Higher-Order Taylor Method and its Applications

Numerical Methods for Dynamical Systems

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Higher-Order Taylor Method

Higher-order Taylor methods are numerical methods used to approximate solutions to an Initial Value Problem (IVP) by extending the idea of Euler's method to higher-order derivatives.

Initial Value Problem (IVP)

$$\dot{x}(t) = f\big(t, x(t)\big), \quad x(t_0) = x_0.$$

Key Idea: Since Euler's method was derived by using Taylor's Theorem with $n = 1$ to approximate the solution of a differential equation, we can improve the accuracy of the approximation by considering higher-order terms of the Taylor series expansion.

Higher-Order Taylor Method

Consider the n^{th} Taylor polynomial about t. Evaluating it at $t+h$, we obtain

$$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \cdots + \frac{h^n}{n!}x^{(n)}(t) + \frac{h^{n+1}}{(n+1)!}x^{(n+1)}(\xi),$$

for some ξ in $(t, t+h)$.

Since $x'(t) = f(t, x(t))$, $\quad x''(t) = f'(t, x(t))$, and, generally,

$$x^{(k)}(t) = f^{(k-1)}(t, x(t)).$$

Substituting these results into the Taylor series expansion, we obtain

$$x(t+h) = x(t) + hf(t, x(t)) + \frac{h^2}{2} f'(t, x(t)) + \dots$$
$$+ \frac{h^n}{n!} f^{(n-1)}(t, x(t)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi),$$

for some $\xi \in (t, t+h)$.

The difference-equation method corresponding to this equation is obtained by deleting the remainder term involving ξ.

Taylor method of order p:

$$
\begin{aligned}
x(t_0) &= x_0, \\
x(t+h) &= x(t) + hT^{(p)}(t, x(t)),
\end{aligned}
$$

for each $t = t_0, t_1, \ldots, t_{n-1}$, where

$$T^{(p)}(t, x(t)) = f(t, x(t)) + \frac{h}{2} f'(t, x(t)) + \cdots + \frac{h^{p-1}}{p!} f^{(p-1)}(t, x(t)).$$

Remark: Euler's method is Taylor's method of order one.

Higher-Order Taylor Method

Step 1: Truncate to Order p

$$x(t+h) = x(t) + hf(t,x(t)) + \frac{h^2}{2}f'(t,x(t)) + \frac{h^2}{3!}f''(t,x(t))\ldots$$

We can choose to truncate the Taylor series at a specific order p.
For example, if we truncate at $p=2$:

$$x(t+h) \;\approx\; x(t) + hf(t,x(t)) + \frac{h^2}{2}f'(t,x(t))$$

Remark:

- ✎ The truncation error depends on the order p of the Taylor series used.
- ✎ Higher p results in a lower truncation error.

Step 2: Discretize the Time Domain

- ✎ Define $t_n = t_0 + nh$, where h is the step size.
- ✎ Let $x_n \approx x(t_n)$.

Step 3: Derive the Update Formula

Applying the approximation:

$$\textbf{(Second-Order):} \quad x_{n+1} \;=\; x_n + h\, f\big(t_n, x_n\big) + \frac{h^2}{2}\, f'\big(t_n, x_n\big).$$

Second-Order Taylor Algorithmic Implementation

1. **Initialize:** x_0, t_0, h.
2. **Loop:** $x_{n+1} = x_n + h\, f(t_n, x_n) + \frac{h^2}{2}\, f'(t_n, x_n), \quad t_{n+1} = t_n + h.$
3. **Stop:** when t_n reaches (or exceeds) final time T.

Higher-Order Taylor Method

Local Truncation Error (LTE):

$$x(t+h) - \left(x(t) + h\, f(t,x) + \frac{h^2}{2}(f_t + f_x f) + \cdots + \frac{h^p}{p!} x^{(p)}(t) \right) = \mathcal{O}(h^{p+1}).$$

Global Error:

- ✎ Over n steps, the total (global) error is $\mathcal{O}(h^p)$.
- ✎ This is why the **Taylor method of order** p is called a p*-th order* method.

Summary:

- ✎ The **Higher-Order Taylor Method** improves accuracy by including more terms from the Taylor series.
- ✎ Increasing the order p significantly reduces global error.
- ✎ However, higher-order methods require computing higher derivatives of $f(t,x)$, which increases complexity.

Example

Solve numerically using Higher-Order Taylor's method:

$$\dot{x} = x - t^2 + 1, \quad x(0) = 0.5, \quad h = 0.2.$$

Solution: Using Apply Taylor's method of orders two ($p = 2$):

$$x_{n+1} = x_n + hf(t_n, x_n) + \frac{h^2}{2} f'(t_n, x_n).$$

we need the first derivative of $f(t, x) = x - t^2 + 1$ with respect to the variable t.

$$\begin{aligned} f'(t,x) &= \frac{d}{dt}(x - t^2 + 1) = x' - 2t \\ &= x - t^2 + 1 - 2t \\ &= x - t^2 - 2t + 1 \end{aligned}$$

Example

Solve numerically using Higher-Order Taylor's method:

$$\dot{x} = x - t^2 + 1, \quad x(0) = 0.5, \quad h = 0.2.$$

So,

$$\begin{aligned} x_{n+1} &= x_n + h\, f(t_n, x_n) + \frac{h^2}{2}\, f'(t_n, x_n) \\ &= x_n + h\big(x_n - t_n^2 + 1\big) + \frac{h^2}{2}\big(x_n - t_n^2 - 2t_n + 1\big) \end{aligned}$$

Starting at $x_0 = 0.5$ and $t_0 = 0$:

$$\begin{aligned} x_1 &= 0.5 + (0.2)\big(0.5 - (0)^2 + 1\big) \\ &\quad + \frac{(0.2)^2}{2}\big((0.5) - (0)^2 - 2(0) + 1\big) \\ &= 0.83. \end{aligned}$$

Example

Solve numerically using Higher-Order Taylor's method:

$$\dot{x} = x - t^2 + 1, \quad x(0) = 0.5, \quad h = 0.2.$$

$$\begin{aligned} x_2 &= 0.83 + (0.2)\big(0.83 - (0.2)^2 + 1\big) \\ &\quad + \frac{(0.2)^2}{2}\big((0.83) - (0.2)^2 - 2(0.2) + 1\big) \\ &= 1.2158. \end{aligned}$$

Compare with exact solution: $y = 2.5e^t - (t^2 + 2t + 2)$

Example

Solve numerically using Higher-Order Taylor's method:

$$\dot{x} = x - t^2 + 1, \quad x(0) = 0.5, \quad h = 0.2.$$

Solution: Using Apply Taylor's method of orders two ($p = 4$):

$$x_{n+1} = x_n + hf(t_n, x_n) + \frac{h^2}{2} f'(t_n, x_n) + \frac{h^3}{6} f''(t_n, x_n) + \frac{h^4}{24} f'''(t_n, x_n).$$

we need the first three derivatives of $f(t, x) = x - t^2 + 1$ with respect to the variable t.

$$\begin{aligned} f'(t, x) &= x - t^2 - 2t \\ f''(t, x) &= \frac{d}{dt}(x - t^2 - 2t + 1) \\ &= x' - 2t - 2 \\ &= x - t^2 + 1 - 2t - 2 \\ &= x - t^2 - 2t - 1. \end{aligned}$$

Example

Solve numerically using Higher-Order Taylor's method:

$$\dot{x} = x - t^2 + 1, \quad x(0) = 0.5, \quad h = 0.2.$$

$$\begin{aligned} f'''(t,x) &= \frac{d}{dt}(x - t^2 - 2t - 1) \\ &= x' - 2t - 2 \\ &= x - t^2 - 2t - 1 \end{aligned}$$

So,

$$\begin{aligned} x_{n+1} &= x_n + h\, f(t_n, x_n) + \frac{h^2}{2}\, f'(t_n, x_n) + \frac{h^3}{6}\, f''(t_n, x_n) + \frac{h^4}{24}\, f'''(t_n, x_n) \\ &= x_n + h\big(x_n - t_n^2 + 1\big) + \frac{h^2}{2}\big(x_n - t_n^2 - 2t_n + 1\big) \\ &\quad + \frac{h^3}{6}\big(x_n - t_n^2 - 2t_n - 1\big) + \frac{h^4}{24}\big(x_n - t_n^2 - 2t_n - 1\big). \end{aligned}$$

Example

Solve numerically using Higher-Order Taylor's method:

$$\dot{x} = x - t^2 + 1, \quad x(0) = 0.5, \quad h = 0.2.$$

Starting at $x_0 = 0.5$ and $t_0 = 0$:

$$\begin{aligned} x_1 &= 0.5 + (0.2)\big(0.5 - (0)^2 + 1\big) \\ &\quad + \frac{(0.2)^2}{2}\big((0.5) - (0)^2 - 2(0) + 1\big) \\ &\quad + \frac{(0.2)^3}{6}\big((0.5) - (0)^2 - 2(0) - 1\big) \\ &\quad + \frac{(0.2)^4}{24}\big((0.5) - (0)^2 - 2(0) - 1\big) \\ &= 0.8293. \end{aligned}$$

Example

Solve numerically using Higher-Order Taylor's method:

$$\dot{x} = x - t^2 + 1, \quad x(0) = 0.5, \quad h = 0.2.$$

Starting at $x_0 = 0.5$:

$$\begin{aligned} x_2 &= 0.8293 + (0.2)\big(0.8293 - (0.2)^2 + 1\big) \\ &\quad + \frac{(0.2)^2}{2}\big((0.8293) - (0.2)^2 - 2(0.2) + 1\big) \\ &\quad + \frac{(0.2)^3}{6}\big((0.8293) - (0.2)^2 - 2(0.2) - 1\big) \\ &\quad + \frac{(0.2)^4}{24}\big((0.8293) - (0.2)^2 - 2(0.2) - 1\big) \\ &= 1.214091. \end{aligned}$$

Compare with exact solution: $y = 2.5e^t - (t^2 + 2t + 2)$.