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Continuous Optimization

Lanchester model for three-way combat

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ABSTRACT

Lanchester (1960) modeled combat situations between two opponents, where mutual attrition occurs continuously in time, by a pair of simple ordinary (linear) differential equations. The aim of the present paper is to extend the model to a conflict consisting of three parties. In particular, Lanchester's main result, i.e. his Square Law, is adapted to a triple fight. However, here a central factor – besides the initial strengths of the forces – determining the long run outcome is the allocation of each opponent's efforts between the other two parties. Depending on initial strengths, (the) solution paths are calculated and visualized in appropriate phase portraits. We are able identify regions in the state space where, independent of the force allocation of the opponents, always the same combatant wins, regions, where a combatant can win if its force allocation is wisely chosen, and regions where a combatant cannot win itself but determine the winner by its forces allocation. As such, the present model can be seen as a forerunner of a dynamic game between three opponents.

1. Introduction

Lanchester (1916) applied a pair of linear ordinary differential equations to understand the dynamics of a battle between two opponents. He was inspired by the attrition and exhaustion of fighters in air combats in World War I. Since then many papers have been published on that and related issues, see, e.g. Morse and Kimball (1951); see also Washburn and Kress (2009), Kress (2012). It is surprising, however, that while Lanchester attrition duels are prevalent in the literature, there are hardly any models for combat situations involving three sides or more sides. An exception is Syms and Solymar (2017), who analyze the Lanchester model with area fire and recruitment. Furthermore, Lin and MacKay (2014) studied the optimal policy for a one-against-many combat in a Lanchester framework. The aim of the present paper is to extend the Lanchester model to three combatants and analyze the more general case where all of the opponents are engaged in combat against each other in a Lanchester framework with aimed fire and without recruitment (which is difficult in an ongoing insurgency).

In the classic Lanchester model two opponents fight each other. Their sizes are considered as state variables. The decrease of their forces over time depends on the size of the forces and their per capita effectiveness measured by their respective attrition rates. There are two main types of Lanchester models corresponding to direct and area fire. The direct fire model results in a quadratic equation (conserved quantity) that is manifested in the Square Law. The area fire model induces a linear state equation and, accordingly, is governed by the Linear Law. Although there exist stochastic versions of the models (e.g., Kress & Talmor, 1999) the commonly used models are deterministic. Deitchman (1962) combined the two types of Lanchester models and defined the "Guerrilla Warfare" model where one side (the guerrillas) utilize direct fire, while the other side (regular forces) use area fire.

Lanchester models are purely attritional and ignore the crucial role of situational awareness and intelligence. Attempts to generalize Lanchester theory by incorporating the effect of information are reported in Kress and Szechtman (2009), Kaplan, Kress, and Szechtman (2010). Schramm and Gaver (2013) combine the Lanchester model with a deterministic epidemic model to account for the impact of information spreading.

Over the years there have been many analyses and extensions of Lanchester models. For example, Bracken (1995) validates the linear Lanchester model with historical data from World War II. Chen and Chu (2001) extend this approach by taking into

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account the timing of a shift between defense and attack. Stochastic aspects in Lanchester models are discussed in Hausken and Moxnes (2000, 2002, 2005). Zero-sum attrition games on a network which differ with respect to the information structure are analyzed in Hohzaki and Higashio (2016). MacKay (2015) combines the Richardson, Lanchester and Deitchman model to find that the typical outcome of such a combined model is not the annihilation of one opponent but a stale-mate, where both parties remain active forever in a steady state.

Of course, there are also many interesting papers outside the Lanchester framework which analyze important aspects of combats such as Peng, Zhai, and Levitin (2016), who analyze a game between an attacker and a defender under the deployment of false targets, and Zhai, Ye, Peng, and Wang (2017), who consider the issue of infrastructure protection within a two-player zero-sum game. See Hausken and Levitin (2012) for a review of papers related to the defense and attack of systems.

The aim of the present paper is to extend the analysis of the classic Lanchester model of direct fire to a three-sided battle. There are many recent and historic examples for three-sided combats such as the Bosnian Civil War and the Iraq Civil War (government vs. Shias vs. Sunnis), see Syms and Solymar (2017) for more examples. The analysis in this paper is motivated by recent events in Syria, where at least six active parties (not to mention the "big" players such as Russia, Turkey and Iran) – al-Qaeda affiliated groups (e.g., Jabahat al Nusra), ISIS, the free Syrian army, Hezbollah, Kurds and Assad regime forces – fight each other to gain control on land, people and national assets. In contrast to a one-on-one engagement, additional parameters are needed to indicate how each side's firepower should be allocated between its two opponents. Compare also the literature on optimal fire distribution where one of the two opponents consists of two heterogeneous forces, see e.g. Taylor (1974), Lin and MacKay (2014). We assume that each party commits to allocate a fixed percentage of its efforts toward each opponent throughout the conflict, e.g., one-third directed against enemy 1 and two-thirds against enemy 2. We will show how the initial force-size of the three opponents together with the attrition rates and the fire-allocation tactics determine the winner of the battle. More complicated, dynamically adjusting strategies are possible in principle, but the fixed proportions problem is interesting in and of itself.

We use eigenvalue analysis to identify surfaces separating regions of initial states that differ in the way the conflict is played out. By restricting the state space to the unit simplex we obtain an illustrative description of the solution paths. Moreover, we are able to identify in that simplex, for each side, its winning regions – initial conditions that guarantee its win.

The paper is organized as follows. In Section 2 we present the model and characterize the solution. In Section 3 we discuss the numerical solution of the problem. Section 4 concludes.

2. Lanchester model with three combatants

We formulate a two-stage Lanchester model in Section 2.1, and introduce some important concepts in Section 2.2. We recapitulate the important properties of the Lanchester model with two sides in Section 2.3, and derive the corresponding properties for the model with three sides in Section 2.4.

2.1. Two-stage model

We consider a situation where each force among three is engaged in combat against the other two (henceforth called also *sides* or *combatants*). The strength of each of the forces at time t is denoted as $I_j(t),\ \ j = 0, 1, 2$. In fact the strength of the forces $I_j,\ \ j = 0, 1, 2$ are normalized by the initial total size $N = \sum_{j=0}^{2} I_j(0)$ and hence denote the relative strengths. Due to the linearity of the ODEs the total strength is given by the multiplication with N. The battle comprises two stages. It is not possible to fight both opponents simultaneously with same forces, therefore it is necessary that each side splits its forces between the two opponents in the first stage of the battle. The fraction of the force of side j that is allocated to engage side i is denoted by the parameter $y_{ij},\ \ i, j = 0, 1, 2$. It is assumed that the opponents do not adapt the allocation of their forces over time. They also cannot form coalitions. The parameters $a_{i,\,j}$ denote the attrition rates when j engages i with $i, j = 0, 1, 2$.

If one of the three forces is annihilated, the two remaining sides continue in a Square Law battle in which all of their forces are engaged. Formally,[1]

$$\dot{I}_0(t) = -a_{01}y_{01}I_1(t) - a_{02}y_{02}I_2(t), \quad t \in [0, \tau_1\rangle \tag{1a}$$

$$\dot{I}_1(t) = -a_{10}y_{10}I_0(t) - a_{12}y_{12}I_2(t), \quad t \in [0, \tau_1\rangle \tag{1b}$$

$$\dot{I}_2(t) = -a_{20}y_{20}I_0(t) - a_{21}y_{21}I_1(t), \quad t \in [0, \tau_1\rangle \tag{1c}$$

where τ_1 is the time when the first force among the three is annihilated. The initial sizes of the forces are given by

$$I_j(0) = I_j^0 \geq 0, \ \ j = 0, 1, 2, \quad \text{and} \quad \sum_{j=0}^{2} I_j^0 = 1. \tag{1d}$$

If the forces of the remaining sides k, l with $k \neq l$ are strictly positive at τ_1, then at the second stage

$$\dot{I}_k(t) = -a_{kl}I_l(t), \quad t \in [\tau_1, \tau_2\rangle \tag{1e}$$

$$\dot{I}_l(t) = -a_{lk}I_k(t), \quad t \in [\tau_1, \tau_2\rangle \tag{1f}$$

$$\dot{I}_j(t) = 0, j = 3 - (k + l), \quad t \in [\tau_1, \tau_2\rangle \tag{1g}$$

where τ_2 is the time when the second stage ends where at least one of the two remaining sides from stage one is annihilated too.

The coefficients in the first stage satisfy

$$0 \leq y_{ij} \leq 1, \quad \sum_{i \neq j} y_{ij} = 1, \quad a_{ij} > 0, \quad i, j = 0, 1, 2, \ \ i \neq j, \tag{1h}$$

and

$$[\tau_1, \tau_2\rangle := \begin{cases} [\tau_1, \tau_2] & \text{if} \tau_2 < \infty \\ [\tau_1, \infty) & \text{if} \tau_2 = \infty. \end{cases}$$

The restriction as in the first stage Eq. (1d) is the normalization mentioned before that allows us to consider the unit tetrahedron as phase space with the initial states (force sizes) lying in the unit 2-simplex, subsequently denoted as Δ.

For the second stage we assume that the combat attrition rates remain the same as in the first stage.

2.2. Extinction times and curves

The next sections address the problem of classifying possible scenarios for the solutions of Eq. (1). Specifically we are interested in determining the first and second extinction times τ_1 and τ_2 and if there exists an opponent $I_k(\cdot)$ who wins in the sense that $I_k(\tau_2) > 0$. Thus, we give the following definitions (Table 1).

Definition 1 (Extinction times, survivors, winner and stages)**.** Let $I(\cdot) = (I_0(\cdot), I_1(\cdot), I_2(\cdot))^\top$ be the solution of Eqs. (1a)–(1d). The time

[1] As usual, the dot denotation refers to the time derivative, i.e. $\dot{I}_i = \frac{dI_i}{dt}$, $i = 0, 1, 2$.

Table 1
All possible cases for the first and the second extinction time.

Cases	*First and second extinction time* τ_1 τ_2
No annihilation in finite time	$\tau_1 = \tau_2 = \infty$
Exactly one annihilationin finite time	$\tau_1 < \tau_2 = \infty$
Two forces are annihilatedat the same time	$\tau_1 = \tau_2 < \infty$
General case	$\tau_1 < \tau_2 < \infty$

τ_1 such that the size of one of the combatants becomes zero is called the *first extinction time*. If none of the combatants becomes zero $\tau_1 = \infty$. A combatant k with $I_k(\tau_1) > 0$ is called *a survivor*. The time τ_2 when one of the survivors becomes zero is called the *second extinction time*. If none of the survivors becomes zero $\tau_2 = \infty$. If $I_j(\tau_2) > 0$ for some j the combatant j is called the *winner of combat scenario* (1).

The solution $I(\cdot)$ on the interval $[0, \tau_1\rangle$ will be called the *solution of the first stage* and on the interval $(\tau_1, \tau_2\rangle$ the solution will be called the *solution of the second stage*.

In Grass, Feichtinger, and Seidl (2016) it is proved that this definition is well defined.

Subsequently we identify six different areas in the initial state space (Δ) with different combinations of survivors and winners. These areas are separated by two types of curves. Before we give a formal definition of these curves we give an informal description of two qualitatively different situations.

Remark 1 (Heuristic explanation of total extinction)**.** Let us assume that for some initial values combatant 0 wins (phase two). Changing the initial states we assume that combatant 1 wins. What happens in the transition between these two cases? In both cases combatant 2 loses, i.e. the first extinction time is finite ($\tau_1 < \infty$). What happens to the second extinction time τ_2 in the transition? The nearer we get to the transition point the longer both opponents remain positive, i.e. τ_2 increases. In the extreme case at the transition the second extinction time becomes infinite ($\tau_2 = \infty$). This can only happen if combatants 0 and 1 end up at the stable path of the second stage. Those initial points that satisfy this condition will be called total extinction curve. See Fig. 1(b).

Remark 2 (Heuristic explanation iso-extinction)**.** Let us consider the situation where the identity of one of the survivors, e.g. combatants 0 and 1, changes. In that case combatant 2 is always the winner of the second stage, thus the second and hence the first extinction times are finite. In the transition combatants 0 and 1 are annihilated at the same time. Thus the first and second extinction time coincide ($\tau_1 = \tau_2$). Those initial points that satisfy this condition will be called the iso-extinction curve. See Fig. 1(a).

Definition 2. Let τ_1 and τ_2 be the first and second extinction times corresponding to an initial point $I^0 = (I^0_0, I^0_1, I^0_2)^\top$. Then

$$\omega^{(1)} := \{I^0 \in \Delta :\ \tau_1 = \tau_2 = \infty\} \tag{2}$$

is called the *total extinction curve of the first kind.*

$$\omega^{(2)} := \{I^0 \in \Delta :\ \tau_1 < \infty,\ \tau_2 = \infty\} \tag{3}$$

is called the *total extinction curve of the second kind.*

$$\gamma := \{I^0 \in \Delta :\ \tau_1 < \infty,\ \tau_1 = \tau_2\} \tag{4}$$

is called the *iso-extinction curve*.

In the next sections we characterize the solution properties of ODEs for the two stages. We note that the Eqs. (1a)–(1c) and Eqs. (1e) and (1f) are linear. Thus, solutions of these ODEs are fully characterized by the eigenvalues and eigenvectors of the corresponding Jacobian matrices. We start with the well-known two-sided Lanchester model of the second stage.

2.3. Subproblem with two combatants

To ease the notation we omit the double indexing for the second stage and set the indices k and l of Eqs. (1e) and (1f) to zero and one. Thus, subproblem Eqs. (1e) and (1f) becomes

$$\dot I_0(t) = -a_1 I_1(t), \quad t \in [0, \tau\rangle \tag{5a}$$

$$\dot I_1(t) = -a_0 I_0(t), \quad t \in [0, \tau\rangle \tag{5b}$$

with

$$I_j(0) = I^0_j \geq 0, \ \ j = 0, 1 \tag{5c}$$

and the coefficients satisfying

$$a_i > 0, \quad i = 0, 1.$$

τ denotes the first time that one of the sides becomes zero.

Definition 3 (Extinction time and winner)**.** Let $(I_0(\cdot), I_1(\cdot))$ be a solution of Eq. (5). The time τ such that one of the combatants becomes zero is called the *extinction time*. If none of the combatants becomes zero, then $\tau = \infty$. If $\tau < \infty$ and $I_k(\tau) > 0$, then combatant k is called the *winner of Eq. (5).*

The eigenvalue analysis yields

Proposition 1. *Let*

$$J = \begin{pmatrix} 0 & -a_1 \\ -a_0 & 0 \end{pmatrix}. \tag{6}$$

be the Jacobian of the Eqs. (5a) and (5b). The eigenvalues $\xi_i,\ i = 0, 1$ of J are given as

$$\xi_{0,1} = \mp\sqrt{a_1 a_0} \tag{7a}$$

with eigenvectors

$$\nu_0 = \begin{pmatrix} a_1 \\ \sqrt{a_1 a_0} \end{pmatrix} \frac{1}{a_1 + \sqrt{a_1 a_0}} \quad \text{and} \quad \nu_1 = \begin{pmatrix} a_1 \\ -\sqrt{a_1 a_0} \end{pmatrix} \tag{7b}$$

The such normalized eigenvector ν_0, corresponding to the negative eigenvalue ξ_0, satisfies

$$\sum_{j=1}^{2} \nu_{0,j} = 1 \quad \text{and} \quad \nu_{0,j} > 0, \ \ j = 1, 2. \tag{8}$$

Proof. Eigenvalues and eigenvectors can be derived from the Jacobian Eq. (6), and simple inspection shows Eq. (8). □

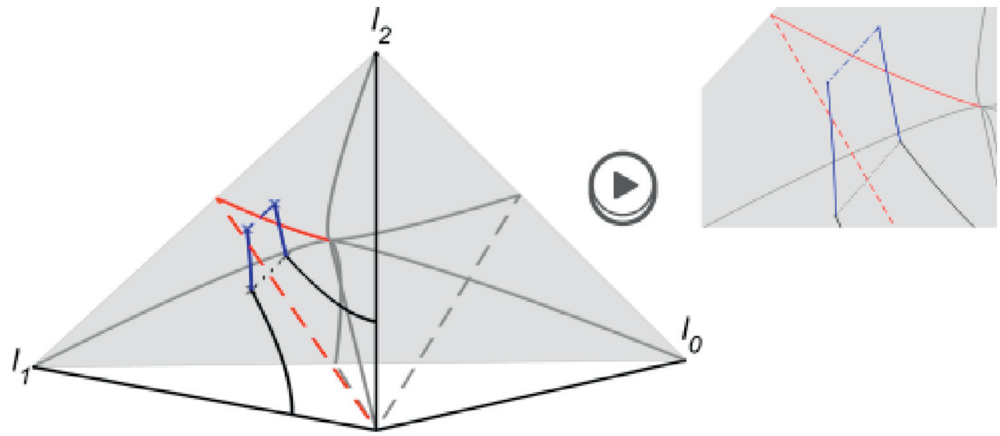
Remark 3. The eigenvector ν_0 corresponding to the negative eigenvalue plays a crucial role. In the second stage of the Lanchester model Eq. (1) three combinations of the a_{kl} parameter values are possible. Subsequently we denote the corresponding (stable) eigenvectors with the normalization Eq.(8) as $\nu_0^{(i)}$, $i = 0, 1, 2$.

The subsequent proposition uniquely characterizes the winner of the Lanchester model Eq. (5).

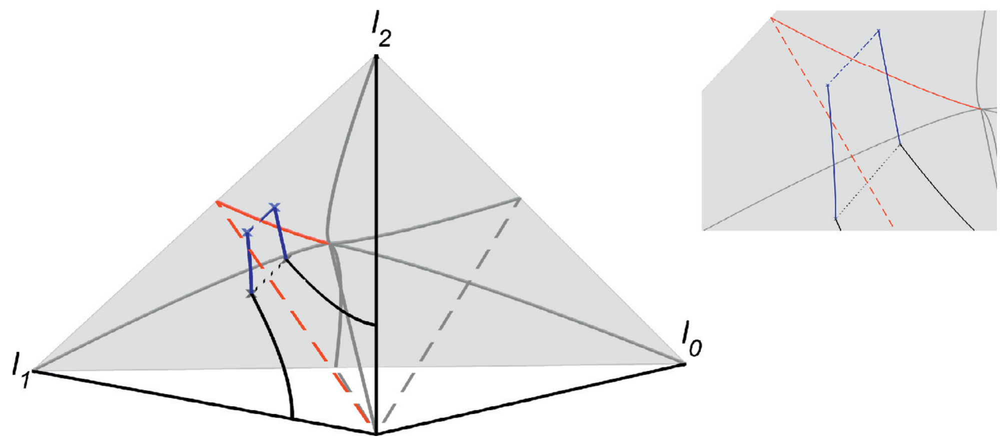
Proposition 2. *If $I_0(0) > 0$ and $I_1(0) > 0$, then combatant 0 or 1, respectively, is the winner iff*

$$\frac{I_1(0)^2}{I_0(0)^2} \lessgtr \frac{a_0}{a_1}. \tag{9}$$

There is no winner, i.e. the extinction time τ is infinite, iff



(a) Crossing the iso-extinction curve



(b) Crossing the total-extinction curve

Fig. 1. The dashed–dotted lines denote the considered initial forces, and the dotted lines show the corresponding forces at the first extinction time. In (a) the initial forces cross the iso-extinction curve (red). At the crossing point the corresponding dotted line hits the I_2 axis and the survivors change, whereas the winner of the combat remains the same. In (b) the initial forces cross the total-extinction curve (red). At the crossing point the dotted lines cross the stable eigenspace (dashed, red) of the second stage. The winner of the combat changes since the solution paths end at different axis. The subplots on the upper right side shows the details near the crossings. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\frac{I_1(0)^2}{I_0(0)^2} = \frac{a_0}{a_1}. \tag{10}$$

Eqs. (9) and (10) are also called Lanchester Square Law. For the proof we use the property that the stable path separates the phase space into invariant regions.

Proof. Using the results of Proposition 1 we find that the line (stable path)

$$I_1 = \frac{\sqrt{a_0}}{\sqrt{a_1}} I_0$$

separates the positive quadrant of the I_0, I_1-space into two invariant regions. That is if

$$I_1(0) \lesseqqgtr \frac{\sqrt{a_0}}{\sqrt{a_1}} I_0(0) \quad \text{then} \quad I_1(t) \lesseqqgtr \frac{\sqrt{a_0}}{\sqrt{a_1}} I_0(t), \ t \geq 0.$$

All occurring values are positive therefore we can square the terms yielding

$$\frac{I_1(0)^2}{I_0(0)^2} \lesseqqgtr \frac{a_0}{a_1} \quad \text{then} \quad \frac{I_1(t)^2}{I_0(t)^2} \lesseqqgtr \frac{a_0}{a_1}, \ t \geq 0.$$

If the inequality is strict, then combatant 1 or 0, respectively, becomes zero in some finite time T. Therefore, the extinction time τ is finite and combatant 0 or 1, respectively, wins. If equality holds, the solution lies on the stable manifold and hence $I_j(t) > 0$, $j = 0, 1$ for all t. Therefore the extinction time τ is infinite and no combatant wins. This finishes the proof. □

Restricting the initial state space of Eq. (5) to the unit 1-simplex (Δ^1), i.e. $I_0(0) + I_1(0) = 1$ we can give a further characterization for the different regions of the winner, cf. Fig. 2. The regions in Δ^1, with combatant $i = 0, 1$ being the winner is denoted as W_i. The separating point $\omega_p \in \Delta^1$ is given by

$$\omega_p = \begin{pmatrix} 1 \\ \Gamma \end{pmatrix} \frac{1}{1+\Gamma}, \quad \text{with} \quad \Gamma := \sqrt{\frac{a_0}{a_1}}$$

and the winning regions are given by

$$W_0 = \left\{ I^0 \in \Delta^1 : \ \frac{1}{1+\Gamma} < I_0^0 \leq 1 \right\}$$
$$W_1 = \left\{ I^0 \in \Delta^1 : \ 0 \leq I_0^0 < \frac{1}{1+\Gamma} \right\}.$$

This simple characterization of the solution structure for the two-side Lanchester model relies on the geometric property that a line separates the plane. Since a line does not separate the three dimensional space we cannot expect such a simple characterization for the three-combatant Lanchester model. Anyhow, a careful inspection of the behavior of solution paths allows at least the formulation of implicit conditions for the characterization of the winning regions. This analysis will be carried out next.

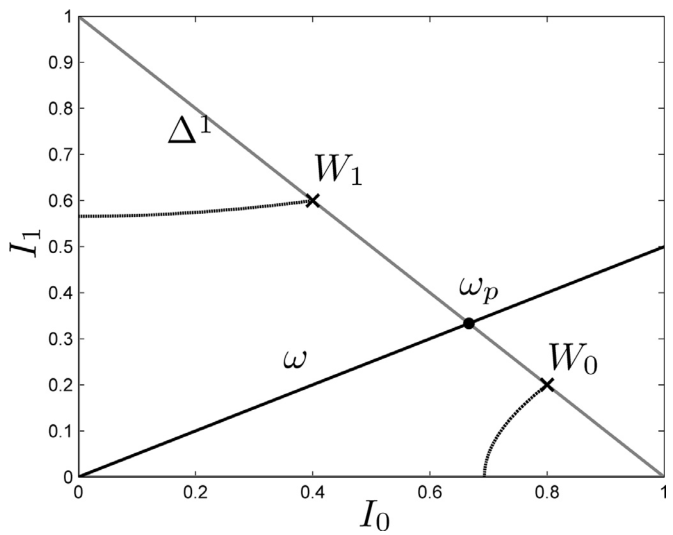


Fig. 2. The figure depicts a typical example of model (5). Two solutions (dashed black curves), starting at the 1-simplex Δ^1 are plotted. For the solution starting in W_1 side 1 is the winner and for the solution starting in W_0 combatant 0 is the winner. The total extinction line (black solid line) separates the entire phase space $I_0 \times I_1$. The total extinction point (black dot) splits the 1-simplex into the two winning regions W_0 and W_1.

2.4. Three combatants analysis

We start characterizing the structure of the eigenspaces of Eqs. (1a)–(1c).

Proposition 3. *Let*

$$J = \begin{pmatrix} 0 & -a_{01}y_{01} & -a_{02}y_{02} \\ -a_{10}y_{10} & 0 & -a_{12}y_{12} \\ -a_{20}y_{20} & -a_{21}y_{21} & 0 \end{pmatrix}. \tag{11}$$

be the Jacobian of the Eqs. (1a)–(1c).

Using the abbreviations

$$D(J) := -\det J = a_{10}\,y_{10}\,a_{20}\,y_{20}\,a_{21}\,y_{21} + a_{01}\,y_{01}\,a_{02}\,y_{02}\,a_{12}\,y_{12} > 0 \tag{12a}$$

$$\Sigma(J) := -(a_{12}\,y_{12}\,a_{21}\,y_{21} + a_{02}\,y_{02}\,a_{20}\,y_{20} + a_{01}\,y_{01}\,a_{10}\,y_{10}) < 0 \tag{12b}$$

$$\Delta(J) := \left(\frac{D(J)}{2}\right)^2 + \left(\frac{\Sigma(J)}{3}\right)^3 \tag{12c}$$

the eigenvalues $\xi_i,\ i = 0, 1, 2$ of J are given as

$$\xi_0 = \sigma_1 + \sigma_2 < 0 \tag{13a}$$

$$\xi_{1,2} = -\frac{\sigma_1 + \sigma_2}{2} \pm \frac{\sigma_1 - \sigma_2}{2}\sqrt{3}\mathrm{i}, \quad \operatorname{Re}\xi_{1,2} > 0 \tag{13b}$$

with

$$\sigma_{1,2} := \sqrt[3]{-\frac{D(J)}{2} \pm \sqrt{\Delta(J)}}. \tag{13c}$$

The eigenvector v_0 corresponding to the negative eigenvalue ξ_0 can be normalized such that

$$\sum_{j=1}^{3} v_{0,j} = 1, \quad v_0 = (v_{0,1}, v_{0,2}, v_{0,3})^\top \quad \text{and} \quad v_{0,j} > 0. \tag{14}$$

A solution $I(\cdot)$ of the Eqs. (1a)–(1c) is given by

$$I(t) = \exp(Jt)I(0), \quad t \geq 0 \tag{15}$$

For a detailed proof see Grass et al. (2016)

We already stated that a comparably simple characterization, like the Square Law, is not be possible for the three-side Lanchester model. In Remarks 1 and 2, we heuristically showed that crossing the total and iso-extinction curves changes the survivor/winner structure. These curves separate the initial state space into areas with different survivors and winners. See Fig. 3 where the various winning regions are shown.

From the arguments given in Remark 1 we see that crossing the total extinction curve (of the second kind) changes the winner of the model. Following the arguments in Remark 2 we find that crossing the iso-extinction curve changes the order of the survivors, while the winner stays the same. Thus, for the determination of the winner the total extinction curves are of more importance.

Let us now have a closer look at the extinction curves introduced in Definition 2. To avoid technicalities we restrict ourselves to an intuitive discussion. For mathematical details we refer to Grass et al. (2016).

Repeating the arguments of Remarks 1 and 2 we find the following procedure to determine the iso- and total-extinction curve (second kind).

A solution $I(\cdot)$ starting at the iso-extinction curve (γ), where two forces are annihilated at the same time, i.e. $I(0) \in \gamma \subset \Delta$ ends at one of the coordinate axes (e_i), (two sides become zero at the same time), i.e. $I(T) \in e_i, i \in \{0, 1, 2\}$.

A solution $I(\cdot)$ starting at the total extinction curve, i.e. $I(0) \in \omega \subset \Delta$ ends at the stable path (see Remark 3) of the second stage lying in one of the coordinate planes, i.e. $I(T) \in v_0^{(i)},\ i \in \{0, 1, 2\}$, where $v_0^{(i)}$ is the stable eigenvector of the second phase with survivors $j, k \neq i$.

In both cases the solution ends at a line going through the origin. Such a line can be written as kx with $k \geq 0$ and $x \in \mathbb{R}^3$. Taking into account that any solution $I(\cdot)$ of the 3-D Lanchester Eqs. (1a)–(1g) is given by $I(T) = \exp(JT)I(0)$, cf. Eq. (15), the corresponding equations are

$$\exp(JT)I(0) = kx, \quad \text{with} \quad k \geq 0,\ x \in \mathbb{R}^3,\ T \geq 0 \tag{16a}$$

$$I_0(0) + I_1(0) + I_2(0) = 1 \tag{16b}$$

This yields four equations in five unknown variables ($I_0(0)$, $I_1(0)$, $I_2(0)$, k, T). Using the implicit function theorem four of the variables can be written as a (differentiable) function of the fifth variable. With T as the free variable we find a unique differentiable curve

$$(c_0(T), c_1(T), c_2(T), k(T))^\top$$

that solves

$$\exp(JT)(c_0(T), c_1(T), c_2(T))^\top = k(T)x, \quad T \geq 0$$
$$c_0(T) + c_1(T) + c_2(T) = 1, \quad T \geq 0.$$

From the previous consideration it follows that we have six choices for the vector x that determine the iso- and total-extinction curves. These are the standard unit vectors (e_i) for the iso-extinction curve and the stable eigenvectors of the second stage $v_0^{(i)}$ for the total extinction curve. Thus we find six curves and a point in the initial state simplex. We identify the vector v_0 with the position vector and hence the point in the $\mathbb{R}^3$ space.

Total-extinction

point of the first kind $\omega^{(1)} = \{v_0\}$, stable eigenvector of the first stage.

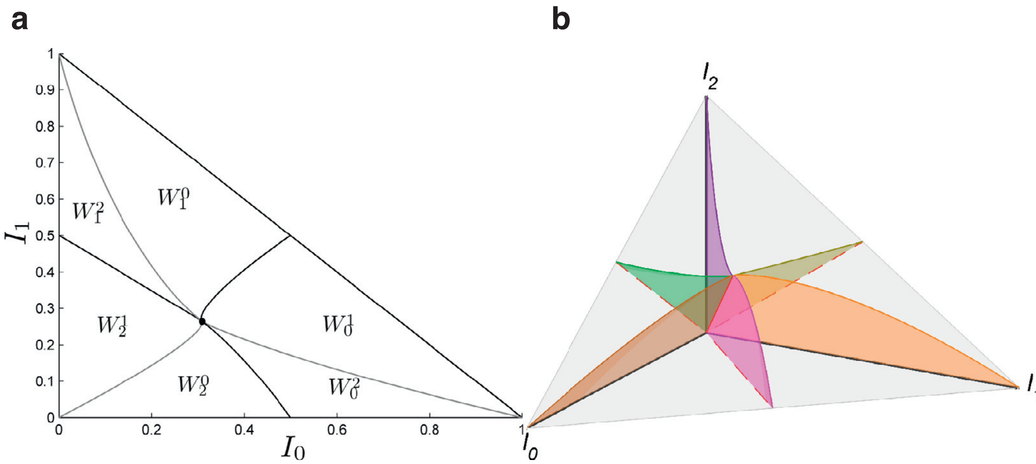


Fig. 3. In (a) the winning regions W_i, $i = 0, 1, 2$ are plotted. These regions are separated by the total-extinction curves (black). Each region is divided into two subregions W_i^j, $j \neq i$, where j is the second survivor of stage one. These are separated by the iso-extinction curves (gray). In (b) the surfaces corresponding to the total- and iso-extinction curves are plotted. These surfaces separate the phase space $I_0 \times I_1 \times I_2$ containing the solution paths for the different regions.

- curve of the second kind $\omega_i^{(2)}(T),\ T \geq 0$ satisfies Eq. (16) for $x = e_i,\ i = 0, 1, 2.$

Iso-extinction curve $\gamma_i(T), T \geq 0$ satisfies Eq. (16) for $x = v_0^{(i)},\ i = 0, 1, 2.$

From these definitions we find that the total extinction curve of the second kind is given by

$$\omega^{(2)} = \bigcup_{i=0}^{2} \{\omega_i^{(2)}(T) :\ T \geq 0\} \tag{17a}$$

and the iso-extinction curve is given by

$$\gamma = \bigcup_{i=0}^{2} \{\gamma_i(T) :\ T \geq 0\}. \tag{17b}$$

The extinction curves start at the boundary of the unit 2-simplex $(\partial\Delta)$

$$\omega_i^{(2)}(0) = v_0^{(i)} \quad \text{and} \quad \gamma_i(0) = e_i,\ i = 0, 1, 2. \tag{17c}$$

With increasing T the extinction curves converge to the total-extinction point of the first kind

$$\lim_{T\to\infty} \omega_i^{(2)}(T) = \lim_{T\to\infty} \gamma_i(T) = \omega^{(1)},\ i = 0, 1, 2. \tag{17d}$$

Thus in total these curves separate the initial state space (Δ) into six areas with different survivors and winners. Considering the solution paths for every initial point lying in the extinction curves we find surfaces that separates the phase space into six regions.

The winning regions W_i, $i = 0, 1, 2$ denote those areas, where combatant i is the winner of model Eq. (1). Taking also the survivors into account, the winning region W_i^j, $i = 0, 1, 2, j \neq i$ denotes those areas, where combatants i, j are survivors and combatant i is the winner of model Eq. (1). See Fig. 3(a).

In what follows we will illustrate these curves and surfaces geometrically. In particular, we will show how they help to solve the central question, namely which opponent will win the three-sided combat.

3. Discussion of the numerical solutions

Fig. 3 depicts an example for the parameter values $y_{10} = 0.3, y_{01} = 0.2$ and $y_{02} = 0.6$ in the first stage. The attrition rates a_{ij}, $i, j = 0, 1, 2,\ i \neq j$, are assumed to be one. Together with the complementary values y_{20}, y_{21} and y_{12} the rates sum up to one, meaning that combatant 0 fights with 30% of his strength against opponent 1 and with 70% of his strength against opponent 2, and so forth. The magenta, green and olive surfaces are the total extinction surfaces, and the violet, brown and orange areas are the iso-extinction surfaces. The corresponding curves illustrate the boundaries of the corresponding surface.

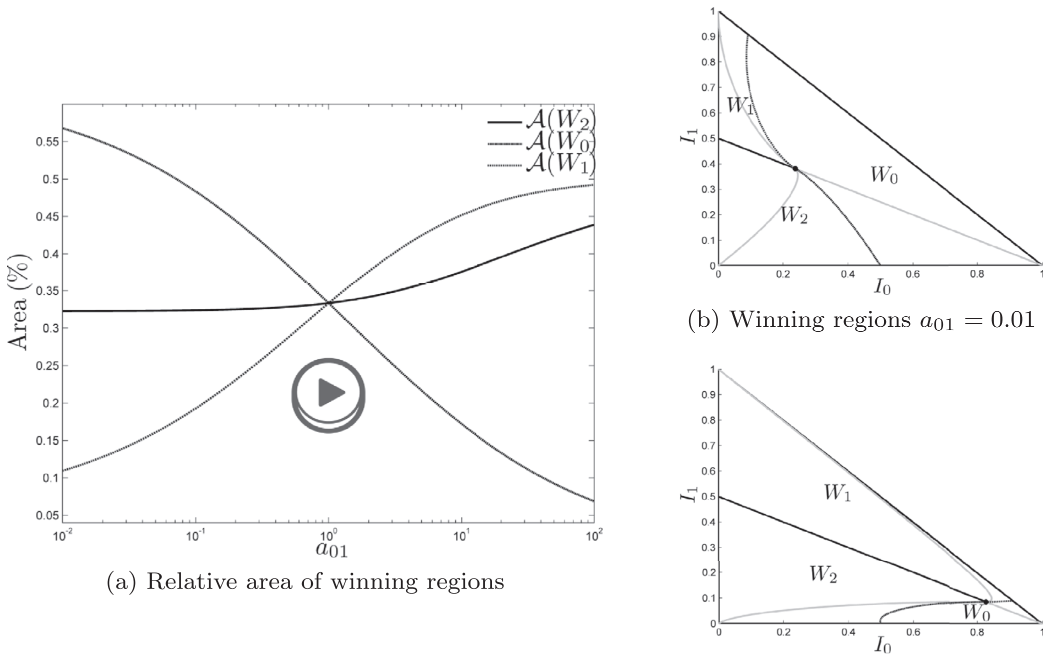
As previously explained six different areas can be calculated, which differ in the winner and/or in the opponent who loses first, see panel Fig. 3(a). Not surprisingly, when the relative size of force I_i, $i = 0, 1, 2,$ is large, this combatant will come off as winner of the battle. If the initial relative size of combatant j, $j = 0, 1, 2, j \neq i$ is large compared to opponent $3 - i - j$, then combatant j survives the first stage, but is eliminated in the second.

Fig. 4 provides a sensitivity analysis with respect to parameter a_{01} for the symmetric case where $y_{ij} = 0.5,\ \ i, j = 0, 1, 2,$ which is the attrition rate when combatant 1 engages opponent 0 (see Eq. (1a)) in the interval [0.01 100]. In the left panel (a) the area (in relative size) for the three winning regions corresponding to the various values of a_{01} is plotted. The figures on the right (b) and (c) show the winning regions for the cases $a_{01} = 0.1$ and $a_{01} = 100$. Obviously, the chances for combatant 0 to come off as winner are much larger if the intensity of the attacks from opponent 1 is relatively low, while the chances for combatant 1 to win are bigger when it is able to cause more damage to opponent 0. But not only combatant 1 profits from a high attack rate, Fig. 4 also clearly shows the extent to which combatant 2 benefits if opponent 1 starts shooting more intensely at combatant 0. When a_{01} increases from 0.01 to 1 the main effect is that opponent 1 increases its chances to win at the cost of opponent 0's chances. But when a_{01} increases further from 1 to 100, then combatant 2 gains almost as much as does opponent 1.

Fig. 5 provides a sensitivity analysis for the parameter y_{10}. Suppose that there are particularly strong animosities between opponents 1 and 2 so that $y_{12} = y_{21} = 0.9$ and $y_{02} = y_{01} = 0.1$. We assume that all combatants are of the same strength, i.e. ($a_{ij} = 1$, i,j=0,1,2), but combatant 0 is assumed to have flexibility over the choice of y_{10} vs. y_{20}. We can distinguish now several scenarios related to the initial state values considering a range of values for $y_{10} \in [0, 1]$ (and, hence y_{20}). For the subsequent description cf. Fig. 5(a).

Region W_i Combatant $i = 0, 1, 2$ always wins, no matter how opponent 0 allocates his forces.

Region I Combatant 0 can win, but only if the forces are allocated accordingly, i.e. the stronger opponent must be primarily fought.

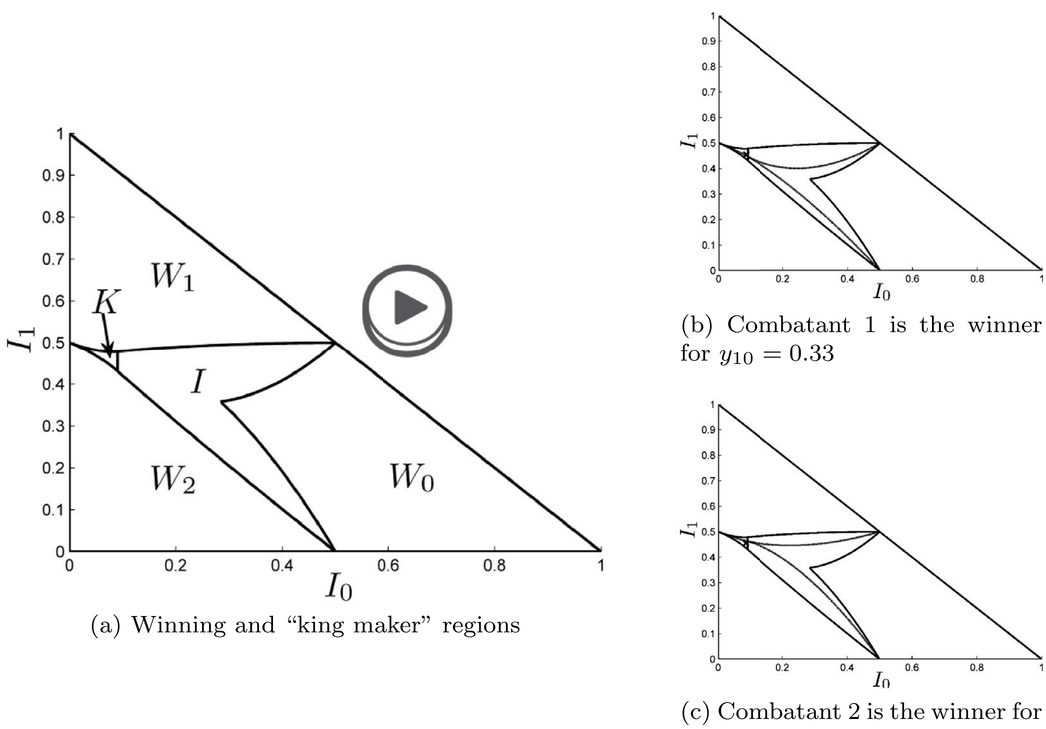


(a) Relative area of winning regions

(b) Winning regions $a_{01} = 0.01$

(c) Winning regions $a_{01} = 100$

Fig. 4. This figure shows a sensitivity analysis carried out for the parameter a_{01} in the interval [0.01, 100].



(a) Winning and "king maker" regions

(b) Combatant 1 is the winner for $y_{10} = 0.33$

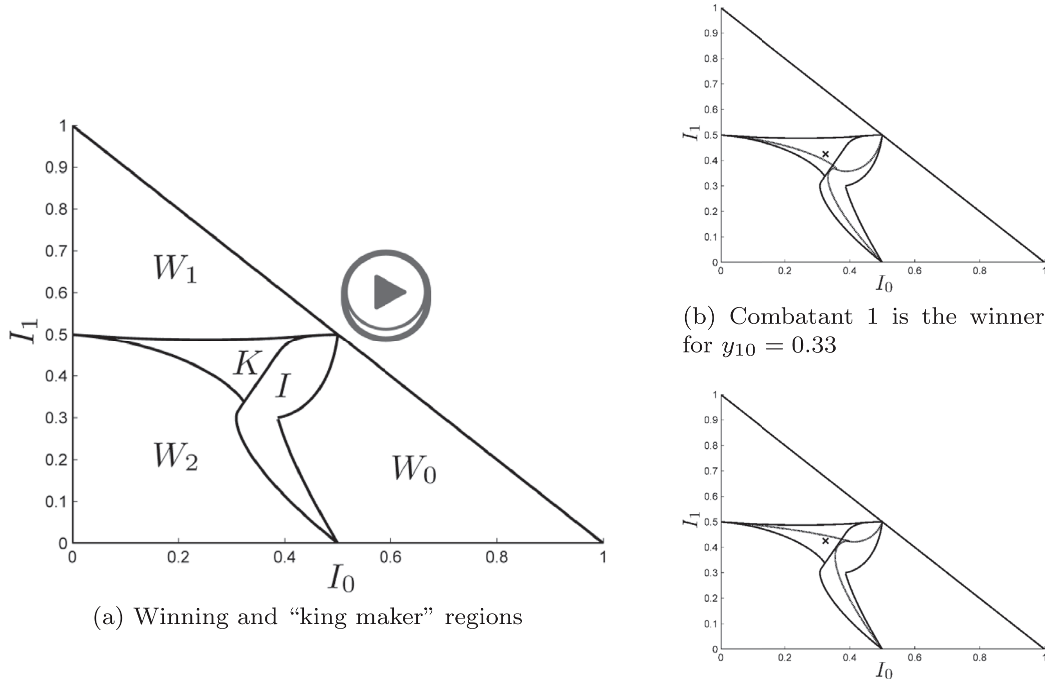
(c) Combatant 2 is the winner for $y_{10} = 0.66$

Fig. 5. This figure shows a sensitivity analysis carried out for the parameter y_{10} in the interval [0, 1], the allocation of combatant 0 forces for opponent 1. Combatants 1 and 2 fight each other with 90% of their forces, i.e. $y_{21} = y_{12} = 0.9$ (symmetric hate).

Region *K* Combatant 0 can be the "king maker" even though its forces are not able to win. If combatant 0 allocates enough of the forces against opponent 2, combatant 1 wins (Fig. 5(b)), otherwise opponent 2 wins (Fig. 5(c)).

In this scenario it is assumed that animosities between opponents 1 and 2 are so strong, that they basically ignore that Combatant 0 can have a substantial influence on the outcome of the conflict no matter whether combatant 0 is able to win the conflict or not.

Assume now that combatant 2 sees opponent 1 as his main threat ($y_{12} = 0.9$), while combatant 1 thinks of combatant 0 as his archenemy ($y_{02} = 0.9$). Here we can analyze how combatant 0 should allocate his forces to be able to win the conflict. Again we



(a) Winning and "king maker" regions

(b) Combatant 1 is the winner for $y_{10} = 0.33$

(c) Combatant 2 is the winner for $y_{10} = 0.66$

Fig. 6. This figure shows a sensitivity analysis carried out for the parameter y_{10} in the interval [0, 1], the allocation of combatant 0 forces for opponent 1. Combatant 2 fights opponent 1 with 90% of its forces; and combatant 1 fights opponent 0 with 90% of its forces ((in)transitive hate).

are able to distinguish the regions described above, see Fig. 6. Due to the severe attacks by opponent 1, the region where combatant 0 can win is significantly smaller than before, however, the region where this combatant can be "king maker" increases.

It is also noteworthy that the region where opponent 2 always wins is larger than the region in the scenario above, where opponent 2 focuses on combatant 1. Thus, it is evident that also opponent 1 and 2 could eventually be better off by a closer consideration regarding which opponent is more dangerous. To wisely choose the appropriate strategy, however, the opponents need information; information about their opponents strength, and information about their opponents strategy. Yet, this information might not be easily accessible or deducible. To analyze the impact of information with respect to strategic interactions, one can use (differential) game theory, but this goes beyond the scope of the present paper.

4. Conclusion

Lanchester's classic models describe duels where two opponents shoot at each other with the goal of annihilating the opponent. While Lanchester's ODE models have never been extended to more than two players, duels have been generalized to (so-called) truels already around the middle of the last century; see Shubik (1987) and Kilgour and Brams (1997) for an introduction and a survey of the issue. Essentially, classical truels have a discrete time structure and include hitting probabilities.

Similar to truels, the purpose of the present paper is to model a three-sided combat where the essential question is which combatant (if any) will be able to win the combat in the sense of being the only survivor. Unlike a two-sided combat, each party has to decide how to allocate its forces between the two opponents. We restrict ourselves to a purely descriptive analysis. While the model is certainly no tool to predict the outcome of any real armed conflict, it may help to better understand the implications of allocation choices in three-sided combats.

In the present paper it has been shown how the Square Law of the two-dimensional Lanchester model can be extended to three dimensions. While the three-dimensional model is significantly more complex than the two-dimensional one, a complete analytical solution of the problem is still possible. While in the 2-D case the stable eigenvector provides the separatrix between the terminal states, in the present case some surfaces take over such a role. We were able to locate areas in the state space which differ in the winner of the combat and areas which differ in which opponent loses first.

We illustrated how the strength and the allocation choices affect the winner of a combat by means of a sensitivity analysis. We saw that in a three-sided combat, it is not always a disadvantage if one of the opponents gains strength, it just depends on which of his opponents this additional strength is mostly directed. In armed conflicts with a strong animosity between two of the parties, a third party might – under certain conditions – be able to take advantage of the situation and determine the outcome of the combat by its force allocation.

In this paper, we showed that the Quadratic Law is based on the geometric fact that the winning regions for the two combatant model are determined by the linear stable manifold which we call a total extinction point/curve. When there are just two combatants, that total extinction rule determines the points where neither of the two combatants wins because they both go extinct at the same moment.

For the three combatant model the total extinction rule generalizes to distinguish initial points where (1) none of the three combatants wins because all go extinct simultaneously, (2) one of the three combatants goes extinct first, but neither of the remaining two combatants wins because they then subsequently go extinct simultaneously. This yields nonlinear curves/surfaces separating the state space into different winning regions as when there were only two combatants at the outset. Unfortunately these curves are only implicitly given and do not follow a “simple” law such as the Quadratic Law.

For a four or general N combatant model the total extinction manifolds separate the regions of different winners. And even though the corresponding implicit equations become more involved no further fundamental complexity is introduced in the sense that a new rule, beside the total extinction rule, has to be considered. For the four combatant model the rule of total extinction is recursively given: (1) none of the four combatants wins because all go extinct simultaneously, (2) one of the four combatants goes extinct first and none of the remaining three combatants wins because one of the two conditions just discussed for the three combatant case pertains and (3) two of the four combatants go extinct simultaneously but before the other two and neither of the remaining two combatants wins because the extinction rule when there are just two original combatants applies.

The extension to an N combatant model is as follows. Whatever number $n < N$ of those combatants goes extinct first and at the same moment, one then just applies the rule when starting with $N - n$ combatants.

There are many further possibilities to extend the model. One could consider the linear Lanchester model with three combatants. Note, however, that in this case it is more difficult to derive results analytically.

Here it was assumed that the opponents have to allocate all of their troops between the opponents. If the engagement of troops is costly it might make sense only to use a certain fraction of the troops for combat.

A further possible extension would be to consider non-constant – possibly optimally determined – attrition rates to get a better understanding about the effects of combatants adapting their strategy over time.

The presented model is only a first step to understand the impact of force allocation in a three-opponent combat. The next step to understand optimal strategies in a combat with three opponents would be to consider allocation which depend on the size of the state variables. This would capture a situation where the opponents adjust their allocation strategy by means of a feedback rule to prevent any of the opponents to become too dominant. The obvious extension then would be to consider the allocation rate as a control variable and determine when it is optimal to attack each opponent. The possibility of a temporary cooperation would lead to many challenges in a differential game setup.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.ejor.2017.07.026.

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