Fixed Point Iterations

Numerical Methods for Dynamical Systems

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Fixed point of a function

Definition

x is a fixed point of a function f if x belongs to both the domain and the codomain of f, and $f(x) = x$.

Example

Let f be defined on the real numbers by

$$f(x) = x^2 - 3x + 4,$$

then 2 is a fixed point of f, because $f(2) = 2$.

Example

A function g given by

$$g(x) = x - 1$$

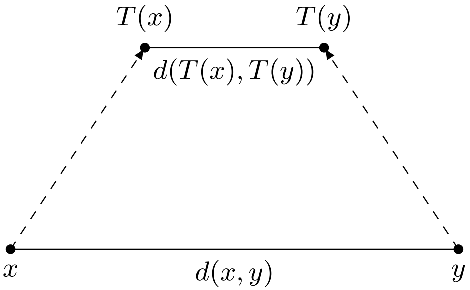
has no fixed points because $x - 1$ is never equal to x for any real number.

Contraction mapping

Definition

Let (X, d) be a metric space. A map $T : X \to X$ is called a **contraction mapping** on X if there exists $L \in [0, 1)$ such that, for all $x, y \in X$, it satisfies

$$d(T(x), T(y)) \leq L \cdot d(x, y)$$



Example (Simple Classic Contraction)

Show that the following map is a contraction:

$$T : \mathbb{R} \to \mathbb{R}, \quad T(x) = \tfrac{1}{2}x.$$

Consider $x, y \in \mathbb{R}$. The usual metric on $\mathbb{R}$ is given by the absolute value of the difference, i.e., $d(x, y) = |x - y|$. Then,

$$d(T(x), T(y)) \;=\; |T(x) - T(y)| \;=\; \left|\tfrac{1}{2}x - \tfrac{1}{2}y\right| \;=\; \tfrac{1}{2}\,|x - y|.$$

Hence, there exists $L = \frac{1}{2} < 1$ such that

$$d(T(x), T(y)) \;\le\; L \cdot |x - y|.$$

Therefore, T is a contraction on $\mathbb{R}$.

Existence and convergence theorems

Theorem (Banach fixed point theorem)

Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point $x^* \in X$, i.e., $T(x^*) = x^*$.

Theorem (Picard Iteration)

If $T : X \to X$ is a contraction on a non-empty complete metric space (X, d), then for any initial point $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = T(x_n)$$

converges to a unique fixed point x^* of T. In other words,

$$\lim_{n\to\infty} x_n = x^*.$$

Example

Find a fixed point of

$$T : \mathbb{R} \to \mathbb{R}, \quad T(x) = \frac{x+2}{2}.$$

T is a contraction on $\mathbb{R}$ since $|T(x) - T(y)| \leq \frac{1}{2}|x - y|$.

Choose an initial seed $x_0 = 0$, and define $x_{n+1} = T(x_n)$. Let's compute the iterates:

$$x_1 = \frac{x_0 + 2}{2} = \frac{0+2}{2} = 1.$$

$$x_2 = \frac{x_1 + 2}{2} = \frac{1+2}{2} = 1.5.$$

$$x_3 = \frac{x_2 + 2}{2} = \frac{1.5+2}{2} = 1.75.$$

$$\vdots$$

n	x_n	x_{n+1}
0	0.00000	1.00000
1	1.00000	1.50000
2	1.50000	1.75000
$\vdots$	$\vdots$	$\vdots$
17	1.99998	1.99999
18	1.99999	2.00000

Example

Find a fixed point of

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = \left(\tfrac{1}{2}x + \tfrac{1}{2},\ \tfrac{1}{2}y + \tfrac{1}{2}\right).$$

Use the standard Euclidean metric on $\mathbb{R}^2$,

$$d\big((x_1, y_1), (x_2, y_2)\big) \;=\; \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Consider

$$\begin{aligned} d(T(x_1, y_1), T(x_2, y_2)) &= \sqrt{\Big(\tfrac{1}{2}(x_1 - x_2)\Big)^2 + \Big(\tfrac{1}{2}(y_1 - y_2)\Big)^2} \\ &= \sqrt{\tfrac{1}{4}\Big(\big((x_1 - x_2)\big)^2 + (y_1 - y_2)\Big)^2} \\ &= \tfrac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \tfrac{1}{2}\, d\big((x_1, y_1), (x_2, y_2)\big). \end{aligned}$$

So, T is a contraction.

Let's illustrate with a simple initial point, say $(x_0, y_0) = (0, 0)$.

Then:

$$(x_1, y_1) = T(0, 0) = (0.5,\ 0.5).$$

$$(x_2, y_2) = T(0.5, 0.5) = (0.5 \times 0.5 + 0.5,\ 0.5 \times 0.5 + 0.5) = (0.75,\ 0.75).$$

$$(x_3, y_3) = T(0.75, 0.75) = (0.5 \times 0.75 + 0.5,\ 0.5 \times 0.75 + 0.5) = (0.875,\ 0.875).$$

$$\vdots$$

We see the sequence $\{(x_n, y_n)\}$ is "closing in" on $(1, 1)$. Indeed,

$$\lim_{n\to\infty} (x_n, y_n) = (1, 1).$$

Theorem

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$, and differentiable on the open interval (a,b). Then there exists some $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

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