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Discrete Combat Models:
Investigating the Solutions to Discrete Forms of Lanchester's Combat Models

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ABSTRACT

Lanchester's equations and their solutions, as continuous differential equations, have been studied for years. This article introduces a new approach with the use of the discrete form of Lanchester's equations, using dynamical systems or difference equations. It begins with Lanchester's square law and develops a generalized analytical solution for the discrete model that can be built by knowing only the kill rates and the initial force sizes of the combatants. It then forms the condition of parity (a draw) to develop a simple relationship of these variables to determine who wins the engagement. This article illustrates these models and their solutions using historic combat examples. It also illustrates that current counter-insurgency combat models can be built and solved using various forms of difference equations.

Keywords: *Combat Modeling, Difference Equations, Historic Battles, Lanchester's Equations*

INTRODUCTION

War is conflict between nations or states carried on by force of considerable duration and magnitude, by land, sea, or air for obtaining superiority and dominion of one over the other. War has been a topic of analysis and researchers for some time. F. W. Lanchester (1916, 1956) originally published his mathematical model for air to air combat in his 1916 book, *Aircraft in Warfare*. These models known as the linear and square law became the basis for much of the analysis of combat. These differential equations models have been the methodology to present and solve many historical combat models. Both Bonder's (1981) and Dolansky's (1964) articles discussed the importance of Lanchester equations in modeling combat. James G. Taylor (1983) alluded to difficulties in solving more difficult "realistic" equations and suggested numerical methods that can easily and conveniently be numerically solved on a computer. The use of computers to analytically solve or numerically solve combat models is the standard method. We suggest using discrete dynamical system (difference equations) as the discrete form of Lanchester's equations in our combat models. We show the discrete forms and their analytical solutions, where applicable. We also show both numerical and graphical solutions to historical conflicts. We compare one of these solutions with the differential equation form to show how closely the discrete form matches the differential equation solution. We also suggest

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uses of these discrete equations in more modern combat models with insurgencies and counter-insurgencies. These models may be used to assist decision making for our nation's leaders.

DISCRETE FORMS OF LANCHESTER'S EQUATIONS

History is filled with examples of the unparalleled heroism and complexities of war. Specific battles like Bunker Hill, the Alamo, Gettysburg, Little Big Horn, Iwo Jima, the Battle of Britain, and the Battle of the Bulge are a part of our culture and heritage. Campaigns like the Cuban Revolution, Vietnam, Panama, and now the conflicts in Afghanistan and Iraq are a part of our personal history. Powers (2008) and Fox (2008) have suggested that we can model the Global War on Terror (GWOT) conflict with Lanchester's equations.

Although combat is fought over continuous time, there are typically discrete starting, pause, and stopping points. Often models of combat employ discrete time simulation. For years, Lanchester's differential equation models were the norm for computer simulations of combat. The diagram of simple combat as modeled by Lanchester is illustrated in Figure 1. We investigate and illustrate the use of a discrete version of these discrete equations. We use models of discrete dynamical systems via difference equations to model conflicts and gain insights by examining the models of "directed fire" historic conflicts such as Nelson's Battle at Trafalgar, the Alamo, and Iwo Jima. We employ difference equations that allow for a complete numerical and graphical solution to be analyzed and do not require the mathematical rigor of differential equations. We further investigate the analytical form of the "direct fire" solutions to provide a solution template, where applicable, to be used in modeling efforts.

Lanchester's equations stated that "under conditions of modern warfare" that combat between two homogeneous forces could be modeled from the state condition of a similar diagram (Taylor, 1980). We will call this state diagram (Figure 1), the change diagram.

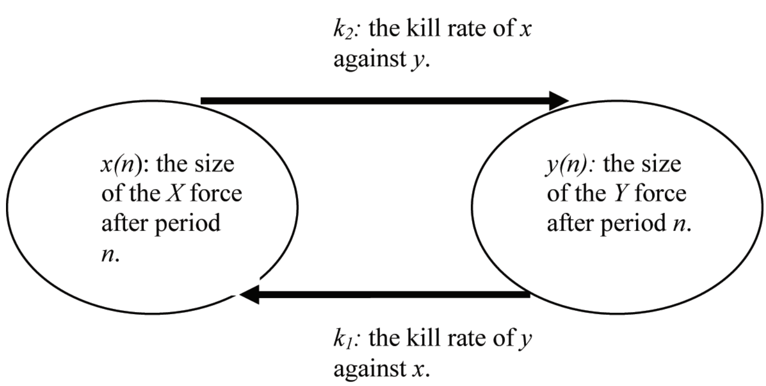
We will use the paradigm, *Future = Present + Change*, to build our mathematical models using discrete dynamical systems. This will be paramount as eventually models will be built that cannot be solved analytically but can be analyzed by numerical (iteration) methods and graphs.

We begin by defining the following variables:

$x(n)$ = the number of combatants in the X-force after period n.

$y(n)$ = the number of combatants in the Y-force after period n.

Figure 1. Change diagram of combat modeled by Lanchester



The future state is then $x(n+1)$ and $y(n+1)$, respectively.

Thus, we have using our paradigm,

x(n+1) = x(n) + Change

y(n+1)=y(n) + Change.

Figure 1 provides the information that reflects change. Our dynamical system of equations is:

$$x(n+1) = x(n) - k_1 y(n) \quad (1)$$
$$y(n+1) = y(n) - k_2 x(n)$$

We define our starting conditions as the size of the combatant forces at time period zero: $x(0)=x_0$ and $y(0)=y_0$.

Dynamical systems can always be solved by iteration, which make them quite attractive for use in both computer modeling and simulations of combat. However, we can gain some powerful insights with those discrete equations that have analytical solutions. This particular dynamical system of equations for Lanchester's direct fire model does have an analytical solution.

ANALYTICAL SOLUTIONS FOR DISCRETE FORM OF LANCHESTER'S DIRECT FIRE EQUATIONS

We return to this system of equations as Lanchester's direct fire equations in difference equation form, equation (1):

$$\begin{aligned} x(n+1) &= x(n) - k_1 y(n) \\ y(n+1) &= -k_2 x(n) + y(n) \end{aligned}$$

Now, we rewrite these in matrix form:

$$X_{n+1} = \begin{bmatrix} 1 & -k_1 \\ -k_2 & 1 \end{bmatrix} X_n, X_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad (2)$$

We will use the matrix solution method using eigenvalues and eigenvectors to find the analytical solution of equation (2). Further, we will characterize the solution we found in terms of only k_1, k_2, x_0, and y_0. This is significant as we can quickly write the solution to this form of Lanchester's equation modeled as in equation (2).

Let's begin by defining eigenvectors and eigenvalues:

Let A be a $n \times n$ matrix. The real number λ is called an eigenvalue of A if there exists a nonzero vector x in R^n such that

$$Ax = \lambda x. \quad (3)$$

The nonzero vector x is called an eigenvector of A associated with the eigenvalue, λ.

Equation (3) is written as,

$$Ax - \lambda x = 0, \; or \; (A - \lambda I)x = 0,$$

where I is a 2×2 identity matrix. The solution to finding λ comes from taking the determinant of the $(A-\lambda I)$ matrix, setting it equal to zero, and solving for λ.

We know that coefficient matrix A can be written as follows:

$$\begin{bmatrix} 1 & -k_1 \\ -k_2 & 1 \end{bmatrix}.$$

We set up the matrix $(A-\lambda I)$ to find the eigenvalues. We take the determinant of the matrix $(A-\lambda I)$:

$$\det \begin{bmatrix} 1-\lambda & -k_1 \\ -k_2 & 1-\lambda \end{bmatrix} = 0.$$

This yields the characteristic equation,

$$(1-\lambda)\cdot(1-\lambda) - k_1 k_2 = \lambda^2 - 2\lambda + 1 - k_1 k_2 = 0.$$

We solve for λ. Although not intuitively obvious to the casual observer, the two eigenvalues are:

$$\lambda_1 = 1 + \sqrt{k_1 k_2}\ . \qquad (4)$$
$$\lambda_2 = 1 - \sqrt{k_1 k_2}$$

Therefore, we have the eigenvalues from the initial form of the equation. We note that the eigenvalues are a function of the kill rates, k_1 and k_2. If you know the kill rates then you can easily obtain the two eigenvalues.

We also note two other characteristics of the eigenvalues:

(1) $\lambda_1 + \lambda_2 = 2$ and (2) $\lambda_1 \geq \lambda_2$.

For most of these combat models one eigenvalue will be > *1* and the other eigenvalue will be < *1*. The equation whose being attrited by the larger value of k_1 or k_2 has the eigenvalue, λ >1.

The general form of the solution for the size of the force *x* after any general discrete time period *t*, called *x(t),* is as follows:

$$x(t) = c_1 V_1 (\lambda_1)^t + c_2 V_2 (\lambda_2)^t\ , \qquad (5)$$

where the vectors V_1 and V_2 are the corresponding eigenvectors.

These eigenvectors, interestingly enough, can be simplified into a ratio of the attrition coefficients, k_1 & k_2. The vector for the dominant eigenvalue always has both a positive and a negative component as its eigenvector while the vector for the other smaller of the two eigenvalues always has two positive entries in this same ratio. This is because the equation for finding the eigenvector comes from:

$$\sqrt{k_1 k_2}\, v_1 - k_1 v_2 = 0\ and\ -\sqrt{k_1 k_2}\, d_1 - k_1 d_2 = 0$$
$$or$$
$$v_1 = k_1,\ \ v_2 = \sqrt{k_1 k_2}\ and\ d_1 = -k_1,\ d_2 = \sqrt{k_1 k_2}$$
$$(6)$$

Having simplified formulas for obtaining eigenvalues and eigenvectors allows us to quickly obtain the general form of the analytical solution. We can then use the initial conditions to obtain the particular solution.

In terms of our parameters of the model, we can easily rewrite equation (5) as follows:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ \sqrt{\frac{k_2}{k_1}} \end{pmatrix} \left(1 + \sqrt{k_1 \cdot k_2}\right)^t + c_2 \begin{pmatrix} 1 \\ \sqrt{\frac{k_2}{k_1}} \end{pmatrix} \left(1 - \sqrt{k_1 \cdot k_2}\right)^t,$$
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$
$$(7)$$

RED AND BLUE FORCE ILLUSTRATIVE EXAMPLE

For example, consider a battle between a Red force, *R(n)*, and a Blue force, *B(n)*, as given below:

$$B(n+1) = B(n) - 0.1 \cdot R(n),\ B(0) = 100$$
$$R(n+1) = R(n) - 0.05 \cdot B(n),\ R(0) = 50$$

We take the ratio of Blue to Red:

$$\frac{B(0)}{R(0)} = \frac{100}{50} = 2\ .$$

We are given the attrition coefficients, k_1 = *-0.1* and k_2 =*-0.050.*

Using the formulas that we just presented, we can quickly obtain the analytical solution.

$$\sqrt{k_1 k_2} = \sqrt{-0.1 \cdot -0.05} = 0.0707\ .$$

We use equation (4) to obtain the two eigenvalues of our combat model. They are *1.0707* and *0.9293*. We build the closed form analytical solution with the ratio of the vectors as ±1 and $\frac{\sqrt{k_1 k_2}}{k_1}$. We find $\frac{\sqrt{k_1 k_2}}{k_1} = 0.7070$. Our general solution would be:

$$X(t) = c_1 \begin{pmatrix} -1 \\ 0.707 \end{pmatrix} (1.0707)^t + c_2 \begin{pmatrix} 1 \\ 0.707 \end{pmatrix} (0.9293)^t$$

With our initial conditions of *(100, 50)* at time period *0*, we solve for the particular solution to obtain:

$$X(t) = -14.64 \begin{pmatrix} -1 \\ 0.707 \end{pmatrix} (1.0707)^t + 85.36 \begin{pmatrix} 1 \\ 0.707 \end{pmatrix} (0.9293)^t$$

We can decompose this into the blue and red equations:

$B(t) = 14.64\ (1.0707)^t + 85.36\ (0.9293)^t$

$R(t) = -10.35(1.0707)^t + 60.35\ (0.9293)^t$

We plot these together in Figure 2 and we observe the behavior illustrating that Blue wins.

These two graphs of the analytical solutions for the Blue and Red forces show that after about *12.5* time periods the Red force (initially at size *50*) approaches *0* and the Blue force (initially at *100*) is slightly below *70*. Thus, we know the Blue force wins and the Red force is annihilated.

We can also develop a relationship for this "win" and quickly see that when $\sqrt{k_1 k_2} \cdot B(0) > k_1 \cdot R(0)$ then the Blue force wins.

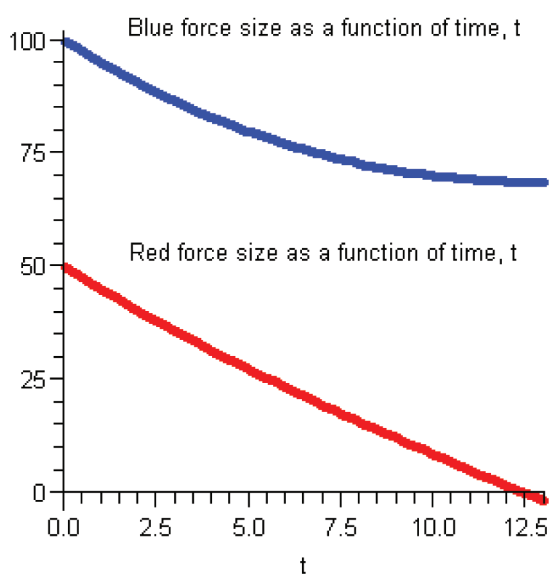
For our example, we find $\sqrt{k_1 k_2} \cdot B(0)\ and\ k_1 \cdot R(0)$.

$$B(0) \cdot \sqrt{k_1 k_2} = 100 \cdot 0.0707 = 7.07$$
$$k_2 \cdot R(0) = .01 \cdot 50 = 5.0$$
$$7.07 > 5.0$$

Since 7.07 is greater than 5.0 then the Blue force wins.

In general, the relationship can be either <, =, or >. So, we state that

Figure 2. Solution plots for B(t) and R(t) showing Blue wins



$$\sqrt{k_1 k_2}\, B(0) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} k_1 R(0) \tag{8}$$

When the relationship is greater than (>) then Blue wins, when the relationship is less than (<) Red wins, and when the relationship is equality (=) then we have a draw.

DEFINING PARITY

The concept of parity in combat modeling is important. We define parity as a fight to finish that ends in a draw—neither side wins. We assume that under the conditions of parity that both sides annihilate themselves (mathematically they both go to zero). No side wins and both sides lose. Mathematically, the parity condition is the easiest to solve and then examine if slight changes can be made in our favor. We can find parity by either manipulating one of the four parameters $\{x_0, y_0, k_1, \text{or } k_2\}$ and hold the other three as constants. These four parameters are the initial conditions, x_0 or y_0, and the attrition coefficients k_1 or k_2.

Again the knowledge of the solution is critical to finding or obtaining these parity values. It turns out under parity that the eigenvectors are in a ratio of the square of the initial conditions.

One eigenvector is $\begin{pmatrix} k_1 \\ \sqrt{k_1 k_2} \end{pmatrix}$. So

$$\frac{k_1}{\sqrt{k_1 k_2}} = \frac{B(0)}{R(0)}$$

or

$$\sqrt{k_1 k_2}\, B(0) = k_1 R(0),$$
$$or$$
$$\frac{k_2}{k_1} = \left(\frac{R(0)}{B(0)}\right)^2. \tag{9}$$

Let's return to our example and assume that Blue force starts with 100 combatants and the Red force with 50 combatants. Recall that Blue wins the engagement. Further let's fix k_2 at 0.05. What value is required for k_1 so that the red force fights to a draw?

We find $\sqrt{(0.05)k_1}\,(100) = k_1 \cdot (50)$

Thus, $k_1 = 0.20$. This means that the Red force must kill the Blue force at a rate twice as great as their current rate (k_1 was 0.1) in order to obtain a draw.

If Blue starts with 100 soldiers and the kill rates are fixed, how many soldiers would Red need? We find that the *R* force needs 70.71 combatants to obtain a draw. This is a 41.42% increase in combatants for Red.

We are not only able to quickly determine who wins the engagement but we can also find values that allow both sides to fight to a draw. This is important because any deviation away from the parity values allows for one side to win the engagement. This helps a force that could be facing defeat to either increase their force enough to win or obtain better weaponry to improve their kill rates enough to win.

QUALITATIVE AND QUANTITATIVE APPROACH

We develop a few qualitative insights with the direct fire approach. First, we return to the forms:

$$\Delta X = X(n+1) - X(n) = -k_1 Y$$
$$\Delta Y = Y(n+1) - Y(n) = -k_2 X.$$

We set both equal to zero and solve for the equations that make both equal to *0*. This yields two lines *x=0* and *y=0* that intersect at *(0,0)* the equilibrium point. Vectors point toward *(0,0)* but analysis shows that the equilibrium value *(0,0)* is not stable. Our assumptions imply trajectories terminates when it reaches either coordinate axis indicating one variable has gone to zero. Figures 3 and 4 illustrate the vectors and the then the regions where the curves result in wins for the

Figure 3. Equilibrium value (0,0) for the Direct Fire Model

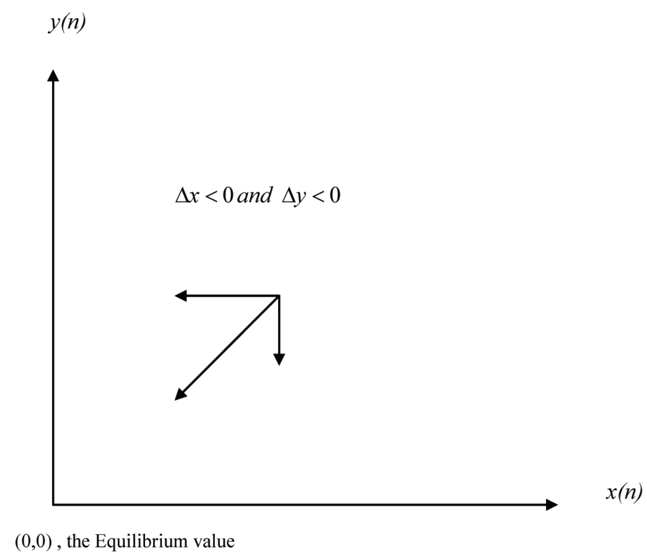
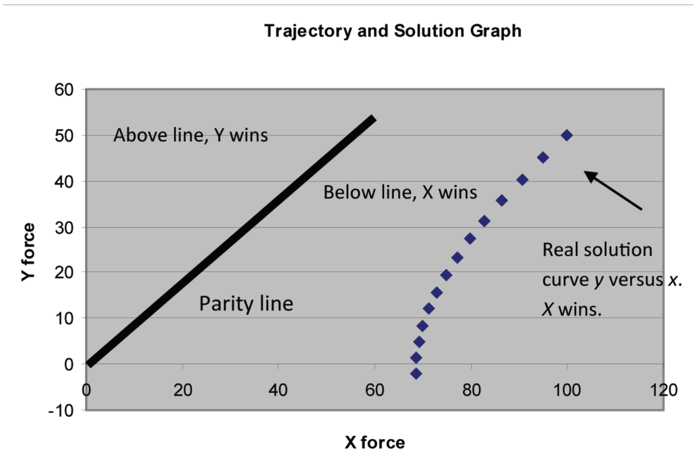


Figure 4. Trajectories for the basic Direct Fire Model illustrating X wins



X force and in wins for the *Y* force, or in a draw (the solid line through the origin).

Recall our parity form allowing x_0 to replace *B(0)* and y_0 to replace *R(0)* in equation (8): $\sqrt{k_1 k_2} \cdot x_0 = k_1 \cdot y_0$. We simplify and solve for *y*. This yields a straight line through the origin of the form: $y = \frac{\sqrt{k_1 k_2}}{k_1} x$.

Along this line we have a draw where both sides go to zero, parity. Above this line, we have the region where *y* wins and below we have the region where x wins. We plot our solution for y versus x and it shows in the next figure that we are in the region where y wins.

ILLUSTRATIVE "DIRECT FIRE" HISTORIC EXAMPLES

Let us use the discrete Lanchester equations and the relationships developed to investigate some historic examples.

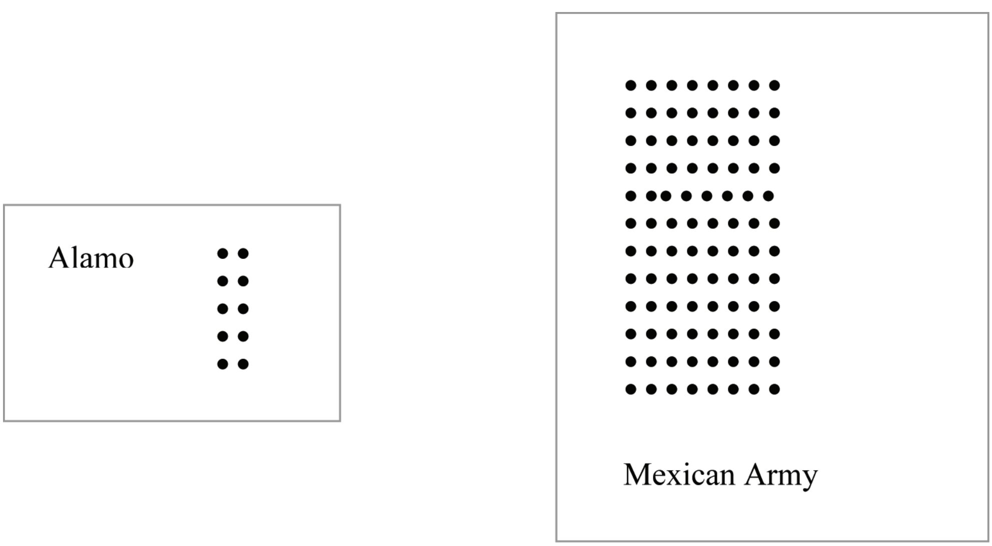
THE BATTLE OF THE ALAMO

First, consider the situation at the Alamo. According to some models created by Thompson (1989), Teague (2005), and Hercilla-Heredia (2008b), approximately 189 Texans were barricaded in the Alamo, being attacked by approximately 3000 Mexicans in the open fields surrounding it. We are interested in describing the loss of combatants in each force over the course of the engagement. We will do this by measuring or defining change. We define *T(n)* to be the number of Texans after period *n* and *M(n)* to be the number of Mexican solders after time period *n*. That is, we want to devise a way to express future = present + change as $T(n+1) = T(n) + \Delta T$ (where ΔT is the loss of Texan combatants over time) and $M(n+1) = M(n) + \Delta M$ (where ΔM is the loss of Mexican combatants over time). The Battle of the Alamo is illustrated as an example of a "directed fire" battle. The combatants on each side can see their opponents and can direct their fire at them. The Texans hiding behind the barricades were the more difficult target, and we need to have our models reflect this fact.

First, consider ΔM. This depends on the number of bullets being fired by the Texan defenders and how accurately they are being fired at the Mexican army. We can use a proportionality model, $\Delta M \propto \left(number\ of\ bullets\right) \cdot \left(probability\ of\ hit\right) \cdot (probability\ kill \mid probability\ hit)$.

The number of bullets capable of being fired depends on how many men are firing and how rapidly each can fire. Given the weaponry at the time, it might be more effective to have only a portion of the combatants firing with the rest reloading the rifles for them. This might increase the intensity of fire. There is also an issue of what portion of the force is in

Figure 5. Mexican army approaching the Alamo



a position to fire on the enemy as well as what portion of the force is exposed to direct fire. If the enemy force is in a rectangular formation, with several lines of combatants one behind the other, only the first one or two rows may be capable of firing freely at the enemy. Thus, $\Delta M = (Texans) \cdot (\%\ firing) \cdot (bullets\ /\ Texan\ /\ \min) \cdot (prob\, hit\ /\ bullet) \cdot (Mexicans\ disabled\ /\ hit)$

All of these variables can be combined into a single proportionality constant k_1. Some of these variables will vary over distance or time. For example, the probability of a hit will likely increase as the Mexican army closes in on the Alamo. However, our model assumes each of these entries, except for the number of combatants, is constant over the course of the battle. Consequently, we can write change term as: $\Delta M = -k_1\, T(n)$, where $T(n)$ is the number of Texans remaining in the battle after period n. The negative sign indicates the number of Mexican combatants is decreasing.

Now, consider ΔT. It is similarly composed of terms like the number of Mexicans, percent firing, number of bullets per Mexican combatant per minute, the probability of a hit, and the number of Texans disabled per hit. According to Teague (2005), we would expect that the rate of fire for the Mexican army to be smaller than the Texans, since they will be reloading while marching instead of reloading while standing still. Similarly, the probability of a hit will also be higher for the Texans shooting from a stance behind a wall than for the Mexicans shooting while marching in the open fields. Thus,

$$\Delta M = -k_1\, T(n) \text{ and } \Delta T = -k_2\, M(n),$$

but the values of k_1 and k_2 will be very different for the two forces. The constants k_1 and k_2 can be referred to as kill rates or as the coefficients of combat effectiveness.

The battle was waged while the Mexicans were in the open field. The effectiveness constant k_2 was very much smaller than k_1 giving a kill advantage to the Texans. Once the Alamo walls were breached, the values of the kill coefficients, k_1 and k_2, probably were vastly altered, and the battle ended is very short time. We model this battle as if it were a fight to the finish. According to historical evidence, the battle of the Alamo lasted for one hour once the siege started. Every Texan was killed at the Alamo and estimates vary for the number of Mexican's killed from 800 to 1500 of the Mexican force of approximately 3000 soldiers. We assume in our model that about 1300 Mexican's were killed in the battle. We calculate the kill coefficients for our model of combat where these numbers represent kills per minute by each side. Through modeling experimentation with the force sizes and length of the battle, we estimated the kill coefficients to be: $k_1 = 0.25$ and $k_2 = 0.00145$.

The situation at the Alamo, as described above, is modeled as:

$$\begin{pmatrix} T(n+1) \\ M(n+1) \end{pmatrix} = \begin{pmatrix} 1 & -0.00145 \\ -0.25 & 1 \end{pmatrix} \begin{pmatrix} T(n) \\ M(n) \end{pmatrix}, \begin{pmatrix} T(0) = 189 \\ M(0) = 3000 \end{pmatrix} \tag{10}$$

From our equation $\sqrt{k_1 k_2} \cdot T_0 < k_1 \cdot M_0$ we have 0.019039 (189) < 0.25(3000). Since 3.598 < 750, we know that the Mexican army wins decisively. In Table 1, we obtained the values to achieve parity so that we can see if under what circumstances the Texans could have won. We can easily see that many of these values are unrealistic for the event. Under parity we can hold three of the four parameters as constants and vary the fourth parameter. Then it can be seen if the new value is feasible or not feasible.

From Table 1, we find the Texans would have to improve their ability to kill the attacking Mexicans by 146.13%. According to Thompson (1989), the Texans already had rifles with a longer range by 200 meters, thus it might not be feasible to improve their killing coefficient by over 146%. The Texans were already inside the Alamo, thus lowering the kill rate of the Mexicans by over 68% may not feasible either. The force size suggests that under these battle conditions set forth in the siege that over 229 or an increase of men of over 21% would be enough to get a draw. However, when the walls of the Alamo are breached by the Mexican army we would need to recalculate the results with different models.

THE BATTLE OF TRAFALGAR

Another historic example of the directed fire model of combat is the Battle of Trafalgar. Trautteur and Virgilio (2003), Giordano, Fox, Horton, and Weir (2009), Giordano and McCormick (2007), Cummings (2001), Fox (2008b), and Kingman (2002) each discussed the applications of Lanchester's equations to this battle. In classical naval warfare, two fleets would sail parallel to each other (see Figure 6) and fire broadside at one another until one fleet was annihilated or gave up. The white fleet represents the British and the black fleet represents the French-Spanish fleet.

In such an engagement, the fleet with superior firepower will inevitably win. To model this battle, we begin with the system of difference equations that models the interaction of two fleets in combat. Suppose we have two opposing forces with A_0 and B_0 ships initially, and $A(t)$ and $B(t)$ ships t units of time after the battle is engaged. Given the style of combat at the time of Trafalgar, the losses for each fleet will be proportional to the effective firepower of the opposing fleet. That is,

$$\Delta A = -k_1 B \text{ and } \Delta B = -k_2 A,$$

where k_1 and k_2 are positive constants that measure the effectiveness of the ship's cannonry and personnel and A and B are both functions of time. In preparing for the Battle at Trafalgar, Admiral Nelson assumed the coefficients of effectiveness of the two fleets were approximately equal. To keep things simple initially, we let $k_1 = k_2 = 0.05$. The figure and numerical listing below allows us to look at many different initial settings and try to ascertain a pattern in the results of the battles.

We iterate these dynamical systems equations to obtain the numbers in the table to determine who wins the engagement. We graph this information as illustrated in Figure 7.

In this example, Admiral Nelson has 27 ships while the allied French and Spanish fleet had 33 ships. As we can see in Table 2, Admiral Nelson is expected to lose all 27 of his ships while the allied fleet will lose only about 14 ships.

Let's return to our equations that we developed earlier.

(0.05)(33) > (0.05)(27)
1.65 > 1.35

Since $\sqrt{k_1 k_2} \cdot FS_0 > k_1 \cdot B_0$ then the French-Spanish Fleet win. The analytical solution can be easily developed as:

Figure 6. The white fleet (British) is outnumbered by the black fleet (French and Spanish fleet)

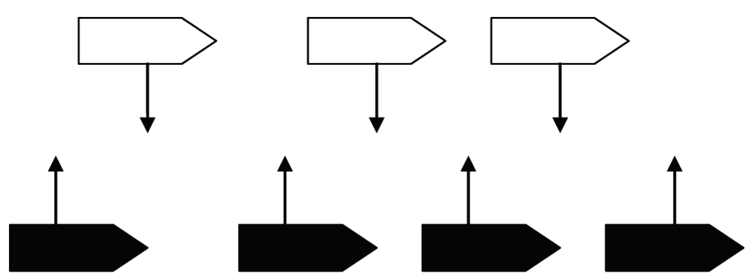
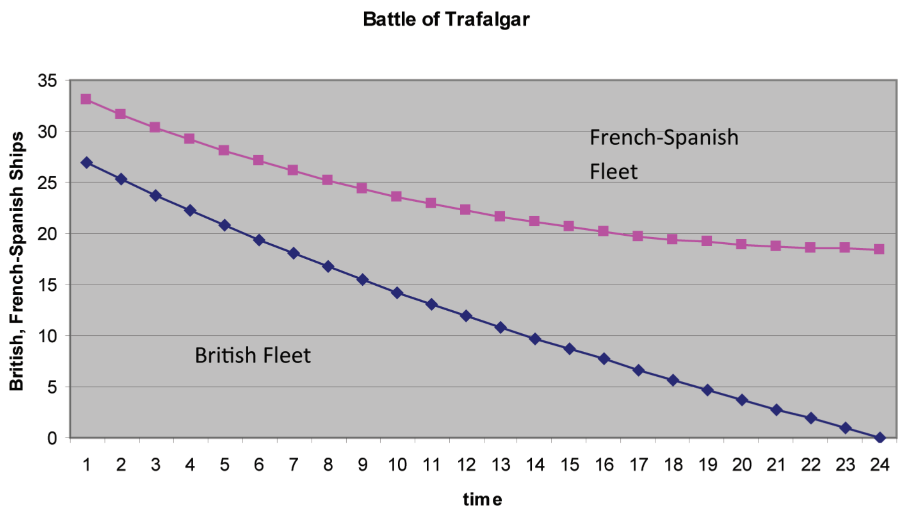


Figure 7. Battle of Trafalgar under normal battle strategies showing the victory of the French-Spanish fleet



$$X(t) = -3\begin{pmatrix} -1 \\ 1 \end{pmatrix}(1.05)^t + 30\begin{pmatrix} 1 \\ 1 \end{pmatrix}(0.95)^t$$

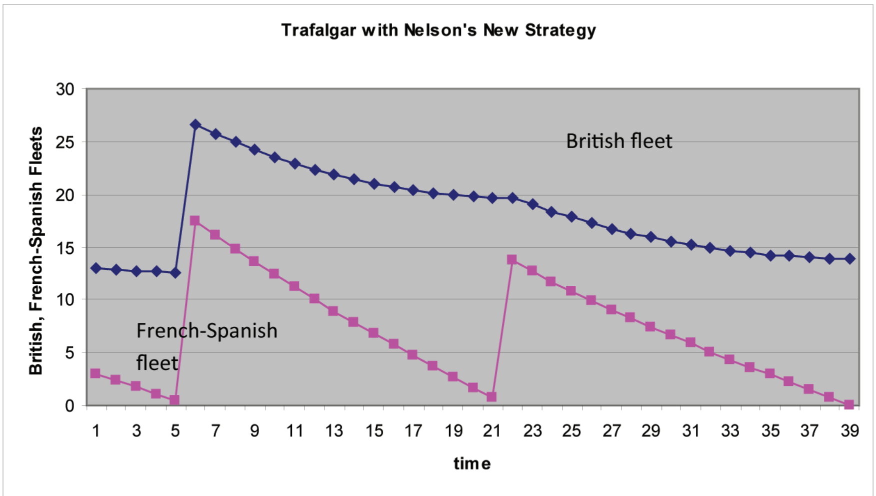
For the British to win, we first find the values that provide them with a draw. We find the British would require 33 ships to have draw. Additionally, we find that the British would have to increase their kill effectiveness to 0.07469 to obtain a draw. Increases just beyond these values, give the British the theoretical edge. However, there were no more ships or armaments to use in the battle. The only option to alter the outcome would be a change in strategy. According to history, Admiral Nelson defied conventional warfare, ordered his captains to split the British fleet and spear the enemy's line, called "crossing the T," to create a "pell-mell battle," which has been called the "Nelson Touch."

We can test this new strategy that was used by Admiral Nelson at the Battle of Trafalgar using our discrete combat model. Admiral Nelson decided to move away from the course of linear battle of the day and use a "divide and conquer" strategy. Nelson decided to break his fleet into two groups of size 13 and size 14. He also divided the enemy fleet into three groups: a force of 17 ships (called *B*), a force of 3 ships (called *A*) and a force of 13 ships (called *C*). We can assume these as the head, middle, and tail of the enemy fleet. His plan was to take the 13 ships and attack the middle 3 ships. Then have his reserve 14 ships rejoin the attack and attack the larger force *B*, and then turn to attack the smaller force *C*. How did Nelson's strategy prevail?

Assuming all other variables remain constant other than the order of the attacks against the differing size forces, we find the Admiral Nelson and the British fleet now win the battle sinking all French-Spanish ships with the British fleet having 13 or 14 ships remaining.

How did we obtain these results? The easiest method was by iteration. We used three battle formulas. We stop each battle when one of the values gets close to zero (before going negative). This is displayed in Figure 8.

Figure 8. British prevail with Nelson's new strategy



DETERMINING THE LENGTH OF THE BATTLE

To determine the length of the battle, we can use our analytical solutions to solve for the appropriate time. Recall the solution to our Blue versus Red example:

$$X(t) = -14.64\begin{pmatrix} -1 \\ 0.707 \end{pmatrix}(1.0707)^t + 83.36\begin{pmatrix} 1 \\ 0.707 \end{pmatrix}(0.9293)^t.$$

This simplifies to

$$X(t) = -\begin{pmatrix} 14.64 \\ -10.35 \end{pmatrix}(1.0707)^t + \begin{pmatrix} 83.36 \\ 58.9355 \end{pmatrix}(0.9293)^t$$

The graph in Figure 2 showed that x wins (as did our other analysis), thus the time parameter we are interested in finding is "when does y go o zero?" If you try to use the x equation, we would end up with trying to take the *ln* of a negative number, which is not possible.

We use only the solution to $y(t)$:

$$y(t) = -10.35 \cdot 1.0707^t + 58.9355 \cdot 0.9293^t$$

and set $y(k)=0$.

We solve for time, t:

$$t = \frac{\ln\left(\dfrac{58.9335}{10.35}\right)}{\ln\left(\dfrac{1.0707}{0.9293}\right)} = 12.28 \text{ time periods.}$$

In general, the time parameter is either of the following two equations:

$$\frac{\ln\left(\dfrac{c_1 v_{11}}{c_2 v_{12}}\right)}{\ln\left(\dfrac{\lambda_1}{\lambda_2}\right)} \quad or \quad \frac{\ln\left(\dfrac{c_1 v_{21}}{c_2 v_{22}}\right)}{\ln\left(\dfrac{\lambda_1}{\lambda_2}\right)} \tag{11}$$

depending on which form yields the *ln* (positive number) in the numerator.

If our Red-Blue combat data was in kills/hour, then the battle lasted for 12.28 hours. Often we are interested in the approximate time or length of the battle. These two formulas in equation (11) allow for a quick computation for time.

BATTLE OF IWO JIMA EXAMPLE

The Battle of Iwo Jima has been studied as another historic example as a Lanchester's differential equation by Engel (1954) and Hercilla-Heredia (2008b). At Iwo Jima during WWII, the Japanese had 21,500 soldiers and the United States had 73,000 soldiers. We assume that all forces were initially in place. The combatants engaged in conventional direct warfare, but the Japanese were fighting from reinforced entrenchments. The coefficient of effectiveness for the Japanese was assumed to be 0.0544 while that of the U.S. side was assumed to be 0.0106 (based on data after the battle). If these values are approximately correct, which side should win? How many should remain on the winning side when the other side has only 1500 remaining?

We can move directly to the parity conditions and the analytical solution to answer these questions. First, we use our parity equations:

$$\sqrt{k_1 k_2} x_0 \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} k_1 y_0 \, .$$

We find that $\sqrt{k_1 k_2} x_0 = 0.02401\ (73,000)$ and $k_1 y_0 = (0.0106)\ (21,500)$. Since 1752.97 > 227.90,we know that the United States wins decisively.

The analytical solution is:

$$X(t) = -12145.67 \begin{pmatrix} -1 \\ 0.4414 \end{pmatrix} (1.024)^t + 60854.33 \begin{pmatrix} 1 \\ 0.4414 \end{pmatrix} (0.976)^t$$

or

$$X(t) = \begin{pmatrix} 12145.67 \\ -5361.1 \end{pmatrix} (1.024)^t + \begin{pmatrix} 60854,33 \\ 26861.1 \end{pmatrix} (0.976)^t$$

We solve the equation for the time it takes for the Japanese force to attrite to 1,500 soldiers. We find it takes 30.922 periods for the Japanese to reach 1,500 soldiers. At that time, the model shows that United States had approximately 53,999 soldiers remaining.

The battle actually ended with 1500 Japanese survivors and 44,314 U.S. survivors and took approximately 33 or 34 days. Our model's approximations are off by about 6% in the length of the battle and by 21.8% in the number of surviving U.S. soldiers. The error in surviving soldiers could cause us to revisit the model's assumptions for the explanation. The United States actually used a phased landing over 15 days of actual combat to reach their final force of 73,000 soldiers. If we model this in a fashion similar to how we did the Battle of Trafalgar with the new strategy and phase in the reinforcements due to the landing over the 15 days, we obtain a slightly more accurate depiction of the action. This solution is strictly done numerically and is illustrated in Figure 9.

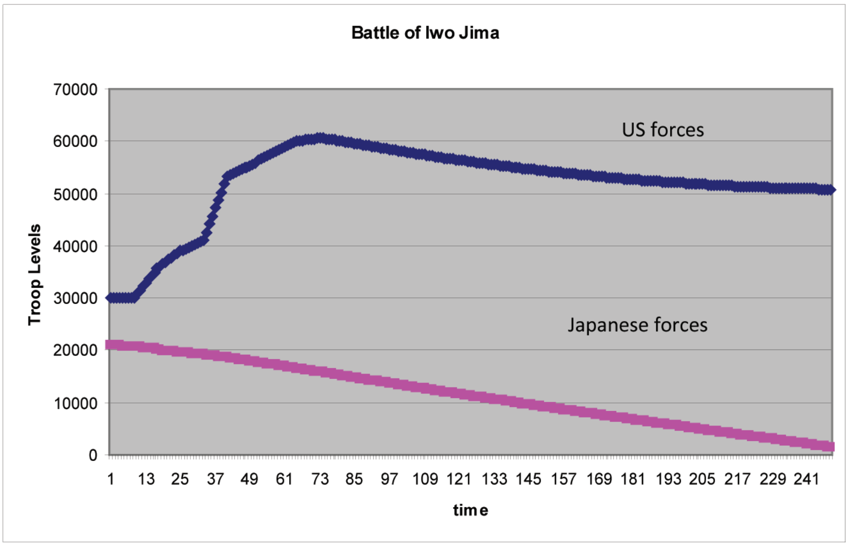
Under these phased landings, the model shows improved accuracy. The models shows that the United States wins in 32 days (underestimated by 3%) and the final U.S. troop strength at the end of the battle is overestimated by approximately 14%.

INSURGENCY AND COUNTER-INSURGENCY OPERATIONS

Today's warfare is considered to be different in that forces from one side are combating insurgents or terrorist factions. According to Powers (2008), Giordano, Fox, Horton, and Weir (2009), and Giordano and McCormick (2007), the dynamics of today's battlefield are quite different but still can be modeled by Lanchester's equations. Consider the later stages of the war in Iraq that has become a multi-ring conflict, according to Kilcullen (2007) and Fox (2008a), as portrayed in Figure 10. Fox (2008a) used dynamical systems to model this conflict.

Insurgency and counter-insurgency operations can be modeled in a simplified sense using a discrete Lanchester's equation model that was modified by Brackney (1959) in Brackney's Mixed Law (also called the Parabolic Law). The Brackney model can be used to repre-

Figure 9. The Battle of Iwo Jima with the phased landings



sent Guerilla warfare and can also be used to represent insurgency and counter-insurgency operations.

We define *Y(n)* to be the insurgent strength after period n. and we define *X(n)* to be the government troop strength after period *n*.

Then we develop the discrete form as,

$$X(n+1)=X(n)-k_1*X(n)*Y(n) \tag{12}$$

$Y(n+1)=Y(n)-k_2*X(n)$ where k_1 and k_2 are kill rates.

Models of this form have been used to model modern guerilla warfare according to Deitchman (1962), Schaffer (1968, 2007), Fowler (2006), and Giordano and McCormick (2007). In the later stages of the Vietnam War, General William Westmoreland requested an increase in the U.S. troop strength of 206,000 to obtain victory. According to Secretary of Defense McNarmara (1968) in a memorandum to President Johnson, he concluded that the requested force increase would not be sufficient for a U.S. victory. President Johnson later rejected Westmoreland's request that eventually led to the American disengagement in 1973. Perhaps the reason for turning down the troop increase was based on analysis that showed over a million more U.S. soldiers would be needed to insure victory.

Let us assume the Viet Cong, *X(n),* used guerilla warfare and the United States, *Y(n),* used conventional techniques. The combat might be models used Brackney's form:

$X(n+1) = X(n)-k_1 *X(n)*Y(n)$
$Y(n+1)=Y(n)-k_2*X(n)$ where k_1 and k_2 are respective kill rates.

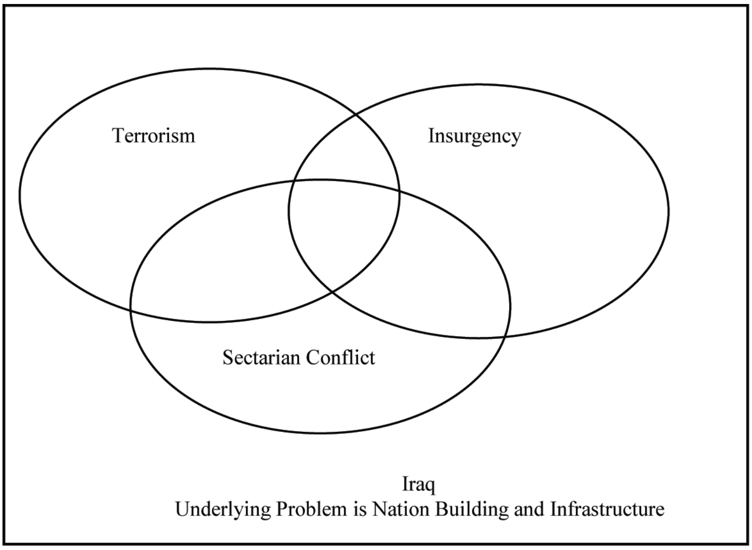
In a fair fight, each the two forces reach 0 at the same time, thus we can infer conditions for a fair fight from these equations.

If $Y = X = 0$,

$$\text{then } Y_0 = \sqrt{\frac{2k_2X_0}{k_1}} \tag{13}$$

During the Vietnam War, the ratio $\dfrac{k_1}{2k_2}$ was estimated after the Tet offensive in 1968 as $\dfrac{1}{1000}$, so $Y_0 = \sqrt{1000X_0}$ at parity. This means

Figure 10. View of the strategic problems in Iraq



that group of $X_0 = 10$ guerillas could be effective against a force *10* times their size fighting a conventional battle (Braun, 1983; Borrelli & Coleman, 1998; Coleman, 1983).

The consequence of having small guerilla units (X_0) in Vietnam is that the total armed forces of the United States and South Vietnamese combined needed to be about 10 times that of the Viet Cong and North Vietnamese for the United States and South Vietnamese to have a reasonable chance of victory. In 1968, the ratio of forces was at 6 to 1. General Westmoreland asked for an additional 206,000 troops, but this request was rejected by President Johnson. Would the additional troops have turned the tide of the war?

In 1968, there were 280,000 guerilla forces of the Viet Cong and North Vietnamese regulars and 1,680,000 forces of the United States (510,000), its allies (70,000) and South Vietnamese soldiers (1,100,000). The ratio of troops at the time of the request was $\frac{1,680,000}{280,000} \approx \frac{6}{1}$.

With the additional 206,000 US troops, the ratio would have been raised to, $\frac{1,866,000}{280,000} \approx \frac{6.7}{1}$. This was not nearly enough to affect the outcome of the conflict. To truly be productive, the United States needed to increase its troop level from 510,000 to more than 1,120,000. Clearly, this was an impossible number.

Schaffer (2007) states that modern, 21st century, terrorist-inspired insurgencies are comparable to the Vietnam experience and can be modeled using Lanchester's equations. It has been pointed out by Schaffer (2007) as well as other documents that the North Vietnamese force size never exceeded approximately 250,000, yet over time they received over 660,000 casualties. This suggests a different model form that allows for growth of the insurgency as well as attrition by combat. We examined the work of Giordano and McCormick (2007) that models the total conflict using insurgent and government growth as well as attrition. We modified the model as a discrete set of equations:

$$\begin{aligned} X(n+1) &= X(n) + a \cdot (L_1 - X(n)) \cdot X(n) - k_1 \cdot X(n) \cdot Y(n) \\ Y(n+1) &= Y(n) + b \cdot (L_2 - Y(n)) \cdot Y(n) - k_2 \cdot X(n) \end{aligned} \tag{14}$$

where k_1 and k_2 are kill rates, *a, b* are positive growth constants, and L_1 and L_2 are carrying capacities.

This model is a combination of both a growth model and a combat model and represents when conflict is on-going and growth is still part of the insurgency operation. This system of equations can only be solved numerically and then analyzed using numerical iteration and graphs. Having laptops with Excel enable soldiers and decision makers to characterize these situations and get quick "results" that they can interpret.

Consider the activities currently on going in Afghanistan. The Taliban is gaining strength recruiting from both Afghanistan and Pakistan. The U.S. forces only grows by a surge in force size that we assume is not forthcoming. Assume we have the following model for the effectiveness of the fighting forces.

Let *x(n)* be the Taliban and *y(n)* be the U.S. forces.

This nonlinear model might be normalized and then simplified as follows:

$$\begin{aligned} x(n+1) &= x(n) + 0.01 \cdot x(n) \cdot (1 - x(n)) - 0.02 \cdot x(n) \cdot y(n) \\ y(n+1) &= y(n) - 0.001 \cdot x(n) \end{aligned} \tag{15}$$

In this nonlinear model, we assume the U.S. forces have not received their surge and that the size of the Taliban is initially slightly larger than the U.S. ground force size. In this unclassified example for analysis, we see that over time the United States will eventually prevail. Figure 11 provides the insights that the United States prevails over the Taliban insurgents.

COMPARISON TO STANDARD LANCHESTER'S EQUATIONS VIA DIFFERENTIAL EQUATIONS

Since we have suggested using the discrete form of Lanchester's equations instead of the differential equation form, let's see how well they compare. Let us revisit the Red and Blue force illustrative example now as a differential equation.

$$\begin{aligned} \frac{dx(t)}{dt} &= -0.1 \cdot y(t) \\ \frac{dy(t)}{dt} &= -0.05 \cdot x(t) \\ x(0) &= 100, y(0) = 50 \end{aligned}$$

This system of differential equations yields the solution accurate to three decimals places:

$$\begin{aligned} x(t) &= 14.644 \cdot e^{0.0707 \cdot t} + 85.355 \cdot e^{-0.707 \cdot t} \\ y(t) &= -10.355 \cdot e^{0.0707 \cdot t} + 60.355 \cdot e^{-0.707 \cdot t} \end{aligned}$$

We provide a plot of the solutions of the differential equations and the solutions by our discrete model using difference equations in Figure 12. Note how close they align.

CONCLUSION

The use of difference equations in combat modeling has practical value. Not only do analytical solutions allow analysts to provide decision makers with quantitative information to quickly analyze potential results but every difference equation has a numerical solution that can be achieved easily. For the decision maker in the field a differential equation is an abstract concept and the tools for analysis are not available to the commander in the field. However, an Excel spreadsheet is a powerful tool for decision makers in that it is available in the field. The systems of difference equations based upon "*Future=Present + Change*" is an intuitive, non-evasive approach for which every combat model has a numerical solution and some combat models such as the direct fire models have analytical solutions that directly lend themselves to analysis and results. The modern combat scenario of terrorist using insurgency tactics, lends itself to more nonlinear

Figure 11. The war in Afghanistan showing the United States prevailing over time

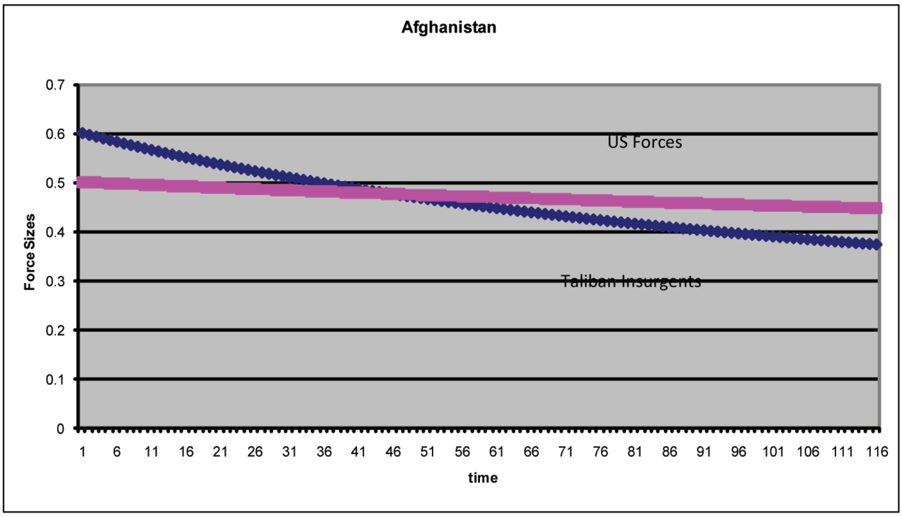
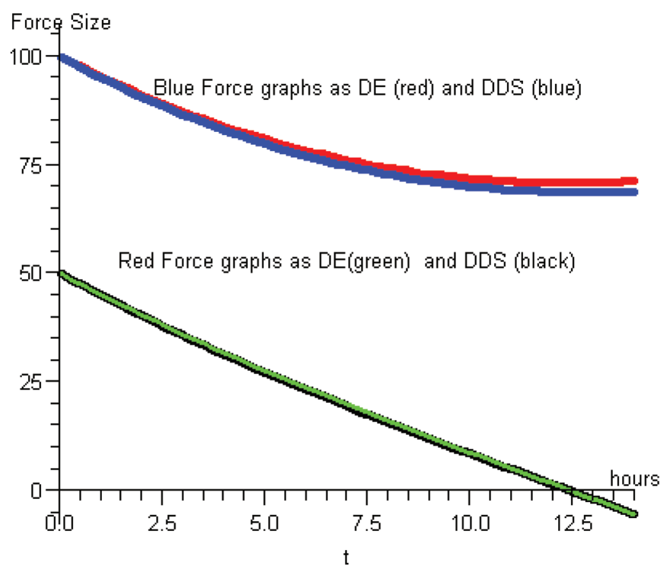


Figure 12. Solution plots of Blue versus Red forces via differential equations and difference equations showing graphically that they are practically the same



models of combat that are more easily analyzed using the numerical and graphical solutions of a difference equation. We are currently teaching this methodology and these models to our military students in our modeling courses in the Defense Analysis department at the Naval Postgraduate School.

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