Solutions to Continuous Dynamical Systems

Numerical Methods for Dynamical Systems

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Outline

Continuous Dynamical Systems

Definition

A continuous dynamical system is a set of ordinary differential equations (ODEs)

$$\dot{x}(t) = f\big(t, x(t)\big),$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

Note:

- ✎ $\dot{x}(t)$ denotes the derivative $\dfrac{\mathrm{d}x}{\mathrm{d}t}(t)$.
- ✎ We often consider autonomous systems: $\dot{x} = f(x)$.

Example (The Logistic equation)

Model Form:

$$\dot{x}(t) = r\,x(t)\,\left(1 - \frac{x(t)}{K}\right),$$

where:

- ✎ $x(t)$ represents the population (or concentration) at time t.
- ✎ $r > 0$ is the **intrinsic growth rate**.
- ✎ $K > 0$ is the **carrying capacity**, i.e., the maximum population the environment can sustain.

Example (The Lotka-Volterra Equations)

Model Form:

$$\begin{cases} \dot{x}(t) = a\,x(t) - b\,x(t)\,y(t), \\ \dot{y}(t) = c\,x(t)\,y(t) - d\,y(t), \end{cases}$$

where:

- ✎ $x(t)$ is the **prey population**.
- ✎ $y(t)$ is the **predator population**.
- ✎ $a > 0$ is the **growth rate** of prey in the absence of predators.
- ✎ $b > 0$ is the **predation rate coefficient**.
- ✎ $c > 0$ is the **growth rate** of predators from consuming prey.
- ✎ $d > 0$ is the **natural mortality rate** of predators.

Solutions

Definition

To a continuous dynamical system

$$\dot{x}(t) \;=\; f\big(t, x(t)\big), \quad t \in I \subseteq \mathbb{R}, \quad x(t_0) = x_0.$$

a function $\varphi : I \to \mathbb{R}^n$ is called a *solution* if:

1. φ is continuously differentiable on I,
2. For every $t \in I$, $\varphi(t)$ satisfies $\dot{\varphi}(t) = f\big(t, \varphi(t)\big)$,
3. φ satisfies the initial condition $\varphi(t_0) = x_0$.

Solutions

Example

Consider the following continuous dynamical system:

$$\dot{x}(t) = \alpha\, x(t), \quad x(t_0) = x_0.$$

Show that $x(t) = x_0\, e^{\alpha(t-t_0)}$ is a solution.

Example (Logistic Equation)

For the logistic model

$$\dot{x}(t) = r\,x(t)\left(1 - \frac{x(t)}{K}\right), \quad x(t_0) = x_0,$$

the solution (for $x_0 > 0$ and $x_0 < K$) can be written as

$$x(t) = \frac{K}{1 + \left(\frac{K}{x_0} - 1\right)e^{-r(t-t_0)}}.$$

Lipschitz Continuity

Definition (Lipschitz continuity)

A function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be **Lipschitz continuous** on D if there exists a constant $L \geq 0$ such that

$$\|f(x) - f(y)\| \ \leq \ L\,\|x - y\| \quad \text{for all} \quad x, y \in D.$$

Remarks:

- ✎ The smallest such L is called the **Lipschitz constant** of f.
- ✎ If $L < 1$, f is said to be a contraction.
- ✎ If f is Lipschitz continuous, it is also uniformly continuous, but not necessarily differentiable.

Example (Constant Function is Lipschitz)

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = c$ for some constant $c \in \mathbb{R}$. Show that f is Lipschitz continuous.

Proof: For any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \; = \; |c - c| \; = \; 0 \; \leq \; 0 \cdot |x - y|.$$

Hence f is Lipschitz continuous with Lipschitz constant $L = 0$.

Example (Linear Function is Lipschitz)

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = ax + b,$$

where $a, b \in \mathbb{R}$. Show that f is Lipschitz continuous.

Proof: For any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| = |(ax + b) - (ay + b)| = |a(x - y)| = |a| \cdot |x - y|.$$

Thus,

$$|f(x) - f(y)| \ \leq \ |a|\,|x - y|.$$

Hence f is Lipschitz continuous with constant $L = |a|$.

Lipschitz Continuity

Example (Square Function is not Lipschitz on $\mathbb{R}$)

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2.$$

Show that f is *not* Lipschitz continuous on $\mathbb{R}$.

Proof (By Contradiction): Suppose there exists a constant $L \geq 0$ such that

$$|x^2 - y^2| \;\leq\; L\,|x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

Observe that

$$|x^2 - y^2| = |(x - y)(x + y)| = |x - y|\,|x + y|.$$

Example (Square Function is not Lipschitz on $\mathbb{R}$)

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2.$$

Show that f is *not* Lipschitz continuous on $\mathbb{R}$.

Proof (By Contradiction): Hence the supposed Lipschitz condition becomes

$$|x + y| \ \le \ L \quad \text{for all } x, y \in \mathbb{R} \text{ with } x \neq y.$$

But if we set $x = t$ and $y = t$ for large t,

$$|x + y| = |2t|.$$

This must be bounded by L for all t, which is impossible as $t \to \infty$. Therefore, f is not Lipschitz on $\mathbb{R}$.

Example (Square Function is not Lipschitz on $\mathbb{R}$)

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2.$$

Show that f is *not* Lipschitz continuous on $\mathbb{R}$.

Note: $f(x) = x^2$ *is* Lipschitz on any bounded interval $[-M, M]$.

Jacobian Matrices

Definition (Jacobian Matrix)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ have components

$$f(x) \;=\; \big(f_1(x),\, f_2(x),\, \ldots,\, f_m(x)\big).$$

If all partial derivatives exist and are continuous in an open region $D \subset \mathbb{R}^n$, then the **Jacobian matrix** of f at $x = (x_1, x_2, \ldots, x_n) \in D$ is

$$\nabla f(x) \;=\; \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

Example (Scalar Function $f : \mathbb{R}^2 \to \mathbb{R}$)

Consider

$$f(x, y) = x^2 + 3xy.$$

All partial derivatives exist, so the Jacobian (which in this case is just the gradient as a row vector) is

$$\nabla f(x, y) = \begin{bmatrix} \dfrac{\partial}{\partial x}(x^2 + 3xy) & \dfrac{\partial}{\partial y}(x^2 + 3xy) \end{bmatrix} = \begin{bmatrix} 2x + 3y & 3x \end{bmatrix}.$$

Jacobian Matrices

Example (Vector Function $f : \mathbb{R}^2 \to \mathbb{R}^2$)

Define
$$f(x,y) = \left(x^2 + y^2,\ x + y^3 \right).$$
Then $f_1(x,y) = x^2 + y^2$ and $f_2(x,y) = x + y^3$. The Jacobian is:
$$\nabla f(x,y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 1 & 3y^2 \end{bmatrix}.$$

Example (Vector Function $f : \mathbb{R}^3 \to \mathbb{R}^2$)

Let

$$f(x, y, z) = \big(e^x + yz,\ x^2 - y^3 + \sin z \big).$$

Then:

$$f_1(x, y, z) = e^x + yz, \quad f_2(x, y, z) = x^2 - y^3 + \sin z.$$

Hence, the Jacobian is a 2×3 matrix:

$$\nabla f(x, y, z) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} e^x & z & y \\ 2x & -3y^2 & \cos z \end{pmatrix}.$$

Theorem (Bounded Gradient Implies Lipschitz Continuity)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function on an open, convex set $D \subseteq \mathbb{R}^n$. If there exists a constant $M \geq 0$ such that

$$\|\nabla f(x)\| \;\leq\; M \quad \text{for all } x \in D,$$

then f is Lipschitz continuous on D with Lipschitz constant M.

Sketch of proof:

Let $x, y \in D$ and consider the straight line segment $\gamma(t) = y + t\,(x - y)$, with $t \in [0, 1]$. By the convexity of D, $\gamma(t) \in D$ for all t.

Using the fundamental theorem of calculus, we have

$$f(x) - f(y) \;=\; \int_0^1 \nabla f\big(\gamma(t)\big)\,\big(x - y\big)\,\mathrm{d}t,$$

$$\|f(x) - f(y)\| \;\leq\; \int_0^1 \big\|\nabla f\big(\gamma(t)\big)\big\|\;\big\|x - y\big\|\,\mathrm{d}t \;\leq\; M\,\|x - y\|.$$

Existence and Uniqueness of Solution

Theorem (Picard–Lindelöf Existence and Uniqueness)

Let $I = [a, b] \subset \mathbb{R}$ *be an interval and* $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous function. If* f *is* **Lipschitz continuous** *in the state variable* x*, then for any* $t_0 \in I$ *and any initial condition* $x_0 \in \mathbb{R}^n$*, the initial value problem*

$$\dot{x}(t) \; = \; f\big(t, x(t)\big), \qquad x(t_0) = x_0,$$

has a unique solution $x(t)$ *on some subinterval* $[t_0 - \delta, \, t_0 + \delta] \subset I$*.*

Example

Show that there exists a solution to the following continuous dynamical system:

$$\dot{x}(t) = f\big(x(t)\big) = 2x(t) - 3\,\sin\big(x(t)\big), \quad x(t) \in \mathbb{R}.$$

Example

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✎ **Check Continuity:** The function $f(x) = 2x - 3\sin(x)$ is composed of polynomials and sine, which are continuous for all $x \in \mathbb{R}$.

Existence and Uniqueness of Solution

Example

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$$\dot{x}(t) = f\big(x(t)\big) = 2x(t) - 3\,\sin\big(x(t)\big), \quad x(t) \in \mathbb{R}.$$

✎ **Check Lipschitz Continuity:**

$$f'(x) = 2 - 3\cos(x).$$

Since $\cos(x)$ is bounded between -1 and 1, we have

$$\big|2 - 3\cos(x)\big| \;\leq\; 2 + 3 \;=\; 5 \quad \text{for all } x \in \mathbb{R}.$$

Hence $|f'(x)| \leq 5$, implying f is globally Lipschitz on $\mathbb{R}$ with Lipschitz constant $L = 5$.

Example

Show that there exists a solution to the following continuous dynamical system:

$$\dot{x}(t) = f\big(x(t)\big) = 2x(t) - 3\,\sin\big(x(t)\big), \quad x(t) \in \mathbb{R}.$$

✎ **Conclusion:** By Picard–Lindelöf theorem, for any initial condition $x(0) = x_0 \in \mathbb{R}$, there is a unique local solution $x(t)$ that satisfies $\dot{x} = f(x)$.
In fact, because f is globally Lipschitz, the solution extends (uniquely) for all time t.

Problem 1: Checking a Solution to a Dynamical System

Consider the system

$$\dot{x}(t) = -2x(t),$$

with the initial condition $x(0) = 3$. The proposed solution is

$$x(t) = 3e^{-2t}.$$

1. Verify that $x(t)$ satisfies the ODE by computing $\dot{x}(t)$ and substituting into the equation.
2. Check that the initial condition $x(0) = 3$ is satisfied.
3. Sketch the solution $x(t)$ and describe its behavior as $t \to \infty$.

Problem 2: Verifying Lipschitz Continuity

Let $f(x) = x^2$ and $g(x) = \sin(x)$.

1. Prove that $g(x)$ is Lipschitz continuous on $\mathbb{R}$. Find its Lipschitz constant.
2. Show that $f(x)$ is not Lipschitz continuous on $\mathbb{R}$, but is Lipschitz on any bounded interval $[-M, M]$.

Problem 3: Applying Picard–Lindelöf

Consider the system

$$\dot{x}(t) = \ln(1 + x^2), \quad x(0) = x_0 \in \mathbb{R}.$$

1. Show that $f(x) = \ln(1 + x^2)$ is differentiable for all $x \in \mathbb{R}$, and compute its derivative $f'(x)$.
2. Prove that $|f'(x)| \leq M$ for some constant M on $\mathbb{R}$, thus showing $f(x)$ satisfies the bounded gradient condition.
3. Use the Picard–Lindelöf theorem to prove that the given system has a unique solution for all $t \in \mathbb{R}$.