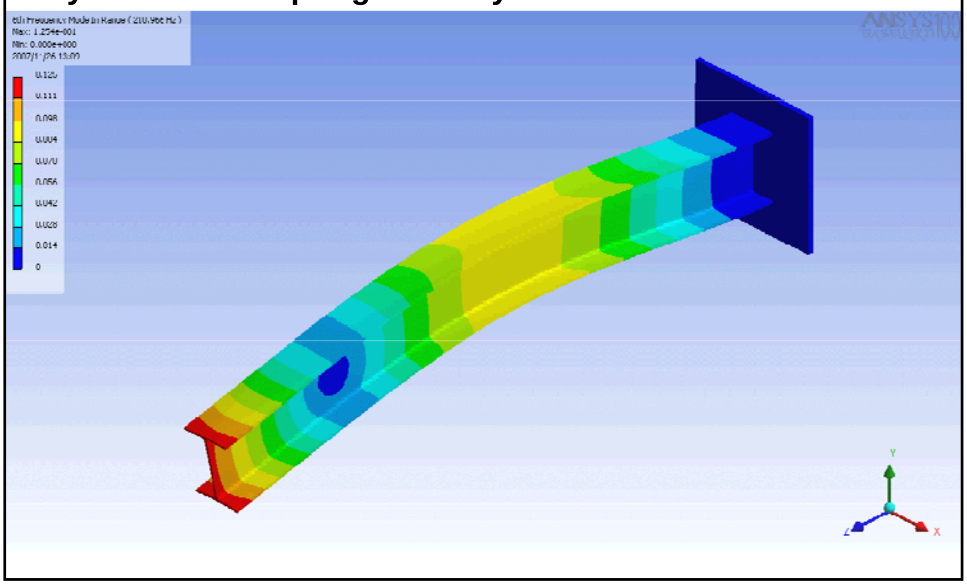
Structural Dynamics

Dynamics of a Spring-Mass System



Structural Dynamics

Dynamics of a Spring-Mass System



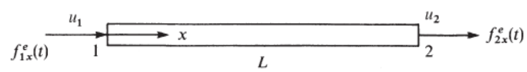
Structural Dynamics

Direct Derivation of the Bar Element

Let's derive the finite element equations for a time-dependent (dynamic) stress analysis of a one-dimensional bar.

Step 1 - Select Element Type

We will consider the linear bar element shown below.



where the bar is of length L, cross-sectional area A, and mass density ρ (with typical units of $lb\text{-}s^2/in^4$), with nodes 1 and 2 subjected to external time-dependent loads: $f_x^e(t)$

Structural Dynamics

Direct Derivation of the Bar Element

Step 2 - Select a Displacement Function

A linear displacement function is assumed in the x direction.

$$u = a_1 + a_2 x$$

The number of coefficients in the displacement function, $\boldsymbol{a_i}$, is equal to the total number of degrees of freedom associated with the element.

We can express the displacement function in terms of the shape functions:

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

Structural Dynamics

Direct Derivation of the Bar Element

Step 3 - Define the Strain/Displacement and Stress/Strain Relationships

The stress-displacement relationship is:

$$\{\varepsilon_x\} = \frac{du}{dx} = [B]\{d\}$$

where: $[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$ $\qquad \{d\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

The stress-strain relationship is given as:

$$\{\sigma_x\} = [D]\{\varepsilon_x\} = [D][B]\{d\}$$

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The bar element is typically not in equilibrium under a time-dependent force; hence, $f_{1x} \neq f_{2x}$.

We must apply Newton's second law of motion, ***f* = *ma***, to each node.

Write the law of motion as the external force f_x^e minus the internal force equal to the nodal mass times acceleration.

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Therefore:

$$f_{1x}^{\ e} = f_{1x} + m_1 \frac{\partial^2 u_1}{\partial t^2} \qquad f_{2x}^{\ e} = f_{2x} + m_2 \frac{\partial^2 u_2}{\partial t^2}$$

where:

$$m_1 = \frac{\rho AL}{2} \qquad m_2 = \frac{\rho AL}{2}$$

$$\begin{Bmatrix} f_{1x}^{\ e} \\ f_{2x}^{\ e} \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 u_1}{\partial t^2} \\ \dfrac{\partial^2 u_2}{\partial t^2} \end{Bmatrix}$$

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

If we replace {*f*} with [*k*]{*d*} we get: $\{f^e(t)\} = [k]\{d\} + [m]\{\ddot{d}\}$

Where the elemental stiffness matrix is:

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \{\ddot{d}\} = \frac{\partial^2 \{d\}}{\partial t^2}$$

and the ***lumped-mass matrix*** is:

$$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Let's derive the ***consistent-mass matrix*** for a bar element.

The typical method for deriving the consistent-mass matrix is the principle of virtual work; however, an even simpler approach is to use *D'Alembert's principle*.

The effective body force is: $\{X^e\} = -\rho\{\ddot{u}\}$

The nodal forces associated with $\{X^e\}$ are found by using the following:

$$\{f_b\} = \int_V [N]^T \{X\}\, dV$$

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting $\{X^e\}$ for $\{X\}$ gives: $\{f_b\} = -\int_V \rho [N]^T \{\ddot{u}\}\, dV$

The second derivative of the u with respect to time is:

$$\{\dot{u}\} = [N]\{\dot{d}\} \qquad \{\ddot{u}\} = [N]\{\ddot{d}\}$$

where $\dot{u}$ and $\ddot{u}$ are the nodal velocities and accelerations, respectively.

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Therefore: $\{f_b\} = -\int_V \rho [N]^T [N]\{\ddot{d}\} dV = -[m]\{\ddot{d}\}$

where: $[m] = \int_V \rho [N]^T [N] dV$

The mass matrix is called the ***consistent mass matrix*** because it is derived using the same shape functions use to obtain the stiffness matrix.

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting the shape functions in the above mass matrix equations give:

$$[m] = \int_V \rho \begin{Bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} dV$$

$$[m] = A\rho \int_0^L \begin{Bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} dx$$

Structural Dynamics

Direct Derivation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting the shape functions in the above mass matrix equations give:

$$[m] = \rho A \int_0^L \begin{bmatrix} \left(1-\frac{x}{L}\right)^2 & \left(1-\frac{x}{L}\right)\frac{x}{L} \\ \left(1-\frac{x}{L}\right)\frac{x}{L} & \left(\frac{x}{L}\right)^2 \end{bmatrix} dx$$

Evaluating the above integral gives:

$$[m] = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Structural Dynamics

Direct Derivation of the Bar Element

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The global stiffness matrix and the global force vector are assembled using the nodal force equilibrium equations, and force/deformation and compatibility equations.

$$\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$$

where

$$[K] = \sum_{e=1}^{N}\left[k^{(e)}\right] \qquad [M] = \sum_{e=1}^{N}\left[m^{(e)}\right] \qquad \{F\} = \sum_{e=1}^{N}\left\{f^{(e)}\right\}$$

Structural Dynamics

Numerical Integration in Time

We now introduce procedures for the discretization of the equations of motion with respect to time.

These procedures will allow the nodal displacements to be determined at different time increments for a given dynamic system.

The general method used is called ***direct integration***. There are two classifications of direct integration: *explicit* and *implicit*.

We will formulate the equations for two direct integration methods.

Structural Dynamics

Numerical Integration in Time

The first, and simplest, is an explicit method known as the ***central difference method***.

The second more complicated but more versatile than the central difference method, is an implicit method known as the ***Newmark-Beta (or Newmark's***) method.

The versatility of Newmark's method is evidenced by its adaptation in many commercially available computer programs.

Structural Dynamics

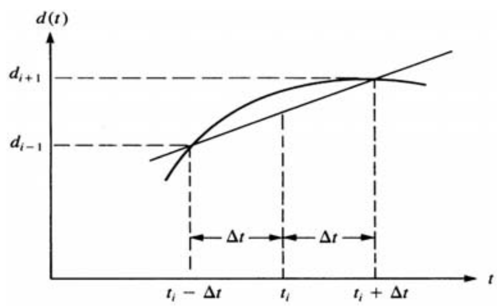
Central Difference Method

The central difference method is based on finite difference expressions for the derivatives in the equation of motion.

For example, consider the velocity and the acceleration at time t:

$$\{\dot{d}_i\} = \frac{\{d_{i+1}\} - \{d_{i-1}\}}{2(\Delta t)}$$

$$\{\ddot{d}_i\} = \frac{\{\dot{d}_{i+1}\} - \{\dot{d}_{i-1}\}}{2(\Delta t)}$$



where the subscripts indicate the time step for a given time increment of Δt.

Structural Dynamics

Central Difference Method

The acceleration can be expressed in terms of the displacements (using a Taylor series expansion) as:

$$\{\ddot{d}_i\} = \frac{\{d_{i+1}\} - 2\{d_i\} + \{d_{i-1}\}}{(\Delta t)^2}$$

We generally want to evaluate the nodal displacements; therefore, we rewrite the above equation as:

$$\{d_{i+1}\} = 2\{d_i\} - \{d_{i-1}\} + \{\ddot{d}_i\}(\Delta t)^2$$

The acceleration can be expressed as:

$$\{\ddot{d}_i\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_i\} - [\mathrm{K}]\{d_i\}\right)$$

Structural Dynamics

Central Difference Method

To develop an expression of d_{i+1}, first multiply the nodal displacement equation by ***M*** and substitute the above equation for $\{\ddot{d}_i\}$ into this equation.

$$[\mathbf{M}]\{d_{i+1}\} = 2[\mathbf{M}]\{d_i\} - [\mathbf{M}]\{d_{i-1}\} + \left(\{\mathbf{F}_i\} - [\mathbf{K}]\{d_i\}\right)(\Delta t)^2$$

Combining terms in the above equations gives:

$$[\mathbf{M}]\{d_{i+1}\} = (\Delta t)^2\{\mathbf{F}_i\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_i\} - [\mathbf{M}]\{d_{i-1}\}$$

To start the computation to determine $\{d_{i+1}\}$, $\{\dot{d}_{i+1}\}$, and $\{\ddot{d}_{i+1}\}$ we need the displacement at time step *i* -1.

Structural Dynamics

Central Difference Method

Using the central difference equations for the velocity and acceleration and solving for $\{d_{i-1}\}$:

$$\{d_{i-1}\} = \{d_i\} - (\Delta t)\{\dot{d}_i\} + \{\ddot{d}_i\}\frac{(\Delta t)^2}{2}$$

Procedure for solution:

1. Given: $\{d_0\}$, $\{\dot{d}_0\}$, and $\{F_i(t)\}$

2. If the acceleration is not given, solve for $\{\ddot{d}_0\}$

$$\{\ddot{d}_0\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_0\} - [\mathrm{K}]\{d_0\}\right)$$

Structural Dynamics

Central Difference Method

Procedure for solution:

3. Solve for $\{d_{-1}\}$ at $t = -\Delta t$

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \{\ddot{d}_0\}\frac{(\Delta t)^2}{2}$$

4. Solve for $\{d_1\}$ at $t = \Delta t$ using the value of $\{d_{-1}\}$ from Step 3

$$[\mathbf{M}]\{d_{i+1}\} = (\Delta t)^2\{\mathbf{F}_i\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_i\} - [\mathbf{M}]\{d_{i-1}\}$$

$$\{d_1\} = [\mathbf{M}]^{-1}\left\{(\Delta t)^2\{\mathbf{F}_0\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_0\} - [\mathbf{M}]\{d_{-1}\}\right\}$$

Structural Dynamics

Central Difference Method

Procedure for solution:

5. With $\{d_0\}$ given and $\{d_1\}$ determined in Step 4 solve for $\{d_2\}$

$$\{d_2\} = [\mathbf{M}]^{-1}\left\{(\Delta t)^2\{\mathbf{F}_1\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_1\} - [\mathbf{M}]\{d_0\}\right\}$$

6. Solve for $\{\ddot{d}_1\}$: $\quad \{\ddot{d}_1\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_1\} - [\mathrm{K}]\{d_1\}\right)$

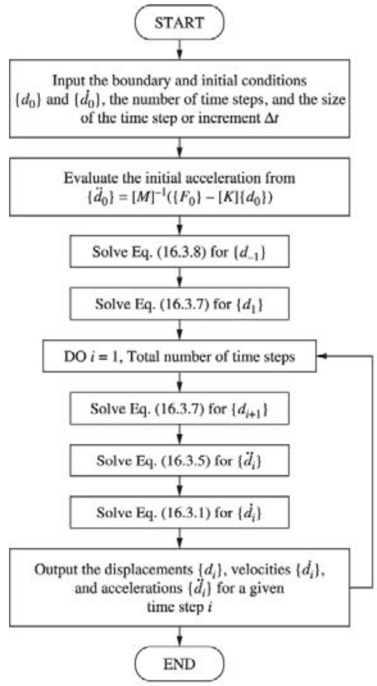
7. Solve for $\{\dot{d}_1\}$: $\quad \{\dot{d}_1\} = \dfrac{\{d_2\} - \{d_0\}}{2(\Delta t)}$

8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

Structural Dynamics

Central Difference Method

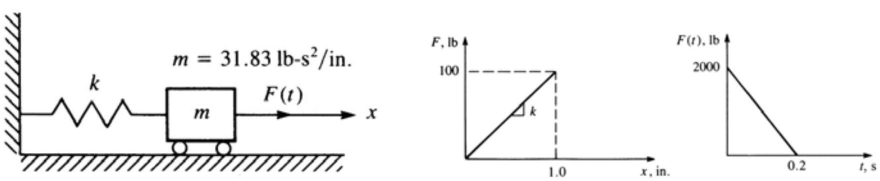
Procedure for solution:



Structural Dynamics

Central Difference Method – Example Problem

Determine the displacement, acceleration, and velocity at 0.05 second time intervals for up to 0.2 seconds for the one-dimensional spring-mass system shown in the figure below.



Consider the above spring-mass system as a single degree of freedom problem represented by the displacement *d*.

Structural Dynamics

Central Difference Method – Example Problem

Procedure for solution:

1. At time t = 0: $\{d_0\} = 0 \quad \{\dot{d}_0\} = 0$

2. If the acceleration is not given, solve for $\{\ddot{d}_0\}$

$$\{\ddot{d}_0\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_0\} - [\mathrm{K}]\{d_0\}\right)$$

$$\{\ddot{d}_0\} = \frac{2{,}000 - 100(0)}{31.83} = 62.83\ {}^{in}\!/\!_{s^2}$$

Structural Dynamics

Central Difference Method – Example Problem

Procedure for solution:

3. Solve for $\{d_{-1}\}$ at $t = -\Delta t$

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \{\ddot{d}_0\}\frac{(\Delta t)^2}{2}$$

$$\{d_{-1}\} = 0 - (0.05)0 + \frac{(0.05)^2}{2}(62.83) = 0.0785\ in$$

4. Solve for $\{d_1\}$ at t = 0.05 s using the value of $\{d_{-1}\}$ from Step 3

$$\{d_1\} = [\mathbf{M}]^{-1}\left\{(\Delta t)^2\{\mathbf{F}_0\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_0\} - [\mathbf{M}]\{d_{-1}\}\right\}$$

$$\{d_1\} = \frac{1}{31.82}\left\{(0.05)^2\,2{,}000 + \left[2(31.83) - (0.05)^2(100)\right]0 - (31.83)(0.0785)\right\}$$

$$= 0.0785\ in$$

Structural Dynamics

Central Difference Method – Example Problem

Procedure for solution:

5. With $\{d_0\}$ given and $\{d_1\}$ determined in Step 4 solve for $\{d_2\}$

$$\{d_2\} = [\mathbf{M}]^{-1}\left\{(\Delta t)^2\{\mathbf{F}_1\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_1\} - [\mathbf{M}]\{d_0\}\right\}$$

$$\{d_2\} = \frac{1}{31.82}\left\{(0.05)^2 1{,}500 + \left[2(31.83) - (0.05)^2(100)\right](0.0785) - (31.83)(0)\right\}$$

$$= 0.274\ in$$

6. Solve for $\{\ddot{d}_1\}$: $\{\ddot{d}_1\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_1\} - [\mathrm{K}]\{d_1\}\right)$

$$\{\ddot{d}_1\} = \frac{1}{31.83}\left[1{,}500 - 100(0.0785)\right] = 46.88\ {in}/{s^2}$$

Structural Dynamics

Central Difference Method – Example Problem

Procedure for solution:

7. Solve for $\{\dot{d}_1\}$: $\{\dot{d}_1\} = \dfrac{\{d_2\} - \{d_0\}}{2(\Delta t)}$

$$\{\dot{d}_1\} = \frac{0.274 - 0}{2(0.05)} = 2.74\ {in}/{s}$$

8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

Structural Dynamics

Central Difference Method – Example Problem

Procedure for solution:

5. With $\{d_1\}$ given and $\{d_2\}$ determined in Step 4 solve for $\{d_3\}$

$$\{d_3\} = [\mathbf{M}]^{-1}\left\{(\Delta t)^2\{\mathbf{F}_2\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_2\} - [\mathbf{M}]\{d_1\}\right\}$$

$$\{d_3\} = \frac{1}{31.82}\left\{(0.05)^2 1{,}000 + \left[2(31.83) - (0.05)^2(100)\right](0.274) - (31.83)(0.0785)\right\}$$

$$= 0.546\ in$$

6. Solve for $\{\ddot{d}_2\}$: $\{\ddot{d}_2\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_2\} - [\mathrm{K}]\{d_2\}\right)$

$$\{\ddot{d}_2\} = \frac{1}{31.83}\left[1{,}000 - 100(0.274)\right] = 30.56\ in/s^2$$

Structural Dynamics

Central Difference Method – Example Problem

Procedure for solution:

7. Solve for $\{\dot{d}_2\}$: $\{\dot{d}_2\} = \dfrac{\{d_3\} - \{d_1\}}{2(\Delta t)}$

$$\{\dot{d}_2\} = \frac{0.546 - 0.0785}{2(0.05)} = 4.68\ in/s$$

8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

Structural Dynamics

Central Difference Method – Example Problem

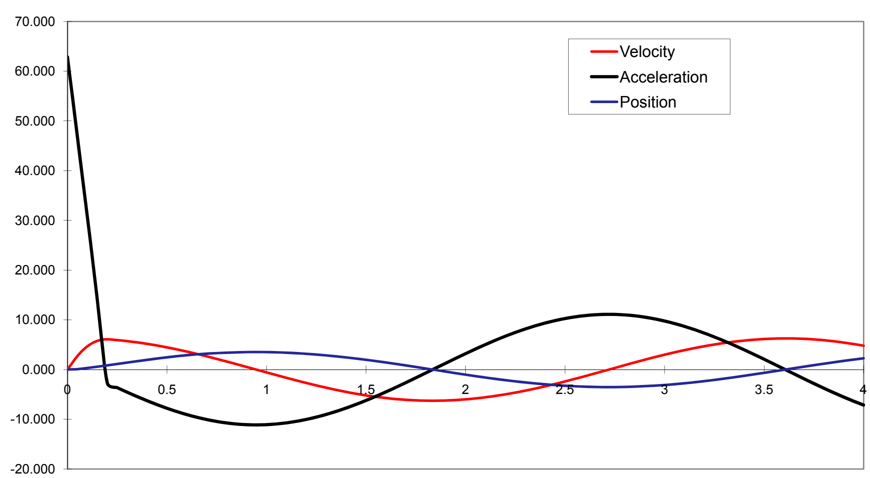
The following table summarizes the results for the remaining time steps as compared with the exact solution.

t (s)	$F(t)$ (lb)	$\ddot{d}_i$ (in/s^2)	$\dot{d}_i$ (in/s)	d_i (in)	d_i (exact)
0.00	2,000	62.83	0.00	0.000	0.0000
0.05	1,500	46.88	2.74	0.0785	0.0718
0.10	1,000	30.56	4.68	0.274	0.2603
0.15	500	13.99	5.79	0.546	0.5252
0.20	0	-2.68	6.07	0.854	0.8250
0.25	0	-3.63	5.91	1.154	1.132

Structural Dynamics

Central Difference Method – Example Problem

Plotting the motion for about 4 s gives:



Structural Dynamics

Newmark's Method

Newmark's equations are given as:

$$\left\{\dot{d}_{i+1}\right\} = \left\{\dot{d}_i\right\} + (\Delta t)\left[(1-\gamma)\left\{\ddot{d}_i\right\} + \gamma\left\{\ddot{d}_{i+1}\right\}\right]$$

$$\left\{d_{i+1}\right\} = \left\{d_i\right\} + (\Delta t)\left\{\dot{d}_i\right\} + (\Delta t)^2\left[\left(\tfrac{1}{2}-\beta\right)\left\{\ddot{d}_i\right\} + \beta\left\{\ddot{d}_{i+1}\right\}\right]$$

where β and γ are parameters.

The parameter β is typically between 0 and ¼, and γ is often taken to be ½.

For example, if β = 0 and γ = ½ the above equation reduce to the central difference method.

Structural Dynamics

Newmark's Method

To find $\{d_{i+1}\}$ first multiply the above equation by the mass matrix [M] and substitute the result into this the expression for acceleration. Recall the acceleration is:

$$\left\{\ddot{d}_i\right\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_i\} - [\mathrm{K}]\{d_i\}\right)$$

The expression $[M]\{d_{i+1}\}$ is:

$$[\mathbf{M}]\{d_{i+1}\} = [\mathbf{M}]\{d_i\} + (\Delta t)[\mathbf{M}]\left\{\dot{d}_i\right\} + (\Delta t)^2[\mathbf{M}]\left(\tfrac{1}{2}-\beta\right)\left\{\ddot{d}_i\right\}$$
$$+\beta(\Delta t)^2\left[\{\mathbf{F}_{i+1}\} - [\mathbf{K}]\{d_{i+1}\}\right]$$

Combining terms gives:

$$\left([\mathbf{M}] + \beta(\Delta t)^2[\mathbf{K}]\right)\{d_{i+1}\} = \beta(\Delta t)^2\{\mathbf{F}_{i+1}\} + [\mathbf{M}]\{d_i\}$$
$$+(\Delta t)[\mathbf{M}]\left\{\dot{d}_i\right\} + (\Delta t)^2[\mathbf{M}]\left(\tfrac{1}{2}-\beta\right)\left\{\ddot{d}_i\right\}$$

Structural Dynamics

Newmark's Method

Dividing the above equation by $\beta(\Delta t)^2$ gives: $[K']\{d_{i+1}\} = \{F'_{i+1}\}$

where:

$$[\mathbf{K}'] = [\mathbf{K}] + \frac{1}{\beta(\Delta t)^2}[\mathbf{M}]$$

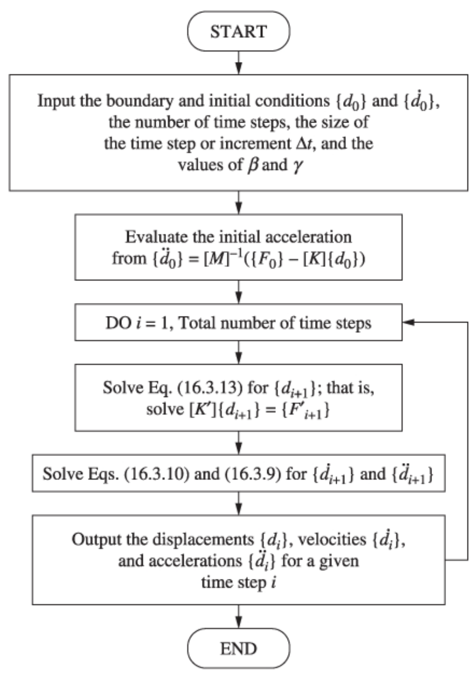
$$\{\mathbf{F}'_{i+1}\} = \{\mathbf{F}_{i+1}\} + \frac{[\mathbf{M}]}{\beta(\Delta t)^2}\left[\{d_i\} + (\Delta t)\{\dot{d}_i\} + \left(\tfrac{1}{2} - \beta\right)(\Delta t)^2\{\ddot{d}_i\}\right]$$

The advantages of using Newmark's method over the central difference method are that Newmark's method can be made unconditionally stable (if β = ¼ and γ = ½) and that larger time steps can be used with better results.

Structural Dynamics

Newmark's Method

Procedure for solution:



Structural Dynamics

Newmark's Method

Procedure for solution:

1. Given: $\{d_0\}, \{\dot{d}_0\}$, and $\{F_i(t)\}$

2. If the acceleration is not given, solve for $\{\ddot{d}_0\}$

$$\{\ddot{d}_0\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_0\} - [\mathrm{K}]\{d_0\}\right)$$

3. Solve for $\{d_1\}$ at $t = 0$

$$[\mathrm{K}']\{d_1\} = \{\mathrm{F}'_1\}$$

Structural Dynamics

Newmark's Method

Procedure for solution:

4. Solve for $\{\ddot{d}_1\}$ (original Newmark equation for $\{d_{i+1}\}$ rewritten for $\{\ddot{d}_{i+1}\}$):

$$\{\ddot{d}_1\} = \frac{1}{\beta(\Delta t)^2}\left[\{d_1\} - \{d_0\} - (\Delta t)\{\dot{d}_0\} - (\Delta t)^2\left(\tfrac{1}{2} - \beta\right)\{\ddot{d}_0\}\right]$$

5. Solve for $\{\dot{d}_1\}$

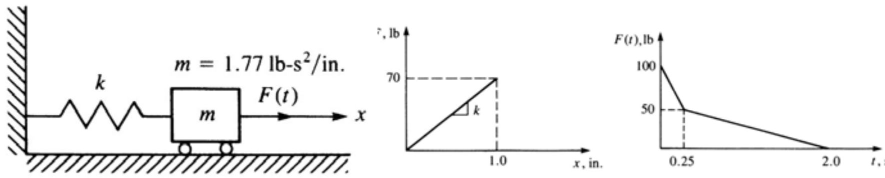
$$\{\dot{d}_1\} = \{\dot{d}_0\} + (\Delta t)\left[(1-\gamma)\{\ddot{d}_0\} + \gamma\{\ddot{d}_1\}\right]$$

6. Repeat Steps 3, 4, and 5 to obtain the displacement, acceleration, and velocity for the next time step.

Structural Dynamics

Newmark's Method – Example Problem

Determine the displacement, acceleration, and velocity at 0.1 second time intervals for up to 0.5 seconds for the one-dimensional spring-mass system shown in the figure below.



Consider the above spring-mass system as a single degree of freedom problem represented by the displacement *d*.

Use Newmark's method with β = 1/6 and γ = ½.

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

1. Given: $\{d_0\}, \{\dot{d}_0\}$, and $\{F_i(t)\}$

2. If the acceleration is not given, solve for $\{\ddot{d}_0\}$

$$\{\ddot{d}_0\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_0\} - [\mathrm{K}]\{d_0\}\right)$$

$$= \frac{1}{1.77}\left[100 - 70(0)\right] = 56.5\ in/s^2$$

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

3. Solve for $\{d_1\}$ at t = 0.1 s $\quad [K']\{d_1\} = \{F'_1\}$

$$[\mathbf{K}'] = [\mathbf{K}] + \frac{1}{\beta(\Delta t)^2}[\mathbf{M}] = 70 + \frac{1}{\frac{1}{6}(0.1)^2}(1.77) = 1{,}132\ lb/in$$

$$\{\mathbf{F}'_1\} = \{\mathbf{F}_1\} + \frac{[\mathbf{M}]}{\beta(\Delta t)^2}\left[\{d_0\} + (\Delta t)\{\dot{d}_0\} + \left(\tfrac{1}{2} - \beta\right)(\Delta t)^2\{\ddot{d}_0\}\right]$$

$$= 80 + \frac{1.77}{\frac{1}{6}(0.1)^2}\left[0 + (0.1)0 + \left(\tfrac{1}{2} - \tfrac{1}{6}\right)(0.1)^2(56.5)\right] = 280\ lb$$

$$\{d_1\} = \frac{\{\mathbf{F}'_1\}}{[\mathbf{K}']} = \frac{280\ lb}{1{,}132\ {}^{lb}\!/_{in}} = 0.248\ in$$

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

4. Solve for $\{\ddot{d}_1\}$ at t = 0.1 s

$$\{\ddot{d}_1\} = \frac{1}{\beta(\Delta t)^2}\left[\{d_1\} - \{d_0\} - (\Delta t)\{\dot{d}_0\} - (\Delta t)^2\left(\tfrac{1}{2} - \beta\right)\{\ddot{d}_0\}\right]$$

$$= \frac{1}{\frac{1}{6}(0.1)^2}\left[0.248 - 0 - (0.1)0 - (0.1)^2\left(\tfrac{1}{2} - \tfrac{1}{6}\right)56.5\right] = 35.4\ {}^{in}\!/_{s^2}$$

5. Solve for $\{\dot{d}_1\}$

$$\{\dot{d}_1\} = \{\dot{d}_0\} + (\Delta t)\left[(1-\gamma)\{\ddot{d}_0\} + \gamma\{\ddot{d}_1\}\right]$$

$$= 0 + (0.1)\left[(1-\tfrac{1}{2})56.5 + \tfrac{1}{2}(35.4)\right] = 4.59\ {}^{in}\!/_{s}$$

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

6. Repeat Steps 3, 4, and 5 to obtain the displacement, acceleration, and velocity for the next time step.

Repeating Steps 3, Solve for {d_1} at t = 0.2 s

$$\{\mathbf{F}'_2\} = \{\mathbf{F}_2\} + \frac{[\mathbf{M}]}{\beta(\Delta t)^2}\left[\{d_1\} + (\Delta t)\{\dot{d}_1\} + \left(\tfrac{1}{2} - \beta\right)(\Delta t)^2\{\ddot{d}_1\}\right]$$

$$= 60 + \frac{1.77}{\frac{1}{6}(0.1)^2}\left[0.248 + (0.1)4.59 + \left(\tfrac{1}{2} - \tfrac{1}{6}\right)(0.1)^2(35.4)\right] = 934\ lb$$

$$\{d_2\} = \frac{\{\mathbf{F}'_2\}}{[\mathbf{K}']} = \frac{934}{1{,}132} = 0.825\ in$$

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

Repeating Step 4: solve for $\{\ddot{d}_1\}$ at t = 0.2 s

$$\{\ddot{d}_2\} = \frac{1}{\beta(\Delta t)^2}\left[\{d_2\} - \{d_1\} - (\Delta t)\{\dot{d}_1\} - (\Delta t)^2\left(\tfrac{1}{2} - \beta\right)\{\ddot{d}_1\}\right]$$

$$= \frac{1}{\frac{1}{6}(0.1)^2}\left[0.825 - 0.248 - (0.1)4.59 - (0.1)^2\left(\tfrac{1}{2} - \tfrac{1}{6}\right)35.4\right]$$

$$= 1.27\ {}^{in}\!/\!_{s^2}$$

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

5. Solve for $\{\dot{d}_2\}$

$$\{\dot{d}_2\} = \{\dot{d}_1\} + (\Delta t)\left[(1-\gamma)\{\ddot{d}_1\} + \gamma\{\ddot{d}_2\}\right]$$

$$= 4.59 + (0.1)\left[(1-\tfrac{1}{2})35.4 + \tfrac{1}{2}(1.27)\right] = 6.42 \ {}^{in}\!/\!_{s}$$

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

The following table summarizes the results for the time steps through t = 0.5 seconds.

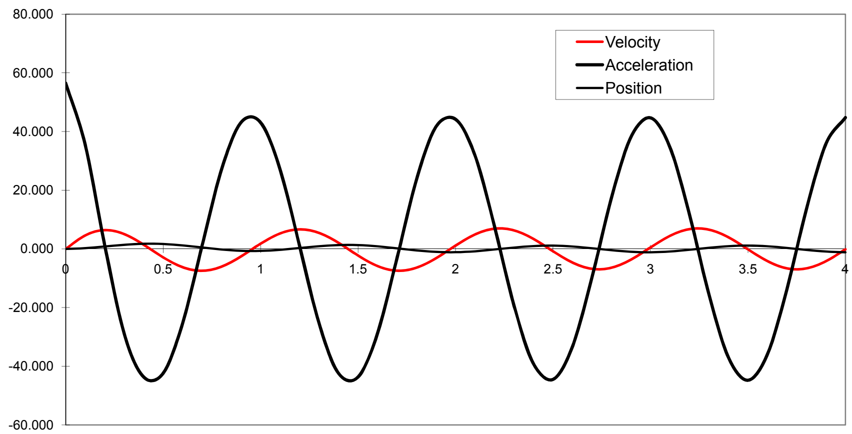
t (s)	$\boldsymbol{F}(t)$ lb	(in/s^2)	(in/s)	d_i (in)
0.0	100	56.6	0.0	0.0
0.1	80	35.4	4.59	0.248
0.2	60	1.27	6.42	0.825
0.3	48.6	-26.2	5.17	1.36
0.4	45.7	-42.2	1.75	1.72
0.5	42.9	-42.2	-2.45	1.68

Structural Dynamics

Newmark's Method – Example Problem

Procedure for solution:

Plotting the motion for about 4 *s* gives:



Structural Dynamics

Natural Frequencies of a One-Dimensional Bar

Before solving the structural stress dynamic analysis problem, let's consider how to determine the natural frequencies of continuous elements.

Natural frequencies are necessary in vibration analysis and important when choosing a proper time step for a structural dynamics analysis.

Natural frequencies are obtained by solving the following equation:

$$[\mathbf{M}]\{\ddot{d}\} + [\mathbf{K}]\{d\} = 0$$

Structural Dynamics

Natural Frequencies of a One-Dimensional Bar

The standard solution for {*d*} is given as: $\{d(t)\} = \{d'\}e^{i\omega t}$

where {*d* ′} is the part of the nodal displacement matrix called ***natural modes*** that is assumed to independent of time, *i* is the standard imaginary number, and ω is a natural frequency.

Differentiating the above equation twice with respect to time gives:

$$\{\ddot{d}\} = \{d'\}\left(-\omega^2\right)e^{i\omega t}$$

Substituting the above expressions for {*d*} and $\{\ddot{d}\}$ into the equation of motion gives:

$$-[\mathbf{M}]\omega^2\{d'\}e^{i\omega t} + [\mathbf{K}]\{d'\}e^{i\omega t} = 0$$

Structural Dynamics

Natural Frequencies of a One-Dimensional Bar

Combining terms gives: $e^{i\omega t}\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right)\{d'\} = 0$

Since $e^{i\omega t}$ is not zero, then: $\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right) = 0$

The above equations are a set of linear homogeneous equations in terms of displacement mode {*d* ′}.

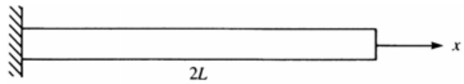
There exists a non-trivial solution if and only if the determinant of the coefficient matrix of is zero.

$$\left|[\mathbf{K}] - \omega^2[\mathbf{M}]\right| = 0$$

Structural Dynamics

One-Dimensional Bar - Example Problem

Determine the first two natural frequencies for the bar shown in the figure below.

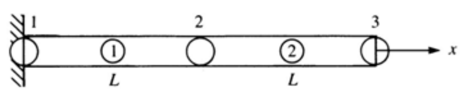


Assume the bar has a length *2L*, modulus of elasticity *E*, mass density ρ, and cross-sectional area *A*.

Structural Dynamics

One-Dimensional Bar - Example Problem

Let's discretize the bar into two elements each of length *L* as shown below.



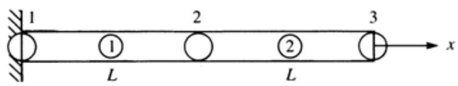
We need to develop the stiffness matrix and the mass matrix (either the lumped- mass of the consistent-mass matrix).

In general, the consistent-mass matrix has resulted in solutions that compare more closely to available analytical and experimental results than those found using the lumped-mass matrix.

Structural Dynamics

One-Dimensional Bar - Example Problem

Let’s discretize the bar into two elements each of length *L* as shown below.

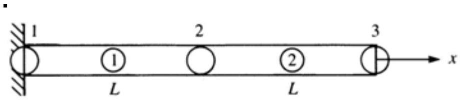


However, when performing a long hand solution, the consistent-mass matrix is more difficult and tedious to compute; therefore, we will use the lumped-mass matrix in this example.

Structural Dynamics

One-Dimensional Bar - Example Problem

Let’s discretize the bar into two elements each of length *L* as shown below.



The elemental stiffness matrices are:

$$\left[k_{(1)}\right] = \frac{AE}{L}\begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad \left[k_{(2)}\right] = \frac{AE}{L}\begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

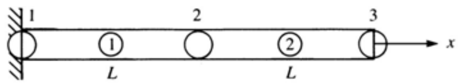
The global stiffness matrix is:

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Structural Dynamics

One-Dimensional Bar - Example Problem

Let's discretize the bar into two elements each of length *L* as shown below.



The lumped-mass matrices are:

$$\left[m_{(1)}\right] = \frac{\rho A L}{2}\begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \left[m_{(2)}\right] = \frac{\rho A L}{2}\begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

The global lumped-mass matrix is:

$$[M] = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Structural Dynamics

One-Dimensional Bar - Example Problem

Substituting the above stiffness and lumped-mass matrices into the natural frequency equation:

$$\left([\mathbf{K}] - \omega^2[\mathbf{M}]\right)\{d'\} = 0$$

and applying the boundary condition $\{u_1\} = 0$ (or $\{d'_1\} = 0$) gives:

$$\left(\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} d'_2 \\ d'_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Set the determinant of the coefficient matrix equal to zero as:

$$\left|\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad \text{where } \lambda = \omega^2$$

Structural Dynamics

One-Dimensional Bar - Example Problem

Dividing the above equation by ρAL and letting $\mu = E/\rho L^2$ gives:

$$\begin{vmatrix} 2\mu - \lambda & -\mu \\ -\mu & \mu - \dfrac{\lambda}{2} \end{vmatrix} = 0$$

Evaluating the determinant of the above equations gives:

$$\lambda = 2\mu \pm \mu\sqrt{2} \qquad \lambda_1 = 0.60\mu \qquad \lambda_2 = 3.41\mu$$

For comparison, the exact solution gives λ = 0.616 *m*, whereas the consistent-mass approach yields λ = 0.648 *m*.

Structural Dynamics

One-Dimensional Bar - Example Problem

Therefore, for bar elements, the lumped-mass approach can yield results as good as, or even better than, the results from the consistent-mass approach.

However, the consistent-mass approach can be mathematically proven to yield an upper bound on the frequencies, whereas the lumped-mass approach has no mathematical proof of boundedness.

The first and second natural frequencies are given as:

$$\omega_1 = \sqrt{\lambda_1} = 0.77\sqrt{\mu} \qquad \omega_2 = \sqrt{\lambda_2} = 1.85\sqrt{\mu}$$

Structural Dynamics

One-Dimensional Bar - Example Problem

The term μ may be computed as:

$$\mu = \frac{E}{\rho L^2} = \frac{30 \times 10^6}{(0.00073)(100)^2} = 4.12 \times 10^6 \ s^{-2}$$

Therefore, first and second natural frequencies are:

$$\omega_1 = 1.56 \times 10^3 \ rad/s \qquad \omega_2 = 3.76 \times 10^3 \ rad/s$$

Structural Dynamics

One-Dimensional Bar - Example Problem

In general, an *n*-degree-of-freedom discrete system has *n* natural modes and frequencies.

A continuous system actually has an infinite number of natural modes and frequencies.

The lowest modes and frequencies are approximated most often; the higher frequencies are damped out more rapidly and are usually less important.

Structural Dynamics

One-Dimensional Bar - Example Problem

Substituting λ_1 into the following equation

$$\left(\frac{AE}{L}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} - \omega^2\frac{\rho AL}{2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}d'_2\\ d'_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Gives: $1.4\mu d'^{(1)}_2 - \mu d'^{(1)}_3 = 0 \qquad -\mu d'^{(1)}_2 + 0.7\mu d'^{(1)}_3 = 0$

where the superscripts indicate the natural frequency.

It is customary to specify the value of one of the natural modes $\{d'\}$ for a given μ_i or ω_i and solve for the remaining values.

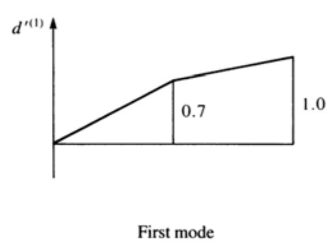
For example, if $\left\{d'^{(1)}_3\right\} = 1$ than the solution for $\left\{d'^{(1)}_2\right\} = 0.7$

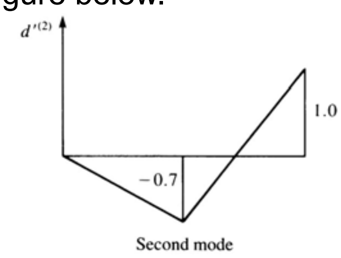
Structural Dynamics

One-Dimensional Bar - Example Problem

Similarly, if we substitute λ_2 and let $\left\{d'^{(2)}_3\right\} = 1$ the solution of the above equations gives $\left\{d'^{(2)}_2\right\} = -0.7$

The modal responses for the first and second natural frequencies are shown in the figure below.





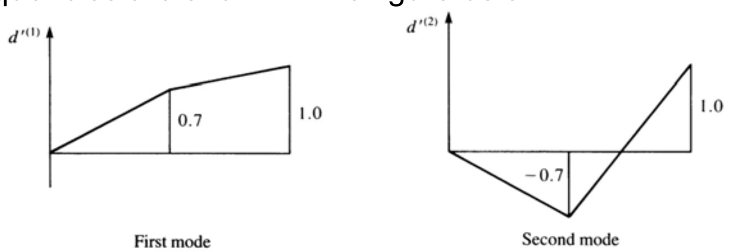
The first mode means that the bar is completely in tension or compression, depending on the excitation direction.

Structural Dynamics

One-Dimensional Bar - Example Problem

Similarly, if we substitute λ_2 and let $\left\{d'^{(2)}_3\right\} = 1$ the solution of the above equations gives $\left\{d'^{(2)}_2\right\} = -0.7$

The modal responses for the first and second natural frequencies are shown in the figure below.

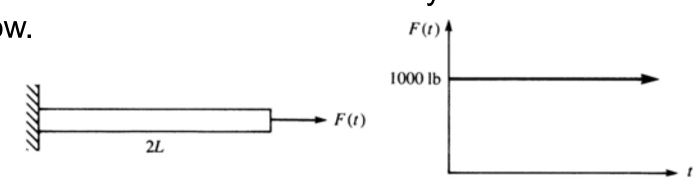


The second mode means that bar is in compression and tension or in tension and compression.

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

Consider the one-dimensional bar system shown in the figure below.



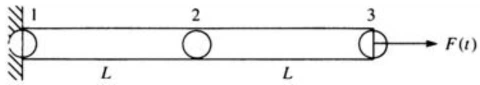
Assume the boundary condition $\{d_{1x}\} = 0$ and the initial conditions $\{d_0\} = 0$ and $\left\{\dot{d}_0\right\} = 0$

Let ρ = 0.00073 *lb-s*^{2}/*in*^{4}, A = 1 *in*^{2}, E = 30 x 10^6 *psi*, and L = 100 *in*.

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

The bar will be discretized into two elements as shown below.



The elemental stiffness matrices are:

$$\left[k_{(1)}\right] = \frac{AE}{L}\begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad \left[k_{(2)}\right] = \frac{AE}{L}\begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

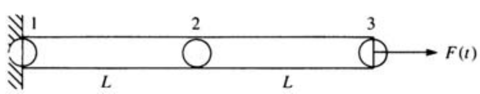
The global stiffness matrix is:

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

The bar will be discretized into two elements as shown below.



The lumped-mass matrices are:

$$\left[m_{(1)}\right] = \frac{\rho AL}{2}\begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \left[m_{(2)}\right] = \frac{\rho AL}{2}\begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

The global lumped-mass matrix is:

$$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

Substitute the global stiffness and mass matrices into the global dynamic equations gives:

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ F_3(t) \end{Bmatrix}$$

where R_1 denotes the unknown reaction at node 1.

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

For this example, we will use the central difference method, because it is easier to apply, for the numerical time integration.

It has been mathematically shown that the time step Δt must be less than or equal to two divided by the highest natural frequency.

$$\Delta t \leq \frac{2}{\omega_{max}}$$

For practical results, we should use a time step defined by:

$$\Delta t \leq \frac{3}{4}\left(\frac{2}{\omega_{max}}\right)$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

An alternative guide (used only for a bar) for choosing the approximate time step is:

$$\Delta t = \frac{L}{c_x}$$

where L is the element length, and $c_x = \sqrt{E_x / \rho}$ is the ***longitudinal wave velocity***.

Evaluating the time step estimates gives:

$$\Delta t = \frac{3}{4}\left(\frac{2}{\omega_{max}}\right) = \frac{1.5}{3.76 \times 10^3} = 0.40 \times 10^{-3} s$$

$$\Delta t = \frac{L}{c_x} = \frac{100}{\sqrt{30 \times 10^6 / 0.00073}} = 0.48 \times 10^{-3} s$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

Guided by these estimates for time step, we will select Δt = 0.25 x 10^{-3} s.

Procedure for solution:

1. At time t = 0: $\{d_0\} = 0 \qquad \{\dot{d}_0\} = 0$

2. If the acceleration is not given, solve for $\{\ddot{d}_0\}$

$$\{\ddot{d}_0\} = [\mathrm{M}]^{-1}\left(\{\mathrm{F}_0\} - [\mathrm{K}]\{d_0\}\right)$$

$$\{\ddot{d}_0\} = \begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_{t=0} = \frac{2}{\rho AL}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{Bmatrix} 0 \\ 1{,}000 \end{Bmatrix} - \frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right)$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

Applying the boundary conditions $u_1 = 0$ and $\ddot{u}_1 = 0$ and simplifying gives:

$$\{\ddot{d}_0\} = \begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_{t=0} = \frac{2000}{\rho AL}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 27,400 \end{Bmatrix} in/s^2$$

3. Solve for d_{-1} at $t = -\Delta t$

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \{\ddot{d}_0\}\frac{(\Delta t)^2}{2}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_{-1} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - (0.25\times 10^{-3})\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \frac{(0.25\times 10^{-3})^2}{2}\begin{Bmatrix} 0 \\ 27,400 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0.856\times 10^{-3} \end{Bmatrix} in$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

4. Solve for d_1 at $t = \Delta t$ using the value of d_{-1} from Step 3

$$\{d_1\} = [\mathbf{M}]^{-1}\left\{(\Delta t)^2\{\mathbf{F}_0\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_0\} - [\mathbf{M}]\{d_{-1}\}\right\}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_1 = \frac{2}{0.073}\begin{bmatrix} ½ & 0 \\ 0 & 1 \end{bmatrix}\left\{(0.25\times 10^{-3})^2\begin{Bmatrix} 0 \\ 1,000 \end{Bmatrix} + \left[\frac{2(0.073)}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right.\right.$$

$$\left.\left. -(0.25\times 10^{-3})^2(30\times 10^4)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right]\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \frac{0.073}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.856\times 10^{-3} \end{Bmatrix}\right\}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_1 = \frac{2}{0.073}\begin{bmatrix} ½ & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0.0625\times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0.0312\times 10^{-3} \end{Bmatrix}\right]$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0.858\times 10^{-3} \end{Bmatrix} in$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

5. With d_0 given and d_1 determined in Step 4 solve for d_2

$$\{d_2\} = [\mathbf{M}]^{-1}\Big\{(\Delta t)^2\{\mathbf{F}_1\} + \big[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\big]\{d_1\} - [\mathbf{M}]\{d_0\}\Big\}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2 = \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\Bigg\{\left(0.25\times10^{-3}\right)^2\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} + \Bigg[\frac{2(0.073)}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-\left(0.25\times10^{-3}\right)^2\left(30\times10^4\right)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\Bigg]\begin{Bmatrix} 0 \\ 0.858\times10^{-3} \end{Bmatrix} - \frac{0.073}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\Bigg\}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2 = \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0.0625\times10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0.0161\times10^{-3} \\ 0.0466\times10^{-3} \end{Bmatrix}\right]$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2 = \begin{Bmatrix} 0.221\times10^{-3} \\ 2.99\times10^{-3} \end{Bmatrix} in$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

6. Solve for $\{\ddot{d}_1\}$

$$\{\ddot{d}_1\} = [\mathbf{M}]^{-1}\left(\{\mathbf{F}_1\} - [\mathbf{K}]\{d_1\}\right)$$

$$\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_1 = \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} - (30\times10^4)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.858\times10^{-3} \end{Bmatrix}\right]$$

$$\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_1 = \begin{Bmatrix} 3{,}526 \\ 20{,}345 \end{Bmatrix} in/s^2$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

7. Solve for $\{\dot{d}_1\}$

$$\{\dot{d}_1\} = \frac{\{d_2\} - \{d_0\}}{2(\Delta t)}$$

$$\begin{Bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}_1 = \frac{\left[\begin{Bmatrix} 0.221\times10^{-3} \\ 2.99\times10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right]}{2\left(0.25\times10^{-3}\right)} = \begin{Bmatrix} 0.442 \\ 5.98 \end{Bmatrix} in/s$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

Repeating Step 5:

$$\{d_3\} = [\mathbf{M}]^{-1}\left\{(\Delta t)^2\{\mathbf{F}_2\} + \left[2[\mathbf{M}] - (\Delta t)^2[\mathbf{K}]\right]\{d_2\} - [\mathbf{M}]\{d_1\}\right\}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_3 = \frac{2}{0.073}\begin{bmatrix} ½ & 0 \\ 0 & 1 \end{bmatrix}\left\{\left(0.25\times10^{-3}\right)^2\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} + \left[\frac{2(0.073)}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right.\right.$$

$$\left.\left. -\left(0.25\times10^{-3}\right)^2\left(30\times10^4\right)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right]\begin{Bmatrix} 0.221\times10^{-3} \\ 2.99\times10^{-3} \end{Bmatrix} - \frac{0.073}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.858\times10^{-3} \end{Bmatrix}\right\}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1.096\times10^{-3} \\ 5.397\times10^{-3} \end{Bmatrix} in$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

Repeating Step 6. Solve for $\{\ddot{d}_2\}$

$$\{\ddot{d}_2\} = [\mathbf{M}]^{-1}\left(\{\mathbf{F}_2\} - [\mathbf{K}]\{d_2\}\right)$$

$$\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_2 = \frac{2}{0.073}\begin{bmatrix} ½ & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} - (30\times 10^4)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.221\times 10^{-3} \\ 2.99\times 10^{-3} \end{Bmatrix}\right]$$

$$\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_1 = \begin{Bmatrix} 10{,}500 \\ 4{,}600 \end{Bmatrix} in/s^2$$

Structural Dynamics

Time-Dependent One-Dimensional Bar - Example

Repeat Step 7: Solve for $\{\dot{d}_2\}$

$$\{\dot{d}_2\} = \frac{\{d_3\} - \{d_1\}}{2(\Delta t)}$$

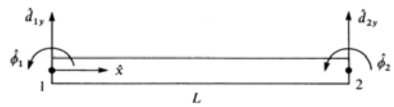
$$\begin{Bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}_2 = \frac{\left[\begin{Bmatrix} 1.096\times 10^{-3} \\ 5.397\times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0.858\times 10^{-3} \end{Bmatrix}\right]}{2\left(0.25\times 10^{-3}\right)} = \begin{Bmatrix} 2.192 \\ 9.078 \end{Bmatrix} in/s$$

Structural Dynamics

Beam Element Mass Matrices and Natural Frequencies

We will develop the lumped- and consistent-mass matrices for time-dependent beam analysis.

Consider the beam element shown in the figure below.



The basic equations of motion are: $\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$

Structural Dynamics

Beam Element Mass Matrices and Natural Frequencies

The stiffness matrix is:

$$[k] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

The lumped-mass matrix is:

$$[m] = \frac{\rho AL}{2}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element Mass Matrices and Natural Frequencies

The mass in lumped equally into each transitional degree of freedom; however, the inertial effects associated with any possible rotational degrees of freedom is assumed to be zero.

A value for these rotational degrees of freedom could be assigned by calculating the mass moment of inertia about each end node using basic dynamics as:

$$I = \frac{1}{3}mL^2 = \frac{1}{3}\left(\frac{\rho AL}{2}\right)\left(\frac{L}{2}\right)^2 = \frac{\rho AL^3}{24}$$

Structural Dynamics

Beam Element Mass Matrices and Natural Frequencies

The consistent-mass matrix can be obtained by applying

$$[m] = \int_V \rho [N]^T [N]\, dV$$

$$[m] = \int_0^L \int_A \rho \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} dA\, dx$$

where

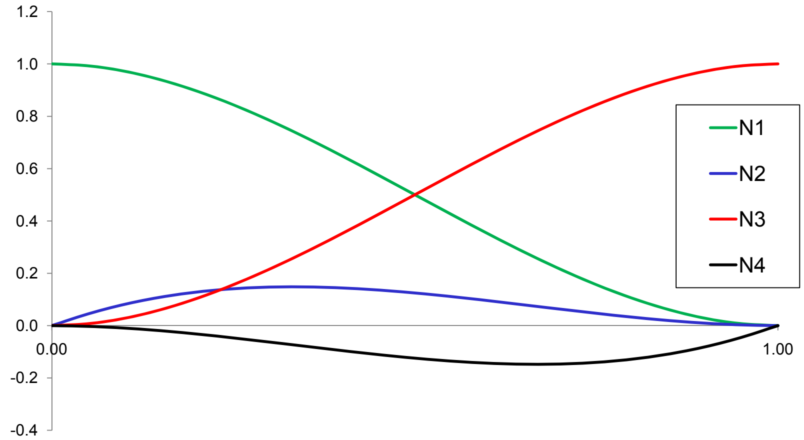
$$N_1 = \frac{1}{L^3}\left(2x^3 - 3x^2L + L^3\right) \qquad N_2 = \frac{1}{L^3}\left(x^3L - 2x^2L^2 + xL^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2x^3 + 3x^2L\right) \qquad N_4 = \frac{1}{L^3}\left(x^3L - x^2L^2\right)$$

Structural Dynamics

Beam Element Mass Matrices and Natural Frequencies

The shape functions are shown below:



Structural Dynamics

Beam Element Mass Matrices and Natural Frequencies

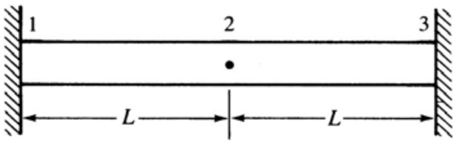
Substituting the shape functions into the above mass expression and integrating gives:

$$[m] = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Structural Dynamics

Beam Element - Example 1

Determine the first natural frequency for the beam shown in the figure below. Assume the bar has a length 2*L*, modulus of elasticity *E*, mass density ρ, and cross-sectional area *A*.



Let's discretize the beam into two elements each of length *L*.

We will use the lumped-mass matrix.

Structural Dynamics

Beam Element - Example 1

We can obtain the natural frequencies by using the following equation.

$$\left|[\mathbf{K}] - \omega^2 [\mathbf{M}]\right| = 0$$

The boundary conditions are $v_1 = \phi_1 = 0$ and $v_3 = \phi_3 = 0$.

Therefore, the global stiffness and lumped-mass matrices are:

$$[\mathbf{K}] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cc} v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \end{array} \qquad [\mathbf{M}] = \frac{\rho AL}{2}\begin{array}{c} \begin{array}{cc} v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element - Example 1

Substituting the global stiffness and mass matrices into the global dynamic equations gives:

$$\left|[\mathbf{K}] - \omega^2 [\mathbf{M}]\right| = 0 \qquad \left| \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} - \omega^2 \rho AL \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right| = 0$$

Dividing by ρAL and simplify

$$\omega^2 = \frac{24EI}{\rho AL^4} \qquad \omega = \frac{4.90}{L^2}\sqrt{\frac{EI}{\rho A}}$$

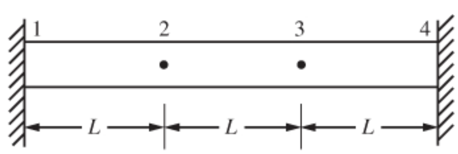
The exact solution for the first natural frequency is:

$$\omega = \frac{5.59}{L^2}\sqrt{\frac{EI}{\rho A}}$$

Structural Dynamics

Beam Element - Example 2

Determine the first natural frequency for the beam shown in the figure below. Assume the bar has a length 3*L*, modulus of elasticity *E*, mass density ρ, and cross-sectional area *A*.



Let’s discretize the beam into three elements each of length *L*.

We will use the lumped-mass matrix.

Structural Dynamics

Beam Element – Example 2

We can obtain the natural frequencies by using the following equation.

$$\left| [\mathbf{K}] - \omega^2 [\mathbf{M}] \right| = 0$$

The boundary conditions are $v_1 = \phi_1 = 0$ and $v_4 = \phi_4 = 0$.

Therefore the elements of the stiffness matrix for element 1 are:

$$\left[k^{(1)}\right] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 2

Element 2:

$$\left[k^{(2)}\right] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Element 3:

$$\left[k^{(3)}\right] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_3 & \phi_3 & v_4 & \phi_4 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 2

Assembling the global stiffness matrix as:

$$[K] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12-12 & 6L+6L & -12 & 6L \\ 6L-6L & 4L^2+2L^2 & -6L & 2L^2 \\ -12 & 6L & 12+12 & -6L+6L \\ -6L & 2L^2 & -6L+6L & 4L^2+4L^2 \end{bmatrix} \end{array}$$

$$[K] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 0 & 12L & -12 & 6L \\ 0 & 6L^2 & -6L & 2L^2 \\ -12 & 6L & 24 & 0 \\ 6L & 2L^2 & 0 & 8L^2 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 2

We can obtain the natural frequencies by using the following equation.

$$\left|[\mathbf{K}] - \omega^2[\mathbf{M}]\right| = 0$$

Therefore the elements of the mass matrix for element 1 are:

$$\left[m^{(1)}\right] = \frac{\rho AL}{2}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 2

Element 2:

$$\begin{array}{cc} & \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \left[m^{(2)}\right] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 3:

$$\begin{array}{cc} & \begin{array}{cccc} v_3 & \phi_3 & v_4 & \phi_4 \end{array} \\ \left[m^{(2)}\right] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The assembled mass matrix is:

$$\begin{array}{cc} & \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ [m] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1+1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ [M] = \rho AL & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 2

We can obtain the Frequency equation as: $\left|[\mathbf{K}] - \omega^2 [\mathbf{M}]\right| = 0$

$$\left| \frac{EI}{L^3} \begin{bmatrix} 0 & 12L & -12 & 6L \\ 0 & 6L^2 & -6L & 2L^2 \\ -12 & 6L & 24 & 0 \\ 6L & 2L^2 & 0 & 8L^2 \end{bmatrix} - \omega^2 \rho AL \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -\omega^2 \rho AL & 12\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 24\frac{EI}{L^3} - \omega^2 \rho AL & 0 \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & 8\frac{EI}{L} \end{vmatrix} = 0$$

Structural Dynamics

Beam Element – Example 2

Simplifying and assuming: $\beta = \rho AL$

$$\begin{vmatrix} -\omega^2\beta & 12\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 24\frac{EI}{L^3}-\omega^2\beta & 0 \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & 8\frac{EI}{L} \end{vmatrix} = 0$$

Evaluating the 4x4 determinate gives:

$$\frac{-1{,}152\omega^2\beta E^3I^3}{L^5} + \frac{48\omega^4\beta^2E^2I^2}{L^2} + \frac{576E^4I^4}{L^8} - \frac{1{,}296E^4I^4}{L^8}$$
$$+\frac{96\omega^2\beta E^3I^3}{L^5} - \frac{4\omega^4\beta^2E^2I^2}{L^2} - \frac{6{,}912E^4I^4}{L^8} = 0$$

Structural Dynamics

Beam Element – Example 2

Simplifying: $\frac{44\omega^4\beta^2E^2I^2}{L^2} - \frac{1{,}056\omega^2\beta E^3I^3}{L^5} - \frac{7{,}632E^4I^4}{L^8} = 0$

$$11\omega^4\beta^2 - \frac{264\omega^2\beta EI}{L^3} - \frac{1{,}908E^2I^2}{L^6} = 0$$

Dividing by $\frac{4E^2I^2}{L^2}$ and solving for the two roots for $\omega_1^2\beta$

$$\omega_1^2\beta = -\frac{5.817254\ EI}{L^3} \qquad \omega_1^2\beta = \frac{29.817254\ EI}{L^3}$$

Structural Dynamics

Beam Element – Example 2

Ignoring the negative root as it is not physically possible and solving explicitly for ω_1 gives:

$$\omega_1^2 = \frac{29.817254\, EI}{\beta L^3} \qquad \omega_1 = \sqrt{\frac{29.817254\, EI}{\beta L^3}} = \frac{5.46}{L^2}\sqrt{\frac{EI}{A\rho}}$$

In summary:

Two elements: $\omega = \frac{4.90}{L^2}\sqrt{\frac{EI}{\rho A}}$

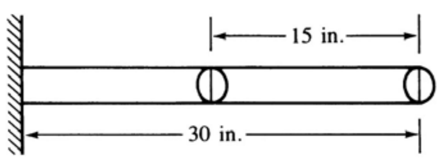
Three elements: $\omega = \frac{5.46}{L^2}\sqrt{\frac{EI}{A\rho}}$

Exact solution: $\omega = \frac{5.59}{L^2}\sqrt{\frac{EI}{\rho A}}$

Structural Dynamics

Beam Element – Example 3

Determine the first natural frequency for the beam shown in the figure below.



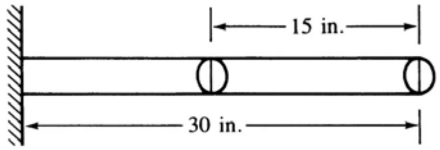
Assume the bar has a length L = 30 *in*, modulus of elasticity $E = 3 \times 10^7$ *psi*, mass density ρ = 0.00073 *lb-s*^{2}/*in*^{4}, and cross-sectional area A = 1 *in*^{2}, moment of inertia I = 0.0833 *in*^{4}, and Poisson's ratio ν = 0.3.

$$\left|[\mathbf{K}] - \omega^2[\mathbf{M}]\right| = 0$$

Structural Dynamics

Beam Element – Example 3

Determine the first natural frequency for the beam shown in the figure below.



Let's discretize the beam into two elements each of length L = 15 *in*. We will use the lumped-mass matrix.

We can obtain the natural frequencies by using the following equation.

$$\left| [\mathbf{K}] - \omega^2 [\mathbf{M}] \right| = 0$$

Structural Dynamics

Beam Element – Example 3

In this example, the elemental stiffness matrices are:

Element 1:

$$[k^{(1)}] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Element 2:

$$[k^{(2)}] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 3

In this example, the elemental mass matrices are:

Element 1:

$$\begin{array}{cc} & \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \left[m^{(1)} \right] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 2:

$$\begin{array}{cc} & \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \left[m^{(2)} \right] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 3

The boundary conditions are $u_1 = \phi_1 = 0$. Therefore the global stiffness and lumped mass matrices is:

$$\begin{array}{cc} & \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ [K] = \dfrac{EI}{L^3} & \begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ M = \dfrac{\rho AL}{2} & \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Structural Dynamics

Beam Element – Example 3

Substituting the global stiffness and mass matrices into the global dynamic equations gives:

$$\left| \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} - \omega^2 \frac{\rho AL}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| = 0$$

$$\left| \mu\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| = 0 \qquad \mu = \frac{EI}{\rho AL^4}$$

Structural Dynamics

Beam Element – Example 3

Evaluating the determinant of the above equations gives:

$$14\mu^4\omega^4 - 240\mu^3\omega^2 + 144\mu^4 = 0 \quad \Rightarrow \quad \begin{aligned} \omega_1 &= 0.7891\ \mu \\ \omega_2 &= 4.0647\ \mu \end{aligned}$$

The solution for the first natural frequency is:

$$\omega_1 = \frac{0.7891}{L^2}\sqrt{\frac{EI}{\rho A}} \qquad \omega_2 = \frac{4.0647}{L^2}\sqrt{\frac{EI}{\rho A}}$$

Structural Dynamics

Beam Element – Example 3

In this example, the length of each element is actual *L*/2, therefore:

$$\omega_1 = \frac{0.7891}{\left(L/2\right)^2}\sqrt{\frac{EI}{\rho A}} = \frac{3.1564}{L^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = \frac{4.0647}{\left(L/2\right)^2}\sqrt{\frac{EI}{\rho A}} = \frac{16.2588}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The exact solution for the first natural frequency is:

$$\omega = \frac{3.516}{L^2}\sqrt{\frac{EI}{\rho A}}$$

Structural Dynamics

Beam Element – Example 3

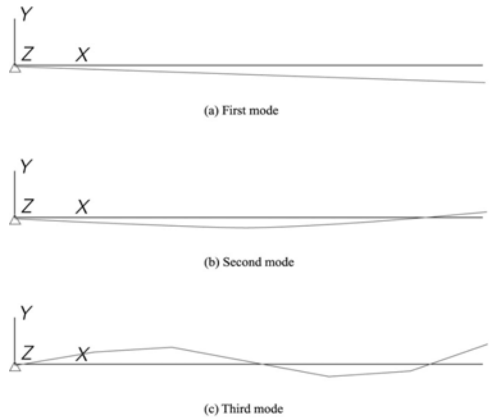
In the exact solution of the vibration of a clamped-free beam, the higher natural frequencies to the first natural frequency can be given as:

$$\frac{\omega_2}{\omega_1} = 6.2669 \qquad \frac{\omega_3}{\omega_1} = 17.5475$$

Structural Dynamics

Beam Element – Example 3

The figure below shows the first, second, and third mode shapes corresponding to the first three natural frequencies for the cantilever beam.



Structural Dynamics

Beam Element – Example 3

The table below shows various finite element solutions compared to the exact solution.

	ω_1 (*rad/s*)	ω_2 (*rad/s*)
Exact Solution	228	1,434
FE Solution		
Using 2 elements	205	1,286
Using 6 elements	226	1,372
Using 10 elements	227.5	1,410
Using 30 elements	228.5	1,430
Using 60 elements	228.5	1,432

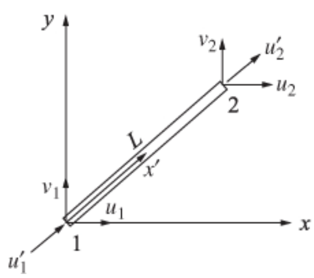
Structural Dynamics

Truss Analysis

The dynamics of trusses and plane frames are preformed by extending the concepts of bar and beam element.

The truss element requires the same transformation of the mass matrix from local to global coordinates as that used for the stiffness matrix given as:

$$[m] = [\mathbf{T}]^T [m'][\mathbf{T}]$$



Structural Dynamics

Truss Analysis

Considering two-dimensional motion, the axial and the transverse displacement are given as:

$$\begin{Bmatrix} u' \\ v' \end{Bmatrix} = \frac{1}{L}\begin{bmatrix} L-x' & 0 & x' & 0 \\ 0 & L-x' & 0 & x' \end{bmatrix}\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix}$$

The shape functions for the matrix are:

$$[N] = \frac{1}{L}\begin{bmatrix} L-x' & 0 & x' & 0 \\ 0 & L-x' & 0 & x' \end{bmatrix}$$

Structural Dynamics

Truss Analysis

The consistent-mass matrix can be obtained by applying:

$$[m'] = \int_V \rho [N]^T [N] dV \qquad [m'] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

The lumped-mass matrix for two-dimensional motion is obtained by simply lumping mass at each node (mass is the same in both the *x* and *y* directions):

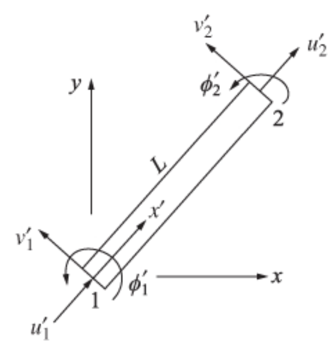
$$[m'] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Structural Dynamics

Plane Frame Analysis

The plane frame element requires combining the bar and beam elements to obtain the local mass matrix.

There are six degrees of freedom associated with a plane frame element.

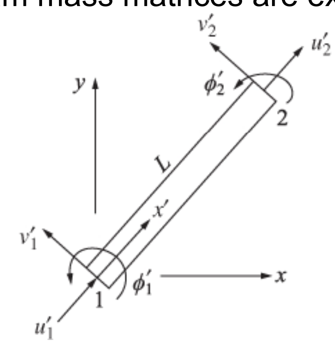


Structural Dynamics

Plane Frame Analysis

The plane frame analysis requires first expanding and then combining the bar and beam mass matrices to obtain the local mass matrix.

The bar and beam mass matrices are expanded to a 6 x 6 and superimposed



Structural Dynamics

Plane Frame Analysis

Combining the local axis consistent-mass matrices for the bar and beam elements gives:

$$[m'] = \rho AL \begin{bmatrix} \frac{2}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{156}{420} & \frac{22L}{420} & 0 & \frac{54}{420} & \frac{-13L}{420} \\ 0 & \frac{22L}{420} & \frac{4L^2}{420} & 0 & \frac{13L}{420} & \frac{-3L^2}{420} \\ \frac{1}{6} & 0 & 0 & \frac{2}{6} & 0 & 0 \\ 0 & \frac{54}{420} & \frac{13L}{420} & 0 & \frac{156}{420} & \frac{-22L}{420} \\ 0 & \frac{-13L}{420} & \frac{-3L^2}{420} & 0 & \frac{-22L}{420} & \frac{4L^2}{420} \end{bmatrix}$$

Structural Dynamics

Plane Frame Analysis

The resulting lumped-mass matrix for a plane frame element is give as:

$$[m'] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Structural Dynamics

Plane Frame Analysis

The global mass matrix for the plane frame element arbitrarily oriented in *x*-*y* coordinates is transformed by:

$$[m] = [\mathbf{T}]^T [m'][\mathbf{T}]$$

where the transformation matrix is given as:

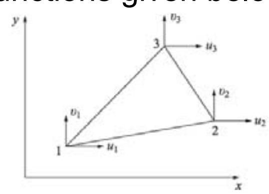
$$[\mathbf{T}] = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Structural Dynamics

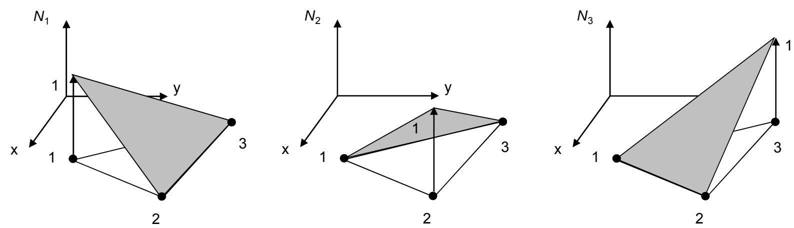
Plane Stress/Strain Elements

The plane stress/strain constant-strain triangle consistent-mass matrix is obtained using the shape functions given below as:

$$[N]=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$



The linear triangular shape functions are illustrated below:



Structural Dynamics

Plane Stress/Strain Elements

The consistent-mass matrix can be obtained by applying:

$$[m]=\int_V \rho [N]^T [N] dV$$

where $dV = tdA$

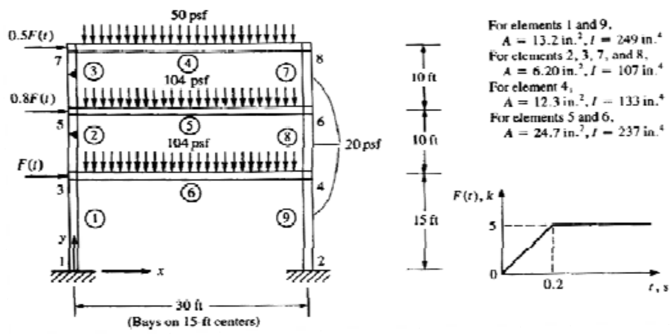
The CST global consistent-mass matrix is:

$$[m]=\frac{\rho t A}{12}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Structural Dynamics

Frame Example Problem 1

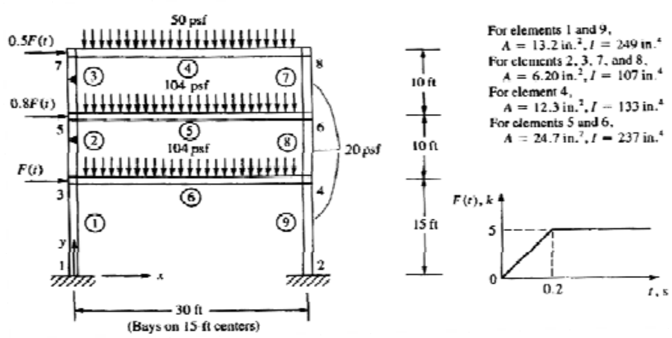
Use SAP2000 to determine the motion of the frame structure below.



Structural Dynamics

Frame Example Problem 1

Assume the modulus of elasticity $E = 3 \times 10^7$ *psi*. The nodal lumped mass values are obtained by dividing the total weight (dead loads included) of each floor or wall section by gravity.



Structural Dynamics

Frame Example Problem 1

For example, compute the total mass of the uniform vertical load on elements 4, 5, and 6:

$$M_4 = \frac{W_4}{g} = \frac{(50\ psf)(30\ ft)(15\ ft)}{386.04\ {}^{in}\!/_{s^2}} = 58.28\ {}^{lb \cdot s^2}\!/_{in}$$

$$M_5 = \frac{W_5}{g} = \frac{(104\ psf)(30\ ft)(15\ ft)}{386.04\ {}^{in}\!/_{s^2}} = 121.23\ {}^{lb \cdot s^2}\!/_{in}$$

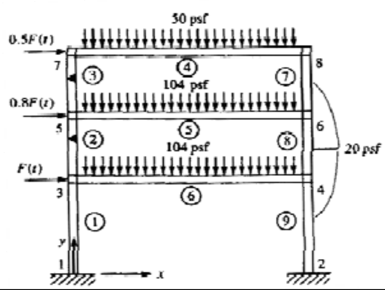
$$M_6 = \frac{W_6}{g} = \frac{(104\ psf)(30\ ft)(15\ ft)}{386.04\ {}^{in}\!/_{s^2}} = 121.23\ {}^{lb \cdot s^2}\!/_{in}$$

Structural Dynamics

Frame Example Problem 1

Next, lump the mass equally to each node of the beam element.

For this example calculation, a lumped mass of 29.14 $lb{\cdot}s^2/in$ should be added to nodes 7 and 8 and a mass of 60.62 $lb{\cdot}s^2/in$ should be added to nodes 3, 4, 5, and 6, all in the x direction.

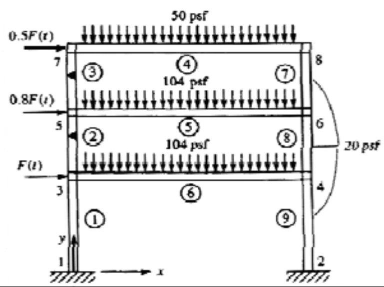


Structural Dynamics

Frame Example Problem 1

In an identical manner, masses for the dead loads for additional wall sections should be added to their respective nodes.

In this example, additional wall loads should be converted to mass added to the appropriate loads.



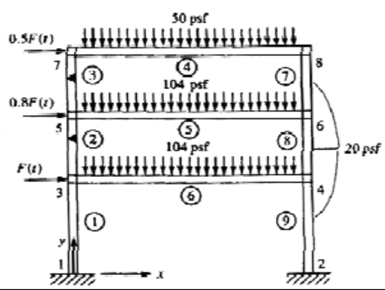
Structural Dynamics

Frame Example Problem 1

For example, the additional mass due to the wall load on element 8 is:

$$M_7 = \frac{W_7}{g} = \frac{(20\ psf)(10\ ft)(15\ ft)}{386.04\ in/s^2} = 7.77\ lb \cdot s^2/in$$

Therefore, an additional 3.88 $lb \cdot s^2/in$ should be added to nodes 6 and 8.



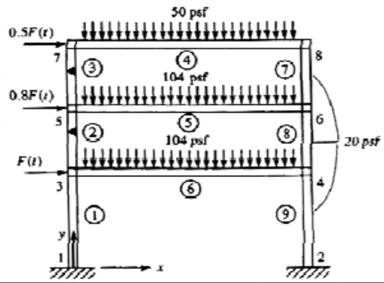
Structural Dynamics

Frame Example Problem 1

For example, the additional mass due to the wall load on element 8 is:

$$M_8 = \frac{W_8}{g} = \frac{(20\ psf)(10\ ft)(15\ ft)}{386.04\ {}^{in}\!/_{s^2}} = 7.77\ {}^{lb \cdot s^2}\!/_{in}$$

Therefore, an additional 3.88 $lb{\cdot}s^2/in$ should be added to nodes 4 and 6.



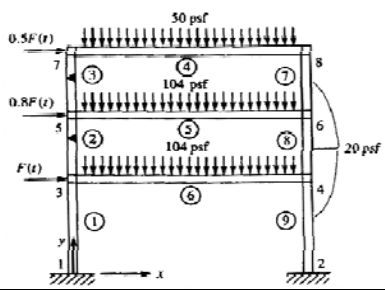
Structural Dynamics

Frame Example Problem 1

For example, the additional mass due to the wall load on element 8 is:

$$M_9 = \frac{W_9}{g} = \frac{(20\ psf)(15\ ft)(15\ ft)}{386.04\ {}^{in}\!/_{s^2}} = 11.66\ {}^{lb \cdot s^2}\!/_{in}$$

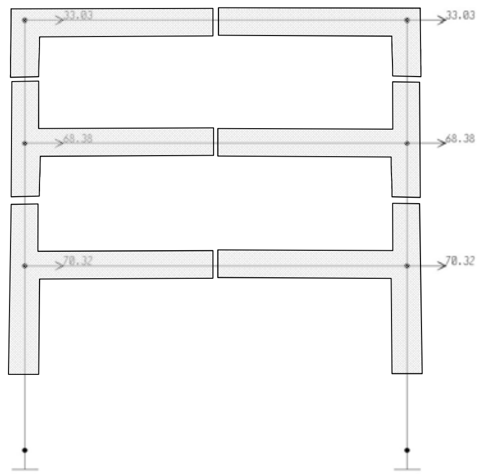
Therefore, an additional 5.83 $lb{\cdot}s^2/in$ should be added to node 4.



Structural Dynamics

Frame Example Problem 1

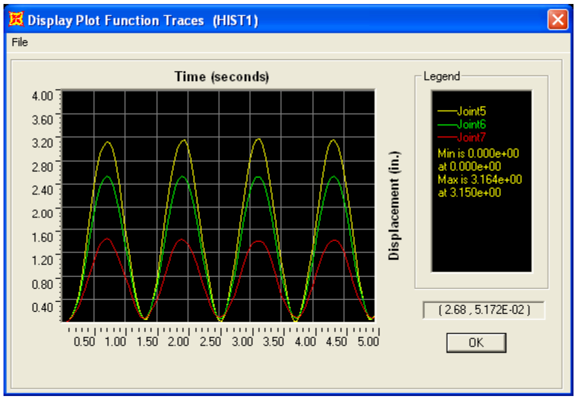
The final values for the nodal masses on this frame are shown below.



Structural Dynamics

Frame Example Problem 1

A trace of the displacements of nodes 8, 6, and 4 as a function of time can be generated using the SAP2000 Display Menu.

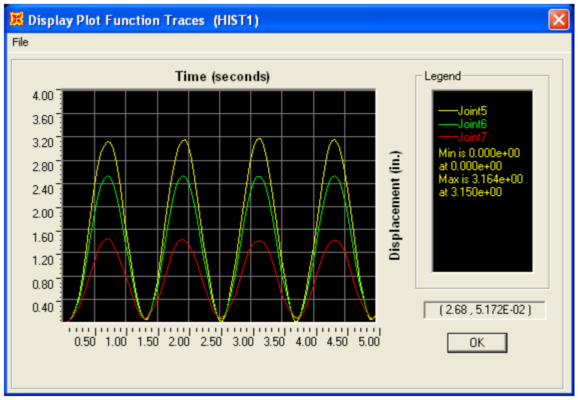


Nodes 8, 6, and 4 are listed in SAP 2000 as 5, 6, and 7.

Structural Dynamics

Frame Example Problem 1

A plot of the displacements of nodes 8, 6, and 4 over the first 5 seconds of the analysis generate by SAP2000 is shown below:

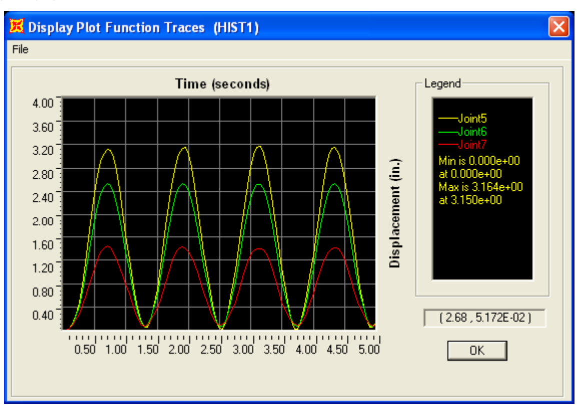


Nodes 8, 6, and 4 are listed in SAP 2000 as 5, 6, and 7.

Structural Dynamics

Frame Example Problem 1

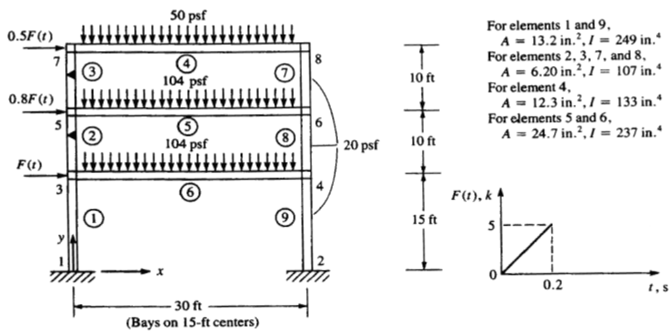
The maximum displacement of node 8 (node 5 in the SAP 2000 plot) is 3.16 in. and the period of the vibration is approximately 3.15 seconds.



Structural Dynamics

Frame Example Problem 2

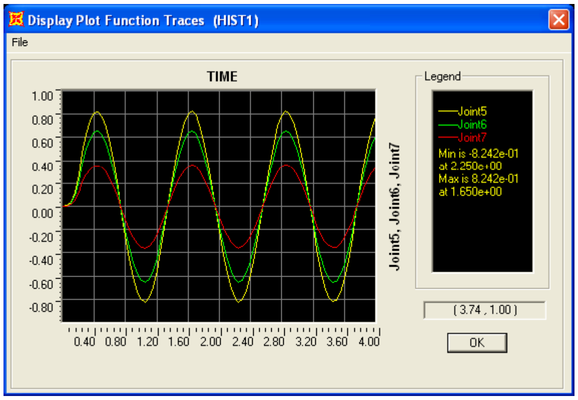
Use SAP2000 to determine the motion of the frame structure below.



Structural Dynamics

Frame Example Problem 2

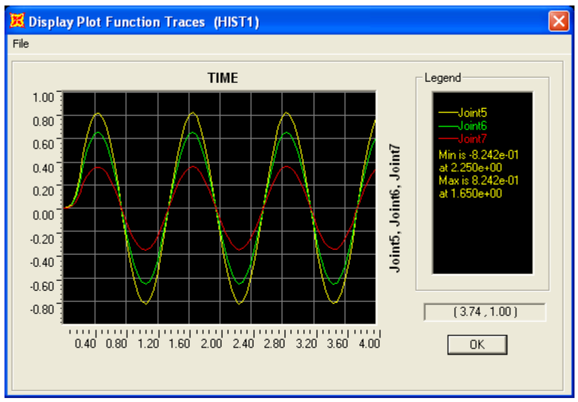
A trace of the displacements as a function of time can be generated using the SAP2000 Display Menu. A plot of the displacements of nodes 8, 6, and 4 over the first 4 seconds of the analysis generate by SAP2000 is shown below:



Structural Dynamics

Frame Example Problem 2

The maximum displacement of node 8 is 0.824 in. and the period of the vibration is approximately 1.65 seconds.



Development of the Plate Bending Element

Problems

23. Do problems ***16.5*** and ***16.11c*** on pages 824-828 in your textbook “A First Course in the Finite Element Method” by D. Logan.

24. Do problems ***16.14*** and ***16.16*** on pages 824-828 in your textbook “A First Course in the Finite Element Method” by D. Logan using SAP2000 or WinFElt.

End of Chapter 16