Euler's Method and its Applications

Numerical Methods for Dynamical Systems

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Taylor Series

Definition

The Taylor series of a real-value function $f(x)$, that is infinitely differentiable at $x = a$, is a power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

or in summation form,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem (Taylor Series Convergence Theorem)

Let $f(x)$ be an infinitely differentiable function on an open interval I containing a. The Taylor series of $f(x)$ centered at a, converges to $f(x)$ for all x in I if and only if the remainder term

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \quad \text{for some } \xi \in (a,x),$$

satisfies

$$\lim_{n\to\infty} R_n(x) = 0.$$

- ✎ If $R_n(x) \to 0$, the Taylor series converges to $f(x)$.
- ✎ If $R_n(x)$ does not vanish, the Taylor series may diverge or approximate another function.

Example (Exponential Function)

Find the Taylor series of $f(x) = e^x$ around $x = 0$.

Solution: The Taylor series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Compute derivatives:

$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x, \ldots$$

Since $f^{(n)}(0) = e^0 = 1$, we substitute:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Example (Sine Function)

Find the Taylor series of $f(x) = \sin x$ around $x = 0$.

Solution: The Taylor series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Compute derivatives:

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x.$$

Evaluating at $x = 0$:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f''''(0) = 0, \ldots$$

Only odd powers of x remain, leading to:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Example (Cosine Function)

Find the Taylor series of $f(x) = \cos x$ around $x = 0$.

Solution: The Taylor series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Compute derivatives:

$$f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x.$$

Evaluating at $x = 0$:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f''''(0) = 1, \ldots$$

Only even powers of x remain, leading to:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

Example (Natural Logarithm Function)

Find the Taylor series of $f(x) = \ln(1+x)$ around $x = 0$.

Solution: The Taylor series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Compute derivatives:

$$f(x) = \ln(1+x), \quad f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}.$$

Evaluating at $x = 0$:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f''''(0) = -6, \ldots$$

The resulting series is:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1.$$

Example (Taylor Expansion of $f(x+h)$)

Derive the Taylor series expansion of a real-valued, infinitely differentiable function f around x for a small increment $h > 0$.

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Firstly, suppose that f is infinitely differentiable at a. Consider its Taylor series expansion:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .$$

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Next, substitute $x - a = h$ (or $x = a + h$) into the above equation to get:

$$f(a+h) = f(a) + f'(a) \cdot h + \frac{f''(a)}{2!} \cdot h^2 + \frac{f'''(a)}{3!} \cdot h^3 + \cdots .$$

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$$f(a+h) = f(a) + f'(a) \cdot h + \frac{f''(a)}{2!} \cdot h^2 + \frac{f'''(a)}{3!} \cdot h^3 + \cdots .$$

Since a is an arbitrary, substitute it with x into the above equation to get:

$$f(x+h) = f(x) + hf'(x) + h^2 \frac{f''(x)}{2!} + h^3 \frac{f'''(a)}{3!} + \cdots .$$

Initial Value Problem (IVP)

$$\dot{x}(t) = f\big(t, x(t)\big), \quad x(t_0) = x_0.$$

Goal: Construct a discrete approximation $\{x_n\}$ to $x(t)$ at times $t_n = t_0 + nh$.

Time	t_0	$t_0 + h$	$t_0 + 2h$	$\dots$	$t_0 + nh$
Solution	$x(t_0)$	$x(t_0 + h)$	$x(t_0 + 2h)$	$\dots$	$x(t_0 + nh)$
Approximation	x_0	x_1	x_2	$\dots$	x_n

Key Idea: Use a *first-order Taylor expansion* to approximate $x(t + h)$ in terms of $x(t)$.

Step 1: Taylor Expansion Around t

$$x(t+h) = x(t) + h\,x'(t) + \frac{h^2}{2!}\,x''(\xi), \quad \text{for some } \xi \in (t,\, t+h).$$

Since $x'(t) = f\big(t, x(t)\big)$, we *truncate* after the first derivative term:

$$x(t+h) \;\approx\; x(t) + h\,f\big(t, x(t)\big).$$

Remark:

- ✎ Higher-order terms are grouped into $\mathcal{O}(h^2)$.
- ✎ This yields a *local* truncation error of order h^2.

Euler's Method

Step 2: Discretize the Time Domain

- ✎ Define $t_n = t_0 + nh$, where h is the step size.
- ✎ Let $x_n \approx x(t_n)$.

Step 3: Derive the Update Formula

Applying the approximation:

$$x_{n+1} \;=\; x_n + h\, f\big(t_n, x_n\big).$$

Euler Algorithmic Implementation

1. **Initialize:** x_0, t_0, h.
2. **Loop:** $x_{n+1} = x_n + h\, f(t_n, x_n), \quad t_{n+1} = t_n + h$.
3. **Stop:** when t_n reaches (or exceeds) final time T.

Local Truncation Error (LTE):

$$x(t+h) - \big(x(t) + h\,f\big(t, x(t)\big)\big) = \mathcal{O}(h^2).$$

Global Error:

- ✎ Over n steps, the total (global) error is $\mathcal{O}(h)$.
- ✎ This is why Euler's method is called a *first-order* method.

Summary:

- ✎ Euler's method is simple and easy to implement.
- ✎ Accuracy can be improved by decreasing h, but too small h increases computational cost.
- ✎ More advanced methods (e.g., Modified Euler, Runge–Kutta) use higher-order terms of the Taylor expansion for better accuracy.

Example (Exponential Growth)

Solve numerically using Euler's method:

$$\dot{x} = x, \quad x(0) = 1, \quad h = 0.1.$$

Solution: Using Euler's method:

$$x_{n+1} = x_n + hx_n.$$

Starting at $x_0 = 1$:

$$\begin{aligned} x_1 &= 1 + 0.1(1) = 1.1. \\ x_2 &= 1.1 + 0.1(1.1) = 1.21. \\ x_3 &= 1.21 + 0.1(1.21) = 1.331. \end{aligned}$$

Compare with exact solution: $x(t) = e^t$.
Observation: Euler's method slightly underestimates the true solution.

Example (Logistic Growth)

Solve numerically using Euler's method:

$$\dot{x} = 2x(1-x), \quad x(0) = 0.1, \quad h = 0.1.$$

Solution: Using Euler's method:

$$x_{n+1} = x_n + h \cdot 2x_n(1 - x_n).$$

Starting at $x_0 = 0.1$:

$$x_1 = 0.1 + 0.1 \cdot 2(0.1)(1 - 0.1) = 0.118.$$
$$x_2 = 0.118 + 0.1 \cdot 2(0.118)(1 - 0.118) = 0.138.$$

Compare with exact solution: $x(t) = \left(1 + 9e^{-2t}\right)^{-1}$.
Observation: Euler's method provides reasonable accuracy for small t but deviates for larger t.

Example (Simple Harmonic Oscillator)

Consider the second-order ODE:

$$\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Step 1: Convert to a First-Order System in Vector Form

Let

$$\mathbf{s} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where } y = \dot{x}.$$

Then,

$$\dot{\mathbf{s}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ -x(t) \end{bmatrix} =: \mathbf{f}\big(\mathbf{y}(t)\big).$$

Example (Simple Harmonic Oscillator)

Consider the second-order ODE:

$$\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Step 2: Apply Euler's Method

$$\mathbf{s}_{n+1} = \mathbf{s}_n + h\,\mathbf{f}(\mathbf{s}_n).$$

In coordinates, this becomes:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} + h \begin{bmatrix} y_n \\ -x_n \end{bmatrix} = \begin{bmatrix} x_n + h\,y_n \\ y_n - h\,x_n \end{bmatrix}.$$

Example (Simple Harmonic Oscillator)

Consider the second-order ODE:

$$\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Step 3: Implementation Example

Initial conditions: $\mathbf{s}_0 = \mathbf{s}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad h = 0.1.$

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix}.$$

$$\mathbf{s}_2 = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix} + 0.1 \begin{bmatrix} -0.1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 - 0.01 \\ -0.1 - 0.1 \end{bmatrix} = \begin{bmatrix} 0.99 \\ -0.2 \end{bmatrix}.$$

Example (Simple Harmonic Oscillator)

Consider the second-order ODE:

$$\ddot{x} + x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Compare with exact solution: $x(t) = \cos(t)$.

Observation: Euler's method introduces numerical damping or growth (energy drift) for oscillatory systems. Over many steps, this leads to inaccurate long-term behavior.