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Persistence and global stability in discrete models of Lotka–Volterra type

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Abstract

In this paper, we establish new sufficient conditions for global asymptotic stability of the positive equilibrium in the following discrete models of Lotka–Volterra type:

$$\begin{cases} N_i(p+1) = N_i(p)\exp\left\{c_i - a_i N_i(p) - \sum_{j=1}^{n} a_{ij} N_j(p - k_{ij})\right\}, & p \geqslant 0,\ 1 \leqslant i \leqslant n, \\ N_i(p) = N_{ip} \geqslant 0, \quad p \leqslant 0, \quad \text{and} \quad N_{i0} > 0, \quad 1 \leqslant i \leqslant n, \end{cases}$$

where each N_{ip} for $p \leqslant 0$, each c_i, a_i and a_{ij} are finite and

$$\begin{cases} a_i > 0, \quad a_i + a_{ii} > 0, \quad 1 \leqslant i \leqslant n, \quad \text{and} \\ k_{ij} \geqslant 0, \quad 1 \leqslant i, j \leqslant n. \end{cases}$$

Applying the former results [Y. Muroya, Persistence and global stability for discrete models of nonautonomous Lotka–Volterra type, J. Math. Anal. Appl. 273 (2002) 492–511] on sufficient conditions for the persistence of nonautonomous discrete Lotka–Volterra systems, we first obtain conditions for the persistence of the above autonomous system, and extending a similar technique to use a nonnegative Lyapunov-like function offered by Y. Saito, T. Hara and W. Ma [Y. Saito, T. Hara, W. Ma, Necessary and sufficient conditions for permanence and global stability of a Lotka–Volterra system with two delays, J. Math. Anal. Appl. 236 (1999) 534–556] for $n = 2$ to the above system for $n \geqslant 2$, we establish new conditions for global asymptotic stability of the positive equilibrium. In some special cases that $k_{ij} = k_{jj}$, $1 \leqslant i, j \leqslant n$, and $\sum_{j=1}^{n} a_{ji} a_{jk} = 0$, $i \neq k$, these conditions become $a_i > \sqrt{\sum_{j=1}^{n} a_{ji}^2}$, $1 \leqslant i \leqslant n$, and improve the well-

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known stability conditions $a_i > \sum_{j=1}^{n} |a_{ji}|$, $1 \leqslant i \leqslant n$, obtained by K. Gopalsamy [K. Gopalsamy, Global asymptotic stability in Volterra's population systems, J. Math. Biol. 19 (1984) 157–168].

Keywords: Persistence; Global asymptotic stability; Discrete model of autonomous Lotka–Volterra type

1. Introduction

Consider the persistence and global asymptotic stability of the following discrete models of Lotka–Volterra type:

$$\begin{cases} N_i(p+1) = N_i(p)\exp\left\{c_i - a_i N_i(p) - \displaystyle\sum_{j=1}^{n} a_{ij} N_j(p-k_{ij})\right\}, \quad p \geqslant 0, \\ N_i(p) = N_{ip} \geqslant 0, \quad p \leqslant 0, \quad \text{and} \quad N_{i0} > 0, \quad 1 \leqslant i \leqslant n, \end{cases} \tag{1.1}$$

where each N_{ip} for $p \leqslant 0$, each c_i, a_i and a_{ij} are finite and

$$\begin{cases} a_i > 0, \quad a_i + a_{ii} > 0, \quad 1 \leqslant i \leqslant n, \quad \text{and} \\ k_{ij} \geqslant 0, \quad 1 \leqslant i, j \leqslant n. \end{cases} \tag{1.2}$$

Recently, making the best use of the symmetry of the system and an extended La Salle's invariance principle, Saito, Hara and Ma [9] has shown necessary and sufficient conditions for permanence and global stability of a symmetrical Lotka–Volterra type predator–prey system with two delays. This improves the well-known sufficient condition on the global asymptotic stability of the positive equilibrium in the system obtained by Gopalsamy [4]. Saito [8] also established the necessary and sufficient condition for global stability of a Lotka–Volterra cooperative or competition system with delays for two species. On the other hand, Xu and Chen [10] has offer new techniques to obtain sufficient conditions of the persistence and global stability for a time-dependent pure-delay-type Lotka–Volterra predator–prey model for three species. On the other hand, Muroya [5,6] established conditions for the persistence and global stability of delay differential system and discrete system for n species, respectively, which are some extensions of the averaged condition offered by Ahmad and Lazer [1,2].

In this paper, applying Lemma 2.2 and Theorem 1.2 in Muroya [6] on sufficient conditions for the persistence of nonautonomous discrete Lotka–Volterra systems to the discrete system (1.1)–(1.2), we first obtain conditions for the persistence of the above autonomous system, and extending a similar technique to use a nonnegative Lyapunov-like function offered by Saito, Hara and Ma [9] for $n = 2$ to the above system for $n \geqslant 2$, we establish new conditions for global asymptotic stability of the positive equilibrium. This is a discrete version of Muroya [7]. In some special cases, these conditions improve the well-known stability result obtained by Gopalsamy [4].

Put

$$a_{ij}^{+} = \max(a_{ij}, 0), \qquad a_{ij}^{-} = \min(a_{ij}, 0), \tag{1.3}$$

and

$$\begin{cases} A_0 = \operatorname{diag}(a_1, a_2, \ldots, a_n), \quad B^{-} = \left[a_{ij}^{-}\right], \quad B^{+} = \left[a_{ij}^{+}\right] \quad \text{and} \\ D^{+} = \operatorname{diag}\left(a_{11}^{+}, a_{22}^{+}, \ldots, a_{nn}^{+}\right) \text{ are } n \times n \text{ matrices,} \quad \text{and} \\ \boldsymbol{c} = [c_i] \text{ is an } n\text{-dimensional vector,} \end{cases} \tag{1.4}$$

and assume that

$$\begin{cases} A_0 + D^+ + B^- \text{ is an } M\text{-matrix}, \quad (A_0 + D^+ + B^-)^{-1}\boldsymbol{c} > \boldsymbol{0} \quad \text{and} \\ \boldsymbol{c} > (B^+ - D^+)(A_0 + D^+ + B^-)^{-1}\boldsymbol{c}, \end{cases} \tag{1.5}$$

where a real $n \times n$ matrix $A = [a_{ij}]$ with $a_{ij} \leqslant 0$ for all $i \neq j$ is called an *M-matrix* if A is nonsingular and $A^{-1} \geqslant \boldsymbol{0}$ (see, for example, Berman and Plemmons [3]).

Applying Lemma 2.2 and Theorem 2.2 in Muroya [6] on the sufficient conditions of the persistence of nonautonomous discrete Lotka–Volterra systems to the system (1.1)–(1.2), we first obtain the following theorem.

Theorem 1.1. (See Muroya [6].) *For the system* (1.1)–(1.2)*, if the condition* (1.5) *is satisfied, then all solutions $N_i(p)$, $1 \leqslant i \leqslant n$, of the system are positive and the system is persistent, that is,*

$$0 < \liminf_{p \geqslant 0} N_i(p) \leqslant \limsup_{p \geqslant 0} N_i(p) < +\infty, \quad 1 \leqslant i \leqslant n. \tag{1.6}$$

In particular, all solutions $N_i(p)$, $1 \leqslant i \leqslant n$, of the system are bounded above, that is,

$$\limsup_{p \to \infty} N_i(p) \leqslant \bar{N}_i, \quad 1 \leqslant i \leqslant n, \tag{1.7}$$

where $\bar{N}_i$, $1 \leqslant i \leqslant n$, are defined by

$$\tilde{c}_i = c_i - \sum_{j=1}^{i-1} a_{ij}^- \bar{N}_j, \quad \tilde{N}_i = \tilde{c}_i / a_i, \quad \bar{N}_i = \begin{cases} \tilde{c}_i / a_i, & \tilde{c}_i \leqslant 1, \\ e^{\tilde{c}_i - 1} / a_i, & \tilde{c}_i > 1. \end{cases} \tag{1.8}$$

By Theorem 1.1 and extending a similar technique to use a nonnegative Lyapunov-like function offered by Saito, Hara and Ma [9] for $n = 2$ to the above system for $n \geqslant 2$, we get the following results.

Theorem 1.2. *For the system* (1.1)–(1.2)*, in addition to* (1.5) *and* (1.7)*, assume*

$$\tilde{c}_i < 1, \quad 1 \leqslant i \leqslant n, \tag{1.9}$$

and suppose that there exists a positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ and

$$a_i > \sqrt{\sum_{j=1}^n |a_{ji}| \left(\sum_{k=1}^n |a_{jk}| \right)}, \quad 1 \leqslant i \leqslant n. \tag{1.10}$$

Then, the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of (1.1) *is globally asymptotically stable for any $k_{ij} \geqslant 0$, $1 \leqslant i, j \leqslant n$.*

In particular, if

$$k_{ij} = k_{jj}, \quad 1 \leqslant i, j \leqslant n, \quad \text{and} \quad a_i > \sqrt{\sum_{k=1}^n \left| \sum_{j=1}^n a_{ji} a_{jk} \right|}, \quad 1 \leqslant i \leqslant n, \tag{1.11}$$

then the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of (1.1) *is globally asymptotically stable for any $k_{ii} \geqslant 0$, $1 \leqslant i \leqslant n$.*

Moreover, if

$$\sum_{j=1}^n a_{ji} a_{jk} = 0, \quad i \neq k, \tag{1.12}$$

then the last inequalities of (1.11) *becomes*

$$a_i > \sqrt{\sum_{j=1}^{n} a_{ji}^2}, \quad 1 \leqslant i \leqslant n. \tag{1.13}$$

Thus, in the cases of (1.11) and (1.12), the condition (1.13) is weaker than the following sufficient condition on the global asymptotic stability of the positive equilibrium of the system

$$a_i > \sum_{j=1}^{n} |a_{ji}|, \quad 1 \leqslant i \leqslant n, \tag{1.14}$$

which was obtained by Gopalsamy [4], and this extends some of results in Saito, Hara and Ma [9] for $n = 2$ to $n \geqslant 2$.

The organization of this paper is as follows. In Section 2, applying the results in Muroya [6], we offer conditions for the persistence of system (1.1)–(1.2), and using a nonnegative Lyapunov-like sequence, we establish conditions for the global asymptotic stability of positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of the system (1.1)–(1.2).

2. Proof of theorems

In this section, we prove Theorems 1.1 and 1.2. Muroya [6] consider the following discrete system of nonautonomous Lotka–Volterra type:

$$\begin{cases} N_i(p+1) = N_i(p)\exp\left\{c_i(p) - a_i(p)N_i(p) - \displaystyle\sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(p)N_j(p-k_l)\right\}, \\ \quad p = 0, 1, 2, \ldots, \\ N_i(p) = N_{ip} \geqslant 0, \quad p \leqslant 0, \quad \text{and} \quad N_{i0} > 0, \quad 1 \leqslant i \leqslant n, \end{cases} \tag{2.1}$$

where each $c_i(p), a_i(p)$ and $a_{ij}^l(p)$ are bounded for $p \geqslant 0$ and

$$\begin{cases} \inf\limits_{p \geqslant 0} a_i(p) > 0, \quad a_{ii}^0(p) \equiv 0, \quad 1 \leqslant i \leqslant n, \\ a_{ij}^l(p) \geqslant 0, \quad 1 \leqslant i \leqslant j \leqslant n,\ 0 \leqslant l \leqslant m, \\ k_0 = 0, \quad \text{integers } k_l \geqslant 0, \quad 1 \leqslant l \leqslant m. \end{cases} \tag{2.2}$$

For a given sequence $\{g(p)\}_{p=0}^{\infty}$, we set

$$\begin{aligned} g_M &= \sup\{g(p) \mid p = 0, 1, 2, \ldots\}, \\ g_L &= \inf\{g(p) \mid p = 0, 1, 2, \ldots\}, \end{aligned} \tag{2.3}$$

and for integers $0 \leqslant p_1 < p_2$, we set

$$A[g, p_1, p_2] = \frac{1}{p_2 - p_1}\sum_{p=p_1}^{p_2-1} g(p). \tag{2.4}$$

The *lower* and *upper averages* of $g(p)$, denoted by $m[g]$ and $M[g]$, respectively, are defined by

$$\begin{aligned} m[g] &= \lim_{q\to\infty} \inf\{A[g, p_1, p_2] \mid p_2 - p_1 \geqslant q\} \quad \text{and} \\ M[g] &= \lim_{q\to\infty} \sup\{A[g, p_1, p_2] \mid p_2 - p_1 \geqslant q\}. \end{aligned} \tag{2.5}$$

Put

$$\begin{aligned}
&a^l_{ijL} = a^{l-}_{ijL} + a^{l+}_{ijL}, \quad a^{l-}_{ijL} \leqslant 0 \leqslant a^{l+}_{ijL},\\
&a^l_{ijM} = a^{l-}_{ijM} + a^{l+}_{ijM}, \quad a^{l-}_{ijM} \leqslant 0 \leqslant a^{l+}_{ijM},\\
&b_{ijL} = \sum_{l=0}^{m} a^l_{ijL}, \qquad b^-_{ijL} = \sum_{l=0}^{m} a^{l-}_{ijL},\\
&b_{ijM} = \sum_{l=0}^{m} a^l_{ijM} \quad \text{and} \quad b^+_{ijM} = \sum_{l=0}^{m} a^{l+}_{ijM}, \quad 1 \leqslant i, j \leqslant n.
\end{aligned} \tag{2.6}$$

Let

$$\begin{aligned}
&A_L = \operatorname{diag}(a_{1L}, a_{2L}, \ldots, a_{nL}), \qquad B^-_L = \left[b^-_{ijL}\right], \qquad B^+_M = \left[b^+_{ijM}\right],\\
&D^+_L = \operatorname{diag}\left(b^+_{11L}, b^+_{22L}, \ldots, b^+_{nnL}\right) \quad \text{and}\\
&D^+_M = \operatorname{diag}\left(b^+_{11M}, b^+_{22M}, \ldots, b^+_{nnM}\right)\\
&\quad \text{are } n \times n \text{ matrices,} \quad \text{and}\\
&\underline{\boldsymbol{c}} = \left[m[c_i]\right] \quad \text{and} \quad \bar{\boldsymbol{c}} = \left[M[c_i]\right]\\
&\quad \text{are } n\text{-dimensional vectors.}
\end{aligned} \tag{2.7}$$

Assume that

$$\left(A_L + D^+_L + B^-_L\right)^{-1} \bar{\boldsymbol{c}} > \mathbf{0} \quad \text{and} \quad \underline{\boldsymbol{c}} > \left(B^+_M - D^+_M\right)\left(A_L + D^+_L + B^-_L\right)^{-1} \bar{\boldsymbol{c}}, \tag{2.8}$$

and put

$$\begin{aligned}
&\tilde{c}_{iM} = c_{iM} - \sum_{j=1}^{i-1} b^-_{ijL} \bar{N}_j, \quad \tilde{N}_i = \tilde{c}_{iM}/a_{iL},\\
&\bar{N}_i = \begin{cases} \tilde{c}_{iM}/a_{iL}, & \tilde{c}_{iM} \leqslant 1,\\ \exp(\tilde{c}_{iM} - 1)/a_{iL}, & \tilde{c}_{iM} > 1. \end{cases}
\end{aligned} \tag{2.9}$$

Muroya [6] obtained the following two results (see Muroya [6, Lemma 2.2 and Theorem 1.2]).

Lemma 2.1. *Assume that for Eq.* (2.7) *and* $c_M = (c_{1M}, c_{2M}, \ldots, c_{nM})^T$,

$$\left(A_L + B^-_L\right)^{-1} \boldsymbol{c}_M > \mathbf{0}. \tag{2.10}$$

Then, any solution of the system (2.1)–(2.2) *is bounded above, and it holds that*

$$\varlimsup_{p \to \infty} N_i(p) \leqslant \bar{N}_i, \quad 1 \leqslant i \leqslant n, \tag{2.11}$$

where $\bar{N}_i$, $1 \leqslant i \leqslant n$, *are defined by* (2.9).

Note that (2.8) implies (2.10).

Lemma 2.2. *For the system* (2.1)–(2.2), *if the condition* (2.8) *is satisfied, then all solutions* $N_i(p)$, $1 \leqslant i \leqslant n$, *of the system are bounded above. Moreover, if there exists a nonempty subset* $Q \in \{1, 2, \ldots, n\}$ *such that*

$$c_{iL} - \sum_{j \notin Q} b^+_{ijM} \bar{N}_j > 0, \quad \text{for any } i \in Q, \tag{2.12}$$

then the system (2.1)–(2.2) *is persistent for solutions, that is,*

$$0 < \liminf_{p \geqslant 0} N_i(p) \leqslant \limsup_{p \geqslant 0} N_i(p) < +\infty, \quad 1 \leqslant i \leqslant n. \tag{2.13}$$

Note that for the system (1.1)–(1.2), (1.5) corresponds to (2.8) in system (2.1)–(2.2) and implies $\boldsymbol{c} > \mathbf{0}$ and for the set $Q = \{1, 2, \dots, n\}$, it holds that

$$c_i - \sum_{j \notin Q} a_{ij}^+ \bar{N}_j > 0, \quad \text{for any } i \in Q, \tag{2.14}$$

which implies (2.12).

Proof of Theorem 1.1. Put

$$l_{ij} = \begin{cases} (i-1) \times (i-1) + j, & i > j, \\ (j-1) \times (j-1) + 2j - i, & i \leqslant j, \end{cases}$$

and

$$\bar{a}_{ij}^l = \begin{cases} a_{ij}, & l = l_{ij}, \\ 0, & \text{otherwise}, \end{cases} \qquad k_l = \begin{cases} k_{ij}, & l = l_{ij}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, we have

$$\sum_{j=1}^{n} a_{ij} N_j(t - k_{ij}) = \sum_{j=1}^{n} \sum_{l=1}^{n^2} \bar{a}_{ij}^l N_j(t - k_l).$$

Thus, the system (1.1)–(1.2) is a special autonomous case of system (2.1)–(2.2). We can apply the results in Lemmas 2.1 and 2.2 to Eqs. (1.1)–(1.2) and obtain the conclusion of the theorem. This completes the proof. □

Proof of Theorem 1.2. Since by Theorem 1.1, the condition (1.9) implies that $\bar{N}_i = \tilde{N}_i < 1/a_i$, $1 \leqslant i \leqslant n$, we have that there is a positive integer p_0 such that for $p \geqslant p_0$, $N_i(p) < \bar{N}_i$, $1 \leqslant i \leqslant n$. Consider a nonnegative Lyapunov-like sequence $\{v(p)\}_{p=0}^{\infty}$ such that for $p \geqslant 0$,

$$\begin{aligned} v(p) = {} & \sum_{i=1}^{n} 2a_i \left\{ \frac{N_i(p)}{N_i^*} - 1 - \ln\bigl(N_i(p)/N_i^*\bigr) \right\} N_i^* \\ & + \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ji}| \left(\sum_{k=1}^{n} |a_{jk}| \right) \sum_{q=p-k_{ji}}^{p-1} \bigl(N_i(q) - N_i^*\bigr)^2. \end{aligned}$$

Then,

$$\begin{aligned} & v(p+1) - v(p) \\ & \quad = \sum_{i=1}^{n} 2a_i \left\{ \bigl(N_i(p+1) - N_i(p)\bigr) - N^* \ln \frac{N_i(p+1)}{N_i(p)} \right\} \\ & \qquad + \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ji}| \left(\sum_{k=1}^{n} |a_{jk}| \right) \bigl\{ \bigl(N_i(p) - N_i^*\bigr)^2 - \bigl(N_i(p - k_{ji}) - N_i^*\bigr)^2 \bigr\}. \end{aligned} \tag{2.15}$$

Since

$$\begin{aligned}&N_i(p+1)-N_i(p)\\&\quad=N_i(p)\left\{\exp\left(\ln\frac{N_i(p+1)}{N_i(p)}\right)-1\right\}\\&\quad=N_i(p)\left\{\ln\frac{N_i(p+1)}{N_i(p)}+\frac{\exp\left(\theta\ln\frac{N_i(p+1)}{N_i(p)}\right)}{2!}\left(\ln\frac{N_i(p+1)}{N_i(p)}\right)^2\right\},\quad 0<\theta<1,\end{aligned}$$

where for $p\geqslant p_0$ and $1\leqslant i\leqslant n$,

$$N_i(p)\exp\left(\theta\ln\frac{N_i(p+1)}{N_i(p)}\right)\leqslant\max\left(N_i(p),N_i(p+1)\right)<\frac{1}{a_i},$$

one can verify that

$$\begin{aligned}&2a_i\left\{\left(N_i(p+1)-N_i(p)\right)-N^*\ln\frac{N_i(p+1)}{N_i(p)}\right\}\\&\quad\leqslant 2a_i\left(N_i(p)-N_i^*\right)\ln\frac{N_i(p+1)}{N_i(p)}+\left(\ln\frac{N_i(p+1)}{N_i(p)}\right)^2,\end{aligned}\tag{2.16}$$

and by (2.1), we have that

$$\ln\frac{N_i(p+1)}{N_i(p)}=-a_i\left(N_i(p)-N_i^*\right)-\sum_{j=1}^n a_{ij}\left(N_j(p-k_{ij})-N_j^*\right).$$

We have that $x-1-\ln x\geqslant 0$, for any $x>0$. By Theorem 1.1, each $N_i(p)$, $1\leqslant i\leqslant n$, are bounded above and below by positive constants for $p\geqslant 0$.

Therefore, it follows from (1.6) that for any $p\geqslant\bar{k}=\max\{k_{ij}\mid k_{ij}\geqslant 0,\ 1\leqslant i,j\leqslant n\}$, $0\leqslant v(p)<+\infty$.

Let

$$p_i=a_i\left(N_i(p)-N^*\right)\quad\text{and}\quad q_{ij}=a_{ij}\left(N_j(p-k_{ij})-N_j^*\right).$$

Then, $\ln\frac{N_i(p+1)}{N_i(p)}=-(p_i+\sum_{j=1}^n q_{ij})$, and

$$\begin{aligned}\ln\frac{N_i(p+1)}{N_i(p)}&=2p_i\left(-p_i-\sum_{j=1}^n q_{ij}\right)\\&=-\left(p_i+\sum_{j=1}^n q_{ij}\right)^2+\sum_{j=1}^n q_{ij}^2+2\sum_{j=2}^n\sum_{k=1}^{j-1}q_{ij}q_{ik}-p_i^2,\end{aligned}$$

and for $r_{ji}=N_j(p-k_{ij})-N_j^*(p)$, we have that

$$\begin{aligned}2\sum_{i=1}^n\sum_{j=2}^n\sum_{k=1}^{j-1}q_{ij}q_{ik}&=2\sum_{i=1}^n\sum_{j=2}^n\sum_{k=1}^{j-1}a_{ij}a_{ik}r_{ji}r_{ki}\leqslant\sum_{i=1}^n\sum_{j=2}^n\sum_{k=1}^{j-1}|a_{ij}a_{ik}|\left(r_{ji}^2+r_{ki}^2\right)\\&=\sum_{j=2}^n\sum_{i=1}^n|a_{ij}|\left(\sum_{k=1}^{j-1}|a_{ik}|\right)r_{ji}^2+\sum_{k=1}^{n-1}\sum_{i=1}^n|a_{ik}|\left(\sum_{j=k+1}^n|a_{ij}|\right)r_{ki}^2\end{aligned}$$

$$= \sum_{i=2}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{i-1}|a_{jk}|\right)r_{ji}^2 + \sum_{i=1}^{n-1}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=i+1}^{n}|a_{jk}|\right)r_{ki}^2$$
$$\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k\neq i}|a_{jk}|\right)r_{ij}^2.$$

Therefore,

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n}q_{ij}^2 + 2\sum_{j=2}^{n}\sum_{k=1}^{j-1}q_{ij}q_{ik}\right) \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)r_{ij}^2$$

and

$$\sum_{i=1}^{n}2a_i\left\{N_i(p+1) - N_i(p) - N_i^*\ln\frac{N_i(p+1)}{N_i(p)}\right\}$$
$$\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)r_{ij}^2 - \sum_{i=1}^{n}p_i^2$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)\left(N_i(p-k_{ji}) - N_i^*\right)^2 - \sum_{i=1}^{n}a_i^2\left(N_i(p) - N_i^*\right)^2.$$

Thus, by (2.15), we obtain

$$v(p+1) - v(p) \leqslant -\sum_{i=1}^{n}\left\{a_i^2 - \sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)\right\}\left(N_i(p) - N_i^*\right)^2$$
$$\leqslant -\delta\sum_{i=1}^{n}\left(N_i(p) - N_i^*\right)^2,$$

where by (1.10),

$$\delta = \min_{1\leqslant i\leqslant n}\left\{a_i^2 - \sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)\right\} > 0.$$

Then,

$$v(p+1) + \delta\sum_{q=0}^{p}\sum_{i=1}^{n}\left(N_i(q) - N_i^*\right)^2 \leqslant v(0), \quad \text{for any } p \geqslant 0,$$

and

$$\sum_{p=0}^{\infty}\sum_{i=1}^{n}\left(N_i(p) - N_i^*\right)^2 \leqslant \frac{v(0)}{\delta} < +\infty,$$

from which we conclude that $\sum_{i=1}^{n}(N_i(p) - N_i^*)^2 = 0$. This result implies that the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of (1.1) is globally asymptotically stable for any $k_{ij} \geqslant 0$, $1 \leqslant i, j \leqslant n$.

In particular, if (1.11) holds, then for $r_j = r_{jj} = N_j(p-k_{jj}) - N_j^*$, $1 \leqslant j \leqslant n$, we have that

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} q_{ij}^2 + 2\sum_{j=2}^{n}\sum_{k=1}^{j-1} q_{ij}q_{ik}\right) = \sum_{i=1}^{n}\left\{\sum_{j=1}^{n} a_{ij}^2 r_j^2 + 2\sum_{j=2}^{n}\sum_{k=1}^{j-1} a_{ij}r_j a_{ik}r_k\right\}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}r_j a_{ik}r_k = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ij}a_{ik}\right) r_j r_k$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left|\left(\sum_{i=1}^{n} a_{ij}a_{ik}\right) r_j r_k\right| \leqslant \sum_{j=1}^{n}\sum_{k=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right| \frac{r_j^2 + r_k^2}{2}$$
$$= \sum_{j=1}^{n}\left(\sum_{k=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right|\right)\frac{r_j^2}{2} + \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right|\right)\frac{r_k^2}{2}$$
$$= \sum_{j=1}^{n}\left(\sum_{k=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right|\right) r_j^2$$
$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} a_{ji}a_{jk}\right|\right) r_i^2.$$

Thus, by (2.15), we obtain

$$v(p+1) - v(p) \leqslant -\sum_{i=1}^{n}\left\{a_i^2 - \sum_{k=1}^{n}\left|\sum_{j=1}^{n} a_{ji}a_{jk}\right|\right\}\big(N_i(p) - N_i^*\big)^2$$
$$\leqslant -\delta_1 \sum_{i=1}^{n}\big(N_i(p) - N_i^*\big)^2,$$

where by (1.11),

$$\delta_1 = \min_{1\leqslant i\leqslant n}\left\{a_i^2 - \sum_{k=1}^{n}\left|\sum_{j=1}^{n} a_{ji}a_{jk}\right|\right\} > 0.$$

Then,

$$v(p+1) + \delta_1 \sum_{q=0}^{p}\sum_{i=1}^{n}\big(N_i(q) - N_i^*\big)^2 \leqslant v(0), \quad \text{for any } p \geqslant 0,$$

and

$$\sum_{p=0}^{\infty}\sum_{i=1}^{n}\big(N_i(p) - N_i^*\big)^2 \leqslant \frac{v(0)}{\delta_1} < +\infty,$$

from which we conclude that $\sum_{i=1}^{n}(N_i(p) - N_i^*)^2 = 0$. This result implies that the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \dots, N_n^*)$ of (1.1) is globally asymptotically stable for any $k_{ii} \geqslant 0$, $1 \leqslant i \leqslant n$.

Moreover, if (1.12) holds, then it is evident that the last inequalities of (1.11) becomes (1.13). $\square$

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