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Novel inertial methods for fixed point problems in reflexive Banach spaces with applications

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Abstract

In this paper, we suggest and analyze four inertial algorithms for solving fixed point problems of Bregman quasi-nonexpansive mappings in the framework of reflexive Banach spaces. Our first two algorithms, we propose inertial-like methods based on Mann-type and Halpern-type iterations, and in the others, we propose relaxed inertial-like methods based on Mann-type and Halpern-type iterations. The weak and strong convergence of the algorithms are established under some appropriate conditions on the parameters. As an application, we utilize our main results to find a zero of the sum of Bregman inverse strongly monotone mappings and maximal monotone operators in real reflexive Banach spaces. Also, we provide several numerical experiments to show the convergence behaviour of our algorithms in both finite-dimensional and infinite-dimensional spaces. Finally, we further utilize our algorithms to numerically solve the data classification problems of lung cancer.

Keywords Reflexive Banach space · Weak convergence · Strong convergence · Bregman quasi-nonexpansive mapping · Fixed point problem

Mathematics Subject Classification 47H09, 47H10, 47J25, 47J05

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1 Introduction

Many problems that occur in various fields of nonlinear analysis can be formulated by the fixed point problem (FPP). The study of FPP has been a rapidly growing area of research over the past until the current year, due to it plays an important role in the study of game theory, optimization problem, integral and differential equations, signal and image processing, machine learning and so on. Up to now, several iterative methods have been constructed and developed for solving the FPP of various classes of nonlinear mappings by many authors (see for instance [20, 28, 32, 42, 45, 50, 54, 56, 57]).

Let E be a real Banach space with norm $\|\cdot\|$ and E^* be its dual space. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ a mapping. A point $x \in C$ is said to be *fixed point* of T if $x = Tx$. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) := \{x \in C : x = Tx\}$. A mapping T is said to be:

(i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C;$$

(ii) *quasi-nonexpansive* if

$$\|Tx - z\| \leq \|x - z\|, \ \forall x \in C, \ z \in F(T);$$

(iii) *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in C.$$

Remark 1.1 It is easy to see that a nonexpansive mapping is a quasi-nonexpansive mapping with nonempty fixed point sets. If $L \in [0, 1)$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping.

One of the classical methods for solving FPP of a nonexpansive mapping is the *Mann's iteration method* which was introduced in 1953 by Mann [33]. This iteration can be given as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \end{cases} \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $T : C \to C$ is a nonexpansive mapping. It was proved that the Mann's iteration method converges weakly to a unique fixed point of T. This method has been extensively studied and applied in many problems of physical science due to a simple construction which is easy to implement in practice. However, the numerical point of view, weak convergence of Mann's iteration may not be enough to make the algorithm more efficient in infinite-dimensional spaces (see [7] and references therein).

So, in order to get the strong convergence result, Halpern [24] introduced the following iterative method, which is known as the *Halpern's iteration*, for finding a fixed point of a nonexpansive mapping defined in real Hilbert spaces:

$$\begin{cases} u, x_1 \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $T : C \to C$ is a nonexpansive mapping. Under appropriate assumptions, he proved that the sequence generated by (2) converges strongly to a fixed point of T. This method is widely used to approximate fixed points for a wider class of nonexpansive type mappings by many authors in various ways (see for instance [29, 41, 54, 63]).

In optimization theory, the inertial technique is a classical tool for the design of algorithms to obtain better performance and faster convergence. This technique was first introduced Polyak [39] in 1964 and it was originated by an implicit time discretization (or heavy ball with friction) of the second-order dynamical systems. Later, Alvarez and Attouch [4] employed the idea of the heavy ball with friction to establish the proximal point algorithm (PPA) for finding a zero point of a set-valued maximal monotone operator A defined in a real Hilbert space H. This method so-called the *inertial PPA* which can be given as follows:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = (I + \lambda_n A)^{-1}(x_n + \theta_n(x_n - x_{n-1})), \end{cases} \tag{3}$$

where $\{\lambda_n\}$ is a positive real sequence, $\theta_n(x_n - x_{n-1})$ is called the *inertial extrapolation term* and θ_n is called the *inertial factor*. They proved that the weak convergence of (3) to a zero point of A under the following conditions:

(C1) $\theta_n \in [0, \theta)$ for some $\theta \in [0, 1)$;

(C2) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$.

In [32], Maingé employed the inertial technique in (3) with Mann's iteration method for approximating fixed points of a nonexpansive mapping T as follows:

$$\begin{cases} x_0, x_1 \in C, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n. \end{cases} \tag{4}$$

He proved the weak convergence of his algorithm under the Conditions (C1) and (C2). The past few years, the inertial technique has successfully been used to enhance the convergence rate and increase the performance of the algorithms for solving the FPP and the related problems (see for instance [21, 23, 36, 48, 49, 51, 52, 55, 59, 61]). However, there have been many modifications of inertial method for solving the FPP have established in Hilbert spaces but only a few these methods established in Banach spaces.

On the other hand, a variant of the inertial method in (3) was suggested by Chbani and Riahi [19]. They suggested the so-called *inertial-like method* which is established as a convex combination of two iterates x_{n-1} and x_n as follows:

$$(1 - \theta_n)x_n + \theta_n x_{n-1} = x_n + \theta_n(x_{n-1} - x_n). \tag{5}$$

In the same time, they proposed two modifications of PPA with the inertial-like technique (5) for solving the Ky Fan equilibrium problems. Also, they proved the weak and strong convergence of these methods in real Hilbert spaces.

Motivated by the results mentioned above, in this paper, we propose four inertial methods for solving the fixed point problems in real reflexive Banach spaces. More specifically, our main contributions in this work are described briefly as follows:

• For the first two inertial methods, we propose inertial-like Mann and Halpern-type iterations, and in the others, we propose relaxed inertial-like Mann and Halpern-type iterations. We also establish the weak and strong convergence of these methods to fixed points of the mappings of Bregman quasi-nonexpansive mappings in reflexive Banach spaces. All methods are based on Bregman distance functions, which are generalized versions of [32, 54] and the previous fixed point algorithms.

• The inertial factors of our two relaxed inertial-like methods are simple and straightforward to choose, which are different from many other works in [21, 26, 26, 59]. As a result, these methods can significantly reduce computation and easy to implement for solving the fixed point problems, and the related problems.

• As a direct sequence of our main results, we obtain the results for solving the quasi-inclusion problems, the variational inequality problem, the equilibrium problems and the convex feasibility problems in reflexive Banach spaces.

• Several numerical tests in finite-dimensional spaces and in infinite-dimensional Banach spaces with a non-Euclidean distance are presented to show the effectiveness of the proposed methods. Also, we perform numerical tests of the proposed methods to show the efficiency of the predictions of lung cancer.

The rest of this paper is organized as follows: In Sect. 2, we collect some preliminaries and lemmas that will be used to prove our main results. In Sect. 3, we analyze and prove the convergence results of the proposed methods. An application of the main results is discussed in Sect. 4 and finally, several numerical experiments of the proposed methods are presented in Sect. 5.

2 Preliminaries

Let E be a real Banach space with the dual space E^* and C be a nonempty subset of E. We denote by $\langle x, j\rangle$ the value of the functional $j \in E^*$ at $x \in E$. The set of real numbers and the set of positive integers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Also, the strong and weak convergence of a sequence $\{x_n\} \subset E$ to a point $v \in E$ are denoted by $x_n \to v$ and $x_n \rightharpoonup v$, respectively.

Let $f : E \to (-\infty, \infty]$ be a function. We denote by $\mathcal{D}(f)$, the domain of f, that is, $\mathcal{D}(f) := \{x \in E : f(x) < \infty\}$ and also denote by $\mathrm{int}(\mathcal{D}(f))$, the interior of the domain of f. The function $f : E \to (-\infty, \infty]$ is said to be:

(1) *proper* if $\mathcal{D}(f) \neq \emptyset$;

(2) *lower semicontinuous* if the set $\{x \in \mathcal{D}(f) : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$;

(3) *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in \mathcal{D}(f)$, $t \in [0, 1]$ and and it is said to be *strictly convex* if the strict inequality holds in a convex function for all $x, y \in \mathcal{D}(f)$ with $x \neq y$;

(4) *uniformly convex* if there exists a continuous and increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-t(1-t)\psi(\|x-y\|)$ for all $x, y \in \mathcal{D}(f)$ and $t \in [0, 1]$, and it is said to be *strongly convex* if f is uniformly convex with $\psi(t) = \sigma t^2$, $\sigma > 0$. It is known that a strongly convex function is also strictly convex;

(5) *uniformly smooth* if there exists a continuous and increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that $f(tx+(1-t)y) \geq tf(x)+(1-t)f(y)-t(1-t)\phi(\|x-y\|)$ for all $x, y \in \mathcal{D}(f)$ and $t \in [0, 1]$;

(6) *bounded on bounded sets* if $f(U)$ is bounded for each bounded subset U of E.

Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. The *Fenchel conjugate* of f is the functional $f^* : E^* \to (-\infty, \infty]$ defined by $f^*(x^*) := \sup_{x\in E}\{\langle x, x^*\rangle - f(x)\}$. A function f on E is said to be *strongly coercive* [62] if $\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty$. A function f is said to be *cofinite* if $\mathcal{D}(f^*) = E^*$. The *directional derivative* of f at $x \in \mathrm{int}(\mathcal{D}(f))$ in the direction $y \in E$ is defined by

$$f'(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{6}$$

If the limit (6) exists for each y, then f is said to be *Gâteaux differentiable at* x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x)\rangle := f'(x, y)$ for all $y \in E$. The function f is said to be *Gâteaux differentiable* if it Gâteaux differentiable at each $x \in \text{int}(\mathcal{D}(f))$. When the limit (6) is attained uniformly for $y \in E$ with $\|y\| = 1$, we say that f is *Fréchet differentiable* at x and it is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (6) is attained uniformly for $x \in C$ and $\|y\| = 1$. From now on, we assume that the function $f : E \to (-\infty, \infty]$ is proper, lower semicontinuous and convex, and $f^* : E^* \to (-\infty, \infty]$ is the Fenchel conjugate of f. If f is Fréchet differentiable, then it is Gâteaux differentiable and if f is Fréchet differentiable, then it is also continuous (see [38, p. 142]).

(i) If f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E (see [46, Proposition 2.1]).

(ii) f is uniformly Fréchet differentiable if and only if f is uniformly smooth (see [62, p. 207]).

(iii) f is uniformly convex if and only if f^* is Fréchet differentiable and ∇f^* is uniformly continuous (see [62, Theorem 3.5.10]).

(iv) f is uniformly convex (uniformly smooth) if and only if f^* is uniformly smooth (uniformly convex) (see [62, Theorem 3.5.5]).

The function $f : E \to (-\infty, \infty]$ is said to be *Legendre* if it satisfies the following two conditions:

(L1) $\text{int}(\mathcal{D}(f)) \neq \emptyset$, f is Gâteaux differentiable on $\text{int}(\mathcal{D}(f))$ and $\mathcal{D}(\nabla f) = \text{int}(\mathcal{D}(f))$;

(L2) $\text{int}(\mathcal{D}(f^*)) \neq \emptyset$, f^* is Gâteaux differentiable on $\text{int}(\mathcal{D}(f^*))$ and $\mathcal{D}(\nabla f^*) = \text{int}(\mathcal{D}(f^*))$.

For a Legendre function f, it is known that ∇f is a bijection from $\text{int}(\mathcal{D}(f))$ into $\text{int}(\mathcal{D}(f^*))$ satisfying $\nabla f = (\nabla f^*)^{-1}$ (see [8, Theorem 5.10]). Several examples of Legendre function can be found in [6, 8]. Another important example of Legendre function when E is a uniformly convex and uniformly smooth Banach space is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. In this case, the gradient of f is coincident with the generalized duality mapping of E, that is, $\nabla f = J_p$, where $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) := \{j^* \in E^* : \langle x, j^*\rangle = \|x\|^p, \|j^*\| = \|x\|^{p-1}\}, \quad \forall x \in E.$$

In particular, if $p = 2$, then $J_p = J$, where J is the normalized duality mapping and if E is a real Hilbert space, then $J_p = I$, where I is the identity mapping. The duality mapping J_p of a smooth Banach space E is said to be *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup^* x$ implies $J_p(x_n) \rightharpoonup^* J_p(x)$.

Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable function. The *Bregman distance* [13] with respect to f is the function $D_f : \mathcal{D}(f) \times \text{int}(\mathcal{D}(f)) \to [0, \infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$$

for all $x \in \mathcal{D}(f)$ and $y \in \text{int}(\mathcal{D}(f))$. It is worth noting that D_f is not a metric because the symmetry and the triangle inequality fail to hold. Note that $D_f(x, x) = 0$, but $D_f(x, y) = 0$ does not imply $x = y$. In this case, when f is Legendre, this indeed holds (see [8, Lemma 7.3 (vi)]).

Define the *negative entropy function* by $f(x) = \sum_{i=1}^m x_i \ln(x_i)$ over the positive orthant $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x_i > 0\}$, we have the *Kullback–Leibler distance* given by

$$D_f(x, y) = \sum_{i=1}^m \left(x_i \ln\left(\frac{x_i}{y_i}\right) + y_i - x_i\right).$$

Note that the Kullback–Leibler distance is a famous example of Bregman distance which used to measure the difference between two probability distributions in statistics. More information on Bregman distances can be found in [40].

Another interesting example of Bregman distance when E is a uniformly convex and uniformly smooth Banach space with $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ is the *p-Lyapunov functional* studied in [12] and it is given by

$$D_f(x, y) := \phi_p(x, y) = \frac{1}{p}\|x\|^p + \frac{1}{q}\|x\|^p - \langle x, J_p(y)\rangle, \ \ \forall x, y \in E,$$

where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and if $p = 2$, then ϕ_p becomes the Lyapunov functional [2, 43] which is given by

$$2D_f(x, y) := \phi(x, y) = \|x\|^2 - \langle x, J(y)\rangle + \|x\|^2, \ \ \forall x, y \in E.$$

Also, if E is a real Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. The Bregman distance has the following useful property called the *three point identity*: for any $x \in \mathcal{D}(f)$ and $y, z \in \text{int}(\mathcal{D}(f))$, it holds that

$$D_f(x, y) = D_f(x, z) - D_f(y, z) + \langle x - y, \nabla f(z) - \nabla f(y)\rangle. \tag{7}$$

The *modulus of total convexity* of f at $x \in \text{int}(\mathcal{D}(f))$ is the function $\upsilon_f : \text{int}(\mathcal{D}(f)) \times [0, \infty) \to [0, \infty]$ defined by

$$\upsilon_f(x, t) := \inf\{D_f(y, x) : y \in \mathcal{D}(f), \|y - x\| = t\}.$$

The function f is called *totally convex* at x if $\upsilon_f(x, t) > 0$ whenever $t > 0$ and f is called *totally convex* if it is totally convex at any point $x \in \text{int}(\mathcal{D}(f))$. More information on totally convex functions can be found in [17]. It is known that if f is Fréchet differentiable and totally convex, then f is cofinite (see [62, Theorem 3.5.10, p.164]).

The *modulus of total convexity* of the function f on the set X is the function $\upsilon_f : \text{int}(\mathcal{D}(f)) \times [0, \infty) \to [0, \infty]$ defined by

$$\upsilon_f(X, t) := \inf\{\upsilon_f(x, t) : x \in X \cap \mathcal{D}(f)\}.$$

The function f is said to be *totally convex on bounded sets* of E if $\upsilon_f(X, t) > 0$ for any nonempty bounded subset X of E and $t > 0$.

Let E be a Banach space. Let $B_r := \{x \in E : \|x\| \leq r\}$ for all $r > 0$. Then a function $f : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets of* E if $\psi_r(t) > 0$ all $r, t > 0$, where $\psi_r : [0, \infty) \to [0, \infty)$ is defined by

$$\psi_r(t) := \inf_{x, y \in B_r, \|x - y\| = t, \alpha \in (0,1)} \frac{\alpha f(x) + (1 - \alpha) f(y) - f(\alpha x + (1 - \alpha) y)}{\alpha(1 - \alpha)}$$

for all $t \geq 0$. The function ψ_r is called the *gauge of the uniform convexity* of f. It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [16, Theorem 2.10]).

The *Bregman projection* with respect to f of $x \in \text{int}(\mathcal{D}(f))$ onto the nonempty, closed and convex set $C \subset \mathcal{D}(f)$ is the minimizer over C defined by

$$\Pi_C^f(x) := \text{argmin}\{D_f(y, x) : y \in C\}.$$

If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then Π_C^f coincides with the generalized projection Π_C (see [1, Definition 7.2]). If E is a Hilbert space, then Π_C^f coincides the metric projection P_C.

Lemma 2.1 ([16, Corollary 4.4]) *Suppose that f is Gâteaux differentiable and totally convex on int($\mathcal{D}(f)$). Let $x \in$ int($\mathcal{D}(f)$) and let C be a nonempty, closed and convex subset of int($\mathcal{D}(f)$). If $z \in C$, then the following statements are equivalent:*

(i) $z = \Pi_C^f(x)$ *is the Bregman projection of x onto C with respect to f;*
(ii) z *is the unique solution of the following variational inequality:*

$$\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leq 0, \quad \forall y \in C;$$

(iii) z *is the unique solution of the following inequality:*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \quad \forall y \in C.$$

Let C be a closed and convex subset of E and T be a mapping from C into itself. We denote $F(T)$ by the set of all fixed points of T, that is, $F(T) := \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is said to be:

(1) *Bregman firmly nonexpansive* if for each $x, y \in C$,

$$\langle Tx - Ty, \nabla f(Tx) - \nabla f(Ty)\rangle \leq \langle Tx - Ty, \nabla f(x) - \nabla f(y)\rangle$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x);$$

(2) *Bregman quasi-firmly nonexpansive* if $F(T) \neq \emptyset$ and for each $x \in C$, $z \in F(T)$,

$$\langle Tx - z, \nabla f(x) - \nabla f(Tx)\rangle \geq 0$$

or, equivalently,

$$D_f(z, Tx) + D_f(Tx, x) \leq D_f(z, x);$$

(3) *Bregman relatively nonexpansive* if $F(T) \neq \emptyset$ and for each $x \in C$, $z \in F(T)$,

$$D_f(z, Tx) \leq D_f(z, x),$$

and $I - T$ is demiclosed at zero, that is, whenever a sequence $\{x_n\}$ in C converges weakly to z and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $z \in F(T)$;

(4) *Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and for each $x \in C$, $z \in F(T)$,

$$D_f(z, Tx) \leq D_f(z, x).$$

Remark 2.2 We remark that the class of Bregman quasi-firmly nonexpansive mappings is the class of Bregman quasi-nonexpansive mappings, the class of Bregman firmly nonexpansive mappings is the class of Bregman quasi-nonexpansive mappings with nonempty fixed point sets and the class of Bregman relatively nonexpansive mappings is the class of Bregman quasi-nonexpansive mappings with $I - T$ is demiclosed at zero. In particular, if $E = H$ is a real Hilbert space and $f(x) = \frac{1}{2}\|x\|^2$, then the class of Bregman quasi-nonexpansive mappings and the class of quasi-nonexpansive mappings are equivalent. Indeed for each $x \in C$ and $z \in F(T)$,

$$D_f(z, Tx) \leq D_f(z, x) \Longleftrightarrow \frac{1}{2}\|Tx - z\|^2 \leq \frac{1}{2}\|x - z\|^2 \Longleftrightarrow \|Tx - z\| \leq \|x - z\|.$$

Lemma 2.3 ([47, Lemma 15.5]) *Let $f : E \to (-\infty, \infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of int($\mathcal{D}(f)$). If $T : C \to C$ is Bregman quasi-nonexpansive, then $F(T)$ is closed and convex.*

Let $f : E \to \mathbb{R}$ be a Legendre function. Let $V_f : E \times E^* \to [0, \infty)$ associated with f be defined by

$$V_f(x, x^*) := f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*.$$

We know the following properties [34, Proposition 1]:
(1) V_f is nonnegative and convex in the second variable;
(2) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$;
(3) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \le V_f(x, x^* + y^*)$ for all $x \in E$ and $x^* \in E^*$ and $y^* \in E^*$.
The following result was proved in [37, Lemma 14].

Lemma 2.4 *Let E be a Banach space and $f : E \to \mathbb{R}$ be a convex function which is bounded and uniformly convex on bounded subsets of E. Let $B_s^* := \{x^* \in E^* : \|x^*\| \le s\}$ for all $s > 0$. Then for any $x \in E$, $y^*, z^* \in B_s$,*

$$V_f(x, \alpha y^* + (1-\alpha)z^*) \le \alpha V_f(x, y^*) + (1-\alpha)V_f(x, z^*) - \alpha(1-\alpha)\psi_s^*(\|y^* - z^*\|),$$

where $\alpha \in (0, 1)$ and ψ_s^* is the gauge of uniform convexity of f^*.

Recall that the function f is called *sequentially consistent* (see [16, p. 9]), if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{D}(f)$ and $\operatorname{int}(\mathcal{D}(f))$, respectively, such that the first one is bounded and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.5 ([15, Lemma 2.1.2]) *The function $f : E \to (-\infty, \infty]$ is sequentially consistent if and only if it is totally convex on bounded subsets of E.*

Lemma 2.6 ([44, Lemma 3.1]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Suppose that $x \in E$. If $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

The following lemmas provide some useful properties of real sequences.

Lemma 2.7 ([58, Lemma 1]) *Assume that $\{s_n\}$ and $\{t_n\}$ are two nonnegative real sequences such that $s_{n+1} \le s_n + t_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.*

Lemma 2.8 ([25, Lemma 7]) *Assume that $\{s_n\}$ is a nonnegative real sequence such that*

$$s_{n+1} \le (1-\delta_n)s_n + \delta_n \tau_n \ \textit{ and } \ s_{n+1} \le s_n - \eta_n + \rho_n,$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$, $\{\eta_n\}$ is a nonnegative real sequence and $\{\tau_n\}$, and $\{\rho_n\}$ are real sequences such that $\sum_{n=1}^{\infty} \delta_n = \infty$, $\lim_{n\to\infty} \rho_n = 0$ and $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\lim_{k\to\infty} \sup \tau_{n_k} \le 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.9 ([30, Lemma 3.1]) *Assume that $\{s_n\}$ and $\{c_n\}$ are nonnegative real sequences such that*

$$s_{n+1} \le (1-a_n)s_n + b_n + c_n, \quad \forall n \ge 1,$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. If $\sum_{n=1}^{\infty} a_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{a_n} \le 0$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.10 ([31, Lemma 3.1]) *Let $\{s_n\}$ be a nonnegative real sequence such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_l\} \subset \mathbb{N}$ such that $\lim_{l\to\infty} m_l = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $l \in \mathbb{N}$:*

$$s_{m_l} \le s_{m_l+1} \ \ and \ \ s_l \le s_{m_l+1}.$$

In fact, $m_l := \max\{j \le l : s_j \le s_{j+1}\}$.

3 Convergence results

In this section, we present inertial algorithms for solving the fixed point problem (FPP) and their convergence analyses in reflexive Banach spaces. In what follows, we adopt the convention $[a]_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. The following conditions are needed in the sequel.
Condition 1 The Banach space E is reflexive.
Condition 2 The function $f : E \to \mathbb{R}$ is strongly coercive, Legendre which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E.
Condition 3 The mapping $T : E \to E$ is Bregman quasi-nonexpansive such that $F(T) \neq \emptyset$ and $I - T$ is demiclosed at zero.
Condition 4 The sequences $\{\xi_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy the following conditions:

(C1) $\sum_{n=1}^{\infty} \xi_n < \infty$;
(C2) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$.

3.1 Weak convergence

In this subsection, we introduce two inertial-like algorithms for solving FPP which are constructed based on Mann-type iteration.

The first algorithm is described as follows.

Algorithm 1: Inertial-like Mann-type iteration for FPP

Initialization: Choose $\theta \in [0, 1)$. Let $x_0, x_1 \in E$ be arbitrary.
Iterative steps: Given the current iterate x_{n-1} and x_n $(n \ge 1)$, calculate x_{n+1} as follows:

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Tu_n)), \quad \forall n \ge 1, \end{cases} \tag{8}$$

where $0 \le \theta_n \le \bar{\theta}_n$ such that

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|\nabla f(x_{n-1}) - \nabla f(x_n)\|}, \theta\right\} & \text{if } x_{n-1} \neq x_n, \\ \theta & \text{otherwise.} \end{cases} \tag{9}$$

Set $n := n + 1$ and go to the iterative step.

Remark 3.1 (1) Note that if $x_{n+1} = x_n = u_n$, then $x_n \in F(T)$.

(2) Some special cases of Algorithm 1 are shown as below.

• If $\theta_n = 0$, then Algorithm 1 reduces to the following Mann-type iteration without the inertial-like term for FPP:

$$x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Tx_n)), \quad \forall n \geq 1. \tag{10}$$

• If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ for all $x \in E$, then Algorithm 1 reduces to the following inertial-like Mann-type iteration for FPP:

$$\begin{cases} u_n = J_q(J_p(x_n) + \theta_n(J_p(x_{n-1}) - J_p(x_n))), \\ x_{n+1} = J_q((1-\alpha_n)J_p(x_n) + \alpha_n J_p(Tu_n)), \quad \forall n \geq 1, \end{cases} \tag{11}$$

where $0 \leq \theta_n \leq \bar{\theta}_n$ such that

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|J_p(x_{n-1}) - J_p(x_n)\|}, \theta\right\} & \text{if } x_{n-1} \neq x_n, \\ \theta \text{ otherwise.} \end{cases} \tag{12}$$

Also, if E is a Hilbert space, then the duality mapping J_p in (11) and (12) are reduced to the identity mapping.

Next, we give the following lemma, which is necessary for proving the our first main result.

Lemma 3.2 *Assume that the Conditions 1–4 are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Suppose that $\{x_n\}$ is bounded, then for each $x \in E$, we have*

$$\sum_{n=1}^{\infty} \theta_n \Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ < \infty.$$

Proof Assume that $\{x_n\}$ is bounded and $x \in E$. From (9), we see that

$$\theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \leq \bar{\theta}_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \leq \xi_n.$$

Since $\sum_{n=1}^{\infty} \xi_n < \infty$, it follows that

$$\sum_{n=1}^{\infty} \theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| < \infty. \tag{13}$$

By the three point identity, we see that

$$\begin{aligned} \Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ &= -D_f(x_{n-1}, x_n) + \langle x - x_{n-1}, \nabla f(x_n) - \nabla f(x_{n-1})\rangle \\ &\leq \langle x - x_{n-1}, \nabla f(x_n) - \nabla f(x_{n-1})\rangle \\ &\leq \|\nabla f(x_n) - \nabla f(x_{n-1})\| M, \end{aligned} \tag{14}$$

where $M = \sup_{n \geq 1}\{\|x_{n-1} - x\|\}$. From (13), we get

$$\sum_{n=1}^{\infty} \theta_n \Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ \leq \sum_{n=1}^{\infty} \theta_n \|\nabla f(x_n) - \nabla f(x_{n-1})\| M < \infty.$$

Therefore, $\sum_{n=1}^{\infty} \theta_n \Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ < \infty$. □

Theorem 3.3 *Assume that the Conditions 1–4 are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Suppose in addition that ∇f is weakly sequentially continuous on E. Then $\{x_n\}$ converges weakly to a point in $F(T)$.*

Proof Let $v \in F(T)$. By the property of V_f, we have

$$\begin{aligned} D_f(v, Tu_n) &\le D_f(v, u_n) \\ &= V_f(v, \nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))) \\ &\le (1-\theta_n)D_f(v, x_n) + \theta_n D_f(v, x_{n-1}). \end{aligned} \tag{15}$$

Hence

$$\begin{aligned} D_f(v, x_{n+1}) &\le (1-\alpha_n)D_f(v, x_n) + \alpha_n D_f(v, Tu_n) \\ &\le (1-\alpha_n)D_f(v, x_n) + \alpha_n(1-\theta_n)D_f(v, x_n) + \alpha_n\theta_n D_f(v, x_{n-1}) \\ &= (1-\alpha_n\theta_n)D_f(v, x_n) + \alpha_n\theta_n D_f(v, x_{n-1}) \\ &\le \max\{D_f(v, x_n), D_f(v, x_{n-1})\} \\ &\le \ldots \le \max\{D_f(v, x_1), D_f(v, x_0)\}. \end{aligned}$$

This implies that $\{D_f(v, x_n)\}$ is bounded and so $\{x_n\}$ is bounded by Lemma 2.6. Using Lemma 2.4 and (15), we obtain

$$\begin{aligned} &D_f(v, x_{n+1}) \\ &= V_f(v, (1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Tu_n)) \\ &\le (1-\alpha_n)D_f(v, x_n) + \alpha_n D_f(v, Tu_n) - \alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|) \\ &\le (1-\alpha_n)D_f(v, x_n) + \alpha_n(1-\theta_n)D_f(v, x_n) \\ &\quad + \alpha_n\theta_n D_f(v, x_{n-1}) - \alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|) \\ &\le D_f(v, x_n) + \alpha_n\theta_n\big[D_f(v, x_{n-1}) - D_f(v, x_n)\big]_+ - \alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|). \end{aligned} \tag{16}$$

This implies that

$$D_f(v, x_{n+1}) \le D_f(v, x_n) + \alpha_n\theta_n\big[D_f(v, x_{n-1}) - D_f(v, x_n)\big]_+. \tag{17}$$

From Lemmas 3.2 and 2.7, we have $\lim_{n\to\infty} D_f(v, x_n)$ exists. Then there exists a nonnegative constant γ such that $\gamma = \lim_{n\to\infty} D_f(v, x_n) = \lim_{n\to\infty} D_f(v, x_{n+r})$ for all $r \in \mathbb{N}$. From (16), we see that

$$\begin{aligned} &\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|) \\ &\quad \le D_f(v, x_n) - D_f(v, x_{n+1}) + \alpha_n\theta_n\big[D_f(v, x_{n-1}) - D_f(v, x_n)\big]_+. \end{aligned} \tag{18}$$

From Lemma 3.2, it easy to see that $\lim_{n\to\infty} \theta_n\Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ = 0$ and since $\lim_{n\to\infty} D_f(v, x_n)$ exists, we have

$$\lim_{n\to\infty} \psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|) = 0.$$

By the property of ψ_s^*, we have

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(Tu_n)\| = 0. \tag{19}$$

From (13), we have $\lim_{n\to\infty} \theta_n\|\nabla f(x_{n-1}) - \nabla f(x_n)\| = 0$. This implies that

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(x_n)\| = \lim_{n\to\infty} \theta_n\|\nabla f(x_{n-1}) - \nabla f(x_n)\| = 0. \tag{20}$$

Thus from (19) and (20), we have

$$\begin{aligned}\|\nabla f(u_n) - \nabla f(Tu_n)\| &\leq \|\nabla f(u_n) - \nabla f(x_n)\| + \|\nabla f(x_n) - \nabla f(Tu_n)\| \\ &\to 0.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0. \tag{21}$$

By the reflexivity of E and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$ for some $z \in E$. From (20), we have $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Thus we also have $u_{n_k} \rightharpoonup z$. This together with (21) and the demiclosedness of $I - T$ yields that $z \in F(T)$. Finally, we show that the sequence $\{x_n\}$ converges weakly to z. In order to do this, it is sufficient to show that $\{x_n\}$ has a unique weak cluster point in $F(T)$. Let $\{x_{m_k}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_k} \rightharpoonup z'$ with $z' \neq z$. As proved in above, we have $z' \in F(T)$. Since $\lim_{n\to\infty} D_f(x, x_n)$ exists for any $x \in F(T)$ and ∇f is weakly sequentially continuous, we have

$$\begin{aligned}\lim_{n\to\infty} D_f(z, x_n) &= \lim_{k\to\infty} D_f(z, x_{m_k}) = \liminf_{k\to\infty} D_f(z, x_{m_k}) \\ &= \liminf_{k\to\infty} \Big(D_f(z, z') + D_f(z', x_{m_k}) + \langle z - z', \nabla f(z') - \nabla f(x_{m_k})\rangle\Big) \\ &\geq \liminf_{k\to\infty} D_f(z, z') + \liminf_{k\to\infty} D_f(z', x_{m_k}) + \liminf_{k\to\infty} \langle z - z', \nabla f(z') - \nabla f(x_{m_k})\rangle \\ &> \liminf_{k\to\infty} D_f(z', x_{m_k}) \\ &= \lim_{n\to\infty} D_f(z', x_n).\end{aligned} \tag{22}$$

In a similar way, we have $\lim_{n\to\infty} D_f(z', x_n) > \lim_{n\to\infty} D_f(z, x_n)$. This is a contradiction with (22). Therefore, $z = z'$ and the proof is finished. □

Next, we propose another Mann-type iteration based on relaxed inertial-like method.

Algorithm 2: Relaxed inertial-like Mann-type iteration for FPP

Initialization: Let $x_0, x_1 \in E$ be arbitrary.
Iterative steps: Given the current iterate x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ x_{n+1} = \nabla f^*((1 - \alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Tu_n)), \quad \forall n \geq 1, \end{cases} \tag{23}$$

where $\{\theta_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$. Set $n := n + 1$ and go to the iterative step.

Remark 3.4 It is interesting to remark that Algorithm 2 (also, Algorithm 4) is simpler and easier for implementation due to the inertial factor θ_n is straightforward to choose an appropriate range in $[a, b] \subset (0, 1)$ for some $a, b > 0$.

Theorem 3.5 *Assume that the Conditions 1–4 are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 2. Suppose in addition that ∇f is weakly sequentially continuous on E. Then $\{x_n\}$ converges weakly to a point in $F(T)$.*

Proof From Theorem 3.3, we know that $\{x_n\}$ is bounded. Let $v \in F(T)$. From Lemma 2.4, we see that

$$\begin{aligned} D_f(v, Tu_n) &\le D_f(v, u_n) \\ &= V_f(v, (1-\theta_n)\nabla f(x_n) + \theta_n \nabla f(x_{n-1})) \\ &\le (1-\theta_n)D_f(v, x_n) + \theta_n D_f(v, x_{n-1}) - \theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|). \end{aligned}$$

Using the same arguments as in the proof of Theorem 3.3, we arrive at

$$\begin{aligned} D_f(v, x_{n+1}) \le (1-\alpha_n)D_f(v, x_n) + \alpha_n D_f(v, Tu_n) \\ -\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|). \end{aligned}$$

Hence

$$\begin{aligned} &D_f(v, x_{n+1}) \\ &\quad \le D_f(v, x_n) + \alpha_n\theta_n(D_f(v, x_{n-1}) \\ &\quad\quad -D_f(v, x_n)) - \alpha_n\theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|) \\ &\quad\quad -\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|). \end{aligned} \tag{24}$$

This implies that

$$D_f(v, x_{n+1}) \le D_f(v, x_n) + \alpha_n\theta_n(D_f(v, x_{n-1}) - D_f(v, x_n)). \tag{25}$$

Now, suppose that $\{D_f(v, x_n)\}$ does not decrease at infinity, in the sense that there exists a subsequence $\{D_f(v, x_{n_k})\}$ of $\{D_f(v, x_n)\}$ such that $D_f(v, x_{n_k}) \le D_f(v, x_{n_k+1})$ for all $k \in \mathbb{N}$. Then there exists an increasing sequence $\{m_l\} \subset \mathbb{N}$ such that

$$D_f(v, x_{m_l}) \le D_f(v, x_{m_l+1}) \tag{26}$$

for all $l \in \mathbb{N}$. From (25), we have

$$D_f(v, x_{m_l+1}) \le D_f(v, x_{m_l}) + \alpha_{m_l}\theta_{m_l}(D_f(v, x_{m_l-1}) - D_f(v, x_{m_l})). \tag{27}$$

By the three point identity, we have

$$D_f(v, x_{m_l-1}) - D_f(v, x_{m_l}) = -D_f(x_{m_l-1}, x_{m_l}) + \langle v - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle.$$

Now, we assume without loss of generality that there exists a subsequence $\{m_l\} \subset \mathbb{N}$ such that

$$\langle x - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle \le 0, \quad \forall x \in E.$$

Then from (27), we have

$$\begin{aligned} &D_f(v, x_{m_l+1}) - D_f(v, x_{m_l}) \\ &\quad \le \alpha_{m_l}\theta_{m_l}(D_f(v, x_{m_l-1}) - D_f(v, x_{m_l})) \\ &\quad = -\alpha_{m_l}\theta_{m_l}D_f(x_{m_l-1}, x_{m_l}) + \alpha_{m_l}\theta_{m_l}\langle v - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle \\ &\quad \le \alpha_{m_l}\theta_{m_l}\langle v - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle \\ &\quad \le 0. \end{aligned} \tag{28}$$

Obviously, there is a contradiction between (26) and (28). Then there exists an integer $n_0' \in \mathbb{N}$ such that $D_f(v, x_{n+1}) \le D_f(v, x_n)$ for all $n \ge n_0'$. By the boundedness of $\{x_n\}$, we have $\lim_{n\to\infty} D_f(v, x_n)$ exists. Again from (24), we have

$$\alpha_n\theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|) + \alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|)$$

$$\leq D_f(v, x_n) - D_f(v, x_{n+1}) + \alpha_n\theta_n(D_f(v, x_{n-1}) - D_f(v, x_n)).$$

Hence

$$\lim_{n\to\infty} \phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|) = 0 \text{ and } \lim_{n\to\infty} \psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|) = 0.$$

By the properties of ϕ_s^* and ψ_s^*, we get

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(x_{n-1})\| = 0 \tag{29}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(Tu_n)\| = 0. \tag{30}$$

From (29), we see that

$$\begin{aligned}\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(x_n)\| &= \lim_{n\to\infty} \theta_n\|\nabla f(x_n) - \nabla f(x_{n-1})\| \\ &\leq \lim_{n\to\infty} b\|\nabla f(x_n) - \nabla f(x_{n-1})\| \\ &= 0. \end{aligned} \tag{31}$$

It follows from (30) and (31) that

$$\begin{aligned}\|\nabla f(u_n) - \nabla f(Tu_n)\| &\leq \|\nabla f(u_n) - \nabla f(x_n)\| + \|\nabla f(x_n) - \nabla f(Tu_n)\| \\ &\to 0.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0.$$

The remainder of the proof is similar to the proof of Theorem 3.3, hence we skip it. □

3.2 Strong convergence

In this subsection, we propose modifications of Algorithm 1 to obtain the strong convergence. These algorithms are constructed based on Halpern-type iteration. In order to prove the strong convergence, we need the following additional condition.

Condition 4* The sequences $\{\xi_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(C3) $\lim_{n\to\infty} \frac{\xi_n}{\beta_n} = 0$.

The algorithm is shown as below.

Remark 3.6 Some special cases of Algorithm 3 are shown as below.

Algorithm 3: Inertial-like Halpern-type iteration for FPP

Initialization: Choose $\theta > 0$. Let $x_0, x_1, u \in E$ be arbitrary.
Iterative steps: Given the current iterate x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ y_n = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Tu_n)), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n)), \quad \forall n \geq 1, \end{cases} \tag{32}$$

where $0 \leq \theta_n \leq \bar{\theta}_n$ such that

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|\nabla f(x_{n-1}) - \nabla f(x_n)\|}, \theta\right\} & \text{if } x_{n-1} \neq x_n, \\ \theta \text{ otherwise.} \end{cases} \tag{33}$$

Set $n := n+1$ and go to the iterative step.

• If $\theta_n = 0$, then Algorithm 3 reduces to the following Halpern-type iteration without the inertial-like term for FPP:

$$\begin{cases} y_n = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Tx_n)), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n)), \quad \forall n \geq 1. \end{cases} \tag{34}$$

• If E is uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ for all $x \in E$, then Algorithm 3 reduces to the following inertial-like Halpern-type iteration for FPP:

$$\begin{cases} u_n = J_q(J_p(x_n) + \theta_n(J_p(x_{n-1}) - J_p(x_n))), \\ y_n = J_q((1-\alpha_n)J_p(x_n) + \alpha_n J_p(Tu_n)), \\ x_{n+1} = J_q(\beta_n J_p(u) + (1-\beta_n)J_p(y_n)), \quad \forall n \geq 1, \end{cases} \tag{35}$$

where $0 \leq \theta_n \leq \bar{\theta}_n$ such that

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|J_p(x_{n-1}) - J_p(x_n)\|}, \theta\right\} & \text{if } x_{n-1} \neq x_n, \\ \theta \text{ otherwise.} \end{cases} \tag{36}$$

Lemma 3.7 *Assume that the Conditions 1, 2, 3 and 4* are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 3. Suppose that $\{x_n\}$ is bounded, then for each $x \in E$, we have*

(i) $\lim_{n\to\infty} \theta_n \Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ = 0$;

(ii) $\lim_{n\to\infty} \frac{\theta_n}{\beta_n} \Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ = 0$.

Proof (i) Assume that $\{x_n\}$ is bounded and $x \in E$. From (33), we see that

$$\theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \leq \xi_n.$$

Since $\beta_n \in (0, 1)$ and $\lim_{n\to\infty} \frac{\xi_n}{\beta_n} = 0$, it follows that

$$\theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \leq \frac{\theta_n}{\beta_n} \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \leq \frac{\xi_n}{\beta_n} \to 0. \tag{37}$$

By using the same argument in (14), we deduce that

$$\Big[D_f(x, x_{n-1}) - D_f(x, x_n)\Big]_+ \leq \|\nabla f(x_n) - \nabla f(x_{n-1})\| M. \tag{38}$$

Then from (37) and (38), we have

$$\theta_n\Big[D_f(x,x_{n-1})-D_f(x,x_n)\Big]_+\le \theta_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|M\to 0.$$

Hence $\lim_{n\to\infty}\theta_n\Big[D_f(x,x_{n-1})-D_f(x,x_n)\Big]_+=0.$

(ii) Using (37) and (38), we can easily show that

$$\lim_{n\to\infty}\frac{\theta_n}{\beta_n}\Big[D_f(x,x_{n-1})-D_f(x,x_n)\Big]_+=0.$$

□

Theorem 3.8 *Assume that the Conditions 1, 2, 3 and 4* are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then $\{x_n\}$ converges strongly to $z=\Pi^f_{F(T)}(u)$, where $\Pi^f_{F(T)}$ is the Bregman projection of E onto $F(T)$.*

Proof Let $v\in F(T)$. As we proved in Theorem 3.3, we have

$$D_f(v,Tu_n)\le(1-\theta_n)D_f(v,x_n)+\theta_nD_f(v,x_{n-1}) \tag{39}$$

and

$$D_f(v,y_n)\le(1-\alpha_n\theta_n)D_f(v,x_n)+\alpha_n\theta_nD_f(v,x_{n-1}). \tag{40}$$

Thus we have

$$\begin{aligned}D_f(v,x_{n+1})&\le\beta_nD_f(v,u)+(1-\beta_n)D_f(v,y_n)\\&\le\beta_nD_f(v,u)+(1-\beta_n)\big[(1-\alpha_n\theta_n)D_f(v,x_n)+\alpha_n\theta_nD_f(v,x_{n-1})\big]\\&\le\beta_nD_f(v,u)+(1-\beta_n)\max\{D_f(v,x_n),D_f(v,x_{n-1})\}\\&\le\max\{D_f(v,u),D_f(v,x_n),D_f(v,x_{n-1})\}\\&\le\ldots\le\max\{D_f(v,u),D_f(v,x_1),D_f(v,x_0)\}.\end{aligned}$$

This implies that $\{D_f(v,x_n\}$ is bounded and, in consequence, $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{Tz_n\}$. Let $z=\Pi^f_{F(T)}(u)$. Using the same arguments as in the proof of Theorem 3.3, we have

$$\begin{aligned}D_f(z,y_n)\le{}&D_f(z,x_n)+\alpha_n\theta_n\big[D_f(z,x_{n-1})-D_f(z,x_n)\big]_+\\&-\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n)-\nabla f(Tu_n)\|).\end{aligned} \tag{41}$$

Also by the properties of V_f and from (41), we have

$$\begin{aligned}&D_f(z,x_{n+1})\\&\quad=V_f(z,\beta_n\nabla f(u)+(1-\beta_n)\nabla f(y_n))\\&\quad\le V_f(z,\beta_n\nabla f(u)+(1-\beta_n)\nabla f(y_n)-\beta_n(\nabla f(u)\\&\qquad-\nabla f(z)))+\beta_n\langle x_{n+1}-z,\nabla f(u)-\nabla f(z)\rangle\\&\quad=V_f(z,\beta_n\nabla f(z)+(1-\beta_n)\nabla f(y_n))+\beta_n\langle x_{n+1}-z,\nabla f(u)-\nabla f(z)\rangle\\&\quad\le\beta_nD_f(z,z)+(1-\beta_n)D_f(z,y_n)+\beta_n\langle x_{n+1}-z,\nabla f(u)-\nabla f(z)\rangle\\&\quad\le(1-\beta_n)D_f(z,x_n)+(1-\beta_n)\alpha_n\theta_n\big[D_f(z,x_{n-1})-D_f(z,x_n)\big]_+\\&\qquad-(1-\beta_n)\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n)-\nabla f(Tu_n)\|)\\&\qquad+\beta_n\langle x_{n+1}-z,\nabla f(u)-\nabla f(z)\rangle.\end{aligned} \tag{42}$$

For each $n \geq 1$, we put $s_n := D_f(z, x_n)$, $\delta_n := \beta_n$,

$$\tau_n := (1-\beta_n)\alpha_n \frac{\theta_n}{\beta_n}\Big[D_f(z, x_{n-1}) - D_f(z, x_n)\Big]_+ + \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle,$$

$$\rho_n := (1-\beta_n)\alpha_n\theta_n\Big[D_f(z, x_{n-1}) - D_f(z, x_n)\Big]_+ + \beta_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle$$

and

$$\eta_n := (1-\beta_n)\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|).$$

Then (42) reduces to the following two inequalities for each $n \geq 1$:

$$s_{n+1} \leq (1-\delta_n)s_n + \delta_n\tau_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n.$$

By our assumption, we have $\sum_{n=1}^{\infty}\delta_n = \infty$. Since $\{x_n\}$ is bounded and $\lim_{n\to\infty}\beta_n = 0$, it follows from Lemma 3.7 (i), that $\lim_{n\to\infty}\rho_n = 0$. In order to complete the proof, using Lemma 2.8, it is sufficient to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. Hence

$$\lim_{k\to\infty}\psi_s^*(\|\nabla f(x_{n_k}) - \nabla f(Tu_{n_k})\|) = 0.$$

By the property of ψ_s^*, we have

$$\lim_{k\to\infty}\|\nabla f(x_{n_k}) - \nabla f(Tu_{n_k})\| = 0. \tag{43}$$

From (37), we note that $\lim_{k\to\infty}\theta_{n_k}\|\nabla f(x_{n_k-1}) - \nabla f(x_{n_k})\| = 0$. Hence

$$\|\nabla f(u_{n_k}) - \nabla f(x_{n_k})\| = \theta_{n_k}\|\nabla f(x_{n_k-1}) - \nabla f(x_{n_k})\| \to 0. \tag{44}$$

It then follows from (43) and (44) that

$$\begin{aligned}\|\nabla f(u_{n_k}) - \nabla f(Tu_{n_k})\| &\leq \|\nabla f(u_{n_k}) - \nabla f(x_{n_k})\| + \|\nabla f(x_{n_k}) - \nabla f(Tu_{n_k})\| \\ &\to 0.\end{aligned}$$

We thus get

$$\lim_{k\to\infty}\|u_{n_k} - Tu_{n_k}\| = 0. \tag{45}$$

On the other hand, from the definition of x_{n_k+1}, we have

$$\lim_{k\to\infty}\|\nabla f(x_{n_k+1}) - \nabla f(y_{n_k})\| = \lim_{k\to\infty}\beta_{n_k}\|\nabla f(u) - \nabla f(y_{n_k})\| = 0. \tag{46}$$

Since $y_{n_k} = \nabla f^*((1-\alpha_{n_k})\nabla f(x_{n_k}) + \alpha_{n_k}\nabla f(Tu_{n_k}))$ and by (30), we have

$$\lim_{k\to\infty}\|\nabla f(y_{n_k}) - \nabla f(x_{n_k})\| = \lim_{k\to\infty}\alpha_{n_k}\|\nabla f(x_{n_k}) - \nabla f(Tu_{n_k})\| = 0. \tag{47}$$

It then follows from (46) and (47) that

$$\begin{aligned}\|\nabla f(x_{n_k+1}) - \nabla f(x_{n_k})\| &\leq \|\nabla f(x_{n_k+1}) - \nabla f(y_{n_k})\| + \|\nabla f(y_{n_k}) - \nabla f(x_{n_k})\| \\ &\to 0.\end{aligned} \tag{48}$$

By the reflexivity of a Banach space E and the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup w \in E$ as $i \to \infty$ and

$$\begin{aligned}&\limsup_{k\to\infty}\langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle \\ &= \lim_{i\to\infty}\langle x_{n_{k_i}} - z, \nabla f(u) - \nabla f(z)\rangle = \langle w - z, \nabla f(u) - \nabla f(z)\rangle.\end{aligned}$$

From (44), we have $\lim_{k\to\infty}\|u_{n_k} - x_{n_k}\| = 0$. From this, we also have $u_{n_{k_i}} \rightharpoonup w$. This together with (45) and the demiclosedness of $I - T$ yields that $w \in F(T)$. Hence

$$\limsup_{k\to\infty}\langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle \le 0.$$

From (48), we have $\lim_{k\to\infty}\|x_{n_k+1} - x_{n_k}\| = 0$. Thus we have

$$\limsup_{k\to\infty}\langle x_{n_k+1} - z, \nabla f(u) - \nabla f(z)\rangle = \limsup_{k\to\infty}\langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle \le 0. \quad (49)$$

This together with Lemma 3.7 (ii) and (36) gives that $\limsup_{k\to\infty}\tau_{n_k} \le 0$. By Lemma 2.8, we conclude that $\lim_{n\to\infty} s_n = 0$. Therefore, $x_n \to z := \Pi^f_{F(T)}(u)$. We thus finish the proof. □

Remark 3.9 For iterative scheme (35), if in particular $p \in (2, \infty)$ and $\theta_n \in [\delta, \bar{\theta}_n]$ for some $\delta > 0$, then $\bar{\theta}_n$ defined in (36) can be weaken to the following condition:

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|x_{n-1} - x_n\|}, \theta\right\} & \text{if } x_{n-1} \ne x_n, \\ \theta & \text{otherwise.}\end{cases} \quad (50)$$

From $0 < \delta \le \theta_n \le \bar{\theta}_n$ and (50), we see that $\theta_n\|x_{n-1} - x_n\| \le \xi_n$. Since $\beta_n \in (0, 1)$ and $\lim_{n\to\infty}\frac{\xi_n}{\beta_n} = 0$, we have

$$\theta_n\|x_{n-1} - x_n\| \le \frac{\theta_n}{\beta_n}\|x_{n-1} - x_n\| \le \frac{\xi_n}{\beta_n} \to 0.$$

Hence

$$\lim_{n\to\infty}\theta_n\|x_{n-1} - x_n\| = \lim_{n\to\infty}\|\theta_n x_{n-1} - \theta_n x_n\| = 0$$

and

$$\lim_{n\to\infty}\frac{\theta_n}{\beta_n}\|x_{n-1} - x_n\| = \lim_{n\to\infty}\left\|\frac{\theta_n}{\beta_n}x_{n-1} - \frac{\theta_n}{\beta_n}x_n\right\| = 0.$$

Since J_p is positively homogeneous of degree $p - 1$ and it is uniformly continuous, we obtain

$$\lim_{n\to\infty}\theta_n\|J_p(x_{n-1}) - J_p(x_n)\| = \lim_{n\to\infty}\frac{1}{\theta_n^{p-2}}\|J_p(\theta_n x_{n-1}) - J_p(\theta_n x_n)\| = 0 \quad (51)$$

and

$$\lim_{n\to\infty}\frac{\theta_n}{\beta_n}\|J_p(x_{n-1}) - J_p(x_n)\| = \lim_{n\to\infty}\left(\frac{\beta_n}{\theta_n}\right)^{p-2}\left\|J_p\left(\frac{\theta_n}{\beta_n}x_{n-1}\right) - J_p\left(\frac{\theta_n}{\beta_n}x_n\right)\right\| = 0. (52)$$

Moreover, we can show that

$$\lim_{n\to\infty}\theta_n\Big[\phi_p(x, x_{n-1}) - \phi_p(x, x_n)\Big]_+ = 0 \text{ and } \lim_{n\to\infty}\frac{\theta_n}{\beta_n}\Big[\phi_p(x, x_{n-1}) - \phi_p(x, x_n)\Big]_+ = 0.$$

Moreover, we can prove the strong convergence of the sequence $\{x_n\}$ generated by (35) by using the same method in the proof of Theorem 3.8.

Also, we propose Halpern-type iteration based on relaxed inertial-like method.

Algorithm 4: Relaxed inertial-like Halpern-type iteration for FPP

Initialization: Let $x_0, x_1, u \in E$ be arbitrary.
Iterative steps: Given the current iterate x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ y_n = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Tu_n)), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n)), \quad \forall n \geq 1, \end{cases} \tag{53}$$

where $\{\theta_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$. Set $n := n + 1$ and go to the iterative step.

Theorem 3.10 *Assume that the Conditions 1, 2, 3 and 4* are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 4. Then $\{x_n\}$ converges strongly to $z = \Pi^f_{F(T)}(u)$, where $\Pi^f_{F(T)}$ is the Bregman projection of E onto $F(T)$.*

Proof As proved in Theorem 3.8, we know that $\{x_n\}$ is bounded. Let $z = \Pi^f_{F(T)}(u)$. By the uniform convexity of f^*, we have

$$\begin{aligned} D_f(z, Tu_n) &\leq D_f(z, u_n) \\ &= D_f(z, \nabla f^*((1-\theta_n)\nabla f(x_n) + \theta_n \nabla f(x_{n-1}))) \\ &\leq (1-\theta_n)D_f(z, x_n) + \theta_n D_f(z, x_{n-1}) - \theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|). \end{aligned}$$

Using the same arguments as in the proof of Theorem 3.3, we arrive at

$$\begin{aligned} D_f(z, y_n) &\leq (1-\alpha_n)D_f(z, x_n) + \alpha_n D_f(z, Tu_n) - \alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|) \\ &\leq (1-\alpha_n)D_f(z, x_n) + \alpha_n(1-\theta_n)D_f(z, x_n) + \alpha_n\theta_n D_f(z, x_{n-1}) \\ &\quad -\alpha_n\theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|) \\ &\quad -\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|) \\ &= D_f(z, x_n) + \alpha_n\theta_n(D_f(z, x_{n-1}) - D_f(z, x_n)) \\ &\quad -\alpha_n\theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|) \\ &\quad -\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|). \end{aligned} \tag{54}$$

Hence

$$\begin{aligned} &D_f(z, x_{n+1}) \\ &\leq \beta_n D_f(z, u) + (1-\beta_n)D_f(z, y_n) \\ &\leq \beta_n D_f(z, u) + (1-\beta_n)D_f(z, x_n) + (1-\beta_n)\alpha_n\theta_n(D_f(z, x_{n-1}) - D_f(z, x_n)) \\ &\quad -(1-\beta_n)\alpha_n\theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|) \\ &\quad -(1-\beta_n)\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n) - \nabla f(Tu_n)\|). \end{aligned} \tag{55}$$

This implies that

$$(1-\beta_n)\alpha_n\theta_n(1-\theta_n)\phi_s^*(\|\nabla f(x_n) - \nabla f(x_{n-1})\|)$$

$$
\begin{aligned}
&+(1-\beta_n)\alpha_n(1-\alpha_n)\psi_s^*(\|\nabla f(x_n)-\nabla f(Tu_n)\|)\\
&\le D_f(z,x_n)-D_f(z,x_{n+1})+(1-\beta_n)\alpha_n\theta_n(D_f(z,x_{n-1})-D_f(z,x_n))+\beta_n M, \quad (56)
\end{aligned}
$$

where $M := \sup_{n\geq 1}\{|D_f(z,u)-D_f(z,x_n)|\}$.

Now, we show that $x_n \to z$. To show this, we need to consider two possible cases as follows:

Case 1. There exists $n_0 \in \mathbb{N}$ such that $D_f(z,x_{n+1}) \le D_f(z,x_n)$ for all $n \ge n_0$. It follows by the boundedness of $\{D_f(z,x_n)\}$ that $\lim_{n\to\infty} D_f(z,x_n)$ exists. Hence

$$
\sum_{n=1}^{\infty}(D_f(z,x_{n-1})-D_f(z,x_n)) = \lim_{n\to\infty}(D_f(z,x_0)-D_f(z,x_n)) < \infty,
$$

which gives

$$
\lim_{n\to\infty}(D_f(z,x_{n-1})-D_f(z,x_n)) = 0.
$$

Then from (56), we get

$$
\lim_{n\to\infty}\phi_s^*(\|\nabla f(x_n)-\nabla f(x_{n-1})\|) = 0 \ \text{ and } \ \lim_{n\to\infty}\psi_s^*(\|\nabla f(x_n)-\nabla f(Tu_n)\|) = 0.
$$

By the properties of ϕ_s^* and ψ_s^*, we get

$$
\lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(x_{n-1})\| = 0 \tag{57}
$$

and

$$
\lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(Tu_n)\| = 0. \tag{58}
$$

From (54), we have

$$
\begin{aligned}
\lim_{n\to\infty}\|\nabla f(u_n)-\nabla f(x_n)\| &= \lim_{n\to\infty}\theta_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|\\
&\le \lim_{n\to\infty} b\|\nabla f(x_n)-\nabla f(x_{n-1})\|\\
&= 0.
\end{aligned} \tag{59}
$$

Hence

$$
\begin{aligned}
\|\nabla f(u_n)-\nabla f(Tu_n)\| &\le \|\nabla f(u_n)-\nabla f(x_n)\|+\|\nabla f(x_n)-\nabla f(Tu_n)\|\\
&\to 0.
\end{aligned}
$$

We thus get

$$
\lim_{n\to\infty}\|u_n-Tu_n\| = 0. \tag{60}
$$

From the definition of x_{n+1}, we get

$$
\lim_{n\to\infty}\|\nabla f(x_{n+1})-\nabla f(y_n)\| = \lim_{n\to\infty}\beta_n\|\nabla f(u)-\nabla f(y_n)\| = 0. \tag{61}
$$

Also, from the definition y_n and (58), we get

$$
\lim_{n\to\infty}\|\nabla f(y_n)-\nabla f(x_n)\| = \lim_{n\to\infty}\alpha_n\|\nabla f(x_n)-\nabla f(Tu_n)\| = 0. \tag{62}
$$

It then follows from (61) and (62) that

$$\begin{aligned}\|\nabla f(x_{n_k+1}) - \nabla f(x_{n_k})\| &\leq \|\nabla f(x_{n_k+1}) - \nabla f(y_{n_k})\| + \|\nabla f(y_{n_k}) - \nabla f(x_{n_k})\| \\ &\to 0. \end{aligned} \tag{63}$$

By the reflexivity of E and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in E$ as $k \to \infty$ and

$$\begin{aligned}&\limsup_{k\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \\ &= \lim_{k\to\infty} \langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle = \langle w - z, \nabla f(u) - \nabla f(z)\rangle.\end{aligned}$$

Since $\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0$, we also have $u_{n_k} \rightharpoonup w$. This together with (60) and the demiclosedness of $I - T$ yields that $w \in F(T)$. From (63), we note that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Hence

$$\limsup_{n\to\infty} \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle = \limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

By the property of V_f and (51), we have

$$\begin{aligned}&D_f(z, x_{n+1}) \\ &\quad = V_f(z, \beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n)) \\ &\quad \leq V_f(z, \beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n) \\ &\qquad -\beta_n(\nabla f(u) - \nabla f(z))) + \beta_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\quad = V_f(z, \beta_n \nabla f(z) + (1-\beta_n)\nabla f(y_n)) \\ &\qquad +\beta_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\quad \leq \beta_n D_f(z, z) + (1-\beta_n) D_f(z, y_n) \\ &\qquad +\beta_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\quad \leq (1-\beta_n) D_f(z, x_n) + (1-\beta_n)\alpha_n\theta_n(D_f(z, x_{n-1}) - D_f(z, x_n)) \\ &\qquad +\beta_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle.\end{aligned}$$

Now, we know that $\limsup_{n\to\infty}\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0$ and

$$\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n\theta_n(D_f(z, x_{n-1}) - D_f(z, x_n)) \leq \sum_{n=1}^{\infty}(D_f(z, x_{n-1}) - D_f(z, x_n)) < \infty.$$

Then from Lemma 2.9, we obtain $\lim_{n\to\infty} D_f(z, x_n) = 0$, Therefore, $x_n \to z := \Pi^f_{F(T)}(u)$.
Case 2. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that $D_f(z, x_{n_i}) < D_f(z, x_{n_i+1})$ for all $i \in \mathbb{N}$. From Lemma 2.10, we know that there exists a nondecreasing sequence $\{m_l\}$ of $\mathbb{N}$ such that $\lim_{l\to\infty} m_l = \infty$ and satisfies the following properties for all numbers $l \in \mathbb{N}$:

$$D_f(z, x_{m_l}) \leq D_f(z, x_{m_l+1}) \ \text{ and } \ D_f(z, x_l) \leq D_f(z, x_{m_l+1}).$$

From (56), we also have

$$\begin{aligned}&(1-\beta_{m_l})\alpha_{m_l}\theta_{m_l}(1-\theta_{m_l})\phi_s^*(\|\nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\|) \\ &\quad +(1-\beta_{m_l})\alpha_{m_l}(1-\alpha_{m_l})\psi_s^*(\|\nabla f(x_{m_l}) - \nabla f(Tu_{m_l})\|) \\ &\quad \leq D_f(z, x_{m_l}) - D_f(z, x_{m_l+1}) + (1-\beta_{m_l})\alpha_{m_l}\theta_{m_l}(D_f(z, x_{m_l-1}) - D_f(z, x_{m_l})) + \beta_{m_l}\tilde{M} \\ &\quad \leq (1-\beta_{m_l})\alpha_{m_l}\theta_{m_l}(D_f(z, x_{m_l-1}) - D_f(z, x_{m_l})) + \beta_{m_l}\tilde{M},\end{aligned} \tag{64}$$

where $\tilde{M} > 0$. By the three point identity, we see that

$$D_f(z, x_{m_l-1}) - D_f(z, x_{m_l}) = -D_f(x_{m_l-1}, x_{m_l}) + \langle z - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle.$$

Now, we assume that there exists a subsequence $\{m_l\} \subset \mathbb{N}$ such that

$$\langle x - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle \leq 0, \quad \forall x \in E.$$

Hence

$$\begin{aligned} D_f(z, x_{m_l-1}) - D_f(z, x_{m_l}) &= -D_f(x_{m_l-1}, x_{m_l}) + \langle z - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle \\ &\leq \langle z - x_{m_l-1}, \nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\rangle \\ &\leq 0. \end{aligned} \tag{65}$$

It then follows from (64) and (65) that

$$\begin{aligned} &(1-\beta_{m_l})\alpha_{m_l}\theta_{m_l}(1-\theta_{m_l})\phi_s^*(\|\nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\|) \\ &\quad +(1-\beta_{m_l})\alpha_{m_l}(1-\alpha_{m_l})\psi_s^*(\|\nabla f(x_{m_l}) - \nabla f(Tu_{m_l})\|) \\ &\leq \beta_{m_l}\tilde{M}. \end{aligned}$$

We thus get

$$\lim_{l\to\infty} \phi_s^*(\|\nabla f(x_{m_l}) - \nabla f(x_{m_l-1})\|) = 0 \text{ and } \lim_{l\to\infty} \psi_s^*(\|\nabla f(x_{m_l}) - \nabla f(Tu_{m_l})\|) = 0.$$

Using a similar proof to that in **Case 1**, we can easily obtain

$$\limsup_{l\to\infty}\langle x_{m_l+1} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0. \tag{66}$$

Also, we have

$$\begin{aligned} D_f(z, x_{m_l+1}) &\leq (1-\beta_{m_l})D_f(z, x_{m_l}) + (1-\beta_{m_l})\alpha_{m_l}\theta_{m_l}(D_f(z, x_{m_l-1}) - D_f(z, x_{m_l})) \\ &\quad +\beta_{m_l}\langle x_{m_l+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq (1-\beta_{m_l})D_f(z, x_{m_l}) + \beta_{m_l}\langle x_{m_l+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq (1-\beta_{m_l})D_f(z, x_{m_l+1}) + \beta_{m_l}\langle x_{m_l+1} - z, \nabla f(u) - \nabla f(z)\rangle. \end{aligned}$$

Hence

$$D_f(z, x_{m_l+1}) \leq \langle x_{m_l+1} - z, \nabla f(u) - \nabla f(z)\rangle.$$

This implies by (66) that $\lim_{l\to\infty} D_f(z, x_{m_l+1}) = 0$. Since $D_f(z, x_l) \leq D_f(z, x_{m_l+1})$, we have $\lim_{l\to\infty} D_f(z, x_l) = 0$. Therefore, $x_n \to z := \Pi^f_{F(T)}(u)$ and we thus finish the proof. □

4 Application to the quasi-inclusion problems

In this section, we utilize the proposed methods to solving quasi-inclusion problems in reflexive Banach spaces.

Let $\mathcal{A} : E \to E^*$ and $\mathcal{B} : E \to 2^{E^*}$ be single and set-valued operators, respectively. The *quasi-inclusion problem* is to find $z \in E$ such that

$$0 \in (\mathcal{A} + \mathcal{B})z, \tag{67}$$

where 0 is the zero vector in E. We denote the solution set of (67) by $(\mathcal{A}+\mathcal{B})^{-1}0$. Many practical problems arising in science and engineering, such as, in image recovery, signal processing and machine learning can be formulated as quasi-inclusion problem in (67) (see, e.g., [18, 53, 60]). The quasi-inclusion problem (67) includes numerous problems in optimization as special cases. For example, consider the minimization problem of the form

$$\min_{x\in E}\{f(x)+g(x)\}, \tag{68}$$

where $f, g : E \to (-\infty, \infty]$ are proper, lower semicontinuous and convex functions. By Fermat's rule, this problem is equivalent to the problem (67) with $\mathcal{A} = \nabla f$ and $\mathcal{B} = \partial g$. In a particular, if g is the indicator function of C, where C is a closed and convex subset of E, then the problem (68) is equivalent to the classical variational inequality problem which is to find $z \in C$ such that $\langle y - z, \mathcal{A}z\rangle \geq 0$ for all $y \in C$.

Now, we denote the domain, range and graph of an operator $\mathcal{A} : E \to 2^{E^*}$ by $\mathcal{D}(\mathcal{A}) := \{x \in E : \mathcal{A}x \neq \emptyset\}$, $\mathcal{R}(\mathcal{A}) := \bigcup\{\mathcal{A}x : x \in \mathcal{D}(\mathcal{A})\}$ and $G(\mathcal{A}) := \{(x, x^*) \in E \times E^* : x^* \in \mathcal{A}x\}$, respectively. An operator $\mathcal{A} : E \to 2^{E^*}$ is said to be *monotone* if for each (x, x^*) and $(y, y^*) \in G(\mathcal{A})$, we have $\langle x - y, x^* - y^*\rangle \geq 0$. A monotone operator $\mathcal{A}$ is called *maximal*, if its graph is not contained in the graph of any other monotone operators on E. It is known that if $f : E \to \mathbb{R}$ is Gâteaux differentiable, strictly convex and cofinite, then $\mathcal{A}$ is maximal monotone if and only if $\mathcal{R}(\nabla f + \lambda\mathcal{A}) = E^*$ for $\lambda > 0$ (see [10, Corollary 2.4]). It is also known that if $\mathcal{A}$ is maximal monotone, then the set $\mathcal{A}^{-1}0 := \{x \in E : 0 \in \mathcal{A}x\}$ is closed and convex.

Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable convex function and $\mathcal{A} : E \to 2^{E^*}$ be a maximal monotone operator. We define the resolvent of $\mathcal{A}$ corresponding to f by $Res^f_{\lambda\mathcal{A}} := (\nabla f + \lambda\mathcal{A})^{-1} \circ \nabla f$. Then $Res^f_{\lambda\mathcal{A}}$ is single-valued and $F(Res^f_{\lambda\mathcal{A}}) = \mathcal{A}^{-1}0$ (see [9]).

A mapping $\mathcal{A} : E \to 2^{E^*}$ satisfying $\mathcal{R}(\nabla f - \lambda\mathcal{A}) \subset \mathcal{R}(\nabla f)$ is called *Bregman inverse strongly monotone* if $\mathcal{D}(\mathcal{A}) \cap$ int$(\mathcal{D}(f)) \neq \emptyset$ and for any $x, y \in$ int$(\mathcal{D}(f))$ and each $u \in \mathcal{A}x$, $v \in \mathcal{A}y$,

$$\langle \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v), u - v\rangle \geq 0. \tag{69}$$

Remark 4.1 Note that in Hilbert spaces, the class of Bregman inverse strongly monotone mappings contains the class of inverse strongly monotone mapping as a special case. Indeed, if E is a Hilbert space and $f(x) = \frac{1}{2}\|x\|^2$, then the class of Bregman inverse strongly monotone mapping becomes the class of inverse strongly monotone mapping.

For any operator $\mathcal{A}^f : E \to 2^E$ associated with $\mathcal{A}$ for $\lambda > 0$ is defined by

$$\mathcal{A}^f := \nabla f^* \circ (\nabla f - \lambda\mathcal{A}). \tag{70}$$

Note that $\mathcal{D}(\mathcal{A}^f) \subset \mathcal{D}(\mathcal{A}) \cap$ int$(\mathcal{D}(f)$ and $\mathcal{R}(\mathcal{A}^f) \subset$ int$(\mathcal{D}(f))$. It is known that the operator $\mathcal{A}$ is Bregman inverse strongly monotone if and only if $\mathcal{A}^f$ is a single-valued mapping (see [14, Lemma 3.5 (c) and (d), p. 2109]).

The following lemma can be found in [35, Theorems 3.1 and3.2].

Lemma 4.2 *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\mathcal{A} : E \to E^*$ be a Bregman inverse strongly monotone mapping and $\mathcal{B} : E \to 2^{E^*}$ be a maximal monotone operator. Then the following statements hold:*

(i) *For each* $x \in E$ *and* $\lambda > 0$, $S_\lambda x := Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f(x)$ *is a Bregman relatively nonexpansive mapping with* $F(S_\lambda) = (\mathcal{A}+\mathcal{B})^{-1}0$.
(ii) $D_f(z, Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f(x)) + D_f(Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f(x), x) \le D_f(z, x)$ *for all* $z \in (\mathcal{A}+\mathcal{B})^{-1}0$, $x \in E$ *and* $\lambda > 0$.

Next, we present the convergence results of the proposed methods for solving the quasi-inclusion problem (67). We make the following additional conditions:
Condition 3* The operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the following conditions:

(B1) $\mathcal{A} : E \to E^*$ is Bregman inverse strongly monotone;
(B2) $\mathcal{B} : E \to 2^{E^*}$ is maximal monotone;
(B3) $(\mathcal{A}+\mathcal{B})^{-1}0 \ne \emptyset$.

From Lemma 4.2, we see that $S_\lambda := Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f$ is a Bregman relatively nonexpansive mapping, which implies that $Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f$ is a Bregman quasi-nonexpansive mapping with $I - S_\lambda$ is demiclosed at zero. In this point of view, we can set $T := Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f$ in Theorems 3.3, 3.5, 3.8 and 3.10. Therefore, we obtain the following results:

Theorem 4.3 *Assume that the Conditions 1, 2, 3** *and 4 are satisfied. Given* $\lambda > 0$. *Let* $x_0, x_1 \in E$ *and* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f(u_n))), \quad \forall n \ge 1, \end{cases} \tag{71}$$

where the sequence $\{\theta_n\}$ *satisfies either one of the following conditions:*

(i) $0 \le \theta_n \le \bar{\theta}_n$, *where* $\bar{\theta}_n$ *is defined in (9);*
(ii) $\{\theta_n\} \subset [a, b] \subset (0, 1)$ *for some* $a, b > 0$.

Suppose in addition that ∇f *is weakly sequentially continuous on* E. *Then the sequence* $\{x_n\}$ *generated by (71) converges weakly to a point in* $(\mathcal{A}+\mathcal{B})^{-1}0$.

Theorem 4.4 *Assume that the Conditions 1, 2, 3** *and 4** *are satisfied. Given* $\lambda > 0$. *Let* $x_0, x_1, u \in E$ *and* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ y_n = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(Res^f_{\lambda \mathcal{B}} \circ \mathcal{A}^f(u_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(u) + (1-\beta_n)\nabla f(y_n)), \quad \forall n \ge 1, \end{cases} \tag{72}$$

where the sequence $\{\theta_n\}$ *satisfies either one of the following conditions:*

(i) $0 \le \theta_n \le \bar{\theta}_n$, *where* $\bar{\theta}_n$ *is defined in (33);*
(ii) $\{\theta_n\} \subset [a, b] \subset (0, 1)$ *for some* $a, b > 0$.

Then $\{x_n\}$ *converges strongly to a point in* $(\mathcal{A}+\mathcal{B})^{-1}0$.

Remark 4.5 We remark that our main results can be further applied to solve the variational inequality problem, the equilibrium problems and the convex feasibility problems (see [45]).

5 Numerical implementations

In this section, we perform numerical results to inspect the behavior of Algorithm 1 (namely, IL-Mann iteration), Algorithm 2 (namely, RIL-Mann iteration), Algorithm 3 (namely, IL-Halpern iteration) and Algorithm 4 (namely, RIL-Halpern iteration). Also, we compare them

Table 1 Numerical results of all algorithms for Example 5.1

Algorithms	Parameters					
	ξ_n	θ	θ_n	α_n	β_n	u
IL-Mann iteration	$\frac{1}{(n+1)^2}$	0.6	–	0.9	–	–
Mann iteration	–	–	0	–	0.4	–
RIL-Mann iteration	–	–	0.1	0.9	–	–
IL-Halpern iteration	$\frac{1}{(n+1)^2}$	0.6	–	0.9	$\frac{1}{n+1}$	0
Halpern iteration	–	–	0	0.4	$\frac{1}{n+1}$	0
RIL-Halpern iteration	–	–	0.1	0.9	$\frac{1}{n+1}$	0

Table 2 Numerical results of all algorithms for Example 5.1

Algorithms	$x_1 = 1$		$x_1 = -0.5$		$x_1 = \pi$		$x_1 = -3\pi$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
IL-Mann iteration	17	0.001	17	0.002	18	0.001	16	0.001
Mann iteration	41	0.007	39	0.003	45	0.004	43	0.004
RIL-Mann iteration	17	0.001	16	0.001	18	0.001	17	0.001
IL-Halpern iteration	15	0.001	15	0.001	16	0.001	16	0.001
Halpern iteration	31	0.002	29	0.002	35	0.003	33	0.004
RIL-Halpern iteration	13	0.001	13	0.001	14	0.001	15	0.001

with the numerical results of Algorithm 1 without inertial-like term (namely, Mann iteration) and Algorithm 3 without inertial-like term (namely, Halpern iteration). We use $E_n := \|x_{n+1} - x_n\| \leq \epsilon$ with $\epsilon = 10^{-6}$ as the stopping criteria of all algorithms.

Example 5.1 In this example, let $E = \mathbb{R}$ endowed with the usual norm $|\cdot|$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{2}|x|^2$, then $\nabla f = \nabla f^* = I$. Define a mapping $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \frac{x}{3}\cos x$ for all $x \in \mathbb{R}$. It easy to see that $x = 0$ is a unique fixed point of T. Then for any $x \in \mathbb{R}$, we have

$$|Tx - 0| = \left|\frac{x}{3}\cos x\right| \leq \left|\frac{x}{3}\right| < |x| = |x - 0|, \quad \forall x \in \mathbb{R}.$$

Hence T is quasi-nonexpansive. Now, let $\{x_n\} \subset \mathbb{R}$ such that $x_n \to 0$. Note that

$$|(I - T)x_n| = \left|x_n\left(1 - \frac{1}{3}\cos x_n\right)\right| \to 0.$$

Moreover, $T0 = 0$, that is, $0 \in F(T)$. Hence $I - T$ is demi-closed at zero. However, T is not nonexpansive. Choose $x = 2\pi$, $y = \frac{3\pi}{2}$ and note that

$$|Tx - Ty| = \left|\frac{3\pi}{2}\cos(2\pi) - \frac{\pi}{2}\cos\left(\frac{3\pi}{2}\right)\right| = \frac{3\pi}{2} > \left|2\pi - \frac{3\pi}{2}\right| = \frac{\pi}{2}.$$

All parameters of each algorithm are chosen as in Table 1.

For all algorithms, we take $x_0 = x_1$ and make comparison of all algorithms with different $x_1 \in \{1, -0.5, \pi, -3\pi\}$. The numerical results are presented in Table 2 and Fig. 1.

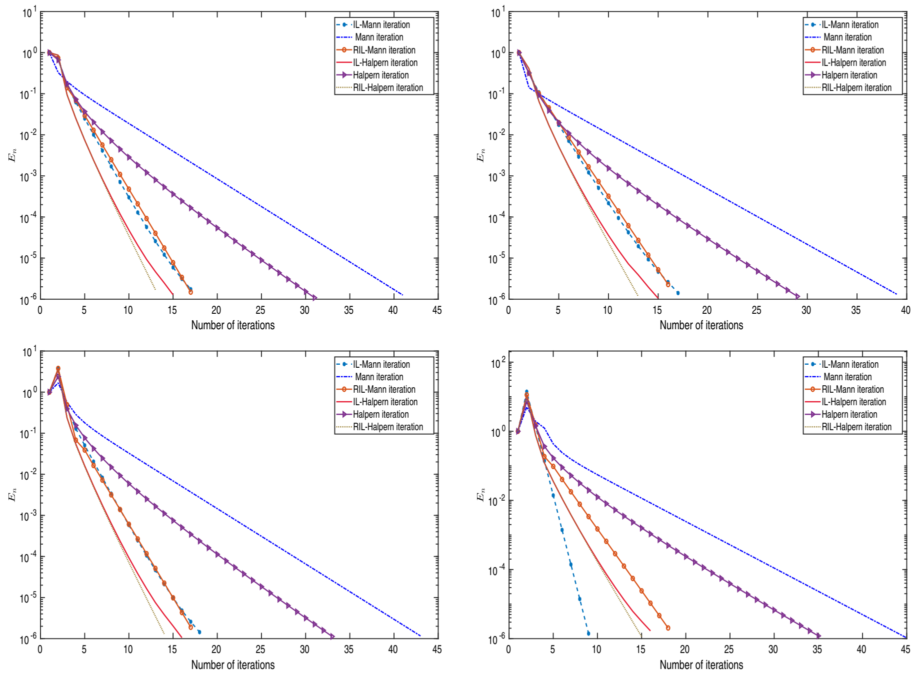


Fig. 1 Numerical results for Example 5.1, Top left: $x_1 = 1$; Top right: $x_1 = -0.5$; Bottom left: $x_1 = \pi$; Bottom right: $x_1 = -3\pi$

For the next two examples, we perform numerical tests in infinite-dimensional Banach spaces which are not Hilbert spaces.

Let $E = \ell_p$ $(1 < p < \infty)$ with the norm $\|x\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |v_k|^p\right)^{1/p}$ and the duality paring $\langle x, y\rangle = \sum_{k=1}^{\infty} v_k w_k$, where $x = (v_1, v_2, \ldots) \in E$ and $y = (w_1, w_2, \ldots) \in E^* = \ell_q$, where $q = \dfrac{p}{p-1}$. Let $f : E \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{p}\|x\|_{\ell_p}^p$, then $\nabla f = J_p$ and $\nabla f^* = J_p^{-1} = J_q$, where J_p and J_q are the generalized duality mappings on E and on E^*, respectively. From [3], J_p and J_q can be computed by the following closed forms:

$$J_p(x) = (|v_1|^{p-2}v_1, |v_2|^{p-2}v_2, \ldots, |v_k|^{p-2}v_k, \ldots), \quad \forall x \in E$$

and

$$J_q(y) = (|w_1|^{q-2}w_1, |w_2|^{q-2}w_2, \ldots, |w_k|^{q-2}w_k, \ldots), \quad \forall y \in E^*.$$

Example 5.2 For each $x \in \ell_p$, define mapping $T : \ell_p \to \ell_p$ by $Tx = \alpha x$ for all $x \in \ell_p$, where $0 < \alpha \leq 1$. It is easy to that $I - T$ is demiclosed at zero with $F(T) = \{\mathbf{0} = (0, 0, \ldots, 0, \ldots)\}$. For each $x \in \ell_p$, we have

$$\begin{aligned} D_f(\mathbf{0}, Tx) &= f(\mathbf{0}) - f(Tx) - \langle \mathbf{0} - Tx, J_p(Tx)\rangle \\ &= f(\mathbf{0}) - f(\alpha x) - \langle \mathbf{0} - Tx, J_p(\alpha x)\rangle \\ &= f(\mathbf{0}) - \alpha^p f(x) - \alpha^p\langle \mathbf{0} - x, J_p(x)\rangle \\ &= \alpha^p\Big(f(\mathbf{0}) - f(x) - \langle \mathbf{0} - x, J_p(x)\rangle\Big) \end{aligned}$$

Table 3 Numerical results of all algorithms for Example 5.2

Algorithms	$m = 100$		$m = 500$		$m = 1000$		$m = 5000$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
IL-Mann iteration	22	0.016	23	0.038	24	0.067	24	0.233
Mann iteration	45	0.015	48	0.047	50	0.067	53	0.205
RIL-Mann iteration	16	0.014	17	0.024	17	0.037	18	0.128
IL-Halpern iteration	19	0.017	20	0.058	20	0.080	21	0.265
Halpern iteration	35	0.017	38	0.048	39	0.078	42	0.234
RIL-Halpern iteration	15	0.010	16	0.035	17	0.053	18	0.175

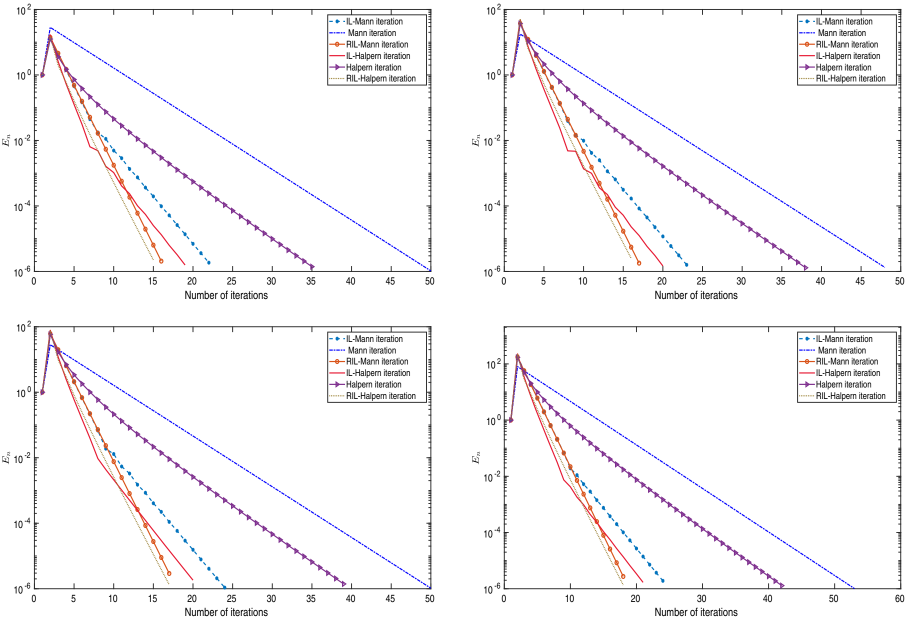


Fig. 2 Numerical results for Example 5.2, Top left: $m = 100$; Top right: $m = 500$; Bottom left: $m = 1000$; Bottom right: $m = 5000$

$$\leq f(\mathbf{0}) - f(x) - \langle \mathbf{0} - x, J_p(x) \rangle = D_f(\mathbf{0}, x).$$

Thus T is a Bregman relatively nonexpansive mapping. For our experimental results, we choose $p = 3/2$ and $\alpha = 1/4$. All parameters of each algorithm are chosen same as in Table 1. The first m-terms of initial points x_0, x_1 are generated randomly for different $m \in \{100, 500, 1000, 5000\}$ with the remaining terms are substituted as zeros. The numerical results are presented in Table 3 and Fig. 2.

Example 5.3 For each $x \in \ell_p$, define mappings $\mathcal{A}, \mathcal{B} : \ell_p \to \ell_q$ by $\mathcal{A}x = \alpha J_p(x)$ and $\mathcal{B}x = \beta J_p(x)$, where $0 < \alpha < 1$ and $\beta > 0$. Now, we show that $\mathcal{A}$ is a Bregman inverse strongly monotone mapping. For each $x, y \in \ell_p$ and by the monotonicity of J_p, we have

$$\langle \nabla f^*(\nabla f(x) - \mathcal{A}x) - \nabla f^*(\nabla f(y) - \mathcal{A}y), \mathcal{A}x - \mathcal{A}y \rangle$$

Table 4 Numerical results of all algorithms for Example 5.3

Algorithms	$m = 1000$		$m = 3000$		$m = 5000$		$m = 9000$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
IL-Mann iteration	133	0.261	137	0.583	139	0.918	141	1.553
Mann iteration	177	0.188	182	0.365	184	0.511	186	0.866
RIL-Mann iteration	41	0.065	42	0.147	43	0.232	43	0.384
IL-Halpern iteration	39	0.105	39	0.269	39	0.433	40	0.825
Halpern iteration	64	0.100	67	0.230	68	0.346	69	0.587
RIL-Halpern iteration	33	0.073	34	0.188	35	0.321	35	0.509

$$
\begin{aligned}
&= \Big\langle J_q\Big(J_p(x) - \alpha J_p(x)\Big) - J_q\Big(J_p(y) - \alpha J_p(y)\Big), \alpha J_p(x) - \alpha J_p(y)\Big\rangle \\
&= \alpha\Big\langle J_q\Big((1-\alpha) J_p(x)\Big) - J_q\Big((1-\alpha) J_p(y)\Big), J_p(x) - J_p(y)\Big\rangle \\
&= \alpha(1-\alpha)^{q-1}\Big\langle x - y, J_p(x) - J_p(y)\Big\rangle \geq 0
\end{aligned}
$$

for all $x, y \in \ell_p$. Thus $\mathcal{A}$ is Bregman inverse strongly monotone. On the other hand, it is easy to see that $\mathcal{B}$ is monotone and $\mathcal{R}(J_p + \lambda\mathcal{B}) = \ell_q$ for all $\lambda > 0$. Thus $\mathcal{B}$ is maximal monotone. The explicit form of the resolvent operator of $\mathcal{B}$ associated to f for $\lambda > 0$ can be written as

$$
\begin{aligned}
Res_{\lambda B}^{f} \circ \mathcal{A}_{\lambda}^{f}(x) &= (\nabla f + \lambda B)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda\mathcal{A})x) \\
&= (J_p + \lambda\mathcal{B})^{-1} \circ J_p \circ (J_q \circ (J_p - \lambda\mathcal{A})x) \\
&= \frac{1}{1+\lambda\beta} J_q\Big((1-\lambda\alpha) J_p(x)\Big) \\
&= \frac{(1-\lambda\alpha)^{q-1}}{1+\lambda\beta} x.
\end{aligned}
$$

We know that $Res_{\lambda\mathcal{B}}^{f} \circ \mathcal{A}_{\lambda}^{f}$ is a Bregman relatively nonexpansive mapping with $(\mathcal{A}+\mathcal{B})^{-1}0 = \{(0, 0, \ldots, 0, \ldots)\}$. Then we can set $T := Res_{\lambda\mathcal{B}}^{f} \circ \mathcal{A}_{\lambda}^{f}$. For our experimental results, we chose $p = 3$, $\alpha = 1/2$ and $\beta = 3$, we have $Tx = \frac{\sqrt{1-\lambda/2}}{1+3\lambda} x$. For all algorithms, we choose $\lambda = 0.1$. All parameters of each algorithm are chosen same as in Table 1. The first m-terms of initial points x_0, x_1 are generated randomly for different $m \in \{1000, 3000, 5000, 9000\}$ with the remaining terms are substituted as zeros. The numerical results are presented in Table 4 and Fig. 3.

Finally, we apply the proposed algorithms to the data classification problems, which based on a learning technique called extreme learning machine (ELM).

Example 5.4 In this example, we are interested to predict datasets for lung cancer. In order to predict lung cancer disease, we use the dataset which is available in [27]. Lung cancer is a type of cancer that reason for death in the world; health systems give reports of over 200,000 new cases of lung cancer each year. About 2 in 5 people with the condition live for at least one year after they're diagnosed, and about 1 in 10 people live for at least ten years. It does not usually cause observable symptoms, only when it's spread through the lungs or into other parts of the body. The effectiveness of a cancer prediction system helps people to know their cancer risk at a low cost, and it also helps people make a practical decision based on their

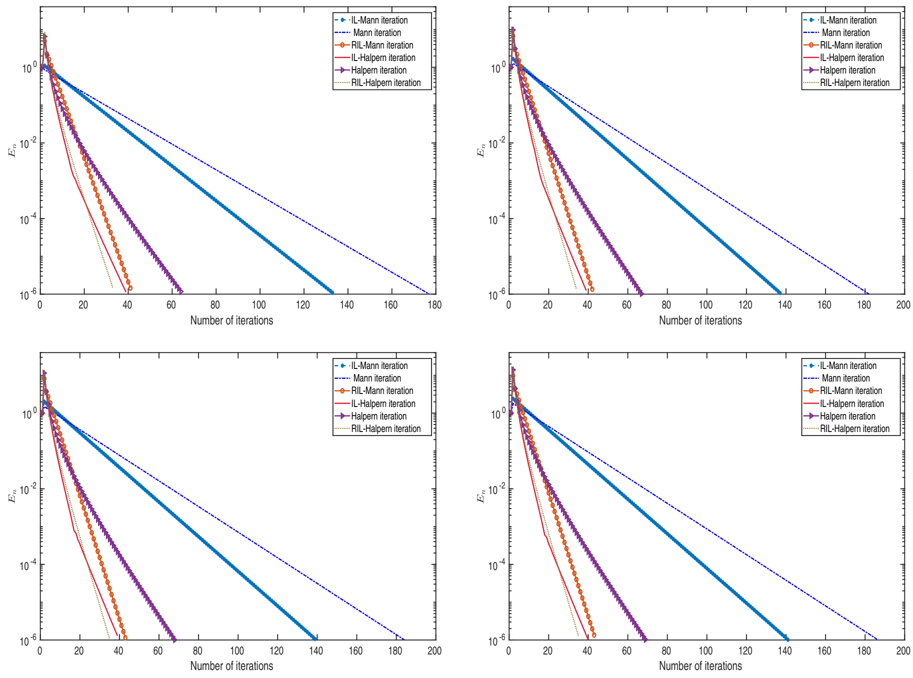


Fig. 3 Numerical results for Example 5.3, Top left: $m = 1000$; Top right: $m = 3000$; Bottom left: $m = 5000$; Bottom right: $m = 9000$

cancer danger status. The data is collected from the website online lung cancer prediction system. The dataset consists of 309 records and 16 characteristics. For the implementation of machine learning algorithms, 60% were used as training, and 40% were used as testing. There are two categories for analyzing the classes: 0 for nonattendance and 1 for the attendance of the disease. We consider the following various algorithms which have been proposed to classify lung cancer as Table 5.

In this case, we use the following extreme learning machine (ELM) for applying our algorithms to solve data classification problems. The training dataset is defined by $S := \{(y_n, b_n) : y_n \in \mathbb{R}^I, b_n \in \mathbb{R}^J, n = 1, 2, ..., K\}$ be the training dataset, where K is distinct samples, y_n is an input data and b_n is a target. For any single hidden layer of ELM with M nodes, the output at the i-th hidden node is

$$O_n = \sum_{i=1}^{M} x_i G(\langle a_i, y_n \rangle + b_i),$$

where G is an activate function, x_i, a_i and b_i are parameters of the optimal output weight, randomly weight and bias at the i-th hidden node, respectively. The hidden layer matrix A is defined as follows:

$$A := \begin{bmatrix} G(\langle a_1, y_1 \rangle + b_1) & \cdots & G(\langle a_M, y_1 \rangle + b_M) \\ \vdots & \ddots & \vdots \\ G(\langle a_1, y_K \rangle + b_1) & \cdots & G(\langle a_M, y_K \rangle + b_M) \end{bmatrix}.$$

Table 5 Distribution of features of the study population

Attribute name	Description	Value type	Variable type
Gender	Patient's gender	(1) women (2) men	Objective
Age	Patient's age in years	int (years)	
Smoking	Whether patient smokes or not	binary	
Yellow fingers	Whether patient yellow fingers or not	binary	
Anxiety	whether patient anxiety or not	binary	
Peer_pressure	Whether patient peer pressure or not	binary	
Chronic disease	Whether patient chronic disease or not	binary	
Fatigue	Whether patient fatigue or not	binary	
Allergy	Whether patient allergy or not	binary	Subjective
Wheezing	Whether patient wheezing or not	binary	
Alcohol	Whether the patient consumes alcohol or not	binary	
Coughing	Whether patient coughing or not	binary	
Shortness of breath	Whether patient shortness of breath or not	binary	
Swallowing difficulty	Whether patient swallowing difficulty or not	binary	
Chest pain	Whether patient chest pain or not	binary	
Lung cancer	Presence or absence of lung cancer	binary	Target

In experiments on regression and classification problems, the main goal of extreme learning machine (ELM) is to find

$$x = (x_1, ..., x_M)^\top \text{ such that } Ax = b, \tag{73}$$

where $b = (b_1, ..., b_K)^\top$ is the training data. The problem (73) can be formulated as the following convex minimization problem:

$$\min_x \{g(x) + h(x)\}, \tag{74}$$

where $g(x) = \frac{1}{2}\|Ax - b\|_2^2$ and $h(x) = \lambda\|x\|_1$, $\lambda > 0$. We denote the set of solutions of (74) by S. It is known from [22, Proposition 3.1 (iii)(b)] that

$$z \in S \Longleftrightarrow 0 \in \nabla g(z) + \partial h(z) \Longleftrightarrow z = Res_{\lambda\partial h}(I - \lambda\nabla g)z, \quad \lambda > 0,$$

where $Res_{\lambda\partial h}$ is the resolvent of ∂h defined by $Res_{\lambda\partial h} := (I + \lambda\partial h)^{-1}$. We know that if $\lambda \in (0, 2/L)$, then $Res_{\lambda\partial h}$ is a nonexpansive mapping (hence it is quasi-nonexpansive), where L is the Lipschitz constant of ∇g. In this point of view, we can set

$$Tx := Res_{\lambda\partial h}(I - \lambda\nabla g)x.$$

To evaluate the quality of the predicted dataset, we use the following matrices [5]:

$$\text{Accuracy (Acc)} = \frac{\text{TP} + \text{TN}}{\text{TP} + \text{FP} + \text{TN} + \text{FN}} \times 100\%,$$
$$\text{Precision (Pre)} = \frac{\text{TP}}{\text{TP} + \text{FP}} \times 100\%,$$
$$\text{Recall (Rec)} = \frac{\text{TP}}{\text{TP} + \text{FN}} \times 100\%$$

and

$$\text{F1} - \text{ score (F1)} = \frac{2 \times \text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}},$$

where TN and TP are the number of negative and positive samples predicted to be negative and positive, respectively. In the meanwhile, FN and FP are the number of positive and negative samples predicted to be negative and positive, respectively. The binary cross entropy loss function calculates the loss of an example by computing the following average:

$$\text{Loss} = -\frac{1}{\text{output size}} \sum_{i=1}^{\text{output size}} \Big(y_i \log \bar{y}_i + (1 - y_i)\log(1 - \bar{y}_i) \Big),$$

where output size is the number of scalar values in the model output, y_i is a corresponding target value and $\bar{y}_i$ is a ith scalar value in the model output.

The sigmoid function is chosen as an activation function for our hidden layer matrix A, we set hidden nodes $M = 270$, $\theta_n = 0.9$, $\beta_n = 1/(50n + 1)$, where $1, 2, ..., N$ and N is a number of iterations, and $u = 0.99$ for RIL-Mann iteration and RIL-Halpern iteration. The calculation is stopped at $N = 500$, or the binary cross entropy loss = 0.15. The different parameter α_n is considered for RIL-Mann iteration and RIL-Halpern iteration, respectively, when $\lambda = \dfrac{0.25}{\max(\text{eigenvalue}(A^\top A))}$. The results are presented in Table 6.

From Table 6, we choose $\alpha_n = 0.9$ for considering the difference parameter $\theta_n = 0.9$. The results are presented in Table 7.

Table 6 Numerical results of training-validation loss and training time of the different parameter α_n

Algorithms	α_n	Training Time	Loss Training	Loss Validation
RIL-Mann iteration	0.9	0.0258	0.3754	0.2571
	0.7	0.0237	0.3878	0.2640
	0.5	0.0222	0.3997	0.2711
	0.1	0.0244	0.4123	0.2787
RIL-Halpern iteration	0.9	0.0231	0.3775	0.2583
	0.7	0.0247	0.3910	0.2659
	0.5	0.0206	0.4001	0.2714
	0.1	0.0234	0.4125	0.2778

Table 7 Numerical results of training-validation loss and training time of the different parameter θ_n

Algorithms	θ_n	Training Time	Loss Training	Loss Validation
RIL-Mann iteration	0.9	0.0217	0.3754	0.2571
	0.7	0.0230	0.3863	0.2631
	0.5	0.0251	0.3922	0.2666
	0.1	0.0262	0.3987	0.2705
RIL-Halpern iteration	0.9	0.0231	0.3775	0.2583
	0.7	0.0236	0.3873	0.2637
	0.5	0.0245	0.3928	0.2670
	0.1	0.0257	0.3990	0.2707

Table 8 Numerical results of training-validation loss and training time of the different parameter β_n

Algorithms	β_n	Training Time	Loss Training	Loss Validation
RIL-Halpern iteration	$1/(2n+1)$	0.0229	0.3973	0.2697
	$1/(50n+1)$	0.0237	0.3775	0.2583
	$1/(2n^{1/2}+1)$	0.0232	0.4194	0.2833
	$1/(50n^{1/2}+1)$	0.0259	0.3951	0.2684

From Tables 6 and 7, we choose $\alpha_n = 0.9$, $u = 0.99$ and $\theta_n = 0.9$ for considering the difference parameter β_n. The results are presented in Table 8.

Next, we provide the several numerical results to show the effectiveness and applicability of the proposed algorithms by numerical experiments on a lung cancer dataset. From Tables 6, 7 and 8, we present choosing the parameters for RIL-Mann iteration and RIL-Halpern iteration to show the efficiency comparing the existing methods. The parameters of each algorithm are chosen in Table 9.

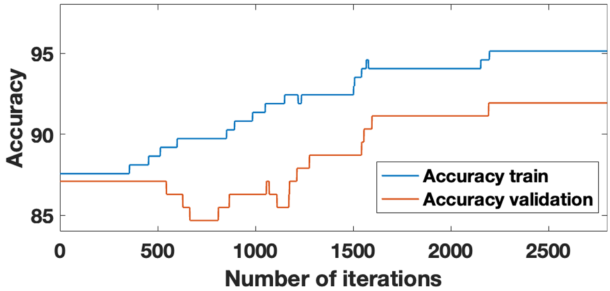
Next, we compare each algorithm with the stopping criteria as accuracy more than 91% or 2800 iterations, the results can be seen in Table 10.

Table 9 All different parameters of each algorithm

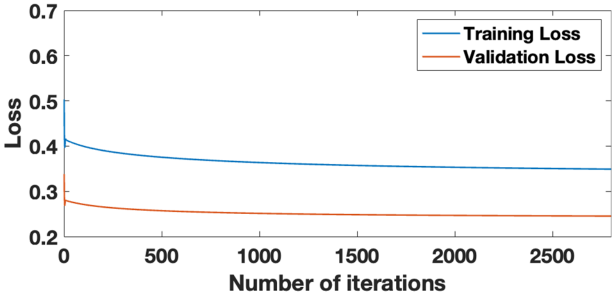
Algorithms	Parameters						
	$\lambda \in (0, 2/L)$	ξ_n	θ	θ_n	α_n	β_n	u
IL-Mann iteration	$\frac{0.25}{\max(\text{eigenvalue}(A^\top A))}$	$1/2n^2$	0.81	–	0.9	–	–
Mann iteration	$\frac{0.25}{\max(\text{eigenvalue}(A^\top A))}$	–	–	0	0.9	–	–
RIL-Mann iteration	$\frac{0.75}{\max(\text{eigenvalue}(A^\top A))}$	–	–	0.9	0.9	–	–
IL-Halpern iteration	$\frac{0.25}{\max(\text{eigenvalue}(A^\top A))}$	$1/2n^2$	0.81	–	0.9	$1/(50n+1)$	0.99
Halpern iteration	$\frac{0.25}{\max(\text{eigenvalue}(A^\top A))}$	–	–	0	0.9	$1/(50n+1)$	0.99
RIL-Halpern iteration	$\frac{0.75}{\max(\text{eigenvalue}(A^\top A))}$	–	–	0.9	0.9	$1/(50n+1)$	0.99

Table 10 The performance of each algorithm

Algorithms	Iteration numbers	Training time	Pre/Rec/F1	Accuracy	Loss		
					Training time	Training	Validation
IL-Mann iteration	2280	0.1995	66.6667/ 62.5000/ 64.5161	91.1290	0.2056	0.3562	0.2480
Mann iteration	2800	0.0949	64.2857/ 56.2500/ 60.0000	90.3226	0.0934	0.3563	0.2480
RIL-Mann iteration	1598	0.0575	66.6667/ 62.5000/ 64.5161	91.1290	0.0589	0.3562	0.2480
IL-Mann iteration	2800	0.3604	66.6667/ 62.5000/ 64.5161	91.1290	0.2397	0.3549	0.2475
Mann iteration	2800	0.1025	58.3333/ 43.7500/ 50.0000	88.7097	0.0954	0.3567	0.2484
RIL-Mann iteration	1920	0.0727	66.6667/ 62.5000/ 64.5161	91.1290	0.0687	0.3559	0.2479



(a) Training and Validation Accuracy



(b) Training and Validation Loss

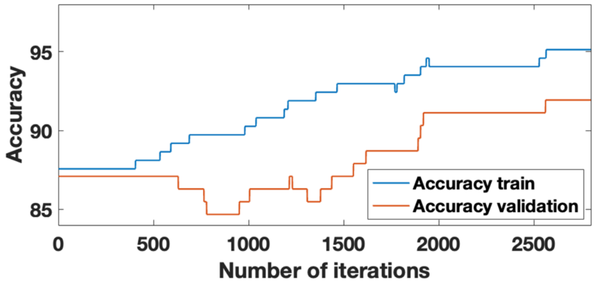
Fig. 4 Training and accuracy, and the validation loss plots of RIL-Mann iteration, respectively

Table 10 shows that RIL-Mann iteration and RIL-Halpern iteration are among those with the highest precision, recall, and accuracy efficiency. Additionally, it has the lowest number of iterations. It has the highest potentiality of classifying lung cancer compared to existing algorithms. We present the training and validation loss with the accuracy of the training to show that RIL-Mann iteration and RIL-Halpern iteration have optimal fitting in training.

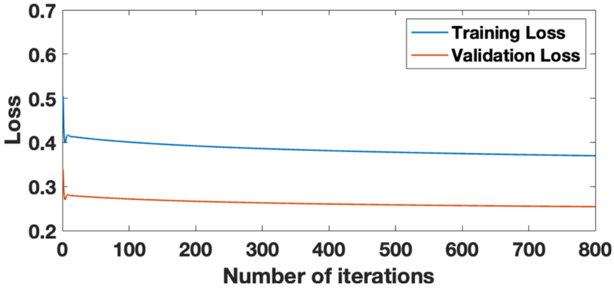
Next, we show the our algorithms with the stopping criteria as 2800 iterations, the results can be seen in Figs. 4 and 5.

Figures 4 and 5 show that RIL-Mann iteration and RIL-Halpern iteration at the 2800 iterations have investigated a precision, recall and F1-score of 68.7500%, and accuracy of 91.9355%. Moreover, we see that RIL-Mann iteration and RIL-Halpern iteration have an optimal fitting model this means that our algorithm suitably learns the training dataset and generalizes well to predict the lung cancer dataset.

Remark 5.5 For our experiments in both finite and infinite-dimensional spaces in Examples 5.1-5.3, we can see that RIL-algorithms (RIL-Mann iteration and RIL-Halpern iteration) have better performances than IL-algorithms and algorithms without inertial factor in terms of number of iterations and elapsed times. This may be because the inertial factor of RIL-algorithms is easy to implement and has low computational costs, and without the prior knowledge of $\bar{\theta}_n$. Moreover, the numerical results show that RIL-algorithms have better efficiency and accuracy than RIL-algorithms and algorithms without inertial factor in the classification and prediction.



(a) Training and Validation Accuracy



(b) Training and Validation Loss

Fig. 5 Training and accuracy, and the validation loss plots of RIL-Halpern iteration, respectively

6 Conclusions

In this paper, we have proposed Mann-type and Halpern-type iterations based on inertial-like and relaxed inertial-like techniques for solving fixed point problems of Bregamn quasi-nonexpansive mappings in reflexive Banach spaces. We have proved both strong and weak convergence theorems of the proposed methods under appropriate conditions. As an application, we have utilized our main results to find a zero of the quasi-inclusion problems in reflexive Banach spaces. Several numerical experiments in finite-dimensional spaces and infinite-dimensional Banach spaces have been provided to show the efficiency of the proposed difference methods in solving some problems including the data classification problems.

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Declarations

Conflict of interest All authors declare that they have no conflicts of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Alber, Y.I.: Generalized projection operators in Banach spaces: Properties and Applications. Proceedings of the Israel Seminar, Vol. 1 of Functional Differential Equation, 1–21, Ariel (1993)
2. Alber, Y.I.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, pp. 15–50. Marcel Dekker, New York (1996)
3. Alber, Y.I., Ryazantseva, I.: Nonlinear Ill-posed Problems of Monotone Type. Springer, Netherlands (2006)
4. Alvarez, F., Attouch, H.: An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Var. Anal. **9**, 3–11 (2001)
5. Azadpour, M., McKay, C.M., Smith, R.L.: Estimating confidence intervals for information transfer analysis of confusion matrices. J. Acoust. Soc. Am. **135**(3), EL140-6 (2014)
6. Bauschke, H.H., Borwein, J.M.: Legendre functions and the method of random Bregman projections. J. Convex Anal. **4**, 27–67 (1997)
7. Bauschke, H.H., Maoušková, E., Reich, S.: Projection and proximal point methods: convergence results and counterexamples. Nonlinear Anal. Theory Methods Appl. **56**, 715–738 (2004)
8. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun. Contemp. Math. **3**, 615–647 (2001)
9. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Bregman monotone optimization algorithms. SIAM J. Control Optim. **42**, 596–636 (2003)
10. Bauschke, H.H., Wang, X., Yao, L.: General resolvents for monotone operators: characterization and extension. In: Cencor, Y., Jiang, M., Wang, G. (eds.) Biomedical Mathematics: Promising Directions in Imagine, Therapy Planning and Inverse Problem (Huangguoshu 2008), pp. 57–74. Medical Physics Plublishing, Madison, WI, USA (2010)
11. Beck, A.: First-Order Methods in Optimization. Society for Industrial and Applied Mathematics, Philadelphia (2017)
12. Bonesky, T., Kazimierski, K.S., Maass, P., Schöpfer, F., Schuster, T.: Minimization of Tikhonov functionals in Banach spaces. Abstr. Appl. Anal. **2008**, 192679 (2008)
13. Bregman, L.M.: The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. **7**, 200–217 (1967)
14. Butnariu, D., Kassay, G.: A proximal-projection method for finding zeroes of set-valued operators. SIAM J. Control Optim. **47**, 2096–2136 (2008)
15. Butnariu, D., Iusem, A.N.: Totally convex functions for fixed points computation and infinite dimensional optimization. Springer, Berlin (2012)
16. Butnariu, D., Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. **2006**, 84919 (2006)
17. Butnariu, D., Reich, S., Zaslavski, A.J.: There are many totally convex functions. J. Convex Anal. **13**, 623–632 (2006)
18. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**, 103–120 (2004)
19. Chbani, Z., Riahi, H.: Weak and strong convergence of an inertial proximal method for solving Ky Fan minimax inequalities. Optim. Lett. **7**, 185–206 (2013)

20. Cholamjiak, P., Suantai, S.: A hybrid method for a family of relatively quasi-nonexpansive mappings and an equilibrium problem in Banach spaces. J. Glob. Optim. **54**, 83–100 (2012)
21. Cholamjiak, W., Cholamjiak, P., Suantai, S.: An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces. J. Fixed Point Theory Appl. **20**, 42 (2018)
22. Combettes, P.L., Wajs, V.R.: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. **4**, 1168–1200 (2005)
23. Dong, Q.L., Cho, Y.J., Rassias, T.M.: General inertial Mann algorithms and their convergence analysis for nonexpansive mappings. Appl. Nonlinear Anal. 175–191 (2018)
24. Halpern, B.: Fixed points of nonexpanding maps. Bull. Am. Math. Soc. **73**, 957–961 (1967)
25. He, S., Yang, C.: Solving the variational inequality problem defined on intersection of finite level sets. Abstr. Appl. Anal. **13**, 8 (2013)
26. Hu, S., Wang, Y., Dong, Q.L.: Convergence analysis of a new Bregman extragradient method for solving fixed point problems and variational inequality problems in reflexive Banach spaces. J. Sci. Comput. **96**, 19 (2023)
27. https://www.kaggle.com/datasets/mysarahmadbhat/lung-cancer (2023). Accessed 1 May 2023
28. Levenshtein, M., Reich, S.: Approximating fixed points of holomorphic mappings in the Hilbert ball. Nonlinear Anal. Theory Methods Appl. **70**, 4145–4150 (2009)
29. Li, Y., Liu, H.B., Zheng, K.: Halpern's iteration for Bregman strongly nonexpansive multi-valued mappings in reflexive Banach spaces with applications. Fixed Point Theory Appl. **2013**, 197 (2013)
30. Maingé, P.E.: Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **325**, 469–479 (2007)
31. Maingé, P.E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899–912 (2008)
32. Maingé, P.E.: Convergence theorems for inertial KM-type algorithms. J. Comput. Appl. Math. **219**, 223–236 (2008)
33. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506–510 (1953)
34. Martín-Márquez, V., Reich, S., Sabach, S.: Iterative Methods for approximating fixed points of Bregman nonexpansive operators. Discrete. Contin. Dyn. Syst. Ser. S. **6**(4), 1043–1063 (2013)
35. Ogbuisi, F.U., Izuchukwu, C.: Approximating a zero of sum of two monotone operators which solves a fixed point problem in reflexive Banach spaces. Numer. Funct. Anal. Optim. **41**, 322–343 (2020)
36. Oyewole, O.K., Reich, S.: An inertial iterative algorithm for approximating solutions to variational inclusion problems in Banach spaces. Pure Appl. Funct. Anal. **8**, 881–902 (2023)
37. Pang, C.T., Naraghirad, E.: Approximating common fixed points of Bregman weakly relatively nonexpansive mappings in Banach spaces. J. Funct. Spaces. **214**, 19 (2014)
38. Pathak, H.K.: An Introduction to Nonlinear Analysis and Fixed Point Theory. Springer Nature Singapore Pte Ltd., Singapore (2018)
39. Polyak, B.T.: Some methods of speeding up the convergence of iterates methods. USSR Comput. Math. Phys. **4**(5), 1–17 (1964)
40. Reem, D., Reich, S., De Pierro, A.: Re-examination of Bregman functions and new properties of their divergences. Optimization. **68**, 279–348 (2019)
41. Reich, S.: Approximating fixed points of nonexpansive mappings. Panam. Math. J. **4**, 23–28 (1994)
42. Reich, S.: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274–276 (1979)
43. Reich, S.: A weak convergence theorem for the alternating method with Bregman distances, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Marcel Dekker, New York (1996)
44. Reich, S., Sabach, S.: Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim. **31**, 2–44 (2010)
45. Reich, S., Sabach, S.: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. Theory Methods Appl. **73**, 122–135 (2010)
46. Reich, S., Sabach, S.: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. **1**, 471–485 (2009)
47. Reich, S., Sabach, S.: Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces. In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, 299–314. Springer, New York (2011)
48. Reich, S., Tuyen, T.M., Sunthrayuth, P., Cholamjiak, P.: Two new inertial algorithms for solving variational inequalities in reflexive Banach spaces. Numer. Funct. Anal. Optim. **42**, 1954–1984 (2021)
49. Reich, S., Thong, D.V., Cholamjiak, P., Van Long, L.: Inertial projection-type methods for solving pseudomonotone variational inequality problems in Hilbert space. Numer. Algorithms. **88**, 813–835 (2021)

50. Sharma, P., Ramos, H., Behl, R., Kanwar, V.: A new three-step fixed point iteration scheme with strong convergence and applications. J. Comput. Appl. Math. **430**, 115242 (2023)
51. Shehu, Y.: Convergence rate analysis of inertial Krasnoselskii-Mann type iteration with applications. Numer. Funct. Anal. Optim. **39**, 1077–1091 (2018)
52. Shehu, Y., Vuong, P.T., Cholamjiak, P.: A self-adaptive projection method with an inertial technique for split feasibility problems in Banach spaces with applications to image restoration problems. J. Fixed Point Theory Appl. **21**, 1–24 (2019)
53. Sra, S., Nowozin, S., Wright, S.J.: Optimization for machine learning. MIT Press, Cambridge (2012)
54. Suantai, S., Cho, Y.J., Cholamjiak, P.: Halpern's iteration for Bregman strongly nonexpansive mappings in reflexive Banach spaces. Comput. Math. Appl. **64**, 489–499 (2012)
55. Suantai, S., Pholasa, N., Cholamjiak, P.: The modified inertial relaxed CQ algorithm for solving the split feasibility problems. J. Ind. Manag. Optim. **14**, 1595–1615 (2018)
56. Sunthrayuth, P., Kumam, P.: A new composite general iterative scheme for nonexpansive semigroups in Banach spaces. Int. J. Math. Math. Sci. **2011**, 18 (2011)
57. Sunthrayuth, P., Kumam, P.: Fixed point solutions of variational inequalities for a semigroup of asymptotically nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2012**, 177 (2012)
58. Tan, K.K., Xu, H.K.: Approximating fixed points of nonexpensive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**(2), 301–308 (1993)
59. Tan, B., Sunthrayuth, P., Cholamjiak, P., Cho, Y.J.: Modified inertial extragradient methods for finding minimum-norm solution of the variational inequality problem with applications to optimal control problem. Int. J. Comput. Math. **100**, 525–545 (2023)
60. Tibshirani, R.: Regression shrinkage and selection via the lasso. J. R. Stat. Soc. B Methodol. **58**, 267–288 (1996)
61. Tuyen, T.M., Sunthrayuth, P., Trang, N.M.: An inertial self-adaptive algorithm for the generalized split common null point problem in Hilbert spaces. Rendiconti del Circolo Matematico di Palermo Series **2**(71), 537–557 (2022)
62. Zălinescu, C.: Convex Analysis in General Vector Spaces, World Scientific, Publishing Co., Inc., River Edge, NJ (2002)
63. Zhu, J.H., Chang, S.S.: Halpern-Mann's iterations for Bregman strongly nonexpansive mappings in reflexive Banach spaces. J. Inequal. Appl. **2013**, 146 (2013)

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RESEARCH ARTICLE

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Inertial-like Bregman projection method for solving systems of variational inequalities

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In this paper, we propose a *self-adaptive inertial-like algorithm* with Bregman distance for approximating a common solution of systems of variational inequalities for a class of monotone and Lipschitz continuous mappings in real reflexive Banach spaces. Our algorithm is constructed without using hybrid projection method and shrinking projection method, and its strong convergence is proved without the prior information of the Lipschitz constant of the mapping. Finally, we provide some numerical experiments to illustrate the performance of the newly proposed method including a comparison with related works in solving signal restoration problems.

KEYWORDS
Bregman distance, monotone mapping, reflexive Banach space, strong convergence, variational inequality problem

MSC CLASSIFICATION
47H09, 47H10, 47J25, 47J05

1 | INTRODUCTION

Let $\mathcal{K}$ be a nonempty, convex, and closed set in a real Banach space; $\mathcal{X}$ and $\mathcal{X}^*$ be the dual of $\mathcal{X}$. We denote the duality pairing between $x \in \mathcal{X}$ and $y^* \in \mathcal{X}^*$ by $\langle x, y^* \rangle$. Let $\mathcal{A} : \mathcal{K} \to \mathcal{X}^*$ be a mapping, the *variational inequality problem* (shortly, VIP) is to find a point $v \in \mathcal{K}$ such that

$$\langle x - v, \mathcal{A}v \rangle \geq 0 \ \forall x \in \mathcal{K}. \tag{1.1}$$

The solution set of VIP is denoted by $VI(\mathcal{K}, \mathcal{A})$. The variational inequality theory is well-known for its crucial roles in many fields of applied science, which include economics, physics, engineering, and other related fields. In particular, it can be applied to mathematical problems, such as optimization problems, game theory, complementarity problems, systems of nonlinear equations, including fixed point problems (see, e.g., earlier studies [1–5]). This explains why a considerable research effort has been wildly devoted in both theory and applications for solving the VIP. Consequently, many researchers have been devoted to finding appropriate numerical algorithms for approximating a solution and common solution of VIP in several settings (see, e.g., earlier studies [6–9] and references therein). A classical and simplest method for solving VIP in a real Hilbert space $\mathcal{H}$ is the projection method, which is defined by $x_{n+1} = P_{\mathcal{K}}(x_n - \lambda \mathcal{A} x_n)$, where $\lambda > 0$ is a suitable stepsize and $P_{\mathcal{K}}$ is the metric (nearest point) projection from $\mathcal{H}$ onto $\mathcal{K}$. The convergence of this method requires a strong monotonicity assumption on $\mathcal{A}$, which is too strong and it may diverge when the strong monotonicity of $\mathcal{A}$ is not satisfied (see He & Liao [10]). However, there are ways to overcome this disadvantage; Korpelevich [11] proposed an iterative method based on the double projection process onto the feasible sets to solve VIP for a class of monotone and Lipschitz continuous mapping in finite-dimensional spaces. This method is known as the *extragradient method* (shortly, EGM). In the last two decades, the EGM has received great interests given by many researchers who modified the method in various ways (see, e.g., [12–15]) and references therein. However, the EGM needs to calculate two projections onto the feasible sets and two values of the mapping at two points in each iteration. As a result, it may be hard to handle, especially if the feasible sets are not explicitly defined or complicated. To reduce the computational cost of the algorithm, *subgradient extragradient method* (shortly, SEGM) has been proposed by Censor et al. [12] (see also earlier works [16, 17]) for solving VIP in a real Hilbert space. Note that the second projection onto $\mathcal{K}$ of this method is replaced by the projection onto a half-space, which can be easier to calculate than the EGM. However, SEGM still required to calculate the value of the mapping at two different points in each iteration. Some novel modifications are proposed in order to reduce the computational cost of the SEGM. Tseng [15] proposed an iterative method called *Tseng's extragradient method* (shortly, TEGM) for solving VIP in a real Hilbert space. This method is more easier to implement than SEGM and EGM due to the fact that TEGM only needs to calculate only one projection in each iteration. Moreover, note that stepsize of the methods mentioned above requires estimating the Lipschitz constant of the mapping before they can be evaluated. This actually limits the applicability of any of these algorithms in practical sense. In order to overcome these disadvantages, Thong and Hieu [18] proposed a modified TEGM with an Armijo-type linesearch technique for solving VIP in a real Hilbert space. Also, Shehu and Iyiola [19] proposed a modified SEGM with an Armijo-type linesearch technique for approximation a solution of the VIP in real Hilbert spaces. However, the linesearch procedure uses an inner iteration, which may take a long time to be computed in each iteration. Another variant of TEGM with a new adaptive stepsize has been proposed by Yang and Liu [9]. This adaptive stepsize is automatically updated over each iteration step by a simple computation, which allows the algorithm to be more easily implemented. Furthermore, it is important to consider generalizing the TEGM to certain Banach spaces since Banach spaces provide a general framework that encompasses a wide range of function spaces, including finite-dimensional spaces and infinite-dimensional spaces. This generality allows for the analysis and treatment of VIP in various contexts. Moreover, many real-world problems naturally lend themselves to the formulation of VIP in Banach spaces. By studying VIP in Banach spaces, one can develop mathematical models and solution techniques that are directly applicable to practical problems in areas such as optimization, economics, physics, and engineering. In this regards, Shehu [20] extended the TEGM with a linesearch procedure for solving the VIP in a two-uniformly convex Banach space. Unfortunately, these result might be too restrictive for some applications in L_p and l_p, for $p > 2$. Recently, some modifications of the original EGM, SEGM, and TEGM have been studied and developed in order to establish the stepsizes that do not require the prior knowledge of the Lipschitz constant of the cost operator (see, e.g., earlier studies [21–23]).

The problem of finding a common solution of system of VIP has become an important research topic in the field of applied science. Censor et al. [24] proposed a modified EGM which is a modification based on hybrid projection method for approximating a solution of systems of VIP in a real Hilbert space. Also, Kitisak et al. [25] proposed a modified hybrid SEGM with Armijo-like stepsize rule for finding a solution of systems of VIP. For solving the system of VIP in a more general setting, Kassay et al. [7] proposed a modified hybrid projection method with Bregman distance for approximating a solution of system of VIP for a class of Bregman inverse with strong monotone mapping in reflexive Banach spaces. It is worthy of noting that any algorithm that uses the hybrid projection method usually yields a strong convergence result which is more desirable especially in infinite-dimensional spaces. However, such algorithms need to calculate a projection onto the intersection of two sets which can be extremely hard to calculate, especially when the structures of the two sets mentioned are complicated. In addition, the inertial technique has recently received much attention from researchers who have applied this technique with the intention to speed up the convergence rate of the algorithms. Applying this technique, Jolaoso et al. [6] proposed an inertial SEGM with adaptive stepsize for solving system of the monotone and Lipschitz continuous VIP in a real Hilbert space. In order to establish an inertial algorithm with Bregman distance, the hybrid projection method (or shrinking projection method) is often used to provide the strong convergence results. Note that constructing algorithms with Bregman distance for solving VIP are useful in the sense that it makes the algorithms easier to implement. The main advantage of this technique is to choose a suitable function so that the projection can be relaxed. Recently, Hieu and Reich [26] proposed a modification of EGM with Bregman projections for solving VIP in a real Hilbert space. Izuchukwu et al. [27] proposed some improvements of EGM in a real reflexive Banach space with the use of Bregman projection methods. Gibali et al. [28] also proposed Bregman inertial hybrid and shrinking projection methods with an Armijo line search technique for solving VIP in a real reflexive Banach space. However, any inertial algorithm with Bregman distance is only applicable for inertial algorithms based on hybrid projection method and shrinking projection method due to structure of Bregman distance and the inertial term in such algorithms. In the past few years, various inertial algorithms have been studied and developed for solving many kinds of optimization problems (see, e.g., previous studies [29–37]). The following questions arise from the results mentioned above:

Question.

(1) Can we extend the TEGM in Shehu [20] to solve the system of VIP in a general Banach space?
(2) Can we modify the TEGM in Shehu [20] that the sequence of stepsizes is chosen without the knowledge of the Lipschitz constant of the mapping and without any linesearch procedure?
(3) Can we modify an inertial TEGM with Bregman distance without using hybrid projection method and shrinking projection method?

The aim of this paper is to give an affirmative answer to the above questions. We propose an *inertial-like algorithm* with Bregman distance for approximating a solution of system of the monotone and Lipschitz continuous VIP in the framework of reflexive Banach spaces. The stepsize of the proposed algorithm employs self-adaptive procedure which is adaptively updated by a simple process without the prior information of the Lipschitz-type constants of the mapping and without any linesearch process. The strong convergence of the proposed algorithm is established without using hybrid projection method and shrinking projection method. Finally, we perform some numerical experiments to demonstrate the effectiveness of our method including comparisons with other related algorithms. The structure of this paper is organized as follows: Some basic results and technical lemmas are given in Section 2. The strong convergence of the proposed algorithm is stated and proved in Section 3, and finally, some numerical tests to illustrate the behaviors of the proposed method are present in Section 4.

2 | PRELIMINARIES

Let $\mathcal{X}$ be a real Banach space with dual space $\mathcal{X}^*$ and $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$. The set of real numbers and the set of positive integers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Also, we use the following notations:

- The weak convergence of $\{x_n\}$ to v is denoted by $x_n \rightharpoonup v$.
- The strong convergence of $\{x_n\}$ to v is denoted by $x_n \to v$.

Throughout this paper, we assume that the function $h : \mathcal{X} \to (-\infty, \infty]$ is proper, convex, and lower semicontinuous, and $h^* : \mathcal{X}^* \to (-\infty, \infty]$ is the Fenchel conjugate of h. We denote by $\operatorname{dom} h$ the domain of h, that is, $\operatorname{dom} h := \{x \in \mathcal{X} : h(x) < \infty\}$. Let $x \in \operatorname{int}(\operatorname{dom} h)$ and $y \in \mathcal{X}$; the *directional derivative* of h at x in the direction $y \in \mathcal{X}$ is defined by

$$h^\circ(x, y) := \lim_{s \downarrow 0} \frac{h(x + sy) - h(x)}{s}. \tag{2.1}$$

The function h is said to be *Gâteaux differentiable* at x if the limit (2.1) exists for each y and $h^\circ(x, y)$ coincides with $\nabla h(x)$ the gradient ∇h at x. If the limit (2.1) is attained uniformly for any $y \in \mathcal{X}$ with $\|y\| = 1$, we say that h is *Fréchet differentiable* at x. We know that if h is Fréchet differentiable function, then it is Gâteaux differentiable (see Pathak [38, p. 142]). The function $h : \mathcal{X} \to \mathbb{R}$ is said to be *uniformly convex* if

$$h(\alpha x + (1 - \alpha)y) \le \alpha h(x) + (1 - \alpha)h(y) - \alpha(1 - \alpha)\varphi(\|x - y\|)$$

for all $x, y \in \operatorname{dom} h$ and $\alpha \in [0, 1]$, where φ is an increasing function vanishing at zero. In particular, if $\varphi(t) = \frac{\kappa}{2}t^2$, then h is *strongly convex* with a constant $\kappa > 0$, and it is also equivalent to the following inequality (see Beck [39, Theorem 5.24]):

$$h(x) \geq h(y) + \langle x - y, \nabla h(y) \rangle + \frac{\kappa}{2}\|x - y\|^2 \tag{2.2}$$

for all $x \in \operatorname{dom} h$ and $y \in \operatorname{int}(\operatorname{dom} h)$. The *Legendre* functions on $\mathcal{X}$ are defined in Reich and Sabach [40, p. 25]. More information on Legendre functions can be found in the recent paper in Reem and Reich [41]. In this case, if h is a Legendre function, then ∇h is a bijection from $\operatorname{int}(\operatorname{dom} h)$ into $\operatorname{int}(\operatorname{dom} h^*)$ with $\nabla h = (\nabla h^*)^{-1}$ (see Bauschke et al. [42, Theorem 5.10]).

Let $h : \mathcal{X} \to (-\infty, \infty]$ be a Gâteaux differentiable function. The bifunction $D_h : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ defined by

$$D_h(x, y) := h(x) - h(y) - \langle x - y, \nabla h(y) \rangle \tag{2.3}$$

is called *Bregman distance with respect to h* [43]. The geometry of Bregman distance is shown in Figure 1.

An important example of Bregman distance when $\mathcal{X}$ is a two-uniformly convex and uniformly smooth Banach space with $h(x) = \frac{1}{2}\|x\|^2$ is the *Lyapunov functional* which was introduced in previous studies [44, 45]. If ϕ is the Lyapunov functional, then it holds that (see Nakajo [46, Lemma 2.3])

$$\phi(x, y) \geq c\|x - y\|^2 \tag{2.4}$$

for some $c > 0$. It is well-known that $\phi(x, y) = \|x - y\|^2$ and $c = 1$ whenever $\mathcal{X}$ is a Hilbert space. More information on Bregman functions and distances can be found in the recent paper in Reem et al. [47]. Note that the Bregman distance is not the usual distance in the sense of the metric, due to it not satisfying the triangle inequality.

The Bregman distance has the following useful property so-called *three-point identity*: for each $x \in \operatorname{dom} h$ and $y, z \in \operatorname{int}(\operatorname{dom} h)$, it holds that

$$D_h(x, y) = D_h(x, z) - D_h(y, z) + \langle x - y, \nabla h(z) - \nabla h(y) \rangle. \tag{2.5}$$

From (2.2) and (2.3), one can see that

$$D_h(x, y) \geq \frac{\kappa}{2}\|x - y\|^2. \tag{2.6}$$

The *total convexity of function h on bounded sets* is defined in Butnariu and Resmerita [48]. It is well-known that h is totally convex on bounded sets if and only if h is uniformly convex on bounded sets (see Butnariu & Resmerita [48, Theorem 2.10]). More information on totally convex functions can be found in Butnariu et al. [49].

Let $h : \mathcal{X} \to (-\infty, \infty]$ be a Gâteaux differentiable function. Recall that the *Bregman projection with respect to h* [48] of $x \in \mathcal{X}$ to a closed and convex subset $\mathcal{K} \subset \mathcal{X}$ is the necessarily unique vector $\Pi^h_{\mathcal{K}}(x) \in \mathcal{K}$ which satisfies

$$\Pi^h_{\mathcal{K}}(x) := \operatorname{argmin}\{D_h(y, x) : y \in \mathcal{K}\}.$$

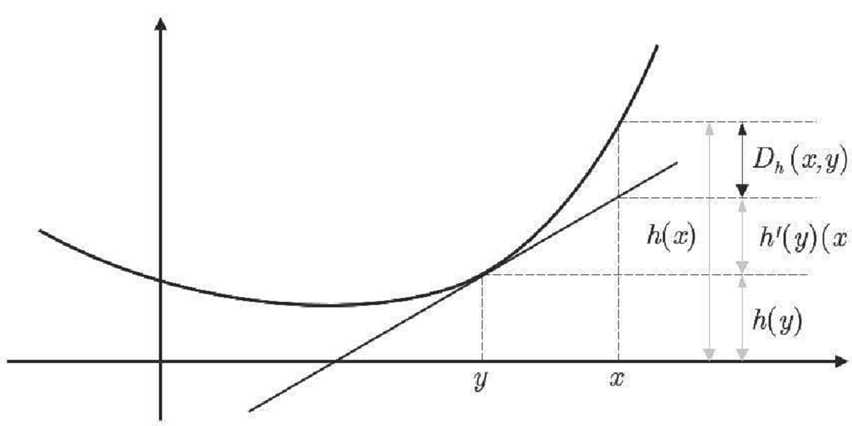

FIGURE 1 The Bregman distance with respect to h.

If h is additionally assumed to be totally convex, then $\Pi^h_{\mathcal{K}}$ satisfies by the following inequalities [48, Corollary 4.4]:

$$\langle y - \Pi^h_{\mathcal{K}}(x), \nabla h(x) - \nabla h(\Pi^h_{\mathcal{K}}(x))\rangle \leq 0 \ \forall y \in \mathcal{K} \tag{2.7}$$

and

$$D_h(y, \Pi^h_{\mathcal{K}}(x)) + D_h(\Pi^h_{\mathcal{K}}(x), x) \leq D_h(y, x) \ \forall y \in \mathcal{K}. \tag{2.8}$$

A mapping $\mathcal{A}$ of $\mathcal{K}$ into $\mathcal{X}^*$ is said to be *monotone* if $\langle x - y, \mathcal{A}x - \mathcal{A}y\rangle \geq 0$ for all $x, y \in \mathcal{K}$ and it is said to be *Lipschitz continuous* if there exists a constant $L > 0$ such that $||\mathcal{A}x - \mathcal{A}y|| \leq L||x - y||$ for all $x, y \in \mathcal{K}$. If $\mathcal{A} : \mathcal{K} \to \mathcal{X}^*$ is a monotone and continuous mapping, then it holds that (see Iiduka & Takahashi [50, Proposition 2.6])

$$\hat{x} \in VI(\mathcal{K}, \mathcal{A}) \Leftrightarrow \langle x - \hat{x}, \mathcal{A}x\rangle \geq 0 \ \forall x \in \mathcal{K}. \tag{2.9}$$

Let $h : \mathcal{X} \to \mathbb{R}$ be a Legendre function. The bifunction $V_h : \mathcal{X} \times \mathcal{X}^* \to [0, \infty)$ is associated with h defined by

$$V_h(x, x^*) := h(x) - \langle x, x^*\rangle + h^*(x^*) \ \forall (x, x^*) \in \mathcal{X} \times \mathcal{X}^*.$$

The following properties of V_h can be found in Martín-Márquez et al. [51, Proposition 1]:

(i) V_h is nonnegative and convex in the second variable;
(ii) $V_h(x, x^*) = D_h(x, \nabla h^*(x^*)) \ \forall (x, x^*) \in \mathcal{X} \times \mathcal{X}^*$;
(iii) $V_h(x, x^*) + \langle \nabla h^*(x^*) - x, y^*\rangle \leq V_h(x, x^* + y^*) \ \forall (x, x^*) \in \mathcal{X} \times \mathcal{X}^*$ and $(x, y^*) \in \mathcal{X} \times \mathcal{X}^*$.

Since V_h is convex in the second variable, it follows that, for all $z \in E$,

$$D_h\left(z, \nabla h^*\left(\sum_{i=1}^N t_i \nabla h(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_h(z, x_i), \tag{2.10}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset [0, 1]$ with $\sum_{i=1}^N t_i = 1$.

The following lemmas provide some useful properties which will be useful in the sequel.

Lemma 2.1 ([52]). *Assume that $\{a_n\}$ is a nonnegative real sequence such that $a_{n+1} \leq (1-\gamma_n)a_n+\gamma_n\delta_n$ for all $n \geq 1$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^\infty \gamma_n = \infty$ and $\{\delta_n\}$ is a real sequence with $\limsup_{n\to\infty}\delta_n \leq 0$. Then $\lim_{n\to\infty}a_n = 0$.*

Lemma 2.2 ([53]). *Assume that $\{a_n\}$ is a real sequence such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_\ell\} \subset \mathbb{N}$ such that $\lim_{\ell\to\infty}m_\ell = \infty$ and the following properties hold for all (sufficiently large) numbers $\ell \in \mathbb{N}$: $a_{m_\ell} \leq a_{m_\ell+1}$ and $a_\ell \leq a_{m_\ell+1}$. In fact, $m_\ell := \max\{j \leq \ell : a_j \leq a_{j+1}\}$.*

3 | ALGORITHM AND ITS CONVERGENCE

In this section, we propose an inertial-like Bregman projection-type algorithm for solving systems of monotone variational inequalities in reflexive Banach spaces. This method is based on TEGM and Halpern-type iteration which stepsize does not need any prior knowledge of the Lipschitz constant of the mapping. For convenience, we simply denote the set $I_N := \{1, 2, \dots, N\}$. In order to establish our strong convergence theorem, the following assumptions are needed.

Assumption 3.1.

(A1) The set $\mathcal{K}_i$ $(i \in I_N)$ is a nonempty, closed, and convex subset of a real reflexive Banach space $\mathcal{X}$.
(A2) The function $h : \mathcal{X} \to \mathbb{R}$ satisfies the following conditions:

- h is strongly convex with $\kappa > 0$;
- h is uniformly Fréchet differentiable on bounded subsets of $\mathcal{X}$;
- h is Legendre which is bounded on bounded subsets of $\mathcal{X}$.

(A3) The mapping $\mathcal{A}_i : \mathcal{X} \to \mathcal{X}^*$ ($i \in I_N$) is monotone and L_i-Lipschitz continuous.
(A4) The solution set of systems of VIP is nonempty; that is, $\Gamma := \bigcap_{i=1}^N VI(\mathcal{K}_i, \mathcal{A}_i) \neq \emptyset$.

Assumption 3.2.

(B1) $\{\eta_n\} \subset (0,1)$ with $\lim_{n\to\infty}\eta_n = 0$ and $\sum_{n=1}^\infty \eta_n = \infty$.
(B2) $\gamma_n^{(i)} \in (0,1)$ with $\liminf_{n\to\infty}\gamma_n^{(i)} > 0$ for $i \in I_N$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$.
(B3) $\epsilon_n = \circ(\eta_n)$, that is, $\lim_{n\to\infty} \frac{\epsilon_n}{\eta_n} = 0$.

Our algorithm is presented as follows.

Algorithm 1: Inertial-like Bregman projection-type algorithm for system of VIP (IBPA)

Initialization: Choose $\lambda_1^{(i)} > 0,\ \mu_i \in (0,\kappa)$ for $i \in I_N$ and $\beta > 0$. Let $x_0, x_1, u \in \mathcal{X}$ be arbitrary.
Iterative steps: Given the current iterate x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Choose β_n such that $0 \leq \beta_n \leq \bar{\beta}_n$, where

$$\bar{\beta}_n = \begin{cases} \min\left\{ \dfrac{\epsilon_n}{\|\nabla h(x_{n-1}) - \nabla h(x_n)\|}, \beta \right\} \text{ if } \nabla h(x_{n-1}) \neq \nabla h(x_n), \\ \beta \text{otherwise.} \end{cases} \tag{3.1}$$

Set $w_n = \nabla h^*(\nabla h(x_n) + \beta_n(\nabla h(x_{n-1}) - \nabla h(x_n)))$ and compute

$$y_n^{(i)} = \Pi_{\mathcal{K}_i}^h \nabla h^*(\nabla h(w_n) - \lambda_n^{(i)}\mathcal{A}_i w_n).$$

Step 2. Compute

$$z_n^{(i)} = \nabla h^*(\nabla h(y_n^{(i)}) - \lambda_n^{(i)}(\mathcal{A}_i y_n^{(i)} - \mathcal{A}_i w_n)).$$

Step 3. Compute

$$x_{n+1} = \nabla h^*\left(\eta_n \nabla h(u) + (1-\eta_n)\left(\gamma_n^{(0)}\nabla h(w_n) + \sum_{i=1}^N \gamma_n^{(i)}\nabla h(z_n^{(i)})\right)\right),$$

where

$$\lambda_{n+1}^{(i)} = \begin{cases} \min\left\{ \min_{i\in I_N}\left\{ \dfrac{\mu_i\|w_n - y_n^{(i)}\|}{\|\mathcal{A}_i w_n - \mathcal{A}_i y_n^{(i)}\|} \right\}, \lambda_n^{(i)} \right\} \text{ if } \mathcal{A}_i w_n - \mathcal{A}_i y_n^{(i)} \neq 0, \\ \lambda_n^{(i)} \text{otherwise.} \end{cases} \tag{3.2}$$

If $w_n = y_n^{(i)}$ for some $n \geq 1$, then stop and $y_n^{(i)}$ is a solution of system of VIP. Otherwise, set $n := n+1$ and return to **Step 1**.

Lemma 3.3. *Assume that Assumptions 3.1 and 3.2 are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 1. If $\{x_n\}$ is bounded, then for each $x \in \Gamma$, we have* $\lim_{n\to\infty} \frac{\beta_n}{\eta_n}(D_h(x, x_{n-1}) - D_h(x, x_n)) = 0$.

Proof. From (3.1), we see that $\beta_n\|\nabla h(x_{n-1}) - \nabla h(x_n)\| \leq \epsilon_n$. Since $\eta_n > 0$ and $\lim_{n\to\infty} \frac{\epsilon_n}{\eta_n} = 0$, it follows that

$$\lim_{n\to\infty} \frac{\beta_n}{\eta_n}\|\nabla h(x_{n-1}) - \nabla h(x_n)\| = 0. \tag{3.3}$$

Let $x \in \Gamma$. Without loss of generality, we may assume that $D_h(x, x_{n-1}) - D_h(x, x_n)$ is nonnegative for all $n \in \mathbb{N}$. From the three-point identity of D_h, we note that

$$\begin{aligned} D_h(x, x_{n-1}) - D_h(x, x_n) &= -D_h(x_{n-1}, x_n) + \langle x - x_{n-1}, \nabla h(x_n) - \nabla h(x_{n-1})\rangle \\ &\leq \langle x - x_{n-1}, \nabla h(x_n) - \nabla h(x_{n-1})\rangle \\ &\leq \|\nabla h(x_n) - \nabla h(x_{n-1})\|K, \end{aligned}$$

where $K = \sup_{n\geq 1}\{\|x_{n-1} - x\|\} < \infty$. Thus by (3.3), we obtain

$$\frac{\beta_n}{\eta_n}(D_h(x, x_{n-1}) - D_h(x, x_n)) \leq \frac{\beta_n}{\eta_n}\|\nabla h(x_n) - \nabla h(x_{n-1})\|K \to 0$$

as $n \to \infty$. This implies that $\lim_{n\to\infty} \frac{\beta_n}{\eta_n}(D_h(x, x_{n-1}) - D_h(x, x_n)) = 0$. □

Next, we prove our main result as follows.

Theorem 3.4. *Assume that Assumptions 3.1 and 3.2 are satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then $\{x_n\}$ converges strongly to $v = \Pi^h_\Gamma(u)$, where Π^h_Γ is the Bregman projection with respect to h from $\mathcal{X}$ onto Γ.*

Proof. By using the similar argument as in the proof of Yang and Liu [9, Lemma 3.1], one can see that

$$\lim_{n\to\infty} \lambda_n^{(i)} = \lambda^{(i)} \geq \min\left\{\min_{i\in I_N}\left\{\frac{\mu_i}{L_i}\right\}, \lambda_1^{(i)}\right\}.$$

Let $v \in \Gamma$. Then for each $i \in I_N$, we have

$$\begin{aligned}
D_h\left(v, z_n^{(i)}\right) &= D_h\left(v, \nabla h^*\left(\nabla h\left(y_n^{(i)}\right) - \lambda_n^{(i)}\left(\mathcal{A}_i y_n^{(i)} - \mathcal{A}_i v_n\right)\right)\right)\\
&= h(v) - h\left(z_n^{(i)}\right) - \left\langle v - z_n^{(i)}, \nabla h\left(y_n^{(i)}\right) - \lambda_n^{(i)}\left(\mathcal{A}_i y_n^{(i)} - \mathcal{A}_i v_n\right)\right\rangle\\
&= h(v) - h\left(z_n^{(i)}\right) - \left\langle v - z_n^{(i)}, \nabla h\left(y_n^{(i)}\right)\right\rangle + \lambda_n^{(i)}\left\langle v - z_n^{(i)}, \mathcal{A}_i y_n^{(i)} - \mathcal{A}_i v_n\right\rangle\\
&= h(v) - h\left(y_n^{(i)}\right) - \left\langle v - y_n^{(i)}, \nabla h\left(y_n^{(i)}\right)\right\rangle + \left\langle v - y_n^{(i)}, \nabla h\left(y_n^{(i)}\right)\right\rangle + h\left(y_n^{(i)}\right) - h\left(z_n^{(i)}\right) - \left\langle v - z_n^{(i)}, \nabla h\left(y_n^{(i)}\right)\right\rangle\\
&\quad + \lambda_n^{(i)}\left\langle v - z_n^{(i)}, \mathcal{A}_i y_n^{(i)} - \mathcal{A}_i v_n\right\rangle\\
&= h(v) - h\left(y_n^{(i)}\right) - \left\langle v - y_n^{(i)}, \nabla h\left(y_n^{(i)}\right)\right\rangle - h\left(z_n^{(i)}\right) + h\left(y_n^{(i)}\right) + \left\langle z_n^{(i)} - y_n^{(i)}, \nabla h\left(y_n^{(i)}\right)\right\rangle\\
&\quad + \lambda_n^{(i)}\left\langle v - z_n^{(i)}, \mathcal{A}_i y_n^{(i)} - \mathcal{A}_i v_n\right\rangle\\
&= D_h\left(v, y_n^{(i)}\right) - D_h\left(z_n^{(i)}, y_n^{(i)}\right) + \lambda_n^{(i)}\left\langle v - z_n^{(i)}, \mathcal{A}_i y_n^{(i)} - \mathcal{A}_i v_n\right\rangle.
\end{aligned} \tag{3.4}$$

Using the three-point identity, we obtain

$$D_h\left(v, y_n^{(i)}\right) = D_h(v, w_n) - D_h\left(y_n^{(i)}, w_n\right) + \left\langle v - y_n^{(i)}, \nabla h(w_n) - \nabla h\left(y_n^{(i)}\right)\right\rangle. \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$\begin{aligned}
D_h\left(v, z_n^{(i)}\right) &= D_h(v, w_n) - D_h\left(y_n^{(i)}, w_n\right) - D_h\left(z_n^{(i)}, y_n^{(i)}\right) + \left\langle v - y_n^{(i)}, \nabla h(w_n) - \nabla h\left(y_n^{(i)}\right)\right\rangle\\
&\quad + \lambda_n^{(i)}\left\langle v - z_n^{(i)}, \mathcal{A}_i y_n^{(i)} - \mathcal{A}_i w_n\right\rangle.
\end{aligned} \tag{3.6}$$

Since $y_n^{(i)} = \Pi^h_{\mathcal{K}_i} \nabla h^*(\nabla h(w_n) - \lambda_n^{(i)} \mathcal{A}_i w_n)$, it follows (2.7) that

$$\left\langle v - y_n^{(i)}, \nabla h(w_n) - \lambda_n^{(i)} \mathcal{A}_i w_n - \nabla h\left(y_n^{(i)}\right)\right\rangle \leq 0.$$

Hence,

$$\left\langle v - y_n^{(i)}, \nabla h(w_n) - \nabla h\left(y_n^{(i)}\right)\right\rangle \leq \lambda_n^{(i)}\left\langle v - y_n^{(i)}, \mathcal{A}_i w_n\right\rangle. \tag{3.7}$$

Combining (3.6) with (3.7), we get

$$\begin{aligned} D_h\left(v,z_n^{(i)}\right) &\leq D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \lambda_n^{(i)}\left\langle v-y_n^{(i)},\mathcal{A}_iw_n\right\rangle \\ &\quad + \lambda_n^{(i)}\left\langle v-z_n^{(i)},\mathcal{A}_iy_n^{(i)}-\mathcal{A}_iw_n\right\rangle \\ &= D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \lambda_n^{(i)}\left\langle v-y_n^{(i)},\mathcal{A}_iw_n\right\rangle + \lambda_n^{(i)}\left\langle v-z_n^{(i)},\mathcal{A}_iy_n^{(i)}\right\rangle \\ &\quad - \lambda_n^{(i)}\left\langle v-z_n^{(i)},Aw_n\right\rangle \\ &= D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \lambda_n^{(i)}\left\langle z_n^{(i)}-y_n^{(i)},\mathcal{A}_iw_n\right\rangle + \lambda_n^{(i)}\left\langle v-z_n^{(i)},\mathcal{A}_iy_n^{(i)}\right\rangle \\ &= D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \lambda_n^{(i)}\left\langle z_n^{(i)}-y_n^{(i)},\mathcal{A}_iw_n\right\rangle - \lambda_n^{(i)}\left\langle y_n^{(i)}-v,\mathcal{A}_iy_n^{(i)}\right\rangle \\ &\quad + \lambda_n^{(i)}\left\langle y_n^{(i)}-z_n^{(i)},\mathcal{A}_iy_n^{(i)}\right\rangle \\ &= D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \lambda_n^{(i)}\left\langle z_n^{(i)}-y_n^{(i)},\mathcal{A}_iw_n-\mathcal{A}_iy_n^{(i)}\right\rangle \\ &\quad - \lambda_n^{(i)}\left\langle y_n^{(i)}-v,\mathcal{A}_iy_n^{(i)}\right\rangle. \end{aligned} \tag{3.8}$$

It is observed that $y_n^{(i)} \in \mathcal{K}_i$ for $i \in I_N$ and $v \in \Gamma$; thus, we have $\left\langle y_n^{(i)}-v,\mathcal{A}_iv\right\rangle \geq 0$ for $i \in I_N$. Since $\left\langle y_n^{(i)}-v,\mathcal{A}_iy_n^{(i)}-\mathcal{A}_iv\right\rangle \geq 0$, we have

$$\left\langle y_n^{(i)}-v,\mathcal{A}_iy_n^{(i)}\right\rangle \geq \left\langle y_n^{(i)}-v,\mathcal{A}_iv\right\rangle \geq 0 \text{ for } i \in I_N. \tag{3.9}$$

By the definition of $\lambda_{n+1}^{(i)}$, we have

$$\|\mathcal{A}_iw_n-\mathcal{A}_iy_n^{(i)}\| \leq \frac{\mu_i}{\lambda_{n+1}^{(i)}}\|w_n-y_n^{(i)}\| \text{ for } i \in I_N. \tag{3.10}$$

Combining (3.8) with (3.9) and (3.10) and using (2.6), we get

$$\begin{aligned} D_h\left(v,z_n^{(i)}\right) &\leq D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \lambda_n^{(i)}\left\langle z_n^{(i)}-y_n^{(i)},\mathcal{A}_iw_n-\mathcal{A}_iy_n^{(i)}\right\rangle \\ &\leq D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \lambda_n^{(i)}\|z_n^{(i)}-y_n^{(i)}\|\|\mathcal{A}_iw_n-\mathcal{A}_iy_n^{(i)}\| \\ &\leq D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \mu_i\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\|z_n^{(i)}-y_n^{(i)}\|\|w_n-y_n^{(i)}\| \\ &\leq D_h(v,w_n) - D_h\left(y_n^{(i)},w_n\right) - D_h\left(z_n^{(i)},y_n^{(i)}\right) + \frac{\mu_i}{2}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\|z_n^{(i)}-y_n^{(i)}\|^2 + \frac{\mu_i}{2}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\|w_n-y_n^{(i)}\|^2 \\ &\leq D_h(v,w_n) - \left(1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right)D_h\left(y_n^{(i)},w_n\right) - \left(1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right)D_h\left(z_n^{(i)},y_n^{(i)}\right). \end{aligned} \tag{3.11}$$

Since $\lim_{n\to\infty}\lambda_n^{(i)} = \lambda^{(i)} > 0$ and $\mu_i \in (0,\kappa)$ for $i \in I_N$, we have $\lim_{n\to\infty}\left(1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right) = 1-\frac{\mu_i}{\kappa} = \frac{\kappa-\mu_i}{\kappa} > 0$. Thus, there exists $n_0^* \in \mathbb{N}$ such that $1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}} > 0$ for all $n \geq n_0^*$ and $i \in I_N$. This implies that

$$\left(1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right)D_h\left(y_n^{(i)},w_n\right) + \left(1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right)D_h\left(z_n^{(i)},y_n^{(i)}\right) \geq 0$$

for all $n \geq n_0^*$ and $i \in I_N$. Then from (3.11), we obtain

$$D_h\left(v, z_n^{(i)}\right) \leq D_h(v, w_n) \text{ for } i \in I_N. \tag{3.12}$$

Now, let $u_n = \nabla h^*\left(\gamma_n^{(0)} \nabla h(w_n) + \sum_{i=1}^N \gamma_n^{(i)} \nabla h\left(z_n^{(i)}\right)\right)$ for all $n \geq 1$. From (2.10), we have

$$\begin{aligned} D_h(v, u_n) &\leq \gamma_n^{(0)} D_h(v, w_n) + \sum_{i=1}^N \gamma_n^{(i)} D_h\left(v, z_n^{(i)}\right) \\ &\leq \gamma_n^{(0)} D_h(v, w_n) + \sum_{i=1}^N \gamma_n^{(i)} D_h(v, w_n) \\ &= D_h(v, w_n). \end{aligned} \tag{3.13}$$

Moreover, we have

$$\begin{aligned} D_h(v, w_n) &= D_h(v, \nabla h^*(\nabla h(x_n) + \beta_n(\nabla h(x_{n-1}) - \nabla h(x_n)))) \\ &= D_h(v, \nabla h^*((1-\beta_n)\nabla h(x_n) + \beta_n \nabla h(x_{n-1}))) \\ &\leq (1-\beta_n) D_h(v, x_n) + \beta_n D_h(v, x_{n-1}). \end{aligned} \tag{3.14}$$

Since $x_{n+1} = \nabla h^*(\eta_n \nabla h(u) + (1-\eta_n)\nabla h(u_n))$, it follows from (3.13) and (3.14) that

$$\begin{aligned} D_h(v, x_{n+1}) &\leq \eta_n D_h(v, u) + (1-\eta_n) D_h(v, u_n) \\ &\leq \eta_n D_h(v, u) + (1-\eta_n) D_h(v, w_n) \\ &\leq \eta_n D_h(v, u) + (1-\eta_n)[(1-\beta_n) D_h(v, x_n) + \beta_n D_h(v, x_{n-1})] \\ &\leq \eta_n D_h(v, u) + (1-\eta_n) \max\{D_h(v, x_n), D_h(v, x_{n-1})\} \\ &\leq \max\{D_h(v, u), D_h(v, x_n), D_h(v, x_{n-1})\} \\ &\leq \ \dots \ \leq \max\{D_h(v, u), D_h(v, x_{n_0^*}), D_h(v, x_{n_0^*-1})\}. \end{aligned}$$

Hence, $\{D_h(v, x_n)\}$ is bounded, which implies that $\{x_n\}$ is bounded by (2.6). We also get that $\{w_n\}$ and $\{u_n\}$ are bounded. Now, utilizing (3.11) and (3.14), we arrive at

$$\begin{aligned} D_h(v, u_n) &\leq \gamma_n^{(0)} D_h(v, w_n) + \sum_{i=1}^N \gamma_n^{(i)} D_h\left(v, z_n^{(i)}\right) \\ &\leq \gamma_n^{(0)} D_h(v, w_n) + \sum_{i=1}^N \gamma_n^{(i)} D_h(v, w_n) - \sum_{i=1}^N \gamma_n^{(i)} \left(1 - \frac{\mu_i}{\kappa} \frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right) D_h\left(y_n^{(i)}, w_n\right) \\ &\quad - \gamma_n^{(i)} \sum_{i=1}^N \left(1 - \frac{\mu_i}{\kappa} \frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right) D_h\left(z_n^{(i)}, y_n^{(i)}\right) \\ &\leq (1-\beta_n) D_h(v, x_n) + \beta_n D_h(v, x_{n-1}) - \sum_{i=1}^N \gamma_n^{(i)} \left(1 - \frac{\mu_i}{\kappa} \frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right) D_h\left(y_n^{(i)}, w_n\right) \\ &\quad - \sum_{i=1}^N \gamma_n^{(i)} \left(1 - \frac{\mu_i}{\kappa} \frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right) D_h\left(z_n^{(i)}, y_n^{(i)}\right). \end{aligned} \tag{3.15}$$

It follows from (3.15) that

$$\begin{aligned} D_h(v, x_{n+1}) &= \eta_n D_h(v, u) + (1-\eta_n) D_h(v, u_n) \\ &\leq \eta_n D_h(v, u) + (1-\eta_n)(1-\beta_n) D_h(v, x_n) + (1-\eta_n)\beta_n D_h(v, x_{n-1}) \\ &\quad - (1-\eta_n) \sum_{i=1}^N \gamma_n^{(i)} \left(1 - \frac{\mu_i}{\kappa} \frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right) D_h\left(y_n^{(i)}, w_n\right) - (1-\eta_n) \sum_{i=1}^N \gamma_n^{(i)} \left(1 - \frac{\mu_i}{\kappa} \frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right) D_h\left(z_n^{(i)}, y_n^{(i)}\right). \end{aligned}$$

This implies that

$$
\begin{aligned}
&(1-\eta_n)\sum_{i=1}^{N}\gamma_n^{(i)}\left(1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right)D_h\left(y_n^{(i)},w_n\right)+(1-\eta_n)\sum_{i=1}^{N}\gamma_n^{(i)}\left(1-\frac{\mu_i}{\kappa}\frac{\lambda_n^{(i)}}{\lambda_{n+1}^{(i)}}\right)D_h\left(z_n^{(i)},y_n^{(i)}\right)\\
&\leq \eta_n D_h(v,u)+(1-\eta_n)D_h(v,x_n)-(1-\eta_n)\beta_n D_h(v,x_n)+(1-\eta_n)\beta_n D_h(v,x_{n-1})-D_h(v,x_{n+1})\\
&\leq D_h(v,x_n)-D_h(v,x_{n+1})+\eta_n(1-\eta_n)\frac{\beta_n}{\eta_n}(D_h(v,x_{n-1})-D_h(v,x_n))+\eta_n M,
\end{aligned}
\tag{3.16}
$$

where $M=\sup_{n\geq 1}\{|D_h(v,u)-D_h(v,x_n)|\}<\infty$. From the property of D_h, (3.13) and (3.14), we have

$$
\begin{aligned}
D_h(v,x_{n+1}) &= V_h(v,\eta_n\nabla h(u)+(1-\eta_n)\nabla h(u_n))\\
&\leq V_h(v,\eta_n\nabla h(u)+(1-\eta_n)\nabla h(u_n)-\eta_n(\nabla h(u)-\nabla h(v)))+\eta_n\langle x_{n+1}-v,\nabla h(u)-\nabla h(v)\rangle\\
&= V_h(v,\eta_n\nabla h(v)+(1-\eta_n)\nabla h(u_n))+\eta_n\langle x_{n+1}-v,\nabla h(u)-\nabla h(v)\rangle\\
&= D_h(v,\nabla h^*(\eta_n\nabla h(v)+(1-\eta_n)\nabla h(u_n))+\eta_n\langle x_{n+1}-v,\nabla h(u)-\nabla h(v)\rangle\\
&\leq \eta_n D_h(v,v)+(1-\eta_n)D_h(v,u_n)+\eta_n\langle x_{n+1}-v,\nabla h(u)-\nabla h(v)\rangle\\
&\leq (1-\eta_n)D_h(v,w_n)+\eta_n\langle x_{n+1}-v,\nabla h(u)-\nabla h(v)\rangle\\
&\leq (1-\eta_n)(1-\beta_n)D_h(v,x_n)+(1-\eta_n)\beta_n D_h(v,x_{n-1})+\eta_n\langle x_{n+1}-v,\nabla h(u)-\nabla h(v)\rangle\\
&= (1-\eta_n)D_h(v,x_n)+(1-\eta_n)\beta_n(D_h(v,x_{n-1})-D_h(v,x_n))\\
&\quad +\eta_n\langle x_{n+1}-v,\nabla h(u)-\nabla h(v)\rangle.
\end{aligned}
\tag{3.17}
$$

We next divide the proof into two cases.

Case 1. There exists $n_0^*\in\mathbb{N}$ such that $\{D_h(v,x_n)\}$ is nonincreasing for all $n\geq n_0^*$. This implies that $\lim_{n\to\infty}D_h(v,x_n)$ exists. Thus, from (3.16) and Lemma 3.3, we obtain $\lim_{n\to\infty}D_h\left(y_n^{(i)},w_n\right)=\lim_{n\to\infty}D_h\left(z_n^{(i)},y_n^{(i)}\right)=0$ for $i\in I_N$ and so $\lim_{n\to\infty}\|y_n^{(i)}-w_n\|=\lim_{n\to\infty}\|z_n^{(i)}-y_n^{(i)}\|=0$ for $i\in I_N$. By the uniform continuity of ∇h on bounded subsets of $\mathcal{X}$ (see Reich and Sabach [54, Proposition 1]), we have

$$
\lim_{n\to\infty}\|\nabla h\left(y_n^{(i)}\right)-\nabla h(w_n)\|=\lim_{n\to\infty}\|\nabla h\left(z_n^{(i)}\right)-\nabla h\left(y_n^{(i)}\right)\|=0 \text{ for } i\in I_N.
$$

We also have

$$
\lim_{n\to\infty}\|\nabla h\left(z_n^{(i)}\right)-\nabla h(w_n)\|=0 \text{ for } i\in I_N. \tag{3.18}
$$

By the definition of x_{n+1}, we see that

$$
\|\nabla h(x_{n+1})-\nabla h(u_n)\|=\eta_n\|\nabla h(u)-\nabla h(u_n)\|\to 0 \tag{3.19}
$$

as $n\to\infty$. Since $u_n=\nabla h^*\left(\gamma_n^{(0)}\nabla h(w_n)+\sum_{i=1}^{N}\gamma_n^{(i)}\nabla h\left(z_n^{(i)}\right)\right)$, it follows from (3.18) that

$$
\|\nabla h(u_n)-\nabla h(w_n)\|\leq\sum_{i=1}^{N}\|\nabla h\left(z_n^{(i)}\right)-\nabla h(w_n)\|\to 0 \tag{3.20}
$$

as $n\to\infty$. Since $\eta_n\in(0,1)$, it follows from (3.3) that

$$
\beta_n\|\nabla h(x_{n-1})-\nabla h(x_n)\|\leq\frac{\beta_n}{\eta_n}\|\nabla h(x_{n-1})-\nabla h(x_n)\|\to 0
$$

as $n\to\infty$. Thus, we have

$$
\|\nabla h(w_n)-\nabla h(x_n)\|=\beta_n\|\nabla h(x_{n-1})-\nabla h(x_n)\|\to 0 \tag{3.21}
$$

as $n \to \infty$. It then follows from (3.19), (3.20), and (3.21) that

$$\|\nabla h(x_{n+1}) - \nabla h(x_n)\| \leq \|\nabla h(x_{n+1}) - \nabla h(u_n)\| + \|\nabla h(u_n) - \nabla h(w_n)\| + \|\nabla h(w_n) - \nabla h(x_n)\| \to 0$$

as $n \to \infty$. By the uniform continuity of ∇h^* on bounded subsets of $\mathcal{X}^*$ (see Zălinescu [55, Theorem 3.5.10]), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.22}$$

Since a Banach space $\mathcal{X}$ is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_\ell}\} \subset \{x_n\}$ which converges weakly to some $\hat{x} \in \mathcal{X}$ and

$$\limsup_{n\to\infty} \langle x_n - v, \nabla h(u) - \nabla h(v)\rangle = \lim_{\ell\to\infty} \langle x_{n_\ell} - v, \nabla h(u) - \nabla h(v)\rangle, \tag{3.23}$$

where $v = \Pi_\Gamma^h(u)$. We now show that $\hat{x} \in \Gamma$. From (3.21), we have $\lim_{\ell\to\infty}\|w_{n_\ell} - x_{n_\ell}\| = 0$ and we also get $w_{n_\ell} \rightharpoonup \hat{x}$. By the definition of y_{n_ℓ}, we see that

$$\left\langle x - y_{n_\ell}^{(i)}, \nabla h(y_{n_\ell}^{(i)}) - \nabla h(w_{n_\ell}) + \lambda_{n_\ell}^{(i)} A w_{n_\ell} \right\rangle \geq 0$$

for all $x \in \ell_i$ and $i \in I_N$. Then by the monotonicity of $\mathcal{A}_i$, we have

$$\begin{aligned} 0 &\leq \left\langle x - y_{n_\ell}^{(i)}, \nabla h(y_{n_\ell}^{(i)}) - \nabla h(w_{n_\ell}) \right\rangle + \lambda_{n_\ell}^{(i)} \left\langle x - y_{n_\ell}^{(i)}, \mathcal{A}_i w_{n_\ell} \right\rangle \\ &= \left\langle x - y_{n_\ell}^{(i)}, \nabla h(y_{n_\ell}^{(i)}) - \nabla h(w_{n_\ell}) \right\rangle + \lambda_{n_\ell}^{(i)} \left\langle x - w_{n_\ell}, \mathcal{A}_i w_{n_\ell} \right\rangle + \lambda_{n_\ell}^{(i)} \left\langle w_{n_\ell} - y_{n_\ell}^{(i)}, \mathcal{A}_i w_{n_\ell} \right\rangle \\ &\leq \left\langle x - y_{n_\ell}^{(i)}, \nabla h(y_{n_\ell}^{(i)}) - \nabla h(w_{n_\ell}) \right\rangle + \lambda_{n_\ell}^{(i)} \left\langle x - w_{n_\ell}, \mathcal{A}_i x \right\rangle + \lambda_{n_\ell}^{(i)} \left\langle w_{n_\ell} - y_{n_\ell}^{(i)}, \mathcal{A}_i w_{n_\ell} \right\rangle. \end{aligned} \tag{3.24}$$

Since $\lim_{\ell\to\infty}\|\nabla h(y_{n_\ell}^{(i)}) - \nabla h(w_{n_\ell})\| = 0$ for $i \in I_N$, we have $\lim_{\ell\to\infty}\|y_{n_\ell}^{(i)} - w_{n_\ell}\| = 0$ for $i \in I_N$. Moreover, we know that $w_{n_\ell} \rightharpoonup \hat{x}$ and $\lim_{\ell\to\infty}\lambda_{n_\ell}^{(i)} = \lambda^{(i)} > 0$ for $i \in I_N$. It follows from (3.24) that

$$\langle x - \hat{x}, \mathcal{A}_i x\rangle \geq 0$$

for all $x \in \mathcal{K}_i$ and $i \in I_N$. Hence, by (2.9), we get $\hat{x} \in VI(\mathcal{K}_i, \mathcal{A}_i)$ for $i \in I_N$ and so $\hat{x} \in \Gamma := \bigcap_{i=1}^N VI(\mathcal{K}_i, \mathcal{A}_i)$. This together with (3.22) and (3.23) implies that

$$\begin{aligned} \limsup_{n\to\infty} \langle x_{n+1} - v, \nabla h(u) - \nabla h(v)\rangle &\leq \limsup_{n\to\infty} \langle x_{n+1} - x_n, \nabla h(u) - \nabla h(v)\rangle + \limsup_{n\to\infty} \langle x_n - v, \nabla h(u) - \nabla h(v)\rangle \\ &= \langle \hat{x} - v, \nabla h(u) - \nabla h(v)\rangle \\ &\leq 0. \end{aligned} \tag{3.25}$$

By (3.17), (3.25), and Lemma 3.3, so in view of Lemma 2.1, we obtain $\lim_{n\to\infty} D_h(v, x_n) = 0$. Therefore, $x_n \to v$.

Case 2. There exists a subsequence $\{n_j\}$ of $\{n\}$ such that $D_h(v, x_{n_j}) \leq D_h(v, x_{n_j+1})$ for all $j \in \mathbb{N}$. By Lemma 2.2, there exists a nondecreasing sequence $\{m_\ell\} \subset \mathbb{N}$ such that $m_\ell \to \infty$ and the following properties are satisfied for all number $\ell \in \mathbb{N}$:

$$D_h(v, x_{m_\ell}) \leq D_h(v, x_{m_\ell+1}) \text{ and } D_h(v, x_\ell) \leq D_h(v, x_{m_\ell+1}). \tag{3.26}$$

From (3.16), we have

$$\begin{aligned}
&(1-\eta_{m_\ell})\sum_{i=1}^{N}\gamma_{m_\ell}^{(i)}\left(1-\frac{\mu_i}{\kappa}\frac{\lambda_{m_\ell}^{(i)}}{\lambda_{m_\ell+1}^{(i)}}\right)D_h\left(y_{m_\ell}^{(i)},w_{m_\ell}\right)+(1-\eta_{m_\ell})\sum_{i=1}^{N}\gamma_{m_\ell}^{(i)}\left(1-\frac{\mu_i}{\kappa}\frac{\lambda_{m_\ell}^{(i)}}{\lambda_{m_\ell+1}^{(i)}}\right)D_h\left(v_{m_\ell}^{(i)},y_{m_\ell}^{(i)}\right)\\
&\le D_h(v,x_{m_\ell})-D_h(v,x_{m_\ell+1})+(1-\eta_{m_\ell})\beta_{m_\ell}(D_h(v,x_{m_\ell-1})-D_h(v,x_{m_\ell}))+\eta_{m_\ell}M\\
&\le \eta_{m_\ell}(1-\eta_{m_\ell})\frac{\beta_{m_\ell}}{\eta_{m_\ell}}(D_h(v,x_{m_\ell-1})-D_h(v,x_{m_\ell}))+\eta_{m_\ell}M,
\end{aligned}$$

where $M<\infty$. Thus, we have $\lim_{\ell\to\infty}D_h\left(y_{m_\ell}^{(i)},w_{m_\ell}\right)=\lim_{\ell\to\infty}D_h\left(z_{m_\ell}^{(i)},y_{m_\ell}^{(i)}\right)=0$ for $i\in I_N$ and so

$$\lim_{\ell\to\infty}\|\nabla h\left(y_{m_\ell}^{(i)}\right)-\nabla h(w_{m_\ell})\|=\lim_{\ell\to\infty}\|\nabla h\left(z_{m_\ell}^{(i)}\right)-\nabla h\left(y_{m_\ell}^{(i)}\right)\|=0 \text{ for } i\in I_N.$$

By using the same way as in the proof of Case 1, we can show that

$$\lim_{\ell\to\infty}\|\nabla h(x_{m_\ell+1})-\nabla h(u_{m_\ell})\|=\lim_{\ell\to\infty}\|\nabla h(u_{m_\ell})-\nabla h(w_{m_\ell})\|=\lim_{\ell\to\infty}\|\nabla h(w_{m_\ell})-\nabla h(x_{m_\ell})\|=0$$

and

$$\lim_{\ell\to\infty}\|\nabla h(x_{m_\ell+1})-\nabla h(x_{m_\ell})\|=0.$$

Moreover, we can show that

$$\lim_{\ell\to\infty}\|x_{m_\ell+1}-x_{m_\ell}\|=0$$

and

$$\limsup_{\ell\to\infty}\langle x_{m_\ell+1}-v,\nabla h(u)-\nabla h(v)\rangle\le 0. \tag{3.27}$$

From (3.17) and (3.26), we have

$$\begin{aligned}
D_h(v,x_{m_\ell+1})&\le(1-\eta_{m_\ell})D_h(v,x_{m_\ell})+(1-\eta_{m_\ell})\beta_{m_\ell}(D_h(v,x_{m_\ell-1})-D_h(v,x_{m_\ell}))\\
&\quad+\eta_{m_\ell}\langle x_{m_\ell+1}-v,\nabla h(u)-\nabla h(v)\rangle\\
&\le(1-\eta_{m_\ell})D_h(v,x_{m_\ell+1})+(1-\eta_{m_\ell})\beta_{m_\ell}(D_h(v,x_{m_\ell-1})-D_h(v,x_{m_\ell}))\\
&\quad+\eta_{m_\ell}\langle x_{m_\ell+1}-v,\nabla h(u)-\nabla h(v)\rangle.
\end{aligned}$$

Since $\eta_{m_\ell}>0$ and $D_h(v,x_\ell)\le D_h(v,x_{m_\ell+1})$, we have

$$D_h(v,x_\ell)\le D_h(v,x_{m_\ell+1})\le(1-\eta_{m_\ell})\frac{\beta_{m_\ell}}{\eta_{m_\ell}}(D_h(v,x_{m_\ell-1})-D_h(v,x_{m_\ell}))+\langle x_{m_\ell+1}-v,\nabla h(u)-\nabla h(v)\rangle.$$

This together with Lemma 3.3 and (3.27) implies that $\limsup_{\ell\to\infty}D_h(v,x_\ell)\le 0$ and so $\lim_{\ell\to\infty}D_h(v,x_\ell)=0$. Therefore, $x_\ell\to v$. This completes the proof. □

The following result is obtained directly from our main result.

Corollary 3.5. *Let $\mathcal{H}$ be a real Hilbert space. Let $\mathcal{K}_i$ be a nonempty, closed, and convex subset of $\mathcal{H}$ and let $\mathcal{A}_i:\mathcal{H}\to\mathcal{H}$ be a monotone and L_i-Lipschitz continuous mapping for $i\in I_N$. Assume that $\Gamma:=\bigcap_{i=1}^{N}VI(\mathcal{K}_i,\mathcal{A}_i)\neq\emptyset$. Choose $\lambda_1^{(i)}>0$, $\mu_i\in(0,1)$ for $i\in I_N$ and $\beta>0$. For given $x_0,x_1,u\in\mathcal{H}$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}
w_n=x_n+\beta_n(x_{n-1}-x_n),\\
y_n^{(i)}=P_{\mathcal{K}_i}\left(w_n-\lambda_n^{(i)}\mathcal{A}_iw_n\right),\\
z_n^{(i)}=y_n^{(i)}-\lambda_n^{(i)}\left(\mathcal{A}_iy_n^{(i)}-\mathcal{A}_iw_n\right),\\
x_{n+1}=\eta_nu+(1-\eta_n)\left(\gamma_n^{(0)}w_n+\sum_{i=1}^{N}\gamma_n^{(i)}z_n^{(i)}\right),
\end{cases} \tag{3.28}$$

where $0 \leq \beta_n \leq \bar{\beta}_n$ *such that*

$$\bar{\beta}_n = \begin{cases} \min\left\{\frac{\epsilon_n}{\|x_{n-1}-x_n\|}, \beta\right\} & \text{if } x_{n-1} \neq x_n, \\ \beta & \text{otherwise} \end{cases}$$

and

$$\lambda_{n+1}^{(i)} = \begin{cases} \min\left\{\min_{i\in I_N}\left\{\frac{\mu_i\|w_n-y_n^{(i)}\|}{\|\mathcal{A}_i w_n-\mathcal{A}_i y_n^{(i)}\|}\right\}, \lambda_n^{(i)}\right\} & \text{if } \mathcal{A}_i w_n - \mathcal{A}_i y_n^{(i)} \neq 0, \\ \lambda_n^{(i)} & \text{otherwise.} \end{cases}$$

Suppose that Assumption 3.2 is satisfied. Then the sequence $\{x_n\}$ *generated by (3.28) converges strongly to* $v = P_\Gamma(u)$*, where* P_Γ^h *is the metric projection from* $\mathcal{H}$ *onto* Γ.

Next, we utilize the proposed method to solve systems of non-Lipschitz variational inequalities. The following additional assumption is needed:

$(A3^*)$ The mapping $\mathcal{A}_i : \mathcal{X} \to \mathcal{X}^*$ $(i \in I_N)$ is monotone and uniformly continuous.

It is known that if $\mathcal{A}$ defined on a convex domain $C \subset E$ is uniformly continuous, then $\mathcal{A}$ is almost Lipschitz continuous [56]; that is, for each $\epsilon > 0$, there exists a $K < \infty$ such that $\|\mathcal{A}x - \mathcal{A}y\| \leq K\|x-y\| + \epsilon$ for all $x, y \in C$.

The following lemma is an important tool to prove the convergence result for solving systems of non-Lipschitz variational inequalities.

Lemma 3.6. *Assume that Assumptions* $(A1), (A2), (A3^*)$*, and*$(A4)$ *and Assumption 3.2 are satisfied. For* $i \in I_N$*, let* $\{\lambda_n^{(i)}\}$ *be a sequence generated by (3.3). Then* $\lim_{n\to\infty}\lambda_n^{(i)} = \lambda^{(i)} \geq \min\left\{\min_{i\in I_N}\left\{\frac{\mu_i}{\mathcal{L}_i}\right\}, \lambda_1^{(i)}\right\}$.

Proof. For $i \in I_N$, by the continuity of $\mathcal{A}_i$, we have

$$\frac{\mu_i\|w_n - y_n^{(i)}\|}{\|\mathcal{A}_i w_n - \mathcal{A}_i y_n^{(i)}\|} \geq \frac{\mu_i\|w_n - y_n^{(i)}\|}{K_i\|w_n - y_n^{(i)}\| + \epsilon_i} = \frac{\mu_i\|w_n - y_n^{(i)}\|}{(K_i + \bar{\epsilon}_i)\|w_n - y_n^{(i)}\|} = \frac{\mu_i}{\mathcal{L}_i},$$

where $\mathcal{L}_i = K_i + \bar{\epsilon}_i$ and $\epsilon_i = \bar{\epsilon}_i\|u_n - y_n^{(i)}\|$ with $\bar{\epsilon}_i > 0$. The rest of the proof is similar to the proof of Yang and Liu [9, Lemma 3.1]. □

Using Lemma 3.6, the uniform continuity of $\mathcal{A}_i$ and following the same argument as in the proof of Theorem 3.4, we obtain the following result.

Theorem 3.7. *Assume that the Assumptions* $(A1), (A2), (A3^*)$*, and* $(A4)$ *and Assumption 3.2 are satisfied. Then the sequences* $\{x_n\}$ *generated by Algorithm 1 converge strongly to* $v = \Pi_\Gamma^h(u)$*, where* Π_Γ^h *is the Bregman projection with respect to h from* $\mathcal{X}$ *onto* Γ.

4 | COMPUTATIONAL EXPERIMENTS

In this section, we present several numerical experiments to verify the performance of our method and compare them with some existing methods. We denote "Iter" and "CPU" by the number of iterations and the CPU time in seconds, respectively.

In the first example, we give a numerical behavior of our Algorithm 1 (IBPA) based on various Bregman distances.

Example 4.1. We consider a Hp-hard problem which is taken from Harker and Pang [57]. Let $\mathcal{A}_i : \mathbb{R}^N \to \mathbb{R}^N$ be defined by $\mathcal{A}_i x := M_i x + q$ with $q \in \mathbb{R}^N$ and

$$M_i := N_i N_i^\top + S_i + D_i \ \text{ for } \ i = 1, 2, \dots, N,$$

where N_i is an $N \times N$, S_i is an $N \times N$ skew-symmetric matrix and D_i is an $N \times N$ positive definite diagonal matrix. It is clear that $\mathcal{A}_i$ is monotone and Lipschitz continuous with $L_i = \|M_i\|$ for $i \in I_N$. In this example, the set of feasible solutions is given by

$$\mathcal{K}_i := \{x = (u_1, u_2, \dots, u_N)^\top \in \mathbb{R}^N : \|x\| \le 1 \text{ and } u_i \ge a \text{ for } i \in I_N\},$$

where $0 < a < \frac{1}{\sqrt{N}}$. The following lists are Bregman distances which are generated by various functions.

(1) The Kullback–Leibler distance

$$D_h^{KL}(x, y) = \sum_{i=1}^{N} \left(u_i \ln\left(\frac{u_i}{v_i}\right) + v_i - u_i \right)$$

generated by Boltzmann–Shannon entropy $h(x) = \sum_{i=1}^{N} u_i \ln u_i$ with

$$\nabla h(x) = (1 + \ln(u_1), 1 + \ln(u_2), \cdots, 1 + \ln(u_N))^\top$$

and

$$(\nabla h)^{-1}(x) = (\exp(u_1 - 1), \exp(u_2 - 1), \dots, \exp(u_N - 1))^\top.$$

(2) The Itakura–Saito distance

$$D_h^{IS}(x, y) = \sum_{i=1}^{N} \left(\frac{u_i}{v_i} - \ln\left(\frac{u_i}{v_i}\right) - 1 \right)$$

generated by Burg entropy $h(x) = -\sum_{i=1}^{N} \ln u_i$ with

$$\nabla h(x) = -\left(\frac{1}{u_1}, \frac{1}{u_2}, \dots, \frac{1}{u_N} \right)^\top$$

and

$$(\nabla h)^{-1}(x) = -\left(\frac{1}{u_1}, \frac{1}{u_2}, \dots, \frac{1}{u_N} \right)^\top.$$

(3) The logistic loss

$$D_h^{LI}(x, y) = \sum_{i=1}^{N} \left(u_i \ln\left(\frac{u_i}{v_i}\right) + (1 - x_i) \ln\left(\frac{1 - u_i}{1 - v_i}\right) \right)$$

generated by function $h(x) = \sum_{i=1}^{N} (u_i \ln u_i + (1 - u_i) \ln(1 - u_i))$ with

$$\nabla h(x) = \left(\ln\left(\frac{u_1}{1 - u_1}\right), \ln\left(\frac{u_2}{1 - u_2}\right), \dots, \ln\left(\frac{u_N}{1 - u_N}\right) \right)^\top$$

and

$$(\nabla h)^{-1}(x) = \left(\frac{\exp(u_1)}{1 + \exp(u_1)}, \frac{\exp(u_2)}{1 + \exp(u_2)}, \dots \frac{\exp(u_N)}{1 + \exp(u_N)} \right)^\top.$$

(4) The Squared Euclidean distance

$$D_h^{SE}(x, y) = \frac{1}{2}\|x - y\|^2$$

generated by Euclidean norm $h(x) = \frac{1}{2}\|x\|^2$ with $\nabla h(x) = x$ and $(\nabla h)^{-1}(x) = x$.

It is clear that each h as above satisfies Assumption 3.1 (A2) with $\kappa = 1$ (see previous studies [58, 59]).

For $q = 0$, the unique common solution of the corresponding system of VIP is $\{0\}$. For $i = 1, 2, \dots, N$, we choose $\gamma_n^{(i)} = \frac{1}{N+1}$, $\lambda_1^{(i)} = 0.4$, $\mu_i = 0.36$, $\beta = 0.09$, $\eta_n = \frac{1}{n+1}$, $\epsilon_n = \frac{1}{\sqrt{n+1}}$ and $u =$ `rand(N,1)` which is generated randomly. Since we have known that the common solution is $v = 0$, we use stopping criterion $E_n := \|x_n - v\| \leq 10^{-4}$ for this experiment. Let x_0, x_1 be generated randomly in $\mathbb{R}^N$ and test IBPA using $N = 10, 30, 50$, and 100.

The numerical behavior of the proposed algorithm for various Bregman distances is shown in Table 1 and Figure 2.

Next, we provide a numerical example in infinite-dimensional Hilbert spaces.

TABLE 1 Numerical results of IBPA with various Bregman distances.

N		D_h^{KL}	D_h^{IS}	D_h^{LI}	D_h^{SE}
10	Iter	21	19	25	36
	CPU	0.0111	0.0141	0.0113	0.0226
30	Iter	15	19	20	29
	CPU	0.0090	0.0102	0.0106	0.0188
50	Iter	28	27	30	44
	CPU	0.0250	0.0213	0.0208	0.0511
100	Iter	15	19	20	29
	CPU	0.0433	0.0462	0.0443	0.0719

Abbreviation: IBPA, inertial-like Bregman projection-type algorithm for system of VIP.

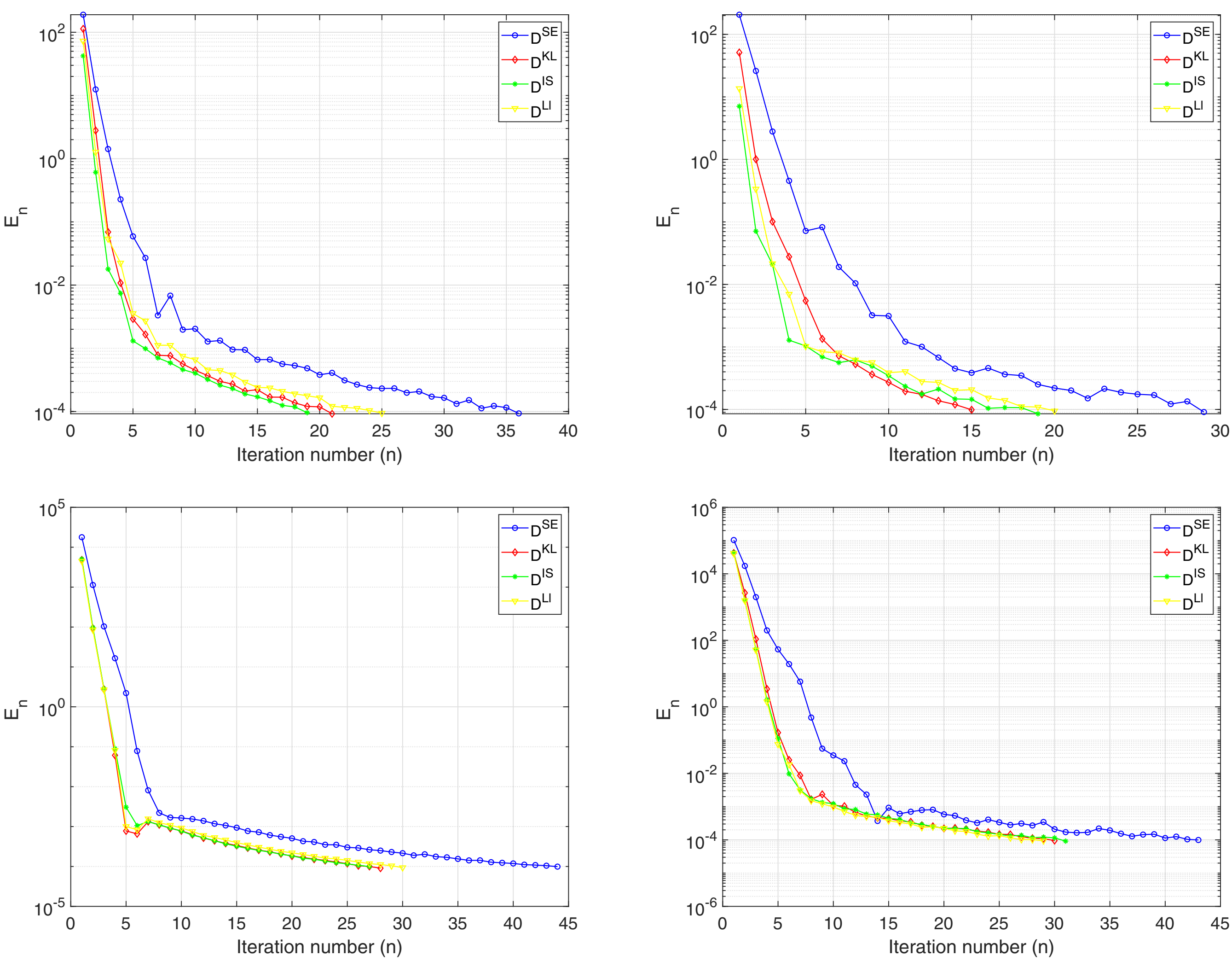

FIGURE 2 Example 4.1. Top left: $N = 10$. Top right: $N = 30$. Bottom left: $N = 50$. Bottom right: $N = 100$. [Colour figure can be viewed at wileyonlinelibrary.com]

Example 4.2. Let $\mathcal{X} := l_2$ with norm $\|x\|_{l_2} = \left(\sum_{k=1}^{\infty} |u_k|^2\right)^{1/2}$ and inner product $\langle x, y\rangle = \sum_{k=1}^{\infty} u_k v_k$ for all $x, y \in l_2$. For $i \in I_N$ ($N = 30$), let $\mathcal{A}_i : l_2 \to l_2$ be defined by

$$\mathcal{A}_i x = \frac{1}{i}x + (1, 1, 1, 0, 0, 0, 0, \dots) \tag{4.1}$$

for all $x \in l_2$. It easy to see that $\mathcal{A}_i$ is monotone and Lipschitz continuous. Let $\mathcal{K}_i := \mathcal{K} = \{x \in l_2 : \|x\|_{l_2} \leq 1\}$. In this experiment, we compare performance of our algorithm (IBPA) with the inertial parallel subextragradient algorithm (IPSA) proposed in Jolaoso et al. [6, Algorithm 4] and modified parallel hybrid subgradient extragradient method (MPHSEM) proposed in Kitisak et al. [25]. In our algorithm, we choose all parameters be the same as in Example 4.1. For IPSEM, we choose $\mu_n = \frac{1}{\sqrt{n+1}}$, $\mu = 0.36$, $\theta = 0.09$, $\lambda_0 = 0.15$, $\delta_n = \frac{1}{n+1}$, $\alpha_{n,i} = \frac{1}{N+1}$. For MPHSEM, we choose $\rho = 0.4$ and $\mu = 0.36$. The stopping criterion $E_n := \|x_{n+1} - x_n\|_{l_2} \leq 10^{-4}$ is used for this experiment. The starting points x_0 and x_1 are chosen as follows:

Case I: $x_0 = (1, 2, -1, 0, 0, 0, \dots)$, $x_1 = (1, -1, 1, 0, 0, 0, \dots)$,
Case II: $x_0 = (1, -1, 0, 0, 0, \dots)$, $x_1 = (1, 1, 0, 0, 0, \dots)$,
Case III: $x_0 = (2, 2, 3, 0, 0, 0, \dots)$, $x_1 = (-2, 4, 1, 0, 0, 0, \dots)$,
Case IV: $x_0 = (1, 1, -1, 0, 0, 0, \dots)$, $x_1 = (-3, -2, -1, 0, 0, 0, \dots)$.

TABLE 2 Numerical results of IBPA, IPSEM and MPHSEM.

x_0, x_1	IBPA		IPSEM		MPHSEM	
	Iter	CPU	Iter	CPU	Iter	CPU
Case I	29	0.0023	45	0.0156	67	0.0195
Case II	27	0.0016	45	0.0140	65	0.0166
Case III	28	0.0020	46	0.0152	63	0.0192
Case IV	29	0.0028	45	0.0149	63	0.0197

Abbreviations: IBPA, Inertial-like Bregman projection-type algorithm for system of VIP; MPHSEM, modified parallel hybrid subgradient extragradient method.

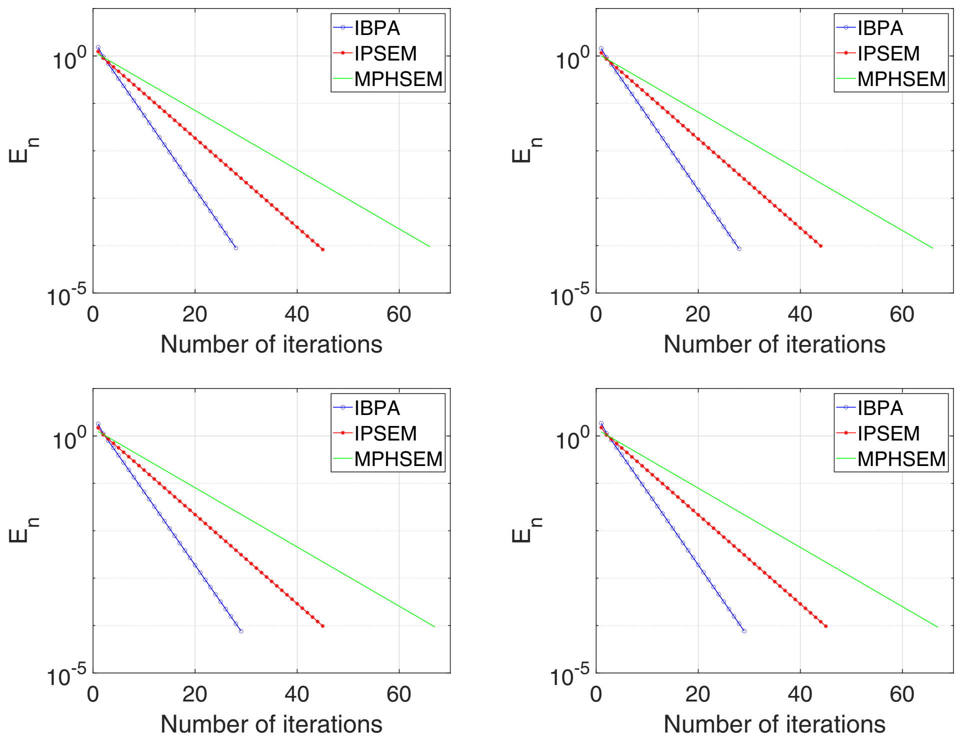

FIGURE 3 Example 4.2. Top left: Case I. Top right: Case II. Bottom left: Case III. Bottom right: Case IV. [Colour figure can be viewed at wileyonlinelibrary.com]

The numerical results of all methods are shown in Table 2 and Figure 3.

Next, we provide a numerical example in infinite-dimensional spaces L_p $(1 < p < 2)$ space, which are not Hilbert spaces.

Example 4.3. Let $\mathcal{X} := L_p([0,1])$ with norm $\|x\|_{L_p} = \left(\int_0^1 |x(t)|^p dt\right)^{1/p}$ and duality pairing $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ for all $x \in L_p([0,1])$ and $y \in L_q([0,1])$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f(x) = \frac{1}{2}\|x\|_{L_p}^2$ for all $x \in L_p([0,1])$. For $i \in I_N$ $(N = 5)$, let $\mathcal{A}_i : L_p([0,1]) \to L_q([0,1])$ be defined by

$$\mathcal{A}_i x(t) = \frac{1}{i} Jx(t) + J(e^{-t} + 1)$$

for all $x \in L_p([0,1])$, where $J : L_p([0,1]) \to L_q([0,1])$ is the duality mapping defined by $Jx := \frac{|x(t)|^{p-2}}{\|x\|_{L_p}^{p-2}} x(t)$ for $x \in L_p([0,1])$ and $t \in [0,1]$ (see Alber & Ryazantseva [60, p. 36]). Now, let $p = 3/2$. It is obvious that $\mathcal{A}_i$ is monotone and uniformly continuous on $L_p([0,1])$. Let $\mathcal{K}_i := \mathcal{K} = \{x \in L_{3/2}([0,1]) : \|x\|_{L_{3/2}} \le 1\}$. For $i \in I_N$, we choose $\gamma_n^{(i)} = \frac{3}{4} + \frac{1}{N4^N}$ for $i = 1, \dots, N$, $\gamma_n^{(0)} = 0$, $\lambda_1^{(i)} = \frac{1}{2^i}$, $\kappa = 1$, $\mu_i = 0.89$, $\beta = 0.01$, $\epsilon_n = \frac{1}{(n+1)^2}$ $\eta_n = \epsilon_n^2$, and $u = t + 1$. We perform the numerical experiment with three different choices of β_n as follows: $\beta_n = \beta_n^{\min} := 0$, $\beta_n = \beta_n^{\mathrm{mid}} := \frac{1}{2}\bar{\beta}_n$, and $\beta_n = \beta_n^{\max} := \bar{\beta}_n$. Note that when $\beta_n = \beta_n^{\min}$, the IBPA is a modified Bregman projection-type algorithm without inertial term. The stopping criterion $E_n := \|x_{n+1} - x_n\| \le 10^{-4}$ is used for this experiment. The stating points x_0 and x_1 are chosen as follows:

TABLE 3 Numerical results of IBPA with various choices of β_n.

x_0, x_1	IBPA ($\beta_n = \beta_n^{\min}$)		IBPA ($\beta_n = \beta_n^{\mathrm{mid}}$)		IBPA ($\beta_n = \beta_n^{\max}$)	
	Iter	CPU	Iter	CPU	Iter	CPU
Case I	18	28.0786	17	26.7291	16	24.9944
Case II	18	27.1716	16	23.3327	16	23.3004
Case III	17	26.4194	15	24.4135	15	24.2376
Case IV	17	26.4871	15	24.3912	3	4.3719

Abbreviation: IBPA, Inertial-like Bregman projection-type algorithm for system of VIP.

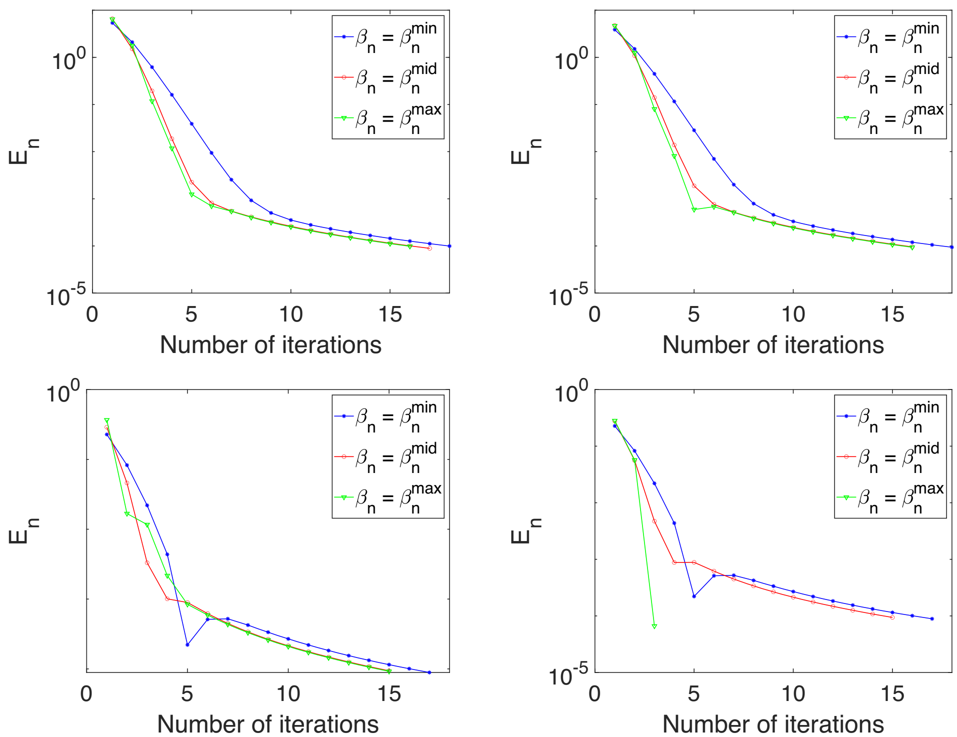

FIGURE 4 Example 4.3. Top left: Case I. Top right: Case II. Bottom left: Case III. Bottom right: Case IV. [Colour figure can be viewed at wileyonlinelibrary.com]

Case I: $x_0 = t + 2, x_1 = 2t^2 - 9,$
Case II: $x_0 = \frac{1}{4}(t^2 + 2t + 1), x_1 = 3t^2 + 7,$
Case III: $x_0 = \exp(2t), x_1 = t^2,$
Case IV: $x_0 = \frac{1}{2}\sin(3t), x_1 = \frac{2t}{3}.$

The numerical behavior of the proposed algorithm for different parameter β_n is shown in Table 3 and Figure 4.

5 | SIGNAL RESTORATION EXPERIMENTS

In this section, we perform numerical tests of our algorithm (IBPA) and compare them with IPSEM and MPHSEM in solving signal recovering problem. A signal recovery problem can be modeled as the following underdetermined linear

TABLE 4 Numerical comparison of algorithms.

Algorithm	Inputting	K-sparse = 10		K-sparse = 50		K-sparse = 100	
		Iter	CPU	Iter	CPU	Iter	CPU
IBPA	$\mathcal{A}_1$	293	0.0478	737	0.1409	924	0.1594
	$\mathcal{A}_1, \mathcal{A}_2$	199	0.0915	304	0.1184	339	0.1695
	$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$	178	0.1155	287	0.1535	323	0.1775
MPHSEM	$\mathcal{A}_1$	1473	1.6911	1495	1.6099	1592	1.5892
	$\mathcal{A}_1, \mathcal{A}_2$	835	2.1185	1388	3.1580	1503	3.2405
	$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$	661	2.5304	931	3.1583	1058	3.4707
IPSEM	$\mathcal{A}_1$	7844	1.7645	15786	3.4830	24397	5.4511
	$\mathcal{A}_1, \mathcal{A}_2$	6321	2.6145	8957	4.0582	10560	5.5268
	$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$	5666	3.3788	7609	5.1811	8744	5.8985

Abbreviations: IBPA, Inertial-like Bregman projection-type algorithm for system of VIP; MPHSEM, modified parallel hybrid subgradient extragradient method.

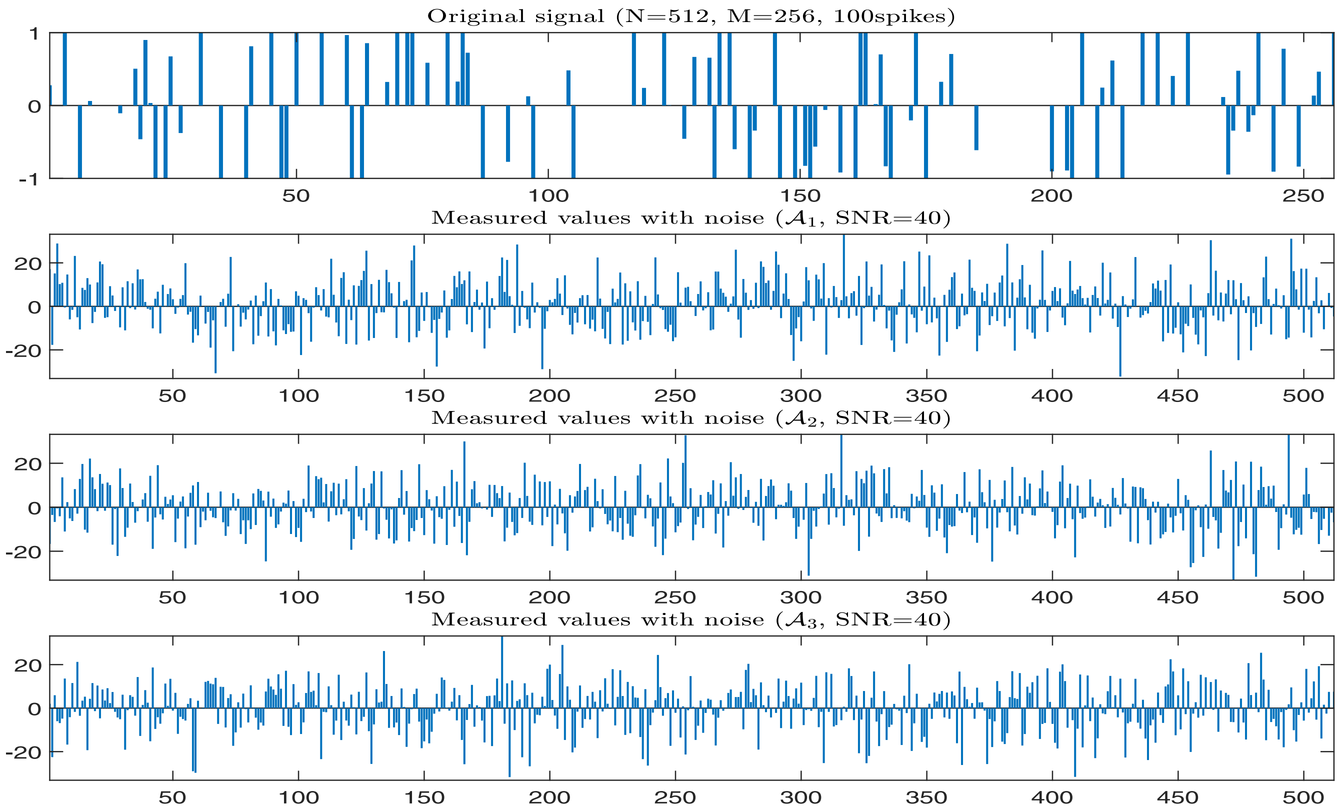

FIGURE 5 Original signal, observation data $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, respectively for $K = 100$. [Colour figure can be viewed at wileyonlinelibrary.com]

equation system:

$$b = \mathcal{A}x + v, \tag{5.1}$$

where $x \in \mathbb{R}^N$ is a K-sparse signal ($K \ll N$), $b \in \mathbb{R}^M$ is an observed or measured data with noisy $v \in \mathbb{R}^M$, and $\mathcal{A} \in \mathbb{R}^{M\times N}$ ($M < N$) is a filter matrix. Finding the solutions of problem (5.1) can be seen as solving the following least squares problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b - \mathcal{A}x\|_2^2, \tag{5.2}$$

where $\|\cdot\|_2$ is the Euclidean norm. This problem can be considered in the system of the following problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b_1 - \mathcal{A}_1x\|_2^2, \min_{x\in\mathbb{R}^N} \frac{1}{2}\|b_2 - \mathcal{A}_2x\|_2^2, \dots , \min_{x\in\mathbb{R}^N} \frac{1}{2}\|b_N - \mathcal{A}_Nx\|_2^2, \tag{5.3}$$

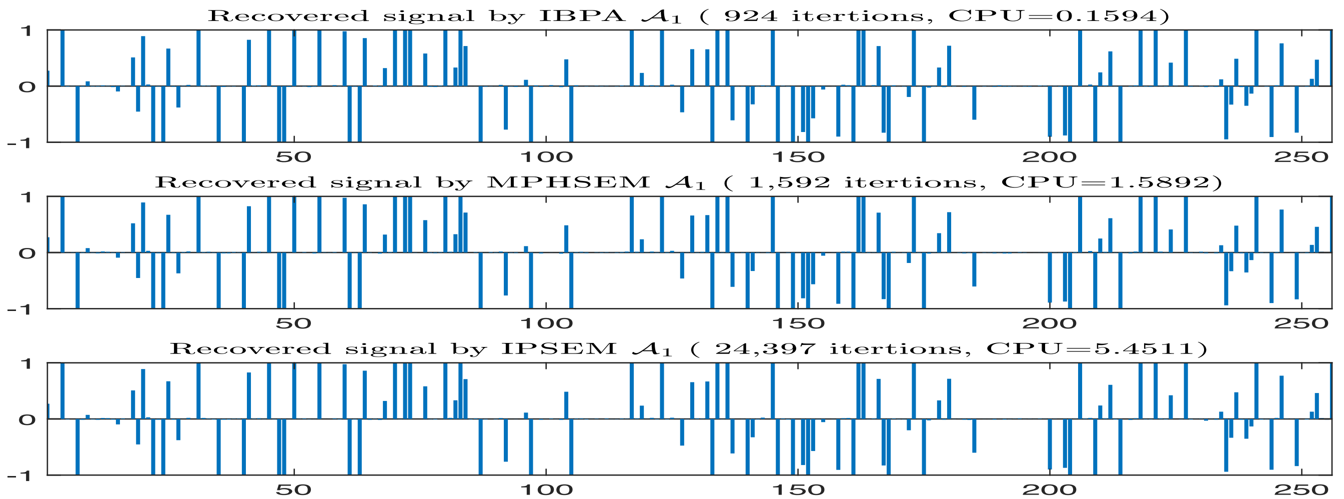

FIGURE 6 Recovered signal by inertial-like Bregman projection-type algorithm for system of VIP (IBPA), modified parallel hybrid subgradient extragradient method (MPHSEM), and IPSEM with $\mathcal{A}_1$, respectively for $K = 100$. [Colour figure can be viewed at wileyonlinelibrary.com]

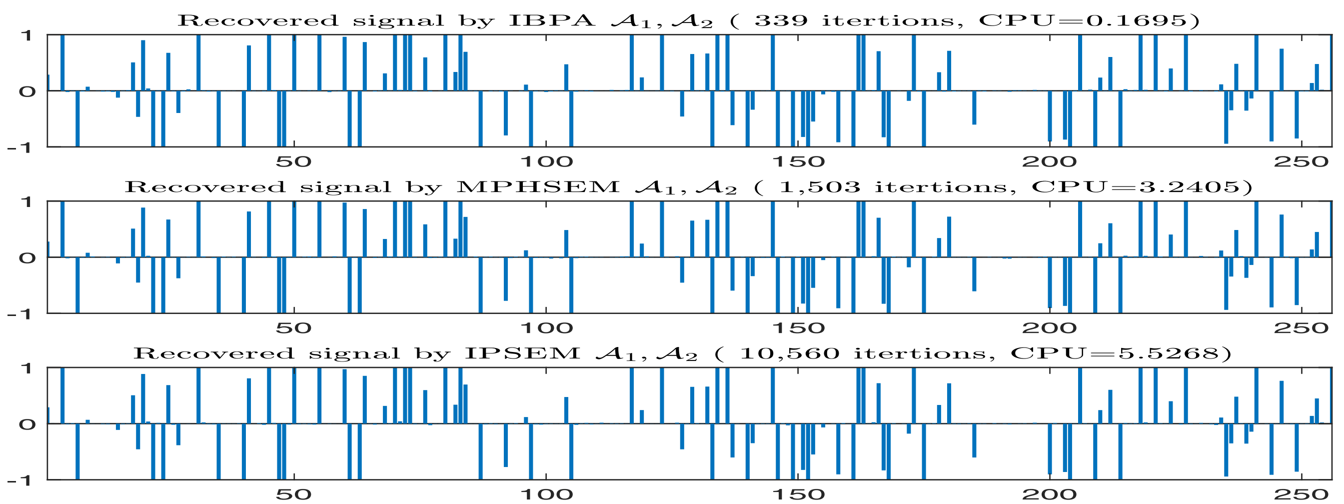

FIGURE 7 Recovered signal by inertial-like Bregman projection-type algorithm for system of VIP (IBPA), modified parallel hybrid subgradient extragradient method (MPHSEM), and IPSEM with $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively for $K = 100$. [Colour figure can be viewed at wileyonlinelibrary.com]

where x is a K-sparse signal, $\mathcal{A}_i$ is a filter matrix, and b_i is an observed signal with noisy for $i = 1, 2, \dots, N$. We can apply the IBPA, IPSEM, and MPHSEM to solve problem (5.3) by setting $\mathcal{A}_i x = \mathcal{A}_i^{\mathsf{T}}(\mathcal{A}_i x - b_i)$ for all $i = 1, 2, \dots, N$. Note that $\mathcal{A}_i$ is monotone and Lipschitz continuous with $L_i = \|\mathcal{A}_i^{\mathsf{T}} \mathcal{A}_i\|$ for $i = 1, 2, \dots, N$.

In this experiment, we choose the signal size to be $N = 512$ and $M = 256$, the original signal x is generated by the uniform distribution in $[-2, 2]$ with K nonzero elements, the matrix $\mathcal{A}_i \in \mathbb{R}^{M \times N}$ is generated from a normal distribution with mean zero and one invariance, and the observation b_i is generated by white Gaussian noise with signal-to-noise ratio $SNR = 40$. Given that the starting points $x_0 = \mathbf{0}$ and $x_1 = \mathbf{1}$ are in $\mathbb{R}^N$. We measure the quality of restored signal mean squared error (MSE), defined by

$$MSE := \frac{1}{N}\|x_n - x\|_2^2 < 10^{-4},$$

where x_n is an estimated signal of x. For IBPA, we choose $u = \mathbf{1} \in \mathbb{R}^N$, $\lambda_n^{(1)} = 0.003$, $\lambda_n^{(2)} = 0.02$, $\lambda_n^{(3)} = 0.001$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\mu_3 = 0.005$, $\beta = 0.25$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\eta_n = \frac{1}{150n+1}$, $\gamma_n^{(1)} = \frac{n^2-n}{3n^2+1}$, $\gamma_n^{(2)} = \frac{n^2-n}{4n^2+1}$ and $\gamma_n^{(3)} = \frac{n^2-n}{5n^2+1}$. For IPSEM, we choose

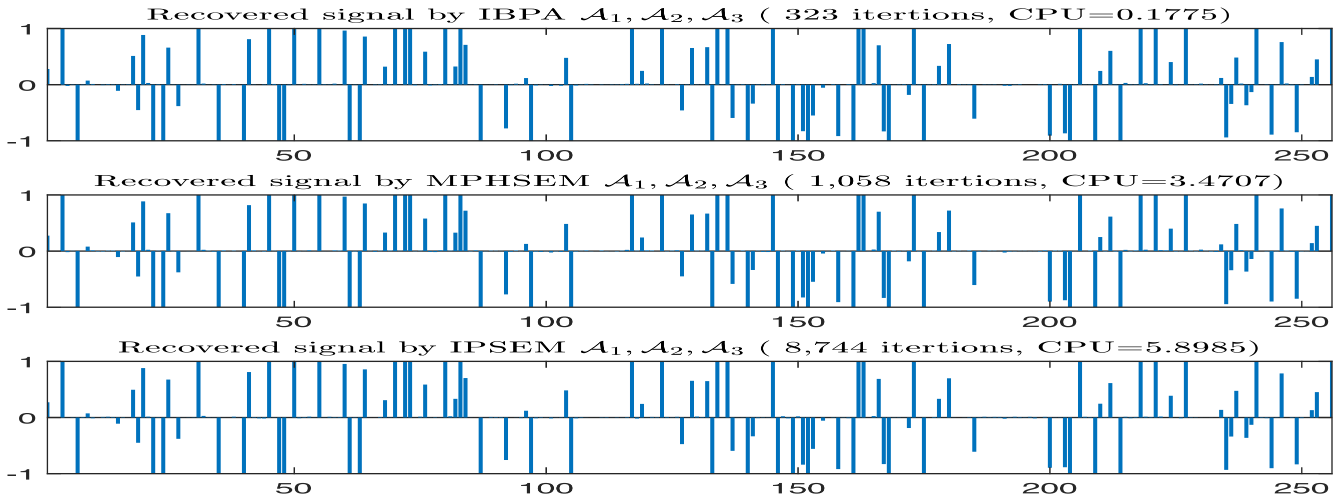

FIGURE 8 Recovered signal by inertial-like Bregman projection-type algorithm for system of VIP (IBPA), modified parallel hybrid subgradient extragradient method (MPHSEM), and IPSEM with $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, respectively for $K = 100$. [Colour figure can be viewed at wileyonlinelibrary.com]

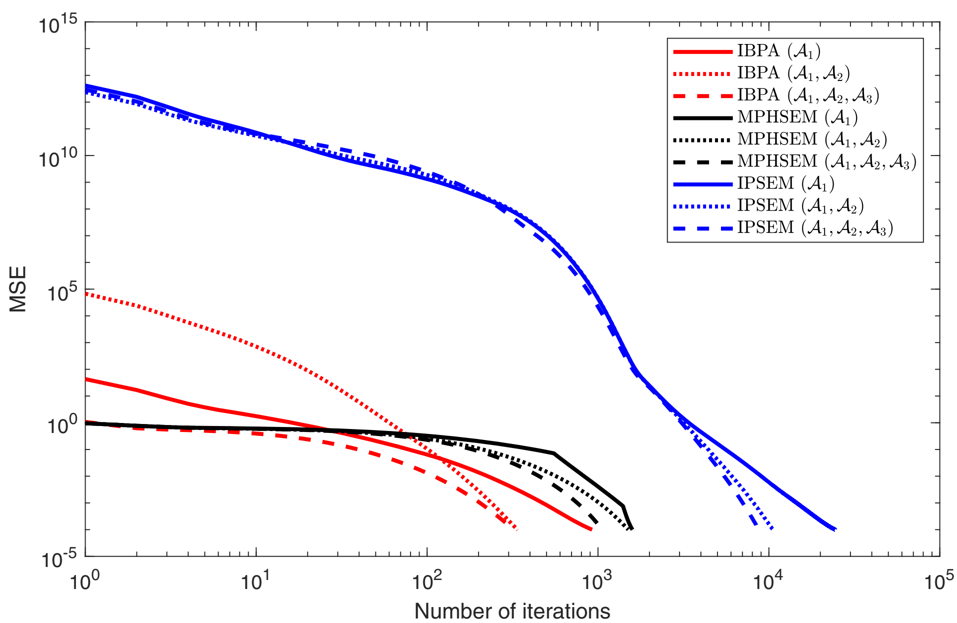

FIGURE 9 Mean squared error (MSE) with $N = 512$, $M = 256$, and $K = 100$. [Colour figure can be viewed at wileyonlinelibrary.com]

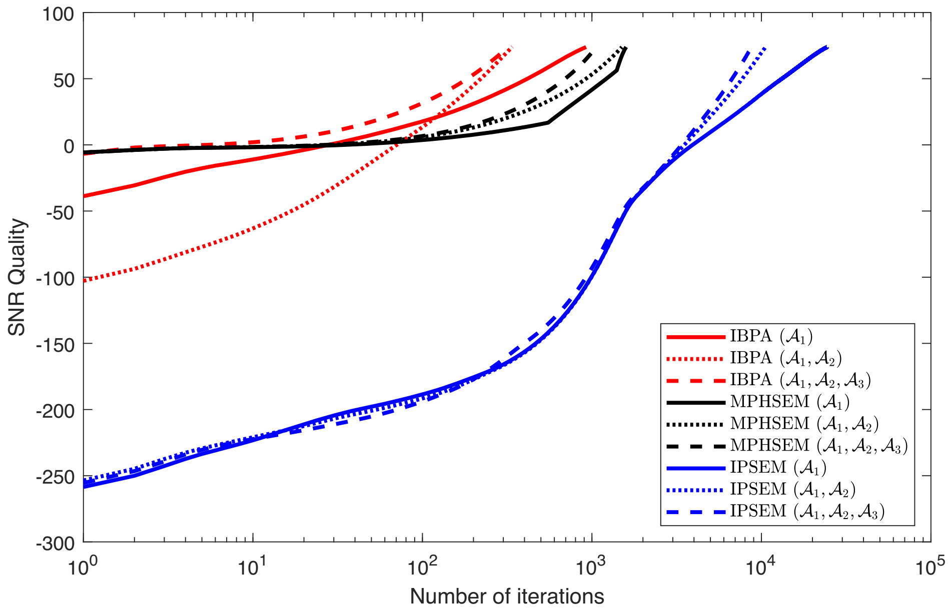

FIGURE 10 Signal-to-noise ratio (SNR) with $N = 512$ and $M = 256$ for $K = 100$. [Colour figure can be viewed at wileyonlinelibrary.com]

$\lambda_n = 1.5$, $\theta = 0.25$, $\mu_n = \frac{1}{(n+1)^2}$, $\mu = 0.005$, $\delta_n = \frac{1}{150n+1}$, $\alpha_n = \frac{1}{150n+1}$, $\alpha_n^{(1)} = \frac{n^2-n}{3n^2+1}$, $\alpha_n^{(2)} = \frac{n^2-n}{4n^2+1}$ and $\alpha_n^{(3)} = \frac{n^2-n}{5n^2+1}$. For MPHSEM, we choose $\rho = 0.25$ and $\mu = 0.02$.

The numerical comparison of all algorithms with different K-sparse is shown in Table 4.

The graphs of original signal, recovered signal, and mean squared error by all algorithms in case $K = 100$ are shown in Figures 5–10.

Remark 5.1. It is clearly observed that recovered signal by IBPA has a good performance in terms of CPU time and number of iterations in comparison with MPHSEM and IPSEM. This shows that our algorithm has better effectiveness than MPHSEM and IPSEM. Physically, it is due to the fact that the proposed algorithm computes only one projection and the stepsize of such method uses self-adaptive procedure which is adaptively updated by a cheap computation and without linesearch process.

6 | CONCLUSIONS

This paper has been devoted to propose a new self-adaptive inertial-like algorithm with Bregman distance for solving the systems of variational inequalities for a class of monotone and Lipschitz continuous mapping in reflexive Banach spaces. Our method used a self-adaptive stepsize which can be easily computed without prior knowledge of the Lipschitz constant of the mapping. Furthermore, without using the hybrid projection and shrinking projection method, the strong convergence of our method has been established under some suitable assumptions. Finally, some numerical behavior of our method in both finite and infinite-dimensional spaces including comparisons with other related works has been presented.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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REFERENCES

1. C. Baiocchi and A. Capelo, Variational and quasivariational inequalities, *Applications to free boundary problems*, Wiley, New York, 1984.
2. R. Glowinski, J. L. Lions, and R. Tremolieres, *Numerical analysis of variational inequalities*, North-Holland, Amsterdam, 1981.
3. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.
4. G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, C. R. Math. Acad. Sci. Paris **258** (1964), 4413–4416.
5. E. Zeidler, *Nonlinear functional analysis and its applications III: variational methods and optimization*, Springer, New York, 1985.
6. L. O. Jolaoso, O. K. Oyewole, and K. O. Aremu, *Inertial self-adaptive parallel extragradient-type method for common solution of variational inequality problems*, Appl. Anal. **102** (2023), 1100–1122.
7. G. Kassay, S. Reich, and S. Sabach, *Iterative methods for solving systems of variational inequalities in reflexive Banach spaces*, SIAM J. Optim. **21** (2011), no. 4, 1319–1344.
8. P. Sunthrayuth and P. Kumam, *Fixed point solutions for variational inequalities in image restoration over q-uniformly smooth Banach spaces*, J. Inequalities Appl. (2014), 473.
9. J. Yang and H. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numerical Algoritm. **80** (2019), 741–752.
10. B. S. He and L. Z. Liao, *Improvements of some projection methods for monotone nonlinear variational inequalities*, J. Optim. Theory Appl. **112** (2002), 111–128.
11. G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ėkon Mat. Metody **12** (1976), 747–756.
12. Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2011), 318–335.
13. B. S. He, *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim. **35** (1997), 69–76.
14. D. F. Sun, *A class of iterative methods for solving nonlinear projection equations*, J. Optim. Theory Appl. **91** (1996), 123–140.
15. P. Tseng, *A modified forward-backward splitting method for maximal monotone mapping*, SIAM J. Control. Optim. **38** (2000), 431–446.
16. Y. Censor, A. Gibali, and S. Reich, *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space*, Optim. Methods Softw. **26** (2011), 827–84.
17. Y. Censor, A. Gibali, and S. Reich, *Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space*, Optimization **61** (2012), 1119–1132.
18. D. V. Thong and D. V. Hieu, *Weak and strong convergence theorems for variational inequality problems*, Numer. Algoritm. **78** (2018), 1045–1060.
19. Y. Shehu and O. S. Iyiola, *Strong convergence result for monotone variational inequalities*, Numer. Algoritm. **76** (2017), 259–282.
20. Y. Shehu, *Single projection algorithm for variational inequalities in Banach spaces with applications to contact problems*, Acta Math. Sci. **40** (2020), 1045–1063.
21. P. Sunthrayuth, L. O. Jolaoso, and P. Cholamjiak, *New Bregman projection methods for solving pseudo-monotone variational inequality problem*, J. Appl. Math. Comput. **68** (2022), 1565–1589.
22. D. V. Thong and P. Vuong, *Improved subgradient extragradient methods for solving pseudomonotone variational inequalities in Hilbert spaces*, Appl. Numer. Math. **163** (2021), 221–238.
23. J. Yang, H. Liu, and G. Li, *Convergence of a subgradient extragradient algorithm for solving monotone variational inequalities*, Numerical Algoritm. **84** (2020), 389–405.
24. Y. Censor, A. Gibali, S. Reich, and S. Sabach, *Common solutions to variational inequalities*, Set-Valued Variational Anal. **20** (2012), 229–247.
25. P. Kitisak, W. Cholamjiak, D. Yambangwai, and R. Jaidee, *A modified parallel hybrid subgradient extragradient method for finding common solutions of variational inequality problems*, Thai J. Math. **18** (2020), no. 1, 261–274.
26. D. V. Hieu and S. Reich, *Two Bregman projection methods for solving variational inequalities*, Optimization **71** (2022), 1777–1802.
27. C. Izuchukwu, S. Reich, and Y. Shehu, *One-step Bregman projection methods for solving variational inequalities in reflexive Banach spaces, Optimization*, 2023, DOI 10.1080/02331934.2023.2168483.
28. A. Gibali, L. O. Jolaoso, O. T. Mewomo, and A. Taiwo, *Fast and simple Bregman projection methods for solving variational inequalities and related problems in Banach spaces*, Results Math. **75** (2020), 179.
29. A. Adamu, D. Kitkuan, A. Padcharoen, C. E. Chidume, and P. Kumam, *Inertial viscosity-type iterative method for solving inclusion problems with applications*, Math. Comput. Simul. **194** (2022), 445–459.
30. A. Adamu, D. Kitkuan, P. Kumam, A. Padcharoen, and T. Seangwattana, *Approximation method for monotone inclusion problems in real Banach spaces with applications*, J. Inequalities Appl. **2022** (2022), 70.
31. A. Gibali and D. V. Hieu, *A new inertial double-projection method for solving variational inequalities*, J. Fixed Point Theory Appl. **21** (2019), 97.
32. C. Izuchukwu, S. Reich, Y. Shehu, and A. Taiwo, *Strong convergence of forward-reflected-backward splitting methods for solving monotone inclusions with applications to image restoration and optimal control*, J. Sci. Computing **94** (2023), 73.

33. S. Reich, T. M. Tuyen, P. Sunthrayuth, and P. Cholamjiak, *Two new inertial algorithms for solving variational inequalities in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **42** (2021), 1954–1984.
34. Y. Shehu and A. Gibali, *New inertial relaxed method for solving split feasibilities*, Optim. Lett. **1** (2021), 2109–2126.
35. Y. Shehu and O. S. Iyiola, *Projection methods with alternating inertial steps for variational inequalities: Weak and linear convergence*, Appl. Numer. Math. **157** (2020), 315–337.
36. Y. Shehu, L. Liu, X. Mu, and Q. L. Dong, *Analysis of versions of relaxed inertial projection and contraction method*, Appl. Numer. Math. **165** (2021), 1–21.
37. Z. B. Wang, P. Sunthrayuth, A. Adamu, and P. Cholamjiak, *Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities*, Optimization (2023), 1–35, DOI 10.1080/02331934.2023.2187663.
38. H. K. Pathak, *An introduction to nonlinear analysis and fixed point theory*, Springer Nature Singapore Pte Ltd, 2018.
39. A. Beck, *First-order methods in optimization, society for industrial and applied mathematics Philadelphia*, 2017.
40. S. Reich and S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **31** (2010), no. 1, 2–44.
41. D. Reem and S. Reich, *Solutions to inexact resolvent inclusion problems with applications to nonlinear analysis and optimization*, Rend. Circolo. Mat. Palermo **67** (2018), 337–371.
42. H. H. Bauschke, J. M. Borwein, and P. L. Combettes, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces*, Commun. Contemp. Math. **3** (2001), 615–647.
43. L. M. Bregman, *The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming*, USSR Comput. Math. Math. Phys. **7** (1967), 200–217.
44. Y. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, *Theory and applications of nonlinear operator of accretive and monotone type*, Marcel Dekker, New York, 1996, pp. 15–50.
45. S. Reich, A weak convergence theorem for the alternating method with Bregman distances, *Theory and applications of nonlinear operators of accretive and monotone type*, Marcel Dekker, New York, 1996, pp. 313–318.
46. K. Nakajo, *Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces*, Appl. Math. Comput. **271** (2015), 251–258.
47. D. Reem, S. Reich, and A. De Pierro, *Re-examination of Bregman functions and new properties of their divergences*, Optimization **68** (2019), 279–348.
48. D. Butnariu and E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal. **1–39** (2006), no. 2006, 84919.
49. D. Butnariu, S. Reich, and A. J. Zaslavski, *There are many totally convex functions*, J. Convex Anal. **13** (2006), 623–632.
50. H. Iiduka and W. Takahashi, *Weak convergence of a projection algorithm for variational inequalities in a Banach space*, J. Math. Anal. Appl. **339** (2008), 668–679.
51. V. Martín-Márquez, S. Reich, and S. Sabach, *Iterative Methods for approximating fixed points of Bregman nonexpansive operators*, Discret. Contin. Dyn. Syst. Series S **6** (2013), no. 4, 1043–1063.
52. H. K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc. **66** (2002), no. 1, 240–256.
53. P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.
54. S. Reich and S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal. **1** (2009), 471–485.
55. C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific, Publishing Co., Inc., River Edge NJ, 2002.
56. R. Vanderbei: *Uniform continuity is almost Lipschitz continuity*. Statistics and Operations Research Series, Princeton University, 1991.
57. P. T. Harker and J. S. Pang, A damped-Newton method for the linear complementarity problem, *Computational solution of nonlinear systems of equations*, Lectures in Applied Mathematics, AMS, Providence, 1990, pp. 265–284.
58. H. H. Bauschke, X. Wang, J. Ye, and X. Yuan, *Bregman distances and Chebyshev sets*, J. Approx. Theory **159** (2009), 3–25.
59. D. V. Hieu and P. Cholamjiak, *Modified extragradient method with Bregman distance for variational inequalities*, Appl. Anal. **101** (2022), 655–670.
60. Y. Alber and I. Ryazantseva, *Nonlinear ill-posed problems of monotone type*, Springer, Netherlands, 2006.

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Inertial projection and contraction methods for solving variational inequalities with applications to image restoration problems

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ABSTRACT. In this paper, we introduce two inertial self-adaptive projection and contraction methods for solving the pseudomonotone variational inequality problem with a Lipschitz-continuous mapping in real Hilbert spaces. The adaptive stepsizes provided by the algorithms are simple to update and their computations are more efficient and flexible. Also we prove some weak and strong convergence theorems without prior knowledge of the Lipschitz constant of the mapping. Finally, we present some numerical experiments to demonstrate the effectiveness of the proposed algorithms by comparisons with related methods and some applications of the proposed algorithms to the image deblurring problem.

1. INTRODUCTION

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H and $A : H \to H$ be a continuous mapping.

The *variational inequality problem* (shortly, VIP) is defined as follows:

$$\text{Find} \quad z \in C \quad \text{such that} \quad \langle Az, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the solution set of the VIP (1.1) by $VI(C, A)$. Several important applications of the VIP (1.1) have been discussed in, for instance, [2, 4, 12, 22, 23, 24, 28]. It is well known that a point z is a solution of the VIP (1.1) if and only if z solves the fixed point equation:

$$z = P_C(z - \lambda Az), \quad \forall \lambda > 0,$$

where P_C is the projection operator from H onto C. One of the earliest projection methods for solving VIP is the *extragradient method* (EGM) introduced independently by Antipin [3] and Korpelevich [25] as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

where $A : H \to H$ is monotone and L-Lipschitz continuous and suitable stepsize $\lambda \in \left(0, \frac{1}{L}\right)$. It was proved that the EGM converges weakly to a solution of VIP in finite dimensional spaces. However, the EGM requires two projections onto the feasible set C which can be computationally costly if A is not simple.

A question of interest in projection-type algorithms is how to reduce the number of projections in the algorithm. This has led to many modifications and improvements of the EGM by many authors.

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In particular, Censor et al. [7] introduced the *subgradient extragradient method* (SEGM) for solving the VIP with a monotone and L-Lipschitz continuous mapping. The algorithm is described as follows:

$$
\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{1.2}
$$

where

$$
T_n := \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \le 0\}.
$$

The authors proved that SEM converges weakly to a solution of the VIP provided the stepsize $\lambda \in \left(0, \frac{1}{L}\right)$. Note that the T_n in (1.2) is a half-space and P_{T_n} can be easily calculated using the closed form formula.

Also, Tseng [38] introduced the following single projection method for solving the VIP. This method is known as the *Tseng extragradient method* (TEGM), which is described as follows:

$$
\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n). \end{cases} \tag{1.3}
$$

It was proved that TEGM converges weakly to a solution of the VIP if the stepsize satisfies $\lambda \in \left(0, \frac{1}{L}\right)$. Note that the TEGM is more efficient than the EGM and its modifications due to its single projection onto the feasible set C per each iteration.

Another method based on the single projection onto C for solving the monotone VIP is the *projection and contraction method* (PCM) introduced by He [14] (see also Sun [33]). The algorithm is stated as follows:

$$
\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = x_n - \gamma \eta_n d_n, \end{cases} \tag{1.4}
$$

where $\gamma \in (0, 2)$, $\lambda_n \in \left(0, \frac{1}{L}\right)$ and

$$
\eta_n := \frac{\langle x_n - y_n, d_n \rangle}{\|d_n\|^2}, \quad d_n := x_n - y_n - \lambda_n(A x_n - A y_n). \tag{1.5}
$$

They proved that PCM converges weakly to a solution of the VIP under appropriate assumptions. Recently, PCM for solving VIP has received great attention from many authors, who improved it in various ways (see, for example, [9, 11, 15, 16]).

However, the stepsizes of the methods SEGM, TEGM and PCM depend on the prior estimate of the Lipschitz constant L of the cost operator which is very difficult to estimate in practice.

In order to modified the method which stepsize does not require prior estimate of the Lipschitz constant and extend to more general class of the monotone VIPs, Thong and Vuong [36] proposed a modification of the TEGM with a linesearch procedure for solving the VIP with a pseudomonotone and Lipschitz continuous mapping in Hilbert spaces. To be more precise, they proposed the following algorithm:

Algorithm A. [The TEGM for the pseudomonotone VIP]
Step 0: Given $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, 1)$. Let $x_1 \in H$ be arbitrary.
Step 1: Calculate

$$
y_n = P_C(x_n - \lambda_n A x_n),
$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$
\gamma l^m \|A x_n - A y_n\| \le \mu \|x_n - y_n\|.
$$

Step 2: Calculate

$$x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n).$$

Update $n := n + 1$ go to **Step 1**.

They proved that, if we assume that $A : H \to H$ is sequentially weakly continuous, then the sequence $\{x_n\}$ generated by Algorithm A converges weakly to a point of $VI(C, A)$.

Very recently, Khanh et al. [21] also proposed the following SEGM for solving the VIP with a pseudomonotone and Lipschitz continuous mapping in Hilbert spaces:

Algorithm B. [The SEGM for the pseudomonotone VIP]
Step 0: Given $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, 1)$. Let $x_1 \in H$ be arbitrary.
Step 1: Calculate

$$y_n = P_C(x_n - \lambda_n Ax_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|.$$

Step 2: Construct the half-space

$$T_n := \{x \in H : \langle x_n - \lambda_n Ax_n - y_n, x - y_n\rangle \leq 0\}$$

and calculate

$$x_{n+1} = P_{T_n}(x_n - \lambda_n Ay_n).$$

Update $n := n + 1$ go to **Step 1**.

The weak convergence of the sequence $\{x_n\}$ generated by Algorithm B was established under assuming the weak sequential continuity of A, which often assumed in many recent works related to the pseudomonotone VIP (see, for example, [5, 8, 17, 21, 26, 36, 41]). In most cases, the strong convergence is also preferable to the weak convergence in many problems that arise in infinite-dimensional spaces because the weak convergence of algorithms does not allow to enable efficient.

On the other hand, the inertial method has been a technique of interest and has received a lot of attention from many researchers. Recently, the inertial technique is often used to accelerated the convergence rate of algorithms to solves many kinds of optimization (see, for example, [1, 8, 9, 16, 18, 27, 30, 35, 39, 41]).

Motivated and inspired by the above work, in this paper, we propose two modified inertial projection and contraction methods with self adaptive stepsize rules to the solve pseudomonotone variational inequality problem in real Hilbert spaces. This adaptive stepsize rules are more efficient and flexible in computations without any linesearch procedure which can be time-consuming and expensive. Also we prove some weak and strong convergence theorems for the proposed methods without any prior knowledge of the Lipschitz constant of the mapping and without assuming the weak sequential continuity of the mapping.

The rest of the paper is divided as follows: In Sect. 2, we provide some preliminary results which are need for our work. In Sect. 3, we prove some weak and strong convergence theorems for the proposed methods. Finally, in Sect. 4, we give some numerical experiments including comparisons with other algorithms and the applications of the proposed algorithms in the image debluring problem.

2. Preliminaries

Let H be a real Hilbert space. For a sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup z$ to indicate that the sequence $\{x_n\}$ converges weakly to a point $x \in H$ and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to a point $x \in H$. A point $x \in H$ is called a *weak cluster point* of a sequence $\{x_n\}$ in H if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point $x \in H$.

For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we know the following inequalities:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \tag{2.6}$$

and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.7}$$

Definition 2.1. Let $A : H \to H$ be a mapping. Then A is said to be:

(1) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H,$$

and, if $L \in [0, 1)$, then A is called *contraction*;

(2) *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in H;$$

(3) *pseudomonotone* if

$$\langle Ax, y - x\rangle \geq 0 \implies \langle Ay, y - x\rangle \geq 0, \quad \forall x, y \in H;$$

(4) *sequentially weakly continuous* if, for each sequence $\{x_n\} \in H$, $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

Remark 2.1. It is observe that every monotone mapping is a pseudomonotone mapping. Indeed, let $A : H \to H$ be a monotone mapping such that $\langle Ax, y - x\rangle \geq 0$ for all $x, y \in H$. It follows that

$$\langle Ay, y - x\rangle = \underbrace{\langle Ay - Ax, y - x\rangle}_{\geq 0} + \underbrace{\langle Ax, y - x\rangle}_{\geq 0} \geq 0$$

for all $x, y \in H$. Hence A is a pseudomonotone mapping, but the converse implication is not true. Several examples of a pseudomonotone mapping which is not necessarily monotone can be found in [5, 20, 32].

Let C be a nonempty closed and convex subset of H. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

Such a mapping P_C is called the *metric projection* of H onto C. The following is well known:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0. \tag{2.8}$$

Let A be a mapping of C into H. Then we know the following property [34]:

$$z \in VI(C, A) \iff z = P_C(z - \lambda Az), \quad \forall \lambda > 0. \tag{2.9}$$

The following are explicit formulas of the metric projection on various feasible sets [6]:

(1) A half-space in H has the form $H_{(a,\beta)} := \{x \in H : \langle a, x\rangle \leq \beta\}$, where $a \in H$, $a \neq 0$ and $\beta \in \mathbb{R}$. Then the projection of x onto $H_{(a,\beta)}$ is given by

$$P_{H_{(a,\beta)}}(x) = \begin{cases} x - \max\left\{\frac{\langle a,x\rangle - \beta}{\|a\|^2}, 0\right\} a & \text{if } \langle a, x\rangle > \beta, \\ x & \text{if } \langle a, x\rangle \leq \beta. \end{cases}$$

(2) A ball $B[p, r] := \{x \in H : \|x - p\| \leq r\}$, where $r > 0$. Then the projection of x onto $B[p, r]$ is given by

$$P_{B[p,r]}(x) = \begin{cases} p + \frac{r}{\max\{\|x-p\|, r\}}(x - p) & \text{if } \|x - p\| > r, \\ x & \text{if } \|x - p\| \leq r. \end{cases}$$

(3) A box constraints in $\mathbb{R}^n$ have the form $\text{Box}[a, b] := \{x \in \mathbb{R}^n : a \leq x \leq b\}$, where $a, b \in \mathbb{R}^n$ and $a \leq b$. Then the projection of x onto $\text{Box}[a, b]$ is given by

$$P_{\text{Box}[a,b]}(x)_i = \min\{b_i, \max\{x, a_i\}\}.$$

We need the following lemmas and facts, which will play an important role in proving our main results.

Lemma 2.1. [29] *Let C be a nonempty set of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each* $x \in C$*;*

(ii) *every weak cluster point of* $\{x_n\}$ *is in* C*.*

Then $\{x_n\}$ converges weakly to a point in C.

Let $\{a_n\}$ be a real sequence. Then we have

$$\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty}(a_n).$$

In particular, if $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then we obtain the following:

(1) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$;

(2) $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty}(a_n) + \liminf_{n\to\infty}(b_n)$.

Lemma 2.2. [1] *Let $\{\varphi_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be three nonnegative real sequences such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \beta_n, \quad \forall n \geq 1,$$

with $\sum_{n=1}^{\infty} \beta_n < \infty$ and there exists a real number α such that $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following results hold:

(1) $\sum_{n=1}^{\infty}[\varphi_n - \varphi_{n-1}]_+ < \infty$, *where* $[t]_+ := \max\{t, 0\}$.

(2) *There exists* $\varphi^* \in [0, \infty)$ *such that* $\lim_{n\to\infty} \varphi_n = \varphi^*$.

Lemma 2.3. [27] *Let $\{a_n\}$ and $\{c_n\}$ be two nonnegative real sequences such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n, \quad \forall n \geq 1.$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(1) *If $b_n \leq \alpha_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*

(2) *If* $\sum_{n1}^{\infty} \alpha_n = \infty$ *and* $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [31] *Let* $\{a_n\}$ *be a nonnegative real sequence,* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{b_n\}$ *be a real sequence. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we introduce two new modified inertial projection and contraction algorithms with adaptive stepsize rule for solving the psuedomonotone VIP. In order to prove the convergence results of the proposed algorithms, we need the following conditions:

Condition 1: The feasible set C is a closed and convex subset of a real Hilbert space H.

Condition 2: The mapping $A : H \to H$ is L-Lipschitz continuous and pseudomonotone on H.

Condition 3: The mapping $A : H \to H$ satisfies the following condition:

$$\text{whenever } \{q_n\} \subset C,\ q_n \rightharpoonup q \text{ one has } \|Aq\| \leq \liminf_{n\to\infty} \|Aq_n\|.$$

Condition 4: The solution set of VIP is nonempty, that is, $VI(C,A) \neq \emptyset$.

Remark 3.2. (1) If H is a finite-dimensional space, then it suffices to assume that the mapping A is continuous pseudomonotone and the Condition 3 is not necessary to assume.

(2) The Condition 3 is weaker than the sequential weak continuity of the mapping A. Indeed, let $A : \ell_2 \to \ell_2$ be a mapping defined by $Ax = x\|x\|$ for all $x \in \ell_2$. Let $\{q_n\} \subset \ell_2$ such that $q_n \rightharpoonup q$. By the weak lower semicontinuity of the norm, we have $\|q\| \leq \liminf_{n\to\infty} \|q_n\|$ and so

$$\|Aq\| = \|q\|^2 \leq (\liminf_{n\to\infty} \|q_n\|)^2 \leq \liminf_{n\to\infty} \|q_n\|^2 = \liminf_{n\to\infty} \|Aq_n\|.$$

To show that A is not sequentially weakly continuous, choose $q_n = e_n + e_1$, where $\{e_n\}$ is a standard basis of ℓ_2, that is, $e_n = (0, 0, \cdots, 1, \cdots)$ with 1 at the n-th position. It is clear that $q_n \rightharpoonup e_1$ and $Aq_n = A(e_n + e_1) = (e_n + e_1)\|e_n + e_1\| \rightharpoonup \sqrt{2}e_1$ but $Ae_1 = e_1\|e_1\| = e_1$. However, if A is monotone, then the Condition 3 is not necessary to assume.

3.1. **The weak convergence.** In this subsection, we propose a modified inertial projection and contraction algorithm for solving the psuedomonotone VIP.

Algorithm 1:

Initialization: Given $\lambda_1 > 0$, $\mu \in (0,1)$ and $\gamma \in \left(1, \frac{2}{\sigma}\right)$, where $\sigma \in (1,2)$. Choose $\{\theta_n\} \subset [0,1)$.

Iterative Steps: Let $x_0, x_1 \in H$ be arbitrary and calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n ($n \geq 1$). Compute

$$u_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2. Compute

$$y_n = P_C(u_n - \lambda_n A u_n).$$

If $u_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of VIP. Otherwise, go to **Step 3**.

Step 3. Compute

$$x_{n+1} = u_n - \gamma \eta_n d_n,$$

where η_n and d_n are defined as follows:

$$\eta_n := (1-\mu)\frac{\|u_n - y_n\|^2}{\|d_n\|^2}, \quad d_n := u_n - y_n - \lambda_n(Au_n - Ay_n), \tag{3.10}$$

and update stepsize by

$$\lambda_{n+1} = \min\left\{\frac{\mu\|u_n - y_n\|}{\|Au_n - Ay_n\|}, \lambda_n\right\}. \tag{3.11}$$

Set $n := n+1$ and return to **Step 1**.

Lemma 3.5. [42] *The sequence $\{\lambda_n\}$ generated by* (3.11) *is nonincreasing and* $\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}$.

Lemma 3.6. *Let $\{u_n\}$, $\{y_n\}$ and $\{d_n\}$ be the sequences generated by Algorithm 1. If there exists $n \geq n_0 \in \mathbb{N}$ such that $u_n = y_n$ or $d_n = 0$, then $y_n \in VI(C, A)$.*

Proof. By the definition of d_n, we have

$$\begin{aligned}
\|d_n\| &= \|u_n - y_n - \lambda_n(Au_n - Ay_n), u_n - y_n\| \\
&\geq \|u_n - y_n\| - \lambda_n\|Au_n - Ay_n\| \\
&\geq \|u_n - y_n\| - \mu\frac{\lambda_n}{\lambda_{n+1}}\|u_n - y_n\| \\
&= \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|u_n - y_n\|.
\end{aligned}$$

Since

$$\lim_{n\to\infty}\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > \frac{1-\mu}{\sigma} > 0,$$

there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0, \quad \forall n \geq n_0$$

and so

$$\|d_n\| \geq \frac{1-\mu}{\sigma}\|u_n - y_n\|, \quad \forall n \geq n_0.$$

It is observe that $\|d_n\| > 0$ for all $n \geq n_0$. Indeed, if there exists $n \geq n_0$ such that $\|d_n\| = 0$ or, equivalently, $d_n = 0$, then $u_n = y_n$. Therefore, y_n is a solution of the VIP. This completes the proof. $\square$

Lemma 3.7. *Suppose that Conditions 1-4 hold. Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then, for each $p \in VI(C,A)$ and $n \geq n_0$, we have*

$$\|x_{n+1}-p\|^2 \leq \|u_n-p\|^2 - \frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|x_{n+1}-u_n\|^2.$$

Proof. For each $p \in VI(C,A)$, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|u_n-\gamma\eta_n d_n - p\|^2 \\ &= \|u_n-p\|^2 - 2\gamma\eta_n\langle u_n-p, d_n\rangle + \gamma^2\eta_n^2\|d_n\|^2. \end{aligned} \tag{3.12}$$

By the definition of d_n, it follows that

$$\begin{aligned} \langle u_n-p,d_n\rangle &= \|u_n-y_n\|^2 - \lambda_n\langle u_n-y_n, Au_n-Ay_n\rangle + \langle y_n-p,d_n\rangle \\ &\geq \|u_n-y_n\|^2 - \lambda_n\|u_n-y_n\|\|Au_n-Ay_n\| + \langle y_n-p,d_n\rangle \\ &\geq \Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|u_n-y_n\|^2 + \langle y_n-p,d_n\rangle. \end{aligned}$$

Since

$$\lim_{n\to\infty}\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big) = 1-\mu > \frac{1-\mu}{\sigma} > 0,$$

there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$1-\mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0.$$

Thus we have

$$\langle u_n-p,d_n\rangle \geq \frac{1-\mu}{\sigma}\|u_n-y_n\|^2 + \langle d_n, y_n-p\rangle, \quad \forall n \geq n_0. \tag{3.13}$$

Since $y_n = P_C(u_n-\lambda_n Au_n)$, it follows from (2.8) that

$$\langle u_n-\lambda_n Au_n - y_n, y_n-p\rangle \geq 0. \tag{3.14}$$

Using the fact that $\langle Ap, y_n-p\rangle \geq 0$ and the pseudomonotonicity of A, we have

$$\langle Ay_n, y_n-p\rangle \geq 0. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{aligned} \langle d_n, y_n-p\rangle &= \langle u_n-y_n-\lambda_n(Au_n-Ay_n), y_n-p\rangle \\ &= \langle u_n-\lambda_n Au_n-y_n, y_n-p\rangle + \lambda_n\langle Ay_n, y_n-p\rangle \\ &\geq 0. \end{aligned} \tag{3.16}$$

Combining (3.13) and (3.16), we obtain

$$\langle u_n-p,d_n\rangle \geq \frac{1-\mu}{\sigma}\|u_n-y_n\|^2, \quad \forall n \geq n_0.$$

By the definition of η_n, we have

$$\langle u_n-p,d_n\rangle \geq \frac{1}{\sigma}\eta_n\|d_n\|^2, \quad \forall n \geq n_0. \tag{3.17}$$

Combining (3.12) and (3.18), we get

$$\|x_{n+1}-p\|^2 \leq \|u_n-p\|^2 - \gamma\Big(\frac{2}{\sigma}-\gamma\Big)\eta_n^2\|d_n\|^2, \quad \forall n \geq n_0.$$

Since $x_{n+1} = u_n - \gamma\eta_n d_n$, we have

$$\eta_n^2\|d_n\|^2 = \frac{1}{\gamma^2}\|x_{n+1}-u_n\|^2.$$

Therefore, it follows that

$$\|x_{n+1}-p\|^2 \le \|u_n-p\|^2 - \frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|x_{n+1}-u_n\|^2, \quad \forall n \ge n_0.$$

This completes the proof. □

Lemma 3.8. *Suppose that Conditions 1-4 hold. Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then we have*

$$\|u_n-y_n\|^2 \le \Big(\frac{1+\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\Big)^2\|x_{n+1}-u_n\|^2.$$

Proof. By the definition of η_n, we have

$$\begin{aligned}\|u_n-y_n\|^2 &= \tfrac{1}{1-\mu}\cdot\eta_n\|d_n\|^2\\ &= \tfrac{1}{1-\mu}\cdot\tfrac{1}{\gamma^2\eta_n}(\gamma^2\eta_n^2\|d_n\|^2)\\ &= \tfrac{1}{1-\mu}\cdot\tfrac{1}{\gamma^2\eta_n}\|x_{n+1}-u_n\|^2.\end{aligned} \tag{3.18}$$

Since $\|d_n\|^2 \le (1+\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|u_n-y_n\|^2$, it follows that

$$\frac{1}{\|d_n\|^2} \ge \frac{1}{(1+\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|u_n-y_n\|^2}.$$

Hence we have

$$\eta_n = (1-\mu)\frac{\|u_n-y_n\|^2}{\|d_n\|^2} \ge \frac{1-\mu}{(1+\mu\frac{\lambda_n}{\lambda_{n+1}})^2}. \tag{3.19}$$

Combining (3.18) and (3.19), we obtain

$$\|u_n-y_n\|^2 \le \Big(\frac{1+\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\Big)^2\|x_{n+1}-u_n\|^2.$$

This completes the proof. □

Lemma 3.9. *Suppose that Conditions 1-4 hold. Let $\{x_n\}$ be the sequence generated by Algorithm 1. If $\{\theta_n\}$ is a nondecreasing sequence, then the following estimate holds: for each $p \in VI(C,A)$ and $n \ge n_0$,*

$$\Gamma_{n+1} \le \Gamma_n - \Big(\Big(\frac{2}{\sigma}-\gamma\Big)\Big(\frac{1-\theta_n}{\gamma}\Big)-\xi_{n+1}\Big)\|x_{n+1}-x_n\|^2,$$

where

$$\Gamma_n := \|x_n-p\|^2 - \theta_n\|x_{n-1}-p\|^2 + \xi_n\|x_n-x_{n-1}\|^2$$

and

$$\xi_n := \theta_n\Big(1+\theta_n+\Big(\frac{2}{\sigma}-\gamma\Big)\Big(\frac{1-\theta_n}{\gamma}\Big)\Big).$$

Proof. Let $p \in VI(C,A)$. From (2.7), we have

$$\begin{aligned}\|u_n-p\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-p\|^2\\ &= \|(1+\theta_n)(x_n-p)-\theta_n(x_{n-1}-p)\|^2\\ &= (1+\theta_n)\|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{3.20}$$

It follows from Lemma 3.7 and (3.20) that, for all $n \ge n_0$,

$$\begin{aligned}\|x_{n+1}-p\|^2 \le\;& (1+\theta_n)\|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2\\ &-\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|x_{n+1}-u_n\|^2.\end{aligned} \tag{3.21}$$

On the other hand, from the equality $\|a-b\|^2 = \|a\|^2 - 2\langle a,b\rangle + \|b\|^2 \geq 0$, we have $2\langle a,b\rangle \leq \|a\|^2 + \|b\|^2$. Hence we have

$$\begin{aligned} \|x_{n+1}-u_n\|^2 &= \|x_{n+1}-x_n-\theta_n(x_n-x_{n-1})\|^2 \\ &= \|x_{n+1}-x_n\|^2 + \theta_n^2\|x_n-x_{n-1}\|^2 - 2\theta_n\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle \\ &\geq \|x_{n+1}-x_n\|^2 + \theta_n^2\|x_n-x_{n-1}\|^2 - \theta_n(\|x_{n+1}-x_n\|^2 + \|x_n-x_{n-1}\|^2) \\ &= (1-\theta_n)\|x_{n+1}-x_n\|^2 + (\theta_n^2-\theta_n)\|x_n-x_{n-1}\|^2. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22), we obtain

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq (1+\theta_n)\|x_n-p\|^2 - \theta_n\|x_{n-1}-p\|^2 - \Big(\frac{2}{\sigma}-\gamma\Big)\Big(\frac{1-\theta_n}{\gamma}\Big)\|x_{n+1}-x_n\|^2 \\ &\quad +\theta_n\Big(1+\theta_n+\Big(\frac{2}{\sigma}-\gamma\Big)\Big(\frac{1-\theta_n}{\gamma}\Big)\Big)\|x_n-x_{n-1}\|^2 \end{aligned} \tag{3.23}$$

for all $n \geq n_0$. We put

$$\xi_n := \theta_n\Big(1+\theta_n+\Big(\frac{2}{\sigma}-\gamma\Big)\Big(\frac{1-\theta_n}{\gamma}\Big)\Big)$$

for all $n \geq n_0$. Since the sequence $\{\theta_n\}$ is nondecreasing, it follows from (3.23) that

$$\begin{aligned} \|x_{n+1}-p\|^2-\theta_{n+1}\|x_n-p\|^2+\xi_{n+1}\|x_{n+1}-x_n\|^2 &\leq \|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2+\xi_n\|x_n-x_{n-1}\|^2 \\ &\quad +\Big(\xi_{n+1}-\Big(\frac{2}{\sigma}-\gamma\Big)\Big(\frac{1-\theta_n}{\gamma}\Big)\Big)\|x_{n+1}-x_n\|^2. \end{aligned} \tag{3.24}$$

By the definition of Γ_n, we can write (3.1) as

$$\Gamma_{n+1} \leq \Gamma_n - \Big(\Big(\frac{2}{\sigma}-\gamma\Big)\Big(\frac{1-\theta_n}{\gamma}\Big) - \xi_{n+1}\Big)\|x_{n+1}-x_n\|^2, \quad \forall n \geq n_0.$$

This completes the proof. □

Lemma 3.10. [37] *Suppose that Conditions 1-4 hold. Let $\{u_n\}$and $\{y_n\}$ be the sequences generated by Algorithm 1. If there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $v \in H$ and $\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\| = 0$, then $v \in VI(C,A)$.*

Now, we prove the weak convergence theorem of Algorithm 1.

Theorem 3.1. *Suppose that Conditions 1-4 hold. Let $\beta := \frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)$. Suppose, in addition, that $\{\theta_n\}$ is a nondecreasing sequence such that $0 \leq \theta_n \leq \theta_{n+1} \leq \theta$ for all $n \geq n_0$, where $\theta < \frac{\sqrt{1+8\beta}-2\beta-1}{2(1-\beta)}$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $VI(C,A)$.*

Proof. Since $0 \leq \theta_n \leq \theta_{n+1} \leq \theta$ with $\theta < \frac{\sqrt{1+8\beta}-2\beta-1}{2(1-\beta)}$, it follows that

$$\begin{aligned} (1-\theta_n)\beta - \xi_{n+1} &= (1-\theta_n)\beta - \theta_{n+1}(1+\theta_{n+1}+(1-\theta_{n+1})\beta) \\ &\geq (1-\theta_{n+1})\beta - \theta_{n+1}(1+\theta_{n+1}+(1-\theta_{n+1})\beta) \\ &\geq (1-\theta)\beta - \theta(1+\theta+(1-\theta)\beta) \\ &= -(1-\beta)\theta^2 - (2\beta+1)\theta + \beta. \end{aligned}$$

Let $\delta := -(1-\beta)\theta^2 - (2\beta+1)\theta + \beta$. It is easy to see that $\delta > 0$. Then it follows from Lemma 3.9 that

$$\Gamma_{n+1} - \Gamma_n \leq -\delta\|x_{n+1}-x_n\|^2, \quad \forall n \geq n_0. \tag{3.25}$$

This implies that $\Gamma_{n+1} - \Gamma_n \leq 0$ for all $n \geq n_0$ and so $\{\Gamma_n\}$ is nonincreasing. Thus, by the definition of Γ_n, it follows that, for all $n \geq n_0$,

$$\begin{aligned}
\|x_n - p\|^2 &= \Gamma_n + \theta_n \|x_{n-1} - p\|^2 - \xi_n \|x_n - x_{n-1}\|^2 \\
&\leq \Gamma_n + \theta_n \|x_{n-1} - p\|^2 \\
&\leq \Gamma_{n_0} + \theta \|x_{n-1} - p\|^2 \\
&\cdots \\
&\leq \theta^{n-n_0} \|x_{n_0} - p\|^2 + \Gamma_{n_0}(1 + \theta + \theta^2 + \ldots + \theta^{n-n_0-1}) \\
&\leq \theta^{n-n_0} \|x_{n_0} - p\|^2 + \frac{\Gamma_{n_0}}{1-\theta}. \qquad (3.26)
\end{aligned}$$

We also observe that

$$\begin{aligned}
\Gamma_{n+1} &= \|x_{n+1} - p\|^2 - \theta_{n+1} \|x_n - p\|^2 + \xi_{n+1} \|x_{n+1} - x_n\|^2 \\
&\geq -\theta_{n+1} \|x_n - p\|^2 \\
&\geq -\theta \|x_n - p\|^2. \qquad (3.27)
\end{aligned}$$

Combining (3.25), (3.26) and (3.27), we have

$$\begin{aligned}
\delta \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 &\leq \delta \sum_{n=n_0}^{k} (\Gamma_n - \Gamma_{n+1}) \\
&= \Gamma_{n_0} - \Gamma_{k+1} \\
&\leq \Gamma_{n_0} + \theta \|x_k - p\|^2 \\
&\leq \Gamma_{n_0} + \theta^{k-n_0+1} \|x_{n_0} - p\|^2 + \frac{\theta \Gamma_{n_0}}{1-\theta}.
\end{aligned}$$

Thus we have

$$\delta \sum_{n=n_0}^{\infty} \|x_{n+1} - x_n\|^2 = \lim_{k\to\infty} \Big(\delta \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 \Big) < \infty. \qquad (3.28)$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Consequently, we have

$$\|x_n - u_n\| = \theta_n \|x_n - x_{n-1}\| \leq \theta \|x_n - x_{n-1}\| \to 0 \quad \text{as} \quad n \to \infty. \qquad (3.29)$$

Now, we see that

$$\begin{aligned}
\|x_{n+1} - u_n\| &= \|x_{n+1} - x_n - \theta_n (x_n - x_{n-1})\| \\
&\leq \|x_{n+1} - x_n\| + \theta_n \|x_n - x_{n-1}\| \\
&\leq \|x_{n+1} - x_n\| + \theta \|x_n - x_{n-1}\|
\end{aligned}$$

and so

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

Also, by Lemma 3.8, we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \qquad (3.30)$$

Since $\{x_n\}$ is bounded, without loss of generality, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$ for some $v \in H$. From (3.29), we also get $u_{n_k} \rightharpoonup v$. This together with (3.29) and Lemma 3.10 concludes that $v \in VI(C, A)$.

On the other hand, from (3.21), it follows that, for all $n \geq n_0$,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) + \theta(1+\theta)\|x_n - x_{n-1}\|^2. \tag{3.31}$$

From (3.28) and Lemma 2.2, we can show that $\lim_{n\to\infty}\|x_n - p\|^2$ exists. In summary, we have shown that

- $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in VI(C, A)$;
- every weak cluster point of $\{x_n\}$ is in $VI(C, A)$.

Therefore, by Lemma 2.1, we conclude that $\{x_n\}$ converges weakly to a point in $VI(C, A)$. This completes the proof. □

3.2. **The strong convergence.** In this subsection, we propose another inertial algorithm which combines the viscosity approximation method and the projection and contraction algorithm with adaptive stepsize rule for solving the psuedomonotone VIP.

In order to obtain the strong convergence, we assume that $f : H \to H$ is a contraction mapping with constant $\alpha \in [0, 1)$. Suppose, in addition, that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0,$$

where $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The algorithm is formulated as follows:

Algorithm 2:

Initialization: Given $\lambda_1 > 0$, $\mu \in (0, 1)$ and $\gamma \in \left(0, \frac{2}{\sigma}\right)$, where $\sigma \in (1, 2)$. Choose $\{\theta_n\} \subset [0, \theta)$ for some $\theta > 0$.

Iterative Steps: Let $x_0, x_1 \in H$ be arbitrary and calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n ($n \geq 1$). Compute

$$u_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2. Compute

$$y_n = P_C(u_n - \lambda_n A u_n).$$

If $u_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of VIP. Otherwise, go to **Step 3**.

Step 3. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(u_n - \gamma\eta_n d_n),$$

where η_n and d_n are defined in (3.10), and update the stepsize by (3.11).

Set $n := n + 1$ and return to **Step 1**.

Theorem 3.2. *Suppose that Conditions 1-4 hold. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z = P_{VI(C,A)}f(z)$.*

Proof. For each $n \geq 1$, let $z_n := u_n - \gamma\eta_n d_n$ and $p \in VI(C, A)$. Following the similar argument in Lemma 3.7, it follows that, for each $n \geq n_0$,

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|z_n - u_n\|^2. \tag{3.32}$$

This gives

$$\|z_n - p\| \leq \|u_n - p\|. \tag{3.33}$$

Moreover, we have

$$\begin{aligned} \|u_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.34}$$

It follows from (3.33) and (3.34) that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(z_n-p)\|\\ &\le \alpha_n\|f(x_n)-p\|+(1-\alpha_n)\|z_n-p\|\\ &\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|z_n-p\|\\ &\le \alpha_n\alpha\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|x_n-p\|+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|\\ &= (1-(1-\alpha)\alpha_n)\|x_n-p\|+(1-\alpha)\alpha_n\Big[\frac{\|f(p)-p\|}{1-\alpha}+\frac{(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|}{(1-\alpha)\alpha_n}\Big].\end{aligned}$$

Put

$$\mu_n := \frac{\|f(p)-p\|}{1-\alpha}+\frac{(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|}{(1-\alpha)\alpha_n}.$$

It is easy to see that $\lim_{n\to\infty}\mu_n$ exists. So there exists $M>0$ such that $\mu_n\le M$ for all $n\in\mathbb{N}$. By Lemma 2.3, we know that $\{\|x_n-p\|\}$ is bounded. Moreover, we see that $\|x_n\|\le\|x_n-p\|+\|p\|$. This implies that $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{y_n\}$ and $\{d_n\}$.

Now, let $z=P_{VI(C,A)}f(z)$. From (2.6), we have

$$\begin{aligned}\|u_n-z\|^2 &= \|x_n-z+\theta_n(x_n-x_{n-1})\|^2\\ &\le \|x_n-z\|^2+2\theta_n\langle x_n-x_{n-1},u_n-z\rangle\\ &\le \|x_n-z\|^2+2\theta_n\|x_n-x_{n-1}\|K,\end{aligned}\tag{3.35}$$

where $K=\sup_{n\ge1}\{\|u_n-z\|\}$. By the convexity of $\|\cdot\|^2$ and (3.32), it follows that, for all $n\ge n_0$,

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n(f(x_n)-z)+(1-\alpha_n)(z_n-p)\|^2\\ &\le \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\|z_n-z\|^2\\ &\le \alpha_n\|f(x_n)-z\|^2+\|z_n-z\|^2\\ &\le \alpha_n\|f(x_n)-z\|^2+\|u_n-z\|^2-\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|z_n-u_n\|^2.\end{aligned}\tag{3.36}$$

Combining (3.35) and (3.36), it follows that, for all $n\ge n_0$,

$$\|x_{n+1}-z\|^2\le\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2+2\theta_n\|x_n-x_{n-1}\|K-\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|z_n-u_n\|^2,$$

which implies that

$$\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|z_n-u_n\|^2\le\|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\|f(x_n)-z\|^2+2\theta_n\|x_n-x_{n-1}\|K.$$

On the other hand, from (2.6), (3.32) and (3.35), it follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n(f(x_n)-z)+(1-\alpha_n)(z_n-p)\|^2\\ &= \|\alpha_n(f(x_n)-f(z))+(1-\alpha_n)(z_n-z)+\alpha_n(f(z)-z)\|^2\\ &\le \|\alpha_n(f(x_n)-f(z))+(1-\alpha_n)(z_n-z)\|^2+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &\le \alpha_n\|f(x_n)-f(z)\|^2+(1-\alpha_n)\|z_n-z\|^2+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &\le \alpha_n\alpha\|x_n-z\|^2+(1-\alpha_n)\big[\|x_n-z\|^2+2\theta_n\|x_n-x_{n-1}\|K\big]\\ &\quad +2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &= (1-(1-\alpha)\alpha_n)\|x_n-z\|^2+2(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|K\\ &\quad +2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle.\end{aligned}$$

Now, we show that the sequence $\{\|x_n - z\|^2\}$ converges to zero. In order to do this, using Lemma 2.4, it is sufficient to show that

$$\limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle \le 0$$

for every subsequence $\{\|x_{n_k} - z\|\}$ of $\{\|x_n - z\|\}$ satisfying

$$\liminf_{k\to\infty}\left[\|x_{n_k+1} - z\| - \|x_{n_k} - z\|\right] \ge 0.$$

Let $\{\|x_{n_k} - z\|\}$ be a subsequence of $\{\|x_n - z\|\}$ such that

$$\liminf_{k\to\infty}\left[\|x_{n_k+1} - z\| - \|x_{n_k} - z\|\right] \ge 0.$$

Thus we have

$$\begin{aligned}
&\liminf_{k\to\infty}\left[\|x_{n_k+1} - z\|^2 - \|x_{n_k} - z\|^2\right]\\
= &\liminf_{k\to\infty}\left[(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|)(\|x_{n_k+1} - z\| + \|x_{n_k} - z\|)\right] \ge 0.
\end{aligned}$$

From (3.37), it follows that

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{1}{\gamma}\Big(\frac{2}{\sigma} - \gamma\Big)\|z_{n_k} - u_{n_k}\|^2\\
\le\ &\limsup_{k\to\infty}\left[\|x_{n_k} - z\|^2 - \|x_{n_k+1} - z\|^2 + \alpha_{n_k}\|f(x_{n_k}) - z\|^2\right.\\
&\left.+2\theta_{n_k}\|x_{n_k} - x_{n_k-1}\|K\right]\\
\le\ &\limsup_{k\to\infty}\left[\|x_{n_k} - z\|^2 - \|x_{n_k+1} - z\|^2\right] + \limsup_{k\to\infty}\alpha_{n_k}\|f(x_{n_k}) - z\|^2\\
&+\limsup_{k\to\infty}2\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\|K\\
=\ &-\liminf_{k\to\infty}\left[\|x_{n_k+1} - z\|^2 - \|x_{n_k} - z\|^2\right]\\
\le\ &0.
\end{aligned}$$

Hence we have

$$\lim_{k\to\infty}\|z_{n_k} - u_{n_k}\| = 0. \tag{3.37}$$

As in the proof lines of Lemma 3.8, we can deduce that

$$\|u_{n_k} - y_{n_k}\|^2 \le \bigg(\frac{1 + \mu\frac{\lambda_{n_k}}{\lambda_{n_k+1}}}{\gamma(1-\mu)}\bigg)^2\|z_{n_k} - u_{n_k}\|^2.$$

Thus it follows from (3.37) that

$$\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0. \tag{3.38}$$

Moreover, we see that

$$\|u_{n_k} - x_{n_k}\| = \theta_{n_k}\|x_{n_k} - x_{n_k-1}\| = \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{3.39}$$

It follows from (3.37) and (3.38) that

$$\|z_{n_k} - x_{n_k}\| \le \|z_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \tag{3.40}$$

and so

$$\begin{aligned}
\|x_{n_k+1} - x_{n_k}\| &= \|\alpha_{n_k}f(x_{n_k}) + (1-\alpha_{n_k})z_{n_k} - x_{n_k}\|\\
&\le \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\|z_{n_k} - x_{n_k}\|\\
&\to 0. \qquad\qquad (3.41)
\end{aligned}$$

Since $\{x_{n_k}\}$ is bounded, without loss of generality, we assume that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup v$ for some $v \in H$ and

$$\limsup_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle = \lim_{j\to\infty}\langle f(z)-z, x_{n_{k_j}}-z\rangle.$$

From (3.39), we also get $u_{n_{k_j}} \rightharpoonup v$. This together with (3.38) and Lemma 3.10 concludes that $v \in VI(C,A)$. Hence we have

$$\limsup_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle = \langle f(z)-z, v-z\rangle \leq 0. \tag{3.42}$$

Combining (3.41) and (3.42), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle f(z)-z, x_{n_k+1}-z\rangle &\leq \limsup_{k\to\infty}\langle f(z)-z, x_{n_k+1}-x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle \\ &\leq 0. \end{aligned}\tag{3.43}$$

From (3.37), we can write it as

$$\begin{aligned}\|x_{n_k+1}-z\|^2 &\leq (1-(1-\alpha)\alpha_{n_k})\|x_{n_k}-z\|^2 \\ &\quad + (1-\alpha)\alpha_{n_k}\Big[\frac{2(1-\alpha_{n_k})\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|K}{(1-\alpha)\alpha_{n_k}} + \frac{2}{1-\alpha}\langle f(z)-z, x_{n_k+1}-z\rangle\Big].\end{aligned}$$

This together with (3.43) and Lemma 2.4 yields that $\lim_{k\to\infty}\|x_n-z\|^2 = 0$. Therefore, $x_n \to z$. This completes the proof. □

4. Numerical experiments

In this section, we give some numerical experiments in two parts. In the first part, we provide a comparison the numerical behaviour of the proposed algorithms and their algorithms with non-inertial terms to illustrate the efficiency and advantages of the proposed algorithms and also compare them with the following:

- Algorithm A: The TEGM for the pseudomonotone VIP [36];
- Algorithm B: The SEGM for the pseudomonotone VIP [21].

In the second part, we apply the proposed algorithms to solve the image restoration problem and compare the computational results with Algorithm A and Algorithm B. In the following, we denote "iter." and "time" by the number of iteration and the running time in seconds, respectively.

4.1. Numerical results.

Problem 4.1. The variational inequality problem in infinite-dimensional spaces

Consider a Hilbert space $H := \ell_2 = \{x = (x_1, x_2, x_3, \cdots) : \sum_{i=1}^{\infty}|x_i|^2 < \infty\}$ with the norm $\|x\| = \left(\sum_{i=1}^{\infty}|x_i|^2\right)^{\frac{1}{2}}$ and the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty}x_i y_i$ for all $x = (x_1, x_2, x_3, \cdots), y = (y_1, y_2, y_3, \cdots) \in \ell_2$. Let $A : \ell_2 \to \ell_2$ be a mapping defined by

$$A(x_1, x_2, x_3, \cdots) = (x_1 e^{-x_1^2}, 0, 0, \cdots).$$

It was shown in [5, Example 2.1] that A is pseudomonotone, Lipschitz continuous and sequentially weakly continuous (hence A satisfies the Condition 3), but not monotone on ℓ_2. The feasible set is $C = \{x = (x_1, x_2, x_3, \cdots) \in \ell_2 : \|x\| \leq 1\}$ and then the projection onto C is easily calculated by the following formula:

$$P_C(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } \|x\| > 1, \\ x & \text{otherwise.}\end{cases}$$

In this experiment, for Algorithm 1, we take $\lambda_1 = 0.36, \mu = 0.54, \sigma = \frac{3}{2}, \gamma = \frac{7}{6}, \theta_n = \frac{9}{10}$ and, in addition, for Algorithm 2, we take $f(x) = \frac{x}{4}$, $\alpha_n = \frac{1}{n+1}$, $\theta_n = \alpha_n^2$. More so, for Algorithm A and Algorithm B, we take $\gamma = 3$, $l = 0.68$, $\mu = 0.34$. We perform numerical test of all algorithms with three different cases of the starting point as follows:

Case A: $x_0 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ and $x_1 = (1, 2, 3, \cdots)$;
Case B: $x_0 = (5, 5, 5, \cdots)$ and $x_1 = (2, 1, 2, \cdots)$;
Case C: $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $x_1 = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \cdots)$.

We use $TOL_n = \|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criteria in each algorithm. The numerical results are shown in Table 1 and Figure 1.

TABLE 1. Numerical results for Problem 4.1.

x_0, x_1	Alg 1		Alg 1 ($\theta_n = 0$)		Alg 2		Alg 2 ($\theta_n = 0$)		Alg A		Alg B	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
Case A	14	0.0065	28	0.0122	10	0.0024	14	0.0031	55	0.0193	87	0.0256
Case B	14	0.0057	33	0.0119	10	0.0041	20	0.0076	56	0.0129	64	0.0229
Case C	13	0.0048	37	0.0097	10	0.0022	20	0.0088	84	0.0209	76	0.0241

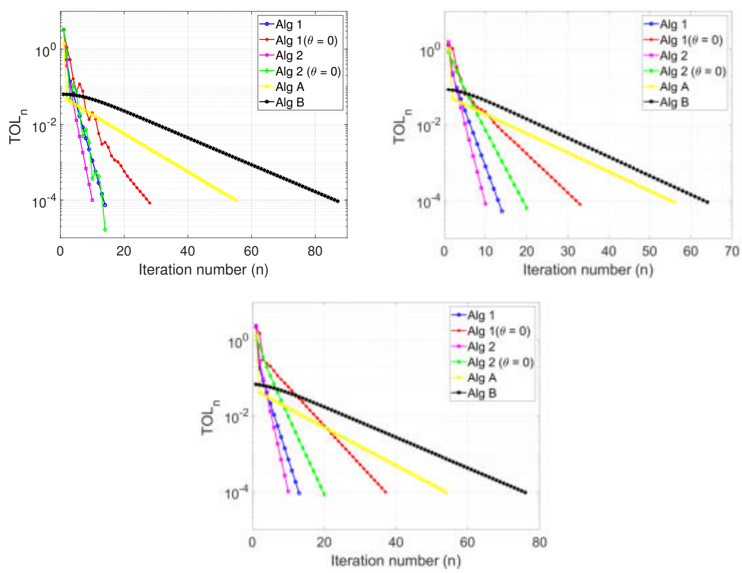

FIGURE 1. Example 4.1, Top Left: Case A; Top Right: Case B; Bottom: Case C.

Problem 4.2. The quadratic fractional programming problem

Consider the following quadratic fractional programming problem:

$$\min_{x \in C} f(x),$$

where $f(x) = \frac{x^T Q x + a^T x + c}{b^T x + d}$ and $C = \{x \in \mathbb{R}^4 : 1 \le x_i \le 10,\ i = 1, 2, 3, 4\}$. Let

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ c = -2,\ d = 4.$$

It is easy to see that Q is symmetric and positive definite on $\mathbb{R}^4$ and so f is pseudoconvex on $\mathbb{R}^4$. It is known that this problem is equivalent to the VIP [13, 23] with

$$Ax := \nabla f(x) = \frac{(b^T x + d)(2Qx + a) - b(x^T Qx + a^T x + c)}{(b^T x + d)^2}.$$

It was shown in [19] that ∇f is continuous pseudomonotone. The VIP has unique solution is $z = (1, 1, 1, 1)^T \in C$.

In this experiment, for Algorithm 1, we take $\lambda_1 = 0.28$, $\mu = 0.45$, $\sigma = \frac{4}{3}$, $\gamma = \frac{5}{4}$, $\theta_n = \frac{3}{5}$ and, in addition, we take $f(x) = \frac{x}{8}$, $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\theta_n = \frac{1}{n+1}$ for Algorithm 2. Also, for Algorithms A and B, we choose $\gamma = 0.33$, $l = 0.66$ and $\mu = 0.64$.. We perform numerical test of all algorithms with three different cases of the starting point as follows:

Case A: $x_0 = (2, 2, 2, 2)^T$ and $x_1 = (4, 4, 4, 4)^T$;
Case B: $x_0 = (3, 3, 3, 3)^T$ and $x_1 = (5, 5, 5, 5)^T$;
Case C: $x_0 = (2, 0, 0, 4)^T$ and $x_1 = (3, 1, 3, 1)^T$.

Since we know the solution of the problem, we use $TOL_n = \|x_n - z\| < 10^{-4}$ as the stopping criteria in each algorithm. The numerical results are shown in Table 2 and Figure 2.

TABLE 2. Numerical results for Problem 4.2.

x_0, x_1	Alg 1		Alg 1 ($\theta_n = 0$)		Alg 2		Alg 2 ($\theta_n = 0$)		Alg A		Alg B	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
Case A	7	0.0038	13	0.0046	8	0.0044	12	0.0059	28	0.0138	37	0.0317
Case B	8	0.0036	14	0.0049	8	0.0038	12	0.0049	25	0.0145	37	0.0220
Case C	8	0.0036	16	0.0083	8	0.0030	12	0.0043	27	0.0132	37	0.0155

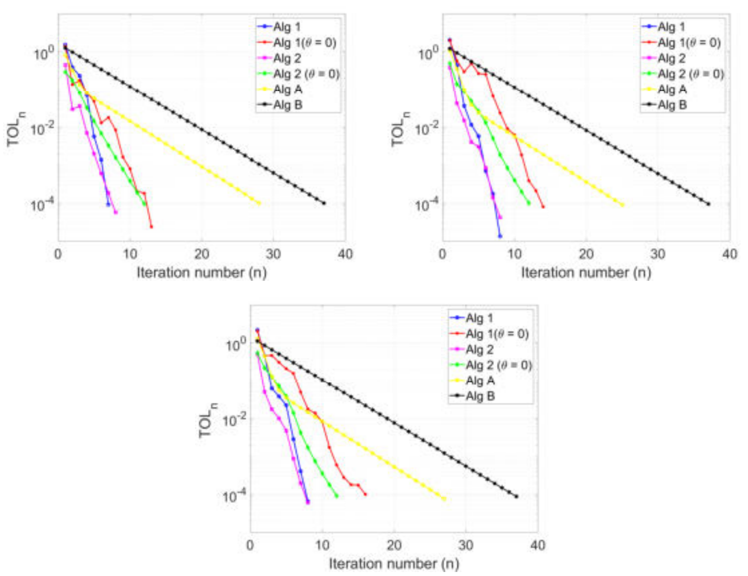

FIGURE 2. Example 4.2, Top Left: Case A; Top Right: Case B; Bottom: Case C.

Remark 4.3. From the above experimental results, we can summarize in the following points:

(1) Algorithm 1 and Algorithm 2 have less iteration numbers and computation times than algorithms without the inertial. It is remarkable that by adding the inertial $\theta_n(x_n - x_{n-1})$ makes convergence faster. This is main advantage of adding inertial term to algorithms in solving the problem.

(2) Algorithm 1 and Algorithm 2 with adaptive stepsizes have a better performance in terms of convergence speed than Algorithm A and Algorithm B with Armijo linesearch procedures. This shows that the proposed algorithms have higher superiority and efficiency than Algorithm A and Algorithm B in solving the pseudomonotone VIP. It is due to the fact that Armijo linesearch procedures use an inner-loop until some stopping criterion is reached. This may takes time-consuming in evaluations of the projections on the feasible set in each iteration.

4.2. **Applications to the image restoration problem.** The image restoration problem is one of the interest topics in image processing and computer vision. This problem has been extensively studied by many authors because of its applications in almost every field such as film restoration, image and video coding, medical and astronomical imaging (see, for example, [10, 40]). Image restoration is a process of recovering images from blurring and noise observation which is to improve the quality of the image. Recall that the image restoration problem can be formulated as the following linear inverse problem:

$$b = Bx + v, \tag{4.44}$$

where $x \in \mathbb{R}^N$ is the original image, $b \in \mathbb{R}^M$ is the degraded image, $B \in \mathbb{R}^{M\times N}$ is the blurring matrix and v is an additive noise. An efficient method for recovering the original image is the ℓ_1-norm regularized least square method given by

$$\min_{x\in\mathbb{R}^N} \left\{ \frac{1}{2}\|Bx - b\|_2^2 + \lambda\|x\|_1 \right\}. \tag{4.45}$$

Our main task is to restore the original image x given the data of the blurred image b. The least square problem (4.45) can be expressed as a variational inequality problem by setting $A := B^T(Bx - b)$. It is known that the operator A in this case is monotone (hence it is pseudomonotone) and Lipschitz continuous with $L = \|B^T B\|$. We consider the grey scale image of M pixels wide and N pixel height, each value is known to be in the range $[0, 255]$. The quality of the restored image is measured by the signal to noise ratio (SNR) which is defined by

$$SNR = 20\log_{10}\left(\frac{\|x\|_2}{\|x - x^*\|_2}\right),$$

where x is the original image and x^* is the restored image. Note that the larger the value of SNR, the better the quality of the restored image. In our experiments, we use the grey test image Tire (291×240) and Cameraman (256×256), each test image is degraded by Gaussian 7×7 blur kernel with standard deviation 4 and the maximum iteration is set to be 1000. We choose $\lambda_1 = 0.5$, $\mu = 0.8$, $\sigma = 1.5$, $\gamma = 1$, $\theta_n = 0.99$, $\alpha_n = \frac{1}{100(n+1)}$, $f(x) = \frac{x}{4}$, $l = 0.3$, $\mu = 0.6$, $x_0 = \mathbf{0} \in \mathbb{R}^{\mathbf{D}}$ and $x_1 = \mathbf{1} \in \mathbb{R}^{\mathbf{D}}$, where $\mathbf{D} = M \times N$.

Figures 3 and 4 show the original, blurred and restored image by using Algorithm 1, Algorithm 2, Algorithm A and Algorithm B. Also, Figure 5 shows the graph of the SNR against number of iterations for each test image using the algorithms. Then we report the time for each algorithm in Table 3.

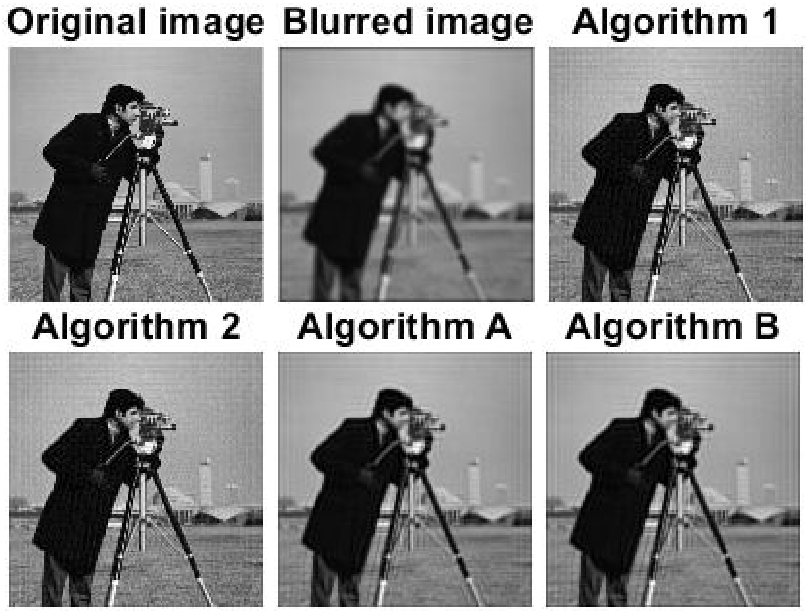

FIGURE 3. Top shows original image of Cameraman (left) and degraded image of Cameraman (right); Bottom shows recovered image by Algorithm 1, Algorithm 2, Algorithm A and Algorithm B.

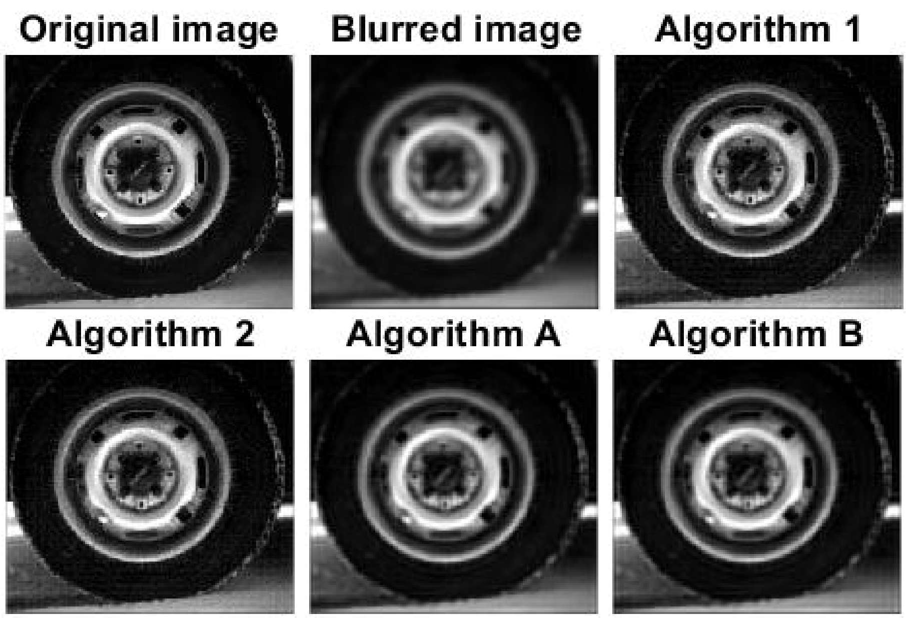

FIGURE 4. Top shows original image of Tire (left) and degraded image of Tire (right); Bottom shows recovered image by Algorithm 1, Algorithm 2, Algorithm A and Algorithm B.

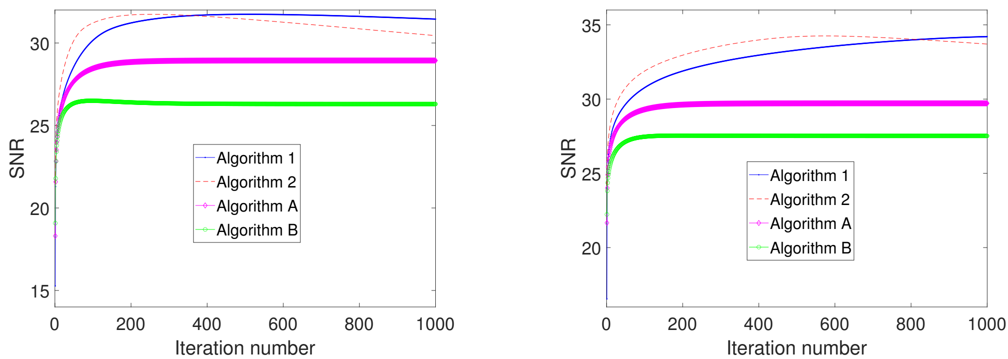

FIGURE 5. Graphs of the SNR values against number of iteration for Cameraman (Left) and Tire (Right).

TABLE 3. Computational results for Deblurring the images

Algorithms	Cameraman		Tire	
	SNR	time	SNR	time
Alg 1	34.2083	42.0504	31.4482	31.3819
Alg 2	38.7060	38.6614	30.4481	29.7643
Alg A	27.5191	91.8467	26.3005	64.4750
Alg B	29.7150	94.2898	28.9397	64.5418

Remark 4.4. From the obtained computational results, we show that both the quality of the restored images and running times of Algorithm 1 and Algorithm 2 are good as compared with Algorithm A and Algorithm B. This shows that the proposed algorithms are more efficient for restoring the degraded image than Algorithm A and Algorithm B.

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REFERENCES

[1] Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces. *SIAM J. Optim.* **14** (2004), 773–782.

[2] Aubin, J. P.; Ekeland, I. *Applied Nonlinear Analysis.* Wiley, New York, 1984.

[3] Antipin, A. S. On a method for convex programs using a symmetrical modification of the Lagrange function. *Ekonomika i Mat. Metody.* **12** (1976), 1164–1173.

[4] Baiocchi, C.; Capelo, A. *Variational and Quasi-variational Inequalities: Applications to Free Boundary Problems.* Wiley, New York, 1984.

[5] Bot, R. I.; Csetnek, E. R.; Vuong, P. T. The forward–backward–forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces. *Eur. J. Oper. Res.* **287** (2020), 49–60.

[6] Cegielski, A. *Iterative Methods for Fixed Point Problems in Hilbert Spaces.* Springer-Verlag, Berlin, Heidelberg, 2010.

[7] Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert spaces. *J. Optim. Theory Appl.* **148** (2011), 318–335.

[8] Cholamjiak, P.; Thong, D. V.; Cho, Y. J. A novel inertial projection and contraction method for solving pseudomonotone variational inequality problems. *Acta Appl. Math.* **169** (2020), 217–245.
[9] Dong, Q. L.; Cho, Y. J.; Zhong, L. L.; Rassias, T. M. Inertial projection and contraction algorithms for variational inequalities, *J. Global Optim.* **70** (2018), 687–704.
[10] Figueiredo, M.; Nowak, R. D.; Wright, S.J. Gradient projection for sparse reconstruction: application to compressed sensing and other inverse problems. *IEEE J. Sel. Top. Signal Process* **1** (2007), 586–597.
[11] Gibali, A.; Thong, D. V.; Tuan, P. A. Two simple projection-type methods for solving variational inequalities. *Anal. Math. Phys.* **9** (2019), 2203–2225.
[12] Glowinski, R.; Lions, J. L.; Tr´emoli'eres, R. *Numerical Analysis of Variational Inequalities.* North-Holland, Amsterdam, 1981.
[13] Harker, P. T.; Pang, J. S. Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Math. Program.* **48** (1990), 161–220.
[14] He, B.S. A class of projection and contraction methods for monotone variational inequalities. *Appl. Math. Optim.* **35** (1997), 69–76.
[15] Jabeen, S.; Bin-Mohsin, B.; Noor, M. A.; Noor, K. I. Inertial projection methods for solving general quasi-variational inequalities. *AIMS Math.* **6** (2021), 1075–1086.
[16] Jolaoso, L. O. An inertial projection and contraction method with a line search technique for variational inequality and fixed point problems. *Optimization* **71** (2022), no. 12, 3485–3514.
[17] Jolaoso, L. O.; Aphane, M.; Khan, S. H. Two Bregman projection methods for solving variational inequality problems in Hilbert spaces with applications to signal processing. *Symmetry* **12** (2020), Art. No. 2007.
[18] Jolaoso, L. O.; Sunthrayuth, P.; Cholamjiak, P.; Cho, Y. J. Analysis of two versions of relaxed inertial algorithms with Bregman divergences for solving variational inequalities. *Comp. Appl. Math.* **41** 300 (2022). https://doi.org/10.1007/s40314-022-02006-x
[19] Karamardian, S.; Schaible, S. Seven kinds of monotone maps. *J. Optim. Theory Appl.* **66** (1990), 37–46.
[20] Khanh, P.D.; Vuong, P.T. Modified projection method for strongly pseudo-monotone variational inequalities. *J. Global Optim.* **58** (2014), 341–350.
[21] Khanh, P. Q.; Thong, D. V.; Vinh, N. T. Versions of the subgradient extragradient method for pseudomonotone variational inequalities. *Acta Appl. Math.* **170** (2020), 319–345.
[22] Khobotov, E.N. Modification of the extragradient method for solving variational inequalities and certain optimization problems. *USSR Comput. Math. Math. Phys.* **27** (1989), 120–127.
[23] Kinderlehrer, D.; Stampacchia, G. *An Introduction to Variational Inequalities and their Applications.* New York, Academic Press, 1980.
[24] Konnov, I. V. *Combined Relaxation Methods for Variational Inequalities.* Springer-Verlag, Berlin, 2001.
[25] Korpelevich, G. M. The extragradient method for finding saddle points and other problems. *Ekon. Mat. Metody.* **12** (1976), 747-–756.
[26] Liu, L.; Qin, X. Strong convergence of an extragradient-like algorithm involving pseudo-monotone mappings. *Numer. Algor.* **83** (2020), 1577–1590.
[27] Maing*é*, P. E. Inertial iterative process for fixed points of certain quasi-nonexpansive mappings. *Set-valued Anal.* **15** (2007), 67–79.
[28] Marcotte, P. Applications of Khobotov's algorithm to variational and network equilibrium problems. *Inform. Syst. Oper. Res.* **29** (1991), 258–270.
[29] Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Amer. Math. Soc.* **73** (1967), 591–597.
[30] Reich, S.; Tuyen, T.M.; Sunthrayuth, P.; Cholamjiak, P. Two new inertial algorithms for solving variational inequalities in reflexive Banach spaces. *Numer. Funct. Anal. Optim.* **42** (2021), no. 16, 1954–1984.
[31] Saejung, S.; Yotkaew, P. Approximation of zeros of inverse strongly monotone operators in Banach spaces. *Nonlinear Anal.* **75** (2012), 742–750.
[32] Shehu, Y.; Dong, Q.L.; Jiang, D. Single projection method for pseudo-monotone variational inequality in Hilbert spaces. *Optimization* **68** (2019), 385–409.
[33] Sun, D. F. A class of iterative methods for solving nonlinear projection equations. *J. Optim. Theory Appl.* **91** (1996), 123–140.
[34] Takahashi, W. *Introduction to Nonlinear and Convex Analysis.* Yokohama publishers, 2009.
[35] Tan, B.; Sunthrayuth, P.; Cholamjiak, P.; Cho, Y. J. Modified inertial extragradient methods for finding minimum-norm solution of the variational inequality problem with applications to optimal control problem. *Int. J. Comput. Math.* (2022) DOI: 10.1080/00207160.2022.2137672
[36] Thong, D. V.; Vuong, P. T. Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities. *Optimization* **68** (2019), no. 11, 2207–2226.
[37] Thong, D.V.; Long, L.V.; Li, X.H.; Dong, Q.L.; Cho, Y.J.; Tuan, P.A. A new self-adaptive algorithm for solving pseudomonotone variational inequality problems in Hilbert spaces. *Optimization* **71** (2021), no. 12, 3669–3693.

[38] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **38** (2000), 431–446.

[39] Tuyen, T. M.; Sunthrayuth, P.; Trang, N. M. An inertial self-adaptive algorithm for the generalized split common null point problem in Hilbert spaces. *Rend. Circ. Mat. Palermo (2)* **71** (2022), 537–557.

[40] Xiao, Y.; Zhu, H. A conjugate gradient method to solve convex constrained monotone equations with applications in compressive sensing. *J. Math. Anal. Appl.* **405** (2013), 310–319.

[41] Yang, J. Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities. *Appl. Anal.* **100** (2021), 1067–1078.

[42] Yang, J.; Liu, H. A modified projected gradient method for monotone variational inequalities. *J. Optim. Theory Appl.* **179** (2018), 197–211.

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Research article

Two-step inertial method for solving split common null point problem with multiple output sets in Hilbert spaces

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Abstract: In this paper, an algorithm with two-step inertial extrapolation and self-adaptive step sizes is proposed to solve the split common null point problem with multiple output sets in Hilbert spaces. Weak convergence analysis are obtained under some easy to verify conditions on the iterative parameters in Hilbert spaces. Preliminary numerical tests are performed to support the theoretical analysis of our proposed algorithm.

Keywords: Hilbert space; metric projection; self-adaptive step size; two-step inertial; split common null point problem
Mathematics Subject Classification: 47H09, 47H10, 49J53, 90C25

1. Introduction

Throughout this paper, $\mathcal{H}$ denotes a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced $\|\cdot\|$, I the identity operator on $\mathcal{H}$, $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers. For a self-operator T on $\mathcal{H}$, $F(T)$ denotes the set of all fixed points of T.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces and let $\mathcal{T}:\mathcal{H}_1\to\mathcal{H}_2$ be bounded linear operator. Let $\{U_j\}_{j=1}^t:\mathcal{H}_1\to\mathcal{H}_1$ and $\{T_i\}_{i=1}^r:\mathcal{H}_2\to\mathcal{H}_2$ be two finite families of operators, where $t,r\in\mathbb{N}$. The split common fixed point problem (SCFPP) is formulated as finding a point $x^*\in\mathcal{H}_1$ such that

$$x^*\in\cap_{j=1}^t F(U_j)\quad\text{such that}\quad \mathcal{T}x^*\in\cap_{i=1}^r F(T_i). \tag{1.1}$$

In particular, if $t = r = 1$, the SCFPP (1.1) reduces to finding a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in F(U) \text{ such that } \mathcal{T}x^* \in F(T). \tag{1.2}$$

The above problem is usually called the two-set SCFPP.

In recent years, the SCFPP (1.1) and the two-set SCFPP (1.2) have been studied and extended by many authors, see for instance [15, 20, 23, 27, 36–40, 47–49]. It is known that the SCFPP includes the multiple-set split feasibility problem and split feasibility problem as a special case. In fact, let $\{C_j\}_{j=1}^t$ and $\{Q_i\}_{i=1}^r$ be two finite families of nonempty closed convex subsets in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $U_j = P_{C_j}$ and $T_i = P_{Q_i}$; then SCFPP (1.1) becomes the multiple-set split feasibility problem (MSSFP) as follows:

$$\text{find } x^* \in \cap_{j=1}^t C_j \text{ such that } \mathcal{T}x^* \in \cap_{i=1}^r Q_i. \tag{1.3}$$

When $t = r = 1$ the MSSFP (1.3) is reduced to the split feasibility problem (SFP) which is described as finding a point $x^* \in \mathcal{H}_1$ satisfying the following property

$$x^* \in C \quad \text{such that } \mathcal{T}x^* \in Q. \tag{1.4}$$

The SFP was first introduced by Censor and Elfving [22] with the aim of modeling certain inverse problems. It has turned out to also play an important role in, for example, medical image reconstruction and signal processing (see [2, 4, 15, 17, 21]). Since then, several iterative algorithms for solving (1.4) have been presented and analyzed. See, for instance [1, 5, 14–16, 18, 19, 23, 24, 27] and references therein.

The CQ algorithm has been extended by several authors to solve the multiple-set split convex feasibility problem. See, for instance, the papers by Censor and Segal [25], Elfving, Kopf and Bortfeld [23], Masad and Reich [35], and by Xu [53, 54].

In 2020, Reich and Tuyen [45] proposed and analyzied the following split feasibility problem with multiple output sets in Hilbert spaces: let $\mathcal{H}, \mathcal{H}_i, i = 1, 2, \ldots, m$ be real Hilbert spaces. Let $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i,\ i = 1, 2, \ldots, m$, be bounded linear operators. Furthermore, let C and Q_i be nonempty, closed and convex subsets of $\mathcal{H}$ and $\mathcal{H}_i, i = 1, 2, \ldots, m$, respectively. Find an element $u^\dagger$, such that:

$$u^\dagger \in \Omega^{SFP} = C \cap \left(\cap_{i=1}^m \mathcal{T}_i^{-1}(Q_i)\right) \neq \emptyset; \tag{1.5}$$

that is, $u^\dagger \in C$ and $\mathcal{T}u^\dagger \in Q_i$, for all $i = 1, 2, \ldots, m$.

To solve problem (1.5), Reich et al. [46] proposed the following iterative methods: for any $u_0, v_0 \in C$, let $\{u_n\}$ and $\{v_n\}$ be two sequence generated by:

$$u_{n+1} = P_C\left[u_n - \gamma \sum_{i=1}^m \mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i u_n\right], \tag{1.6}$$

$$v_{n+1} = \alpha_n f(v_n) + (1 - \alpha_n)P_C\left[v_n - \gamma_n \sum_{i=1}^m \mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i v_n\right], \tag{1.7}$$

where $f : C \to C$ is a strict k-contraction with $k \in [0, 1)$, $\{\gamma_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$. They established the weak and strong convergence of iterative methods (1.6) and (1.7), respectively.

In 2021, Reich and Tuyen [44] considered the following split common null point problem with multiple output sets in Hilbert spaces: let $\mathcal{H}, \mathcal{H}_i, i = 1, 2, \ldots, N$, be real Hilbert spaces and let $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i, i = 1, 2, \ldots, N$, be bounded linear operators. Let $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}_i},\ i = 1, 2, \ldots, N$ be maximal monotone operators. Given $\mathcal{H}, \mathcal{H}_i$ and $\mathcal{T}_i$ as defined above, the split common null point problem with multiple output sets is to find a point $u^\dagger$ such that

$$u^\dagger \in \Omega := \mathcal{B}^{-1}(0) \cap \left(\cap_{i=1}^{N} \mathcal{T}_i^{-1}(\mathcal{B}_i^{-1}(0))\right) \neq \emptyset. \tag{1.8}$$

To solve problem (1.8), Reich and Tuyen [44] proposed the following iterative method:

Algorithm 1.1. *For any $u_0 \in \mathcal{H}$, Let $\mathcal{H}_0 = \mathcal{H}, \mathcal{T}_0 = I^{\mathcal{H}}, \mathcal{B}_0 = \mathcal{B}$, and let $\{u_n\}$ be the sequence generated by:*

$$\begin{aligned} v_n &= \sum_{i=0}^{N} \beta_{i,n} \left[u_n - \tau_{i,n} \mathcal{T}_i^* (\mathcal{I}^{\mathcal{H}_i} - J_{r_{i,n}}^{\mathcal{B}_i}) \mathcal{T}_i u_n\right] \\ u_{n+1} &= \alpha_n f(u_n) + (1 - \alpha_n) v_n,\ n \geq 0, \end{aligned}$$

where $\{\alpha_n\} \subset (0, 1)$, and $\{\beta_{i,n}\}$ and $\{r_{i,n}\}, i = 0, 1, \ldots, N$, are sequences of positive real numbers, such that $\{\beta_{i,n}\} \subset [a, b] \subset (0, 1)$ and $\sum_{i=0}^{N} \beta_{i,n} = 1$ for each $n \geq 0$, and $\tau_{i,n} = \rho_{i,n} \frac{\|(\mathcal{I}^{\mathcal{H}_i} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i u_n\|^2}{\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_i} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i u_n\|^2 + \theta_{i,n}}$, where $\{\rho_{i,n}\} \subset [c, d] \subset (0, 2)$ and $\{\theta_{i,n}\}$ is a sequence of positive real numbers for each $i = 0, 1, \ldots, N$, and $f : \mathcal{H} \to \mathcal{H}$ is a strict contraction mapping $\mathcal{H}$ into itself with the contraction coefficient $k \in [0, 1)$.

They established the strong convergence of the sequence $\{u_n\}$ generated by Algorithm 1.1 which is a solution of the Problem (1.8)

Alvarez and Attouch [7] applied the following inertial technique to develop an inertial proximal method for finding the zero of a monotone operator, i.e.,

$$\text{find } x \in H \text{ such that } 0 \in G(x). \tag{1.9}$$

where $G : H \to 2^H$ is a set-valued monotone operator. Given $x_{n-1},\ x_n \in H$ and two parameters $\theta_n \in [0, 1), \lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \lambda_n G(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}). \tag{1.10}$$

Here, the inertia is induced by the term $\theta_n(x_n - x_{n-1})$. The equation (1.10) may be thought as coming from the implicit discretization of the second-other differential system

$$0 \in \frac{d^2 x}{dt^2}(t) + \rho \frac{dx}{dt}(t) + G(x(t)) \text{ a.e. } t > 0, \tag{1.11}$$

where $\rho > 0$ is a damping or a friction parameter. This point of view inspired various numerical methods related to the inertial terminology which has a nice convergence property [6–8, 28, 29, 33] by incorporating second order information and helps in speeding up the convergence speed of an algorithm (see, e.g., [3, 7, 9–13, 51, 52] and the references therein).

Recently, Thong and Hieu [50] introduced an inertial algorithm to solve split common fixed point problem (1.1). The algorithm is of the form

$$\begin{cases} x_0, x_1 \in \mathcal{H}_1, \\ y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{j=1}^{r} w_j U_j \left(I + \sum_{i=1}^{t} \eta_i \gamma \mathcal{T}^*(T_i - \mathcal{I})\mathcal{T}\right) y_n. \end{cases} \tag{1.12}$$

Under approximate conditions, they show that the sequence $\{x_n\}$ generated by (1.12) converges weakly to some solution of SCFPP (1.1).

It was shown in [43, Section 4] by example that one-step inertial extrapolation $w_n = x_n + \theta(x_n - x_{n-1})$, $\theta \in [0, 1)$ may fail to provide acceleration. It was remarked in [32, Chapter 4] that the use of inertial of more than two points x_n, x_{n-1} could provide acceleration. For example, the following two-step inertial extrapolation

$$y_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) \tag{1.13}$$

with $\theta > 0$ and $\delta < 0$ can provide acceleration. The failure of one-step inertial acceleration of ADMM was also discussed in [42, Section 3] and adaptive acceleration for ADMM was proposed instead. Polyak [41] also discussed that the multi-step inertial methods can boost the speed of optimization methods even though neither the convergence nor the rate result of such multi-step inertial methods was established in [41]. Some results on multi-step inertial methods have recently be studied in [26].

Our Contributions. Motivated by [44, 50], in this paper, we consider the following split common null point problem with multiple output sets in Hilbert spaces: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces. Let $\{U_j\}_{j=1}^r : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of quasi-nonexpansive operators and $\mathcal{B}_i : \mathcal{H}_2 \to 2^{\mathcal{H}_2}, i = 1, 2, \ldots, t.$ be maximal monotone operators and $\{\mathcal{T}_i\}_{i=1}^t : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split common null point problem with multiple output set is to find a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in \cap_{j=1}^r F(U_j) \cap \left(\cap_{i=1}^t \mathcal{T}_i^{-1}(\mathcal{B}_i^{-1}0)\right) \neq \emptyset. \tag{1.14}$$

Let Υ be the solution set of (1.14). We propose a two-step inertial extrapolation algorithm with self-adaptive step sizes for solving problem (1.14) and give the weak convergence result of our problem in real Hilbert spaces. We give numerical computations to show the efficiency of our proposed method.

2. Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert spaces $\mathcal{H}$. We know that for each point $u^* \in \mathcal{H}$, there is a unique element $P_C u^* \in C$, such that:

$$\|u^* - P_C u^*\| = \inf_{v \in C} \|u^* - v\|. \tag{2.1}$$

We recall that the mapping $P_C : \mathcal{H} \to C$ defined by (2.1) is said to be metric projection of $\mathcal{H}$ onto C. Moreover, we have (see, for instance, Section 3 in [31]):

$$\langle u^* - P_C u^*, v - P_C u^* \rangle \le 0, \ \ \forall\, u^* \in \mathcal{H},\ v \in C. \tag{2.2}$$

Definition 2.1. Let $T : \mathcal{H} \to \mathcal{H}$ be an operator with $F(T) \neq \emptyset$. Then

- $T : \mathcal{H} \to \mathcal{H}$ is called nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall\, u, v \in \mathcal{H}, \tag{2.3}$$

- $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive if

$$\|Tu - v\| \leq \|u - v\|, \quad \forall\, v \in F(T),\ u \in \mathcal{H}. \tag{2.4}$$

We denote by $F(T)$ the set of fixed points of mapping T; that is, $F(T) = \{u^* \in C : Tu^* = u^*\}$. Given an operator $\mathcal{E} : \mathcal{H} \to 2^{\mathcal{H}}$, its domain, range, and graph are defined as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &:= \{u^* \in \mathcal{H} : \mathcal{E}(u^*) \neq \emptyset\}, \\ \mathcal{R}(\mathcal{E}) &:= \cup\{\mathcal{E}(u^*) : u^* \in \mathcal{D}(\mathcal{E})\} \end{aligned}$$

and

$$\mathcal{G}(\mathcal{E}) := \{(u^*, v^*) \in \mathcal{H} \times \mathcal{H} : u^* \in \mathcal{D}(\mathcal{E}),\ v^* \in \mathcal{E}(u^*)\}.$$

The inverse operator $\mathcal{E}^{-1}$ of $\mathcal{E}$ is defined by:

$$u^* \in \mathcal{E}^{-1}(v^*) \text{ if and only if } v^* \in \mathcal{E}(u^*).$$

Recall that the operator $\mathcal{E}$ is said to be monotone if, for each $u^*, v^* \in \mathcal{D}(\mathcal{E})$, we have $\langle f - g, u^* - v^* \rangle \geq 0$ for all $f \in \mathcal{E}(u^*)$ and $g \in \mathcal{E}(v^*)$. We denote by $\mathcal{I}^{\mathcal{H}}$ the identity mapping on $\mathcal{H}$. A monotone operator $\mathcal{E}$ is said to be maximal monotone if there is no proper monotone extension of $\mathcal{E}$ or, equivalently, by Minty's theorem, if $\mathcal{R}(\mathcal{I}^{\mathcal{H}} + \lambda\mathcal{E}) = \mathcal{H}$, for all $\lambda > 0$. If $\mathcal{E}$ is maximal monotone, then we can define, for each $\lambda > 0$, a nonexpansive single-valued operator $J_\lambda^{\mathcal{E}} : \mathcal{R}(\mathcal{I}^{\mathcal{H}} + \lambda\mathcal{E}) \to \mathcal{D}(\mathcal{E})$ by

$$J_\lambda^{\mathcal{E}} = (\mathcal{I}^{\mathcal{H}} + \lambda\mathcal{E})^{-1}.$$

This operator is called the resolvent of $\mathcal{E}$. It is easy to see that $\mathcal{E}^{-1}(0) = F(J_\lambda^{\mathcal{E}})$, for all $\lambda > 0$.

Lemma 2.2. *[45] Suppose that $\mathcal{E} : \mathcal{D}(\mathcal{E}) \subset \mathcal{H} \to 2^{\mathcal{H}}$ is a monotone operator. Then, we have the following statements:*

(i) For $r \geq s > 0$, we have:

$$\|u - J_s^{\mathcal{E}} u\| \leq 2\|u - J_r^{\mathcal{E}} u\|,$$

for all elements $u \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E}) \cap \mathcal{R}(\mathcal{I}^{\mathcal{H}} + s\mathcal{E})$.

(ii) For all numbers $r > 0$ and for all points $u, v \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E})$, we have:

$$\langle u - v, J_r^{\mathcal{E}} u - J_r^{\mathcal{E}} v \rangle \geq \|J_r^{\mathcal{E}} u - J_r^{\mathcal{E}} v\|^2.$$

(iii) For all numbers $r > 0$ and for all points $u, v \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E})$, we have:

$$\langle (\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})u - (\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})v, u - v \rangle \geq \|(\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})u - (\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})v\|^2.$$

(iv) If $S = \mathcal{E}^{-1}(0) \neq \emptyset$, then for all elements $u^* \in S$ and $u \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E})$, we have:

$$\|J_r^{\mathcal{E}} u - u^*\|^2 \leq \|u - u^*\|^2 - \|u - J_r^{\mathcal{E}} u\|^2.$$

Lemma 2.3. *[30] Suppose that $\mathcal{T}$ is a nonexpansive mapping from a closed and convex subset of a Hilbert space $\mathcal{H}$ into $\mathcal{H}$. Then, the mapping $\mathcal{I}^{\mathcal{H}} - \mathcal{T}$ is demiclosed on C; that is, for any $\{u_n\} \subset C$, such that $u_n \rightharpoonup u \in C$ and the sequence $(\mathcal{I}^{\mathcal{H}} - \mathcal{T})(u_n) \to v$, we have $(\mathcal{I}^{\mathcal{H}} - \mathcal{T})(u) = v$.*

Lemma 2.4. *[34] Given an integer $N \geq 1$. Assume that for each $i = 1, \dots, N$, $T_i : \mathcal{H} \to \mathcal{H}$ is a k_i-demicontractive operator such that $\cap_{i=1}^N F(T_i) \neq \emptyset$. Assume that $\{w_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N w_i = 1$. Setting $U = \sum_{i=1}^N w_i T_i$, then the following results hold:*

(i) $F(U) = \cap_{i=1}^N F(T_i)$.
(ii) *U is λ-demicontractive operator, where $\lambda = \max\{k_i | i = 1, \dots, N\}$.*
(iii) $\langle x - Ux, x - z\rangle \geq \frac{1-\lambda}{2} \sum_{i=1}^N w_i \|x - T_i x\|^2$ *for all $x \in \mathcal{H}$ and $z \in \cap_{i=1}^N F(T_i)$.*

Lemma 2.5. *Let $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\begin{aligned}\|(1+\alpha)x - (\alpha-\beta)y - \beta z\|^2 &= (1+\alpha)\|x\|^2 - (\alpha-\beta)\|y\|^2 - \beta\|z\|^2 + (1+\alpha)(\alpha-\beta)\|x-y\|^2 \\ &\quad + \beta(1+\alpha)\|x-z\|^2 - \beta(\alpha-\beta)\|y-z\|^2.\end{aligned}$$

3. Main result

We give the following assumptions in order to obtain our convergence analysis.

Assumptions 3.1. *We assume that the inertial parameters $\theta \in [0, \frac{1}{2})$, $\rho \in (0,1)$ and $\delta \in (-\infty, 0]$ satisfies the following conditions.*

(a)
$$0 \leq \theta < \frac{1-\rho}{1+\rho};$$

(b)
$$\max\left\{\frac{2\theta\rho}{1-\rho} - (1-\theta), \frac{\theta-1}{\theta+1}[\rho(2\theta+1) - (\theta-1)]\right\} < \delta \leq 0;$$

(c)
$$(2\rho - 1)(\theta^2 + \delta^2) + (2-\rho)(\theta - \delta) + \rho - 2\delta\theta - 1 < 0.$$

Now, we present our proposed method and our convergence analysis as follows:

Algorithm 3.2. *Two-Step Inertial for Split Common Null Point Problem*

Step 1. *Choose $\delta \in (-\infty, 0]$ and $\theta \in [0, 1/2)$ such that Assumption 3.1 is fulfilled. Pick $x_{-1}, x_0, x_1 \in \mathcal{H}_1$ and set $n = 1$.*

Step 2. *Given x_{n-2}, x_{n-1} and x_n, compute x_{n+1} as follows*

$$\begin{cases} y_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}), \\ x_{n+1} = (1-\rho)y_n + \rho\sum_{j=1}^r w_j U_j\left(\mathcal{I} - \sum_{i=1}^t \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i\right)y_n, \end{cases} \tag{3.1}$$

where $\{\delta_{i,n}\}$ and $\{r_{i,n}\}$, $i = 1, 2, \dots t$, are sequences of positive real numbers, such $\{\delta_{i,n}\} \subset [a,b] \subset (0,1)$ and $\sum_{i=1}^t \delta_{i,n} = 1$, for each $n \geq 1$ and

$$\tau_{i,n} = \rho_{i,n}\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i y_n\|^2}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i y_n\|^2 + \theta_{i,n}}, \tag{3.2}$$

where $\{\rho_{i,n}\} \subset [c,d] \subset (0,2)$ *and* $\{\theta_{i,n}\}$ *are sequences of positive real numbers for each* $i = 1,2,\dots,t$, *and* $\{U_j\}_{j=1}^{r}$ *is a finite family of quasi-nonexpansive operators.*

Step 3. *Set* $n \leftarrow n+1$ *and go to* ***Step 2.***

Lemma 3.3. *For* $t \in \mathbb{N}$, *let* $\{\mathcal{B}_i\}_{i=1}^{t} : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ *be a finite family maximal monotone operators. Let* $\{T_i\}_{i=1}^{t} : \mathcal{H}_1 \to \mathcal{H}_2$, *be a finite family of bounded linear operators. Define the operator* $V : \mathcal{H}_1 \to \mathcal{H}_1$ *by*

$$V := \mathcal{I} - \sum_{i=1}^{t} \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i, \tag{3.3}$$

where $\tau_{i,n}$ *is as defined in* (3.2), $\{\delta_{i,n}\}_{i=1}^{t} \subset (0,1)$ *and* $\sum_{i=1}^{t}\delta_{i,n} = 1$. *Then we have the following results:*

(1)

$$\|Vx - z\|^2 \le \|x-z\|^2 - \sum_{i=1}^{t}\delta_{i,n}\rho_{i,n}(2-\rho_{i,n})\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}.$$

(2) $x \in F(V)$ *if and only if* $T_i x \in \cap_{i=1}^{t} F(J_{r_{i,n}}^{\mathcal{B}_i})$, *for* $i = 1,2,\dots,t$.

Proof. (1) Given a point $z \in \Upsilon$, it follows from the convexity of the function $\|\cdot\|^2$ that:

$$\begin{aligned}
\|Vx - z\|^2 &= \left\| x - \sum_{i=1}^{t}\delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z \right\|^2 \\
&= \left\| \sum_{i=1}^{t}\delta_{i,n}(x - \tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z) \right\|^2 \\
&\le \sum_{i=1}^{t}\delta_{i,n}\|x - \tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z\|^2.
\end{aligned} \tag{3.4}$$

Using $J_{r_{i,n}}^{\mathcal{B}_i}(\mathcal{T}_i z) = \mathcal{T}_i z$ and Lemma 2.2(iii), for each $i = 1,2,\dots,t$, we see that

$$\begin{aligned}
&\|x - \tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z\|^2 \\
= &\ \|x-z\|^2 - 2\tau_{i,n}\langle \mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x, x - z\rangle + \tau_{i,n}^2\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 \\
= &\ \|x-z\|^2 - 2\tau_{i,n}\langle (\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x, \mathcal{T}_i x - \mathcal{T}_i z\rangle + \tau_{i,n}^2\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 \\
= &\ \|x-z\|^2 - 2\tau_{i,n}\langle (\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - (\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i z, \mathcal{T}_i x - \mathcal{T}_i z\rangle \\
&+\tau_{i,n}^2\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 \\
\le &\ \|x-z\|^2 - 2\tau_{i,n}\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \tau_{i,n}^2(\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}) \\
= &\ \|x-z\|^2 - \rho_{i,n}(2-\rho_{i,n})\frac{\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}.
\end{aligned} \tag{3.5}$$

Hence, from (3.4) and (3.5), we get

$$\|Vx - z\|^2 \le \|x-z\|^2 - \sum_{i=1}^{t}\delta_{i,n}\rho_{i,n}(2-\rho_{i,n})\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}.$$

(2) It is obvious that if $\mathcal{T}_i x \in \cap_{i=1}^t F(J_{r_{i,n}}^{\mathcal{B}_i})$ then $x \in F(V)$. We show the converse, let $x \in F(V)$ and $z \in \mathcal{T}_i^{-1}\left(F(J_{r_{i,n}}^{\mathcal{B}_i})\right)$ we have

$$\|x - z\|^2 = \|Vx - z\|^2 \leq \|x - z\|^2 - \sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}. \tag{3.6}$$

Since $\rho_{i,n} \subset (0, 2)$, we obtain

$$(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x = 0, \quad \forall\, i = 1, 2, \ldots, t.$$

That is, $\mathcal{T}_i x \in \cap_{i=1}^t F(J_{r_{i,n}}^{\mathcal{B}_i})$.

□

Lemma 3.4. *For $t, r \in \mathbb{N}$, let $\{\mathcal{B}_i\}_{i=1}^t : \mathcal{H}_2 \to \mathcal{H}_2$ be maximal monotone operators such that $\cap_{i=1}^t F(J_{r_{i,n}}^{\mathcal{B}_i}) \neq \emptyset$ and $\{U_j\}_{j=1}^r : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of quasi-nonexpansive operators such that $\cap_{j=1}^r F(U_j) \neq \emptyset$. Assume that $\{\mathcal{I} - U_j\}_{j=1}^r$ and $\{\mathcal{I} - J_{r_{i,n}}^{\mathcal{B}_i}\}_{i=1}^t$ are demiclosed at zero. Let $\{\mathcal{T}_i\}_{i=1}^t : \mathcal{H}_1 \to \mathcal{H}_2$, be bounded linear operators suppose $\Upsilon \neq \emptyset$. Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by*

$$Sx = \sum_{j=1}^{r} w_j U_j \left(\mathcal{I} - \sum_{i=1}^{t} \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i\right)x,$$

where $\{\tau_{i,n}\}$, is as defined in (3.2), $\{w_j\}_{j=1}^r$ and $\{\delta_{i,n}\}_{i=1}^t$ are in $(0, 1)$ with $\sum_{j=1}^r w_j = 1$ and $\sum_{i=1}^t \delta_{i,n} = 1$. Assume that the following conditions are satisfied

(A1) $\min_{i=1,2,\ldots,t}\{\inf_n\{r_{i,n}\}\} = r > 0$;
(A2) $\max_{i=1,2,\ldots,t}\{\sup_n\{\theta_{i,n}\}\} = K < \infty$.

Then the following hold:

(a) The operator S is quasi-nonexpansive.
(b) $F(S) = \Upsilon$.
(c) $\mathcal{I} - S$ is demiclosed at zero.

Proof. From the definition of V we can rewrite the operator S as

$$Sx = \sum_{j=1}^{r} w_j U_j V x.$$

We show the following

(i) $\{U_j V\}_{j=1}^r$ is a finite family of quasi-nonexpansive operator,
(ii) $\cap_{j=1}^r F(U_j V) = \Upsilon$,
(iii) for each $j = 1, 2, \ldots, r$ then $\mathcal{I} - U_j V$ is demiclosed at zero.

By Lemma 3.3, V is quasi-nonexpansive. Therefore, for each $j = 1, 2, \ldots, r$ the operator $U_j V$ is quasi-nonexpansive. Next, we show that for each $j = 1, 2, \ldots, r$, then

$$F(U_j V) = F(U_j) \cap F(V).$$

Indeed, it suffices to show that for each $j = 1, 2, \ldots, r$ $F(U_jV) \subset F(U_j) \cap F(V)$. Let $p \in F(U_jV)$. It is enough to show that $p \in F(V)$. Now, taking $z \in F(U_j) \cap F(V)$; we have

$$\begin{aligned} \|p - z\|^2 &= \|U_jVp - z\|^2 \leq \|Vp - z\|^2 \\ &\leq \|p - z\|^2 - \sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x\|^2 + \theta_{i,n}}. \end{aligned}$$

This implies that

$$\sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x\|^2 + \theta_{i,n}} = 0.$$

That is $J^{\mathcal{B}_i}_{r_{i,n}}(\mathcal{T}_i p) = \mathcal{T}_i p, \quad \forall\ i = 1, 2, \ldots, t$. This implies that $\mathcal{T}_i p \in \cap_{i=1}^{t} F(J^{\mathcal{B}_i}_{r_{i,n}})$. Thus, $p \in F(V)$. Therefore, $F(U_j) \cap F(V) = F(U_jV), \quad \forall j = 1, \ldots, r$. We now show that

$$\begin{aligned} \Upsilon &= \{p \in \cap_{j=1}^{r} F(U_j) \quad \text{such that} \quad \mathcal{T}_i p \in \cap_{i=1}^{t} F(J^{\mathcal{B}_i}_{r_{i,n}})\} \\ &= \cap_{j=1}^{t} F(U_jV). \end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned} \Upsilon &= \{x \in \cap_{j=1}^{r} F(U_j) | \mathcal{T}_i x \in \cap_{i=1}^{r} F(J^{\mathcal{B}_i}_{r_{i,n}})\} \\ &= \{x \in \cap_{i=1}^{r} F(U_j) | x \in F(V)\} \\ &= \cap_{j=1}^{r} F(U_j) \cap F(V) \\ &= \cap_{j=1}^{r} F(U_jV). \end{aligned}$$

Finally, we show that for each $j = 1, ..r$, $\mathcal{I} - U_jV$ is demiclosed at zero. Let $\{x_n\} \subset \mathcal{H}_1$ be a sequence such that $x_n \rightharpoonup z \in \mathcal{H}_1$ and $U_jVx_n - x_n \to 0$ we have

$$0 \leq \|x_n - z\| - \|U_jVx_n - z\| \leq \|x_n - U_jVx_n\| \to 0.$$

This implies that

$$\|x_n - z\|^2 - \|U_jVx_n - z\|^2 \to 0.$$

By Lemma 3.3, we have

$$\begin{aligned} \|U_jVx_n - z\|^2 &\leq \|Vx_n - z\|^2 \\ &\leq \|x_n - z\|^2 - \sum_{i=0}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x_n\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x_n\|^2 + \theta_{i,n}}. \end{aligned} \tag{3.7}$$

This implies that

$$\sum_{i=0}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x\|^2 + \theta_{i,n}} \leq \|x_n - z\|^2 - \|U_jVx_n - z\|^2.$$

Since $\{\delta_{i,n}\} \subset [a, b] \subset (0, 1)$, $\{\rho_{i,n}\} \subset [c, d] \subset (0, 2)$, and (3.7) implies

$$\frac{\|(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x_n\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x_n\|^2 + \theta_{i,n}} \to 0, \quad \text{as} \quad n \to \infty, \tag{3.8}$$

$\forall \quad i = 0,1,2\ldots t$. It follows from the boundedness of the sequence $\{x_n\}$ that $L := max_{i=0,1,\ldots,N}\{\sup\{\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^2\}\} < \infty$. Thus from Condition (A2), It follows that

$$\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^2 + \theta_{i,n}} \geq \frac{\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^4}{L + K}.$$

Combining this with (3.8), we deduce that

$$\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\| \to 0 \tag{3.9}$$

$\forall\ i = 0, 1, \ldots, N$, Lemma 2.2(i) and Condition (A1) now imply that

$$\|(\mathcal{I}^{\mathcal{H}_2} - J_r^{\mathcal{B}_i})\mathcal{T}_i x_n\| \leq 2\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|, \tag{3.10}$$

$\forall\ i = 0, 1, \ldots, N$. Thus using (3.9) and (3.10), we are able to deduce that

$$\|(\mathcal{I}^{\mathcal{H}_2} - J_r^{\mathcal{B}_i})\mathcal{T}_i x_n\| \to 0, \tag{3.11}$$

$\forall i = 0, 1, \ldots, N$.

From $\|Vx_n - x_n\| = \|\sum_{i=0}^t \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|$, the assumptions on $\{\delta_{i,n}\}$ and $\{\tau_{i,n}\}$ and (3.10), it follows that

$$\|Vx_n - x_n\| \to 0.$$

On the other hand

$$\|U_j V x_n - V x_n\| \leq \|U_j V x_n - x_n\| + \|V x_n - x_n\| \to 0. \tag{3.12}$$

Since $x_n \rightharpoonup z$, we have $Vx_n \rightharpoonup z$ and by the demiclosedness of U_j we have $z \in F(U_j)$. Since, for each $i = 1, 2, \ldots, N$, $\mathcal{T}_i$ is a bounded linear operator, it follows that $\mathcal{T}_i x_{n_k} \rightharpoonup \mathcal{T}_i z$. Thus by Lemma 2.3 and (3.11) implies that $\mathcal{T}_i z \in F(J_r^{\mathcal{B}_i})\ \ \forall i = 1, \ldots, t$ that is $\mathcal{T}_i z \in \cap_{i=1}^t F(J_r^{\mathcal{B}_i})$. By Lemma 3.3 we get $z \in F(V)$. Therefore, $z \in F(U_j) \cap F(V) = F(U_j V)$.

By Claim (i) and Lemma 2.4, we obtain $Sx = \sum_{j=1}^r w_j U_j V x$ is quasi-nonexpansive and $F(S) = \cap_{j=1}^t F(U_j V) = \Upsilon$.

Finally, we show that $\mathcal{I} - S$ is demiclosed at zero. Indeed, Let $\{x_n\} \subset \mathcal{H}_1$ be a sequence such that $x_n \rightharpoonup z \in \mathcal{H}_1$ and $\|x_n - Sx_n\| \to 0$. Let $p \in F(S)$ by Lemma 2.4, we have

$$\langle x_n - Sx_n, x_n - p\rangle \geq \frac{1}{2}\sum_{j=1}^t \|x_n - U_j V x_n\|^2.$$

This imples that, for each $j = 1, \ldots, t$ we have

$$\|x_n - U_j V x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

By the demiclosedness of $\mathcal{I} - U_j V$ we have $z \in F(U_j V)$. Therefore $z \in \cap_{j=1}^t F(U_j V) = F(S)$. □

Theorem 3.5. *For $t, r \in \mathbb{N}$. Let $\{\mathcal{B}_i\}_{i=1}^t : \mathcal{H}_2 \to \mathcal{H}_2$ be a finite family of maximal monotone operators such that $\cap_{i=1}^t F(J_r^{\mathcal{B}_i}) \neq \emptyset$ and $\{U_j\}_{j=1}^r : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of quasi-nonexpansive operators such that $\cap_{j=1}^r F(U_j) \neq \emptyset$. Assume that $\{\mathcal{I} - U_j\}_{j=1}^r$ and $\{\mathcal{I} - J_r^{\mathcal{B}_i}\}_{i=1}^t$ are demiclosed at zero. Let $\mathcal{T}_i : \mathcal{H}_1 \to \mathcal{H}_2$, $i = 1, 2, \ldots, N$ be bounded linear operators. Suppose $\Upsilon \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. and suppose that Assumptions* (3.1) *(a)-(c) are fulfilled. Then $\{x_n\}$ converges weakly to an element of Υ.*

Proof. Let $S = \sum_{j=1}^r w_j U_j \left(\mathcal{I} - \sum_{i=1}^t \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i\right)$, then the sequence $\{x_{n+1}\}$ can be rewritten as follows

$$x_{n+1} = (1-\rho)y_n + \rho S y_n. \tag{3.13}$$

By Lemma 3.3, we have that S is quasi-nonexpansive. Let $z \in \Upsilon$, from (3.13), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1-\rho)(y_n - z) + \rho(S y_n - z)\|^2 \\ &= (1-\rho)\|y_n - z\|^2 + \rho\|S y_n - z\|^2 - \rho(1-\rho)\|y_n - S y_n\|^2 \\ &\leq \|y_n - z\|^2 - \rho(1-\rho)\|y_n - S y_n\|^2. \end{aligned} \tag{3.14}$$

Observe that

$$\begin{aligned} y_n - z &= x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) - z \\ &= (1+\theta)(x_n - z) - (\theta - \delta)(x_{n-1} - z) - \delta(x_{n-2} - z). \end{aligned}$$

Hence by Lemma 2.5, we have

$$\begin{aligned} \|y_n - z\|^2 &= \|(1+\theta)(x_n - z) - (\theta - \delta)(x_{n-1} - z) - \delta(x_{n-2} - z)\|^2 \\ &= (1+\theta)\|x_n - z\|^2 - (\theta - \delta)\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 \\ &\quad +(1+\theta)(\theta - \delta)\|x_n - x_{n-1}\|^2 + \delta(1+\theta)\|x_n - x_{n-2}\|^2 \\ &\quad -\delta(\theta - \delta)\|x_{n-1} - x_{n-2}\|^2. \end{aligned} \tag{3.15}$$

Note also that

$$\begin{aligned} 2\theta\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle &= 2\langle \theta(x_{n+1} - x_n), x_n - x_{n-1}\rangle \\ &\leq 2|\theta|\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &= 2\theta\|x_{n+1} - x_n\|\|x_n - x_{n-1}\|, \end{aligned}$$

and so,

$$-2\theta\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \geq -2\theta\|x_{n+1} - x_n\|\|x_n - x_{n-1}\|. \tag{3.16}$$

Also,

$$\begin{aligned} 2\delta\langle x_{n+1} - x_n, x_{n-1} - x_{n-2}\rangle &= 2\langle \delta(x_{n+1} - x_n), x_{n-1} - x_{n-2}\rangle \\ &\leq 2|\delta|\|x_{n+1} - x_n\|\|x_{n-1} - x_{n-2}\|, \end{aligned}$$

which implies that

$$-2\delta\langle x_{n+1}-x_n, x_{n-1}-x_{n-2}\rangle \geq -2|\delta|\|x_{n+1}-x_n\|\|x_{n-1}-x_{n-2}\|. \tag{3.17}$$

Similarly, we have

$$2\delta\theta\langle x_{n-1}-x_n, x_{n-1}-x_{n-2}\rangle \leq 2|\delta|\theta\|x_n-x_{n-1}\|\|x_{n-1}-x_{n-2}\|,$$

and thus

$$2\delta\theta\langle x_n-x_{n-1}, x_{n-1}-x_{n-2}\rangle \geq -2|\delta|\theta\|x_n-x_{n-1}\|\|x_{n-1}-x_{n-2}\|. \tag{3.18}$$

By (3.16)–(3.18) and Cauchy-Schwartz inequality one has

$$\begin{aligned}
\|x_{n+1}-y_n\|^2 &= \|x_{n+1}-(x_n+\theta(x_n-x_{n-1})+\delta(x_{n-1}-x_{n-2}))\|^2\\
&= \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1}-x_{n-2})\|^2\\
&= \|x_{n+1}-x_n\|^2-2\theta\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle-2\delta\langle x_{n+1}-x_n, x_{n-1}-x_{n-2}\rangle\\
&\quad +\theta^2\|x_n-x_{n-1}\|^2+2\delta\theta\langle x_n-x_{n-1}, x_{n-1}-x_{n-2}\rangle+\delta^2\|x_{n-1}-x_{n-2}\|^2\\
&\geq \|x_{n+1}-x_n\|^2-2\theta\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|-2|\delta|\|x_{n+1}-x_n\|\|x_{n-1}-x_{n-2}\|\\
&\quad +\theta^2\|x_n-x_{n-1}\|^2-2|\delta|\theta\|x_n-x_{n-1}\|\|x_{n-1}-x_{n-2}\|+\delta^2\|x_{n-1}-x_{n-2}\|^2\\
&\geq \|x_{n+1}-x_n\|^2-\theta\|x_{n+1}-x_n\|^2-\theta\|x_n-x_{n-1}\|^2-|\delta|\|x_{n+1}-x_n\|^2\\
&\quad -|\delta|\|x_{n-1}-x_{n-2}\|^2+\theta^2\|x_n-x_{n-1}\|^2-|\delta|\theta\|x_n-x_{n-1}\|^2\\
&\quad -|\delta|\theta\|x_{n-1}-x_{n-2}\|^2+\delta^2\|x_{n-1}-x_{n-2}\|^2\\
&= (1-|\delta|-\theta)\|x_{n+1}-x_n\|^2+(\theta^2-\theta-|\delta|\theta)\|x_n-x_{n-1}\|^2\\
&\quad +(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2.
\end{aligned} \tag{3.19}$$

Observe that

$$\|Sy_n-y_n\|^2=\frac{1}{\rho^2}\|x_{n+1}-y_n\|^2. \tag{3.20}$$

Putting (3.20) in (3.14), we get

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \|y_n-z\|^2-\rho(1-\rho)\|Sy_n-y_n\|^2\\
&= \|y_n-z\|^2-\frac{1-\rho}{\rho}\|x_{n+1}-y_n\|^2.
\end{aligned} \tag{3.21}$$

Combining (3.15) and (3.19) in (3.21) with noting that $\delta \leq 0$ we obtain

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= (1+\theta)\|x_n-z\|^2-(\theta-\delta)\|x_{n-1}-z\|^2-\delta\|x_{n-2}-z\|^2\\
&\quad +(1+\theta)(\theta-\delta)\|x_n-x_{n-1}\|^2+\delta(1+\theta)\|x_n-x_{n-2}\|^2-\delta(\theta-\delta)\|x_{n-1}-x_{n-2}\|^2\\
&\quad -\frac{(1-\rho)}{\rho}(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2-\frac{(1-\rho)}{\rho}(\theta^2-\theta-|\delta|\theta)\|x_n-x_{n-1}\|^2\\
&\quad -\frac{(1-\rho)}{\rho}(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1+\theta)\|x_n - z\|^2 - (\theta-\delta)\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 + \delta(1+\theta)\|x_n - x_{n-2}\|^2 \\
&\quad + \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - \theta - |\delta|\theta)\right]\|x_n - x_{n-1}\|^2 \\
&\quad - \left[\delta(\theta-\delta) + \frac{(1-\rho)}{\rho}(\delta^2 - |\delta| - |\delta|\theta)\right]\|x_{n-1} - x_{n-2}\|^2 \\
&\quad - \frac{(1-\rho)}{\rho}(1 - |\delta| - \theta)\|x_{n+1} - x_n\|^2 \\
&\leq (1+\theta)\|x_n - z\|^2 - (\theta-\delta)\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 \\
&\quad + \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - \theta + \delta\theta)\right]\|x_n - x_{n-1}\|^2 \\
&\quad - \left[\delta(\theta-\delta) + \frac{(1-\rho)}{\rho}(\delta^2 + \delta + \delta\theta)\right]\|x_{n-1} - x_{n-2}\|^2 \\
&\quad - \frac{(1-\rho)}{\rho}(1 + \delta - \theta)\|x_{n+1} - x_n\|^2. \qquad (3.22)
\end{aligned}
$$

By rearranging we get

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 - \theta\|x_n - z\|^2 - \delta\|x_{n-1} - z\|^2 + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_{n+} - x_n\|^2 \\
&\leq \|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2 \\
&+ \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - 2\theta + \delta\theta + \delta + 1)\right] \\
&- \left[\delta(\theta-\delta) + \frac{(1-\rho)}{\rho}(\delta^2 + \delta + \delta\theta)\right]\|x_{n-1} - x_{n-2}\|^2. \qquad (3.23)
\end{aligned}
$$

Define

$$
\Upsilon_n := \|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\| + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2.
$$

Let us show that $\Upsilon_n \geq 0, \forall n \geq 1$. Now

$$
\begin{aligned}
\Upsilon_n &= \|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2 \\
&\geq \|x_n - z\|^2 - 2\theta\|x_n - x_{n-1}\|^2 - 2\theta\|x_n - z\|^2 - \delta\|x_{n-2} - z\|^2 \\
&\quad + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2 \\
&= (1-2\theta)\|x_n - z\|^2 + \left[\frac{(1-\rho)}{\rho}(1+\delta-\theta) - 2\theta\right]\|x_n - x_{n-1}\|^2 - \delta\|x_{n-2} - z\|^2. \qquad (3.24)
\end{aligned}
$$

Since $\theta < 1/2$, $\delta \leq 0$, $\frac{2\theta\rho}{1-\rho} - (1-\theta) < \delta$ and $0 \leq \theta < \frac{1-\rho}{1+\rho}$, it follows from (3.24) that $\Upsilon_n \geq 0$. Furthermore, we drive from (3.23)

$$
\Upsilon_{n+1} - \Upsilon_n \leq \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - 2\theta + \delta\theta + \delta + 1)\right]\|x_n - x_{n-1}\|^2
$$

$$
\begin{aligned}
& -\left[\delta(\theta-\delta)+\frac{(1-\rho)}{\rho}(\delta^2+\delta+\delta\theta)\right]\|x_{n-1}-x_{n-2}\|^2 \\
= & -\left[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1)\right] \\
& \times\left(\|x_{n-1}-x_{n-2}\|-\|x_n-x_{n-1}\|^2\right) \\
& +[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1) \\
& -\delta(\theta-\delta)-\frac{(1-\rho)}{\rho}(\delta^2+\delta+\delta\theta)]\|x_{n-1}-x_{n-2}\|^2 \\
= & \; q_1[\|x_{n-1}-x_{n-2}\|^2-\|x_n-x_{n-1}\|^2]-q_2\|x_{n-1}-x_{n-2}\|^2, \qquad (3.25)
\end{aligned}
$$

where

$$
q_1 := -\left[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1)\right]
$$

and

$$
\begin{aligned}
q_2: \; = & -[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1) \\
& -\delta(\theta-\delta)-\frac{(1-\rho)}{\rho}(\delta^2+\delta+\delta\theta)]\|x_{n-1}-x_{n-2}\|^2. \qquad (3.26)
\end{aligned}
$$

By our assumption, it holds that

$$
\frac{\theta-1}{\theta+1}[\rho(2\theta+1)-(\theta-1)]<\delta. \qquad (3.27)
$$

As a result $q_1>0$. Also $q_2>0$ by Assumption 3.1 (c). Then by (3.25) we have

$$
\Upsilon_{n+1}+q_1\|x_n-x_{n-1}\|^2 \leq \Upsilon_n+q_1\|x_{n-1}-x_{n-2}\|^2-q_2\|x_{n-1}-x_{n-2}\|^2. \qquad (3.28)
$$

Letting $\bar{\Upsilon}_n := \Upsilon_n+q_1\|x_{n-1}-x_{n-2}\|^2$. Then $\bar{\Upsilon}_n \geq 0,\ \forall n \geq 1$. Also, it follows from (3.28) that

$$
\bar{\Upsilon}_{n+1} \leq \bar{\Upsilon}_n. \qquad (3.29)
$$

These facts imply that the sequence $\{\bar{\Upsilon}_n\}$ is decreasing and bounded from below and thus $\lim_{n\to\infty}\bar{\Upsilon}_n$ exists. Consequently, we get from (3.28) and the squeeze theorem that

$$
\lim_{n\to\infty} q_1\|x_{n-1}-x_{n-2}\|^2=0. \qquad (3.30)
$$

Hence

$$
\lim_{n\to\infty} \|x_{n-1}-x_{n-2}\|^2=0. \qquad (3.31)
$$

As a result

$$
\|x_{n+1}-y_n\| \;=\; \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1})\|
$$

$$\leq \quad \|x_{n+1} - x_n\| + \theta\|x_n - x_{n-1}\| + |\delta|\|x_{n-1} - x_{n-2}\| \to 0$$

as $n \to \infty$. By $\lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = 0$, one has

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0, \text{ as } n \to \infty.$$

By (3.31) and the existence of $\lim\limits_{n\to\infty}\bar{\Upsilon}_n$, we have that $\lim\limits_{n\to\infty}\Upsilon_n$ exists and hence $\{\Upsilon_n\}$ is bounded. Now, since $\lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have from the definition of Υ_n that

$$\lim_{n\to\infty}[\|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2] \tag{3.32}$$

exists. Using the boundedness of $\{\Upsilon_n\}$, we obtain from (3.24) that $\{x_n\}$ is bounded. Consequently $\{y_n\}$ is bounded. From (3.8), we obtain

$$\rho(1-\rho)\|Sy_n - y_n\| \leq \|y_n - z\|^2 - \|x_{n+1} - z\|^2.$$

This implies that

$$\lim_{n\to\infty}\|Sy_n - y_n\| = 0. \tag{3.33}$$

Finally, we show that the sequence $\{x_n\}$ converges weakly to $x^* \in \Upsilon$. Indeed, since $\{x_n\}$ is bounded we assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^* \in H$. Since $\|x_n - y_n\| \to 0$, we also have $y_{n_j} \rightharpoonup x^*$. Then by the demiclosedness of $I - S$, we obtain $x^* \in F(S) = \Upsilon$.

Now, we show that $\{x_n\}$ has unique weak limit point in Υ. Suppose that $\{x_{m_j}\}$ is another subsequence of $\{x_n\}$ such that $x_{m_j} \rightharpoonup v^*$ as $j \to \infty$. Observe that

$$2\langle x_n, x^* - v^*\rangle = \|x_n - v^*\|^2 - \|x_n - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2 \tag{3.34}$$

$$2\langle x_{n-1}, x^* - v^*\rangle = \|x_{n-1} - v^*\|^2 - \|x_{n-1} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2$$

and

$$2\langle x_{n-2}, x^* - v^*\rangle = \|x_{n-2} - v^*\|^2 - \|x_{n-2} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2.$$

Therefore

$$2\langle -\theta x_{n-1}, x^* - v^*\rangle = -\theta\|x_{n-1} - v^*\|^2 + \theta\|x_{n-1} - x^*\|^2 + \theta\|v^*\|^2 - \theta\|x^*\|^2. \tag{3.35}$$

and

$$2\langle -\delta x_{n-2}, x^* - v^*\rangle \;=\; -\delta\|x_{n-2} - v^*\|^2 + \delta\|x_{n-2} - x^*\|^2 + \delta\|v^*\|^2 - \delta\|x^*\|^2. \tag{3.36}$$

Addition of (3.34)–(3.36) gives

$$\begin{aligned} & 2\langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle \\ = \; & \Big(\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \delta\|x_{n-2} - v^*\|^2\Big) \end{aligned} \tag{3.37}$$

$$-\left(\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 - \delta\|x_{n-2} - x^*\|^2\right) + (1 - \theta - \delta)(\|x^*\| - \|v^*\|^2). \quad (3.38)$$

According to (3.32), we get

$$\lim_{n\to\infty}[\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 - \delta\|x_{n-2} - x^*\|^2] \quad (3.39)$$

exists and

$$\lim_{n\to\infty}[\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \delta\|x_{n-2} - v^*\|^2] \quad (3.40)$$

exists. This implies from (3.37) that $\lim_{n\to\infty}\langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle$ exists. Consequently,

$$\begin{aligned}\langle v^* - \theta v^* - \delta v^*, x^* - v^*\rangle &= \lim_{j\to\infty}\langle x_{n_j} - \theta x_{n_{j-1}} - \beta x_{n_{j-2}}, x^* - v^*\rangle \\ &= \lim_{n\to\infty}\langle x_{n-1} - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle \\ &= \lim_{j\to\infty}\langle x_{m_j} - \theta x_{m_{j-1}} - \beta x_{m_{j-2}}, x^* - v^*\rangle \\ &= \langle x^* - \theta x^* - \beta x^*, x^* - v^*\rangle.\end{aligned}$$

Hence

$$(1 - \theta - \delta)\|x^* - v^*\|^2 = 0.$$

Since $\delta \le 0 < 1 - \theta$, we obtain that $x^* = v^*$. Therefore, the sequence $\{x_n\}$ converges weakly to $x^* \in \Upsilon$. This completes the proof. □

4. Numerical examples

In this section, we give a numerical description to illustrate how our proposed algorithm can be implemented in the setting of the real Hilbert space $\mathbb{R}$. Furthermore, we shall show the effect of the double inertia in the fast convergence of the sequence generated by our proposed Algorithm 3.2. First, we give the set of parameters that satisfy the conditions given in assumption 3.1. To this end, fix $\rho = \frac{1}{2}$ and take

$$\theta = \frac{1}{4},\ \delta = -\frac{1}{5} \text{ and } -\frac{1}{1000}; \quad \theta = \frac{1}{5},\ \delta = -\frac{1}{10} \text{ and } -\frac{1}{1000}; \quad \theta = \frac{1}{6},\ \delta = -\frac{1}{100} \text{ and } -\frac{1}{1000}.$$

Clearly, these parameters satisfy the conditions given in assumption 3.1. Next, we define the operators to be used in the implementation of Algorithm 3.2. In Algorithm 3.2, fix $t = N = r = 3$. Set $H_1 = H_2 = H_3 = \mathbb{R}$. Let $\delta_{i,n} = \frac{1}{3}, \rho_{i,n} = \theta_{i,n} = \frac{2}{3}, r_{i,n} = \frac{1}{2}$ and $w_j = \frac{1}{3}$, where $i, j \in \{1, 2, 3\}$ and $n \ge 1$. Let $B_i,\ T_i,\ U_j : \mathbb{R} \to \mathbb{R}$ be defined by

$$B_i x = 2ix, \text{ then } J^{B_i}_{r_{i,n}} x = \frac{x}{1 + 2ir_{i,n}}, \quad T_i x = ix, \quad U_j x = jx.$$

Then,

$$\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x = \frac{2i^3 r_{i,n}}{1 + 2ir_{i,n}}, \quad \text{and} \quad \tau_{i,n} = \frac{2}{3}\frac{(2i^2 r_{i,n} y_n)^2}{(2i^3 r_{i,n})^2 + 2(1 + 2ir_{i,n})^2}.$$

With this, we are ready to implement our proposed Algorithm 3.2 on MATLAB. Choosing $x_0 = 1$, $x_1 = -2$ and $x_2 = 0.5$, and setting maximum number of iterations to 150 or 10^{-16}, as our stopping criteria, we varied the double inertial parameters as given above. We obtained the following successive approximations:

Table 1. Results of the numerical simulations.

No. Iter	Inertia Para.	$\lvert x_{n+1} - x_n \rvert$
120	$\theta = \frac{1}{4}$ δ =-$\frac{1}{5}$	1.11E-16
80	$\theta = \frac{1}{4}$ δ =-$\frac{1}{1000}$	1.11E-16
107	$\theta = \frac{1}{5}$ δ =-$\frac{1}{10}$	1.11E-16
88	$\theta = \frac{1}{5}$ δ =-$\frac{1}{1000}$	1.11E-16
95	$\theta = \frac{1}{6}$ δ =-$\frac{1}{100}$	1.11E-16
116	$\theta = \frac{1}{6}$ δ =-$\frac{1}{1000}$	1.11E-16

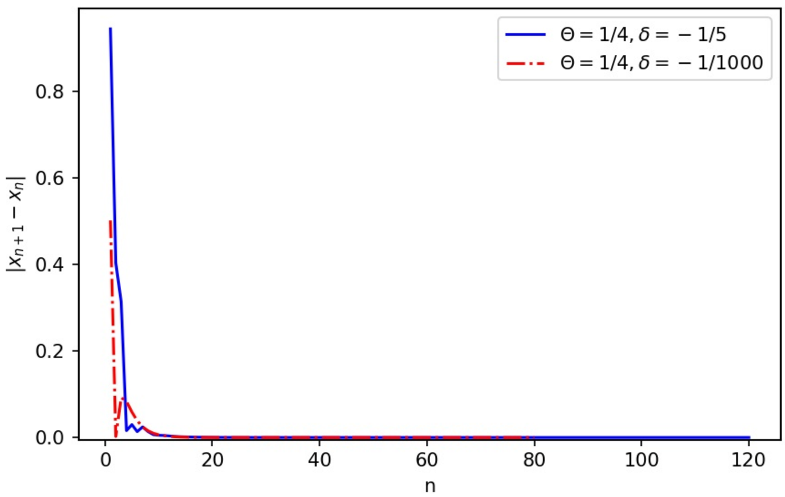

Figure 1. Graph of the iterates of Algorithm 3.2 when $\theta = \frac{1}{4}$, δ =-$\frac{1}{5}$ and δ =-$\frac{1}{1000}$.

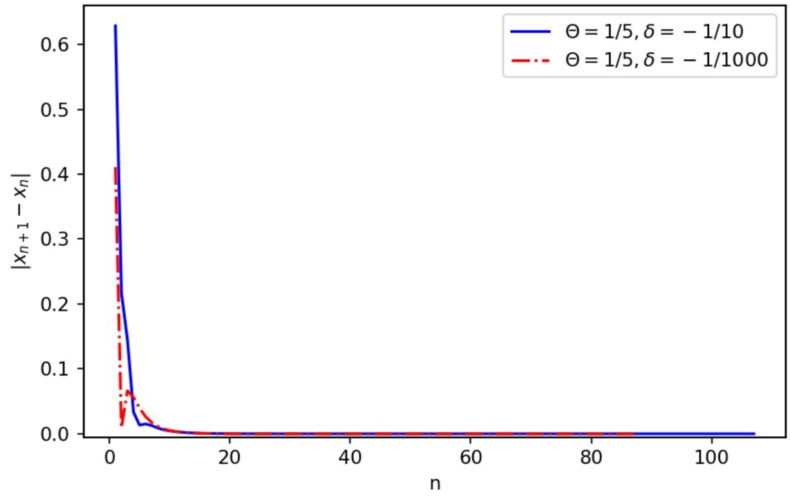

Figure 2. Graph of the iterates of Algorithm 3.2 when $\theta = \frac{1}{5}$, δ =-$\frac{1}{10}$ and δ =-$\frac{1}{1000}$.

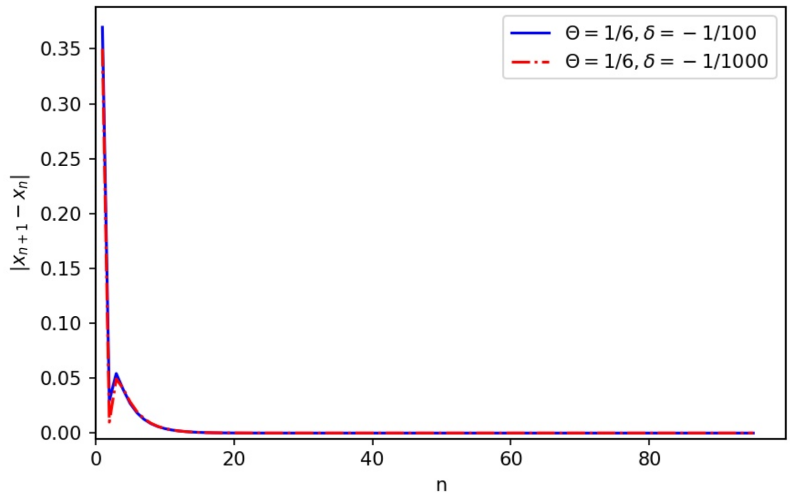

Figure 3. Graph of the iterates of Algorithm 3.2 when $\theta = \frac{1}{6}$, δ =-$\frac{1}{100}$ and δ =-$\frac{1}{1000}$.

5. Discussion

From the numerical simulations presented in Table 1 and Figures 1–3, we saw that in this example, the best choice for the double inertial parameters is $\theta = \frac{1}{4}$ and $\delta = -\frac{1}{1000}$. Furthermore, we observed that as θ decreases and δ approaches 0, the number of iterations required to satisfy the stopping criteria increases.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. A. Adamu, A. A. Adam, Approximation of solutions of split equality fixed point problems with applications, *Carpathian J. Math.*, **37** (2021), 381–392. https://doi.org/10.37193/CJM.2021.03.02
2. A. Adamu, D. Kitkuan, P. Kumam, A. Padcharoen, T. Seangwattana, Approximation method for monotone inclusion problems in real Banach spaces with applications, *J. Inequal. Appl.*, **2022** (2022), 70. https://doi.org/10.1186/s13660-022-02805-0
3. A. Adamu, P. Kumam, D. Kitkuan, A. Padcharoen, Relaxed modified Tseng algorithm for solving variational inclusion problems in real Banach spaces with applications, *Carpathian J. Math.*, **39** (2023), 1–26. https://doi.org/10.37193/CJM.2023.01.01
4. A. Adamu, D. Kitkuan, A. Padcharoen, C. E. Chidume, P. Kumam, Inertial viscosity-type iterative method for solving inclusion problems with applications, *Math. Comput. Simul.*, **194** (2022), 445–459. https://doi.org/10.1016/j.matcom.2021.12.007
5. B. Ali, A. A. Adam, A. Adamu, An accelerated algorithm involving quasi-ϕ-nonexpansive operators for solving split problems, *J. Nonlinear Model. Anal.*, **5** (2023), 54–72. https://doi.org/10.12150/jnma.2023.54
6. F. Alvarez, On the minimizing of a second other dissipative dynamical system in Hilbert space, *SIAM J. Control Optim.*, **38** (2000), 1102–1119. https://doi.org/10.1137/S0363012998335802
7. F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operator via discretization of nonlinear oscillator with damping, *Set-Valued Anal.*, **9** (2001), 3–11. https://doi.org/10.1023/A:1011253113155
8. F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *SIAM J. Optim.*, **14** (2004), 773–782. https://doi.org/10.1137/S1052623403427859
9. H. Attouch, J. Peypouquet, P. Redont, A dynamical approach to an inertial forward-backward algorithm for convex minimization, *SIAM J. Optim.*, **24** (2014), 232–256. https://doi.org/10.1137/130910294

10. A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM J. Imaging Sci.*, **2** (2009), 183–202. https://doi.org/10.1137/080716542
11. R. I. Bot, E. R. Csetnek, An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems, *Numer. Algor.*, **71** (2016), 519–540. https://doi.org/10.1007/s11075-015-0007-5
12. R. I. Bot, E. R. Csetnek, An inertial alternating direction method of multipliers, *Minimax Theory Appl.*, **1** (2016), 29–49.
13. R. I. Bot, E. R. Csetnek, A hybrid proximal-extragradient algorithm with inertial effects, *Numer. Funct. Anal. Optim.*, **36** (2015), 951–963. https://doi.org/10.1080/01630563.2015.1042113
14. D. Butnariu, E. Resmerita, Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.*, **2006** (2006), 084919. https://doi.org/10.1155/AAA/2006/84919
15. C. Byrne, Iterative oblique projection onto convex sets and split feasibility, *Inverse Probl.*, **18** (2002), 441–453. https://doi.org/10.1088/0266-5611/18/2/310
16. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Probl.*, **20** (2004), 103–120. https://doi.org/10.1088/0266-5611/20/1/006
17. C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, *J. Nonlinear Convex Anal.*, **13** (2012), 759–775.
18. C. E. Chidume, A. Adamu, Solving split equality fixed point problem for quasi-phi-nonexpansive mappings, *Thai J. Math.*, **19** (2021), 1699–1717.
19. C. E. Chidume, A. Adamu, A new iterative algorithm for split feasibility and fixed point problem, *J. Nonlinear Var. Anal.*, **5** (2021), 201–210. https://doi.org/10.23952/jnva.5.2021.2.02
20. C. E. Chidume, A. A. Adam, A. Adamu, An algorithm for split equality fixed point problems for a class of quasi-phi-nonexpansive mappings in certain real Banach spaces, *Creat. Math. Inform.*, **32** (2023), 29–40. https://doi.org/10.37193/CMI.2023.01.04
21. C. E. Chidume, A. Adamu, P. Kumam, D. Kitkuan, Generalized hybrid viscosity-type forward-backward splitting method with application to convex minimization and image restoration problems, *Numer. Funct. Anal. Optim.*, **42** (2021), 1586–1607. https://doi.org/10.1080/01630563.2021.1933525
22. Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algor.*, **8** (1994), 221–239. https://doi.org/10.1007/BF02142692
23. Y. Censor, T. Elfving, N. Kopf, T. Bortfield, The multiple-set split feasibility problem and its application, *Inverse Probl.*, **21** (2005), 2071–2084. https://doi.org/10.1088/0266-5611/21/6/017
24. Y. Censor, A. Gibali, S. Reich, Algorithms for split variational inequality problems, *Numer. Algor.*, **59** (2012), 301–323. https://doi.org/10.1007/s11075-011-9490-5
25. Y. Censor, A. Segal, The split common fixed point problem for directed operators, *J. Convex Anal.*, **16** (2009), 587–600.
26. P. L. Combettes, L. E. Glaudin, Quasi-nonexpansive iterations on the affine hull of orbits: from Mann's mean value algorithm to inertial methods, *SIAM J. Optim.*, **27** (2017), 2356–2380. https://doi.org/10.1137/17M112806X

27. V. Dadashi, Shrinking projection algorithms for split common null point problem, *Bull. Aust. Math. Soc.*, **96** (2017), 299–306. https://doi.org/10.1017/S000497271700017X

28. P. Dechboon, A. Adamu, P. Kumam, A generalized Halpern-type forward-backward splitting algorithm for solving variational inclusion problems, *AIMS Mathematics*, **8** (2023), 11037–11056. https://doi.org/10.3934/math.2023559

29. J. Deepho, A. Adamu, A. H. Ibrahim, A. B. Abubakar, Relaxed viscosity-type iterative methods with application to compressed sensing, *J. Anal.*, 2023. https://doi.org/10.1007/s41478-022-00547-2

30. K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, New York: Cambridge University Press, 1990. https://doi.org/10.1017/CBO9780511526152

31. K. Goebel, S. Reich, *Uniform convexity, Hyperbolic Geometry, and Nonexpansive mappings*, New York: Marcel Dekker Inc, 1984.

32. J. Liang, Convergence rates of first-order operator splitting methods, PhD thesis, Normandie Université, 2016.

33. P. E. Maingé, Convergence theorems for inertial KM-type algorithms, *J. Comput. Appl. Math.*, **219** (2008), 223–236. https://doi.org/10.1016/j.cam.2007.07.021

34. P. E. Maingé, Regularized and inertial algorithms for common points of nonlinear operators, *J. Math. Anal. Appl.*, **34** (2008), 876–887. https://doi.org/10.1016/j.jmaa.2008.03.028

35. E. Masad, S. Reich, A note on the multiple-set split convex feasibility problem in Hilbert space, *J. Nonlinear Convex Anal.*, **8** (2007), 367–371.

36. F. U. Ogbuisi, The projection method with inertial extrapolation for solving split equilibrium problems in Hilbert spaces, *Appl. Set-Valued Anal. Optim.*, **3** (2021), 239–255. https://doi.org/10.23952/asvao.3.2021.2.08

37. F. U. Ogbuisi, O. S. Iyiola, J. M. T. Ngnotchouye, T. M. M. Shumba, On inertial type self-adaptive iterative algorithms for solving pseudomonotone equilibrium problems and fixed point problems, *J. Nonlinear Funct. Anal.*, **2021** (2021), 4. https://doi.org/10.23952/jnfa.2021.4

38. C. C. Okeke, A. U. Bello, L. O. Jolaoso, K. C. Ukandu, Inertial method for split null point problems with pseudomonotone variational inequality problems, *Numer. Algebra Control Optim.*, **12** (2022), 815–836. https://doi.org/10.3934/naco.2021037

39. C. C. Okeke, L. O. Jolaoso, R. Nwokoye, A Self-Adaptive Shrinking Projection Method with an Inertial Technique for Split Common Null Point Problems in Banach Spaces, *Axioms*, **9** (2020), 140. https://doi.org/10.3390/axioms9040140

40. C. C. Okeke, L. O. Jolaoso, Y. Shehu, Inertial accelerated algorithms for solving split feasibility with multiple output sets in Hilbert spaces, *Int. J. Nonlinear Sci. Numer. Simul.*, **24** (2023), 769–790. https://doi.org/10.1515/ijnsns-2021-0116

41. B. T. Polyak, *Introduction to optimization. Optimization Software*, New York: Publications Division, 1987.

42. C. Poon, J. Liang, Trajectory of Alternating Direction Method of Multiplier and Adaptive Acceleration, *Advances In Neural Information Processing Systems*, 2019.

43. C. Poon, J. Liang, Geometry of First-order Methods and Adaptive Acceleration, 2003. https://doi.org/10.48550/arXiv.2003.03910
44. S. Reich, T. M. Tuyen, Two new self-adaptive algorithms for solving the split common null point problem with multiple output sets in Hilbert spaces, *J. Fixed Point Theory Appl.*, **23** (2021), 16. https://doi.org/10.1007/s11784-021-00848-2
45. S. Reich, T. M. Tuyen, Iterative methods for solving the generalized split common null point problem in Hilbert spaces, *Optimization,* **69** (2020), 1013–1038. https://doi.org/10.1080/02331934.2019.1655562
46. S. Reich, T. M. Tuyen, The split feasibility problem with multiple output sets in Hilbert spaces, *Optim. Lett.*, **14** (2020), 2335–2353. https://doi.org/10.1007/s11590-020-01555-6
47. W. Takahashi, The split feasibility problem and the shrinking projection in Banach spaces, *J. Nonlinear Convex Anal.*, **16** (2015), 1449–1459.
48. W. Takahashi, The split common null point problem in Banach spaces, *Arch. Math. (Basel)*, **104** (2015), 357–365. https://doi.org/10.1007/s00013-015-0738-5
49. S. Takahashi, W. Takahashi, The split common null point problem and the shrinking projection method in Banach spaces, *Optimization* **65** (2016), 281–287. https://doi.org/10.1080/02331934.2015.1020943
50. D. V. Thong, D. V. Hieu, An inertial method for solving split common fixed point problems, *J. Fixed Point Theory Appl.*, **19** (2017), 3029–3051. https://doi.org/10.1007/s11784-017-0464-7
51. T. M. Tuyen, P. Sunthrayuth, N. M. Trang, An inertial self-adaptive algorithm for the generalized split common null point problem in Hilbert spaces, *Rend. Circ. Mat. Palermo, II. Ser.*, **71** (2022), 537–557. https://doi.org/10.1007/s12215-021-00640-8
52. Z. B. Wang, P. Sunthrayuth, A. Adamu, P. Cholamjiak, Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities, *Optimization*, 2023. https://doi.org/10.1080/02331934.2023.2187663
53. H. K. Xu, A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem, *Inverse Probl.*, **22** (2006), 2021–2034. https://doi.org/10.1088/0266-5611/22/6/007
54. H. K. Xu, Iterative methods for split feasibility problem in infinite dimensional Hilbert spaces, *Inverse Probl.*, **26** (2010), 105018. https://doi.org/10.1088/0266-5611/26/10/105018

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An intermixed algorithm for solving fixed point problems of proximal operators in Hilbert Spaces

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ABSTRACT. The aim of this paper is to modify proximal operators in Hilbert spaces. We introduce an intermixed algorithm with viscosity technique to find the solution of fixed point problem of two proximal operators in a real Hilbert space, utilizing the modified proximal operators. Under some mild conditions, a strong convergence theorem is established for the proposed algorithm. We also apply our main result to the split feasibility problem. Finally we provide numerical examples for supporting the main result.

1. INTRODUCTION

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$ and let $\Gamma_0(H)$ be a class of convex, lower semicontinuous, and proper functions from a Hilbert space H to $(-\infty, +\infty]$. Let C be closed convex subset of H. Let $S : C \to C$ be a nonlinear mapping, a point $x \in C$ is called a fixed point of S if $Sx = x$. We denote by $Fix(S)$, the set of all fixed points of S, i.e. $Fix(S) = \{x \in C : S(x) = x\}$. Consider the following convex minimization problem

$$\min_{x \in H}(f(x) + g(x)), \tag{1.1}$$

where $f \in \Gamma_0(H)$, $g : H \to \mathbb{R}$ is convex and differentiable with the Lipschitz continuous gradient denoted by ∇g. The solution set of (1.1) will be denoted by $\mathrm{Argmin}(f + g)$. In 2014, Xu [29] presented an important mathematical tool to demonstrate that the solution set of (1.1) is equivalent to the fixed point equation as follows:

$$\tilde{x} = \mathrm{Prox}_{\gamma f}(\tilde{x} - \gamma \nabla g(\tilde{x})), \tag{1.2}$$

where $\gamma > 0$ and $\mathrm{Prox}_{\gamma f}\, x := \mathrm{argmin}_{u \in H}\{f(u) + \frac{1}{2\gamma}\|u - x\|^2\}$ is the proximal mapping of f (see [2] for more informations on the proximal mapping). The most widely used algorithm for solving the convex minimization problem (1.1) is the so-called proximal-gradient algorithm. This proximal-gradient algorithm is given by: $x_1 \in H$ and

$$x_{n+1} = \mathrm{Prox}_{\gamma f}(I - \gamma \nabla g)(x_n), \quad \forall n \geq 1. \tag{1.3}$$

where Prox_f is the proximal operator of f, $\gamma \in (0, 2/L)$ and L is the Lipschitz constant of ∇g, then the sequence $\{x_n\}$ generated by algorithm (1.3) converges weakly to an element of $\mathrm{Argmin}(f + g)$ [see [2], Corollary 28.9]. This method is sometimes called the forward-backward algorithm. The proximal-gradient algorithm can be used in real-world applications, for example, in signal recovery, in image deblurring, and in machine learning (regression on highdimensional datasets) (see, [4], [20], [14], [22]). Recently there are extensive works in studying proximal gradient algorithm, see [19], [13], [28], [27], [10], [1], [25] and the references therein. For a set C, we denote by δ_C the indicator function of the set, that is, $\delta_C(x) = 0$ if $x \in C$ and ∞ otherwise. We denote the metric projection

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onto C as P_C. Clearly, by definition, $P_C x = \operatorname{Prox}_{\delta_C} x$. When $f = \delta_C$, the algorithm (1.3) becomes the popular gradient projection algorithm, which is defined as follows. For an initial guess $x_1 \in H$,

$$x_{n+1} = P_C(I - \gamma\nabla g)(x_n), \quad \forall n \geq 1. \tag{1.4}$$

Recently, Guo and Cui [8] modified the proximal-gradient algorithm with viscosity technique as follows: For arbitrarily given $x_1 \in H$, let the sequences $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)\operatorname{Prox}_{\mu_n f}(I - \mu_n \nabla g)(x_n) + e(x_n), \quad \forall n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a real number sequence in $[0,1]$, $0 < a = \inf_n \mu_n \leq \mu_n < \frac{2}{L}$, $h : H \to H$ a ρ-contractive operator with $\rho \in (0,1)$, and $e : H \to H$ represents a perturbation operator and satisfies $\sum_{n=1}^{\infty} \|e(x_n)\| < +\infty$. Under some appropriate conditions, they proved that the algorithm (1.5) converges strongly to a solution of (1.1) in a real Hibert space.

Currently, one of the best methods for solving the fixed point problem of nonlinear mapping is to use the intermixed algorithm, proposed by Yao et al.[32]. This algorithm has the following features: the definition of the sequence $\{x_n\}$ is involved in the sequence $\{y_n\}$ and the definition of the sequence $\{y_n\}$ is also involved in the sequence $\{x_n\}$. In recent years, the intermixed algorithm has attracted a significant amount of interest from authors, who improved it in various ways (see, e.g., [26], [23], [11]). In particular, Yao et al.[32] introduced the intermixed algorithm for two strict pseudo-contractions as follows: For arbitrarily given $x_1 \in C$, $y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1-\delta_n)x_n + \delta_n P_C[\alpha_n h_1(y_n) + (1-k-\alpha_n)x_n + k\mathcal{T}x_n], & \forall n \geq 1, \\ y_{n+1} = (1-\delta_n)y_n + \delta_n P_C[\alpha_n h_2(x_n) + (1-k-\alpha_n)y_n + k\mathcal{S}y_n], & \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\delta_n\}$ are two real number sequences in $(0,1)$, $\mathcal{S}, \mathcal{T} : C \to C$ are λ-strictly pseudo-contractions with $k \in (0, 1-\lambda)$, and $h_1, h_2 : C \to H$ are ρ_1, ρ_2-contractions, respectively. Moreover, Yao et al. also proved in [32] that the sequence $\{x_n\}$ generated by (1.6) weakly converges to a fixed point of two strict pseudo-contractions under some appropriate conditions.

The Krasnoselskii-Mann algorithm (see, [15], [16], [21]) is one of the most well-known fixed point algorithms. In recent years, several researchers have increasingly investigated the Krasnoselskii-Mann algorithm in various directions, for example [24], [7], [33], [9] and the references therein. In particular, Kanzow and Shehu [12] proposed the following inexact Krasnoselskii–Mann algorithm for finding a fixed point of a nonexpansive mapping T in a real Hilbert space: For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence generated iteratively by

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + r_n, \quad \forall n \geq 1, \tag{1.7}$$

where $T : H \to C$ is a nonexpansive mapping, r_n denotes the residual vector and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $[0,1]$ such that $\alpha_n + \beta_n \leq 1$. They proved that if $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, $\sum_{n=1}^{\infty} \|r_n\| < \infty$, and $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$, then the sequence $\{x_n\}$ generated by (1.7) converges weakly to a fixed point of T.

In this paper, based on the problems (1.1) and (1.2), we modify a proximal operator and introduce a new mathematical tool relevant to the modified proximal operator in Hilbert spaces. Inspired and motivated by previous works, we introduce an intermixed algorithm with viscosity technique to find the solution of fixed point problem of two proximal operators in a real Hilbert space. Using the mathematical tool above, a strong convergence theorem for the proposed algorithm is established under some mild conditions. Applications to the split feasibility problems are also considered. Finally, we provide some numerical experiments to verify the theoretical results of this paper. In summary,

- Applying the convex minimization problem (1.1) and the fixed point equation (1.2), we propose a new mathematical tool related to two proximal operators;
- We propose an intermixed algorithm to find the solution of fixed point problem of two proximal operators in a real Hilbert space and prove a strong convergence theorem for the proposed algorithm under some mild conditions;
- Our algorithm combine the proximal-gradient algorithm with viscosity technique in Guo and Cui [8], the intermixed algorithm in Yao et al.[32] and the Krasnoselskii–Mann algorithm in Kanzow and Shehu [12].

The organization of our paper is as follows: In section 2, we first recall some basic definitions and lemmas. We also give a new lemma related to two proximal operators (see Lemma 2.4 below). In section 3, we prove the strong convergence theorem of our proposed algorithm under some mild conditions. In section 4, we consider the application of our main result to solve the split feasibility problems. In section 5, we provide numerical examples to support the main result.

2. Preliminary

For the purpose of proving our theorem, we provide several definitions and lemmas in this section. For convenience, the following notations are used throughout the paper:

- H denotes a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$;
- C denotes a nonempty closed convex subset of H;
- $\Gamma_0(H)$ denotes a class of convex, lower semicontinuous, and proper functions from a Hilbert space H to $(-\infty, +\infty]$;
- $x_n \to q$ ($x_n \rightharpoonup q$) denote the strong (weak) convergence of a sequence $\{x_n\}$ to q in H, respectively;
- $Fix(S)$ denotes the set of all fixed points of S.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.1. [18] *For given $x \in H$ and let $P_C : H \to C$ be a metric projection. Then*
(a) $w = P_C x$ *if and only if* $\langle x - w, y - w \rangle \leq 0, \quad \forall y \in C;$
(b) $w = P_C x$ *if and only if* $\|x - w\|^2 \leq \|x - y\|^2 - \|y - w\|^2, \quad \forall y \in C;$
(c) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$

Definition 2.1. A mapping $S : C \to C$ is called nonexpansive, if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

Definition 2.2. A mapping $A : C \to H$ is called

(i) Monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-Strongly monotone if there exists a positive real number η such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C;$$

(iii) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \quad \forall x, y \in C;$$

(iv) α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Clearly, If A is η-Strongly monotone and L-Lipschitz continuous, then A is $\frac{\eta}{L^2}$-inverse strongly monotone. If A is an α-inverse strongly monotone mapping, then $\frac{1}{\alpha}$-Lipschitz continuous.

Lemma 2.2. [12] *Let X be a real inner product space.Then*

(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X;$

(b) $\|px + qy\|^2 = p(p + q)\|x\|^2 + q(p + q)\|y\|^2 - pq\|x - y\|^2, \quad \forall x, y \in X, \forall p, q \in \mathbb{R}.$

The following Lemma, which comes from [17], [30], will be used to prove our strong convergence result.

Lemma 2.3. [17], [30] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n + \mu_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} |\mu_n| < \infty$. Then, the following results hold:

(i) If $\delta_n \leq \alpha_n M$ for some $M \geq 0$ then $\{a_n\}$ is a bounded sequence;

(ii) If $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup\limits_{n\to\infty} \dfrac{\delta_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Let the function $f \in \Gamma_0(H)$. The set

$$\partial f(x) = \{z \in H : \langle z, y - x\rangle + f(x) \leq f(y),\ \forall y \in H\}$$

is called a *subdifferential* of f at $x \in H$. The function f is said to be *subdifferentiable* at x if $\partial f(x) \neq \emptyset$. The elements of $\partial f(x)$ are called the *subgradients* of f at x. If the function f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$; this is the gradient of f. It is well known that the subdifferential ∂f is a maximal monotone operator. It is notable that a point $x^* \in H$ minimizes f if and only if $0 \in \partial f(x^*)$. Let x and p be in H. The proximity operator of f is characterized by the inclusion

$$p = \operatorname{Prox}_{\gamma f} x \Leftrightarrow x - p \in \gamma\partial f(p).$$

In other words,

$$\operatorname{Prox}_{\gamma f} = (I + \gamma\partial f)^{-1}.$$

Moreover, the proximity operator of f is firmly nonexpansive, namely,

$$\langle \operatorname{Prox}_{\gamma f}(x) - \operatorname{Prox}_{\gamma f}(y), x - y\rangle \geq \|\operatorname{Prox}_{\gamma f}(x) - \operatorname{Prox}_{\gamma f}(y)\|^2 \tag{2.8}$$

for all $x, y \in H$, which is equivalent to

$$\|\operatorname{Prox}_{\gamma f}(x) - \operatorname{Prox}_{\gamma f}(y)\|^2 \leq \|x - y\|^2 - \|(I - \operatorname{Prox}_{\gamma f})(x) - (I - \operatorname{Prox}_{\gamma f})(y)\|^2 \tag{2.9}$$

for all $x, y \in H$. For general information on proximal operator, see Combettes and Pesquet [3].

Proposition 2.1. [2] *Let the function $f \in \Gamma_0(H)$ and let $x, p \in H$. Then*

$$p = Prox_f x \Leftrightarrow \langle y - p, x - p\rangle + f(p) \leq f(y),$$

for all $y \in H$.

Lemma 2.4. *Let the function $f \in \Gamma_0(H)$. Let $A, B : C \to H$ be δ^A and δ^B- inverse strongly monotone operators, respectively, with $\delta = \min\{\delta^A, \delta^B\}$ and $Fix(Prox_{\gamma f}(I - \gamma A)) \cap Fix(Prox_{\gamma f}(I - \gamma B)) \neq \emptyset$. Then*

$$Fix(Prox_{\gamma f}(I-\gamma A))\cap Fix(Prox_{\gamma f}(I-\gamma B)) = Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B))), \tag{2.10}$$

for all $a \in (0,1)$ and $\gamma \in (0, 2\delta)$.

Proof. From Proposition 2.1, it is easy to see that

$$Fix(Prox_{\gamma f}(I-\gamma A))\cap Fix(Prox_{\gamma f}(I-\gamma B)) \subseteq Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B))). \tag{2.11}$$

Let $x_0 \in Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B)))$ and $x^* \in Fix(Prox_{\gamma f}(I-\gamma A))\cap Fix(Prox_{\gamma f}(I-\gamma B))$. So, we have

$$x^* \in Fix(Prox_{\gamma f}(I - \gamma(aA + (1-a)B))).$$

By the definitions of A, B, we have

$$\begin{aligned}
\|x_0 - x^*\|^2 =& \|Prox_{\gamma f}(I - \gamma(aA + (1-a)B))x_0 - Prox_{\gamma f}(I - \gamma(aA + (1-a)B))x^*\|^2 \\
\leq& \|x_0 - x^* - \gamma(a(Ax_0 - Ax^*) + (1-a)(Bx_0 - Bx^*))\|^2 \\
=& \|x_0 - x^*\|^2 - 2\gamma\langle a(Ax_0 - Ax^*) + (1-a)(Bx_0 - Bx^*), x_0 - x^*\rangle \\
&+ \gamma^2\|a(Ax_0 - Ax^*) + (1-a)(Bx_0 - Bx^*)\|^2 \\
\leq& \|x_0 - x^*\|^2 - 2\gamma a\langle Ax_0 - Ax^*, x_0 - x^*\rangle - 2\gamma(1-a)\langle Bx_0 - Bx^*, x_0 - x^*\rangle \\
&+ \gamma^2\left(a\|Ax_0 - Ax^*\|^2 + (1-a)\|Bx_0 - Bx^*\|^2\right) \\
\leq& \|x_0 - x^*\|^2 - \gamma a(2\delta - \gamma)\|Ax_0 - Ax^*\|^2 - \gamma(1-a)(2\delta - \gamma)\|Bx_0 - Bx^*\|^2. \qquad (2.12)
\end{aligned}$$

It implies that

$$Ax_0 = Ax^*, Bx_0 = Bx^*.$$

Let $y \in H$. By applying Proposition 2.1 and $x^* \in Fix(Prox_{\gamma f}(I - \gamma A))$, we obtain

$$\langle y - x^*, (I - \gamma A)x^* - x^*\rangle + \gamma f(x^*) \leq \gamma f(y),$$

which implies that

$$f(y) - f(x^*) + \langle y - x^*, Ax^*\rangle \geq 0. \tag{2.13}$$

Since $x^* \in Fix(Prox_{\gamma f}(I - \gamma B))$, we also get

$$f(y) - f(x^*) + \langle y - x^*, Bx^*\rangle \geq 0. \tag{2.14}$$

Since $x_0 \in Fix(Prox_{\gamma f}(I - \gamma(aA + (1-a)B))$ and by Proposition 2.1, we get

$$\langle y - x_0, (I - \gamma(aA + (1-a)B))x_0 - x_0\rangle + \gamma f(x_0) \leq \gamma f(y),$$

which implies that

$$f(y) - f(x_0) + \langle y - x_0, (aA + (1-a)B)x_0\rangle \geq 0. \tag{2.15}$$

From (2.13) and $Ax_0 = Ax^*$, we obtain

$$\begin{aligned}
\langle y - x_0, Ax_0\rangle + f(y) - f(x_0) &= \langle y - x^*, Ax^*\rangle + \langle x^* - x_0, Ax_0\rangle + f(y) - f(x^*) \\
&\quad + f(x^*) - f(x_0) \\
&= \langle y - x^*, Ax^*\rangle + f(y) - f(x^*) + \langle x^* - x_0, Ax_0\rangle \\
&\quad + f(x^*) - f(x_0) \\
&\geq \langle x^* - x_0, Ax_0\rangle + f(x^*) - f(x_0). \qquad (2.16)
\end{aligned}$$

From $Bx_0 = Bx^*$, (2.14), and (2.15), we get

$$\begin{aligned}\langle x^* - x_0, aAx_0\rangle + &af(x^*) - af(x_0)\\ &= \langle x^* - x_0, aAx_0 + (1-a)Bx_0\rangle - \langle x^* - x_0, (1-a)Bx_0\rangle\\ &\quad + af(x^*) - af(x_0)\\ &= \langle x^* - x_0, aAx_0 + (1-a)Bx_0\rangle + f(x^*) - f(x_0)\\ &\quad - f(x^*) + f(x_0) - \langle x^* - x_0, (1-a)Bx_0\rangle\\ &\quad + af(x^*) - af(x_0)\\ &\geq \langle x_0 - x^*, (1-a)Bx^*\rangle + (1-a)f(x_0) - (1-a)f(x^*)\\ &= (1-a)\left(\langle x_0 - x^*, Bx^*\rangle + f(x_0) - f(x^*)\right)\\ &\geq 0.\end{aligned}$$

Since $a \in (0,1)$, we have

$$\langle x^* - x_0, Ax_0\rangle + f(x^*) - f(x_0) \geq 0. \tag{2.17}$$

From (2.16) and (2.17), we obtain

$$\langle y - x_0, Ax_0\rangle + f(y) - f(x_0) \geq 0, \forall y \in H. \tag{2.18}$$

It follows from (2.18) and Proposition 2.1,

$$\langle y - x_0, x_0 - (I - \gamma A)x_0\rangle + \gamma f(y) - \gamma f(x_0) \geq 0, \forall y \in H.$$

It implies that

$$x_0 \in Fix(Prox_{\gamma f}(I - \gamma A)). \tag{2.19}$$

Using the same method as (2.19), we also have

$$x_0 \in Fix(Prox_{\gamma f}(I - \gamma B)).$$

So, we can conclude that

$$Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B))) \subseteq Fix(Prox_{\gamma f}(I-\gamma A)) \cap Fix(Prox_{\gamma f}(I-\gamma B)). \tag{2.20}$$

From (2.11) and (2.20), we deduce that

$$Fix(Prox_{\gamma f}(I-\gamma A)) \cap Fix(Prox_{\gamma f}(I-\gamma B)) = Fix(Prox_{\gamma f}(I-\gamma(aA+(1-a)B))). \tag{2.21}$$

□

Lemma 2.5. [31] *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

3. Main results

In this section, we introduce an intermixed algorithm and prove a strong convergence of the proposed algorithm to find the solution of fixed point problem of two proximal operators.

For every $i = 1, 2$, let the functions $f_i \in \Gamma_0(H)$, let $A_i, B_i : C \to H$ be δ_i^A and δ_i^B-inverse strongly monotone operators, respectively, with $\delta_i = \min\{\delta_i^A, \delta_i^B\}$. Assume that $\Omega_i = Fix(Prox^i_{\gamma f}(I - \gamma_i A_i)) \cap Fix(Prox^i_{\gamma f}(I - \gamma_i B_i)) \neq \emptyset$, for all $i = 1, 2$. Let $\xi_1, \xi_2 : H \to H$ be σ_1 and σ_2-contraction mappings with $\sigma_1, \sigma_2 \in (0, 1)$ and $\sigma = \max\{\sigma_1, \sigma_2\}$.

Now, we introduce an intermixed algorithm with viscosity technique for solving a common fixed point of proximal operators as follows:

Algorithm 1: An intermixed algorithm with viscosity technique for solving a common fixed point of two proximal operators.

Initialization: Given $x_1, y_1 \in C$ be arbitrary.
Iterative Steps: Given the current iterate $\{x_n\}$, $\{y_n\}$, calculate $\{x_{n+1}\}$, $\{y_{n+1}\}$ as follows:
Step 1: Compute

$$\begin{cases} v_n = Prox^2_{\gamma f}(y_n - \gamma_2(a_2 A_2 + (1-a_2)B_2)y_n) \\ u_n = Prox^1_{\gamma f}(x_n - \gamma_1(a_1 A_1 + (1-a_1)B_1)x_n) \end{cases}$$

Step 2: Compute

$$\begin{cases} y_{n+1} = \mu_n y_n + \beta_n P_C(\alpha_n \xi_2(x_n) + (1-\alpha_n)v_n) \\ x_{n+1} = \mu_n x_n + \beta_n P_C(\alpha_n \xi_1(y_n) + (1-\alpha_n)u_n), \end{cases}$$

where $\{\mu_n\}, \{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\mu_n + \beta_n \leq 1$, $\gamma_i \in (0, 2\delta_i)$, $a_i \in (0,1)$ and $Prox^i_{\gamma f} : H \to H$ is the proximity operator, for all $i = 1, 2$.
Set $n := n+1$ and go to Step 1.

Theorem 3.1. *For every $i = 1, 2$, let the functions $f_i \in \Gamma_0(H)$, let $A_i, B_i : C \to H$ be δ_i^A and δ_i^B- inverse strongly monotone operators, respectively, with $\delta_i = \min\{\delta_i^A, \delta_i^B\}$. Assume that $\Omega_i = Fix(Prox^i_{\gamma f}(I - \gamma_i A_i)) \cap Fix(Prox^i_{\gamma f}(I - \gamma_i B_i)) \neq \emptyset$, for all $i = 1, 2$. Let $\xi_1, \xi_2 : H \to H$ be σ_1 and σ_2-contraction mappings with $\sigma_1, \sigma_2 \in (0,1)$ and $\sigma = \max\{\sigma_1, \sigma_2\}$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 1, satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) *there are* $\overline{\varepsilon}, l > 0$ *with* $0 < \overline{\varepsilon} \leq \mu_n, \beta_n \leq l < 1$ *for all* $n \in \mathbb{N}$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} (1 - \mu_n - \beta_n) < \infty.$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{\Omega_1}\xi_1(y^*)$ and $y^* = P_{\Omega_2}\xi_2(x^*)$, respectively.

Proof. Putting $K_i = \mathrm{Prox}^i_{\gamma f}(I - \gamma_i(a_i A_i + (1-a_i)B_i)))$ for all $i = 1, 2$. First, we will show that K_i is nonexpansive mapping for all $i = 1, 2$. For every $i = 1, 2$, let $x^*, x_0 \in C$. Using the same method as (2.12), we have

$$\begin{aligned} \|K_i x^* - K_i x_0\| =& \| \mathrm{Prox}^i_{\gamma f}(I - \gamma_i(a_i A_i + (1-a_i)B_i)))x^* \\ & - \mathrm{Prox}^i_{\gamma f}(I - \gamma_i(a_i A_i + (1-a_i)B_i)))x_0\|^2 \\ \leq& \|x^* - x_0\|^2 + a_i\gamma_i(\gamma_i - 2\delta_i)\|A_i x^* - A_i x_0\|^2 \\ & + (1-a_i)\gamma_i(\gamma_i - 2\delta_i)\|B_i x^* - B_i x_0\|^2 \\ \leq& \|x^* - x_0\|^2. \end{aligned} \tag{3.22}$$

Thus, $\mathrm{Prox}^i_{\gamma f}(I - \gamma_i(a_i A_i + (1-a_i)B_i)))$ is nonexpansive mapping for all $i = 1, 2$. Assume that $x^* \in \Omega_1$ and $y^* \in \Omega_2$.

From the definition of x_n, u_n and the nonexpansiveness of K_i, we have

$$
\begin{aligned}
\|x_{n+1}-x^*\| =&\|\mu_n x_n + \beta_n P_C(\alpha_n \xi_1(y_n) + (1-\alpha_n)u_n) - x^*\| \\
=&\|\mu_n(x_n - x^*) + \beta_n(P_C(\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n) - x^*) \\
&- (1-\mu_n-\beta_n)x^*\| \\
\leq&\mu_n\|x_n - x^*\| + \beta_n\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
\leq&\mu_n\|x_n - x^*\| + \beta_n(\alpha_n\|\xi_1(y_n) - x^*\| + (1-\alpha_n)\|u_n - x^*\|) \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
=&\mu_n\|x_n - x^*\| + \beta_n(\alpha_n\|\xi_1(y_n) - x^*\| + (1-\alpha_n)\|K_1x_n - x^*\|) \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
\leq&\mu_n\|x_n - x^*\| + \beta_n(\alpha_n\|\xi_1(y_n) - x^*\| + (1-\alpha_n)\|x_n - x^*\|) \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
\leq&(1-\beta_n)\|x_n - x^*\| + \beta_n\alpha_n\|\xi_1(y_n) - x^*\| + \beta_n(1-\alpha_n)\|x_n - x^*\| \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
=&(1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\|\xi_1(y_n) - x^*\| \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
\leq&(1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n(\|\xi_1(y_n) - \xi_1(y^*)\| + \|\xi_1(y^*) - x^*\|) \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
\leq&(1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\sigma_1\|y_n - y^*\| + \alpha_n\beta_n\|\xi_1(y^*) - x^*\| \\
&+ (1-\mu_n-\beta_n)\|x^*\| \\
\leq&(1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\sigma\|y_n - y^*\| + \alpha_n\beta_n\|\xi_1(y^*) - x^*\| \\
&+ (1-\mu_n-\beta_n)\|x^*\|. \qquad (3.23)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\|y_{n+1}-y^*\| \leq&(1-\alpha_n\beta_n)\|y_n - y^*\| + \alpha_n\beta_n\sigma\|x_n - x^*\| + \alpha_n\beta_n\|\xi_2(x^*) - y^*\| \\
&+ (1-\mu_n-\beta_n)\|y^*\|. \qquad (3.24)
\end{aligned}
$$

Combining (3.23) and (3.24), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\| + \|y_{n+1}-y^*\| \leq&(1-\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\sigma\|y_n - y^*\| \\
&+ \alpha_n\beta_n\|\xi_1(y^*) - x^*\| + (1-\mu_n-\beta_n)\|x^*\| \\
&+ (1-\alpha_n\beta_n)\|y_n - y^*\| + \alpha_n\beta_n\sigma\|x_n - x^*\| \\
&+ \alpha_n\beta_n\|\xi_2(x^*) - y^*\| + (1-\mu_n-\beta_n)\|y^*\| \\
=&(1-\alpha_n\beta_n)\left(\|x_n - x^*\| + \|y_n - y^*\|\right) \\
&+ \alpha_n\beta_n\sigma\left(\|x_n - x^*\| + \|y_n - y^*\|\right) \\
&+ \alpha_n\beta_n(\|\xi_1(y^*) - x^*\| + \|\xi_2(x^*) - y^*\|) \\
&+ (1-\mu_n-\beta_n)(\|x^*\| + \|y^*\|)
\end{aligned}
$$

$$
\begin{aligned}
&=(1-\alpha_n\beta_n(1-\sigma))(\|x_n-x^*\|+\|y_n-y^*\|)\\
&\quad+\alpha_n\beta_n(\|\xi_1(y^*)-x^*\|+\|\xi_2(x^*)-y^*\|)\\
&\quad+(1-\mu_n-\beta_n)(\|x^*\|+\|y^*\|)\\
&=(1-\alpha_n\beta_n(1-\sigma))(\|x_n-x^*\|+\|y_n-y^*\|)\\
&\quad+\alpha_n\beta_n(1-\sigma)\left(\frac{\|\xi_1(y^*)-x^*\|+\|\xi_2(x^*)-y^*\|}{1-\sigma}\right)\\
&\quad+(1-\mu_n-\beta_n)(\|x^*\|+\|y^*\|).
\end{aligned}
$$

By Lemma 2.3, we get that $\{x_n\}$ and $\{y_n\}$ are bounded. Next, we will show that $\{u_n\}$, $\{v_n\}$, $\{\xi_1(y_n)\}$, and $\{\xi_2(x_n)\}$ are bounded. From the definition of u_n and the nonexpansiveness of K_1, we have

$$
\begin{aligned}
\|u_n-x^*\| &=\|K_1x_n-x^*\|\\
&\leq\|x_n-x^*\|.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, then $\{u_n\}$ is bounded. Using the same method, we establish that the sequence v_n is bounded. Observe that

$$
\begin{aligned}
\|\xi_1(y_n)-x^*\| &\leq\|\xi_1(y_n)-\xi_1(x^*)\|+\|\xi_1(x^*)-x^*\|\\
&\leq\sigma_1\|x_n-x^*\|+\|\xi_1(x^*)-x^*\|.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, then $\{\xi_1(y_n)\}$ is bounded. Using the same method, we show that the sequence $\{\xi_2(x_n)\}$ is bounded.

Setting $T_n=P_C(\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n)$ and $T_n^*=P_C(\alpha_n\xi_2(x_n)+(1-\alpha_n)v_n)$. We will show that $\{T_n\}$ and $\{T_n^*\}$ are bounded. Observe that

$$
\begin{aligned}
\|T_n-x^*\| &=\|P_C(\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n)-x^*\|\\
&\leq\|\alpha_n\xi_1(y_n)+(1-\alpha_n)u_n-x^*\|\\
&\leq\alpha_n\|\xi_1(y_n)-x^*\|+(1-\alpha_n)\|u_n-x^*\|\\
&\leq\alpha_n\|\xi_1(y_n)-\xi_1(x^*)\|+\alpha_n\|\xi_1(x^*)-x^*\|+(1-\alpha_n)\|u_n-x^*\|\\
&\leq\alpha_n\sigma_1\|x_n-x^*\|+\alpha_n\|\xi_1(x^*)-x^*\|+(1-\alpha_n)\|x_n-x^*\|\\
&=(1-\alpha_n(1-\sigma_1))\|x_n-x^*\|+\alpha_n\|\xi_1(x^*)-x^*\|.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, then $\{T_n\}$ is bounded. Using the same method, we show that the sequence $\{T_n^*\}$ is bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.
By the nonexpansiveness of K_i, we have

$$\begin{aligned}
\|T_n - T_{n-1}\| =&\|P_C(\alpha_n \xi_1(y_n) + (1-\alpha_n)u_n) - P_C(\alpha_{n-1}\xi_1(y_{n-1}) + (1-\alpha_{n-1})u_{n-1})\| \\
\leq&\|(\alpha_n \xi_1(y_n) + (1-\alpha_n)u_n) - (\alpha_{n-1}\xi_1(y_{n-1}) + (1-\alpha_{n-1})u_{n-1})\| \\
=&\|\alpha_n \xi_1(y_n) - \alpha_n \xi_1(y_{n-1}) + \alpha_n \xi_1(y_{n-1}) + (1-\alpha_n)u_n - (1-\alpha_n)u_{n-1} \\
&+ (1-\alpha_n)u_{n-1} - \alpha_{n-1}\xi_1(y_{n-1}) - (1-\alpha_{n-1})u_{n-1}\| \\
=&\|\alpha_n(\xi_1(y_n) - \xi_1(y_{n-1})) + (\alpha_n - \alpha_{n-1})\xi_1(y_{n-1}) + (1-\alpha_n)(u_n - u_{n-1}) \\
&+ (\alpha_{n-1} - \alpha_n)u_{n-1}\| \\
\leq&\alpha_n\|\xi_1(y_n) - \xi_1(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|\xi_1(y_{n-1})\| \\
&+ (1-\alpha_n)\|K_1 x_n - K_1 x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u_{n-1}\| \\
\leq&\alpha_n \sigma_1\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\xi_1(y_{n-1})\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\|u_{n-1}\| \\
\leq&\alpha_n \sigma\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\xi_1(y_{n-1})\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\|u_{n-1}\|.
\end{aligned} \tag{3.25}$$

From the definition of x_n and (3.25), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| =&\|\mu_n x_n + \beta_n T_n - (\mu_{n-1}x_{n-1} + \beta_{n-1}T_{n-1})\| \\
\leq&\mu_n\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|x_{n-1}\| \\
&+ \beta_n\|T_n - T_{n-1}\| + |\beta_n - \beta_{n-1}|\|T_{n-1}\| \\
\leq&\mu_n\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|x_{n-1}\| \\
&+ \beta_n\Big(\alpha_n\sigma\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\xi_1(y_{n-1})\| \\
&+ (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u_{n-1}\|\Big) \\
&+ |\beta_n - \beta_{n-1}|\|T_{n-1}\| \\
\leq&(1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|x_{n-1}\| \\
&+ \alpha_n\beta_n\sigma\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\Big(\|\xi_1(y_{n-1})\| + \|u_{n-1}\|\Big) \\
&+ |\beta_n - \beta_{n-1}|\|T_{n-1}\|.
\end{aligned} \tag{3.26}$$

Using the same method as derived in (3.26), we have

$$\begin{aligned}
\|y_{n+1} - y_n\| \leq&(1-\alpha_n\beta_n)\|y_n - y_{n-1}\| + |\mu_n - \mu_{n-1}|\|y_{n-1}\| + \alpha_n\beta_n\sigma\|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\Big(\|\xi_2(x_{n-1})\| + \|v_{n-1}\|\Big) + |\beta_n - \beta_{n-1}|\|T^*_{n-1}\|.
\end{aligned} \tag{3.27}$$

From (3.26) and (3.27), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq&(1-(1-\sigma)\beta_n\alpha_n)\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right) \\
&+ |\mu_n - \mu_{n-1}|\left(\|x_{n-1}\| + \|y_{n-1}\|\right) \\
&+ |\beta_n - \beta_{n-1}|\left(\|T_{n-1}\| + \|T^*_{n-1}\|\right) \\
&+ |\alpha_n - \alpha_{n-1}|\big(\|\xi_1(y_{n-1})\| + \|u_{n-1}\| \\
&+ \|\xi_2(x_{n-1})\| + \|v_{n-1}\|\big).
\end{aligned}$$

Applying Lemma 2.3 and the condition (iii), we can conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0 \text{ and } \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0. \tag{3.28}$$

Next, we show that $\|x_n - U_n\| \to 0$ as $n \to \infty$ where $U_n = \alpha_n \xi_1(y_n) + (1-\alpha_n)u_n$, $\|y_n - V_n\| \to 0$ where $V_n = \alpha_n \xi_2(x_n) + (1-\alpha_n)v_n$ as $n \to \infty$.
From the definition of x_n, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|\mu_n x_n + \beta_n P_C U_n - x^*\|^2 \\
=& \|\mu_n(x_n - x^*) + \beta_n(P_C U_n - x^*) - (1-\mu_n-\beta_n)x^*\|^2 \\
\leq& \|\mu_n(x_n - x^*) + \beta_n(P_C U_n - x^*)\|^2 \\
& - 2(1-\mu_n-\beta_n)\langle x^*, x_{n+1} - x^*\rangle \\
\leq& \|\mu_n(x_n - x^*) + \beta_n(P_C U_n - x^*)\|^2 \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\|P_C U_n - x^*\|^2 \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\|U_n - x^*\|^2 \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 \\
& + \beta_n(\mu_n+\beta_n)\|\alpha_n(\xi_1(y_n) - u_n) + (u_n - x^*)\|^2 \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big(\|u_n - x^*\|^2 \\
& + 2\alpha_n\langle \xi_1(y_n) - u_n, \alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\rangle\Big) \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big(\|u_n - x^*\|^2 \\
& + 2\alpha_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\|\Big) \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \|x_n - x^*\|^2 + 2\alpha_n\beta_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
& - \mu_n\beta_n\|x_n - P_C U_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|. \qquad (3.29)
\end{aligned}$$

It follows from (3.29) that

$$\begin{aligned}
\mu_n\beta_n\|x_n - P_C U_n\|^2 \leq& \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& + 2\alpha_n\beta_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
& + 2\alpha_n\beta_n\|\xi_1(y_n) - u_n\|\|\alpha_n\xi_1(y_n) + (1-\alpha_n)u_n - x^*\| \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|.
\end{aligned}$$

By (3.28), the condition (i)- (iii), we get

$$\lim_{n\to\infty}\|P_C U_n - x_n\| = 0. \qquad (3.30)$$

From definition of y_n and applying the same method as (3.30), we have

$$\lim_{n\to\infty}\|P_C V_n - y_n\| = 0. \qquad (3.31)$$

From Lemma 2.1, we obtain

$$\|P_CU_n - x^*\|^2 \leq \|U_n - x^*\|^2 - \|U_n - P_CU_n\|^2. \tag{3.32}$$

From the definition of U_n, we get

$$\begin{aligned}\|U_n - x^*\|^2 =&\|\alpha_n(\xi_1(y_n) - x^*) + (1-\alpha_n)(u_n - x^*)\|^2\\ \leq&\alpha_n\|\xi_1(y_n) - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2\\ \leq&\alpha_n\|\xi_1(y_n) - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2.\end{aligned} \tag{3.33}$$

From (3.32), (3.33), and Lemma 2.2, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 =&\|\mu_n x_n + \beta_n P_CU_n - x^*\|^2\\ \leq&\mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\|P_CU_n - x^*\|^2\\ &- \mu_n\beta_n\|x_n - P_CU_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|\\ \leq&\mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)(\|U_n - x^*\|^2 - \|U_n - P_CU_n\|^2)\\ &- \mu_n\beta_n\|x_n - P_CU_n\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|\\ \leq&\mu_n(\mu_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)(\alpha_n\|\xi_1(y_n) - x^*\|^2\\ &+ (1-\alpha_n)\|x_n - x^*\|^2 - \|U_n - P_CU_n\|^2) - \mu_n\beta_n\|x_n - P_CU_n\|^2\\ &+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|\\ =&\mu_n\|x_n - x^*\|^2 + \beta_n\alpha_n\|\xi_1(y_n) - x^*\|^2 + \beta_n(1-\alpha_n)\|x_n - x^*\|^2\\ &- \beta_n(\mu_n+\beta_n)\|U_n - P_CU_n\|^2 - \mu_n\beta_n\|x_n - P_CU_n\|^2\\ &+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|\\ =&(\mu_n + \beta_n(1-\alpha_n))\|x_n - x^*\|^2 + \beta_n\alpha_n\|\xi_1(y_n) - x^*\|^2\\ &- \beta_n(\mu_n+\beta_n)\|U_n - P_CU_n\|^2 - \mu_n\beta_n\|x_n - P_CU_n\|^2\\ &+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|\\ \leq&\|x_n - x^*\|^2 + \beta_n\alpha_n\|\xi_1(y_n) - x^*\|^2 - \beta_n(\mu_n+\beta_n)\|U_n - P_CU_n\|^2\\ &+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|,\end{aligned}$$

it follows that

$$\begin{aligned}\beta_n(\mu_n+\beta_n)\|U_n - P_CU_n\|^2 \leq&\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\beta_n\|\xi_1(y_n) - x^*\|^2\\ &+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|\\ \leq&\|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|)\\ &+ \alpha_n\beta_n\|\xi_1(y_n) - x^*\|^2\\ &+ 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|.\end{aligned}$$

From $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and the condition (i), (ii), we have

$$\lim_{n\to\infty}\|U_n - P_CU_n\| = 0. \tag{3.34}$$

Applying the same argument as (3.34) to the definition of V_n also yields

$$\lim_{n\to\infty}\|V_n - P_CV_n\| = 0. \tag{3.35}$$

Consider

$$\begin{aligned}\|x_n - U_n\| =& \|x_n - P_C U_n + P_C U_n - U_n\|\\ \le& \|x_n - P_C U_n\| + \|P_C U_n - U_n\|.\end{aligned}$$

From (3.30) and (3.34), we have

$$\lim_{n\to\infty} \|x_n - U_n\| = 0. \tag{3.36}$$

Using the same methodology as (3.36) and the definition of y_n, we also have

$$\lim_{n\to\infty} \|y_n - V_n\| = 0. \tag{3.37}$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$ and $\|y_n - v_n\| \to 0$ as $n \to \infty$. Observe that

$$U_n - x_n = \alpha_n(\xi_1(y_n) - x_n) + (1-\alpha_n)(u_n - x_n),$$

this implies that

$$(1-\alpha_n)\|u_n - x_n\| \le \|U_n - x_n\| + \alpha_n\|\xi_1(y_n) - x_n\|. \tag{3.38}$$

From (3.36), (3.38), and the condition (i), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \| \operatorname{Prox}^1_{\gamma f}(I - \gamma_1(a_1 A_1 + (1-a_1)B_1))x_n - x_n\| = 0. \tag{3.39}$$

Applying the same argument as (3.39), we also obtain

$$\lim_{n\to\infty} \|v_n - y_n\| = \lim_{n\to\infty} \| \operatorname{Prox}^2_{\gamma f}(I - \gamma_2(a_2 A_2 + (1-a_2)B_2))y_n - y_n\| = 0. \tag{3.40}$$

Next, we show that $\limsup\limits_{n\to\infty}\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \le 0$, where $x^* = P_{\Omega_1}\xi_1(y^*)$ and $\limsup\limits_{n\to\infty}\langle \xi_2(x^*) - y^*, V_n - y^*\rangle \le 0$, where $y^* = P_{\Omega_2}\xi_2(x^*)$. Indeed, take a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that

$$\limsup_{n\to\infty}\langle \xi_1(y^*) - x^*, U_n - x^*\rangle = \lim_{k\to\infty}\langle \xi_1(y^*) - x^*, U_{n_k} - x^*\rangle.$$

Since $\{x_n\}$, $\{y_n\}$ are bounded, without loss of generality, we may assume that $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k} \rightharpoonup \bar{y}$ as $k \to \infty$, respectively. Since C is closed and convex, C is weakly closed. So, we obtain $\bar{x}, \bar{y} \in C$.
Since $\operatorname{Prox}^i_{\gamma f}(I - \gamma_i(a_i A_i + (1-a_i)B_i))$ is nonexpansive, for all $i = 1, 2$, (3.39), and (3.40), it follows from Lemma 2.5 that $\bar{x} \in Fix(\operatorname{Prox}^1_{\gamma f}(I - \gamma_1(a_1 A_1 + (1-a_1)B_1)))$ and $\bar{y} \in Fix(\operatorname{Prox}^2_{\gamma f}(I - \gamma_2(a_2 A_2 + (1-a_2)B_2)))$.
By Lemma 2.4, we have

$$\bar{x} \in Fix(Prox^1_{\gamma f}(I - \gamma_1 A_1)) \cap Fix(Prox^1_{\gamma f}(I - \gamma_1 B_1)) = \Omega_1. \tag{3.41}$$

and

$$\bar{y} \in Fix(Prox^2_{\gamma f}(I - \gamma_2 A_2)) \cap Fix(Prox^2_{\gamma f}(I - \gamma_2 B_2)) = \Omega_2. \tag{3.42}$$

From (3.36), we obtain $U_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Since $U_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$, $\bar{x} \in \Omega_1$ and Lemma 2.1, we can derive that

$$\begin{aligned}\limsup_{n\to\infty}\langle \xi_1(y^*) - x^*, U_n - x^*\rangle =& \lim_{k\to\infty}\langle \xi_1(y^*) - x^*, U_{n_k} - x^*\rangle\\ =& \langle \xi_1(y^*) - x^*, \bar{x} - x^*\rangle\\ \le& 0. \end{aligned} \tag{3.43}$$

Similarly, indeed, take a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\limsup_{n\to\infty}\langle \xi_2(x^*) - y^*, V_n - y^*\rangle = \lim_{k\to\infty}\langle \xi_2(x^*) - y^*, V_{n_k} - y^*\rangle.$$

From (3.37), we obtain $V_{n_k} \rightharpoonup \bar{y}$ as $k \to \infty$.
Following the same method as (3.43), we easily obtain that

$$\limsup_{n\to\infty}\langle \xi_2(x^*) - y^*, V_n - y^*\rangle \leq 0. \tag{3.44}$$

Finally, we show that $\{x_n\}$ converses strongly to x^*, where $x^* = P_{\Omega_1}\xi_1(y^*)$ and $\{y_n\}$ converses strongly to y^*, where $y^* = P_{\Omega_2}\xi_2(x^*)$.
Since $U_n = \alpha_n\xi_1(y_n) + (1-\alpha_n)u_n$ and $V_n = \alpha_n\xi_2(x_n) + (1-\alpha_n)v_n$ and the definition of x_n, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|\mu_n x_n + \beta_n P_C U_n - x^*\|^2 \\
\leq& \mu_n(\mu_n + \beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\|\alpha_n(\xi_1(y_n) - x^*) \\
& + (1-\alpha_n)(u_n - x^*)\|^2 + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n + \beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big((1-\alpha_n)\|u_n - x^*\|^2 \\
& + 2\alpha_n\langle \xi_1(y_n) - x^*, U_n - x^*\rangle\Big) \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n(\mu_n + \beta_n)\|x_n - x^*\|^2 + \beta_n(\mu_n+\beta_n)\Big((1-\alpha_n)\|x_n - x^*\|^2 \\
& + 2\alpha_n\langle \xi_1(y_n) - x^*, U_n - x^*\rangle\Big) \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& \mu_n\|x_n - x^*\|^2 + \beta_n(1-\alpha_n)\|x_n - x^*\|^2 \\
& + 2\alpha_n\beta_n(\mu_n+\beta_n)\langle \xi_1(y_n) - x^*, U_n - x^*\rangle \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& (\mu_n + \beta_n(1-\alpha_n))\|x_n - x^*\|^2 + 2\alpha_n\beta_n(\mu_n+\beta_n)\langle \xi_1(y_n) - x^*, U_n - x^*\rangle \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
=& ((\mu_n + \beta_n) - \alpha_n\beta_n)\|x_n - x^*\|^2 \\
& + 2\alpha_n\beta_n(\mu_n+\beta_n)\Big(\langle \xi_1(y_n) - \xi_1(y^*), U_n - x^*\rangle + \langle \xi_1(y^*) - x^*, U_n - x^*\rangle\Big) \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
& + 2\alpha_n\beta_n(\mu_n+\beta_n)(\|\xi_1(y_n) - \xi_1(y^*)\|\|U_n - x^*\| + \langle \xi_1(y^*) - x^*, U_n - x^*\rangle) \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\| \\
\leq& (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
& + 2\alpha_n\beta_n\|\xi_1(y_n) - \xi_1(y^*)\|(\|U_n - x_{n+1}\| + \|x_{n+1} - x^*\|) \\
& + 2\alpha_n\beta_n(\mu_n+\beta_n)\langle \xi_1(y^*) - x^*, U_n - x^*\rangle \\
& + 2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1} - x^*\|
\end{aligned}$$

$$
\begin{aligned}
\leq&(1-\alpha_n\beta_n)\|x_n-x^*\|^2\\
&+2\alpha_n\beta_n\sigma\|y_n-y^*\|\|U_n-x_{n+1}\|+2\alpha_n\beta_n\sigma\|y_n-y^*\|\|x_{n+1}-x^*\|\\
&+2\alpha_n\beta_n(\mu_n+\beta_n)\langle\xi_1(y^*)-x^*,U_n-x^*\rangle\\
&+2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|\\
\leq&(1-\alpha_n\beta_n)\|x_n-x^*\|^2\\
&+2\alpha_n\beta_n\sigma\|y_n-y^*\|\|U_n-x_{n+1}\|+\alpha_n\beta_n\sigma\left(\|y_n-y^*\|^2+\|x_{n+1}-x^*\|^2\right)\\
&+2\alpha_n\beta_n(\mu_n+\beta_n)\langle\xi_1(y^*)-x^*,U_n-x^*\rangle\\
&+2(1-\mu_n-\beta_n)\|x^*\|\|x_{n+1}-x^*\|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2\leq&\frac{1-\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\|x_n-x^*\|^2+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|\|U_n-x_{n+1}\|\\
&+\frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle\xi_1(y^*)-x^*,U_n-x^*\rangle\\
&+\frac{2(1-\mu_n-\beta_n)}{1-\alpha_n\beta_n\sigma}\|x^*\|\|x_{n+1}-x^*\|\\
=&\Big(1-\frac{\alpha_n\beta_n-\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\Big)\|x_n-x^*\|^2+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|\|U_n-x_{n+1}\|\\
&+\frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle\xi_1(y^*)-x^*,U_n-x^*\rangle\\
&+\frac{2(1-\mu_n-\beta_n)}{1-\alpha_n\beta_n\sigma}\|x^*\|\|x_{n+1}-x^*\|\\
=&\Big(1-\frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\Big)\|x_n-x^*\|^2+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|\|U_n-x_{n+1}\|\\
&+\frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle\xi_1(y^*)-x^*,U_n-x^*\rangle\\
&+\frac{2(1-\mu_n-\beta_n)}{1-\alpha_n\beta_n\sigma}\|x^*\|\|x_{n+1}-x^*\|,
\end{aligned}
$$

there exists $\hat{M}>0$, such that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2\leq&\Big(1-\frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\Big)\|x_n-x^*\|^2+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|\|U_n-x_{n+1}\|\\
&+\frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle\xi_1(y^*)-x^*,U_n-x^*\rangle\\
&+(1-\mu_n-\beta_n)\hat{M}\|x^*\|\|x_{n+1}-x^*\|.
\end{aligned}
\tag{3.45}
$$

Similarly, as previously stated, there exists $\bar{M}>0$, such that

$$
\begin{aligned}
\|y_{n+1}-y^*\|^2\leq&\Big(1-\frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\Big)\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|x_n-x^*\|\|V_n-y_{n+1}\|\\
&+\frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|x_n-x^*\|^2+\frac{2\alpha_n\beta_n(\mu_n+\beta_n)}{1-\alpha_n\beta_n\sigma}\langle\xi_2(x^*)-y^*,V_n-y^*\rangle\\
&+(1-\mu_n-\beta_n)\bar{M}\|y^*\|\|y_{n+1}-y^*\|.
\end{aligned}
\tag{3.46}
$$

From (3.45), (3.46), and choose $M = \max\{\hat{M}, \bar{M}\}$, we get

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\
\leq &\Big(1 - \frac{\alpha_n \beta_n (1-\sigma)}{1 - \alpha_n \beta_n \sigma}\Big)(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&+ \frac{2\alpha_n \beta_n \sigma}{1 - \alpha_n \beta_n \sigma}(\|y_n - y^*\|\|U_n - x_{n+1}\| + \|x_n - x^*\|\|V_n - y_{n+1}\|) \\
&+ \frac{\alpha_n \beta_n \sigma}{1 - \alpha_n \beta_n \sigma}(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&+ \frac{2\alpha_n \beta_n (\mu_n + \beta_n)}{1 - \alpha_n \beta_n \sigma}(\langle \xi_1(y^*) - x^*, U_n - x^* \rangle + \langle \xi_2(x^*) - y^*, V_n - y^* \rangle) \\
&+ (1 - \mu_n - \beta_n) M \left(\|x^*\|\|x_{n+1} - x^*\| + \|y^*\|\|y_{n+1} - y^*\|\right) \\
= &\Big(1 - \frac{\alpha_n \beta_n (1-2\sigma)}{1 - \alpha_n \beta_n \sigma}\Big)(\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&+ \frac{2\alpha_n \beta_n \sigma}{1 - \alpha_n \beta_n \sigma}(\|y_n - y^*\|\|U_n - x_{n+1}\| + \|x_n - x^*\|\|V_n - y_{n+1}\|) \\
&+ \frac{2\alpha_n \beta_n (\mu_n + \beta_n)}{1 - \alpha_n \beta_n \sigma}(\langle \xi_1(y^*) - x^*, U_n - x^* \rangle + \langle \xi_2(x^*) - y^*, V_n - y^* \rangle) \\
&+ (1 - \mu_n - \beta_n) M \left(\|x^*\|\|x_{n+1} - x^*\| + \|y^*\|\|y_{n+1} - y^*\|\right).
\end{aligned}
\tag{3.47}
$$

By (3.28), (3.36), (3.37), (3.43), (3.44), the condition (i), (iii) and Lemma 2.3, we have $\lim_{n\to\infty} (\|x_n - x^*\| + \|y_n - y^*\|) = 0$. It implies that the sequences $\{x_n\}$, $\{y_n\}$ converge to $x^* = P_{\Omega_1}\xi_1(y^*)$, $y^* = P_{\Omega_2}\xi_2(x^*)$, respectively. This completes the proof. □

Remark 3.1. We have the following observations for the offered Algorithms 1.

(1) It should be noted that we use a new mathematical tool (Lemma 2.4) related to two proximal operators that exploits the information of v_n and u_n, which actually draws inspiration from Xu [29] and Guo and Cui [8].

(2) By combining the proximal-gradient algorithm with viscosity technique in Guo and Cui [8], the intermixed algorithm in Yao et al.[32] and the Krasnoselskii–Mann algorithm in Kanzow and Shehu [12], the algorithm presented in this paper provide a strong convergence theorem in real Hilbert spaces.

4. Applications

In this section, we reduce our main problem to the following the split feasibility problems.

4.1. **The Split Feasibility Problem.** Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{4.48}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The set of all solution (SFP) is denoted by $\Gamma = \{x \in C; Ax \in Q\}$. The split feasibility problem is the first example of the split inverse problem, which was first introduced by Censor and Elfving [5] in Euclidean spaces.

Proposition 4.2. *([6]) Given $x^* \in \mathcal{H}_1$, the following statements are equivalent.*

(i) x^* solves the Γ;

(ii) $P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$, where A^* is the adjoint of A;

(iii) x^* *solves the variational inequality problem of finding* $x^* \in C$ *such that*

$$\langle \nabla \mathcal{Q}(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in C, \tag{4.49}$$

where $\nabla \mathcal{Q} = A^*(I - P_Q)A$.

If C is a closed convex subset of H and the function f is the indicator function of C then it is well known that $\mathrm{Prox}_{\gamma f} = P_C$, the projection operator of H, onto the closed convex set C and putting $A_i = B_i$ for all $i = 1, 2$ in Theorem 3.1. Consequently, the following result can be obtain from Theorem 3.1.

Algorithm 2: An intermixed algorithm with viscosity technique for solving the split feasibility problems.

Initialization: Given $x_1, y_1 \in C$ be arbitrary.

Iterative Steps: Given the current iterate $\{x_n\}$, $\{y_n\}$, calculate $\{x_{n+1}\}$, $\{y_{n+1}\}$ as follows:

Step 1: Compute

$$\begin{cases} v_n = P_C(I - \gamma_2 \nabla \mathcal{Q}_2) y_n) \\ u_n = P_C(I - \gamma_1 \nabla \mathcal{Q}_1) x_n) \end{cases}$$

Step 2: Compute

$$\begin{cases} y_{n+1} = \mu_n y_n + \beta_n P_C(\alpha_n \xi_2(x_n) + (1 - \alpha_n) v_n) \\ x_{n+1} = \mu_n x_n + \beta_n P_C(\alpha_n \xi_1(y_n) + (1 - \alpha_n) u_n), \end{cases}$$

where $\nabla \mathcal{Q}_i = A_i^*(I - P_Q)A_i$, $\gamma_i \in (0, \frac{2}{\|A_i\|^2})$, $\{\mu_n\}$, $\{\beta_n\}, \{\alpha_n\} \subseteq [0, 1]$, with $\mu_n + \beta_n \leq 1$.

Set $n := n + 1$ and go to Step 1.

From Proposition 4.2, it is clear that the solution of the problem (4.49) is the same as the problem (4.48). By applying the aforementioned technique, it is possible to find the solution to the two-split feasibility problem, as demonstrated in the following theorem.

Theorem 4.2. *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C*,* Q *be two nonempty closed convex subsets of real Hilbert spaces* H_1 *and* H_2*, respectively. Let* $A_1, A_2 : H_1 \to H_2$ *be bounded linear operators with* A_1^*, A_2^* *are adjoint of* A_1 *and* A_2*, respectively. Assume that* $\Gamma_1 = \{x^* \in C; A_1 x^* \in Q\} \neq \emptyset$ *and* $\Gamma_2 = \{y^* \in C; A_1 y^* \in Q\} \neq \emptyset$*. Let* $\xi_1, \xi_2 : H \to H$ *be* σ_1 *and* σ_2*-contraction mappings with* $\sigma_1, \sigma_2 \in (0, 1)$ *and* $\sigma = \max\{\sigma_1, \sigma_2\}$*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences generated by Algorithm 2, satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) there are $\overline{\varepsilon}, l > 0$ *with* $0 < \overline{\varepsilon} \leq \mu_n, \beta_n \leq l < 1$ *for all* $n \in \mathbb{N}_+$*;*

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum\limits_{n=1}^{\infty} (1 - \mu_n - \beta_n) < \infty.$

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\Gamma_1}\xi_1(y^*)$ *and* $y^* = P_{\Gamma_2}\xi_2(x^*)$*, respectively.*

Proof. Let $x^*, x_0 \in C$ and $\nabla \mathcal{Q}_i = A_i^*(I - P_Q)A_i$, for all $i = 1, 2$. First, we show that $\nabla \mathcal{Q}_i$ is $\dfrac{1}{\|A_i\|^2}$-inverse strongly monotone for all $i = 1, 2$. Since P_Q is firmly nonexpansive, then

P_Q is $\frac{1}{2}$-averaged mapping, thus $I - P_Q$ is 1-inverse strongly monotone. Observe that

$$\begin{aligned}\langle \nabla \mathcal{Q}_i(x^*) - \nabla \mathcal{Q}_i(x_0), x^* - x_0 \rangle =& \langle A_i^*(I - P_Q)A_i x^* - A_i^*(I - P_Q)A_i x_0, x^* - x_0 \rangle \\ =& \langle (I - P_Q)A_i x^* - (I - P_Q)A_i x_0, A_i x^* - A_i x_0 \rangle \\ \geq& \|(I - P_Q)A_i x^* - (I - P_Q)A_i x_0\|^2 \\ \geq& \frac{1}{\|A_i\|^2} \cdot \|A_i^*(I - P_Q)A_i x^* - A_i^*(I - P_Q)A_i x_0\|^2 \\ =& \frac{1}{\|A_i\|^2} \cdot \|\nabla \mathcal{Q}_i(x^*) - \nabla \mathcal{Q}_i(x_0)\|^2.\end{aligned}$$

Then $\nabla \mathcal{Q}_i$ is $\frac{1}{\|A_i\|^2}$-inverse strongly monotone, for all $i = 1, 2$. So, we can conclude of Theorem 4.2 from Proposition 4.2 and Theorem 3.1. $\square$

5. Numerical examples

In this section, we give some numerical examples to support our main theorem. All the numerical results are completed on Apple MacBook Pro with 2 GHz Quad-Core Intel Core i5. The program is implemented in Python 3.10.4.

Example 5.1. We consider our problem in the infinite-dimensional Hilbert space $H = L_2([0, 1])$ with the inner product defined by

$$\langle x, y \rangle := \int_0^1 x(t)y(t)dt, \ \forall x, y \in H$$

and the induced norm by

$$\|x\|_2 := \left(\int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}}, \ \forall x \in H.$$

Let $C := \{x \in L_2([0, 1]) : \|x\| \leq 1\}$ be the unit ball. Then, we have

$$P_C(x(t)) = \begin{cases} x(t), & \text{if } \|x(t)\|_2 \leq 1, \\ \dfrac{x(t)}{\|x(t)\|_2}, & \text{if } \|x(t)\|_2 > 1. \end{cases} \tag{5.50}$$

Now take $f = \|\cdot\|_2$, the norm in $L_2([0, 1])$. Then, the proximal operator is given by

$$\mathrm{Prox}_{\gamma f}(x(t)) = \begin{cases} \left(1 - \dfrac{\gamma}{\|x(t)\|_2}\right) x(t), & \text{if } \|x(t)\|_2 \geq \gamma, \\ 0, & \text{if } \|x(t)\|_2 < \gamma. \end{cases} \tag{5.51}$$

This proximal operator is also known as the block soft thresholding operator.
For every $i = 1, 2$, let $A_i, B_i : C \to H$ defined by

$$A_1(x(t)) = x(t), A_2(x(t)) = \frac{x(t)}{3}, B_1(x(t)) = 5x(t) \text{ and } B_2(x(t)) = \frac{22x(t)}{7},$$

for all $t \in [0, 1]$, $x \in C$. For every $i = 1, 2$, we take the operator $\xi_i : H \to H$ to be defined as $\xi_1(x(t)) = \dfrac{x(t)}{25}$ and $\xi_2(x(t)) = \dfrac{x(t)}{36}$, for all $t \in [0, 1]$, $x \in H$. In Algorithm 1, choose $\alpha_n = \dfrac{1}{5n}$, $\mu_n = \dfrac{1}{(n+1)^2} - \dfrac{1}{(n+1)^4}$, $\beta_n = 1 - \dfrac{1}{(n+1)^2}$, $a_1 = 0.70$ and $a_2 = 0.20$, for all

$n \in \mathbb{N}$. So our Algorithm 1 has the following form:

(5.52)

$$\begin{cases} v_n = \mathrm{Prox}^2_{\gamma f}(y_n - \gamma_2(0.2A_2 + 0.8B_2)y_n) \\ u_n = \mathrm{Prox}^1_{\gamma f}(x_n - \gamma_1(0.7A_1 + 0.3B_1)x_n) \\ y_{n+1} = \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}\right) y_n + \left(1 - \frac{1}{(n+1)^2}\right) P_C\left(\frac{1}{5n}\xi_2(x_n) + \left(1 - \frac{1}{5n}\right) v_n\right) \\ x_{n+1} = \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}\right) x_n + \left(1 - \frac{1}{(n+1)^2}\right) P_C\left(\frac{1}{5n}\xi_1(y_n) + \left(1 - \frac{1}{5n}\right) u_n\right). \end{cases}$$

We test the Algorithm 1 for three different starting points and use $\|x_{n+1} - x_n\|_2 < 10^{-10}$ and $\|y_{n+1} - y_n\|_2 < 10^{-10}$ as stopping criterion.

Case 1: $x_1 = \frac{t}{5}$ and $y_1 = \frac{t}{7}$;

Case 2: $x_1 = e^{-5t}$ and $y_1 = \frac{t^2}{2}$;

Case 3: $x_1 = \sin(2t)$ and $y_1 = \cos(2t)$.

According to the definition of A_i, B_i, f_i, for all $i = 1, 2$, then the solution of the problem is $x^*(t) = \{\mathbf{0}\}$. The computational experiments, using our Algorithm 1, for each case are reported in Tables 1, 2, 3, and Figures 1, 2, 3. The convergence behavior of the error $\|x_n - x_{n-1}\|_2$ and $\|y_n - y_{n-1}\|_2$ for each case is shown in Figure 4.

TABLE 1. Computational result of **Case 1** for Example 5.1.

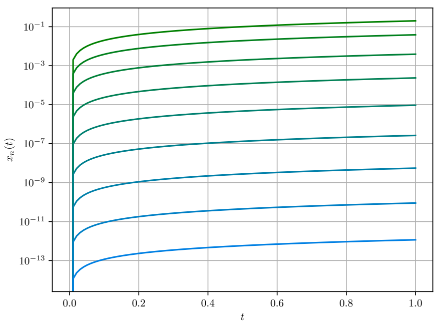
n	$x_n(t)$	$y_n(t)$	$\|x_{n+1} - x_n\|_2$	$\|y_n - y_{n-1}\|_2$
1	$0.2t$	$0.14286t$	-	-
2	$0.038357t$	$0.027619t$	0.093325	0.066533
3	$0.0038866t$	$0.0028225t$	0.019902	0.014316
4	$0.00023478t$	$0.00017213t$	0.0021084	0.0015302
5	$9.3462 \cdot 10^{-6}t$	$6.9228 \cdot 10^{-6}t$	0.00013016	$9.5382 \cdot 10^{-5}$
6	$2.6317 \cdot 10^{-7}t$	$1.9705 \cdot 10^{-7}t$	$5.2441 \cdot 10^{-6}$	$3.8831 \cdot 10^{-6}$
7	$5.5187 \cdot 10^{-9}t$	$4.1782 \cdot 10^{-9}t$	$1.4876 \cdot 10^{-7}$	$1.1136 \cdot 10^{-7}$
8	$8.9582 \cdot 10^{-11}t$	$6.8575 \cdot 10^{-11}t$	$3.1345 \cdot 10^{-9}$	$2.3727 \cdot 10^{-9}$
9	$1.16 \cdot 10^{-12}t$	$8.976 \cdot 10^{-13}t$	$5.1051 \cdot 10^{-11}$	$3.9074 \cdot 10^{-11}$

TABLE 2. Computational result of **Case 2** for Example 5.1.

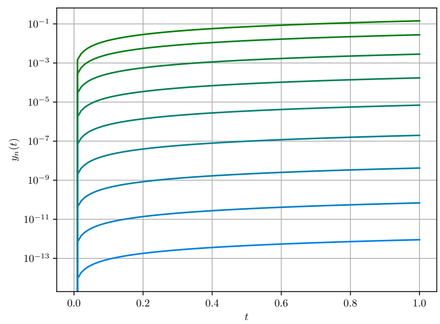
n	$x_n(t)$	$y_n(t)$	$\|x_n - x_{n-1}\|_2$	$\|y_n - y_{n-1}\|_2$
1	e^{-5t}	$0.2t^2$	-	-
2	$0.0012t^2 + 0.1875e^{-5t}$	$0.0375t^2 + 0.0041667e^{-5t}$	0.25688	0.072554
3	$0.00025185t^2 + 0.018533e^{-5t}$	$0.0037067t^2 + 0.00087449e^{-5t}$	0.053474	0.015251
4	$2.4024 \cdot 10^{-5}t^2 + 0.0010881e^{-5t}$	$0.00021762t^2 + 8.3415 \cdot 10^{-5}e^{-5t}$	0.0055276	0.0016045
5	$1.3403 \cdot 10^{-6}t^2 + 4.1944 \cdot 10^{-5}e^{-5t}$	$8.3888 \cdot 10^{-6}t^2 + 4.654 \cdot 10^{-6}e^{-5t}$	0.00033198	$9.9186 \cdot 10^{-5}$
6	$4.9247 \cdot 10^{-8}t^2 + 1.14 \cdot 10^{-6}e^{-5t}$	$2.28 \cdot 10^{-7}t^2 + 1.71 \cdot 10^{-7}e^{-5t}$	$1.2973 \cdot 10^{-5}$	$4.044 \cdot 10^{-6}$
7	$1.2823 \cdot 10^{-9}t^2 + 2.3014 \cdot 10^{-8}e^{-5t}$	$4.6027 \cdot 10^{-9}t^2 + 4.4525 \cdot 10^{-9}e^{-5t}$	$3.5598 \cdot 10^{-7}$	$1.1746 \cdot 10^{-7}$
8	$2.4901 \cdot 10^{-11}t^2 + 3.5898 \cdot 10^{-10}e^{-5t}$	$7.1796 \cdot 10^{-11}t^2 + 8.6463 \cdot 10^{-11}e^{-5t}$	$7.2412 \cdot 10^{-9}$	$2.5624 \cdot 10^{-9}$
9	$3.7454 \cdot 10^{-13}t^2 + 4.4625 \cdot 10^{-12}e^{-5t}$	$8.925 \cdot 10^{-13}t^2 + 1.3005 \cdot 10^{-12}e^{-5t}$	$1.1372 \cdot 10^{-10}$	$4.3587 \cdot 10^{-11}$
10	$4.4933 \cdot 10^{-15}t^2 + 4.5323 \cdot 10^{-14}e^{-5t}$	$9.0647 \cdot 10^{-15}t^2 + 1.5602 \cdot 10^{-14}e^{-5t}$	$1.4228 \cdot 10^{-12}$	$5.9411 \cdot 10^{-13}$

TABLE 3. Computational result of **Case 3** for Example 5.1.

n	$x_n(t)$	$y_n(t)$	$\|x_n - x_{n-1}\|_2$	$\|y_n - y_{n-1}\|_2$
1	$\sin(2t)$	$\cos(2t)$	-	-
2	$0.1875\sin(2t) + 0.006\cos(2t)$	$0.0041667\sin(2t) + 0.4676\cos(2t)$	0.62492	0.33764
3	$0.018533\sin(2t) + 0.0022552\cos(2t)$	$0.0012487\sin(2t) + 0.088194\cos(2t)$	0.13131	0.24253
4	$0.0010891\sin(2t) + 0.00035263\cos(2t)$	$0.00010534\sin(2t) + 0.0051715\cos(2t)$	0.014005	0.053239
5	$4.2022 \cdot 10^{-5}\sin(2t) + 2.347 \cdot 10^{-5}\cos(2t)$	$5.4972 \cdot 10^{-6}\sin(2t) + 0.00019906\cos(2t)$	0.00091556	0.0031992
6	$1.1434 \cdot 10^{-6}\sin(2t) + 9.4349 \cdot 10^{-7}\cos(2t)$	$1.9385 \cdot 10^{-7}\sin(2t) + 5.4012 \cdot 10^{-6}\cos(2t)$	$3.975 \cdot 10^{-5}$	0.00012508
7	$2.3112 \cdot 10^{-8}\sin(2t) + 2.5917 \cdot 10^{-8}\cos(2t)$	$4.9126 \cdot 10^{-9}\sin(2t) + 1.0883 \cdot 10^{-7}\cos(2t)$	$1.2299 \cdot 10^{-6}$	$3.4336 \cdot 10^{-6}$
8	$3.6101 \cdot 10^{-10}\sin(2t) + 5.2106 \cdot 10^{-10}\cos(2t)$	$9.3615 \cdot 10^{-11}\sin(2t) + 1.6942 \cdot 10^{-9}\cos(2t)$	$2.8427 \cdot 10^{-8}$	$6.9863 \cdot 10^{-8}$
9	$4.4943 \cdot 10^{-12}\sin(2t) + 8.0267 \cdot 10^{-12}\cos(2t)$	$1.3891 \cdot 10^{-12}\sin(2t) + 2.1015 \cdot 10^{-11}\cos(2t)$	$5.0783 \cdot 10^{-10}$	$1.0972 \cdot 10^{-9}$
10	$4.5716 \cdot 10^{-14}\sin(2t) + 9.7958 \cdot 10^{-14}\cos(2t)$	$1.6498 \cdot 10^{-14}\sin(2t) + 2.1296 \cdot 10^{-13}\cos(2t)$	$7.1996 \cdot 10^{-12}$	$1.3724 \cdot 10^{-11}$

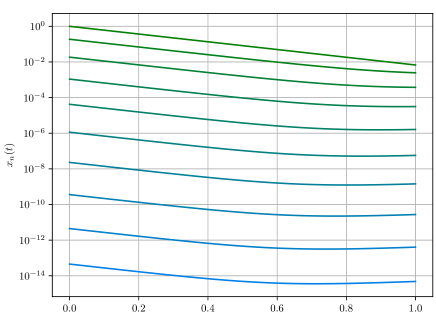


(A) **Case 1** : $x_1 = \frac{t}{5}$ for $n = 1, 2, 3, ..., 8.$ and

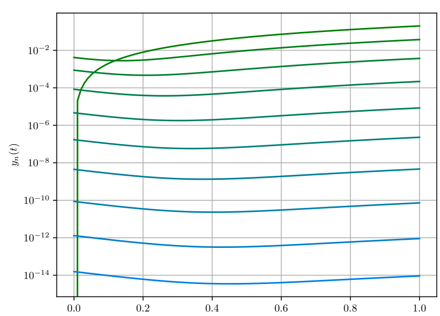


(B) **Case 1** : $y_1 = \frac{t}{7}$ for $n = 1, 2, 3, ..., 8.$

FIGURE 1. The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with **Case 1** in Example 5.1 and y-axis is illustrated in Log scale.

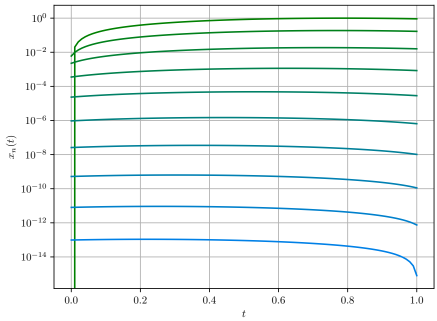


(A) **Case 2** : $x_1 = e^{-5t}$ for $n = 1, 2, 3, ..., 9.$

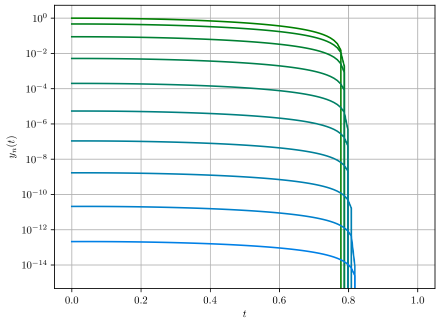


(B) **Case 2** : $y_1 = \frac{t^2}{2}$ for $n = 1, 2, 3, ..., 9.$

FIGURE 2. The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with **Case 2** in Example 5.1 and y-axis is illustrated in logscale.

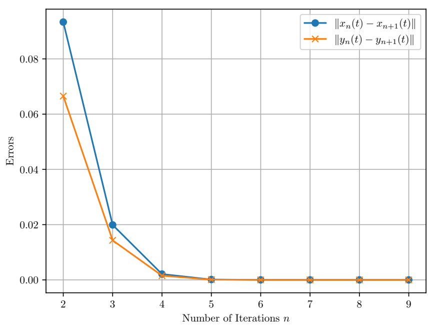


(A) **Case 3** : $x_1 = \sin(2t)$ for $n = 1, 2, 3, ..., 9$.

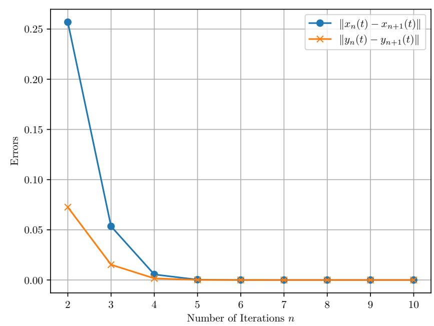


(B) **Case 3** : $y_1 = \cos(2t)$ for $n = 1, 2, 3, ..., 9$.

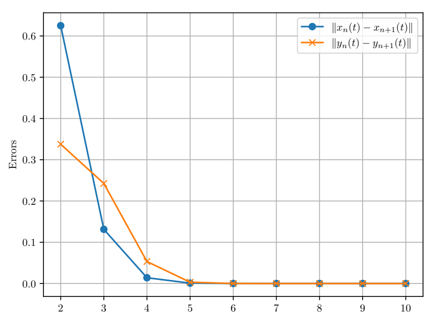
FIGURE 3. The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with **Case 3** in Example 5.1 and y-axis is illustrated in logscale.



(A) Case 1 : $x_1 = 0.2t$ and $y_1 = 0.8t$.



(B) Case 2 : $x_1 = e^{-2t}$ and $y_1 = \dfrac{t^2}{2}$.



(C) Case 3 : $x_1 = \sin(2t)$ and $y_1 = \cos(2t)$

FIGURE 4. Error plotting of $\|x_n - x_{n-1}\|_2$ and $\|y_n - y_{n-1}\|_2$ in Example 5.1.

Moreover, we also provide the comparison (in terms of convergence and the CPU time) of the sequences μ_n and β_n on Algorithm 1 by choosing different μ_n and β_n with $\mu_n + \beta_n \leq$

1 satisfying the conditions (ii), (iii) in the following choices.

Choice 1: $\mu_n = \dfrac{1}{(n+1)^2} - \dfrac{1}{(n+1)^4}$ and $\beta_n = 1 - \dfrac{1}{(n+1)^2}$;

Choice 2: $\mu_n = \dfrac{n}{(8n+9)} - \dfrac{1}{(n^2+1)}$ and $\beta_n = 1 - \dfrac{1}{(n^2+1)}$;

Choice 3: $\mu_n = \dfrac{1}{n+1} - \dfrac{1}{(n+1)^{20}}$ and $\beta_n = 1 - \dfrac{1}{n+1}$;

Choice 4: $\mu_n = \dfrac{1}{n^2+1}$ and $\beta_n = 1 - \dfrac{1}{n^2+1}$.

It is emphasized that all these sequences of μ_n and β_n are to satisfy conditions (ii) and (iii). The results are reported in Table 4.

TABLE 4. Comparison of Algorithm 1 for Example 5.1 with different cases of μ_n and β_n.

Starting point		Choice 1	Choice 2	Choice 3	Choice 4
$x_1 = \frac{t}{5}$	No. of Iter.	9	10	14	16
$y_1 = \frac{t}{7}$	CPU Time (s)	0.56213975	0.69141006	0.92488122	1.04418993

Remark 5.2. By testing the convergence behavior of Algorithm 1, we see in Example 5.1 that

(1) Tables 1, 2, 3 and Figures 1, 2, 3, 4 show that $\{x_n\}$ and $\{y_n\}$ converge to $x(t) = \mathbf{0}$, where $\mathbf{0} \in Fix(\mathrm{Prox}_{\gamma f}(I - \gamma A)) \cap Fix(\mathrm{Prox}_{\gamma f}(I - \gamma B))$, for all $i = 1, 2$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 5.1 can be guaranteed by Theorem 3.1.
(2) From the discussion of Tables 1, 2, and 3, we see that the sequences $\{x_n\}$ and $\{y_n\}$ in **Case 1** on algorithm 1 converge the fastest.
(3) The sequences $\mu_n = \frac{1}{(n+1)^2} - \frac{1}{(n+1)^4}$ and $\beta_n = 1 - \frac{1}{(n+1)^2}$ with $\mu_n + \beta_n \leq 1$ satisfy the conditions (ii), (iii) in Theorem 3.1.
(4) From the discussion of Table 4, we see that the sequences $\{x_n\}$ and $\{y_n\}$ in **Choice 1** on algorithm 1 converge the fastest and the least time.

Next, we use the Algorithm 2 in Theorem 4.2 to solve a system of linear equations. Systems of linear equations are used in a wide range of fields, including traffic analysis, economics, and electrical engineering.

Example 5.2. We assume that $H_1 = H_2 = \mathbb{R}^4$. Solving a system of linear equations $A_i\mathbf{x} = b_i$ for all $i = 1, 2$. In the following, we take:

$$A_1 = \begin{pmatrix} 2 & -1 & 3 & -1 \\ 1 & -2 & 1 & -3 \\ 2 & -1 & -1 & 1 \\ 2 & 0 & -2 & 3 \end{pmatrix}, \; b_1 = \begin{pmatrix} 1 \\ -15 \\ 7 \\ 10 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -1 & -3 & 0 & -3 \\ 1 & -1 & 2 & -2 \\ 1 & 2 & 1 & 0 \\ 2 & -2 & 8 & 6 \end{pmatrix}, \; b_2 = \begin{pmatrix} -22 \\ -6 \\ 9 \\ 36 \end{pmatrix}.$$

Then the split feasibility problem can be formulated as the problem of finding a point $\mathbf{x}^*$ with the property $\mathbf{x}^* \in C$ and $A_i\mathbf{x}^* \in Q$, where $C = \mathbb{R}^4$, $Q = \{b_i\}$, for all $i = 1, 2$. That is,

$\mathbf{x}^*$ is the solution of the system of linear equations $A_i\mathbf{x} = b_i$, and

$$\mathbf{x}^* = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}.$$

For every $i = 1, 2$, we take the operator $\xi_i : \mathbb{R}^4 \to \mathbb{R}^4$ to be defined as $\xi_1(\mathbf{x}) = \dfrac{\mathbf{x}}{49}$ and $\xi_2(\mathbf{x}) = \dfrac{\mathbf{x}}{64}$. In Algorithm 2, choose $\alpha_n = \dfrac{1}{5n}$, $\mu_n = \dfrac{1}{(n+1)^2} - \dfrac{1}{(n+1)^4}$ and $\beta_n = 1 - \dfrac{1}{(n+1)^2}$ for all $n \in \mathbb{N}$. So our Algorithm 2 becomes

$$\begin{cases} \mathbf{v}_n = \mathbf{y}_n - \frac{1}{100} A_2^*(I - P_Q)A_2\mathbf{y}_n \\ \mathbf{u}_n = \mathbf{x}_n - \frac{2}{100} A_1^*(I - P_Q)A_1\mathbf{x}_n \\ \mathbf{y}_{n+1} = \left(\dfrac{1}{(n+1)^2} - \dfrac{1}{(n+1)^4}\right)\mathbf{y}_n + \left(1 - \dfrac{1}{(n+1)^2}\right)\left(\dfrac{1}{5n}\xi_2(\mathbf{x}_n) + \left(1 - \dfrac{1}{5n}\right)\mathbf{v}_n\right) \\ \mathbf{x}_{n+1} = \left(\dfrac{1}{(n+1)^2} - \dfrac{1}{(n+1)^4}\right)\mathbf{x}_n + \left(1 - \dfrac{1}{(n+1)^2}\right)\left(\dfrac{1}{5n}\xi_1(\mathbf{y}_n) + \left(1 - \dfrac{1}{5n}\right)\mathbf{u}_n\right). \end{cases} \tag{5.53}$$

According to the definition of A_i, for all $i = 1, 2$, then the solution of the problem is $\mathbf{x}^* = (1, 3, 2, 4)^T$. From Theorem 4.2, we can conclude that the sequences x_n and y_n converge strongly to $\mathbf{x}^*$. The numerical results, using our Algorithm 2, for the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ are reported in Tables 5 and 6.

TABLE 5. The numerical results for the sequence $\{\mathbf{x}_n\}$ of Example 5.2.

n	$\mathbf{x}_n = (x_n^1, x_n^2, x_n^3, x_n^4)^T$	$\lVert\mathbf{x}_n - \mathbf{x}_{n-1}\rVert_2$
1	(10, 10, 10, 10)	-
100	(1.01101939, 2.96006399, 1.9079274, 3.93186252)	2.0827e-03
500	(0.99900056, 2.98427758, 1.98454128, 3.9924486)	4.7872e-05
1000	(0.99952084, 2.99227651, 1.99233583, 3.99621442)	1.1675e-05
5000	(0.99990674, 2.99847538, 1.99847759, 3.9992422)	4.5827e-07
10000	(0.99995351, 2.99923891, 1.99923946, 3.99962108)	1.1430e-07

TABLE 6. The numerical results for the sequence $\{\mathbf{y}_n\}$ of Example 5.2.

n	$\mathbf{y}_n = (y_n^1, y_n^2, y_n^3, y_n^4)^T$	$\lVert\mathbf{y}_n - \mathbf{y}_{n-1}\rVert_2$
1	(-10, -10, -10, -10)	-
100	(0.74657296, 3.05961817, 2.10596020, 3.95701678)	1.9015e-03
500	(0.97220178, 3.00338976, 2.01114426, 3.99416767)	2.3602e-04
1000	(1.02507278, 2.98613767, 1.98783547, 4.00236647)	5.3877e-05
5000	(1.01697453, 2.99245005, 1.99212248, 4.00209563)	4.7330e-06
10000	(1.00763210, 2.99656705, 1.99645073, 4.00093167)	1.0058e-06

Remark 5.3. By testing the convergence behavior of Algorithm 2, we see in Example 5.2 that

(1) Tables 5 and 6 show that $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ converge to $\mathbf{x}^* = (1, 3, 2, 4)^T$. The convergence of $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ of Example 5.2 can be guaranteed by Theorem 4.2.

(2) Tables 5 and 6, we see that $\mathbf{x}_{10000} = (0.99995351, 2.99923891, 1.99923946, 3.99962108)^T$ is an approximation of the system of linear equations with an error $1.1430e-07$ and $\mathbf{y}_{10000} = (1.00763210, 2.99656705, 1.99645073, 4.00093167)^T$ is an approximation of the system of linear equations with an error $1.0058e-06$, respectively.

6. Conclusion

In this paper, we introduce an intermixed algorithm with viscosity technique for solving a common fixed point of two proximal operators in a real Hilbert space. The strong convergence theorem of our proposed algorithm, Theorem 3.1, has been established and proven under some mild conditions. However, we should like remark the following:

(1) We modify the results of Yao et al.[32] from strict pseudo-contraction mappings to proximal operators of in Hilbert spaces. Further, we also give the new mathematical tool related to proximal operators by using the concept of the convex minimization problem and the fixed point equation (1.2) (see Lemma 2.4).

(2) Our result is proved with a new assumption on the control conditions $\{\mu_n\}$ and $\{\beta_n\}$ such that $\mu_n + \beta_n \leq 1$.

(3) We apply our theorem to solve the split feasibility problem by using an intermixed algorithm with viscosity technique.

(4) We give a numerical example that shows the efficiency and implementation of our main result in the space L_2 as shown in Example 5.1. Moreover, we present a numerical example of the algorithm 2 for solving the system of linear equations in Example 5.2.

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References

[1] Abbas, M.; AlShahrani, M.; Ansari, Q. H. et al.: Iterative methods for solving proximal split minimization problems. *Numer Algor* **78** (2018), 193–215.

[2] Bauschke, H. H.; Combettes, P. L. *Convex Analysis and Monotone Operators Theory in Hilbert Spaces*. CMS Books in Mathematics. Springer-Verlag, second edition (2017).

[3] Combettes, P. L.; Pesquet, J. C. *Proximal splitting methods in signal processing.* In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, 185–212. Springer, Berlin (2011).

[4] Combettes, L.; Wajs, V. R. Signal recovery by proximal forward-backward splitting. *Multiscale Model. Simul.* **4** (2005), 1168–1200.

[5] Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithm.* **8** (1994), 221–239.

[6] Ceng, L. C.; Ansari, Q. H.; Yao, J. C. An extragradient method for solving split feasibility and fixed point problems. *Comput. Math. Appl.* **64** (2012), 633–642.

[7] Dong, Y. New inertial factors of the Krasnoselskii-Mann iteration. *Set-Valued Var. Anal.* **29** (2021), 145–161.

[8] Guo, Y.; Cui, W. Strong convergence and bounded perturbation resilience of a modified proximal gradient algorithm. *J Inequal Appl.* **2018** (2018), 103.

[9] Iyiola, O. S.; Shehu, Y. New Convergence Results for Inertial Krasnoselskii–Mann Iterations in Hilbert Spaces with Applications. *Results Math.* **76** (2021), 75.

[10] Jolaoso, L. O.; Abass, H. A. A viscosity-proximal gradient method with inertial extrapolation for solving certain minimization problems in Hilbert space. *Arch Math.* **55** (2019), no. 3, 167-194.

[11] Jolaoso, L. O.; Ogbuisi, F.; Mewomo, O. An intermixed algorithm for two strict pseudocontractions in q-uniformly smooth banach space. *Nonlinear Studies.* **26** (2019), no. 1, 27–41.
[12] Kanzow, C.; Shehu, Y. Generalized Krasnoselskii-Mann-type iterations for nonexpansive mappings in Hilbert spaces. *Comput. Optim. Appl.* **67** (2017), 595–620.
[13] Khuangsatung, W.; Jailoka, P.; Suantai, S. An iterative method for solving proximal split feasibility problems and fixed point problems. *Comp. Appl. Math.* **38** (2019), 177.
[14] Kesornprom, S.; Cholamjiak, P. A modified inertial proximal gradient method for minimization problems and applications. *AIMS Mathematics.* **7** (2022), no. 5, 8147–8161.
[15] Krasnoselskii, M. A. Two remarks on the method of successive approximations.*Uspekhi Mat. Nauk.* **10** (1955), 123-127.
[16] Mann, W. R. Mean value methods in iteration. *Bull. Amer. Math. Soc.* **4** (1953), 506–510.
[17] Maingé, P.E. Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **325** (2007), no. 1, 469–479.
[18] Ming, T.; Liu, L. General iterative methods for equilibrium and constrained convex minimization problem. *J. Optim. Theory Appl.* **63** (2014), 1367–1385.
[19] Moudafi, A.; Thakur, B. S. Solving proximal split feasibility problems without prior knowledge of operator norms. *Optim Lett.* **8** (2014), 2099–2110.
[20] Pakkaranang, N.; Kumam, P.; Berinde, V.; Suleiman, Y. I. Superiorization methodology and perturbation resilience of inertial proximal gradient algorithm with application to signal recovery. *J Supercomput.* **76** (2020), 9456–9477.
[21] Reich, S. Weak convergence theorems for nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* **67** (1979), 274–276.
[22] Sahu, D. R.; Yao, J. C.; Verma, M.; Shukla, K. K. Convergence rate analysis of proximal gradient methods with applications to composite minimization problems. *Optimization*. **70** (2021), no. 1, 75–100.
[23] Saechou, K.; Kangtunyakarn, A. Modified intermixed iteration for solving the split general system of variational inequality problems and applications. *Comp. Appl. Math.* **40** (2021), 264.
[24] Shehu, Y. Convergence rate analysis of inertial Krasnoselskii–mann type iteration with applications. *Numer. Funct. Anal. Optim.* **39** (2018), no. 10, 1077–1091.
[25] Suantai, S.; Jailoka, P.; Hanjing, A. An accelerated viscosity forward-backward splitting algorithm with he linesearch process for convex minimization problems. *J Inequal Appl.* **2021** (2021), 42.
[26] Sripattanet, A.; Kangtunyakarn, A. Convergence theorem for solving a new concept of the split variational inequality problems and application. *RACSAM.* **114** (2020), 177.
[27] Thongpaen P.; Wattanataweekul R. A fast fixed-point algorithm for convex minimization problems and its application in image restoration problems. *Mathematics.* **9** (2021), no. 20, 2619.
[28] Wairojjana, N.; Pakkaranang, N.; Uddin, I.; Kumam, P.; Awwal A. M. Modified proximal point algorithms involving convex combination technique for solving minimization problems with convergence analysis. *Optimization.* **69:(7-8)** (2020), 1655–1680.
[29] Xu, H. K. Properties and iterative methods for the lasso and its variants. *Chin. Ann. Math. Ser.* **B35** (2014), no. 3, 501–518.
[30] Xu, H. K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66** (2002), no. 1, 240–256.
[31] Xu, H. K. Viscosity approximation methods for non-expansive mappings. *J. Math. Anal. Appl.* **298** (2004), 279–291.
[32] Yao, Z.; Kang, S. M.; Li, H. J. An intermixed algorithm for strict pseudo-contractions in Hilbert spaces. *Fixed Point Theory and Appl.* **2015**, 206 (2015).
[33] Zhang, Y. C.; Guo, K.; Wang, T. Generalized Krasnoselskii–Mann-Type Iteration for Nonexpansive Mappings in Banach Spaces. *J. Oper. Res. Soc. China.* **9** (2021), 195–206.

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A regularization method for solving the G-variational inequality problem and fixed-point problems in Hilbert spaces endowed with graphs

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Abstract

This article considers and investigates a variational inequality problem and fixed-point problems in real Hilbert spaces endowed with graphs. A regularization method is proposed for solving a G-variational inequality problem and a common fixed-point problem of a finite family of G-nonexpansive mappings in the framework of Hilbert spaces endowed with graphs, which extends the work of Tiammee et al. (Fixed Point Theory Appl. 187, 2015) and Kangtunyakarn, A. (Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 112:437–448, 2018). Under certain conditions, a strong convergence theorem of the proposed method is proved. Finally, we provide numerical examples to support our main theorem. The numerical examples show that the speed of the proposed method is better than some recent existing methods in the literature.

Mathematics Subject Classification: Primary 47H09; 47H10; secondary 90C33

Keywords: G-variational inequality problem; G-inverse strongly monotone mapping; G-nonexpansive mapping; Regularization method; Directed graph

1 Introduction

Assume that H is a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H and $\mathcal{T}: C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of $\mathcal{T}$ if $\mathcal{T}x = x$. Let $F(\mathcal{T}) := \{x \in C : \mathcal{T}x = x\}$ be the set of fixed points of $\mathcal{T}$. The mapping $\mathcal{T}$ is *nonexpansive* if $\|\mathcal{T}x - \mathcal{T}y\| \le \|x - y\|$ for all $x, y \in C$.

Denote by $G = (V(G), E(G))$ a directed graph, where $V(G)$ and $E(G)$ are the set of its vertices and edges, respectively. Assuming that G has no parallel edges, we denote G^{-1} as the directed graph derived from G by reversing the direction of its edges, i.e.,

$$E(G^{-1}) = \{(x,y) : (y,x) \in E(G)\}.$$

In 2008, Jachymski [1] studied fixed-point theory in a metric space endowed with a directed graph by combining the concepts of fixed-point theory and graph theory. The following contractive-type mapping with a directed graph was proposed. Given a metric space (X,d), let G be a directed graph such that the set of its vertices $V(G) = X$ and the set of its edges $E(G)$ consists of all loops, i.e., $\triangle = \{(x,x) : x \in X\} \subseteq E(G)$. A mapping $T : X \to X$ is said to be a G-contraction if it preserves the edges of G, i.e.,

$$x,y \in X, \quad (x,y) \in E(G) \quad \Rightarrow \quad \bigl(T(x),T(y)\bigr) \in E(G)$$

and there exists $\alpha \in (0,1)$ such that for any $x,y \in X$,

$$(x,y) \in E(G) \quad \Rightarrow \quad d\bigl(T(x),T(y)\bigr) \leq \alpha d(x,y).$$

The generalized Banach contraction principle in a metric space endowed with a directed graph was also established.

Given a nonempty convex subset C of a Banach space X and a directed graph G with $V(G) = C$, then $T : C \to C$ is said to be G-nonexpansive if the following conditions hold:

1. T is edge preserving, i.e., $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)$ for any $x,y \in C$;
2. $(x,y) \in E(G) \Rightarrow \|Tx - Ty\| \leq \|x - y\|$ for any $x,y \in C$.

This mapping was proposed by Tiammee *et al.* [2] in 2015. Moreover, Tiammee *et al.* [2] also introduced Property G and the following Halpern iteration process for finding the set of fixed points of G-nonexpansive mappings in Hilbert spaces endowed with a directed graph. Suppose C has Property G. Let $\{x_n\}$ be a sequence generated by $x_0 = u \in C$ and

$$x_{n+1} = \beta_n u + (1 - \beta_n)Tx_n, \quad n \geq 0, \tag{1}$$

where $\{\beta_n\} \subseteq [0,1]$ and $T : C \to C$ is a G-nonexpansive mapping. If $\{x_n\}$ is dominated by $P_{F(T)}x_0$ and $\{x_n\}$ dominates x_0, then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$ under some suitable control conditions.

In 2017, Kangtunyakarn [3] suggested G-S-mapping generated by a finite family of G-nonexpansive mappings and finite real numbers and introduced the following Halpern iteration associated with G-S-mapping for solving the fixed-point problem of a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs. Let $\{x_n\}$ be a sequence generated by $x_0 = u \in C$ and

$$x_{n+1} = \beta_n u + (1 - \beta_n)Sx_n, \quad n \geq 0, \tag{2}$$

where $\{\beta_n\} \subseteq [0,1]$, and S is a G-S-mapping. He showed that the sequence $\{x_n\}$ generated by (2) converges strongly to a point in $F(S) = \bigcap_{i=1}^{N} F(T_i)$ under some suitable control conditions. Furthermore, in the past few years, several iterative methods have been introduced for solving the fixed-point problem of G-nonexpansive mappings; see [4–8] and the references therein.

For a given nonlinear operator $\bar{A} : C \to H$, we consider the following variational inequality problem of solving $x \in C$ such that

$$\langle y - x, \bar{A}x \rangle \geq 0, \tag{3}$$

for all $y \in C$. Denote by $VI(C,\bar{A})$ the set of solutions of the variational inequality (3). The variational inequalities were introduced in [9, 10], which has been extensively studied in the literature; see [11–13]. It is well known that $\tilde{u}$ solves the problem (3) if and only if $\tilde{u}$ solves the equation

$$\tilde{u} = P_C(I - \lambda\bar{A})\tilde{u}, \quad \forall\lambda > 0.$$

This work focuses on the following G-variational inequality problem in Hilbert spaces endowed with graphs, which Kangtunyakarn introduced [14] in 2020. In order to propose this problem, he combined the concept of problem (3) with graph theory. Given a directed graph G with $V(G) = C$, the *G-variational inequality problems* is to find a point $x^* \in C$ such that

$$\langle y - x^*, Ax^*\rangle \geq 0, \tag{4}$$

for all $y \in C$ with $(x^*, y) \in E(G)$, where A is a mapping from C to H. We denote by $G\text{-}VI(C, A)$ the set of all solutions of (4).

Moreover, he also introduced the following G-α-inverse strongly monotone in Hilbert spaces endowed with graphs: A mapping $A : C \to H$ is said to be *G-α-inverse strongly monotone* if there exists a positive number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2,$$

for all $x, y \in C$ with $(x, y) \in E(G)$. For more information on the G-variational inequality problem and G-α-inverse strongly monotone, see [14].

Furthermore, the following method for solving the G-variational inequality problems and the fixed-point problem of a G-nonexpansive mapping in Hilbert spaces endowed with graphs were also introduced in [14]. Let $\{x_n\}$ be a sequence generated by $x_0 = u \in C$ and

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda A)x_n + \gamma_n Sx_n, \quad n \geq 0, \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda \in (0, 2\alpha)$, $S : C \to C$ is a G-nonexpansive mapping, and $A : C \to H$ is a G-α-inverse strongly monotone operator with $A^{-1}(0) \neq \emptyset$. Under certain conditions, a strong convergence result of the algorithm (5) in Hilbert spaces endowed with graphs was shown.

In this paper, motivated by Tiammee *et al.* [2], Kangtunyakarn [3], and Kangtunyakarn [14], we study the G-variational inequality problem (4) and introduce a new method for solving the G-variational inequality problem (4) and fixed-point problems of a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs as follows: Given $u = x_0 \in C$, let the sequences $\{x_n\}$ be defined by

$$x_{n+1} = P_C(I - \lambda A)\big(\beta_n u + (1 - \beta_n)Sx_n\big), \quad n \geq 0, \tag{6}$$

where $\{\beta_n\} \subseteq [0, 1]$, $\lambda \in (0, 2\alpha)$, $A : C \to H$ is a G-α-inverse strongly monotone operator with $A^{-1}(0) \neq \emptyset$, and S is a G-S-mapping generated by a finite family of G-nonexpansive mappings and finite real numbers. We note that the proposed method (6) reduces to the iteration process (2) when $A = 0$, $P_C = I$ and reduces to the iteration process (1) when $A = 0$,

$P_C = I$, $N = 1$ and $\alpha_1^N = 1$. Under suitable conditions, a strong convergence theorem of the proposed method is proved. In the last section, we provide numerical examples to support our main theorem. The main result extends and improves the corresponding results. We made the following contributions to this research.

- The proposed method is constructed around the Halpern iteration process in [15] and the regularization technique in [16]. In this case, our main results is to solve a common solution of the G-variational inequality problem (G-$VI(C,A)$) and the fixed-point problems of a finite family of G-nonexpansive mappings ($\bigcap_{i=1}^N F(T_i)$), while the results in Tiammee *et al.* [2] and Kangtunyakarn [3] are to solve a solution of the fixed-point problem of a G-nonexpansive mapping ($F(T)$) and a common solution of the fixed-point problems of a finite family of G-nonexpansive mappings ($\bigcap_{i=1}^N F(T_i)$), respectively.
- Under certain mild conditions, the strong convergence of the iterative sequences generated by the proposed method is established in Hilbert spaces endowed with graphs.
- Numerical examples in finite- and infinite-dimensional spaces are provided to demonstrate the convergence behavior of our proposed method and the comparison to the Halpern-type algorithms proposed in Algorithm 5 of Kangtunyakarn [14]. It is shown that the proposed iterative method has a faster convergence speed (in terms of CPU time and the number of iterations) than Algorithm 5 of Kangtunyakarn [14] (see Sect. 4).

This paper is organized as follows. In Sect. 2, we first recall some basic definitions and lemmas. In Sect. 3, we propose a modified regularization method and analyze its convergence. In Sect. 4, some numerical experiments are provided.

2 Preliminaries

For the purpose of proving our theorem, we provide several definitions and lemmas in this section. For convenience, the following notations are used throughout the paper:

- H denotes a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\|\cdot\|$;
- C denotes a nonempty, closed, and convex subset of H;
- $x_n \to q$ denotes the strong convergence of a sequence $\{x_n\}$ to q in H;
- $x_n \rightharpoonup q$ denotes the weak convergence of a sequence $\{x_n\}$ to q in H;
- $G = (V(G), E(G))$ denotes a directed graph with $V(G) = C$ and $E(G)$ is convex.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, there exists the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The fact that H satisfies *Opial's condition* is well known, *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1 ([17]) *For any $u \in H$ and $v \in C$, $P_C u = v$ if and only if the inequality $\langle u - v, v - w\rangle \geq 0$ holds for all $w \in C$.*

Lemma 2.2 *For every $p,q \in H$. If $\|p+q\| = \|p\| + \|q\|$, there exists $s > 0$ such that $q = sp$ or $p = sq$.*

Lemma 2.3 ([18]) *Let $\{z_n\}$ be a sequence of nonnegative real numbers satisfying*

$$z_{n+1} \leq (1-a_n)z_n + \tau_n, \quad \forall n \geq 0,$$

where $\{a_n\}$ is a sequence in $(0,1)$ and $\{\tau_n\}$ is a sequence of real numbers. If the following conditions hold:

(1) $\sum_{n=1}^{\infty} a_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\tau_n}{a_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\tau_n| < \infty$,

then, $\lim_{n\to\infty} z_n = 0$.

The following basic definitions of domination in graphs ([19, 20]) are needed to prove the main theorem.

Given G a directed graph, a set $X \subseteq V(G)$ is called a dominating set if there exists $x \in X$ such that $(x,z) \in E(G)$ for every $z \in V(G) \setminus X$, and we say that x dominates z or z is dominated by x. Let $z \in V(G)$, a set $X \subseteq V(G)$ is dominated by z if $(z,x) \in E(G)$ for any $x \in X$ and we say that X dominates z if $(x,z) \in E(G)$ for all $x \in X$. This work assumes that $E(G)$ contains all loops.

Definition 2.4 (Property G [2]) Let X be a normed space. A nonempty $C \subset X$ is said to have the Property G if every sequence $\{x_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Theorem 2.5 ([2]) *Let X be a normed space, and G be a directed graph with $V(G) = X$. Let $T : X \to X$ be a G-nonexpansive mapping. If X has a Property G, then T is continuous.*

Theorem 2.6 ([2]) *Let C have the Property G. If $T : C \to C$ is a G-nonexpansive mapping, and $F(T) \times F(T) \subseteq E(G)$, then $F(T)$ is closed and convex.*

Definition 2.7 ([2]) A graph G is called transitive if for any $x,y,z \in V(G)$ such that (x,y) and (y,z) are in $E(G)$, then $(x,z) \in E(G)$.

Lemma 2.8 ([14]) *Let G be transitive with $E(G) = E(G^{-1})$ and let $A : C \to H$ be a G-α-inverse strongly monotone mapping with $A^{-1}(0) \neq \emptyset$. Then, G-$VI(C,A) = A^{-1}(0) = F(P_C(I - \lambda A))$, for all $\lambda > 0$.*

Lemma 2.9 ([14]) *Let C have a property G and let $A : C \to H$ be a G-α-inverse strongly monotone mapping with $F(P_C(I-\lambda A)) \times F(P_C(I-\lambda A)) \subseteq E(G)$, for all $\lambda \in (0, 2\alpha)$. Then, $F(P_C(I-\lambda A))$ is closed and convex.*

In 2017, Kangtunyakarn [3] introduced the G-S-mapping generated by a finite family of nonlinear mappings and finite real numbers as follows.

Definition 2.10 ([3]) For every $i = 1,2,\dots,N$, let T_i be a mapping of C into itself. For each $k = 1,2,\dots,N$, let $\alpha_k = (\alpha_1^k, \alpha_2^k, \alpha_3^k)$ where $\alpha_1^k, \alpha_2^k, \alpha_3^k \in [0,1]$ and $\alpha_1^k + \alpha_2^k + \alpha_3^k = 1$. Define the mapping $S : C \to C$ as follows:

$$
\begin{aligned}
&\mathcal{L}_0 = I\\
&\mathcal{L}_1 = \alpha_1^1 T_1 \mathcal{L}_0 + \alpha_2^1 \mathcal{L}_0 + \alpha_3^1 I,\\
&\mathcal{L}_2 = \alpha_1^2 T_2 \mathcal{L}_1 + \alpha_2^2 \mathcal{L}_1 + \alpha_3^2 I,\\
&\mathcal{L}_3 = \alpha_1^3 T_3 \mathcal{L}_2 + \alpha_2^3 \mathcal{L}_2 + \alpha_3^3 I,\\
&\vdots\\
&\mathcal{L}_{N-1} = \alpha_1^{N-1} T_{N-1} \mathcal{L}_{N-2} + \alpha_2^{N-1} \mathcal{L}_{N-2} + \alpha_3^{N-1} I,\\
&S = \mathcal{L}_N = \alpha_1^N T_N \mathcal{L}_{N-1} + \alpha_2^N \mathcal{L}_{N-1} + \alpha_3^N I.
\end{aligned}
$$

This mapping is called the *G-S-mapping* generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.11 ([3]) *Let $\{T_i\}_{i=1}^N : C \to C$ be a G-nonexpansive mapping with $\bigcap_{i=1}^N F(T_i)$ the dominating set. Let $\alpha_k = (\alpha_1^k, \alpha_2^k, \alpha_3^k)$, where $\alpha_1^k, \alpha_2^k, \alpha_3^k \in [0,1]$ with $\alpha_1^k + \alpha_2^k + \alpha_3^k = 1$ for all $k = 1, 2, \dots, N$ and $\alpha_1^k \in (0,1)$ for all $k = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_2^k, \alpha_3^k \in [0,1]$ for all $k = 1, 2, \dots, N$. Let $S : C \to C$ be a G-S-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then, $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a G-nonexpansive mapping.*

Lemma 2.12 ([3]) *Let C have the Property G. If $T : C \to C$ is a G-nonexpansive mapping, then $I - T$ is demiclosed at zero.*

3 Main results

In this section, we establish a strong convergence theorem of a regularization algorithm designed to solve the G-variational inequality problem and the fixed-point problem of a finite family of G-nonexpansive mappings in a Hilbert space endowed with graphs.

Theorem 3.1 *Let H be a Hilbert space and $C \subset H$ be nonempty, closed, and convex. Suppose a directed graph G with $V(G) = C$ has Property G, and it is transitive with $E(G) = E(G^{-1})$ is convex. Let $A : C \to H$ be a G-α-inverse strongly monotone operator with $A^{-1}(0) \neq \emptyset$. Let $\{T_i\}_{i=1}^N : C \to C$ be a G-nonexpansive mapping with $\bigcap_{i=1}^N F(T_i)$ the dominating set. Let $\alpha_k = (\alpha_1^k, \alpha_2^k, \alpha_3^k)$, where $\alpha_1^k, \alpha_2^k, \alpha_3^k \in [0,1]$ with $\alpha_1^k + \alpha_2^k + \alpha_3^k = 1$ for all $k = 1, 2, \dots, N$ and $\alpha_1^k \in (0,1)$ for all $k = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_2^k, \alpha_3^k \in [0,1]$ for all $k = 1, 2, \dots, N$. Let $S : C \to C$ be a G-S-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Assume that*

$$\Omega = \bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C, A) \neq \emptyset$$

with $\bigcap_{i=1}^N F(T_i) \times \bigcap_{i=1}^N F(T_i) \subseteq E(G)$ and $G\text{-}VI(C,A) \times G\text{-}VI(C,A) \subseteq E(G)$, and there exists $x_0 \in C$ such that $(x_0, T_1 x_0) \in E(G)$. Let $\{x_n\}$ be a sequence generated by $x_0 = u \in C$ and

$$x_{n+1} = P_C(I - \lambda A)\big(\beta_n u + (1 - \beta_n) S x_n\big), \quad n \geq 0, \tag{7}$$

where $\{\beta_n\} \subseteq [0,1]$ and $\lambda \in (0, 2\alpha)$.

If the following conditions hold:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $(T_i x, T_{i+1}x) \in E(G)$ *for all* $i = 1, 2, \dots, N-1$ *and* $x \in C$,

then, $\{x_n\}$ *converges strongly to* $P_\Omega x_0$, *where* P_Ω *is a metric projection on* Ω, $P_\Omega x_0$ *is dominated by* $\{x_n\}$, $P_\Omega x_0$ *is dominated by* x_0, *and* $\{x_n\}$ *dominates* x_0.

Proof First, we show that

$$\left\|P_C(I - \lambda A)x - P_C(I - \lambda A)y\right\| \le \|x - y\|, \tag{8}$$

for all $x, y \in C$ with $(x, y) \in E(G)$. Indeed, letting $x, y \in C$ with $(x, y) \in E(G)$, we have

$$\begin{aligned} \left\|P_C(I - \lambda A)x - P_C(I - \lambda A)y\right\|^2 &\le \left\|x - y - \lambda(Ax - Ay)\right\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle \\ &\quad + \lambda^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 - \lambda(2\alpha - \lambda)\|Ax - Ay\|^2 \\ &\le \|x - y\|^2. \end{aligned} \tag{9}$$

From Lemmas 2.8 and 2.9, we have G-$VI(C, A)$ is closed and convex. From Lemmas 2.11 and 2.12, we have $F(S) = \bigcap_{i=1}^{N} F(T_i)$ is closed and convex. Then, Ω is closed and convex. Moreover, from $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and Lemma 2.11, we have $\bigcap_{i=1}^{N} F(T_i) = F(S)$ and S is a G-nonexpansive mapping.

Next, we will show that $(x_n, x_{n+1}) \in E(G)$ for all $n \ge 0$.

Put $x^* = P_\Omega x_0$. Since x^* is dominated by $\{x_n\}$, we have $(x_n, x^*) \in E(G)$ for all $n \ge 0$.

Since x^* is dominated by $\{x_0\}$, we have $(x_0, x^*) \in E(G)$.

Since $\{x_n\}$ dominates x_0, we have $(x_n, x_0) \in E(G)$ for all $n \ge 0$.

Since $E(G) = E(G^{-1})$ and $(x_n, x_0) \in E(G)$, then $(x_0, x_n) \in E(G)$ for all $n \ge 0$.

By the transitivity of $E(G)$ and since (x_n, x^*), (x^*, x_0), (x_0, x_{n+1}) are in $E(G)$, then $(x_n, x_{n+1}) \in E(G)$ for all $n \ge 0$.

Putting $y_n = \beta_n u + (1 - \beta_n)Sx_n$ for all $n \ge 0$, it follows that $x_{n+1} = P_C(I - \lambda A)y_n$ for all $n \ge 0$.

We now claim that (x_0, Sx_n), (x_n, Sx_n), and (x_n, y_n) are in $E(G)$ for all $n \ge 0$.

Since $E(G) = E(G^{-1})$ and $(x_0, x^*) \in E(G)$, we have $(x^*, x_0) \in E(G)$.

We now prove this result by using mathematical induction. By continuing in the same direction as in Theorem 3.1 [3], we have $(x_0, Sx_0) \in E(G)$.

Since (x_0, x_0), $(x_0, Sx_0) \in E(G)$, and $E(G)$ is convex, we have $(x_0, y_0) \in E(G)$.

Since S is G-nonexpansive and $(x_0, x_1) \in E(G)$, we obtain $(Sx_0, Sx_1) \in E(G)$.

By the transitivity of $E(G)$ and since (x_0, Sx_0), (Sx_0, Sx_1) are in $E(G)$, we obtain $(x_0, Sx_1) \in E(G)$.

As $E(G)$ is convex and (x_0, x_0), (x_0, Sx_1) are in $E(G)$, we have $(x_0, y_1) \in E(G)$.

By the transitivity of $E(G)$ and since (x_1, x_0), (x_0, Sx_1) are in $E(G)$, we have $(x_1, Sx_1) \in E(G)$.

By the transitivity of $E(G)$ and since (x_1, x_0), (x_0, y_1) are in $E(G)$, we have $(x_1, y_1) \in E(G)$.

Suppose that $(x_0, Sx_k) \in E(G)$ for all $k \ge 0$. Since (x_0, x_0), $(x_0, Sx_k) \in E(G)$ and $E(G)$ is convex, we have $(x_0, y_k) \in E(G)$ for all $k \ge 0$.

Since S is G-nonexpansive and $(x_k, x_{k+1}) \in E(G)$ for all $k \geq 0$, we obtain $(Sx_k, Sx_{k+1}) \in E(G)$ for all $k \geq 0$.

By the transitivity of $E(G)$ and since (x_0, Sx_k), (Sx_k, Sx_{k+1}) are in $E(G)$ for all $k \geq 0$, we obtain $(x_0, Sx_{k+1}) \in E(G)$ for all $k \geq 0$.

As $E(G)$ is convex and (x_0, x_0), (x_0, Sx_{k+1}) are in $E(G)$ for all $k \geq 0$, we have $(x_0, y_{k+1}) \in E(G)$ for all $k \geq 0$.

By the transitivity of $E(G)$ and since (x_{k+1}, x_0), (x_0, y_{k+1}) are in $E(G)$ for all $k \geq 0$, we have $(x_{k+1}, y_{k+1}) \in E(G)$ for all $k \geq 0$.

By the transitivity of $E(G)$ and since (x_{k+1}, x_0), (x_0, Sx_{k+1}) are in $E(G)$ for all $k \geq 0$, we have $(x_{k+1}, Sx_{k+1}) \in E(G)$ for all $k \geq 0$.

From induction, we obtain that (x_0, Sx_n), (x_n, Sx_n), and (x_n, y_n), are in $E(G)$ for all $n \geq 0$. Moreover, By the transitivity of $E(G)$ and since (x^*, x_0), (x_0, y_n) are in $E(G)$, we have $(x^*, y_n) \in E(G)$ for all $n \geq 0$.

From Lemma 2.8, we obtain $G\text{-}VI(C,A) = A^{-1}(0)$. Then, $x^* \in A^{-1}(0)$. Since $Ax^* = 0$, we have

$$\begin{aligned}
\|P_C(I-\lambda A)y_n - x^*\|^2 &\leq \|y_n - x^* - \lambda A y_n\|^2 \\
&= \|y_n - x^*\|^2 - 2\lambda\langle y_n - x^*, Ay_n - Ax^*\rangle + \lambda^2\|Ay_n\|^2 \\
&\leq \|y_n - x^*\|^2 - 2\lambda\|Ay_n - Ax^*\|^2 + \lambda^2\|Ay_n\|^2 \\
&= \|y_n - x^*\|^2 - \lambda(2\alpha - \lambda)\|Ay_n\|^2 \\
&\leq \|y_n - x^*\|^2.
\end{aligned} \tag{10}$$

From the definition of x_n, (10), and since S is a G-nonexpansive mapping, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|y_n - x^*\| \\
&\leq \|\beta_n u + (1-\beta_n)Sx_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1-\beta_n)\|Sx_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1-\beta_n)\|x_n - x^*\| \\
&\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}.
\end{aligned} \tag{11}$$

By using mathematical induction, we conclude that the sequences $\{x_n\}$, $\{P_C(I-\lambda A)y_n\}$, and $\{Sx_n\}$ are all bounded.

From the definition of x_n and (9), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|P_C(I-\lambda A)y_n - P_C(I-\lambda A)y_{n-1}\| \\
&\leq \|y_n - y_{n-1}\| \\
&\leq \|\beta_n u + (1-\beta_n)Sx_n - \alpha_{n-1}u - (1-\alpha_{n-1})Sx_{n-1}\| \\
&\leq |\beta_n - \alpha_{n-1}|\|u\| + (1-\beta_n)\|Sx_n - Sx_{n-1}\| + |\beta_n - \alpha_{n-1}|\|Sx_{n-1}\| \\
&\leq (1-\beta_n)\|x_n - x_{n-1}\| + 2M|\beta_n - \alpha_{n-1}|,
\end{aligned}$$

where $M = \max_{n\in\mathbb{N}}\{\|u\|, \|Sx_n\|\}$. Applying Lemma 2.3, and conditions (i) and (ii), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{12}$$

Since $x_{n+1} = P_C(I - \lambda A)y_n$, then we also have

$$\lim_{n\to\infty} \left\|P_C(I - \lambda A)y_n - x_n\right\| = 0. \tag{13}$$

By the nonexpansiveness of P_C, we obtain

$$\begin{aligned}\left\|x_{n+1} - x^*\right\|^2 &= \left\|P_C(I - \lambda A)y_n - P_C(I - \lambda A)x^*\right\|^2 \\ &\leq \left\|(I - \lambda A)y_n - (I - \lambda A)x^*\right\|^2 \\ &= \left\|y_n - x^* - \lambda\big(Ay_n - Ax^*\big)\right\|^2 \\ &= \left\|y_n - x^*\right\|^2 - 2\lambda\big\langle y_n - x^*, Ay_n - Ax^*\big\rangle + \lambda^2\left\|Ay_n - Ax^*\right\|^2.\end{aligned} \tag{14}$$

From the definition of y_n and since S is G-nonexpansive, we have

$$\begin{aligned}\left\|y_n - x^*\right\|^2 &\leq \beta_n\left\|u - x^*\right\|^2 + (1 - \beta_n)\left\|Sx_n - x^*\right\|^2 \\ &\leq \beta_n\left\|u - x^*\right\|^2 + (1 - \beta_n)\left\|x_n - x^*\right\|^2 \\ &\leq \beta_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2.\end{aligned} \tag{15}$$

From (14) and (15), we obtain

$$\begin{aligned}\left\|x_{n+1} - x^*\right\|^2 &\leq \beta_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - 2\lambda\big\langle y_n - x^*, Ay_n - Ax^*\big\rangle \\ &\quad + \lambda^2\left\|Ay_n - Ax^*\right\|^2 \\ &\leq \beta_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - 2\lambda\alpha\left\|Ay_n - Ax^*\right\|^2 \\ &\quad + \lambda^2\left\|Ay_n - Ax^*\right\|^2 \\ &\leq \beta_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \lambda(2\alpha - \lambda)\left\|Ay_n - Ax^*\right\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\lambda(2\alpha - \lambda)\left\|Ay_n - Ax^*\right\|^2 &\leq \beta_n\left\|u - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\ &\leq \beta_n\left\|u - x^*\right\|^2 + \big(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\big)\|x_{n+1} - x_n\|.\end{aligned}$$

From the condition (i) and (12), we obtain

$$\lim_{n\to\infty} \left\|Ay_n - Ax^*\right\| = 0. \tag{16}$$

From the definition of $P_C(I - \lambda A)$, we have

$$\left\|P_C(I - \lambda A)y_n - x^*\right\|^2 \leq \left\|P_C(I - \lambda A)y_n - P_C(I - \lambda A)x^*\right\|^2$$

$$\begin{aligned}
&\le \langle (I-\lambda A)y_n - (I-\lambda A)x^*, P_C(I-\lambda A)y_n - x^*\rangle \\
&= \frac{1}{2}\Big[\big\|(I-\lambda A)y_n - (I-\lambda A)x^*\big\|^2 + \big\|P_C(I-\lambda A)y_n - x^*\big\|^2 \\
&\quad - \big\|(I-\lambda A)y_n - (I-\lambda A)x^* - \big(P_C(I-\lambda A)y_n - x^*\big)\big\|^2\Big] \\
&\le \frac{1}{2}\Big[\big\|y_n - x^*\big\|^2 + \big\|P_C(I-\lambda A)y_n - x^*\big\|^2 \\
&\quad - \big\|y_n - P_C(I-\lambda A)y_n - \lambda\big(Ay_n - Ax^*\big)\big\|^2\Big] \\
&\le \frac{1}{2}\Big[\beta_n\big\|u - x^*\big\|^2 + \big\|x_n - x^*\big\|^2 + \big\|P_C(I-\lambda A)y_n - x^*\big\|^2 \\
&\quad - \big\|y_n - P_C(I-\lambda A)y_n\big\|^2 - \lambda^2\big\|Ay_n - Ax^*\big\|^2 \\
&\quad + 2\lambda\langle y_n - P_C(I-\lambda A)y_n, Ay_n - Ax^*\rangle\Big] \\
&\le \frac{1}{2}\Big[\beta_n\big\|u - x^*\big\|^2 + \big\|x_n - x^*\big\|^2 + \big\|P_C(I-\lambda A)y_n - x^*\big\|^2 \\
&\quad - \big\|y_n - P_C(I-\lambda A)y_n\big\|^2 - \lambda^2\big\|Ay_n - Ax^*\big\|^2 \\
&\quad + 2\lambda\big\|y_n - P_C(I-\lambda A)y_n\big\|\big\|Ay_n - Ax^*\big\|\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
\big\|x_{n+1} - x^*\big\|^2 &= \big\|P_C(I-\lambda A)y_n - x^*\big\|^2 \\
&\le \beta_n\big\|u - x^*\big\|^2 + \big\|x_n - x^*\big\|^2 - \big\|y_n - P_C(I-\lambda A)y_n\big\|^2 \\
&\quad - \lambda^2\big\|Ay_n - Ax^*\big\|^2 + 2\lambda\big\|y_n - P_C(I-\lambda A)y_n\big\|\big\|Ay_n - Ax^*\big\|. \qquad (17)
\end{aligned}$$

From (17), we have

$$\begin{aligned}
\big\|y_n - P_C(I-\lambda A)y_n\big\|^2 &\le \beta_n\big\|u - x^*\big\|^2 + \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\|x_{n+1} - x_n\| \\
&\quad + 2\lambda\big\|y_n - P_C(I-\lambda A)y_n\big\|\big\|Ay_n - Ax^*\big\|. \qquad (18)
\end{aligned}$$

From the condition (i) and (12), (16), and (18), we have

$$\lim_{n\to\infty}\big\|P_C(I-\lambda A)y_n - y_n\big\| = 0. \qquad (19)$$

Since

$$\|y_n - x_n\| \le \big\|y_n - P_C(I-\lambda A)y_n\big\| + \big\|P_C(I-\lambda A)y_n - x_n\big\|,$$

from (13) and (19), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \qquad (20)$$

From the definition of y_n, condition (i), and (20), we obtain

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \qquad (21)$$

Since $\{x_n\}$ is bounded in a Hilbert space H and C has the property G, without loss of generality, we may assume that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\{x_{n_k}\} \rightharpoonup \omega$ for some $\omega \in C$ and $(x_{n_k}, \omega) \in E(G)$. Since $\lim_{n\to\infty} \|y_n - x_n\| = 0$, then we obtain $y_{n_k} \to \omega$ as $k \to \infty$. By Lemma 2.12 and (21), we obtain $\omega \in F(S)$. This implies that

$$\omega \in \bigcap_{i=1}^{N} F(T_i). \tag{22}$$

By the transitivity of $E(G)$ and (y_{n_k}, x_{n_k}), (x_{n_k}, ω) being in $E(G)$, we obtain $(y_{n_k}, \omega) \in E(G)$ for all $k \in \mathbb{N}$. Assume that $\omega \neq P_C(I - \lambda A)\omega$. From Opial's condition, (9), (19), and $(y_{n_k}, \omega) \in E(G)$, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|y_{n_k} - \omega\| &< \liminf_{k\to\infty} \left\|y_{n_k} - P_C(I - \lambda A)\omega\right\| \\
&\leq \liminf_{k\to\infty} \left(\left\|y_{n_k} - P_C(I - \lambda A)y_{n_k}\right\| + \left\|P_C(I - \lambda A)y_{n_k} - P_C(I - \lambda A)\omega\right\| \right) \\
&\leq \liminf_{k\to\infty} \|y_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction. Then, $\omega \in F(P_C(I - \lambda A))$. Therefore, from Lemma 2.8, we have

$$w \in G\text{-}VI(C, A). \tag{23}$$

From (22) and (23), we can conclude that $\omega \in \Omega$.

Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \Omega$, we have

$$\limsup_{n\to\infty} \langle x_n - x^*, x_0 - x^* \rangle = \lim_{k\to\infty} \langle x_{n_k} - x^*, x_0 - x^* \rangle = \langle \omega - x^*, x_0 - x^* \rangle \leq 0, \tag{24}$$

where $x^* = P_\Omega x_0$. From the definition of x_n and (10), we have

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &= \left\|P_C(I - \lambda A)y_n - P_C(I - \lambda A)x^*\right\|^2 \\
&\leq \left\|y_n - x^*\right\|^2 \\
&\leq (1 - \beta_n)\left\|x_n - x^*\right\|^2 + 2\beta_n \langle x_{n+1} - x^*, x_0 - x^* \rangle.
\end{aligned}$$

Applying Lemma 2.3, (24), and the condition (i), we can conclude that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega x_0$. This completes the proof. □

In our main results, if we choose $N = 1$ and $\alpha_1^N = 1$, then we obtain the following result.

Corollary 3.2 *Let C have the Property G and G be transitive with $E(G) = E(G^{-1})$. Let $A : C \to H$ be a G-α-inverse strongly monotone operator with $A^{-1}(0) \neq \emptyset$. Let $T : C \to C$ be a G-nonexpansive mapping. Assume that*

$$\Omega = F(T) \cap G\text{-}VI(C, A) \neq \emptyset$$

with $F(T) \times F(T) \subseteq E(G)$ and G-$VI(C,A) \times G$-$VI(C,A) \subseteq E(G)$, and there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Let $\{x_n\}$ be a sequence generated by $x_0 = u \in C$ and

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n u + (1-\beta_n)Tx_n \\ x_{n+1} = P_C(I-\lambda A)y_n, \quad n \geq 0, \end{cases} \tag{25}$$

where $\{\beta_n\} \subseteq [0,1]$ and $\lambda \in (0,2\alpha)$.

If the following conditions hold:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then, $\{x_n\}$ converges strongly to $P_\Omega x_0$, where P_Ω is a metric projection on Ω, $P_\Omega x_0$ is dominated by $\{x_n\}$, $P_\Omega x_0$ is dominated by x_0, and $\{x_n\}$ dominates x_0.

Taking $A = 0$ and $P_C = I$ in Theorem 3.1, then we obtain the following result.

Corollary 3.3 *Let C have the Property G and G be transitive with $E(G) = E(G^{-1})$. Let $\{T_i\}_{i=1}^N : C \to C$ be a G-nonexpansive mapping with $\bigcap_{i=1}^N F(T_i)$ the dominating set. Let $\alpha_k = (\alpha_1^k, \alpha_2^k, \alpha_3^k)$, where $\alpha_1^k, \alpha_2^k, \alpha_3^k \in [0,1]$ with $\alpha_1^k + \alpha_2^k + \alpha_3^k = 1$ for all $k = 1,2,\dots,N$ and $\alpha_1^k \in (0,1)$ for all $k = 1,2,\dots,N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_2^k, \alpha_3^k \in [0,1]$ for all $k = 1,2,\dots,N$. Let $S : C \to C$ be a G-S-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \beta_n$. Assume that*

$$\Omega = \bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A) \neq \emptyset$$

with $\bigcap_{i=1}^N F(T_i) \times \bigcap_{i=1}^N F(T_i) \subseteq E(G)$, and there exists $x_0 \in C$ such that (x_0, T_1x_0). Let $\{x_n\}$ be a sequence generated by $x_0 = u \in C$ and

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \beta_n u + (1-\beta_n)Sx_n, \quad n \geq 0, \end{cases} \tag{26}$$

where $\{\beta_n\} \subseteq [0,1]$.

If the following conditions hold:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $(T_ix, T_{i+1}x) \in E(G)$ *for all* $i = 1,2,\dots,N-1$ *and* $x \in C$,

then, $\{x_n\}$ converges strongly to $P_\Omega x_0$, where P_Ω is a metric projection on Ω, $P_\Omega x_0$ is dominated by $\{x_n\}$, $P_\Omega x_0$ is dominated by x_0, and $\{x_n\}$ dominates x_0.

Taking $A = 0$, $P_C = I$, $N = 1$, and $\alpha_1^N = 1$ in Theorem 3.1, then we obtain the following result.

Corollary 3.4 *Let C have the Property G and G be transitive with $E(G) = E(G^{-1})$. Let $T : C \to C$ be a G-nonexpansive mapping. Assume that $F(T) \neq \emptyset$ with $F(T) \times F(T) \subseteq E(G)$, and there exists $x_0 \in C$ such that (x_0, Tx_0). Let $\{x_n\}$ be a sequence generated by $x_0 = u \in C$*

and

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \beta_n u + (1-\beta_n)Tx_n, \quad n \geq 0, \end{cases} \tag{27}$$

where $\{\beta_n\} \subseteq [0,1]$.

If the following conditions hold:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then, $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$, *where* $P_{F(T)}$ *is a metric projection on* $F(T)$, $P_{F(T)}x_0$ *is dominated by* $\{x_n\}$, $P_{F(T)}x_0$ *is dominated by* x_0, *and* $\{x_n\}$ *dominates* x_0.

Remark 1 We have the following consequences of Theorem 3.1:

1. The results of Kangtunyakarn [3] is a special case of Theorem 3.1 by taking $A = 0$ and $P_C = I$.
2. The results of Tiammee *et al.* [2] is a special case of Theorem 3.1 by taking $A = 0$, $P_C = I$, $N = 1$, and $\alpha_1^N = 1$.

4 Examples and numerical results

In this section, we provide some numerical examples to support our obtained result. To obtain these results, we recall some lemmas as follows.

Lemma 4.1 [14] *Let* $G = (V(G), E(G))$ *be a directed graph with* $V(G) = C$ *dominating* z *for all* $z \in C$. *Let* $E(G)$ *be convex and* G *be a transitive with* $E(G) = E(G^{-1})$. *Let* $S : C \to C$ *be a* G*-nonexpansive mapping with* $F(S) \neq \emptyset$ *and* $F(S) \times F(S) \subseteq E(G)$. *Then*,

(i) $I - S$ *is* G-$\frac{1}{2}$*-inverse strongly monotone*;

(ii) G-$VI(C, I - S) = F(S)$.

We now provide an example to support our main result.

Example 4.2 Let $H = \mathbb{R}$ and $C = [0, 1.5]$ with the usual norm $\|x - y\| = |x - y|$ and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G) = \{(x, y) : x, y \in [0, 1]$ with $|x - y| < 1\}$. For every $i = 1, 2, \dots, N$, define the mapping $T_i : [0, 1.5] \to [0, 1.5]$ by

$$T_i x = \begin{cases} \frac{x}{2^i} + (1 - \frac{1}{2^i})\frac{1}{2} & \text{if } x \in [0,1], \\ 0 & \text{if } x \in (1, 1.5]. \end{cases} \tag{28}$$

Let $S : [0, 1.5] \to [0, 1.5]$ be a G-S-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_n$, where

$$\alpha_i = \left(\frac{1}{5^i}, \frac{5^i - 1}{5^i}\left(\frac{6^i - 1}{6^i}\right), \frac{5^i - 1}{5^i}\left(\frac{1}{6^i}\right)\right)$$

for all $i = 1, 2, \dots, N$ and let $A : [0, 1.5] \to \mathbb{R}$ be a mapping defined by

$$Ax = x - \frac{x^2}{4} - \frac{7}{16}, \tag{29}$$

for all $x \in [0, 1.5]$.

Suppose that the sequence $\{x_n\}$ is generated by $u = x_0 = \frac{1}{6}$ and

$$x_{n+1} = P_{[0,1.5]}\left(I - \frac{1}{6}A\right)\left(\beta_n u + (1-\beta_n)Sx_n\right), \tag{30}$$

where $\beta_n = \frac{1}{2n+2}$ for all $n \geq 0$.

Then, the sequence $\{x_n\}$ converges strongly to $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}x_0 = \{\frac{1}{2}\}$.

Solution. It is clear that $A^{-1}(0) \neq \emptyset$, since $\frac{1}{2} \in A^{-1}(0)$ and $E(G) = E(G^{-1})$. Let $x, y \in [0, 1.5]$ with $(x,y) \in E(G)$. It is easy to check that T_i is a G-nonexpansive mapping for all $i = 1, 2, \dots, N$ such that $\bigcap_{i=1}^N F(T_i) = \{\frac{1}{2}\}$. However, it is not nonexpansive, as $|x-y| < |T_i x - T_i y|$ for all $i = 1, 2, \dots, N$, where $x = 1$, $y = 1.1$. Let $v \in V(G) - \bigcap_{i=1}^N F(T_i)$. Since $\frac{1}{2} \in \bigcap_{i=1}^N F(T_i)$, we have $|\frac{1}{2} - v| < 1$. It follows that $\bigcap_{i=1}^N F(T_i)$ is the dominating set. Let $x \in V(G) = [1, 1.5] = [0,1] \cup (1, 1.5]$. Then,

Case $x \in [0,1]$. Then,

$$T_i x = \frac{x}{2^i} + \left(1 - \frac{1}{2^i}\right)\frac{1}{2} \quad \text{and} \quad T_{i+1}x = \frac{x}{2^{i+1}} + \left(1 - \frac{1}{2^{i+1}}\right)\frac{1}{2},$$

for all $i = 1, 2, \dots, N-1$. Since $[0,1]$ is convex, we have $T_i x, T_{i+1}x \in [0,1]$ for all $i = 1, 2, \dots, N-1$. Observe that

$$|T_{i+1}x - T_i x| = \frac{1}{2^{i+1}}\left|\frac{1}{2} - x\right| \leq 1.$$

Then, $(T_{i+1}x, T_i x) \in E(G)$ for all $i = 1, 2, \dots, N-1$.

Case $x \in (1, 1.5]$. It is obvious that $|T_i x - T_{i+1}x| < 1$. Then, $(T_i x, T_{i+1}x) \in E(G)$ for all $i = 1, 2, \dots, N-1$.

It is easy to check that $\mathcal{T}$ is a G-nonexpansive mapping, where $\mathcal{T}x = \frac{x^2}{4} + \frac{7}{16}$ for all $x \in [0,1]$.

Since $Ax = x - \frac{x^2}{4} - \frac{7}{16} = (I - \mathcal{T})x$ for all $x \in [0,1]$, $\mathcal{T}$ is a G-nonexpansive mapping and from Lemma 4.1, we have A is G-$\frac{1}{2}$-inverse strongly monotone. Then, $\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A) = \{\frac{1}{2}\}$.

For every $z \in [0,1]$, we have

$$\begin{aligned}
\left(I - \frac{1}{6}A\right)z &= z - \frac{1}{6}Az \\
&= z - \frac{1}{6}\left(z - \frac{z^2}{4} - \frac{7}{16}\right) \\
&= \frac{5z}{6} + \frac{1}{6}\left(\frac{z^2}{4} + \frac{7}{16}\right) \\
&= \frac{5z}{6} + \frac{1}{6}\left(\frac{z^2}{4} + \frac{3}{4}\cdot\frac{28}{48}\right) \in [0,1].
\end{aligned}$$

From the definition of P_C, we have

$$P_{[0,1.5]}\left(I - \frac{1}{6}A\right)z \in [0,1], \tag{31}$$

for all $z \in [0,1]$.

Putting $x_0 = \frac{1}{6}$ and $y_n = \beta_n u + (1-\beta_n)Sx_n$ for all $n \geq 0$, from (31), we have $P_{[0,1.5]}(I - \frac{1}{4}A)x_0 \in [0,1]$. This implies that $T_1x_0 \in [0,1]$ and $|x_0 - T_1x_0| < 1$. This implies that $(x_0, T_1x_0) \in E(G)$.

From the definition of S and $x_0 = \frac{1}{6}$, we have $Sx_0 \in [0,1]$. It follows that $(x_0, Sx_0) \in E(G)$.

From $x_0, Sx_0 \in [0,1]$ and the definition of y_n, we have $y_0 \in [0,1]$ for all $n \geq 0$.

From (31) and $y_0 \in [0,1]$, we have $P_{[0,1.5]}(I - \frac{1}{6}A)y_0 \in [0,1]$. This implies that $x_1 \in [0,1]$.

Continuing in this way, we have $x_n \in [0,1]$ for all $n \geq 0$.

Since $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}x_0 = \{\frac{1}{2}\}$ and $x_n \in [0,1]$ for all $n \geq 0$, we have $|x_n - \frac{1}{2}| < 1$. This implies that $(x_n, P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}x_0) \in E(G)$. Then, $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}x_0$ is dominated by $\{x_n\}$.

It is obvious that $\{x_n\}$ dominates x_0 and also $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}x_0$ is dominated by x_0, where $x_0 = \frac{1}{6}$. From Theorem 3.1, we have the sequence $\{x_n\}$ converging strongly to $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}x_0 = \{\frac{1}{2}\}$.

We first start with the initial point $x_0 = \frac{1}{6}$. Our testing procedure takes $|x_{n+1} - x_n| \leq 1E-12$ as the stopping condition. Now, a convergence of the algorithm (30) is shown in Table 1 and visualized in Figs. 1 and 2.

Next, a comparison of algorithm (30) and algorithm (5) of [14] is provided, focusing on CPU time and the number of iterations for different initial points, as detailed in Table 2. Moreover, Our testing procedure takes $|x_{n+1} - x_n| \leq 1E-6$ as the stopping condition.

Remark 2 By observing the convergence behavior of Algorithm (30) in Example 4.2, we conclude that

1. Table 1 and Figs. 1 and 2 show that $\{x_n\}$ converges to a solution, i.e., $x_n \to 1/2 \in \Omega$. The convergence of $\{x_n\}$ of Example 4.2 can be guaranteed by Theorem 3.1.
2. The values of the sequence $\{x_n\}$ with respect to n are also plotted in Fig. 1, demonstrating that $(x_n, x_0), (x_{n+1}, x_n) \in E(G)$.

Table 1 Convergence of the algorithm (30) in Example 4.2

n	x_n	x_{n+1}	$\lvert x_n - x_{n+1}\rvert$
0	0.166666666667	0.234169560185	0.067502893519
1	0.234169560185	0.279645516164	0.045475955979
2	0.279645516164	0.313633206903	0.033987690739
3	0.313633206903	0.340160992255	0.026527785352
4	0.340160992255	0.361372598770	0.021211606515
5	0.361372598770	0.378608112538	0.017235513769
⋮	⋮	⋮	⋮
396	0.498263926837	0.498268307737	0.000004380900
397	0.498268307737	0.498272666581	0.000004358844
398	0.498272666581	0.498277003536	0.000004336955
399	0.498277003536	0.498281318767	0.000004315230
400	0.498281318767	0.498285612435	0.000004293668
⋮	⋮	⋮	⋮
828,361	0.499999171527	0.499999171528	0.000000000001
828,362	0.499999171528	0.499999171529	0.000000000001
828,363	0.499999171529	0.499999171530	0.000000000001
828,364	0.499999171530	0.499999171531	0.000000000001
828,365	0.499999171531	0.499999171532	0.000000000001
828,366	0.499999171532	0.499999171533	0.000000000001
828,367	0.499999171533	0.499999171533	0.000000000000

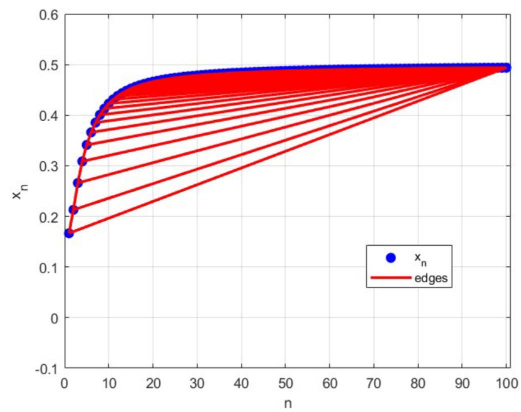

Figure 1 Visualization of the first one hundred rounds of algorithm (30) in Example 4.2

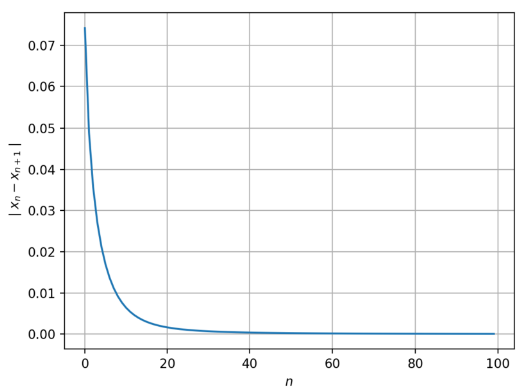

Figure 2 Visualization of the error $|x_n - x_{n+1}|$ of algorithm (30) in Example 4.2

Table 2 Numerical values of algorithm (30) and algorithm (5) of [14]

Starting point		Algorithm (30)	Algorithm (5) of [14]
$x_0 = 1/6$	No. of Iter.	781	1143
	CPU Time (s)	6.662671	9.472705
$x_0 = 1/8$	No. of Iter.	895	1294
	CPU Time (s)	7.451446	11.2578180
$x_0 = 3/4$	No. of Iter.	297	601
	CPU Time (s)	2.443946	5.194785

3. For every $i = 1, 2, \ldots, N$, T_i are G-nonexpansive mappings but not nonexpansive.
4. From Table 2, we see that the sequence generated by our algorithm (30) has better convergence than algorithm (5) of [14] in terms of the number of iterations and the CPU time.

Next, we give an example in the infinite-dimensional space l_2 to support some results as follows.

Example 4.3 Let $C := \{\mathbf{x} = (x_1, x_2, x_3, \ldots) \in l_2 : \|\mathbf{x}\|_{l_2} \leq 1 \text{ and } x_i \in [0,1] \text{ for } i = 1, 2, 3, \ldots\}$ with the norm $\|\mathbf{x}\|_{l_2} = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2}$ and the inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{\infty} x_i y_i$ for $\mathbf{y} =$

Table 3 Convergence of the algorithm (33) in Example 4.3

n	$\mathbf{x}_n$	$\|\mathbf{x}_{n+1}-\mathbf{x}_n\|_{l_2}$
0	(0.1666666667, 0.1250000000, 0, 0, 0, ...)	–
1	(0.0520833333, 0.0356445312, 0, 0, 0, ...)	0.145305678049104
2	(0.0289080584, 0.0212303748, 0, 0, 0, ...)	0.027292146725652
3	(0.0211989415, 0.0157728954, 0, 0, 0, ...)	0.009445346214247
4	(0.0168688945, 0.0125839647, 0, 0, 0, ...)	0.005377600440776
5	(0.0140193120, 0.0104711016, 0, 0, 0, ...)	0.003547437162869
6	(0.0119960131, 0.0089667510, 0, 0, 0, ...)	0.002521271327788
7	(0.0104841218, 0.0078407665, 0, 0, 0, ...)	0.001885114422034
8	(0.0093111644, 0.0069662178, 0, 0, 0, ...)	0.001463101033630
9	(0.0083745148, 0.0062672882, 0, 0, 0, ...)	0.001168680937519
10	(0.0076092299, 0.0056958782, 0, 0, 0, ...)	0.000955076040639
⋮	⋮	⋮
10,198	(0.0000081708, 0.0000061281, 0, 0, 0, ...)	0.000000001001519
10,199	(0.0000081700, 0.0000061275, 0, 0, 0, ...)	0.000000001001323
10,200	(0.0000081692, 0.0000061269, 0, 0, 0, ...)	0.000000001001126
10,201	(0.0000081684, 0.0000061263, 0, 0, 0, ...)	0.000000001000930
10,202	(0.0000081676, 0.0000061257, 0, 0, 0, ...)	0.000000001000734
10,203	(0.0000081668, 0.0000061251, 0, 0, 0, ...)	0.000000001000538
10,204	(0.0000081660, 0.0000061245, 0, 0, 0, ...)	0.000000001000342
10,205	(0.0000081652, 0.0000061239, 0, 0, 0, ...)	0.000000001000145
10,206	(0.0000081644, 0.0000061233, 0, 0, 0, ...)	0.000000000999950

$(y_1, y_2, y_3, \dots) \in l_2 : \|\mathbf{y}\|_{l_2} \le 1$ and $y_i \in [0,1]$. Let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(\mathbf{x},\mathbf{y}) : x_i, y_i \in [0, \frac{1}{3}]$ with $\|\mathbf{x}-\mathbf{y}\|_{l_2} \le \frac{1}{5}$ for $i = 1,2,3,\dots\}$. Define the mapping $T : C \to C$ by

$$T\mathbf{x} = \left(\frac{1}{2}x_1^2, \frac{3}{8}x_2^2, 0, 0, 0, \dots\right), \quad \forall \mathbf{x} \in C. \tag{32}$$

Suppose that the sequence $\{\mathbf{x}_n\}$ is generated by $\mathbf{u} = \mathbf{x}_0 = (\frac{1}{6}, \frac{1}{8}, 0, 0, 0, \dots)$ and

$$\mathbf{x}_{n+1} = \beta_n \mathbf{u} + (1-\beta_n) T\mathbf{x}_n, \tag{33}$$

where $\beta_n = \frac{1}{2n+2}$ for all $n \ge 0$. Then, the sequence $\{\mathbf{x}_n\}$ converges strongly to $P_{F(T)}\mathbf{x}_0$.

Solution. We can easily show that T is a G-nonexpansive mapping with $F(T) = \{\mathbf{0}\}$, where $\mathbf{0} = (0,0,0,0,0,\dots)$ is the null vector on l_2. From the definition of T and $\mathbf{u} = \mathbf{x}_0 = (\frac{1}{6}, \frac{1}{8}, 0, 0, 0, \dots)$, we have $(\mathbf{x}_0, T\mathbf{x}_0) \in E(G)$. Since $P_{F(T)}\mathbf{x}_0 = \{\mathbf{0}\}$ and the definition of $\mathbf{x}_n$, we have $\|\mathbf{x}_n - \mathbf{0}\|_{l_2} \le \frac{1}{5}$. It follows that $(\mathbf{x}_n, P_{F(T)}\mathbf{x}_0) \in E(G)$. Then, $P_{F(T)}\mathbf{x}_0$ is dominated by $\{\mathbf{x}_n\}$. It is obvious that $\{\mathbf{x}_n\}$ dominates $\mathbf{x}_0$ and also $P_{F(T)}\mathbf{x}_0$ is dominated by $\{\mathbf{x}_0\}$. From Corollary 3.4, we have the sequence $\{\mathbf{x}_n\}$ converging strongly to $P_{F(T)}\mathbf{x}_0 = \{\mathbf{0}\}$. We first start with the initial point $\mathbf{x}_0 = (\frac{1}{6}, \frac{1}{8}, 0, 0, 0, \dots)$. The stopping criterion for our testing method is taken as $\|\mathbf{x}_{n+1} - \mathbf{x}_n\|_{l_2} \le 1\text{E}-9$. Now, a convergence of the algorithm (33) is shown in Table 3 and visualized in Fig. 3.

Remark 3 By observing the convergence behavior of Algorithm (30) in Example 4.3, we conclude that it converges to a solution, i.e., $\mathbf{x}_n \to \mathbf{0} \in F(T)$.

Next, we provide a numerical example to support our results in a two-dimensional space.

Example 4.4 Let $H = \mathbb{R}^2$ and $C = [-2,2] \times [-2,2]$. Let $G = (V(G), E(G))$ be a directed graph, where $V(G) = C$ and $E(G) = \{(\mathbf{x},\mathbf{y}) = ((x_1,x_2),(y_1,y_2)) : \mathbf{x},\mathbf{y} \in [-1,1] \times [-1,1]\}$. Let

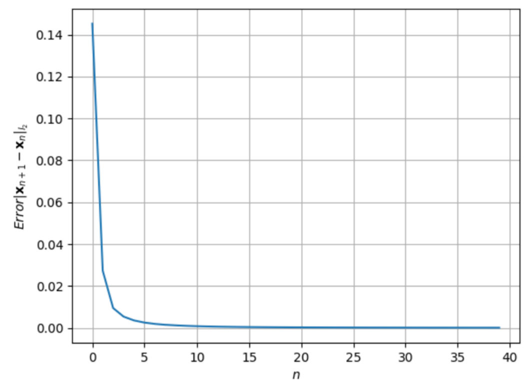

Figure 3 Visualization of the convergence and error $\|x_{n+1} - x_n\|_{l_2}$ of algorithm (33) in Example 4.3

$P_C : H \to C$ be a metric projection defined by

$$P_C(z_1, z_2) = \big(\max\{\min\{z_1, 2\}, -2\}, \max\{\min\{z_2, 2\}, -2\}\big), \tag{34}$$

for all $\mathbf{z} = (z_1, z_2) \in H$.

For every $i = 1, 2, \ldots, N$, let $T_i : C \to C$ be mappings defined by

$$T_i(x_1, x_2) = \left(\frac{x_1}{3^i} + \left(1 - \frac{1}{3^i}\right)\frac{1}{2}, \frac{x_2}{2^{i+1}} + \left(1 - \frac{1}{2^{i+1}}\right)\frac{1}{4}\right), \tag{35}$$

for all $x_1, x_2 \in C$.

Let $S : C \to C$ be a G-S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$, where

$$\alpha_i = \left(\frac{1}{5^i}, \frac{5^i - 1}{5^i}\left(\frac{6^i - 1}{6^i}\right), \frac{5^i - 1}{5^i}\left(\frac{1}{6^i}\right)\right),$$

for all $i = 1, 2, \ldots, N$

and let $A : C \to H$ be a mapping defined by

$$A(x_1, x_2) = \left(x_1 - \frac{x_1^3}{4} - \frac{15}{32}, \frac{x_2}{5} - \frac{1}{20}\right), \tag{36}$$

for all $(x_1, x_2) \in C$.

Suppose that the sequence $\{\mathbf{x}^n\}$ is generated by $\mathbf{u} = \mathbf{x}^0 = (x_1^0, x_2^0) = (1, 0)$ and

$$\mathbf{x}^{n+1} = P_C\left(I - \frac{1}{2}A\right)\left(\beta_n \mathbf{u} + (1 - \beta_n)S\mathbf{x}^n\right), \tag{37}$$

where $\beta_n = \frac{1}{2n+4}$ for all $n \geq 0$.

Then, the sequence $\{x^n\}$ converges strongly to $P_{\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A)}\mathbf{x}^0 = \{(\frac{1}{2}, \frac{1}{4})\}$.

Solution. It is clear that $A^{-1}(0,0) \neq \emptyset$, since $(\frac{1}{2}, \frac{1}{4}) \in A^{-1}(0,0)$ and $E(G) = E(G^{-1})$. Let $\mathbf{x}, \mathbf{y} \in C$ with $(\mathbf{x}, \mathbf{y}) \in E(G)$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Then, we have $\mathbf{x}, \mathbf{y} \in [-1, 1] \times [-1, 1]$. It is easy to verify that T_i are G-nonexpansive mappings for all $i = 1, 2, \ldots, N$ such that $\bigcap_{i=1}^N F(T_i) = \{(\frac{1}{2}, \frac{1}{4})\}$.

From the definition of the mapping A, it is obvious that A is G-$\frac{1}{2}$-inverse strongly monotone and $G\text{-}VI(C,A) = \{(\frac{1}{2},\frac{1}{4})\}$. Therefore, $\bigcap_{i=1}^{N} F(T_i) \cap G\text{-}VI(C,A) = \{(\frac{1}{2},\frac{1}{4})\}$.

From the definition of T_i and $\mathbf{x} \in [-1,1] \times [-1,1]$, we have

$$T_i\mathbf{x} = \left(\frac{x_1}{3^i} + \left(1 - \frac{1}{3^i}\right)\frac{1}{2}, \frac{x_2}{2^{i+1}} + \left(1 - \frac{1}{2^{i+1}}\right)\frac{1}{4}\right)$$

and

$$T_{i+1}\mathbf{x} = \left(\frac{x_1}{3^{i+1}} + \left(1 - \frac{1}{3^{i+1}}\right)\frac{1}{2}, \frac{x_2}{2^{i+2}} + \left(1 - \frac{1}{2^{i+2}}\right)\frac{1}{4}\right),$$

for all $i = 1,2,\dots,N-1$. Since $[-1,1]$ is convex, we have $T_i\mathbf{x}, T_{i+1}\mathbf{x} \in [-1,1] \times [-1,1]$ for all $i = 1,2,\dots,N-1$. Then, $(T_{i+1}\mathbf{x}, T_i\mathbf{x}) \in E(G)$ for all $i = 1,2,\dots,N-1$.

Putting $\mathbf{u} = \mathbf{x}^0 = (0,1) \in [-1,1] \times [-1,1]$ and the definition of T_1, we obtain $T_1\mathbf{x}^0 \in [-1,1] \times [-1,1]$. This implies that $(\mathbf{x}^0, T_1\mathbf{x}^0) \in E(G)$. From the definition of S, we have, $S\mathbf{x}^0 \in [-1,1] \times [-1,1]$.

Since $\mathbf{u} = \mathbf{x}^0 = (1,0)$, $S\mathbf{x}^0 = (1,0)$, $\beta_n = \frac{1}{2n+4}$ for all $n \geq 0$, we have

$$\left(\beta_0\mathbf{u} + (1-\beta_0)S\mathbf{x}^0\right) = (1,0)$$

and it follows that

$$\left(I - \frac{1}{2}A\right)\left(\beta_0\mathbf{u} + (1-\beta_0)S\mathbf{x}^0\right) = (0.859375, 0.025000). \tag{38}$$

From (34) and (38), we have

$$\mathbf{x}^1 = \left(x_1^1, x_1^2\right) = P_C\left(I - \frac{1}{2}A\right)\left(\beta_0\mathbf{u} + (1-\beta_0)S\mathbf{x}^0\right) = (0.859375, 0.025000). \tag{39}$$

It follows from (39) that $\mathbf{x}^1 \in [-1,1] \times [-1,1]$.

Since $\mathbf{u} = (1,0)$, $S\mathbf{x}^1 = (0.8114583, 0.0587500)$, $\beta_n = \frac{1}{2n+4}$ for all $n \geq 0$, we have

$$\left(\beta_1\mathbf{u} + (1-\beta_1)S\mathbf{x}^1\right) = (0.8428819, 0.0489583)$$

and it follows that

$$\left(I - \frac{1}{2}A\right)\left(\beta_1\mathbf{u} + (1-\beta_1)S\mathbf{x}^1\right) = (0.7306692, 0.0690625). \tag{40}$$

From (34) and (40), we have

$$\mathbf{x}^2 = \left(x_1^2, x_2^2\right) = P_C\left(I - \frac{1}{2}A\right)\left(\beta_1\mathbf{u} + (1-\beta_1)S\mathbf{x}^1\right) = (0.7306692, 0.0690625). \tag{41}$$

It follows from (41) that $\mathbf{x}^2 \in [-1,1] \times [-1,1]$.

Continuing in this way, we have $\mathbf{x}^n = (x_1^n, x_2^n) \in [-1,1] \times [-1,1]$ for all $n \geq 0$. This implies that $(\mathbf{x}^n, \mathbf{x}^0) \in E(G)$.

Table 4 Convergence of the algorithm (37) in Example 4.2

n	x_1^n	x_2^n	$\|\mathbf{x}_n - \mathbf{x}_{n+1}\|$
0	1	0	–
1	0.859375	0.025	0.1428299
2	0.7306692	0.0690625	0.1360393
3	0.6493484	0.104429	0.0886784
4	0.601678	0.1309444	0.0545484
⋮	⋮	⋮	⋮
50	0.5058484	0.2413978	0.0002141
51	0.505734	0.2415687	0.0002057
52	0.505624	0.2417329	0.0001976
53	0.5055182	0.2418908	0.0001901
⋮	⋮	⋮	⋮
94	0.5031153	0.2454498	0.0000595
95	0.5030826	0.2454979	0.0000582
96	0.5030505	0.2455451	0.0000570
97	0.5030191	0.2455912	0.0000558
98	0.5029884	0.2456365	0.0000547

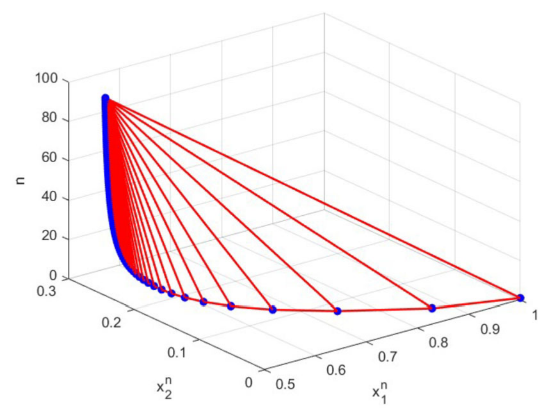

Figure 4 Visualization of the convergence of the algorithm (37) in Example 4.4

From $\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A) = \{(\frac{1}{2}, \frac{1}{4})\}$, it is easy to see that $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}\mathbf{x}^0 \in [-1,1] \times [-1,1]$.

Since $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}\mathbf{x}^0 \in [-1,1] \times [-1,1]$ and $\mathbf{x}^n \in [-1,1] \times [-1,1]$ for all $n \geq 0$, we have $(\mathbf{x}^n, P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}\mathbf{x}^0) \in E(G)$. Then, $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}\mathbf{x}^0$ is dominated by $\{\mathbf{x}^n\}$.

It is obvious that $\{\mathbf{x}^n\}$ dominates $\mathbf{x}^0$ and also $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}\mathbf{x}^0$ is dominated by $\mathbf{x}^0$, where $\mathbf{x}^0 = \{(\frac{1}{2}, \frac{1}{4})\}$. From Theorem 3.1, we have the sequence $\{\mathbf{x}^n\}$ converging strongly to $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}\mathbf{x}^0 = \{(\frac{1}{2}, \frac{1}{4})\}$.

Now, a convergence of the algorithm (37) is shown in Table 4 and visualized in Figs. 4 and 5.

Remark 4 For the provided Example 4.4, we have the following observations:

1. Table 4 and Figs. 4 and 5 show that $\{\mathbf{x}^n\}$ converges to $(\frac{1}{2}, \frac{1}{4})$. The convergence of $\{\mathbf{x}^n\}$ in Example 4.4 can be guaranteed by Theorem 3.1.
2. The values of the sequence $\{\mathbf{x}^n\}$ with respect to n are also plotted in Fig. 4, showing that $(\mathbf{x}^n, \mathbf{x}^0), (\mathbf{x}^{n+1}, \mathbf{x}^n) \in E(G)$.

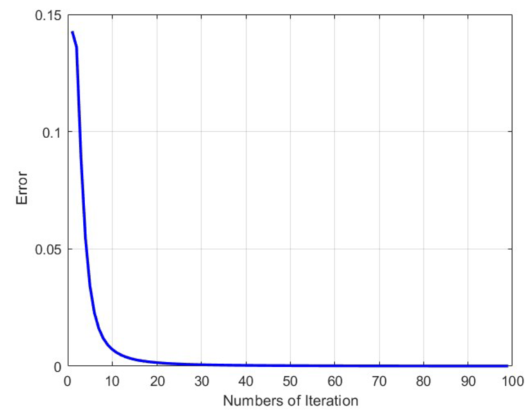

Figure 5 Error plotting of algorithm (37) in Example 4.4

In the following example, we investigate the metric projection onto a half-space $H\_(\alpha,\beta):\{z\in H:\langle\alpha,z\rangle\leq\beta\}$, where $\alpha\in H$, $\alpha\neq 0$ and $\beta\in\mathbb{R}$. It is obvious that

$$P_{H\_(\alpha,\beta)}x=\begin{cases}x-\frac{\langle\alpha,x\rangle-\beta}{\|\alpha\|^2}\alpha, & \text{if}\langle\alpha,x\rangle>\beta,\\ x, & \text{if}\langle\alpha,x\rangle\leq\beta.\end{cases}\tag{42}$$

Equality (42) is clear if $x\in H\_(\alpha,\beta)$, i.e., $\langle\alpha,x\rangle\leq\beta$ (see, [21] for more details).

Example 4.5 Let $H=\mathbb{R}^2$ and $C=\{(x_1,x_2)\in\mathbb{R}^2:-3x_1+x_2\leq 9\}$. Then, we obtain

$$P_C(x_1,x_2)=\begin{cases}(\frac{x_1+3x_2-27}{10},\frac{3x_1+9x_2+9}{10}), & \text{if}-3x_1+x_2>9,\\ (x_1,x_2), & \text{if}-3x_1+x_2\leq 9,\end{cases}\tag{43}$$

for all $(x_1,x_2)\in\mathbb{R}^2$.

Let $G=(V(G),E(G))$ be a directed graph, where $V(G)=C$ and $E(G)=\{(\mathbf{x},\mathbf{y})=((x_1,x_2),(y_1,y_2)):\mathbf{x},\mathbf{y}\in[0,1]\times[0,1]\}$.

For every $i=1,2,\dots,N$, let $T_i:C\to C$ be mappings defined by

$$T_i(x_1,x_2)=\left(\frac{x_1^2}{5^i},\frac{x_2}{2^i}+\left(1-\frac{1}{2^i}\right)\frac{1}{2}\right),\tag{44}$$

for all $(x_1,x_2)\in C$.

Let $S:C\to C$ be a G-S-mapping generated by $T_1,T_2,\dots,T_N$ and $\alpha_1,\alpha_2,\dots,\alpha_n$, where

$$\alpha_i=\left(\frac{1}{5^i},\frac{5^i-1}{5^i}\left(\frac{6^i-1}{6^i}\right),\frac{5^i-1}{5^i}\left(\frac{1}{6^i}\right)\right),$$

for all $i=1,2,\dots,N$ and let $A:C\to H$ be a mapping defined by

$$A(x_1,x_2)=\left(x_1-\frac{x_1^3}{4},\frac{4x_2}{5}-\frac{2}{5}\right),\tag{45}$$

for all $(x_1,x_2)\in C$.

Suppose that the sequence $\{\mathbf{x}^n\}$ is generated by $\mathbf{u} = \mathbf{x}^0 = (x_1^0, x_2^0) = (1,0)$ and

$$\mathbf{x}^{n+1} = P_C\left(I - \frac{1}{2}A\right)\left(\beta_n\mathbf{u} + (1-\beta_n)S\mathbf{x}^n\right), \tag{46}$$

where $\beta_n = \frac{1}{2n+4}$ for all $n \geq 0$.

Then, the sequence $\{x^n\}$ converges strongly to $P_{\bigcap_{i=1}^N F(T_i)\cap G\text{-}VI(C,A)}\mathbf{x}^0 = \{(0, \frac{1}{2})\}$.

Solution. It is clear that $A^{-1}(0,0) \neq \emptyset$, since $(0, \frac{1}{2}) \in A^{-1}(0,0)$ and $E(G) = E(G^{-1})$. Let $\mathbf{x}, \mathbf{y} \in C$ with $(\mathbf{x}, \mathbf{y}) \in E(G)$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Then, we have $\mathbf{x}, \mathbf{y} \in [0,1] \times [0,1]$. It is easy to verify that T_i are G-nonexpansive mappings for all $i = 1, 2, \ldots, N$ such that $\bigcap_{i=1}^N F(T_i) = \{(0, \frac{1}{2})\}$.

From the definition of the mapping A, it is obvious that A is G-$\frac{1}{2}$-inverse strongly monotone and $G\text{-}VI(C,A) = \{(0, \frac{1}{2})\}$. Therefore, $\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A) = \{(0, \frac{1}{2})\}$.

From the definition of T_i, it is obvious that $T_i\mathbf{x}, T_{i+1}\mathbf{x} \in [0,1] \times [0,1]$ for all $i = 1, 2, \ldots, N-1$. Then, $(T_{i+1}\mathbf{x}, T_i\mathbf{x}) \in E(G)$ for all $i = 1, 2, \ldots, N-1$.

Putting $\mathbf{u} = \mathbf{x}^0 = (1,0) \in [0,1] \times [0,1]$ and the definition of T_i, we obtain $T_1\mathbf{x}^0 \in [0,1] \times [0,1]$. This implies that $(\mathbf{x}^0, T_1\mathbf{x}^0) \in E(G)$. From the definition of S, we have, $S\mathbf{x}^0 \in [0,1] \times [0,1]$.

Since $\mathbf{u} = \mathbf{x}^0 = (1,0)$, $S\mathbf{x}^0 = (1,0)$, and $\beta_n = \frac{1}{2n+4}$ for all $n \geq 0$, we have

$$\left(\beta_0\mathbf{u} + (1-\beta_0)S\mathbf{x}^0\right) = (1,0)$$

and it follows that

$$\left(I - \frac{1}{2}A\right)\left(\beta_0\mathbf{u} + (1-\beta_0)S\mathbf{x}^0\right) = (0.6250000, 0.2000000). \tag{47}$$

From (43) and (47), we have

$$\mathbf{x}^1 = \left(x_1^1, x_1^2\right) = P_C\left(I - \frac{1}{2}A\right)\left(\beta_0\mathbf{u} + (1-\beta_0)S\mathbf{x}^0\right) = (0.6250000, 0.2000000). \tag{48}$$

It follows from (48) that $\mathbf{x}^1 \in [0,1] \times [0,1]$.

Since $\mathbf{u} = (1,0)$, $S\mathbf{x}^1 = (0.5156250, 0.2300000)$, and $\beta_n = \frac{1}{2n+4}$ for all $n \geq 0$, we have

$$\left(\beta_1\mathbf{u} + (1-\beta_1)S\mathbf{x}^1\right) = (0.5963542, 0.1916667)$$

and it follows that

$$\left(I - \frac{1}{2}A\right)\left(\beta_1\mathbf{u} + (1-\beta_1)S\mathbf{x}^1\right) = (0.3246879, 0.3150000). \tag{49}$$

From (43) and (49), we have

$$\mathbf{x}^2 = \left(x_1^2, x_2^2\right) = P_C\left(I - \frac{1}{2}A\right)\left(\beta_1\mathbf{u} + (1-\beta_1)S\mathbf{x}^1\right) = (0.3246879, 0.3150000). \tag{50}$$

It follows from (50) that $\mathbf{x}^2 \in [0,1] \times [0,1]$.

Continuing in this way, we have $\mathbf{x}^n = (x_1^n, x_2^n) \in [0,1] \times [0,1]$ for all $n \geq 0$. This implies that $(\mathbf{x}^n, \mathbf{x}^0) \in E(G)$.

From $\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A) = \{(0, \frac{1}{2})\}$, it is easy to see that $P_{\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A)}\mathbf{x}^0 \in [0,1] \times [0,1]$.

Since $P_{\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A)}\mathbf{x}^0 \in [0,1] \times [0,1]$ and $\mathbf{x}^n \in [0,1] \times [0,1]$ for all $n \geq 0$, we have $(\mathbf{x}^n, P_{\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A)}\mathbf{x}^0) \in E(G)$. This implies that $P_{\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A)}\mathbf{x}^0$ is dominated by $\{\mathbf{x}^n\}$.

It is obvious that $\{\mathbf{x}^n\}$ dominates $\mathbf{x}^0$ and also $P_{\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A)}\mathbf{x}^0$ is dominated by $\mathbf{x}^0$, where $\mathbf{x}^0 = \{(0, \frac{1}{2})\}$.

From Theorem 3.1, we have the sequence $\{\mathbf{x}^n\}$ converging strongly to

$P_{\bigcap_{i=1}^N F(T_i) \cap G\text{-}VI(C,A)}\mathbf{x}^0 = \{(0, \frac{1}{2})\}$.

Now, a convergence of the algorithm (46) is shown in Table 5 and visualized in Figs. 6 and 7.

Table 5 Convergence of the algorithm (46) in Example 4.2

n	x_1^n	x_2^n	$\\|\mathbf{x}_n - \mathbf{x}_{n+1}\\|$
0	1.0000000	0.0000000	–
1	0.6250000	0.2000000	0.4250000
2	0.3246879	0.3150000	0.3215779
3	0.1794026	0.3774509	0.1581389
4	0.1139924	0.4124193	0.0741707
⋮	⋮	⋮	⋮
50	0.0080038	0.4937886	0.0002045
51	0.0078488	0.4939094	0.0001965
52	0.0076997	0.4940257	0.0001890
53	0.0075561	0.4941375	0.0001820
⋮	⋮	⋮	⋮
94	0.0042831	0.4966832	0.0000579
95	0.0042383	0.4967179	0.0000567
96	0.0041945	0.4967520	0.0000555
97	0.0041515	0.4967853	0.0000544
98	0.0041094	0.4968180	0.0000533

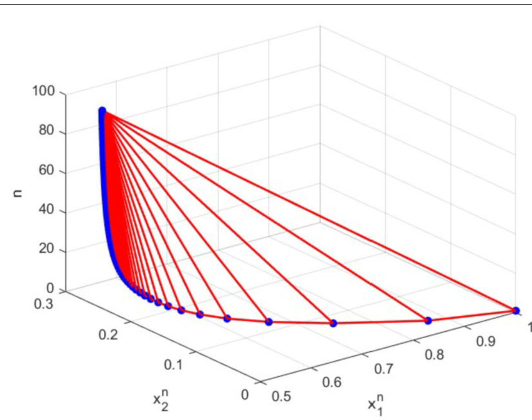

Figure 6 Visualization of the convergence of the algorithm (46) in Example 4.4

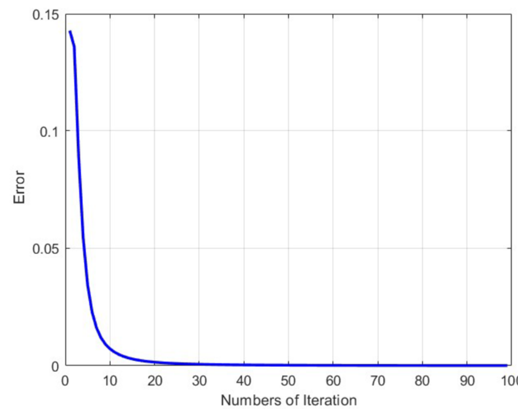

Figure 7 Error plotting of algorithm (46) in Example 4.4

Remark 5 For the provided Example 4.5, we have the following observations:

1. Table 5 and Figs. 6 and 7 show that $\{\mathbf{x}^n\}$ converges to $(0, \frac{1}{2})$. The convergence of $\{\mathbf{x}^n\}$ in Example 4.5 can be guaranteed by Theorem 3.1.
2. The values of the sequence $\{\mathbf{x}^n\}$ with respect to n are also plotted in Fig. 6, showing that $(\mathbf{x}^n, \mathbf{x}^0), (\mathbf{x}^{n+1}, \mathbf{x}^n) \in E(G)$.

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Data availability

Not applicable.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

Conceptualization, AK and WK; Formal analysis, AK and WK; Investigation, WK and AS; Methodology, WK, AS, and AK; Supervision, AK and WK; Writing—original draft, WK and AS; Writing—review and editing, AK and WK and Software, WK and AS. All the authors read and approved the final manuscript.

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References

1. Jachymski, J.: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **136**(4), 1359–1373 (2008)
2. Tiammee, J., Kaewkhao, A., Suantai, S.: On Browder's convergence theorem and Halpern iteration process for *G*-nonexpansive mappings in Hilbert spaces endowed with graphs. Fixed Point Theory Appl. **187** (2015)
3. Kangtunyakarn, A.: Modified Halpern's iteration for fixed point theory of a finite family of *G*-nonexpansive mappings endowed with graph. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**, 437–448 (2018)

4. Tripak, O.: Common fixed points of *G*-nonexpansive mappings on Banach spaces with a graph. Fixed Point Theory Appl. **87** (2016). https://doi.org/10.1186/s13663-016-0578-4
5. Suparatulatorn, R., Cholamjiak, W., Suantai, S.: A modified S-iteration process for *G*-nonexpansive mappings in Banach spaces with graphs. Numer. Algorithms **77**, 479–490 (2018)
6. Suantai, S., Donganont, M., Cholamjiak, W.: Hybrid methods for a countable family of *G*-nonexpansive mappings in Hilbert spaces endowed with graphs. Mathematics **7**(936) (2019)
7. Suantai, S., Kankam, K., Cholamjiak, W., Yajai, W.: Parallel hybrid algorithms for a finite family of *G*-nonexpansive mappings and its application in a novel signal recovery. Mathematics **10**(12), 2140 (2022)
8. Wattanataweekul, R., Janngam, K.: An accelerated common fixed point algorithm for a countable family of *G*-nonexpansive mappings with applications to image recovery. J. Inequal. Appl. **68** (2022)
9. Lions, J.L., Stampacchia, G.: Variational inequalities. Commun. Pure Appl. Math. **20**, 493–517 (1967)
10. Kinderlehrer, D., Stampaccia, G.: An Iteration to Variational Inequalities and Their Applications. Academic Press, New York (1990)
11. Thong, D.V., Hieu, D.V.: Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. Numer. Algorithms **80**, 1283–1307 (2018)
12. Thong, D.V., Hieu, D.V.: Strong convergence of extragradient methods with a new step size for solving variational inequality problems. Comput. Appl. Math. **38**(136) (2019). https://doi.org/10.1007/s40314-019-0899-0
13. Jolaoso, L.O., Taiwo, A., Alakoya, T.O., Mewomo, T.O.: A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem. Comput. Appl. Math. **39**(38) (2020). https://doi.org/10.1007/s40314-019-1014-2
14. Kangtunyakarn, A.: The variational inequality problem in Hilbert spaces endowed with graphs. J. Fixed Point Theory Appl. **22**(4) (2020)
15. Halpern, B.: Fixed points of nonexpansive maps. Bull. Am. Math. Soc. **73**, 957–961 (1967)
16. Alber, Y.I., Ryazantseva, I.: Nonlinear Ill-Posed Problems of Monotone Type. Springer, New York (2006)
17. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
18. Xu, H.K.: An iterative approach to quadric optimization. J. Optim. Theory Appl. **116**, 659–678 (2003)
19. Pang, C., Zhang, R., Zhang, Q., Wang, J.: Dominating sets in directed graphs. Inf. Sci. **180**, 3647–3652 (2010)
20. Bang-Jensen, J., Gutin, G.: Digraphs Theory, Algorithms and Applications, 2nd edn. Springer, Heidelberg (2009)
21. Cegielski, A.: Iterative methods for fixed point problems. In: Hilbert Spaces. Lecture Notes in Mathematics, vol. 2057. Springer, Heidelberg (2012)

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Self-adaptive CQ-type algorithms for the split feasibility problem involving two bounded linear operators in Hilbert spaces

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ABSTRACT. In this article, we consider and investigate a split convex feasibility problem involving two bounded linear operators in Hilbert spaces. We introduce a self-adaptive CQ-type algorithm by selecting the stepsize which is independent of the operator norms and establish a strong convergence result of the proposed algorithm under some mild control conditions. Moreover, we propose a self-adaptive relaxed CQ-type algorithm for solving the problem constrained by sub-level sets of convex functions. A numerical example and an application in compressed sensing are also given to illustrate the convergence behaviour of our proposed algorithms. Our results in this paper improve and generalize some existing results in the literature.

1. INTRODUCTION

Let C and Q be two nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (shortly, SFP) is to find a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP is the first instance of the split inverse problem (referred to [13, Sect. 2]), which was first introduced by Censor and Elfving [11] in Euclidean spaces. The SFP model can be applied to solving many mathematical problems such as the constrained least-squares problem, the linear split feasibility problem, and the linear programming problem and it can be used in real-world applications, for example, in signal processing, in image recovery, in intensity-modulated therapy, in pattern recognition and in data prediction (see [3, 5, 10, 12, 20, 22]). Consequently, the SFP has been widely studied and various methods for solving such a problem have been invented and developed by many authors, see [2, 9, 17, 24, 25, 35, 36, 37, 38, 41, 43, 44] and the references therein. One of the powerful methods for approximating solutions of (1.1) is known as the *CQ algorithm* introduced by Byrne [2] as follows:

$$\begin{cases} x_1 \in H_1, \\ x_{k+1} = P_C\left(x_k - \lambda A^*(I - P_Q)Ax_k\right), \quad k \geq 1, \end{cases} \tag{1.2}$$

where $\lambda \in \left(0, 2/\|A\|^2\right)$, P_C and P_Q are the metric projections onto C and Q, respectively, and A^* stands for the adjoint operator of A. After that, various kinds of the split inverse problem, which are generalizations of the SFP were introduced and studied, see [4, 12, 13, 14, 28, 32] for instance.

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In this paper, we focus on a generalization of the SFP (1.1) in which two bounded linear operators $A, B : H_1 \to H_2$ are involved – Finding a point

$$x \in C \quad \text{such that} \quad Ax \in Q \text{ and } Bx \in \tilde{Q}, \tag{1.3}$$

where $C \subseteq H_1$ and $Q, \tilde{Q} \subseteq H_2$ are nonempty closed convex subsets. We call (1.3) the *two-operator split feasibility problem* (two-operator SFP), see [28, 32] for the general versions of this problem. The two-operator SFP (1.3) can be reduced to the convexly constrained linear problem ([15, 29]) involving two linear operators, that is, finding a point $x \in C$ such that $Ax = y, Bx = \tilde{y}$ in H_2.

In 2019, Kangtunyakarn [21] studied the two-operator SFP (1.3) in case that $Q = \tilde{Q}$ and introduced a viscosity-based algorithm with a given contraction $f : C \to C$ as follows:
(1.4)

$$\begin{cases} x_1 \in C, \\ x_{k+1} = \beta_k f(x_k) + \delta_k x_k + \gamma_k P_C\Big[x_k - \dfrac{\lambda}{2}\Big(A^*(I - P_Q)Ax_k + B^*(I - P_Q)Bx_k\Big)\Big], \quad k \geq 1, \end{cases}$$

where $\lambda \in \big(0, 2/\max\{\|A\|^2, \|B\|^2\}\big)$ and $\{\beta_k\}, \{\delta_k\}$, and $\{\gamma_k\}$ are real sequences in $(0, 1)$. A strong convergence theorem of (1.4) was proved under some suitable conditions on the control sequences, see [21, Theorem 3.1].

It is noted that the parameters λ in (1.2) and in (1.4) depend on the norms of bounded linear operators, so these algorithms have a drawback in the sense that the implementation of them requires to calculate or estimate the operator norms, which is not an easy task in general practice (see [25, Subsection 6.1.2] for instance). To overcome this, in [2, Proposition 4.1], it was presented a helpful method for estimating operator (matrix) norms but its conditions seem restrictive. López et al. [25] proposed an alternative way that is to select the stepsize λ_k which does not need any prior knowledge of the operator norm for replacing the parameter λ in (1.2) as follows:

$$\lambda_k := \frac{\mu_k \|(I - P_Q)Ax_k\|^2}{2\,\|A^*(I - P_Q)Ax_k\|^2}, \tag{1.5}$$

where $\mu_k \in (0, 4)$. We can see that the choice of the stepsize λ_k in (1.5) is independent of the operator norm $\|A\|$. This stepsize was widely employed in optimization methods and was also modified for use in fixed point methods, see [8, 18, 19, 27, 33, 35]. The CQ algorithm with the self-adaptive stepsize defined by (1.5) [25, Algorithm 3.1] guarantees only weak convergence for the SFP (1.1), see [25, Theorem 3.5]. However, strong convergence gives more desirable theoretical result in the setting of Hilbert spaces. To get strong convergence, Vinh et al. [35] employed a modification of the CQ algorithm ([37, Algorithm 4.1]) with the stepsize (1.5) for solving the SFP (1.1) as follows:

$$\begin{cases} x_1 \in H_1, \\ x_{k+1} = P_C\Big[(1 - \beta_k)\big(x_k - \lambda_k A^*(I - P_Q)Ax_k\big)\Big], \quad k \geq 1, \end{cases} \tag{1.6}$$

where the stepsize λ_k is defined by (1.5) and $\{\beta_k\} \subset (0, 1)$. They proved that the sequence generated by (1.6) converges strongly to the minimum-norm solution to (1.1) under some suitable control conditions, see [35, Theorem 3.1].

Here, the above review leads us to the following natural questions.

1. Can we design a CQ-type algorithm whose stepsize does not depend on the operator norm $\|A\|$ or $\|B\|$ to solve the two-operator SFP (1.3)?
2. How do we adapt the algorithm designed from Question 1 to be a strongly convergent method?

Motivated and inspired by the above questions and the results of Kangtunyakarn [21], López et al. [25], and Vinh et al. [35], we aim to invent a self-adaptive CQ-type algorithm whose stepsize does not depend on any operator norms for solving the two-operator SFP in the setting of Hilbert spaces. Moreover, we will prove that the sequence generated by the proposed algorithm converges strongly to the minimum-norm solution. The rest of the paper is organized as follows. In Sect. 2, some basic facts and useful lemmas for proving our main results are given. Our main result is in Sect. 3. In this section, we introduce a self-adaptive CQ-type algorithm using the stepsize which is independent of the bounded linear operator norms for finding a solution of (1.3). A strong convergence theorem of the proposed algorithm is analyzed and established. In Sect. 4, we propose a self-adaptive relaxed CQ-type algorithm for solving the two-operator SFP in case of sub-level sets of convex functions and also prove its strong convergence result. Finally, in Sect. 5, we provide numerical experiments of our proposed algorithms in the setting of a Euclidean space and in the signal recovery problem with two different blurring operations, and also compare the efficiency of our algorithms with that of some methods depending on the operator norms.

2. Preliminaries

Throughout this paper, we suppose that H, H_1 and H_2 are real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and the induced norms $\| \cdot \|$ (in particular, in Euclidean spaces, $\| \cdot \|_1$ denotes the l_1-norm and $\| \cdot \|_2$ denotes the Euclidean norm). The notation I stands for the identity operator on a Hilbert space. Let $\{x_k\}$ be a sequence in H. Weak and strong convergence of $\{x_k\}$ to $x \in H$ are denoted by $x_k \rightharpoonup x$ and $x_k \to x$, respectively. The set of all weak-cluster points of $\{x_k\}$ is denoted by $\omega_w(x_k)$.

Let $f : H \to \mathbb{R}$ be a function and $x \in H$. We say that f is weakly lower semi-continuous at x if for every sequence $\{x_k\} \subset H$, $x_k \rightharpoonup x$ implies $f(x) \leq \liminf\limits_{k\to\infty} f(x_k)$. A subdifferential ∂f of f at x is defined by

$$\partial f(x) = \{u \in H : f(x) + \langle u, z - x\rangle \leq f(z),\ \forall z \in H\}.$$

The function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$. One can see that if f is subdifferentiable at x, then f is weakly lower semi-continuous at x. We denote the gradient of f by ∇f if f is differentiable.

Let K be a nonempty closed convex subset of H. Recall that the metric projection P_K from H onto K assigns to each $x \in H$ the unique point $P_K x$ in K satisfying $\|x - P_K x\| = \inf\limits_{z\in K} \|x - z\|$. Some properties of the metric projection are listed below.

Lemma 2.1. *The metric projection P_K has the following properties:*

(1) $\langle x - P_K x, z - P_K x\rangle \leq 0, \quad \forall x \in H, \forall z \in K;$
(2) $\langle x - P_K x, x - z\rangle \geq \|x - P_K x\|^2, \quad \forall x \in H, \forall z \in K;$
(3) P_K *is firmly nonexpansive, i.e.,*

$$\|P_K x - P_K y\|^2 \leq \|x - y\|^2 - \|(x - P_K x) - (y - P_K y)\|^2, \quad \forall x, y \in H,$$

in particular,

$$\|P_K x - z\|^2 \leq \|x - z\|^2 - \|x - P_K x\|^2, \quad \forall x \in H, \forall z \in K.$$

Let Q be a nonempty closed convex subset of H_2 and let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. Define a function $f : H_1 \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2.$$

We know that f is weakly lower semi-continuous on H_1 and differentiable with the gradient $\nabla f : H_1 \to H_1$ given by

$$\nabla f(x) = A^*(I - P_Q)Ax.$$

Moreover, ∇f is Lipschitz continuous with the Lipschitz constant $\|A\|^2$, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2 \|x - y\|, \quad \forall x, y \in H_1.$$

For more details, the reader is referred to optimization books, see [1, 31] for instance.

We end this section with the following useful lemmas for proving our strong convergence results.

Lemma 2.2 ([39]). *Let $\{t_k\}$ be a sequence of nonnegative real numbers satisfying*

$$t_{k+1} \leq (1 - \beta_k)t_k + \beta_k \delta_k, \quad \forall k \in \mathbb{N},$$

where $\{\beta_k\}$ is a sequence in $(0, 1)$ and $\{\delta_k\}$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} \beta_k = \infty$ and $\limsup_{k\to\infty} \delta_k \leq 0$. Then, $\lim_{k\to\infty} t_k = 0$.

Lemma 2.3 ([26]). *Let $\{s_k\}$ be a sequence of real numbers such that there exists a subsequence $\{k_j\}$ of $\{k\}$ which satisfies $s_{k_j} < s_{k_j+1}$ for all $j \in \mathbb{N}$. Let $\{\tau(k)\}$ be a sequence of positive integers defined by*

$$\tau(k) := \max\{n \leq k : s_n < s_{n+1}\}$$

for all $k \geq k_0$ (for some k_0 large enough). Then $\{\tau(k)\}$ is a nondecreasing sequence such that $\tau(k) \to \infty$ as $k \to \infty$, and it holds that

$$s_{\tau(k)} \leq s_{\tau(k)+1} \text{ and } s_k \leq s_{\tau(k)+1}, \quad \forall k \geq k_0.$$

3. Self-adaptive CQ-type algorithm and its convergence result

This main section provides positive answers to the questions raised in the introduction section, namely that we introduce a self-adaptive CQ-type algorithm (Algorithm 1) whose stepsize does not depend on the bounded linear operator norms to solve the two-operator SFP (1.3). Subsequently, we analyze and establish strong convergence of the proposed algorithm. The following assumptions are set throughout the section:

- H_1 and H_2 are real Hilbert spaces,
- $C \subseteq H_1$ and $Q, \tilde{Q} \subseteq H_2$ are nonempty closed convex subsets,
- $A, B : H_1 \to H_2$ are two bounded linear operators,
- $\Gamma = \left\{x \in C : Ax \in Q, Bx \in \tilde{Q}\right\} \neq \emptyset$.

We begin with the following result that will be helpful in designing our algorithm.

Lemma 3.4. *Let $x^* \in C$. Then, $x^* \in \Gamma$ if and only if $\left\|A^*(I - P_Q)Ax^* + B^*(I - P_{\tilde{Q}})Bx^*\right\| = 0$.*

Proof. Let $x^* \in C$. If $x^* \in \Gamma$, then $Ax^* \in Q, Bx^* \in \tilde{Q}$ and so $(I - P_Q)Ax^* = (I - P_{\tilde{Q}})Bx^* = 0$. It is obvious that $\left\|A^*(I - P_Q)Ax^* + B^*(I - P_{\tilde{Q}})Bx^*\right\| = 0$. Conversely, we assume that

$\left\| A^*(I-P_Q)Ax^* + B^*(I-P_{\tilde{Q}})Bx^* \right\| = 0$. Pick $p \in \Gamma$. By Lemma 2.1(2), we have

$$\begin{aligned} 0 &= \left\| A^*(I-P_Q)Ax^* + B^*(I-P_{\tilde{Q}})Bx^* \right\| \|x^* - p\| \\ &\geq \left\langle A^*(I-P_Q)Ax^* + B^*(I-P_{\tilde{Q}})Bx^*, x^* - p \right\rangle \\ &= \langle A^*(I-P_Q)Ax^*, x^* - p\rangle + \left\langle B^*(I-P_{\tilde{Q}})Bx^*, x^* - p \right\rangle \\ &= \langle (I-P_Q)Ax^*, Ax^* - Ap\rangle + \left\langle (I-P_{\tilde{Q}})Bx^*, Bx^* - Bp \right\rangle \\ &\geq \|(I-P_Q)Ax^*\|^2 + \|(I-P_{\tilde{Q}})Bx^*\|^2, \end{aligned}$$

which implies that $(I-P_Q)Ax^* = (I-P_{\tilde{Q}})Bx^* = 0$. Hence, $Ax^* \in Q$ and $Bx^* \in \tilde{Q}$, that is, $x^* \in \Gamma$. □

Here, our iterative algorithm for solving the two-operator SFP (1.3) is designed as follows.

Algorithm 1: Self-adaptive CQ-type algorithm for the two-operator SFP

Initialization: Take two real sequences $\{\beta_k\} \subset (0,1)$ and $\{\mu_k\} \subset (0,4)$. Choose $x_0 \in H_1$ arbitrarily. Set $x_1 = P_C x_0$ and $k = 1$.

Iterative Step: Given x_k, if $\left\| A^*(I-P_Q)Ax_k + B^*(I-P_{\tilde{Q}})Bx_k \right\| = 0$, then $x_{k+1} = x_k$ (in this case, x_k solves (1.3) by Lemma 3.4) and the iterative process stops. Otherwise, calculate

$$\lambda_k = \mu_k \frac{\|(I-P_Q)Ax_k\|^2 + \|(I-P_{\tilde{Q}})Bx_k\|^2}{\left\| A^*(I-P_Q)Ax_k + B^*(I-P_{\tilde{Q}})Bx_k \right\|^2}, \tag{3.7}$$

$$x_{k+1} = P_C\left[(1-\beta_k)\left(x_k - \frac{\lambda_k}{2}\left(A^*(I-P_Q)Ax_k + B^*(I-P_{\tilde{Q}})Bx_k\right)\right)\right]. \tag{3.8}$$

Update $k := k+1$ and return to Iterative Step.

For the sake of simplicity, we let $g : H_1 \to \mathbb{R}$ be defined by

$$g(x) := \frac{1}{4}\left(\|(I-P_Q)Ax\|^2 + \|(I-P_{\tilde{Q}})Bx\|^2\right) \tag{3.9}$$

with the gradient given by

$$\nabla g(x) = \frac{1}{2}\Big(A^*(I-P_Q)Ax + B^*(I-P_{\tilde{Q}})Bx\Big), \quad x \in H_1.$$

Note that (1.3) is equivalent to the problem of finding $x \in C$ such that $g(x) = 0$. In other words, (3.7) and (3.8) can be rewritten in the form of the following modified gradient-projection method:

$$x_{k+1} = P_C\Big[(1-\beta_k)\big(x_k - \lambda_k \nabla g(x_k)\big)\Big], \text{ where } \lambda_k = \frac{\mu_k g(x_k)}{\|\nabla g(x_k)\|^2}.$$

To verify the convergence of Algorithm 1, the following two lemmas are required.

Lemma 3.5. *Let $\{x_k\}$ be a sequence generated by Algorithm 1. If $\nabla g(x_k) \neq 0$, then the following two inequalities hold for all $x^* \in \Gamma$,*

$$\|x_{k+1} - x^*\|^2 \leq \beta_k \|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2 - \mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2} \tag{3.10}$$

and

$$\begin{aligned}\|x_{k+1} - x^*\|^2 \leq (1-\beta_k)\|x_k - x^*\|^2 + \beta_k\Big[&\beta_k\|x^*\|^2 + 2(1-\beta_k)\langle x_k - x^*, -x^*\rangle \\ &+ 2(1-\beta_k)\lambda_k \langle \nabla g(x_k), x^*\rangle\Big].\end{aligned} \tag{3.11}$$

Proof. Let $x^* \in \Gamma$. Using (3.8) and Lemma 2.1(3), we have

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \Big\|P_C\Big[(1-\beta_k)\big(x_k - \lambda_k \nabla g(x_k)\big)\Big] - P_C x^*\Big\|^2 \\ &\leq \big\|(1-\beta_k)\big(x_k - \lambda_k\nabla g(x_k)\big) - x^*\big\|^2 \\ &= \big\|\beta_k(-x^*) + (1-\beta_k)\big(x_k - \lambda_k \nabla g(x_k) - x^*\big)\big\|^2 && (3.12) \\ &\leq \beta_k\|x^*\|^2 + (1-\beta_k)\left\|x_k - \lambda_k\nabla g(x_k) - x^*\right\|^2. && (3.13)\end{aligned}$$

By Lemma 2.1(2), we get

$$\begin{aligned}\langle \nabla g(x_k), x_k - x^*\rangle &= \frac{1}{2}\Big\langle A^*(I-P_Q)Ax_k + B^*(I-P_{\tilde{Q}})Bx_k, x_k - x^*\Big\rangle \\ &= \frac{1}{2}\Big[\langle A^*(I-P_Q)Ax_k, x_k - x^*\rangle + \langle B^*(I-P_{\tilde{Q}})Bx_k, x_k - x^*\rangle\Big] \\ &= \frac{1}{2}\Big[\langle (I-P_Q)Ax_k, Ax_k - Ax^*\rangle + \langle (I-P_{\tilde{Q}})Bx_k, Bx_k - Bx^*\rangle\Big] \\ &\geq \frac{1}{2}\Big[\|(I-P_Q)Ax_k\|^2 + \|(I-P_{\tilde{Q}})Bx_k\|^2\Big] = 2g(x_k). && (3.14)\end{aligned}$$

Now using (3.7) and (3.14), we obtain

$$\begin{aligned}\|x_k - \lambda_k\nabla g(x_k) - x^*\|^2 &= \|x_k - x^*\|^2 + \lambda_k^2\left\|\nabla g(x_k)\right\|^2 - 2\lambda_k\left\langle \nabla g(x_k), x_k - x^*\right\rangle \\ &\leq \|x_k - x^*\|^2 + \lambda_k^2\left\|\nabla g(x_k)\right\|^2 - 4\lambda_k g(x_k) \\ &= \|x_k - x^*\|^2 + \frac{\mu_k^2 g^2(x_k)}{\|\nabla g(x_k)\|^2} - \frac{4\mu_k g^2(x_k)}{\|\nabla g(x_k)\|^2} \\ &= \|x_k - x^*\|^2 - \mu_k(4-\mu_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}. && (3.15)\end{aligned}$$

Consequently, substituting (3.15) into (3.13) yields

$$\|x_{k+1} - x^*\|^2 \leq \beta_k\|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2 - \mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}$$

and (3.10) is obtained. We next show that (3.11) is true. From (3.15), we also have

$$\begin{aligned}\|x_k - \lambda_k\nabla g(x_k) - x^*\|^2 &\leq \|x_k - x^*\|^2 - \mu_k(4-\mu_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2} \\ &\leq \|x_k - x^*\|^2. && (3.16)\end{aligned}$$

By using (3.12) and (3.16), we obtain

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\le \left\|\beta_k(-x^*)+(1-\beta_k)\big(x_k-\lambda_k\nabla g(x_k)-x^*\big)\right\|^2\\ &= \beta_k^2\|x^*\|^2+(1-\beta_k)^2\,\|x_k-\lambda_k\nabla g(x_k)-x^*\|^2\\ &\quad+2\beta_k(1-\beta_k)\,\langle x_k-\lambda_k\nabla g(x_k)-x^*,-x^*\rangle\\ &\le \beta_k^2\|x^*\|^2+(1-\beta_k)^2\,\|x_k-x^*\|^2+2\beta_k(1-\beta_k)\,\langle x_k-x^*,-x^*\rangle\\ &\quad+2\beta_k(1-\beta_k)\lambda_k\,\langle\nabla g(x_k),x^*\rangle\\ &\le (1-\beta_k)\|x_k-x^*\|^2+\beta_k\Big[\beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle\\ &\quad+2(1-\beta_k)\lambda_k\,\langle\nabla g(x_k),x^*\rangle\Big].\end{aligned}$$

This completes the proof. □

Lemma 3.6. *The sequence $\{x_k\}$ generated by Algorithm 1 is bounded.*

Proof. If $\nabla g(x_m)=0$ for some $m\in\mathbb{N}$, then $x_k=x_m$ for all $k>m$ and hence $\{x_k\}$ is bounded. Assume that $\nabla g(x_k)\neq 0$ for all $k\in\mathbb{N}$. Let $x^*\in\Gamma$. Using (3.10), we get

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\le \beta_k\|x^*\|^2+(1-\beta_k)\|x_k-x^*\|^2-\mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}\\ &\le \beta_k\|x^*\|^2+(1-\beta_k)\|x_k-x^*\|^2\\ &\le \max\left\{\|x^*\|^2,\|x_k-x^*\|^2\right\}.\end{aligned}$$

By mathematical induction, we deduce that

$$\|x_{k+1}-x^*\|^2 \;\le\; \max\{\|x^*\|^2,\|x_1-x^*\|^2\},\quad \forall k\in\mathbb{N},$$

it follows that $\{x_k\}$ is bounded. □

Now, we are ready to prove a strong convergence theorem of Algorithm 1.

Theorem 3.1. *The sequence $\{x_k\}$ generated by Algorithm 1 converges strongly to a solution x^* to* (1.3) *provided that the control sequences $\{\beta_k\}$ and $\{\mu_k\}$ satisfy the following conditions:*

(C1) (1) $\lim\limits_{k\to\infty}\beta_k=0$ *and* (2) $\sum\limits_{k=1}^{\infty}\beta_k=\infty$;

(C2) $\inf\limits_k \mu_k(4-\mu_k)>0$.

Proof. If $\nabla g(x_m)=0$ for some $m\in\mathbb{N}$, then the result is obtained directly by Lemma 3.4. So, we assume that $\nabla g(x_k)\neq 0$ for all $k\in\mathbb{N}$. Let $x^*:=P_\Gamma 0$. Using (3.10), we get

$$\|x_{k+1}-x^*\|^2 \le \beta_k\|x^*\|^2+(1-\beta_k)\|x_k-x^*\|^2-\mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2},$$

it follows that

$$\mu_k(4-\mu_k)(1-\beta_k)\frac{g^2(x_k)}{\|\nabla g(x_k)\|^2}\le \beta_k\|x^*\|^2+\|x_k-x^*\|^2-\|x_{k+1}-x^*\|^2. \tag{3.17}$$

The rest of the proof into will be divided into two cases:

Case 1. Assume that there exists $k_0\in\mathbb{N}$ such that $\{\|x_k-x^*\|\}_{k\ge k_0}$ is either nonincreasing or nondecreasing. In this case, $\{\|x_k-x^*\|\}$ is convergent because it is bounded. It follows that $\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \to 0$ as $k \to \infty$. Then, in view of (3.17) with (C1)(1) and (C2), we obtain

$$\lim_{k\to\infty} \frac{g^2(x_k)}{\|\nabla g(x_k)\|^2} = 0, \tag{3.18}$$

which implies that

$$\lim_{k\to\infty} \lambda_k \|\nabla g(x_k)\| = \lim_{k\to\infty} \frac{g(x_k)}{\|\nabla g(x_k)\|} = 0. \tag{3.19}$$

Let $y_k = (1-\beta_k)\big(x_k - \lambda_k \nabla g(x_k)\big)$. Consider

$$\|x_k - y_k\| = \|(1-\beta_k)\lambda_k \nabla g(x_k) + \beta_k x_k\| \le \lambda_k \|\nabla g(x_k)\| + \beta_k \|x_k\|,$$

it follows from (3.19) and (C1)(1) that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.20}$$

By the same computation as the proof of (3.10), we get

$$\|y_k - x^*\|^2 \le \beta_k \|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2. \tag{3.21}$$

For $x \in H_1$, we let

$$h(x) := \|(I - P_C)x\|^2,\ g^A(x) := \frac{1}{2}\|(I - P_Q)Ax\|^2 \text{ and } g^B(x) := \frac{1}{2}\|(I - P_{\tilde{Q}})Bx\|^2.$$

Using Lemma 2.1(3) and (3.21), we have

$$\begin{aligned} h(y_k) &= \|y_k - P_C y_k\|^2 \\ &\le \|y_k - x^*\|^2 - \|P_C y_k - x^*\|^2 \\ &\le \beta_k \|x^*\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2, \end{aligned}$$

which implies that

$$\lim_{k\to\infty} h(y_k) = 0. \tag{3.22}$$

Since ∇g^A and ∇g^B are Lipschitz continuous with coefficients $\|A\|^2$ and $\|B\|^2$, respectively, one is able to show that ∇g is Lipschitz continuous with a coefficient $L := \max\{\|A\|^2, \|B\|^2\}$. Thus, we have

$$\|\nabla g(x_k)\| = \|\nabla g(x_k) - \nabla g(x^*)\| \le L\|x_k - x^*\|, \quad \forall k \in \mathbb{N}.$$

By the boundedness of $\{x_k - x^*\}$, the above inequality yields that $\{\nabla g(x_k)\}$ is bounded. This together with (3.18) implies that $g(x_k) \to 0$ as $k \to \infty$ and hence

$$\lim_{k\to\infty} g^A(x_k) = \lim_{k\to\infty} g^B(x_k) = 0. \tag{3.23}$$

We next show that $\omega_w(x_k) \subseteq \Gamma$. Since $\{x_k\}$ is bounded, $\omega_w(x_k) \neq \emptyset$. Let $\hat{x} \in \omega_w(x_k)$. Then, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \hat{x}$. Since g^A is weakly lower semi-continuous on H_1, it follows from (3.23) that

$$0 \le g^A(\hat{x}) \le \liminf_{j\to\infty} g^A(x_{k_j}) = 0.$$

Hence, $g^A(\hat{x}) = 0$, that is, $A\hat{x} \in Q$. Similarly, by using the weakly lower semicontinuity of g^B and (3.23), we get $g^B(\hat{x}) = 0$, that is, $B\hat{x} \in \tilde{Q}$. Since $x_{k_j} \rightharpoonup \hat{x}$, it also follows from (3.20) that $y_{k_j} \rightharpoonup \hat{x}$. By using the weakly lower semicontinuity of h and (3.22), we then deduce that $\hat{x} \in C$. Therefore, $\hat{x} \in \Gamma$ and this means that $\omega_w(x_k) \subseteq \Gamma$. Since $x^* = P_\Gamma 0$, it follows from Lemma 2.1(1) that

$$\limsup_{k\to\infty} \langle x_k - x^*, -x^* \rangle = \max_{\hat{x}\in\omega_w(x_k)} \langle \hat{x} - x^*, -x^* \rangle \le 0. \tag{3.24}$$

Now using (3.11), we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\le (1-\beta_k)\|x_k-x^*\|^2+\beta_k\Big[\beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle\\ &\quad +2(1-\beta_k)\lambda_k\left\langle \nabla g(x_k),x^*\right\rangle\Big]\\ &\le (1-\beta_k)\|x_k-x^*\|^2+\beta_k\Big[\beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle\\ &\quad +2(1-\beta_k)\lambda_k\left\|\nabla g(x_k)\right\|\|x^*\|\Big]\\ &= (1-\beta_k)\|x_k-x^*\|^2+\beta_k\delta_k,\quad \forall k\in\mathbb{N}, \qquad (3.25)\end{aligned}$$

where

$$\delta_k := \beta_k\|x^*\|^2+2(1-\beta_k)\langle x_k-x^*,-x^*\rangle+2(1-\beta_k)\lambda_k\left\|\nabla g(x_k)\right\|\|x^*\|.$$

Using (C1)(1), (3.19), and (3.24), we get $\limsup_{k\to\infty}\delta_k\le 0$. Recall from (C1)(2) that $\sum_{k=1}^{\infty}\beta_k=\infty$. Consequently, by applying Lemma 2.2 to (3.25), we immediately obtain that $x_k\to x^*$ as $k\to\infty$.

Case 2. Assume that $\{\|x_k-x^*\|\}$ is not monotone. There exists a subsequence $\{k_j\}$ of $\{k\}$ such that $\|x_{k_j}-x^*\|<\|x_{k_j+1}-x^*\|$ for all $j\in\mathbb{N}$. Define a positive interger sequence $\tau(k)$ by

$$\tau(k):=\max\{n\le k:\|x_n-x^*\|<\|x_{n+1}-x^*\|\}$$

for all $k\ge k_0$ (for some k_0 large enough). By Lemma 2.3, $\{\tau(k)\}$ is nondecreasing such that $\tau(k)\to\infty$ as $k\to\infty$ and

$$\left\|x_{\tau(k)}-x^*\right\|^2-\left\|x_{\tau(k)+1}-x^*\right\|^2\le 0 \qquad (3.26)$$

for all $k\ge k_0$. From (3.17) and (3.26), we have

$$\begin{aligned}\mu_{\tau(k)}\left(4-\mu_{\tau(k)}\right)\left(1-\beta_{\tau(k)}\right)\frac{g^2\left(x_{\tau(k)}\right)}{\left\|\nabla g\left(x_{\tau(k)}\right)\right\|^2} &\le \beta_{\tau(k)}\|x^*\|^2+\left\|x_{\tau(k)}-x^*\right\|^2-\left\|x_{\tau(k)+1}-x^*\right\|^2\\ &\le \beta_{\tau(k)}\|x^*\|^2.\end{aligned}$$

In view of the above inequity with (C1)(1) and (C2), we get

$$\lim_{k\to\infty}\frac{g^2\left(x_{\tau(k)}\right)}{\left\|\nabla g\left(x_{\tau(k)}\right)\right\|^2}=0.$$

By the same way as the proof in Case 1, we obtain

$$\limsup_{k\to\infty}\left\langle x_{\tau(k)}-x^*,-x^*\right\rangle=\max_{\hat{x}\in\omega_w\left(x_{\tau(k)}\right)}\langle\hat{x}-x^*,-x^*\rangle\le 0$$

and

$$\left\|x_{\tau(k)+1}-x^*\right\|^2\le\left(1-\beta_{\tau(k)}\right)\left\|x_{\tau(k)}-x^*\right\|^2+\beta_{\tau(k)}\delta_{\tau(k)}, \qquad (3.27)$$

where

$$\delta_{\tau(k)}:=\beta_{\tau(k)}\|x^*\|^2+2\left(1-\beta_{\tau(k)}\right)\langle x_{\tau(k)}-x^*,-x^*\rangle+2\left(1-\beta_{\tau(k)}\right)\lambda_{\tau(k)}\left\|\nabla g\left(x_{\tau(k)}\right)\right\|\|x^*\|$$

such that $\limsup_{k\to\infty}\delta_{\tau(k)}\le 0$. By looking at (3.27) with the fact that $\|x_{\tau(k)}-x^*\|\le\|x_{\tau(k)+1}-x^*\|$, we have $\left\|x_{\tau(k)+1}-x^*\right\|^2\le\delta_{\tau(k)}$. This implies that $\limsup_{k\to\infty}\left\|x_{\tau(k)+1}-x^*\right\|^2\le 0$. Consequently, by utilizing Lemma 2.3, we have

$$0\le\|x_k-x^*\|\le\left\|x_{\tau(k)+1}-x^*\right\|\to 0 \text{ as } k\to\infty.$$

Therefore, in both cases we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Gamma 0$. The proof is complete. □

Remark 3.1. It is worth mentioning that there are some advantages of our main result as follows:

(1) If $\{x_k\}$ is a sequence generated by Algorithm 1 such that $\nabla g(x_k) \neq 0$ for all $k \in \mathbb{N}$, then $\{x_k\}$ converges to the minimum-norm solution x^* to (1.3), where $x^* = P_\Gamma 0$.
(2) The choice of stepsize λ_k defined by (3.7) depends on x_k and hence Algorithm 1 does not need to know the value of $\|A\|$ or $\|B\|$.
(3) A result in [35, Theorem 3.1] for solving the SFP is a consequence of Theorem 3.1, namely that if $A = B$ and $Q = \tilde{Q}$ in our problem, then Algorithm 1 is immediately reduced to (1.6) [35, Algorithm 3.1].

Remark 3.2. We note that the concept of choosing the stepsizes λ_k in (1.5) and (3.7) can be extended to the case of the finite familes of operators A_j and sets Q_j $(j = 1, 2, \ldots, n)$ in such a way:

$$\lambda_k = \frac{\mu_k g(x_k)}{\|\nabla g(x_k)\|^2}, \tag{3.28}$$

where $\mu_k \in (0, 4)$ and $g : H_1 \to \mathbb{R}$ is defined by $g(x) := \dfrac{1}{2n} \displaystyle\sum_{j=1}^{n} \left\|(I - P_{Q_j})A_j x\right\|^2$ with the gradient given by $\nabla g(x) = \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} A_j^*(I - P_{Q_j})A_j x$. It would be interesting to modify the gradient-projection method with the stepsize (3.28) to solve the *constrained multiple-set split feasibility problem* (CMSSFP) [28] which is formulated as finding a point

$$x \in \bigcap_{i=1}^{m} C_i \text{ such that } A_j x \in Q_j,$$

where $C_i \subseteq H_1$ $(i = 1, 2, \ldots, m)$ and $Q_j \subseteq H_2$ $(j = 1, 2, \ldots, n)$ are nonempty closed convex subsets and $\{A_j : H_1 \to H_2\}$ is a finite family of bounded linear operators.

4. Self-adaptive relaxed CQ-type algorithm

Due to our main result in Sect. 3, we consider the two-operator SFP (1.3) for general closed convex subsets C, Q, and $\tilde{Q}$; however, finding the explicit forms of the metric projections P_C, P_Q, and $P_{\tilde{Q}}$ in Algorithm 1 may not be easy when these closed convex subsets are complicated. Fortunately, one of the ways for calculating the metric projection onto a sub-level set of a convex function suggested by Fukushima [16] is to compute the sequence of metric projections onto half-spaces containing such a sub-level set. By this idea, Yang [41] considered the SFP (1.1) in the case of two sub-level sets

$$C = \{x \in H_1 : f_1(x) \leq 0\} \text{ and } Q = \{y \in H_2 : f_2(y) \leq 0\}, \tag{4.29}$$

where $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ are two convex functions. Also, assume that f_1 and f_2 are subdifferentiable on H_1 and H_2, respectively, and both ∂f_1 and ∂f_2 are bounded operators (i.e., bounded on bounded sets). Yang [41] then introduced the so-called *relaxed CQ algorithm* for solving the SFP (1.1) constrained by (4.29) as follows:

$$\begin{cases} x_1 \in H_1, \\ x_{k+1} = P_{C_k}(x_k - \lambda A^*(I - P_{Q_k})Ax_k), \quad k \geq 1, \end{cases} \tag{4.30}$$

where $\lambda \in (0, 2/\|A\|^2)$ and C_k and Q_k are half-spaces given as

$$C_k = \{x \in H_1 : f_1(x_k) + \langle c_k, x - x_k \rangle \leq 0\},$$

where $c_k \in \partial f_1(x_k)$ and

$$Q_k = \{y \in H_2 : f_2(Ax_k) + \langle q_k, y - Ax_k \rangle \leq 0\},$$

where $q_k \in \partial f_2(Ax_k)$. It follows from the definition of the subdifferential that $C \subseteq C_k$ and $Q \subseteq Q_k$ for all $k \geq 1$. Since P_{C_k} and P_{Q_k} have closed forms (see [6, 16]), then the implementation of the relaxed CQ algorithm (4.30) is easier than that of the CQ algorithm (1.2) (in situations that P_C and P_Q have no closed forms). In addition, López et al. [25, Algorithm 4.1] modified (4.30) by using the self-adaptive stepsize λ_k (1.5). Vinh et al. [35, Algorithm 4.1] also introduced a relaxation version of the self-adaptive CQ-type algorithm (1.6) to solve this problem.

This section was motivated by the above-mentioned notions and results. We now focus on the two-operator SFP (1.3) in which closed convex subsets C, Q, and $\tilde{Q}$ are sub-level sets of convex functions. In what follows, we set the following hypotheses:

- H_1 and H_2 are real Hilbert spaces,
- $\emptyset \neq C \subseteq H_1$ and $\emptyset \neq Q, \tilde{Q} \subseteq H_2$ are given as:
$$C = \{x \in H_1 : f_1(x) \leq 0\},$$
$$Q = \{y \in H_2 : f_2(y) \leq 0\},$$
$$\tilde{Q} = \left\{y \in H_2 : \tilde{f}_2(y) \leq 0\right\},$$
where $f_1 : H_1 \to \mathbb{R}$ and $f_2, \tilde{f}_2 : H_2 \to \mathbb{R}$ are subdifferentiable and convex functions such that their subdifferential operators are bounded,
- $A, B : H_1 \to H_2$ are two bounded linear operators,
- $\Gamma = \left\{x \in C : Ax \in Q, Bx \in \tilde{Q}\right\} \neq \emptyset$.

Let $x_k \in H_1$. Denote

$$C_k := \{x \in H_1 : f_1(x_k) + \langle c_k, x - x_k \rangle \leq 0\}, \tag{4.31}$$

where $c_k \in \partial f_1(x_k)$,

$$Q_k := \{y \in H_2 : f_2(Ax_k) + \langle q_k, y - Ax_k \rangle \leq 0\}, \tag{4.32}$$

where $q_k \in \partial f_2(Ax_k)$, and

$$\tilde{Q}_k := \left\{y \in H_2 : \tilde{f}_2(Bx_k) + \langle \tilde{q}_k, y - Bx_k \rangle \leq 0\right\}, \tag{4.33}$$

where $\tilde{q}_k \in \partial \tilde{f}_2(Bx_k)$.

Lemma 4.7. *If there exists $x_k \in C$ such that $\left\|A^*(I - P_{Q_k})Ax_k + B^*(I - P_{\tilde{Q}_k})Bx_k\right\| = 0$, then $x_k \in \Gamma$.*

Proof. Let $x_k \in C$ be such that $\left\|A^*(I - P_{Q_k})Ax_k + B^*(I - P_{\tilde{Q}_k})Bx_k\right\| = 0$. Pick any $p \in \Gamma$. Since $Q \subseteq Q_k$ and $\tilde{Q} \subseteq \tilde{Q}_k$, then $Ap \in Q_k$ and $Bp \in \tilde{Q}_k$. By the same computation as the proof in Lemma 3.4, we get

$$\begin{aligned} 0 &= \left\|A^*(I - P_{Q_k})Ax_k + B^*(I - P_{\tilde{Q}_k})Bx_k\right\| \|x_k - p\| \\ &\geq \|(I - P_{Q_k})Ax_k\|^2 + \|(I - P_{\tilde{Q}_k})Bx_k\|^2, \end{aligned}$$

which follows that $(I-P_{Q_k})Ax_k = (I-P_{\tilde{Q}_k})Bx_k = 0$ and hence $Ax_k \in Q_k$ and $Bx_k \in \tilde{Q}_k$. By (4.32) and (4.33), we have $f_2(Ax_k) \le 0$ and $\tilde{f}_2(Bx_k) \le 0$. Thus, $Ax_k \in Q$ and $Bx_k \in \tilde{Q}$, i.e., $x_k \in \Gamma$. $\square$

Using (4.31)–(4.33), a relaxation version of Algorithm 1 is presented as follows.

Algorithm 2: Self-adaptive relaxed CQ-type algorithm for the two-operator SFP

Initialization: Take two real sequences $\{\beta_k\} \subset (0,1)$ and $\{\mu_k\} \subset (0,4)$. Choose an initial point $x_1 \in H_1$ arbitrarily and set $k = 1$.

Iterative Step: Given x_k, if $\left\| A^*(I-P_{Q_k})Ax_k + B^*(I-P_{\tilde{Q}_k})Bx_k \right\| = 0$, then $x_{k+1} = x_k$ and the iterative process stops. Otherwise, calculate

$$\lambda_k = \mu_k \frac{\|(I-P_{Q_k})Ax_k\|^2 + \|(I-P_{\tilde{Q}_k})Bx_k\|^2}{\left\| A^*(I-P_{Q_k})Ax_k + B^*(I-P_{\tilde{Q}_k})Bx_k \right\|^2}, \tag{4.34}$$

$$x_{k+1} = P_{C_k}\left[(1-\beta_k)\left(x_k - \frac{\lambda_k}{2}\Big(A^*(I-P_{Q_k})Ax_k + B^*(I-P_{\tilde{Q}_k})Bx_k\Big)\right)\right]. \tag{4.35}$$

Update $k := k+1$ and go on to Iterative Step.

For the sake of simplicity, we define a function $g_k : H_1 \to \mathbb{R}$ by

$$g_k(x) := \frac{1}{4}\Big(\|(I-P_{Q_k})Ax\|^2 + \|(I-P_{\tilde{Q}_k})Bx\|^2\Big)$$

with the gradient given by

$$\nabla g_k(x) = \frac{1}{2}\Big(A^*(I-P_{Q_k})Ax + B^*(I-P_{\tilde{Q}_k})Bx\Big), \quad x \in H_1.$$

So, (4.34) and (4.35) become

$$\lambda_k = \frac{\mu_k g_k(x_k)}{\|\nabla g_k(x_k)\|^2} \text{ and } x_{k+1} = P_{C_k}\Big[(1-\beta_k)\big(x_k - \lambda_k \nabla g_k(x_k)\big)\Big].$$

Below we prove a strong convergence result of Algorithm 2 which extends a result in [35, Theorem 4.1].

Theorem 4.2. *Let $\{x_k\}$ be a sequence generated by Algorithm 2 with the control sequences $\{\beta_k\}$ and $\{\mu_k\}$ satisfying:*

(C1) (1) $\lim_{k\to\infty} \beta_k = 0$ *and* (2) $\sum_{k=1}^{\infty} \beta_k = \infty$;

(C2) $\inf_k \mu_k(4-\mu_k) > 0$.

If $\nabla g_k(x_k) \neq 0$ for all $x_k \notin C$, then $\{x_k\}$ converges strongly to a point $x^* \in \Gamma$.

Proof. If $\nabla g_m(x_m) = 0$ for some $x_m \in C$, then the result is done by Lemma 4.7. So, we suppose that $\nabla g_k(x_k) \neq 0$ for all $k \in \mathbb{N}$. Let $x^* := P_\Gamma 0$. In view of the proof of Lemma 3.5 with replacing g and C by g_k and C_k, respectively, we deduce that

$$\|x_{k+1} - x^*\|^2 \le \beta_k\|x^*\|^2 + (1-\beta_k)\|x_k - x^*\|^2 - \mu_k(4-\mu_k)(1-\beta_k)\frac{g_k^2(x_k)}{\|\nabla g_k(x_k)\|^2} \tag{4.36}$$

and

$$\|x_{k+1}-x^*\|^2 \le (1-\beta_k)\|x_k-x^*\|^2 + \beta_k\Big[\beta_k\|x^*\|^2 + 2(1-\beta_k)\langle x_k-x^*, -x^*\rangle \\ + 2(1-\beta_k)\lambda_k \langle \nabla g_k(x_k), x^*\rangle\Big]. \tag{4.37}$$

By (4.36), we obtain that $\{x_k\}$ is bounded and

$$\mu_k(4-\mu_k)(1-\beta_k)\frac{g_k^2(x_k)}{\|\nabla g_k(x_k)\|^2} \le \beta_k\|x^*\|^2 + \|x_k-x^*\|^2 - \|x_{k+1}-x^*\|^2. \tag{4.38}$$

Now we consider the rest of the proof into two cases:

Case 1. Suppose that $\{\|x_k-x^*\|\}_{k\ge k_0}$ is either nonincreasing or nondecreasing (for some k_0). We then have $\{\|x_k-x^*\|\}$ is a convergent sequence and so $\|x_k-x^*\|^2-\|x_{k+1}-x^*\|^2 \to 0$ as $k\to\infty$. From (4.38), we get

$$\lim_{k\to\infty}\frac{g_k^2(x_k)}{\|\nabla g_k(x_k)\|^2} = 0, \tag{4.39}$$

which implies that

$$\lim_{k\to\infty}\lambda_k\|\nabla g_k(x_k)\| = \lim_{k\to\infty}\frac{g_k(x_k)}{\|\nabla g_k(x_k)\|} = 0. \tag{4.40}$$

Set $y_k = (1-\beta_k)\big(x_k - \lambda_k\nabla g_k(x_k)\big)$. By the same computation as in the proof of Theorem 3.1, we deduce that

$$\lim_{k\to\infty}\|x_k-y_k\| = 0 \tag{4.41}$$

and

$$\lim_{k\to\infty}\|(I-P_{C_k})y_k\| = 0. \tag{4.42}$$

Since $P_{C_k}y_k \in C_k$, it follows from (4.31) and using (4.41), (4.42), and the boundedness assumption on ∂f_1 that

$$\begin{aligned} f_1(x_k) &\le \langle c_k, x_k - P_{C_k}y_k\rangle \\ &= \langle c_k, x_k - y_k + y_k - P_{C_k}y_k\rangle \\ &\le \|c_k\|\left(\|x_k-y_k\| + \|(I-P_{C_k})y_k\|\right) \to 0 \text{ as } k\to\infty. \end{aligned} \tag{4.43}$$

Note that $\nabla g_k(x^*) = 0$ for all $k\in\mathbb{N}$. Since ∇g_k is Lipschitz continuous with a coefficient $L := \max\{\|A\|^2, \|B\|^2\}$, we have

$$\|\nabla g_k(x_k)\| = \|\nabla g_k(x_k) - \nabla g_k(x^*)\| \le L\|x_k - x^*\|, \quad \forall k\in\mathbb{N}.$$

So, $\{\nabla g_k(x_k)\}$ is bounded. This together with (4.39) yields that $g_k(x_k)\to 0$ as $k\to\infty$ and hence

$$\lim_{k\to\infty}\|(I-P_{Q_k})Ax_k\| = \lim_{k\to\infty}\|(I-P_{\tilde{Q}_k})Bx_k\| = 0. \tag{4.44}$$

Since $P_{Q_k}(Ax_k) \in Q_k$, it follows from (4.32) and using (4.44) and the boundedness assumption on ∂f_2 that

$$\begin{aligned} f_2(Ax_k) &\le \langle q_k, (I-P_{Q_k})Ax_k\rangle, \\ &\le \|q_k\|\,\|(I-P_{Q_k})Ax_k\| \to 0 \text{ as } k\to\infty. \end{aligned} \tag{4.45}$$

Similarly, since $P_{\tilde{Q}_k}(Bx_k) \in \tilde{Q}_k$, it follows from (4.33) and using (4.44) and the boundedness assumption on $\partial \tilde{f}_2$ that

$$\begin{aligned} \tilde{f}_2(Bx_k) &\leq \langle \tilde{q}_k, (I - P_{\tilde{Q}_k})Bx_k \rangle, \\ &\leq \|\tilde{q}_k\| \, \|(I - P_{\tilde{Q}_k})Bx_k\| \ \to 0 \ \text{ as } \ k \to \infty. \end{aligned} \tag{4.46}$$

Now let $\hat{x} \in \omega_w(x_k)$. Thus, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \hat{x}$. By the weakly lower semicontinuity of f_1 and using (4.43), we get

$$f_1(\hat{x}) \leq \liminf_{j \to \infty} f_1(x_{k_j}) \leq 0.$$

This means that $\hat{x} \in C$. Since A and B are bounded linear operators, we also have $Ax_{k_j} \rightharpoonup A\hat{x}$ and $Bx_{k_j} \rightharpoonup B\hat{x}$. By the weakly lower semicontinuity of f_2 and $\tilde{f}_2$ and using (4.45), (4.46), we obtain

$$f_2(A\hat{x}) \leq \liminf_{j \to \infty} f_2(Ax_{k_j}) \leq 0 \ \text{ and } \ \tilde{f}_2(B\hat{x}) \leq \liminf_{j \to \infty} \tilde{f}_2(Bx_{k_j}) \leq 0,$$

which imply that $A\hat{x} \in Q$ and $B\hat{x} \in \tilde{Q}$. Hence, $\hat{x} \in \Gamma$ and so we obtain that $\omega_w(x_k) \subseteq \Gamma$. Now, using the characterization of the projection, Lemma 2.1(1) with $P_\Gamma 0 = x^*$, we have

$$\limsup_{k \to \infty} \langle x_k - x^*, -x^* \rangle = \max_{\hat{x} \in \omega_w(x_k)} \langle \hat{x} - x^*, -x^* \rangle \leq 0. \tag{4.47}$$

From (4.37), we get

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leq (1 - \beta_k)\|x_k - x^*\|^2 + \beta_k \Big[\beta_k \|x^*\|^2 + 2(1 - \beta_k)\langle x_k - x^*, -x^* \rangle \\ &\qquad + 2(1 - \beta_k)\lambda_k \, \|\nabla g_k(x_k)\| \, \|x^*\| \Big] \\ &= (1 - \beta_k)\|x_k - x^*\|^2 + \beta_k \delta_k, \quad \forall k \in \mathbb{N}, \end{aligned}$$

where $\delta_k := \beta_k \|x^*\|^2 + 2(1 - \beta_k)\langle x_k - x^*, -x^* \rangle + 2(1 - \beta_k)\lambda_k \, \|\nabla g_k(x_k)\| \, \|x^*\|$. It follows from (4.40) and (4.47) that $\limsup_{k \to \infty} \delta_k \leq 0$. Finally, utilizing Lemma 2.2 with the above inequality, we can conclude that $x_k \to x^*$ as $k \to \infty$.

Case 2. Assume that $\{\|x_k - x^*\|\}$ is not monotone. Using Lemma 2.3 and following the similar argument to the proof in Case 1, one can prove that $\{x_k\}$ also converges strongly to $x^* = P_\Gamma 0$. So, we omit the proof for this case. □

5. Numerical experiments

To illustrate the convergence performance of our proposed algorithms and to support our main results, we first employ Algorithm 1 for solving (1.3) in the setting of a Euclidean space (see Example 5.1). After that, we use Algorithm 2 to solve the problem of recovering a sparse signal from a limited number of sampling with two different blurring operations (see Example 5.2). In both examples, we also compare the efficiency of our algorithms with that of some methods based on the operator norms. All the numerical experiments are completed on Apple MacBook Pro with 2 GHz Quad-Core Intel Core i5 with 16 GB memory. The program is implemented in MATLAB R2023a.

Example 5.1. Let $H_1 = H_2 = \mathbb{R}^2$ with the Euclidean norm. Consider a ball C and a half-space $Q = \tilde{Q}$ given by

$$C = \Big\{ (a, b) \in \mathbb{R}^2 : \sqrt{(a-2)^2 + b^2} \leq 2 \Big\} \ \text{ and } \ Q = \big\{ (a, b) \in \mathbb{R}^2 : 3a + 2b \leq -3 \big\}$$

and two operators $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ given by $A(a,b) = (-a,0)$ and $B(a,b) = (0,b)$ for all $(a,b) \in \mathbb{R}^2$. One can see that $\Gamma = \{x \in C : Ax, Bx \in Q\} \neq \emptyset$. We will find the minimum-norm element x^* in Γ by using our self-adaptive CQ-type algorithm, Algorithm 1. To do this, we arrange the following explicit forms of the metric projections:

$$P_C(a,b) = \begin{cases} (2,0) + \dfrac{2}{\sqrt{(a-2)^2+b^2}}(a-2,b), & \text{if } (a,b) \notin C, \\ (a,b), & \text{otherwise,} \end{cases}$$

and

$$P_Q(a,b) = \begin{cases} (a,b) - \dfrac{3a+2b+3}{13}(3,2), & \text{if } (a,b) \notin Q, \\ (a,b), & \text{otherwise,} \end{cases}$$

for all $(a,b) \in \mathbb{R}^2$. Firstly, we test the convergence behavior of Algorithm 1 by taking $\beta_k = \frac{1}{k+1}$ and $\mu_k = \frac{2k}{k+1}$ with the starting point $x_0 = (4,2)$ as shown in Figure 1. It is observed that $x_k \to (1,-1.5) \in \Gamma$ where $\|(1,-1.5)\| = \min_{p \in \Gamma} \|p\|$.

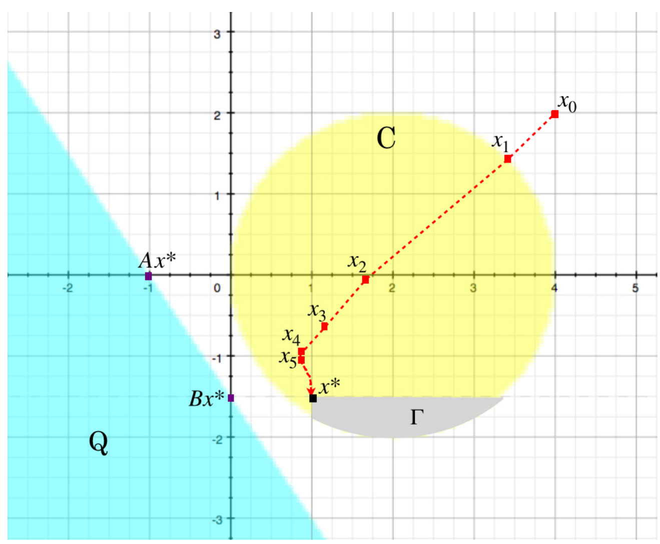

FIGURE 1. Illustration of the convergence behavior of Algorithm 1

Next, we analyze the convergence performance of Algorithm 1 by choosing different accelerating sequences $\{\mu_k\}$ and also compare with that of the following algorithms depending on the operator norms.

Algorithm 3: Let $\{x_k\}$ be a sequence generated by (3.8) where

$$\lambda_k := \lambda \in \left(0, \frac{2}{\max\{\|A\|^2, \|B\|^2\}}\right).$$

Algorithm 4: ([21]) Let $\{x_k\}$ be a sequence generated by (1.4) where $f := 0$.

Each algorithm is equipped with the parameters in Table 1.

TABLE 1. Setting parameters for each algorithm

Parameters	Algorithm 1	Algorithm 3	Algorithm 4
$\beta_k = \dfrac{1}{k+1}$	$\surd$	$\surd$	$\surd$
$\mu_k = \dfrac{\rho k}{k+1}$, $0 < \rho < 4$	$\surd$	–	–
$\delta_k = \gamma_k = \dfrac{k}{2k+2}$	–	–	$\surd$
$0 < \lambda < 2$	–	$\surd$	$\surd$

TABLE 2. Numerical experiments with the different choices of the stepsizes

	Choice of the stepsizes	k (No. of iter.)	CPU time (s)	x_k	E_k
Algorithm 1	$\rho = 0.5$	2198	0.2279830	$(0.9990240, -1.4967059)$	9.996E-07
	$\rho = 1$	1100	0.1576130	$(0.9990244, -1.4967074)$	9.987E-07
	$\rho = 2$	551	0.1091000	$(0.9990253, -1.4967104)$	9.968E-07
	$\rho = 3.5$	198	0.0890400	$(0.9991338, -1.4968926)$	8.726E-07
	$\rho = 3.9$	111	0.0785620	$(0.9993079, -1.4966262)$	9.585E-07
Algorithm 3	$\lambda = 0.5$	5919	0.5656190	$(0.9990239, -1.4967055)$	9.998E-07
	$\lambda = 1$	2960	0.3619410	$(0.9990240, -1.4967061)$	9.995E-07
	$\lambda = 1.9$	1558	0.2533760	$(0.9990241, -1.4967063)$	9.993E-07
Algorithm 4 ([21])	$\lambda = 0.5$	11837	1.1435740	$(0.9990238, -1.4967052)$	9.999E-07
	$\lambda = 1$	5919	0.6078450	$(0.9990239, -1.4967055)$	9.998E-07
	$\lambda = 1.9$	3115	0.3736660	$(0.9990238, -1.4967052)$	9.998E-07

We choose the starting point $x_0 = x_1 = (2,2)$ and use the stopping criterion for the iterative process as: $E_k := g(x_k) < 10^{-6}$, where g is defined by (3.9). Now the comparision of the numerical experiments of Algorithms 1, 3, and 4 are shown in Table 2.

Remark 5.3. By testing the performance of Algorithms 1, 3, and 4 and from Table 2, we observe that

(1) All studied algorithms give the approximate solutions close to $(1, -1.5)$ which is the minimum-norm solution.
(2) Algorithm 1 converges the fastest and takes the least time.
(3) The choice of the stepsizes influencses the convergence behavior of all studied algorithms. Namely that if $\{\mu_k\}$ is taken close to 4 (for Algorithm 1) and λ is taken close to 2 (for Algorithms 3 and 4), then the number of iterations and the CPU time have reduction. Meanwhile, choosing different starting points has no significant impact on their convergence behavior.

Example 5.2. (*Compressed Sensing* [25, 30]). Here, we consider the problem of recovering a sparse signal $x \in \mathbb{R}^N$ from the observation of two signals $y, \tilde{y} \in \mathbb{R}^M$ $(M < N)$ via the linear equation systems:

$$y = Ax + \varepsilon \quad \text{and} \quad \tilde{y} = Bx + \tilde{\varepsilon}, \tag{5.48}$$

where $A, B : \mathbb{R}^N \to \mathbb{R}^M$ are two bounded linear observation operators (they are often ill-condition) and $\varepsilon, \tilde{\varepsilon}$ are additive noises. The problem (5.48) can be solved by using the *LASSO* technique ([34]) in the froms of the constrained least-squares problem:

$$\text{minimize} \quad \frac{1}{2}\|Ax - y\|_2^2 \quad \text{and} \quad \frac{1}{2}\|Bx - \tilde{y}\|_2^2 \tag{5.49}$$

with respect to $x \in C := \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$, where $t > 0$ is a given constant. If (5.49) has a solution, we see that (5.49) is a particular case of the two-operator SFP (1.3) where $Q = \{y\}$ and $\tilde{Q} = \{\tilde{y}\}$. Since C is the closed l_1 ball in $\mathbb{R}^N$ with the radius t, we will employ the relaxation version of our self-adaptive CQ-type algorithm, Algorithm 2 to solve (5.49). Define $f_1(x) = \|x\|_1 - t$ and consider the half-space C_k denoted by (4.31). The closed form of the metric projection from $\mathbb{R}^N$ onto C_k is as follows:

$$P_{C_k}(x) = \begin{cases} x, & \text{if } f_1(x_k) + \langle c_k, x - x_k\rangle \leq 0, \\ x - \dfrac{f_1(x_k) + \langle c_k, x - x_k\rangle}{\|c_k\|^2} c_k, & \text{otherwise,} \end{cases}$$

where $c_k \in \partial f_1(x_k)$ is chosen as

$$c_k^{(i)} = \begin{cases} 1, & \text{if } x_k^{(i)} > 0, \\ 0, & \text{if } x_k^{(i)} = 0, \\ -1, & \text{if } x_k^{(i)} < 0, \end{cases}$$

see [17, Section 5].

In our experiment, two sampling matrices $A, B \in \mathbb{R}^{M\times N}$ are generated randomly from normal distributions with $N = 2048$ and $M = 1024$. The sparse signal $x^* \in \mathbb{R}^N$ is generated from a uniform distribution in $[-2, 2]$ with m nonzero components. The measured values y and $\tilde{y}$ are generated by white Gaussian noise with the signal-to-noise ratio (SNR) as 40 and 50 decibles, respectively. Set $t = m$. We test three cases as follows:

Case 1: $m = 10$, *Case 2:* $m = 50$, *Case 3:* $m = 100$.

We compare the signal recovery performance of Algorithm 2 with that of the following algorithm depending on $\|A\|$ and $\|B\|$.

Algorithm 5: Let $\{x_k\}$ be a sequence generated by (4.35) where

$$\lambda_k := \lambda \in \left(0, \frac{2}{\max\{\|A\|^2, \|B\|^2\}}\right).$$

Let $\beta_k = \frac{1}{k+1}$ and $\mu_k = \frac{2k}{k+1}$ for Algorithm 2 and $\beta_k = \frac{1}{k+1}$ and $\lambda = \frac{1}{\max\{\|A\|^2,\|B\|^2\}}$ for Algorithm 5. The process is started with the initial signal $x_1 = 0$. The restoration accuracy is measured by the mean squared error (MSE), i.e.,

$$\text{MSE}(k) = \frac{1}{N}\|x^* - x_k\|^2 < 10^{-4},$$

where x^* is the original signal and x_k is an estimated signal of x^*. Now, the numerical results of recovering the signal x^* are reported as Figures 2–7.

Remark 5.4. By the simple experiments as shown in Figures 2–7, we note that

(1) The original signals x^* can be recovered by Algorithms 2 and 5.
(2) If the number of spikes of x^* increases, then both methods also require an increase in the number of iterations and the CPU time. However, the number of iterations and the CPU time of using Algorithm 2 are less than those of using Algorithm 5.

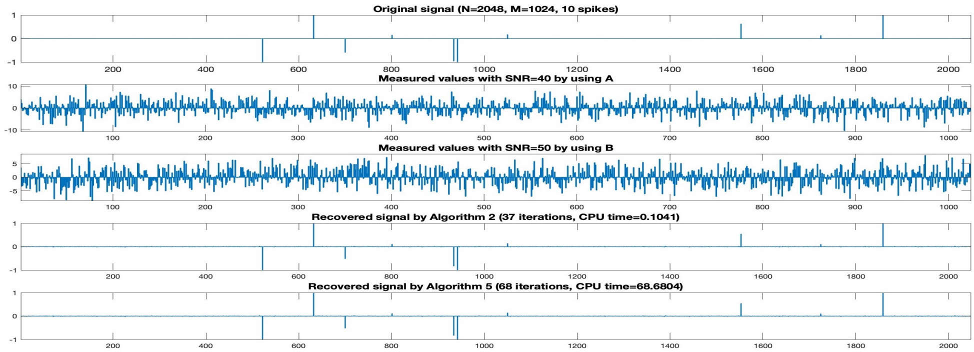

FIGURE 2. Signal recovery experiment in Case 1.
From top to bottom: original signal; observation data using A; observation data using B; recovered signal by Algorithm 2; recovered signal by Algorithm 5

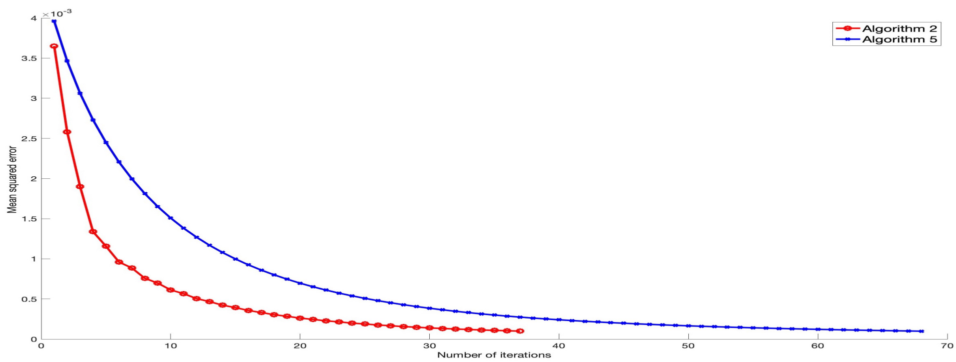

FIGURE 3. The mean squared error versus the number of iterations in Case 1

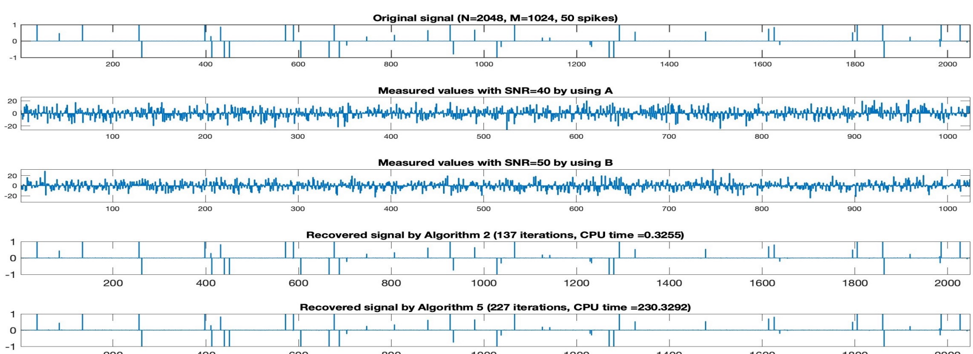

FIGURE 4. Signal recovery experiment in Case 2.
From top to bottom: original signal; observation data using A; observation data using B; recovered signal by Algorithm 2; recovered signal by Algorithm 5

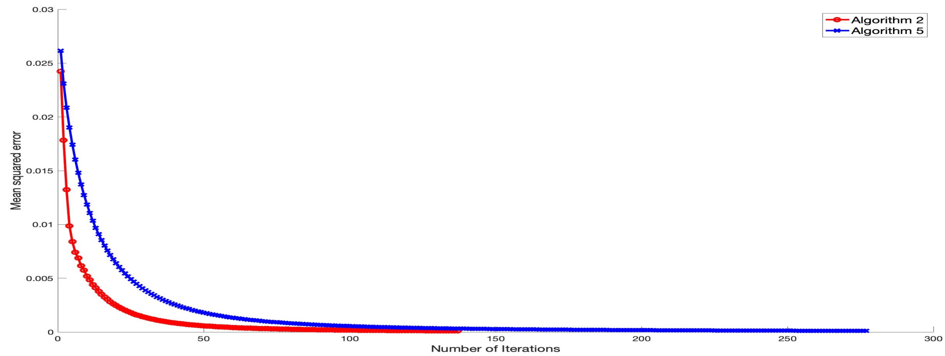

FIGURE 5. The mean squared error versus the number of iterations in Case 2

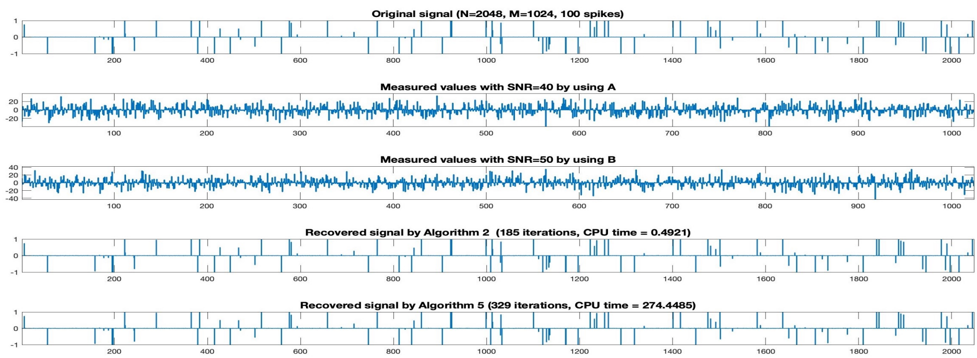

FIGURE 6. Signal recovery experiment in Case 3.
From top to bottom: original signal; observation data using A; observation data using B; recovered signal by Algorithm 2; recovered signal by Algorithm 5

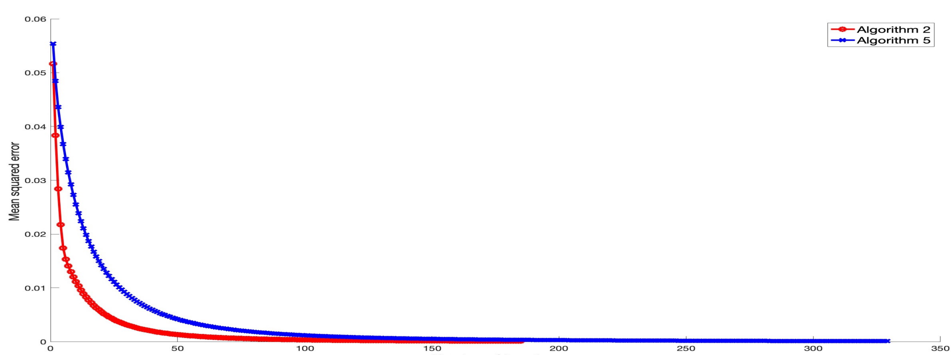

FIGURE 7. The mean squared error versus the number of iterations in Case 3

For the closed forms of some metric projections onto simple closed convex subsets in Hilbert spaces, the reader is referred to [6, Chapter 4]. There are also some examples for the split feasibility problem and related problems in the infinite-dimensional Hilbert spaces, see [23, 33, 35].

Conclusion

This paper discusses and analyzes the convergence results on the two-operator split feasibility problem (two-operator SFP) in Hilbert spaces, namely finding a point of a closed convex subset of a Hilbert space such that each of its images under two given bounded linear operators belongs to a closed convex subset of another Hilbert space. We introduce a self-adaptive CQ-type algorithm where the stepsize does not depend on such bounded linear operator norms. Under some mild conditions, we then prove that the sequence generated by the proposed algorithm converges strongly to the minimum-norm solution of the two-operator SFP. A relaxation version of our proposed algorithm is also introduced for solving the problem constrained by sub-level sets of convex functions. Our main results improve the result of Kangtunyakarn [21, Theorems 3.1] in terms of selecting the stepsize in the algorithm and generalize the results of Vinh et al. [35, Theorems 3.1 and 4.1] for the split feasibility problem (also improve the results of Xu [40], Wang and Xu [37], Yao et al. [43] and Chuang [7]). In addition, it is observed from our numerical experiments that our self-adaptive CQ-type algorithms (without any operator norms) are more efficient than the CQ-type algorithms based on the operator norms.

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References

[1] Aubin, A. J. *Optima and Equilibria: An Introduction to Nonlinear Analysis*. Springer, Berlin, 1993.

[2] Byrne, C. Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse Probl.* **18** (2002), 441–453.

[3] Byrne, C. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Probl.* **20** (2004), 103–120.

[4] Byrne, C.; Censor, Y.; Gibali, A.; Reich, S. The split common null point problem. *J. Nonlinear Convex Anal.* **13** (2012), 759–775.

[5] Bussaban, L.; Suantai, S.; Kaewkhao, A. A parallel inertial S-iteration forward-backward algorithm for regression and classification problems. *Carpathian J. Math.* **36** (2020), 35–44.

[6] Cegielski, A. *Iterative Methods for Fixed Point Problems in Hilbert Spaces.* Lecture Notes in Mathematics 2057, Springer, Berlin, Heidelberg, 2012.

[7] Chuang, S. S. Strong convergence theorems for the split variational inclusion problem in Hilbert spaces. *Fixed Point Theory Appl.* **2013** (2013), Article no. 350.

[8] Cegielski, A. Landweber-type operator and its properties. *Contemp. Math.* **658** (2016), 139–148.

[9] Ceng, L. C.; Ansari, Q. H.; Yao, J. C. An extragradient method for solving split feasibility and fixed point problems. *Comput. Math. Appl.* **64** (2012), 633–642.

[10] Censor, Y.; Bortfeld, T.; Martin, B.; Trofimov, A. A unified approach for inversion problems in intensity modulated radiation therapy. *Phys. Med. Biol.* **51** (2003), 2353–2365.

[11] Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithm.* **8** (1994), 221–239.

[12] Censor, Y.; Elfving, T.; Kopf, N.; Bortfeld, T. The multiple-sets split feasibility problem and its applications for inverse problems. *Inverse Probl.* **21** (2005), 2071–2084.

[13] Censor, Y.; Gibali, A.; Reich, S. Algorithms for the split variational inequality problem. *Numer. Algorithms* **59** (2012), 301–323.

[14] Censor, Y.; Segal, A. The split common fixed point problem for directed operators. *J. Convex Anal.* **16** (2009), 587–600.

[15] Eicke, B. Iteration methods for convexly constrained ill-posed problems in Hilbert space. *Numer. Funct. Anal. Optim.* **13** (1992), 413–419.

[16] Fukushima, M. A relaxed projection method for variational inequalities. *Math. Program.* **35** (1986), 58–70.

[17] Gibali, A.; Mai, D. T.; Vinh, N. T. A new relaxed CQ algorithm for solving split feasibility problems in Hilbert spaces and its applications. *J. Ind. Manag. Optim.* **15** (2019), 963–984.

[18] Jailoka, P.; Suantai, S. Viscosity approximation methods for split common fixed point problems without prior knowledge of the operator norm. *Filomat* **34** (2020), 761–777.

[19] Jailoka, P.; Suantai, S. On split fixed point problems for multi-valued mappings and designing a self-adaptive method. *Results Math.* **76** (2021), Article no. 133.

[20] Jailoka, P.; Suantai, S.; Hanjing, A. A fast viscosity forward-backward algorithm for convex minimization problems with an application in image recovery. *Carpathian J. Math.* **37** (2021), 449–461.

[21] Kangtunyakarn, A. Iterative scheme for finding solutions of the general split feasibility problem and the general constrained minimization problems. *Filomat* **33** (2019), 233–243.

[22] Kotzer, T.; Cohen, N.; Shamir, J. Extended and alternative projections onto convex sets: theory and applications. *Technical Report No. EE 900, Dept. of Electrical Engineering, Technion, Haifa, Israel* (1993).

[23] Khuangsatung, W.; Jailoka, P.; Suantai, S. An iterative method for solving proximal split feasibility problems and fixed point problems. *Comput. Appl. Math.* **38** (2019), no. 177.

[24] Kesornprom, S.; Pholasa, N.; Cholamjiak, P. On the convergence analysis of the gradient-CQ algorithms for the split feasibility problem. *Numer. Algor.* **84** (2020), 997–1017.

[25] López, G.; Martín-Márquez, V.; Wang, F.; Xu, H. K. Solving the split feasibility problem without prior knowledge of matrix norms. *Inverse Probl.* **28** (2012), https://doi.org/10.1088/0266- 5611/28/8/085004.

[26] Mainge, P. E. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. *Set-Valued Anal.*, **16** (2008), 899–912.

[27] Maingé, P. E. A viscosity method with no spectral radius requirements for the split common fixed point problem. *European J. Oper. Res.* **235** (2014), 17–27.

[28] Masad, E.; Reich, S. A note on the multiple-set split convex feasibility problem in Hilbert space. *J. Nonlinear Convex Anal.* **8** (2007), 367–371.

[29] Neubauer, A. Tikhonov-regularization of ill-posed linear operator equations on closed convex sets. *J. Approx. Theory* **53** (1988), 304–320.

[30] Nguyen, T. L. N.; Shin, Y. Deterministic sensing matrices in compressive sensing: A survey. *Sci. World J.* **2013** (2013), Article ID 192795, 6 pp.

[31] Polyak, B. T. *Introduction to Optimization*. Optimization Software, New York, 1987.

[32] Reich, S.; Truong, M. T.; Mai, T. N. H. The split feasibility problem with multiple output sets in Hilbert spaces. *Optim. Lett.* **14** (2020), 2335–2353.

[33] Suantai, S.; Jailoka, P. A self-adaptive algorithm for split null point problems and fixed point problems for demicontractive multivalued mappings. *Acta Appl. Math.* **170** (2020), 883–901.

[34] Tibshirani, R. Regression shrinkage and selection via the lasso. *J. R. Stat. Soc. B Methodol.* **58** (1996), 267–288.

[35] Vinh, V. T.; Cholamjiak, P.; Suantai, S. A new CQ algorithm for solving split feasibility problems in Hilbert spaces. *Bull. Malays. Math. Sci. Soc.* **42** (2019), 2517–2534.

[36] Vinh, N. T.; Hoai, P. T. Some subgradient extragradient type algorithms for solving split feasibility and fixed point problems. *Math. Meth. Appl. Sci.* **39** (2016), 3808–3823.

[37] Wang, F.; Xu, H. K. Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem. *J. Inequal. Appl.* **2010** (2010), Article ID 102085.

[38] Wang, P.; Zhou, J.; Wang, R.; Chen, J. New generalized variable stepsizes of the CQ algorithm for solving the split feasibility problem. *J. Inequal. Appl.* **2017** (2017), no. 135.

[39] Xu, H. K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66** (2002), 240–256.

[40] Xu, H. K. Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. *Inverse Probl.* **26** (2010), 105018.

[41] Yang, Q. The relaxed CQ algorithm for solving the split feasibility problem. *Inverse Problems* **20** (2004), 1261–1266.

[42] Yang, Q. On variable-step relaxed projection algorithm for variational inequalities. *J. Math. Anal. Appl.* **302** (2005), 166–179

[43] Yao, Y. H.; Gang, W. J.; Liou, Y. C. Regularized methods for the split feasibility problem. *Abstr. Appl. Anal.* **2012** (2012), Article ID 140679.

[44] Yao, Y.; Postolache, M.; Liou, Y. Strong convergence of a self-adaptive method for the split feasibility problem. *Fixed Point Theory Appl.* **2013** (2013), no. 201.

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On the intermixed method for mixed variational inequality problems: another look and some corrections

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Abstract

We explore the intermixed method for finding a common element of the intersection of the solution set of a mixed variational inequality and the fixed point set of a nonexpansive mapping. We point out that Khuangsatung and Kangtunyakarn's statement [J. Inequal. Appl. 2023:1, 2023] regarding the resolvent utilized in their paper is not correct. To resolve this gap, we employ the epigraphical projection and the product space approach. In particular, we obtain a strong convergence theorem with a weaker assumption.

Keywords: Nonexpansive mapping; Variational inequality; Fixed point; Epigraphical projection

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let $C \subset \mathcal{H}$, $S : C \to \mathcal{H}$, and $\alpha > 0$. We say that

- S is *α-Lipschitzian* if $\|Sx - Sy\| \le \alpha\|x - y\|$ for all $x, y \in C$;
- S is *α-inverse strongly monotone* if $\langle Sx - Sy, x - y\rangle \ge \alpha\|Sx - Sy\|^2$ for all $x, y \in C$.

An α-Lipschitzian mapping with $\alpha \in (0,1)$ ($\alpha = 1$, resp.) is called a *contraction* (a *nonexpansive* mapping, resp.). The following two classical nonlinear problems have been widely studied:

Fixed Point Problem: Find $x \in C$ such that $x = Sx$ (see [2]).

Variational Inequality Problem: Find $x \in C$ such that $\langle Sx, y - x\rangle \ge 0$ for all $y \in C$ (see [3]).

The solution sets of the preceding two problems are denoted by $\mathrm{Fix}(S)$ and $\mathrm{VI}(C,S)$, respectively. The following two observations are well known.

- If $S : C \to C$ is any mapping and $\mathrm{Id} : C \to C$ is the identity mapping, then $\mathrm{Fix}(S) = \mathrm{VI}(C, \mathrm{Id} - S)$. In fact, if $x = Sx$, then $\langle(\mathrm{Id} - S)x, y - x\rangle = 0$ for all $y \in C$. Hence $\mathrm{Fix}(S) \subset \mathrm{VI}(C, \mathrm{Id} - S)$. On the other hand, let $x \in C$ be such that $\langle(\mathrm{Id} - S)x, y - x\rangle \ge 0$ for all $y \in C$. Let $y := Sx \in C$. It follows that $-\|x - Sx\|^2 = \langle x - Sx, Sx - x\rangle \ge 0$, and hence $x = Sx$. This implies that reverse inclusion, and the statement is proved.
- If C is a closed convex subset of $\mathcal{H}$ and $S : C \to \mathcal{H}$ is any mapping, then $\mathrm{VI}(C,S) = \mathrm{Fix}(P_C \circ (\mathrm{Id} - S))$, where P_C is the metric projection onto C. Note that for

$x \in \mathcal{H}$ and $z \in C$, $z = P_C x$ if and only if $\langle z - x, y - z\rangle \geq 0$ for all $y \in C$ (for example, see [4]). To see this, let $x \in C$. It follows that

$$\langle Sx, y - x\rangle \geq 0 \text{ for all } y \in C \iff \langle x - (\mathrm{Id} - S)x, y - x\rangle \geq 0 \text{ for all } y \in C.$$

Hence $x \in \mathrm{VI}(C, S) \iff x = P_C(\mathrm{Id} - S)x \iff x \in \mathrm{Fix}(P_C \circ (\mathrm{Id} - S))$, and the statement is proved.

Recently, Khuangsatung and Kangtunyakarn [1] studied the following problem:

Let $f : \mathcal{H} \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. Let $C \subset \mathcal{H}$ be a closed convex set, and let $S : C \to C$. The *mixed variational inequality* problem is to find an element $x \in C$ such that

$$\langle Sx, y - x\rangle + f(y) - f(x) \geq 0 \quad \text{for all } y \in C.$$

The solution of this problem is denoted by $\mathrm{VI}(C, S, f)$. If $f \equiv 0$, then the mixed variational inequality problem becomes the (classical) variational inequality problem. They claimed in their Lemma 2.6 that

$$\mathrm{VI}(C, S, f) = \mathrm{Fix}\bigl((\mathrm{Id} + \gamma \partial f)^{-1} \circ (\mathrm{Id} - \gamma S)\bigr) \quad (\text{for all } \gamma > 0),$$

where ∂f is the *subdifferential* operator of f, that is,

$$\partial f(x) := \bigl\{z \in \mathcal{H} : \langle z, y - x\rangle + f(x) \leq f(y) \text{ for all } y \in \mathcal{H}\bigr\}.$$

Unfortunately, their claim is *not* correct. To see this, let $C := [1, 2] \subset \mathbb{R}$, $Sx := 2x$ for all $x \in C$, and $f(x) := 0$ for all $x \in \mathbb{R}$. It follows that $\mathrm{VI}(C, S, f) = \{1\}$ and $\mathrm{Fix}((\mathrm{Id} + \gamma \partial f)^{-1} \circ (\mathrm{Id} - \gamma S)) = \mathrm{Fix}(\mathrm{Id} - \gamma S) = \varnothing$ for all $\gamma > 0$. In this paper, we propose an alternative way to address this gap. Moreover, we use the product space approach to deduce the *intermixed method* [5] and show that the convergence result can be established under a *weaker* assumption.

Let us recall their main result.

Theorem KK *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose that $A_1, A_2, B_1, B_2 : C \to \mathcal{H}$ are α-inverse strongly monotone operators and $T_1, T_2 : C \to C$ are nonexpansive mappings. Suppose that $f_1, f_2 : \mathcal{H} \to (-\infty, \infty]$ are proper, convex, and lower semicontinuous functions. Assume that for $i = 1, 2$,*

$$\Omega_i := \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A_i, f_i) \cap \mathrm{VI}(C, B_i, f_i) \neq \varnothing.$$

Suppose that $g_1, g_2 : \mathcal{H} \to \mathcal{H}$ are contractions and $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are iterative sequences generated by the following scheme:

$x_1, y_1 \in C$ *are arbitrarily chosen,*

$x_n' := b_1 x_n + (1 - b_1) T_1 x_n,$

$y_n' := b_2 y_n + (1 - b_2) T_2 y_n,$

$$x_n'' := J_{\gamma_1 f_1}\big(x_n - \gamma_1\big(a_1A_1 + (1-a_1)B_1\big)x_n\big),$$
$$y_n'' := J_{\gamma_2 f_2}\big(y_n - \gamma_2\big(a_2A_2 + (1-a_2)B_2\big)y_n\big),$$
$$x_{n+1} := (1-\beta_n)x_n' + \beta_n P_C\big(\alpha_n g_2(y_n) + (1-\alpha_n)x_n''\big),$$
$$y_{n+1} := (1-\beta_n)y_n' + \beta_n P_C\big(\alpha_n g_1(x_n) + (1-\alpha_n)y_n''\big),$$

where $\gamma_1, \gamma_2 \in (0, 2\alpha)$, $a_1, a_2, b_1, b_2 \in (0,1)$, *and the sequences* $\{\alpha_n\}_{n=1}^\infty$ *and* $\{\beta_n\}_{n=1}^\infty \subset [0,1]$ *satisfy the following conditions*:

- (C1) $\lim_n \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$,
- (C2) $\beta_n \in [k,l] \subset (0,1)$ *for all* $n \geq 1$,
- (C3) $\sum_n |\alpha_n - \alpha_{n+1}| < \infty$ *and* $\sum_n |\beta_n - \beta_{n+1}| < \infty$.

Then there are two elements x^* *and* y^* *such that* $x^* = P_{\Omega_1} g_2(y^*)$, $y^* = P_{\Omega_2} g_1(x^*)$, *and the iterative sequences* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *converge strongly to* x^* *and* y^*, *respectively.*

We need the following lemma.

Lemma 1 ([6]) *Let* $\{s_n\}_{n=1}^\infty$ *be a sequence of nonnegative real numbers, let* $\{t_n\}_{n=1}^\infty$ *be a sequence of real numbers, and let* $\{\alpha_n\}_{n=1}^\infty$ *be a sequence in* $[0,1]$ *such that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n \quad \text{for all } n \geq 1.$$

If $\sum_n \alpha_n = \infty$ *and* $\limsup_n t_n \leq 0$, *then* $\lim_n s_n = 0$.

Lemma 2 *Let* $C \subset \mathcal{H}$ *and* $S : C \to \mathcal{H}$. *Then*:

- (a) *If* C *is closed and convex and* S *is nonexpansive, then* $\mathrm{Fix}(S)$ *is closed and convex.*
- (b) *If* S *is* α*-inverse strongly monotone, then* $\mathrm{Id} - \lambda S$ *is nonexpansive for all* $\lambda \in [0, 2\alpha]$.

2 Main results

2.1 A Halpern-type method

Recall that a nonexpansive mapping $S : C \to C$ is *r-strongly quasi-nonexpansive* ($r > 0$) if $\mathrm{Fix}(S) \neq \varnothing$ and

$$\|Sx - p\|^2 \leq \|x - p\|^2 - r\|x - Sx\|^2 \quad \text{for all } x \in C \text{ and } p \in \mathrm{Fix}(S).$$

It is well known that every nonexpansive mapping $S : C \to C$ satisfies the *Browder demiclosedness principle*: $p \in \mathrm{Fix}(S)$ whenever $\{x_n\}_{n=1}^\infty$ is a sequence in C such that $\lim_n \|x_n - Sx_n\| = 0$ and $\{x_n\}_{n=1}^\infty$ converges weakly to $p \in C$ (see [7]). The technique we used in the following result is taken from Wang et al. [8].

Theorem 3 *Let* $C \subset \mathcal{H}$ *be closed and convex, and let* $S, U : C \to C$ *be nonexpansive mappings such that* $F := \mathrm{Fix}(S) \cap \mathrm{Fix}(U) \neq \varnothing$. *Suppose that* S *is* r*-strongly quasinonexpansive, where* $r > 0$. *Suppose that* $u \in \mathcal{H}$ *and* $\{x_n\}_{n=1}^\infty$ *is an iterative sequence generated by the following scheme*:

$$x_1 \in C \text{ is arbitrarily chosen},$$
$$x_{n+1} := (1-\beta_n)Sx_n + \beta_n P_C\big(\alpha_n u + (1-\alpha_n)Ux_n\big) \quad (n \geq 1),$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0,1]$ *satisfy the following conditions*:

$$\lim_n \frac{\alpha_n}{1-\beta_n} = 0 \quad \text{and} \quad \sum_n \alpha_n \beta_n = \infty.$$

Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $P_F u$.

Proof Note that F is closed and convex. Let $z := P_F u$. It follows that $z = P_F z = Sz = Uz$ and

$$\begin{aligned}
\|x_{n+1} - z\| &\le (1-\beta_n)\|Sx_n - z\| + \beta_n \big\| P_C\big(\alpha_n u + (1-\alpha_n)Ux_n\big) - P_F z\big\| \\
&\le (1-\beta_n)\|x_n - z\| + \beta_n\alpha_n\|u - z\| + \beta_n(1-\alpha_n)\|Ux_n - z\| \\
&\le (1-\beta_n\alpha_n)\|x_n - z\| + \beta_n\alpha_n\|u - z\| \\
&\le \max\big\{\|x_n - z\|, \|u - z\|\big\}.
\end{aligned}$$

It follows by induction that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. In particular, the sequences $\{Sx_n\}_{n=1}^{\infty}$, $\{Ux_n\}_{n=1}^{\infty}$, and $\{P_C(\alpha_n u + (1-\alpha_n)Ux_n)\}_{n=1}^{\infty}$ are all bounded. For convenience, we denote

$$u_n := \alpha_n u + (1-\alpha_n)Ux_n.$$

We refine the preceding estimates by considering $\|\cdot\|^2$ as follows:

$$\|Sx_n - z\|^2 \le \|x_n - z\|^2 - r\|x_n - Sx_n\|^2,$$

and

$$\begin{aligned}
\|u_n - z\|^2 &= \big\|\alpha_n(u - z) + (1-\alpha_n)(Ux_n - z)\big\|^2 \\
&\le \big\|(1-\alpha_n)(Ux_n - z)\big\|^2 + 2\big\langle\alpha_n(u - z), u_n - z\big\rangle \\
&\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, u_n - z\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad = \big\|(1-\beta_n)(Sx_n - z) + \beta_n(P_C u_n - z)\big\|^2 \\
&\quad = (1-\beta_n)\|Sx_n - z\|^2 + \beta_n\|P_C u_n - z\|^2 - \beta_n(1-\beta_n)\|Sx_n - P_C u_n\|^2 \\
&\quad \le (1-\beta_n)\big(\|x_n - z\|^2 - r\|x_n - Sx_n\|^2\big) + \beta_n\big((1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, u_n - z\rangle\big) \\
&\qquad - \beta_n(1-\beta_n)\|Sx_n - P_C u_n\|^2 \\
&\quad = (1-\alpha_n\beta_n)\|x_n - z\|^2 + 2\alpha_n\beta_n\langle u - z, u_n - z\rangle \\
&\qquad - r(1-\beta_n)\|x_n - Sx_n\|^2 - \beta_n(1-\beta_n)\|Sx_n - P_C u_n\|^2.
\end{aligned}$$

Since $\sum_n \alpha_n\beta_n = \infty$, we have

$$\limsup_n \|x_n - z\|^2 \le L,$$

where

$$L := \limsup_n \left(2\langle u - z, u_n - z\rangle - \frac{r(1-\beta_n)}{\alpha_n \beta_n} \|x_n - Sx_n\|^2 - \frac{1-\beta_n}{\alpha_n} \|Sx_n - P_C u_n\|^2 \right).$$

Note that $L \le 2\limsup_n \langle u - z, u_n - z\rangle < \infty$ because $\{u_n\}_{n=1}^\infty$ is bounded. If $L = -\infty$, then it follows that $\limsup_n \|x_n - z\|^2 \le 0$, and we are done. We now assume that L is finite. Let $\{n_k\}_{k=1}^\infty$ be a strictly increasing sequence such that $\{u_{n_k}\}_{k=1}^\infty$ converges weakly to some element $q \in C$ and

$$\lim_k \left(2\langle u - z, u_{n_k} - z\rangle - \frac{r(1-\beta_{n_k})}{\alpha_{n_k} \beta_{n_k}} \|x_{n_k} - Sx_{n_k}\|^2 - \frac{1-\beta_{n_k}}{\alpha_{n_k}} \|Sx_{n_k} - P_C u_{n_k}\|^2 \right) = L.$$

In particular, the sequences

$$\left\{ \frac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}} \|x_{n_k} - Sx_{n_k}\|^2 \right\}_{k=1}^\infty \quad \text{and} \quad \left\{ \frac{1-\beta_{n_k}}{\alpha_{n_k}} \|Sx_{n_k} - P_C u_{n_k}\|^2 \right\}_{k=1}^\infty$$

are both bounded. Note that $\lim_n \frac{\alpha_n\beta_n}{1-\beta_n} = \lim_n \frac{\alpha_n}{1-\beta_n} = 0$. It follows that

$$\lim_k \|x_{n_k} - Sx_{n_k}\|^2 = \lim_k \|Sx_{n_k} - P_C u_{n_k}\|^2 = 0.$$

Moreover, we have $\lim_k \|u_{n_k} - Ux_{n_k}\| = 0$. In particular, $\lim_k \|P_C u_{n_k} - Ux_{n_k}\| = 0$ and $x_{n_k} \rightharpoonup q$. Then it follows that

$$\lim_k \|x_{n_k} - Ux_{n_k}\| \le \lim_k \big(\|x_{n_k} - Sx_{n_k}\| + \|Sx_{n_k} - P_C u_{n_k}\| + \|P_C u_{n_k} - Ux_{n_k}\| \big) = 0.$$

In particular, it follows from the Browder demiclosedness principle that $q \in F$, and hence $\langle z - u, q - z\rangle \ge 0$. This implies that $\limsup_n \|x_n - z\|^2 \le L \le 2\lim_k \langle u - z, u_{n_k} - z\rangle = 2\langle u - z, q - z\rangle \le 0$. □

Corollary 4 *Suppose that $C, S, U, F, r, \{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are as in the preceding theorem. Suppose that $h : \mathcal{H} \to \mathcal{H}$ is a contraction and $\{x_n\}_{n=1}^\infty$ is an iterative sequence generated by the following scheme*:

$$\begin{aligned} &x_1 \in C \text{ is arbitrarily chosen}, \\ &x_{n+1} := (1-\beta_n)Sx_n + \beta_n P_C\big(\alpha_n h(x_n) + (1-\alpha_n)Ux_n\big) \quad (n \ge 1). \end{aligned}$$

Proof Note that $P_F \circ h : \mathcal{H} \to \mathcal{H}$ is a contraction, and thus it follows that there exists a unique element $z \in \mathcal{H}$ such that $z = (P_F \circ h)(z)$. It is clear that $z \in F$. We let $u := h(z)$ and define

$$\begin{aligned} &y_1 := x_1, \\ &y_{n+1} := (1-\beta_n)Sy_n + \beta_n P_C\big(\alpha_n u + (1-\alpha_n)Uy_n\big) \quad (n \ge 1). \end{aligned}$$

It follows from the preceding theorem that $\lim_n \|y_n - z\| = 0$. Suppose that h is γ-Lipschitzian with $\gamma \in (0,1)$. We have the following estimate:

$$\|x_{n+1} - y_{n+1}\| \le (1-\beta_n)\|Sx_n - Sy_n\| + \beta_n \big\| P_C\big(\alpha_n h(x_n) + (1-\alpha_n)Ux_n\big)$$

$$
\begin{aligned}
&\quad - P_C\big(\alpha_n u + (1-\alpha_n)Uy_n\big)\big\| \\
&\le (1-\beta_n)\|x_n - y_n\| + \beta_n\big\|\alpha_n h(x_n) + (1-\alpha_n)Ux_n \\
&\quad - \big(\alpha_n h(z) + (1-\alpha_n)Uy_n\big)\big\| \\
&\le (1-\beta_n)\|x_n - y_n\| + \alpha_n\beta_n\big\|h(x_n) - h(z)\big\| + (1-\alpha_n)\beta_n\|Ux_n - Uy_n\| \\
&\le (1-\alpha_n\beta_n)\|x_n - y_n\| + \alpha_n\beta_n\big\|h(x_n) - h(z)\big\|.
\end{aligned}
$$

It follows from $\sum_n \alpha_n\beta_n = \infty$ that

$$
\begin{aligned}
\limsup_n \|x_n - y_n\| &\le \limsup_n \big\|h(x_n) - h(z)\big\| \\
&\le \limsup_n \gamma\|x_n - z\| \\
&\le \gamma \limsup_n \big(\|x_n - y_n\| + \|y_n - z\|\big) \\
&= \gamma \limsup_n \|x_n - y_n\|.
\end{aligned}
$$

In particular, since $\gamma < 1$, we have $\lim_n \|x_n - y_n\| = 0$, and hence $\lim_n \|x_n - z\| = 0$. The proof is complete. □

Let $S :=$ Id and $u \in C$. We immediately obtain the following result.

Corollary 5 *Let $C \subset \mathcal{H}$ be closed and convex, and let $U : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(U) \neq \emptyset$. *Suppose that $u \in C$ and $\{x_n\}_{n=1}^{\infty}$ is an iterative sequence generated by the following scheme*:

$$
\begin{aligned}
&x_1 \in C \text{ is arbitrarily chosen,} \\
&x_{n+1} := (1-\beta_n)x_n + \beta_n\big(\alpha_n u + (1-\alpha_n)Ux_n\big) \quad (n \ge 1),
\end{aligned}
$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0,1]$ satisfy the following conditions:

$$
\lim_n \frac{\alpha_n}{1-\beta_n} = 0 \quad \textit{and} \quad \sum_n \alpha_n\beta_n = \infty.
$$

Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_{\mathrm{Fix}(U)}u$.

2.2 Comments and remarks on the mixed variational inequality problem

Let $C \subset \mathcal{H}$ be closed and convex, let $A : C \to \mathcal{H}$, and let $f : \mathcal{H} \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. The *mixed variational inequality problem* is to find $x \in C$ such that

$$
\langle Ax, y - x\rangle + f(y) - f(x) \ge 0 \quad \text{for all } y \in C. \tag{$\star$}
$$

As pointed out in the introduction of the paper, the resolvent proposed by Khuangsatung and Kangtunyakarn [1] is not correct. Moreover, without any further assumption on C and $\mathrm{dom} f$, it is possible to encounter the experession $\infty - \infty$ in ($\star$). For example, let $Ax := 0$

for all $x \in C := [1,2] \subset \mathbb{R}$. and let $f(x) := 0$ if $x \in [3,4]$ and $f(x) := \infty$ if $x \notin [3,4]$. To be on the right track, we discuss the problem with an additional assumption.

This mixed type problem was also considered by Mosco [9] in 1969. From now on, we also assume that $\operatorname{dom} f \subset C$ is as in Mosco's setting. In particular, we also have $\mathrm{VI}(C,A,f) \subset \operatorname{dom} f$.

Mosco proved that the mixed and the classical variational inequality problems are equivalent. To see this, let $\widehat{\mathcal{H}} := \mathcal{H} \times \mathbb{R}$ with $\langle\!\langle \widehat{x}, \widehat{y} \rangle\!\rangle := \langle x, y\rangle + rs$ for all $\widehat{x} := (x,r)$ and $\widehat{y} := (y,s) \in \widehat{\mathcal{H}}$, and let $\widehat{C} := C \times \mathbb{R}$. Note that $|||\widehat{x}|||^2 = \langle\!\langle \widehat{x}, \widehat{x} \rangle\!\rangle = \|x\|^2 + r^2$. Define $\widehat{A} : \widehat{C} \to \widehat{\mathcal{H}}$ by

$$\widehat{A}(x,r) := (Ax, 1) \quad \text{for all } (x,r) \in \widehat{C}.$$

Here $\operatorname{epi} f := \{(x,r) \in \widehat{\mathcal{H}} : f(x) \le r\}$ is the *epigraph* of f, which is closed and convex because of the lower semicontinuity and convexity of f.

Theorem 6 *Suppose that* $\operatorname{dom} f \subset C$. *The following statements are true*:

(1) $\mathrm{VI}(C,A,f) = \{x \in C : \langle Ax, y - x\rangle + f(y) - f(x) \ge 0$ *for all* $y \in \operatorname{dom} f\}$;

(2) $(x,r) \in \mathrm{VI}(\operatorname{epi} f, \widehat{A}) \iff x \in \mathrm{VI}(C,A,f)$ *and* $r = f(x)$;

(3) *If* A *is* α*-inverse strongly monotone, then so is* $\widehat{A}$, *and hence* $\mathrm{Id} - \lambda\widehat{A}$ *is nonexpansive for all* $\lambda \in (0, 2\alpha]$.

Proof (1) is straight forward. (2) was proved by Mosco. For completeness, we give a proof of (2).

($\Rightarrow$) Let $(x,r) \in \mathrm{VI}(\operatorname{epi} f, \widehat{A})$, and let $y \in \operatorname{dom} f$. This implies that $(y, f(y)) \in \operatorname{epi} f$ and

$$\langle Ax, y - x\rangle + f(y) - r = \langle\!\langle \widehat{A}(x,r), (y, f(y)) - (x,r) \rangle\!\rangle \ge 0.$$

Note that $f(x) \le r$. This implies that $\langle Ax, y - x\rangle + f(y) - f(x) \ge 0$. Moreover, we have

$$f(x) - r = \langle\!\langle \widehat{A}(x,r), (x, f(x)) - (x,r) \rangle\!\rangle \ge 0.$$

This implies that $f(x) \ge r$, and hence $r = f(x)$. In particular, we have

$$\langle Ax, y - x\rangle + f(y) - f(x) \ge 0.$$

($\Leftarrow$) Suppose that $x \in \mathrm{VI}(C,A,f)$. We prove that $(x, f(x)) \in \mathrm{VI}(\operatorname{epi} f, \widehat{A})$. To see this, let $(y,s) \in \operatorname{epi} f$. It follows that $f(y) \le s$ and

$$\langle\!\langle \widehat{A}(x, f(x)), (y,s) - (x, f(x)) \rangle\!\rangle = \langle Ax, y - x\rangle + s - f(x) \ge \langle Ax, y - x\rangle + f(y) - f(x) \ge 0.$$

(3) Suppose that A is α-inverse strongly monotone. We show that $\widehat{A} : \widehat{C} \to \widehat{\mathcal{H}}$ is also α-inverse strongly monotone. To see this, let $\widehat{x} := (x,r), \widehat{y} := (y,s) \in \widehat{C}$. It follows that

$$\langle\!\langle \widehat{A}\widehat{x} - \widehat{A}\widehat{y}, \widehat{x} - \widehat{y} \rangle\!\rangle = \langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2 = \alpha\left|\left|\left|\widehat{A}\widehat{x} - \widehat{A}\widehat{y}\right|\right|\right|^2.$$

In particular, $\mathrm{Id} - \lambda\widehat{A}$ is nonexpansive for $\lambda \in (0, 2\alpha]$. □

Because of the error of the resolvent proposed by the authors of [1], we cannot infer the closedness and the convexity of $\mathrm{VI}(C,A,f)$. However, the conclusion remains true as follows.

Corollary 7 *Let $A : C \to \mathcal{H}$ be α-inverse strongly monotone, and let $f : \mathcal{H} \to (-\infty,\infty]$ be a proper convex and lower semicontinuous function. Suppose that* $\operatorname{dom} f \subset C$. *Then* $\mathrm{VI}(C,A,f)$ *is closed and convex.*

Proof We assume that $\mathrm{VI}(C,A,f)$ is nonempty. Note that $\mathrm{VI}(\operatorname{epi} f,\widehat{A}) = \mathrm{Fix}(P_{\operatorname{epi} f} \circ (\mathrm{Id} - \alpha\widehat{A}))$ is closed and convex. To prove the closedness of $\mathrm{VI}(C,A,f)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{VI}(C,A,f)$ and assume that $\{x_n\}_{n=1}^{\infty}$ is strongly convergent to a point $x \in C$. It suffices to show that $(x,f(x)) \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$. Put $r := f(x)$ and $r_n := f(x_n)$. From the lower semicontinuity of f it follows that $r \le \liminf_n r_n$. Note that for $(y,s) \in \widehat{C} := C \times \mathbb{R}$, we have

$$\langle Ax_n, y - x_n\rangle + s - r_n = \langle\langle \widehat{A}(x_n,r_n),(y,s) - (x_n - r_n)\rangle\rangle \ge 0.$$

Since A is $(1/\alpha)$-Lipschitzian and hence continuous, we obtain that $\lim_n \langle Ax_n, y - x_n\rangle = \langle Ax, y - x\rangle$. In particular, $\langle Ax, y - x\rangle + s \ge \limsup_n r_n \ge r$. Hence $\langle\langle \widehat{A}(x,r),(y,s) - (x,r)\rangle\rangle = \langle Ax, y - x\rangle + s - r \ge 0$, that is, $(x,f(x)) = (x,r) \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$.

Finally, we prove that $\mathrm{VI}(C,A,f)$ is convex. To this end, let $x,x' \in \mathrm{VI}(C,A,f)$ and $t \in (0,1)$. It follows that $(x,r),(x',r') \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$, where $r := f(x)$ and $r' := f(x')$. Put $x'' := (1-t)x + tx'$. Since $\mathrm{VI}(C,A,f)$ is convex, it follows that $(x'',(1-t)r + tr') \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$. In particular, for $(y,s) \in \widehat{C} := C \times \mathbb{R}$ and $r'' := f(x'')$, we have $r'' \le (1-t)r + tr'$ and

$$\begin{aligned}\big\langle\big\langle \widehat{A}\big(x'',r''\big),(y,s) - \big(x'',r''\big)\big\rangle\big\rangle &= \big\langle Ax'', y - x''\big\rangle + s - r'' \\ &\ge \big\langle Ax'', y - x''\big\rangle + s - \big((1-t)r + tr'\big) \\ &= \big\langle\big\langle \widehat{A}\big(x'',(1-t)r + tr'\big),(y,s) - \big(x'',(1-t)r + tr'\big)\big\rangle\big\rangle \ge 0.\end{aligned}$$

It follows that $(x'',r'') \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$, and hence $x'' \in \mathrm{VI}(C,A,f)$. □

2.3 Another look at the intermixed method via a product space approach

Suppose that C, $\mathcal{H}$, A_i, B_i, T_i, f_i, g_i $(i = 1,2)$ are as in Theorem KK. Note that we can show that $\mathrm{VI}(C,A_1,f_1) \cap \mathrm{VI}(C,B_1,f_1) = \mathrm{VI}(C,a_1A_1 + (1-a_1)B_1,f_1)$ for $0 < a_1 < 1$ if $\mathrm{VI}(C,A_1,f_1) \cap \mathrm{VI}(C,B_1,f_1) \ne \emptyset$ and if A_1 and B_1 are α-inverse strongly monotone. Corresponding to this note, we assume for simplicity that $A_1 = B_1$ and $A_2 = B_2$. We also assume that

$$\Omega_i := \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A_i,f_i) \ne \emptyset \quad \text{for } i = 1,2.$$

To deduce and correct the conclusion in Theorem KK, let us fix the following notation.

Let

$$\boldsymbol{\mathcal{H}} := \widehat{\mathcal{H}} \times \widehat{\mathcal{H}} \quad \text{and} \quad \boldsymbol{C} := \widehat{C} \times \widehat{C},$$

where $\widehat{\mathcal{H}} := \mathcal{H} \times \mathbb{R}$ and $\widehat{C} := C \times \mathbb{R}$. Note that $\boldsymbol{\mathcal{H}}$ is a Hilbert space endowed with the inner product $[\cdot,\cdot]$ defined by

$$\big[\boldsymbol{x},\boldsymbol{x'}\big] := \big\langle x,x'\big\rangle + rr' + \big\langle y,y'\big\rangle + ss'$$

for all $\boldsymbol{x} := ((x,r),(y,s))$ and $\boldsymbol{x'} := ((x',r'),(y',s')) \in \boldsymbol{\mathcal{H}}$. Moreover, the induced norm of each element $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{\mathcal{H}}$ is given by

$$\|\boldsymbol{x}\| := \left(\|x\|^2 + r^2 + \|y\|^2 + s^2\right)^{1/2}.$$

Define $\boldsymbol{A} : \boldsymbol{C} \to \boldsymbol{\mathcal{H}}$ and $\boldsymbol{S} : \boldsymbol{C} \to \boldsymbol{C}$ by

$$\boldsymbol{Ax} := \big((A_1 x, 1), (A_2 y, 1)\big)$$

and

$$\boldsymbol{Sx} := \Big(\big(b_1 x + (1-b_1)T_1 x, r\big), \big(b_2 y + (1-b_2)T_2 y, s\big)\Big)$$

for $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{C}$.

Using the preceding setting, we obtain the following results.

Proposition 8 (Properties of $\boldsymbol{A}$) *Let* $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{\mathcal{H}}$ *and* $\boldsymbol{E} := \operatorname{epi} f_1 \times \operatorname{epi} f_2$. *Then the following two statements are equivalent*:

(a) $\boldsymbol{x} \in \mathrm{VI}(\boldsymbol{E}, \boldsymbol{A})$;

(b) $x \in \mathrm{VI}(C, A_1, f_1)$, $y \in \mathrm{VI}(C, A_2, f_2)$, $r = f_1(x)$, *and* $s = f_2(x)$.

If, in addition, $A_1, A_2 : C \to \mathcal{H}$ *are* α*-inverse strongly monotone, then* $\boldsymbol{A} : \boldsymbol{C} \to \boldsymbol{\mathcal{H}}$ *is* α*-inverse strongly monotone.*

Proof (a) $\Longrightarrow$ (b) Let $\boldsymbol{x} := ((x,r),(y,s)) \in \mathrm{VI}(\boldsymbol{E}, \boldsymbol{A})$. Let $\boldsymbol{x'} := ((x',r'),(y,s))$, where $(x',r') \in \operatorname{epi} f_1$. It follows that $\boldsymbol{x'} \in \boldsymbol{E}$, and hence $\langle\langle \widehat{A_1}(x,r), (x',r') - (x,r) \rangle\rangle = [\boldsymbol{Ax}, \boldsymbol{x'} - \boldsymbol{x}] \geq 0$. This means that $(x,r) \in \mathrm{VI}(\operatorname{epi} f_1, \widehat{A_1})$. It follows from Theorem 6 that $x \in \mathrm{VI}(C, A_1, f_1)$ and $r = f_1(x)$. Using a similar technique, we obtain the remaining conclusion.

(b) $\Longrightarrow$ (a) is trivial.

Suppose that $A_1, A_2 : C \to \mathcal{H}$ are α-inverse strongly monotone. To see that $\boldsymbol{A} : \boldsymbol{C} \to \boldsymbol{\mathcal{H}}$ is α-inverse strongly monotone, let $\boldsymbol{x} := ((x,r),(y,s))$ and $\boldsymbol{x'} := ((x',r'),(y',s')) \in \boldsymbol{C}$. It follows that

$$\begin{aligned}
&\big[\boldsymbol{Ax} - \boldsymbol{Ax'}, \boldsymbol{x} - \boldsymbol{x'}\big] \\
&\quad = \Big[\big((A_1 x - A_1 x', 0), (A_2 y - A_2 y', 0)\big), \big((x - x', r - r'), (y - y', s - s')\big)\Big] \\
&\quad = \langle A_1 x - A_1 x', x - x' \rangle + \langle A_2 y - A_2 y', y - y' \rangle \\
&\quad \geq \alpha\big(\|A_1 x - A_1 x'\|^2 + \|A_2 y - A_2 y'\|^2\big) = \alpha\|\boldsymbol{Ax} - \boldsymbol{Ax'}\|^2.
\end{aligned}$$

This completes the proof. □

Proposition 9 (Properties of $\boldsymbol{S}$) *Let* $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{C}$. *Then the following two statements are equivalent*:

(a) $\boldsymbol{x} \in \mathrm{Fix}(\boldsymbol{S})$;

(b) $x \in \mathrm{Fix}(T_1)$ *and* $y \in \mathrm{Fix}(T_2)$.

If, in addition, T_1, T_2 *are nonexpansive and* $\mathrm{Fix}(T_1) \times \mathrm{Fix}(T_2) \neq \emptyset$, *then* $\boldsymbol{S}$ *is nonexpansive and* r*-strongly quasinonexpansive where* $r := \min\{b_1(1-b_1), b_2(1-b_2)\}$.

Proof (a) $\iff$ (b) is trivial. Now we suppose that T_1 and T_2 are nonexpansive and $\mathrm{Fix}(T_1) \times \mathrm{Fix}(T_2) \neq \emptyset$. It is clear that $\boldsymbol{S}$ is nonexpansive. Let $r := \min\{b_1(1-b_1), b_2(1-b_2)\}$. We show that $\boldsymbol{S}$ is r-strongly quasinonexpansive. To see this, let $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{C}$ and $\boldsymbol{p} := ((p,r'),(q,s')) \in \mathrm{Fix}(\boldsymbol{S})$. It follows that

$$\begin{aligned}\left\|\left(b_1x + (1-b)T_1x\right) - p\right\|^2 &\le b_1\|x-p\|^2 + (1-b_1)\|T_1x - p\|^2 - b_1(1-b_1)\|x - T_1x\|^2 \\ &\le \|x-p\|^2 - r\|x - T_1x\|^2.\end{aligned}$$

Similarly, $\|(b_2y + (1-b_2)T_2y) - q\|^2 \le \|y-q\|^2 - r\|y - T_2y\|^2$. This implies that

$$\begin{aligned}\|\boldsymbol{Sx} - \boldsymbol{p}\|^2 &= \left\|\left(b_1x + (1-b)T_1x\right) - p\right\|^2 + \left(r - r'\right)^2 \\ &\quad + \left\|\left(b_2y + (1-b_2)T_2y\right) - q\right\|^2 + \left(s - s'\right)^2 \\ &\le \|x-p\|^2 + \left(r-r'\right)^2 + \|y-q\|^2 + \left(s-s'\right)^2 - r\left(\|x - T_1x\|^2 + \|y - T_2y\|^2\right) \\ &= \|\boldsymbol{x} - \boldsymbol{p}\|^2 - r\|\boldsymbol{x} - \boldsymbol{Sx}\|^2.\end{aligned}$$

The proof is complete. □

Proposition 10 *If $g_1, g_2 : \mathcal{H} \to \mathcal{H}$ are α-Lipschitzian, then $\boldsymbol{h} : \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ defined by*

$$\boldsymbol{h}(\boldsymbol{x}) := \left(\left(g_2(y), \alpha s\right), \left(g_1(x), \alpha r\right)\right) \quad \textit{for all } \boldsymbol{x} := \left((x,r),(y,s)\right) \in \boldsymbol{\mathcal{H}}$$

is also α-Lipschitzian.

Proof To see this, let $\boldsymbol{x} := ((x,r),(y,s))$, $\boldsymbol{x'} := ((x',r'),(y',s')) \in \boldsymbol{\mathcal{H}}$. It follows that

$$\begin{aligned}\left\|\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}\left(\boldsymbol{x'}\right)\right\|^2 &= \left\|\left(\left(g_2(y) - g_2\left(y'\right), \alpha\left(s-s'\right)\right), \left(g_1(x) - g_1\left(x'\right), \alpha\left(r - r'\right)\right)\right)\right\|^2 \\ &= \left\|g_2(y) - g_2\left(y'\right)\right\|^2 + \alpha^2\left(s-s'\right)^2\big) + \left\|g_1(x) - g_1\left(x'\right)\right\|^2 + \alpha^2\left(r-r'\right)^2 \\ &\le \alpha^2\left(\left\|y - y'\right\|^2 + \left(s-s'\right)^2 + \left\|x - x'\right\|^2 + \left(r-r'\right)^2\right) \\ &= \alpha^2\left\|\boldsymbol{x} - \boldsymbol{x'}\right\|^2.\end{aligned}$$

This completes the proof. □

The intermixed algorithm can be regarded as a classical algorithm of Theorem 3, and we obtain the following convergence theorem.

Theorem 11 *Let $\boldsymbol{U} := \boldsymbol{P_E}(\mathbf{Id} - \lambda\boldsymbol{A})$ and $\boldsymbol{F} := \mathrm{VI}(\boldsymbol{E}, \boldsymbol{A}) \cap \mathrm{Fix}(\boldsymbol{S})$. Suppose that $\boldsymbol{x}_1 \in \boldsymbol{C}$ is arbitrarily chosen and*

$$\boldsymbol{x_{n+1}} := (1-\beta_n)\boldsymbol{Sx_n} + \beta_n\boldsymbol{P_C}\left(\alpha_n\boldsymbol{h}(\boldsymbol{x_n}) + (1-\alpha_n)\boldsymbol{Ux_n}\right),$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0,1]$ satisfy the following conditions:

$$\lim_n \frac{\alpha_n}{1-\beta_n} = 0 \quad \textit{and} \quad \sum_n \alpha_n\beta_n = \infty.$$

Then the iterative sequence $\{\boldsymbol{x_n}\}_{n=1}^{\infty}$ converges to $\boldsymbol{z} = \boldsymbol{P_F} \circ \boldsymbol{h}(\boldsymbol{z})$.

Remark 12 Our result is simultaneously a correction and an improvement of Theorem KK in the following ways.

(1) We use a product space approach to consider the mixed variational inequality problem and the intermixed algorithm.
(2) The resolvent proposed for the mixed variational inequality problem in the original work is not correct, and we propose a correction.
(3) The assumptions on the parameters $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are more general than those in Theorem KK. Moreover, Condition (C3) is superfluous. The choice $\alpha_n = \beta_n := 1/\sqrt{n}$ is applicable in our result, but it is not in Theorem KK.

Finally, we express the iterative sequence in our Theorem 11 as follows:

$$\begin{aligned}
&(x_1, r_1), (y_1, s_1) \in C \times \mathbb{R} \text{are arbitrarily chosen,} \\
&\bigl(x'_n, r'_n\bigr) := \bigl(b_1 x_n + (1-b_1)T_1 x_n, r_n\bigr), \\
&\bigl(y'_n, s'_n\bigr) := \bigl(b_2 y_n + (1-b_2)T_2 y_n, s_n\bigr), \\
&\bigl(x''_n, r''_n\bigr) := P_{\operatorname{epi} f_1}(x_n - \lambda_n A_1 x_n, r_n - \lambda_n), \\
&\bigl(y''_n, s''_n\bigr) := P_{\operatorname{epi} f_2}(x_n - \lambda_n A_2 x_n, s_n - \lambda_n), \\
&(x_{n+1}, r_{n+1}) := \bigl((1-\beta_n)x'_n + \beta_n P_C\bigl(\alpha_n h(x_n) + (1-\alpha_n)x''_n\bigr), \\
&\qquad\qquad (1-\beta_n)r'_n + \beta_n\bigl(\alpha_n \alpha r_n + (1-\alpha_n)r''_n\bigr)\bigr), \\
&(y_{n+1}, s_{n+1}) := \bigl((1-\beta_n)y'_n + \beta_n P_C\bigl(\alpha_n h(y_n) + (1-\alpha_n)y''_n\bigr), \\
&\qquad\qquad (1-\beta_n)s'_n + \beta_n\bigl(\alpha_n \alpha s_n + (1-\alpha_n)s''_n\bigr)\bigr).
\end{aligned}$$

For more detail on epigraphical projection, we refer to the book of Bauschke and Combettes [4]. It follows from our Theorem 11 that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ converge strongly to x^* and y^*, respectively, where $x^* = P_{\mathrm{VI}(C,A_1,f_1)\cap \mathrm{Fix}(T_1)} g_2(y^*)$ and $y^* = P_{\mathrm{VI}(C,A_2,f_2)\cap \mathrm{Fix}(T_2)} g_1(x^*)$.

Author contributions
I am the sole author of the manuscript.

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No datasets were generated or analysed during the current study.

Declarations

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The authors declare no competing interests.

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References

1. Khuangsatung, W., Kangtunyakarn, A.: An intermixed method for solving the combination of mixed variational inequality problems and fixed-point problems. J. Inequal. Appl. **2023**, Article ID 1 (2023)
2. Shashkin, Y.A.: Fixed Points. Mathematical World, vol. 2. Am. Math. Soc., Providence (1991)
3. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Pure and Applied Mathematics, vol. 88. Academic Press, New York (1980)
4. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics. Springer, Cham (2017)

5. Yao, Z., Kang, S.M., Li, H.J.: An intermixed algorithm for strict pseudo-contractions in Hilbert spaces. Fixed Point Theory Appl. **2015**, Article ID 206 (2015)
6. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240–256 (2002)
7. Browder, F.E.: Semicontractive and semiaccretive nonlinear mappings in Banach spaces. Bull. Am. Math. Soc. **74**, 660–665 (1968)
8. Wang, Y., Wang, F., Xu, H.K.: Error sensitivity for strongly convergent modifications of the proximal point algorithm. J. Optim. Theory Appl. **168**, 901–916 (2016)
9. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)

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New inertial self-adaptive algorithms for the split common null-point problem: application to data classifications

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Abstract

In this paper, we propose two inertial algorithms with a new self-adaptive step size for approximating a solution of the split common null-point problem in the framework of Banach spaces. The step sizes are adaptively updated over each iteration by a simple process without the prior knowledge of the operator norm of the bounded linear operator. Under suitable conditions, we prove the weak-convergence results for the proposed algorithms in p-uniformly convex and uniformly smooth Banach spaces. Finally, we give several numerical results in both finite- and infinite-dimensional spaces to illustrate the efficiency and advantage of the proposed methods over some existing methods. Also, data classifications of heart diseases and diabetes mellitus are presented as the applications of our methods.

Mathematics Subject Classification: 47H09; 47H10; 47J25; 47J05

Keywords: p-uniformly convex Banach spaces; Weak convergence; Split common null-point problem; Maximal monotone operator; Self-adaptive algorithm

1 Introduction

In this paper, we consider the following *split common null-point problem* [13] (see also [29]): find $z \in H_1$ such that

$$z \in A^{-1}0 \cap T^{-1}(B^{-1}0), \tag{1.1}$$

where $A : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$ are set-valued maximal monotone operators, $T : H_1 \to H_2$ is a bounded linear operator, and H_1 and H_2 are real Hilbert spaces. We denote the solution set of the split common null-point problem (1.1) by Ω. The split common null-point problem can be applied to solving many real-life problems, for instance, in practices as a model in intensity-modulated radiation-therapy treatment planning [14, 15] and in sensor networks in computerized tomography and data compression [19]. In addition, the split common null-point problem also generalizes several split-type problems that is the core the modeling of many inverse problems such as the split feasibility problem, the split equilibrium problem, and the split minimization problem as special cases.

Byrne et al. [13] introduced the following iterative scheme for solving the split common null-point problem: for given $x_0 \in H_1$ and the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = R_\tau\big(x_n - \lambda T^*(I - Q_\mu)Tx_n\big), \tag{1.2}$$

where $R_\tau = (I + \tau A)^{-1}$ and $Q_\mu = (I + \mu B)^{-1}$ are the resolvent operators of A for $\tau > 0$ and of B for $\mu > 0$, respectively. They proved the weak-convergence theorem for solving the split common null-point problem provided the step size $\lambda \in (0, \frac{2}{\|T\|^2})$.

Alofi et al. [5] introduced the following iterative scheme based on a modified Halpern's iteration for solving the split common null-point problem in the case that H_1 is a Hilbert space and F is a uniformly convex and smooth Banach space: for given $x_1 \in H_1$ and the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\big(\alpha_n u_n + (1 - \alpha_n)R_{\tau_n}\big(x_n - \tau_n T^* J(I - Q_{\mu_n})Tx_n\big)\big), \tag{1.3}$$

where R_τ is the resolvent of A for $\tau > 0$ and Q_μ is the metric resolvent of B for $\mu > 0$, $\{\tau_n\}, \{\mu_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, and $\{\beta_n\} \subset (0, 1)$ that satisfies some appropriate assumptions on the parameters, J is the duality mapping on F, T is the bounded linear operator from H_1 to F, and $\{u_n\}$ is the sequence in H_1 such that $u_n \to u$. They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point of Ω provided τ_n satisfies the following inequality:

$$0 < a \le \tau_n \|T\|^2 \le b < 2 \tag{1.4}$$

for some $a, b > 0$.

Later, Suantai et al. [39] generalized the result of Alofi et al. [5] in the case that E is a p-uniformly convex and uniformly smooth Banach space, and F is a uniformly convex and smooth Banach space. To be more precise, they introduced the following scheme: for given $x_1 \in E$ and the sequence $\{x_n\}$ generated iteratively by

$$\begin{cases} z_n = J_q^{E^*}(J_p^E(x_n) - \tau_n T^* J_p^F(I - Q_{\mu_n})Tx_n), \\ y_n = J_q^{E^*}(\alpha_n J_p^E(u_n) + (1 - \alpha_n)J_p^E(R_{\tau_n} z_n)), \\ x_{n+1} = J_q^{E^*}(\beta_n J_p^E(x_n) + (1 - \beta_n)J_p^E(y_n)), \end{cases} \tag{1.5}$$

where J_p^E and $J_q^{E^*}$ are the generalized duality mapping of E into E^* and the duality mapping of E^* into E, respectively, where $1 < q \le 2 \le p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and T is the bounded linear operator from E to F. They also proved the strong convergence of the sequence $\{x_n\}$ generated by (1.3) to a point of Ω provided τ_n satisfies the following inequality:

$$0 < a \le \tau_n \le b < \left(\frac{q}{\kappa_q \|T\|^q}\right)^{\frac{1}{q-1}} \tag{1.6}$$

for some $a, b > 0$.

However, several iterative methods involve a step size that requires to compute the norm of the bounded linear operator $\|T\|$ prior to choosing τ_n. In general, it may not be easy to compute $\|T\|$. In particular, it makes the algorithms not easily implemented when the computation of $\|T\|$ is complicated. To overcome this drawback, a new step-size strategy without the prior knowledge of the operator norm of the bounded linear operator has been proposed by López et al. [28]. This method is known as a *self-adaptive method* that was first used to solve the split feasibility problem when the step-size criterion is independent of the operator norm of the bounded linear operator.

In optimization theory, the inertial technique has been widely used to accelerate the rate of convergence of the algorithms. This technique was motivated by the implicit time discretization of second-order dynamical systems (or a heavy ball with friction). Based on the inertial technique, Alvarez and Attouch [7] proposed the following so-called *inertial proximal point algorithm* for finding a zero point of a set-valued maximal monotone operator A: for given $x_0, x_1 \in H$ and the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = R_{\tau_n}\big(x_n + \theta_n(x_n - x_{n-1})\big), \tag{1.7}$$

where R_{τ_n} is the resolvent of A and $x_n + \theta_n(x_n - x_{n-1})$ is called the *inertial term*. They also proved that the sequence $\{x_n\}$ generated by (1.7) converges weakly to a zero point of A provided $\{\tau_n\}$ is increasing and $\theta_n \in [0,1)$ is chosen so that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$. In recent years, the inertial method was further studied intensively and it also has been used to solve some other optimization problems (see, for example, [16, 21, 32, 37, 38, 41, 45]).

In 2019, Tang [43] proposed the following inertial algorithm for solving the split common null-point problem in the case that H_1 is a Hilbert space, and F is a 2-uniformly convex and smooth Banach space: for given $x_1 \in H_1$ and $\alpha \in [0,1)$, choose θ_n such that $0 < \theta_n < \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\alpha, \epsilon_n(\max\{\|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\|\})^{-1}\}, & x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise}, \end{cases} \tag{1.8}$$

where $\{\epsilon_n\} \subset (0,\infty)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Compute the sequence $\{x_n\}$ generated iteratively by

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = R_r(I - \tau_n T^* J(I - Q_\mu)T)u_n \end{cases} \tag{1.9}$$

with the step size

$$\tau_n = \rho_n \frac{f(u_n)}{\|F(w_n)\|^2 + \|H(u_n)\|^2}, \tag{1.10}$$

where $\{\rho_n\} \subset (0,4)$, $f(u_n) = \frac{1}{2}\|J(I - Q_\mu)Tu_n\|^2$, $F(u_n) = T^* J(I - Q_\mu)Tw_n$, and $H(u_n) = (I - R_r)u_n$. The weak convergence of the sequence $\{x_n\}$ is established without the prior knowledge of the operator norm of the bounded linear operator.

Recently, several inertial algorithms for solving the split common null-point problem in Hilbert spaces have been studied by many authors (see, for example, [8, 17, 24, 30, 51]). However, such methods have been studied in Banach spaces by a few authors (see, for example, [43, 44]).

Inspired and motivated by the works mentioned above, in this paper, we introduce two new inertial self-adaptive algorithms that are based on the classical inertial method and relaxed inertial method for finding a solution of the split common null-point problem in Banach spaces. The weak-convergence theorems are proved without the prior knowledge of the operator norm of the bounded linear operator. We provide numerical implementations to show that our algorithms are efficient and competitive with some related algorithms. Our results are new and complement some previous results in the literature.

The contributions of this paper can be summarized as follows:

(1) The weak-convergence result of iterative scheme (1.9) of Tang [43] is proved in a Hilbert space and a 2-uniformly convex smooth Banach space where this result can only be implemented in ℓ_p for $p \in (1, 2]$ exclude the case of $p > 2$. This is limited in practical applications of such a method. In this paper, our results generalize the weak-convergence result of Tang [43] from between two of those spaces to p-uniformly convex and uniformly smooth Banach spaces, and as a result our results can be implemented in ℓ_p for $p > 1$.

(2) Even though the step size of the iterative scheme (1.9) of Tang [43] is computed without the prior knowledge of the operator norms it requires calculation of $\|T^*J(I - Q_\mu)u_n\|^2$ and $\|(I - R_\tau)u_n\|^2$ in order to choose the step size τ_n. This could be computationally expensive during implementations, especially in the case where the resolvent of A and the metric resolvent of B are difficult to compute. In this paper, our step size τ_n defined by (3.3), is adaptively updated by a cheap computation without the prior knowledge of the operator norms and only requires us to compute one metric resolvent of B.

(3) For the iterative scheme (1.5) of Suantai et al. [39], the choice of the sequence of step size depends on the bounded linear operator $\|T\|$, which is a difficult task during the implementation of the algorithm. In this paper, the choice of the sequence of our step size τ_n defined by (3.3), is independent of the operator norm of the bounded linear operator. As a result, we do not require to calculate the norm $\|T\|$ in order to choose the step size τ_n, which is easier to implement than such a method.

(4) We use the inertial and relaxed inertial techniques to improve the rate of convergence of our algorithms that makes the algorithms converge faster and computationally more efficient for solving the split common null-point problem in Banach spaces. Note that these inertial techniques in this paper are studied outside Hilbert spaces for solving such a problem.

(5) We present numerical results of our algorithms in Banach spaces to illustrate the efficiency and advantage over iterative scheme (1.5) of Suantai et al. [39] that gives the strong convergence and we also present several numerical results of our algorithms in finite-dimensional spaces. Moreover, we apply our results to data classifications for two datasets of heart diseases and diabetes mellitus.

Our paper is organized as the following four parts. In Sect. 2, we give some of the basic facts and notation that will be used in the paper. In Sect. 3, we propose two new inertial self-adaptive algorithms and prove our convergence results, and finally, in Sect. 4, we present several numerical results to verify the advantages and efficiency of the proposed algorithms.

2 Preliminaries

In this section, we give some definitions and preliminary results that will be used in proving the main results.Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a real Banach space with norm $\|\cdot\|$ with its the dual space E^*. We denote $\langle u,j\rangle$ by the value of a functional j in E^* at $u \in E$, that is, $\langle u,j\rangle = j(u)$ for all $u \in E$. We write $u_n \to x$ to indicate that a sequence $\{u_n\}$ converges strongly to u. Similarly, $u_n \rightharpoonup u$ and $u_n \rightharpoonup^* u$ will symbolize the weak and weak* convergence, respectively. Let $S_E = \{u \in E : \|u\| = 1\}$ and $B_E = \{u \in E : \|u\| \leq 1\}$ be a unit sphere and unit ball of E, respectively.

Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The *modulus of convexity* of E is the function $\delta_E : [0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|u+v\|}{2} : u,v \in B_E, \|u-v\| \geq \epsilon\right\}.$$

The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{\|u+tv\| + \|u-tv\|}{2} - 1 : u,v \in S_E\right\}.$$

Definition 2.1 A Banach space E is said to be:

(1) *strictly convex* if $\frac{\|u+v\|}{2} < 1$ for all $u,v \in S_E$ and $u \neq v$;
(2) *smooth* if $\lim_{t\to 0} \frac{\|u+tv\|-\|u\|}{t}$ exists for each $u,v \in S_E$;
(3) *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$;
(4) *p-uniformly convex* if there is a $\kappa_p > 0$ such that $\delta_E(\epsilon) \geq \kappa_p \epsilon^p$ for all $\epsilon \in (0,2]$;
(5) *uniformly smooth* if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$;
(6) *q-uniformly smooth* if there exists a $\kappa_q > 0$ such that $\rho_E(t) \leq \kappa_q t^q$ for all $t > 0$.

Remark 2.2 It is known that if E is uniformly convex, then E is reflexive and strictly convex; if E is uniformly smooth, then E is reflexive and smooth (see [2]). From the Definition 2.1, one can see that every p-uniformly convex (q-uniformly smooth) space is a uniformly convex (uniformly smooth) space. Moreover, it is also known that E is p-uniformly convex (q-uniformly smooth) if and only if E^* is q-uniformly smooth (p-uniformly convex) (see [2, 49]).

For the Lebesgue spaces L_p, sequence spaces l_p, and Sobolev spaces W_p^m, it is also known that [23, 50]

$$L_p(l_p) \text{ or } W_p^m \text{ is } \begin{cases} p\text{-uniformly smooth, 2-uniformly convex} & \text{for } 1 < p \leq 2, \\ p\text{-uniformly convex, 2-uniformly smooth} & \text{for } 2 \leq p < \infty. \end{cases} \tag{2.1}$$

For $p > 1$. The mapping $J_p : E \to 2^{E^*}$ defined by

$$J_p(u) = \left\{u^* \in E^* : \langle u,u^*\rangle = \|u\|^p, \|u^*\| = \|u\|^{p-1}\right\}$$

is called the *generalized duality mapping* of E. In particular, $J_2 = J$ is called the *normalized duality mapping* and if E is a Hilbert space, then $J_p = I$, where I is the identity mapping. The duality mapping J_p of a smooth Banach space E is said to be *weakly sequentially continuous* if for any sequence $\{u_n\} \subset E$ such that $u_n \rightharpoonup u$ implies $J_p(u_n) \rightharpoonup^* J_p(u)$. For the generalized duality mapping, the following facts are known [2, 18, 34]:

(i) J_p is homogeneous degree $p-1$, that is, $J_p(\alpha u) = |\alpha|^{p-1} \operatorname{sign}(\alpha) J_p(u)$ for all $u \in E$, $\alpha \in \mathbb{R}$. In particular, $J_p(-u) = -J_p(u)$ for all $u \in E$.

(ii) If E is smooth, then J_p is monotone, that is, $\langle u - v, J_p(u) - J_p(v)\rangle \geq 0$ for all $u, v \in E$. Moreover, if E is strictly convex, then J_p is strictly monotone.

(iii) If E is uniformly smooth, then J_p is single valued from E into E^* and it is uniformly continuous on bounded subsets of E.

(iv) If E is reflexive, smooth, and strictly convex, then the inverse $J_p^{-1} = J_q^*$ is single valued, one-to-one, and surjective, where J_q^* is the duality mapping from E^* into E.

Lemma 2.3 ([49]) *If E is a q-uniformly smooth Banach space, then there is a constant $\kappa_q > 0$ such that*

$$\|u - v\|^q \leq \|u\|^q - q\langle v, J_q(u)\rangle + \kappa_q \|v\|^q, \quad \forall u, v \in E,$$

where κ_q is called the q-uniform smoothness coefficient of E.

Remark 2.4 The exact values of the constant κ_q can be found in [35, 50].

We next recall the definition of Bregman distance. Let E be a real smooth Banach space and f be a convex and Gâteaux differentiable function on E. The bifunction $D_f : E \times E \to [0, \infty)$ defined by

$$D_f(u, v) = f(u) - f(v) - \langle u - v, \nabla f(v)\rangle$$

is called the *Bregman distance with respect to f*. Note that the Bregman distance is not a metric due to its lack of symmetry and failure to satisfy the triangle inequality. If $f_p(x) = \frac{1}{p}\|x\|^p$ for $p > 1$, then $\nabla f = J_p$. Hence, we have the Bregman distance with respect to $f = f_p$ given by

$$\begin{aligned} D_{f_p}(u, v) &= \frac{1}{p}\|u\|^p - \frac{1}{p}\|v\|^p - \langle u - v, J_p(v)\rangle \\ &= \frac{1}{p}\|u\|^p + \frac{1}{q}\|v\|^p - \langle u, J_p(v)\rangle. \end{aligned}$$

Moreover, if $p = 2$, then $2D_{f_2}(u, v) = \|u\|^2 - \|v\|^2 - 2\langle u, J(v)\rangle = \phi(u, v)$, where ϕ is called the *Lyapunov function* studied in [4, 31]. Also, if E is a Hilbert space, then $\phi(u, v) = \|u - v\|^2$. The following properties of the Bregman distance are well known: for each $u, v, w \in E$,

$$D_{f_p}(u, v) = D_{f_p}(u, w) - D_{f_p}(v, w) + \langle u - v, J_p(w) - J_p(v)\rangle \tag{2.2}$$

and

$$D_{f_p}(u, v) + D_{f_p}(v, u) = \langle u - v, J_p(u) - J_p(v)\rangle. \tag{2.3}$$

For a p-uniformly convex space, it holds that [36]

$$\tau\|u-v\|^p \le D_{f_p}(u,v) \le \langle u-v, J_p(u)-J_p(v)\rangle, \tag{2.4}$$

where $\tau > 0$ is some fixed number.

Also, we define a function $V_{f_p} : E \times E^* \to [0,\infty)$ by

$$V_{f_p}(u,u^*) = \frac{1}{p}\|u\|^p - \langle u,u^*\rangle + \frac{1}{q}\|u^*\|^q$$

for all $u \in E$ and $u^* \in E^*$. Note that V_{f_p} is nonnegative, convex in the second variable and $V_{f_p}(u,u^*) = D_{f_p}(u, J_q(u^*))$ for all $u \in E$ and $u^* \in E^*$. Moreover, the following property is known:

$$V_{f_p}(u,u^*) + \langle J_p^{-1}(u^*) - u, v^*\rangle \le V_{f_p}(u,u^*+v^*) \tag{2.5}$$

for all $u \in E$ and $u^*, v^* \in E^*$.

Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space. Then, for any $u \in E$, there exists a unique element $w \in C$ such that

$$\|u-w\| = \min_{v\in C}\|u-v\|.$$

The mapping P_C defined by $w = P_C(u)$ is called the *metric projection* of E onto C. We know the following property [36]:

$$\langle v - P_C(u), J_p(u-P_C(u))\rangle \le 0, \quad \forall v \in C.$$

Recall that the *Bregman projection with respect to* f_p is defined by

$$\Pi_C^{f_p}(u) = \operatorname*{argmin}_{v\in C} D_{f_p}(u,v), \quad \forall u \in E.$$

If $p = 2$, then $\Pi_C^{f_p}$ becomes the generalized projection and denoted by Π_C. Also, in this case, if E is a Hilbert space, then Π_C becomes the metric projection denoted by P_C. We also know the following property [11]:

$$\langle v - \Pi_C^{f_p}(u), J_p(u) - J_p(\Pi_C^{f_p}(u))\rangle \le 0, \quad \forall v \in C. \tag{2.6}$$

Let C be a nonempty subset of E and $T : C \to C$ be a mapping. We denote the fixed-point set of T by $F(T) = \{u \in C : u = Tu\}$. Let $A : E \to 2^{E^*}$ be a set-valued mapping. The domain of A is denoted by $\mathcal{D}(A) = \{u \in E : Au \neq \emptyset\}$ and the range of A is also denoted by $\mathcal{R}(A) = \bigcup\{Au : u \in \mathcal{D}(A)\}$. The set of zeros of A is defined by $A^{-1}0 = \{u \in \mathcal{D}(A) : 0 \in Au\}$. It is known that $A^{-1}0$ is closed and convex (see [40]). A set-valued mapping A is said to be *monotone* if

$$\langle x-y, u-v\rangle \ge 0, \quad \forall x,y \in \mathcal{D}(A), u \in Ax \text{ and } v \in Ay.$$

A monotone operator A on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.

Let E be a p-uniformly convex and uniformly smooth Banach space and $A : E \to 2^{E^*}$ be a maximal monotone operator. Following [10], for each $u \in E$ and $\tau > 0$, we define the *resolvent* of A by

$$R_\tau(u) = (J_p + \tau A)^{-1} J_p(u), \quad \forall u \in E.$$

One can see that $A^{-1}0 = F(R_\tau)$ for $\tau > 0$. We also know the following property [26]:

$$D_{f_p}\big(v, R_\tau(u)\big) + D_{f_p}\big(R_\tau(u), u\big) \le D_{f_p}(v, u) \tag{2.7}$$

for all $u \in E$ and $v \in A^{-1}0$.

For each $u \in E$ and $\mu > 0$, we define the *metric resolvent* of A for $\mu > 0$ by

$$Q_\mu(u) = \big(I + \mu J_p^{-1} A\big)^{-1}(u), \quad \forall u \in E. \tag{2.8}$$

It is clear that in a Hilbert space, the metric resolvent operator is equivalent to the resolvent operator. From (2.8), one can see that $0 \in J_p(Q_\mu(u) - u) + \mu A Q_\mu(u)$ and $A^{-1}0 = F(Q_\mu)$ for $\mu > 0$. The monotonicity of A implies that

$$\big\langle Q_\mu(u) - Q_\mu(v), J_p\big(u - Q_\mu(u)\big) - J_p\big(v - Q_\mu(v)\big)\big\rangle \ge 0 \tag{2.9}$$

for all $u, v \in E$. If $A^{-1}0 \neq \emptyset$, then

$$\big\langle Q_\mu(u) - v, J_p\big(u - Q_\mu(u)\big)\big\rangle \ge 0 \tag{2.10}$$

for all $u \in E$ and $v \in A^{-1}0$ (see [9]). For any sequence $\{x_n\}$ in E, we see that

$$\begin{aligned} \|x_n - v\| \big\|x_n - Q_\mu(x_n)\big\|^{p-1} &\ge \big\langle x_n - v, J_p\big(x_n - Q_\mu(x_n)\big)\big\rangle \\ &\ge \big\langle x_n - Q_\mu(x_n), J_p\big(x_n - Q_\mu(x_n)\big)\big\rangle \\ &= \big\|x_n - Q_\mu(x_n)\big\|^p. \end{aligned} \tag{2.11}$$

This implies that $\|x_n - Q_\mu(x_n)\| \le \|x_n - v\|$. If $\{x_n\}$ is bounded, then $\{x_n - Q_\mu(x_n)\}$ is also bounded.

Let E be a p-uniformly convex and uniformly smooth Banach space and $f : E \to \mathbb{R} \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function. The *subdifferential* of f at x is defined by

$$\partial f(x) = \big\{z \in E^* : f(x) + \langle y - x, z\rangle \le f(y), \forall y \in E\big\}.$$

Let C be a closed and convex subset of E. The indicator function δ_C of C at x is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

The subdifferentiable $\partial\delta_C$ is a maximal monotone operator since δ_C is a proper, convex, and lower semicontinuous function (see [33]). Moreover, we also know that

$$\partial\delta_C(x) = N_C(x) = \{z \in E^* : \langle y - x, z\rangle \leq 0, \forall y \in C\},$$

where N_C is the normal cone of C. In particular, if we define the resolvent of $\partial\delta_C$ for $\tau > 0$ by $R_\tau(u) = (J_p + \tau\partial\delta_C)^{-1}J_p(u)$ for all $u \in E$, then $R_\tau = \Pi_C^{f_p}$, where $\Pi_C^{f_p}$ is the Bregman projection with respect to f_p (see [48]). Moreover, we also have $(\partial\delta_C)^{-1}0 = C$. Also, if we define the metric resolvent of $\partial\delta_C$ for $\mu > 0$ by $Q_\mu(u) = (I + \mu J_p^{-1}\partial\delta_C)^{-1}(u)$ for all $u \in E$, then

$$\begin{aligned} z = Q_\mu(u) \quad &\Leftrightarrow \quad 0 \in J_p(z-u) + \mu Az \\ &\Leftrightarrow \quad \frac{J_p(u-z)}{\mu} \in Az = \partial\delta_C(z) = N_C(z) \\ &\Leftrightarrow \quad \langle y - z, J_p(u-z)\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad z = P_C(u), \end{aligned}$$

where P_C is the metric projection of E onto C.

Throughout this paper, we adopt the notation $[a]_+ := \max\{a, 0\}$, where $a \in \mathbb{R}$.

Lemma 2.5 ([6]) *Let $\{\varphi_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ be three nonnegative real sequences such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \beta_n, \quad \forall n \geq 1,$$

with $\sum_{n=1}^\infty \beta_n < \infty$ and there exists a real number α such that $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then, the following results hold:

(i) $\sum_{n=1}^\infty [\varphi_n - \varphi_{n-1}]_+ < \infty$;

(ii) *There exists $\varphi^* \in [0, \infty)$ such that $\lim_{n\to\infty} \varphi_n = \varphi^*$.*

Lemma 2.6 ([42]) *Assume that $\{s_n\}$ and $\{t_n\}$ are two nonnegative real sequences such that $s_{n+1} \leq s_n + t_n$ for all $n \geq 1$. If $\sum_{n=1}^\infty t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.*

3 Main results

In this paper, we propose two weakly convergent inertial self-adaptive algorithms to solve the split common null-point problem in Banach spaces. In what follows, we denote J_p^E and $J_q^{E^*}$ by the generalized duality mapping of E into E^* and the duality mapping of E^* into E, respectively, where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In order to prove the results, the following assumptions are needed in the following.

(A1) Let E be a p-uniformly convex and uniformly smooth Banach space and F be a uniformly convex and smooth Banach space.

(A2) Let $A : E \to 2^{E^*}$ and $B : F \to 2^{F^*}$ be maximal monotone operators.

(A3) Let $T : E \to F$ be a bounded linear operator with $T \neq 0$ and $T^* : F^* \to E^*$ be the adjoint operator of T.

(A4) Let R_τ be a resolvent operator associated with A for $\tau > 0$ and Q_μ be a metric resolvent associated with B for $\mu > 0$.

(A5) The set solution $\Omega := A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$.

The following conditions are also assumed:

(C1) Let $\{\alpha_n\} \subset (0,1]$ with $\liminf_{n\to\infty} \alpha_n > 0$;
(C2) Let $\{\mu_n\} \subset (0,\infty)$ with $\liminf_{n\to\infty} \mu_n > 0$.
The first algorithm is stated as follows:

Algorithm 1 (Inertial self-adaptive algorithm for the split common null-point problem)
Step 0. Given $\tau_1 > 0$, $\beta \in (0,1)$ and $\mu \in (0, \frac{q}{\kappa_q})$. Choose $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\{\beta_n\} \subset (0,\infty)$ such that $\sum_{n=1}^{\infty} \beta_n < \infty$. Let $x_0, x_1 \in E$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n ($n \geq 1$). Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\beta, \frac{\beta_n}{\|J_p^E(x_n)-J_p^E(x_{n-1})\|^q}, \frac{\beta_n}{D_{f_p}(x_n,x_{n-1})}\}, & \text{if } x_n \neq x_{n-1}, \\ \beta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute

$$\begin{cases} u_n = J_q^{E^*}(J_p^E(x_n) + \theta_n(J_p^E(x_n) - J_p^E(x_{n-1}))), \\ y_n = J_q^{E^*}(J_p^E(u_n) - \tau_n T^* J_p^F(I - Q_{\mu_n})Tu_n), \\ x_{n+1} = J_q^{E^*}((1-\alpha_n)J_p^E(u_n) + \alpha_n J_p^E(R_{\tau_n} y_n)), \end{cases} \tag{3.2}$$

where

$$\tau_{n+1} = \begin{cases} \min\{(\frac{\mu\|(I-Q_{\mu_n})Tu_n)\|^p}{\|T^*J_p^F(I-Q_{\mu_n})Tu_n\|^q})^{\frac{1}{q-1}}, \tau_n + s_n\}, & \text{if } T^*J_p^F(I-Q_{\mu_n})Tu_n \neq 0, \\ \tau_n + s_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

Remark 3.1 If $x_{n+1} = y_n = u_n$ for some n, then y_n is a solution in Ω. Indeed, if $y_n = u_n$, we see that $y_n = J_q^{E^*}(J_p^E(y_n) - \tau_n T^* J_p^F(I - Q_{\mu_n})Tu_n)$ for $\tau_n > 0$. This implies that $(I - Q_{\mu_n})Ty_n = 0$, that is, $Ty_n = Q_{\mu_n} Ty_n$. In addition, if $x_{n+1} = y_n$, then $y_n = J_q^{E^*}((1-\alpha_n)J_p^E(y_n) + \alpha_n J_p^E(R_{\tau_n} y_n))$. This implies that $y_n = R_{\tau_n} y_n$. Now, since $y_n = R_{\tau_n} y_n$ and $Ty_n = Q_{\mu_n} Ty_n$, we have $y_n \in A^{-1}0$ and $y_n \in T^{-1}(B^{-1}0)$. Therefore, $y_n \in \Omega := A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Remark 3.2 From (3.1), we observe that $0 \leq \theta_n \leq \beta < 1$ for all $n \geq 1$. Also, we obtain $\theta_n \|J_p^E(x_n) - J_p^E(x_{n-1})\|^q \leq \beta_n$ and $\theta_n D_{f_p}(x_n, x_{n-1}) \leq \beta_n$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \beta_n < \infty$, we have

$$\sum_{n=1}^{\infty} \theta_n \left\|J_p^E(x_n) - J_p^E(x_{n-1})\right\|^q < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \theta_n D_{f_p}(x_n, x_{n-1}) < \infty. \tag{3.4}$$

Lemma 3.3 *Let $\{\tau_n\}$ be a sequence generated by* (3.3). *Then, we have* $\lim_{n\to\infty} \tau_n = \tau$, *where*

$$\tau \in \left[\min\left\{\left(\frac{\mu}{\|T\|^q}\right)^{\frac{1}{q-1}}, \tau_1\right\}, \tau_1 + s\right] \quad \textit{and} \quad s = \sum_{n=1}^{\infty} s_n.$$

Proof In the case of $T^*J_p^F(I-Q_{\mu_n})Tu_n \neq 0$, we see that

$$\begin{aligned}
&\frac{\mu\|(I-Q_{\mu_n})Tu_n\|^p}{\|T^*J_p^F(I-Q_{\mu_n})Tu_n\|^q} \\
&\quad\geq \frac{\mu\|(I-Q_{\mu_n})Tu_n\|^p}{\|T^*\|^q\|J_p^F(I-Q_{\mu_n})Tu_n\|^q} \\
&\quad= \frac{\mu\|(I-Q_{\mu_n})Tu_n\|^p}{\|T\|^q\|(I-Q_{\mu_n})Tu_n\|^{(p-1)q}} \\
&\quad= \frac{\mu}{\|T\|^q}.
\end{aligned} \tag{3.5}$$

By the definition of τ_n and induction, we have

$$\tau_n \leq \tau_{n-1} + s_{n-1} \leq \tau_{n-2} + s_{n-2} + s_{n-1} \leq \cdots \leq \tau_1 + \sum_{i=1}^{n-1} s_i \leq \tau_1 + \sum_{n=1}^{\infty} s_n.$$

Thus, $\tau_n \leq \tau_1 + \sum_{n=1}^\infty s_n$ for all $n \geq 1$. From (3.5), we see that

$$\begin{aligned}
\tau_{n+1} &= \min\left\{\left(\frac{\mu\|(I-Q_{\mu_n})Tu_n)\|^p}{\|T^*J_p^F(I-Q_{\mu_n})Tu_n\|^q}\right)^{\frac{1}{q-1}}, \tau_n + s_n\right\} \\
&\geq \min\left\{\left(\frac{\mu}{\|T\|^q}\right)^{\frac{1}{q-1}}, \tau_n + s_n\right\} \\
&\geq \min\left\{\left(\frac{\mu}{\|T\|^q}\right)^{\frac{1}{q-1}}, \tau_n\right\} \\
&\geq \cdots \geq \min\left\{\left(\frac{\mu}{\|T\|^q}\right)^{\frac{1}{q-1}}, \tau_1\right\}.
\end{aligned}$$

Hence $\tau_n \geq \min\{(\frac{\mu}{\|T\|^q})^{\frac{1}{q-1}}, \tau_1\}$ for all $n \geq 1$. Therefore, $\min\{(\frac{\mu}{\|T\|^q})^{\frac{1}{q-1}}, \tau_1\} \leq \tau_n \leq \tau_1 + \sum_{n=1}^\infty s_n$ for all $n \geq 1$. Since $\tau_{n+1} \leq \tau_n + s_n$ for all $n \geq 1$, we have $\lim_{n\to\infty}\tau_n$ exists by Lemma 2.6. In this case, we denote $\tau = \lim_{n\to\infty}\tau_n$. Obviously, $\tau \in [\min\{(\frac{\mu}{\|T\|^q})^{\frac{1}{q-1}}, \tau_1\}, \tau_1 + s]$. □

Remark 3.4 The adaptive step size τ_n generated by (3.3) is different from many adaptive step sizes as studied in [43, 44]. Note that τ_n is allowed to increase when the iteration increases. Therefore, it reduces the dependence on the initial step size τ_1. Since $\sum_{n=1}^\infty s_n < \infty$, we have $\lim_{n\to\infty} s_n = 0$. As a result, τ_n may not increase when n is large.

Lemma 3.5 *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then, for each $n \geq 1$, the following inequality holds for all $v \in \Omega$:*

$$D_{f_p}(v, x_{n+1}) \leq D_{f_p}(v, x_n) + \theta_n\big(D_{f_p}(v, x_n) - D_{f_p}(v, x_{n-1})\big) + \xi_n(p,q) - \delta_n(p,q),$$

where $\xi_n(p,q) := \theta_n D_{f_p}(x_n, x_{n-1}) + \frac{\kappa_q\theta_n^q}{q}\|J_p^E(x_n) - J_p^E(x_{n-1})\|^q$, $\delta_n(p,q) := \alpha_n\tau_n(1 - \frac{\kappa_q\mu}{q}(\frac{\tau_n}{\tau_{n+1}})^{q-1})\|w_n\|^p + \alpha_n D_{f_p}(y_n, R_{\tau_n}y_n)$ *and* $w_n := Tu_n - Q_{\mu_n}Tu_n$.

Proof Let $v \in \Omega := A^{-1}0 \cap T^{-1}(B^{-1}0)$. From Lemma 2.3, we have

$$\begin{aligned}
D_{f_p}(v,y_n) &= V_{f_p}\big(v, J_p^E(u_n) - \tau_n T^* J_p^F(I-Q_{\mu_n})Tu_n\big) \\
&= \frac{1}{p}\|v\|^p - \big\langle v, J_p^E(u_n) - \tau_n T^* J_p^F(w_n)\big\rangle + \frac{1}{q}\big\|J_p^E(u_n) - \tau_n T^* J_p^F(w_n)\big\|^q \\
&= \frac{1}{p}\|v\|^p - \big\langle v, J_p^E(u_n)\big\rangle + \tau_n\big\langle Tv, J_p^F(w_n)\big\rangle + \frac{1}{q}\big\|J_p^E(u_n) - \tau_n T^* J_p^F(w_n)\big\|^q \\
&\leq \frac{1}{p}\|v\|^p - \big\langle v, J_p^E(u_n)\big\rangle + \tau_n\big\langle Tv, J_p^F(w_n)\big\rangle + \frac{1}{q}\big\|J_p^E(u_n)\big\|^q - \tau_n\big\langle u_n, T^* J_p^F(w_n)\big\rangle \\
&\quad + \frac{\kappa_q \tau_n^q}{q}\big\|T^* J_p^F(w_n)\big\|^q \\
&= \frac{1}{p}\|v\|^p - \big\langle v, J_p^E(u_n)\big\rangle + \frac{1}{q}\|u_n\|^q + \tau_n\big\langle Tv - Tu_n, J_p^F(w_n)\big\rangle + \frac{\kappa_q \tau_n^q}{q}\big\|T^* J_p^F(w_n)\big\|^q \\
&= D_{f_p}(v,u_n) + \tau_n\big\langle Tv - Tu_n, J_p^F(w_n)\big\rangle + \frac{\kappa_q \tau_n^q}{q}\big\|T^* J_p^F(w_n)\big\|^q. \qquad (3.6)
\end{aligned}$$

Note that $w_n := Tu_n - Q_{\mu_n}Tu_n$ and $v \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, we have $v \in A^{-1}0$ and $Tv \in B^{-1}0$. It then follows from (2.10) that

$$\begin{aligned}
\big\langle Tv - Tu_n, J_p^F(w_n)\big\rangle &= \underbrace{\big\langle Tv - Q_{\mu_n}Tu_n, J_p^F(w_n)\big\rangle}_{\leq 0} - \big\langle Tu_n - Q_{\mu_n}Tu_n, J_p^F(w_n)\big\rangle \\
&\leq -\|Tu_n - Q_{\mu_n}Tu_n\|^p = -\|w_n\|^p. \qquad (3.7)
\end{aligned}$$

From the definition of τ_{n+1}, we have

$$\big\|T^* J_p^F(I-Q_{\mu_n})Tu_n\big\|^q \leq \frac{\mu}{\tau_{n+1}^{q-1}}\big\|(I-Q_{\mu_n})Tu_n\big\|^p. \qquad (3.8)$$

Combining (3.6), (3.7), and (3.8), we obtain

$$D_{f_p}(v,y_n) \leq D_{f_p}(v,u_n) - \tau_n\left(1 - \frac{\kappa_q \mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p. \qquad (3.9)$$

Now, we estimate $D_{f_p}(v,u_n)$. From Lemma 2.3, we have

$$\begin{aligned}
D_{f_p}(v,u_n) &= V_{f_p}\big(v, J_p^E(x_n) + \theta_n\big(J_p^E(x_n) - J_p^E(x_{n-1})\big)\big) \\
&= \frac{1}{p}\|v\|^p - \big\langle v, J_p^E(x_n) + \theta_n\big(J_p^E(x_n) - J_p^E(x_{n-1})\big)\big\rangle \\
&\quad + \frac{1}{q}\big\|J_p^E(x_n) + \theta_n\big(J_p^E(x_n) - J_p^E(x_{n-1})\big)\big\|^q \\
&\leq \frac{1}{p}\|v\|^p - \big\langle v, J_p^E(x_n)\big\rangle - \theta_n\big\langle v, J_p^E(x_n) - J_p^E(x_{n-1})\big\rangle \\
&\quad + \frac{1}{q}\big\|J_p^E(x_n)\big\|^q + \theta_n\big\langle x_n, J_p^E(x_n) - J_p^E(x_{n-1})\big\rangle + \frac{\kappa_q}{q}\big\|\theta_n\big(J_p^E(x_n) - J_p^E(x_{n-1})\big)\big\|^q \\
&= \frac{1}{p}\|v\|^p - \big\langle v, J_p^E(x_n)\big\rangle + \frac{1}{q}\|x_n\|^p + \frac{\kappa_q \theta_n^q}{q}\big\|J_p^E(x_n) - J_p^E(x_{n-1})\big\|^q
\end{aligned}$$

$$
\begin{aligned}
&+\theta_n\langle x_n-v,J_p^E(x_n)-J_p^E(x_{n-1})\rangle\\
&=D_{f_p}(v,x_n)+\frac{\kappa_q\theta_n^q}{q}\|J_p^E(x_n)-J_p^E(x_{n-1})\|^q+\theta_n\langle x_n-v,J_p^E(x_n)-J_p^E(x_{n-1})\rangle. \quad (3.10)
\end{aligned}
$$

We observe that

$$\theta_n\langle x_n-v,J_p^E(x_n)-J_p^E(x_{n-1})\rangle=\theta_nD_{f_p}(v,x_n)-\theta_nD_{f_p}(v,x_{n-1})+\theta_nD_{f_p}(x_n,x_{n-1}). \quad (3.11)$$

Combining (3.10) and (3.11), we obtain

$$
\begin{aligned}
D_{f_p}(v,u_n)\le{}& D_{f_p}(v,x_n)+\theta_n\big(D_{f_p}(v,x_n)-D_{f_p}(v,x_{n-1})\big)+\theta_nD_{f_p}(x_n,x_{n-1})\\
&+\frac{\kappa_q\theta_n^q}{q}\|J_p^E(x_n)-J_p^E(x_{n-1})\|^q. \quad (3.12)
\end{aligned}
$$

Then, from (2.7) and (3.10), we obtain

$$
\begin{aligned}
D_{f_p}(v,x_{n+1})&\le(1-\alpha_n)D_{f_p}(v,u_n)+\alpha_nD_{f_p}(v,R_{\tau_n}y_n)\\
&\le(1-\alpha_n)D_{f_p}(v,u_n)+\alpha_nD_{f_p}(v,y_n)-\alpha_nD_{f_p}(y_n,R_{\tau_n}y_n)\\
&\le(1-\alpha_n)D_{f_p}(v,u_n)+\alpha_n\left[D_{f_p}(v,u_n)-\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p\right]\\
&\quad-\alpha_nD_{f_p}(y_n,R_{\tau_n}y_n)\\
&=D_{f_p}(v,u_n)-\alpha_n\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p-\alpha_nD_{f_p}(y_n,R_{\tau_n}y_n)\\
&\le D_{f_p}(v,x_n)+\theta_n\big(D_{f_p}(v,x_n)-D_{f_p}(v,x_{n-1})\big)+\theta_nD_{f_p}(x_n,x_{n-1})\\
&\quad+\frac{\kappa_q\theta_n^q}{q}\|J_p^E(x_n)-J_p^E(x_{n-1})\|^q\\
&\quad-\alpha_n\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p-\alpha_nD_{f_p}(y_n,R_{\tau_n}y_n). \quad (3.13)
\end{aligned}
$$

From the definitions of $\xi_n(p,q)$ and $\delta_n(p,q)$, then (3.13) can be written in a short form as follows:

$$D_{f_p}(v,x_{n+1})\le D_{f_p}(v,x_n)+\theta_n\big(D_{f_p}(v,x_n)-D_{f_p}(v,x_{n-1})\big)+\xi_n(p,q)-\delta_n(p,q).$$

Thus, this lemma is proved. □

Theorem 3.6 *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Suppose, in addition, that J_p^E is weakly sequentially continuous on E. Then, $\{x_n\}$ converges weakly to a point in Ω.*

Proof Using the fact that $\lim_{n\to\infty}\tau_n$ exists and $\mu\in(0,\frac{q}{\kappa_q})$, we have

$$\lim_{n\to\infty}\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)=\frac{\mu}{q}\left(\frac{q}{\mu}-\kappa_q\right)>0.$$

Then, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\kappa_q \mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1} > 0, \quad \forall n \geq n_0,$$

and, in consequence,

$$\delta_n(p,q) := \alpha_n \tau_n \left(1 - \frac{\kappa_q \mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p + \alpha_n D_{f_p}(y_n, R_{\tau_n} y_n) > 0, \quad \forall n \geq n_0.$$

Then, from Lemma 3.5, we can deduce that

$$D_{f_p}(v, x_{n+1}) \leq D_{f_p}(v, x_n) + \theta_n\big(D_{f_p}(v, x_n) - D_{f_p}(v, x_{n-1})\big) + \xi_n(p,q).$$

Since $\xi_n(p,q) := \theta_n D_{f_p}(x_n, x_{n-1}) + \frac{\kappa_q \theta_n^q}{q}\|J_p^E(x_n) - J_p^E(x_{n-1})\|^q$, it follows from (3.4) that

$$\sum_{n=1}^{\infty} \xi_n(p,q) \leq \sum_{n=1}^{\infty}\left(\theta_n D_{f_p}(x_n, x_{n-1}) + \frac{\kappa_q \theta_n}{q}\left\|J_p^E(x_n) - J_p^E(x_{n-1})\right\|^q\right) < \infty. \tag{3.14}$$

From (3.14), we also have $\lim_{n\to\infty} \xi_n(p,q) = 0$. From Lemma 2.5, we can conclude that $\lim_{n\to\infty} D_{f_p}(v, x_n)$ exists and

$$\sum_{n=1}^{\infty}\big[D_{f_p}(v, x_n) - D_{f_p}(v, x_{n-1})\big]_+ < \infty.$$

Thus, we have $\{D_{f_p}(v, x_n)\}$ is bounded and so $\{x_n\}$ is also bounded by (2.4). Moreover, we obtain

$$\lim_{n\to\infty}\big[D_{f_p}(v, x_n) - D_{f_p}(v, x_{n-1})\big]_+ = 0. \tag{3.15}$$

From Lemma 3.5, we see that

$$\delta_n(p,q) \leq D_{f_p}(v, x_n) - D_{f_p}(v, x_{n+1}) + \theta_n\big(D_{f_p}(v, x_n) - D_{f_p}(v, x_{n-1})\big) + \xi_n(p,q).$$

Since $\lim_{n\to\infty} D_{f_p}(v, x_n)$ exists and $\lim_{n\to\infty} \xi_n(p,q) = 0$, we have

$$\lim_{n\to\infty} \delta_n(p,q) = \lim_{n\to\infty}\left[\alpha_n \tau_n\left(1 - \frac{\kappa_q \mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p + \alpha_n D_{f_p}(y_n, R_{\tau_n} y_n)\right] = 0.$$

Consequently,

$$\lim_{n\to\infty}\|w_n\| = \lim_{n\to\infty}\|Tu_n - Q_{\mu_n} Tu_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} D_{f_p}(y_n, R_{\tau_n} y_n) = 0.$$

By the continuity of J_p^F, we have

$$\lim_{n\to\infty}\left\|J_p^F(w_n)\right\| = \lim_{n\to\infty}\left\|J_p^F(Tu_n - Q_{\mu_n} Tu_n)\right\| = 0. \tag{3.16}$$

Moreover, we have

$$\lim_{n\to\infty} \|y_n - R_{\tau_n} y_n\| = 0. \tag{3.17}$$

By the definition of y_n, the continuity of T^*, and from (3.16), we have

$$\lim_{n\to\infty} \left\|J_p^E(y_n) - J_p^E(u_n)\right\| = \lim_{n\to\infty} \tau_n \left\|T^* J_p^F(I - Q_{\mu_n})Tu_n\right\| = 0. \tag{3.18}$$

On the other hand, by the definition of u_n and from (3.4), we obtain

$$\begin{aligned}\lim_{n\to\infty} \left\|J_p^E(u_n) - J_p^E(x_n)\right\|^q &= \lim_{n\to\infty} \theta_n^q \left\|J_p^E(x_n) - J_p^E(x_{n-1})\right\|^q \\ &\leq \lim_{n\to\infty} \beta^{q-1}\theta_n \left\|J_p^E(x_n) - J_p^E(x_{n-1})\right\|^q \\ &= 0.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \left\|J_p^E(u_n) - J_p^E(x_n)\right\| = 0. \tag{3.19}$$

From (3.19), we also have $\lim_{n\to\infty} \|u_n - x_n\| = 0$ by the uniform continuity of $J_q^{E^*}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in E$ and so $u_{n_k} \rightharpoonup w$. Put $v_n = R_{\tau_n} y_n$ for all $n \in \mathbb{N}$. From (3.17) and (3.18), we see that

$$\begin{aligned}\left\|J_p^E(u_n) - J_p^E(v_n)\right\| &= \left\|J_p^E(u_n) - J_p^E(R_{\tau_n} y_n)\right\| \\ &\leq \left\|J_p^E(u_n) - J_p^E(y_n)\right\| + \left\|J_p^E(y_n) - J_p^E(R_{\tau_n} y_n)\right\| \\ &\to 0.\end{aligned} \tag{3.20}$$

Consequently, $\lim_{n\to\infty} \|u_n - v_n\| = 0$. Thus,

$$\|v_n - x_n\| \leq \|v_n - u_n\| + \|u_n - x_n\| \to 0.$$

Using the above inequality, we also obtain $v_{n_k} \rightharpoonup w$. Since R_{τ_n} is the resolvent of A for $\tau_n > 0$, we have

$$\begin{aligned} v_n = R_{\tau_n}\left(J_q^{E^*}\left(J_p^E(u_n) - \tau_n T^* J_p^F(u_n)\right)\right) \quad &\Leftrightarrow \quad \left(J_p^E + \tau_n A\right)^{-1}\left(J_p^E(u_n) - \tau_n T^* J_p^F(u_n)\right)\right) \\ &\Leftrightarrow \quad J_p^E(u_n) - \tau_n T^* J_p^F(u_n) \in J_p^E(v_n) + \tau_n A v_n \\ &\Leftrightarrow \quad \frac{1}{\tau_n}\left(J_p^E(u_n) - J_p^E(v_n) - \tau_n T^* J_p^F(w_n)\right) \in Av_n. \end{aligned}$$

Replacing n by n_k and using the fact that A is monotone, thus

$$\left\langle s - v_{n_k}, s^* - \frac{1}{\tau_{n_k}}\left(J_p^E(u_{n_k}) - J_p^E(v_{n_k}) - \tau_{n_k} T^* J_p^F(w_{n_k})\right)\right\rangle \geq 0$$

for all $(s, s^*) \in A$. Now, T^* is continuous, which is due to the fact that T^* is a bounded and linear operator. Then, from (3.16), (3.20), and $\lim_{k\to\infty} \tau_{n_k} = \tau > 0$, we obtain $\langle s - w, s^* -$

$0\rangle \geq 0$ for all $(s, s^*) \in A$. Note that A is maximal monotone, thus $w \in A^{-1}0$. On the other hand, we know that T is also continuous. This fact, together with $\|u_{n_k} - x_{n_k}\| \to 0$ and $\|Tu_{n_k} - Q_{\mu_{n_k}} Tu_{n_k}\| \to 0$, means that we have $Tu_{n_k} \rightharpoonup Tw$ and $Q_{\mu_{n_k}} u_{n_k} \rightharpoonup Tw$. Since Q_{μ_n} is the metric resolvent of B for $\mu_n > 0$, we have

$$\frac{J_p^F(Tu_n - Q_{\mu_n} Tu_n)}{\mu_n} \in BQ_{\mu_n} Tu_n$$

for all $n \in \mathbb{N}$. Replacing n by n_k, it then follows from the monotonicity of B that

$$\left\langle u - Q_{\mu_{n_k}} Tu_{n_k}, u^* - \frac{J_p^F(Tu_{n_k} - Q_{\mu_{n_k}} Tu_{n_k})}{\mu_{n_k}} \right\rangle \geq 0$$

for all $(u, u^*) \in B$. Then, from (3.16) and $\liminf_{k\to\infty} \mu_{n_k} > 0$, we obtain $\langle u - Tw, u^* - 0\rangle \geq 0$ for all $(u, u^*) \in B$. Note that B is maximal monotone, thus $Tw \in B^{-1}0$ and so $w \in T^{-1}(B^{-1}0)$. We thus obtain $w \in \Omega := A^{-1}0 \cap T^{-1}(B^{-1}0)$. In order to prove the weak convergence of the sequence $\{x_n\}$, it is sufficient to show that $\{x_n\}$ has a unique weak limit point in Ω. In this case, we can assume that $\{x_{m_k}\}$ is another subsequence of $\{x_n\}$ such that $x_{m_k} \rightharpoonup w' \in \Omega$. Note that $x_{n_k} \rightharpoonup w \in \Omega$. Indeed, suppose by contradiction that with $w' \neq w$. Since $\lim_{n\to\infty} D_{f_p}(v, x_n)$ exists for any $v \in \Omega$, it then follows from (2.2) and the weak sequential continuity of J_p^E that

$$\begin{aligned}
\lim_{n\to\infty} D_{f_p}(w, x_n) &= \lim_{k\to\infty} D_{f_p}(w, x_{m_k}) = \liminf_{k\to\infty} D_{f_p}(w, x_{m_k}) \\
&= \liminf_{k\to\infty} \bigl(D_{f_p}(w, w') + D_{f_p}(w', x_{m_k}) + \langle w - w', J_p^E(w') - J_p^E(x_{m_k})\rangle\bigr) \\
&\geq \liminf_{k\to\infty} D_{f_p}(w, w') + \liminf_{k\to\infty} D_{f_p}(w', x_{m_k}) \\
&\quad + \liminf_{k\to\infty} \langle w - w', J_p^E(w') - J_p^E(x_{m_k})\rangle \\
&> \liminf_{k\to\infty} D_{f_p}(w', x_{m_k}) \\
&= \lim_{n\to\infty} D_{f_p}(w', x_n).
\end{aligned} \tag{3.21}$$

In the same way as above, we can show that

$$\lim_{n\to\infty} D_{f_p}(w', x_n) > \lim_{n\to\infty} D_{f_p}(w, x_n),$$

which is a contradiction with (3.21). Hence, $w = w'$ and therefore, the sequence $\{x_n\}$ converges weakly to a point in Ω. This finishes the proof. □

Next, we present a second algorithm that is slightly different from the first proposed algorithm.

Algorithm 2 (Relaxed inertial self-adaptive algorithm for the split common null-point problem)

Step 0. Given $\tau_1 > 0$, $\beta \in (0,1)$ and $\mu \in (0, \frac{q}{\kappa_q})$. Choose $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\{\beta_n\} \subset (0,\infty)$ such that $\sum_{n=1}^{\infty} \beta_n < \infty$. Let $x_0, x_1 \in E$ be arbitrary and calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n ($n \geq 1$). Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\beta, \frac{\beta_n}{\|J_p^E(x_n)-J_p^E(x_{n-1})\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \beta, & \text{otherwise.} \end{cases} \tag{3.22}$$

Step 2. Compute

$$\begin{cases} u_n = J_q^{E^*}(J_p^E(x_n) + \theta_n(J_p^E(x_{n-1}) - J_p^E(x_n))), \\ y_n = J_q^{E^*}(J_p^E(u_n) - \tau_n T^* J_p^F(I - Q_{\mu_n})Tu_n), \\ x_{n+1} = J_q^{E^*}((1-\alpha_n)J_p^E(u_n) + \alpha_n J_p^E(R_{\tau_n} y_n)), \end{cases} \tag{3.23}$$

where τ_n is defined the same as in (3.3).

Remark 3.7 It should be noted that Algorithm 2 is slightly different from Algorithm 1 but $\bar{\theta}_n$ of this algorithm is simpler to compute than $\bar{\theta}_n$ of Algorithm 1, that is, it is chosen without any prior knowledge of the Bregman distance D_{f_p} at two points x_n and x_{n-1}, which is flexible and easy to implement in solving the problem. This is why we call the technique proposed in this case a "relaxed inertial algorithm".

Remark 3.8 From (3.22), we observe that $0 \leq \theta_n \leq \beta < 1$ for all $n \geq 1$. Also, we obtain $\theta_n \|J_p^E(x_n) - J_p^E(x_{n-1})\| \leq \beta_n$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \beta_n < \infty$, we have $\sum_{n=1}^{\infty} \theta_n \|J_p^E(x_n) - J_p^E(x_{n-1})\| < \infty$ and so

$$\lim_{n\to\infty} \theta_n \big\|J_p^E(x_n) - J_p^E(x_{n-1})\big\| = 0.$$

Theorem 3.9 *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Suppose, in addition, that J_p^E is weakly sequentially continuous on E. Then, $\{x_n\}$ converges weakly to a point in Ω.*

Proof Let $v \in \Omega := A^{-1}0 \cap T^{-1}(B^{-1}0)$. Similarly, by using the same argument as in the proof of Theorem 3.6, we have

$$D_{f_p}(v, y_n) \leq D_{f_p}(v, u_n) - \tau_n \left(1 - \frac{\kappa_q \mu}{q} \left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right) \|w_n\|^p. \tag{3.24}$$

By the definition of u_n in (3.23), we see that

$$\begin{aligned} D_{f_p}(v, u_n) &= D_{f_p}\big(v, J_q^{E^*}\big((1-\theta_n)J_p^E(x_n) + \theta_n J_p^E(x_{n-1})\big)\big) \\ &\leq (1-\theta_n)D_{f_p}(v, x_n) + \theta_n D_{f_p}(v, x_{n-1}). \end{aligned} \tag{3.25}$$

Thus, we have

$$\begin{aligned} D_{f_p}(v, x_{n+1}) &\leq (1-\alpha_n)D_{f_p}(v, u_n) + \alpha_n D_{f_p}(v, y_n) - \alpha_n D_{f_p}(y_n, R_{\tau_n} y_n) \\ &\leq (1-\alpha_n)D_{f_p}(v, u_n) + \alpha_n \left[D_{f_p}(v, u_n) - \tau_n \left(1 - \frac{\kappa_q \mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p\right] \\ &\quad - \alpha_n D_{f_p}(y_n, R_{\tau_n} y_n) \end{aligned}$$

$$
\begin{aligned}
&= D_{f_p}(v,u_n) - \alpha_n\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p - \alpha_n D_{f_p}(y_n,R_{\tau_n}y_n) \\
&\le (1-\theta_n)D_{f_p}(v,x_n) + \theta_n D_{f_p}(v,x_{n-1}) - \alpha_n\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p \\
&\quad -\alpha_n D_{f_p}(y_n,R_{\tau_n}y_n). \qquad (3.26)
\end{aligned}
$$

From Theorem 3.6, we know that

$$
\delta_n(p,q) := \alpha_n\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p + \alpha_n D_{f_p}(y_n,R_{\tau_n}y_n) > 0, \quad \forall n \ge n_0.
$$

Thus, we can deduce that

$$
\begin{aligned}
D_{f_p}(v,x_{n+1}) &\le (1-\theta_n)D_{f_p}(v,x_n) + \theta_n D_{f_p}(v,x_{n-1}) \\
&\le \max\bigl\{D_{f_p}(v,x_n), D_{f_p}(v,x_{n-1})\bigr\} \\
&\le \cdots \max\bigl\{D_{f_p}(v,x_{n_0}), D_{f_p}(v,x_{n_0-1})\bigr\}.
\end{aligned}
$$

Hence, $\{D_{f_p}(v,x_n)\}$ is bounded. From (2.4), we also obtain $\{x_n\}$ is bounded. From (3.26), we have

$$
\begin{aligned}
D_{f_p}(v,x_{n+1}) &\le D_{f_p}(v,x_n) + \theta_n\bigl[D_{f_p}(v,x_{n-1}) - D_{f_p}(v,x_n)\bigr]_+ \\
&\quad - \alpha_n\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p \\
&\quad - \alpha_n D_{f_p}(y_n,R_{\tau_n}y_n), \qquad (3.27)
\end{aligned}
$$

which implies that

$$
D_{f_p}(v,x_{n+1}) \le D_{f_p}(v,x_n) + \theta_n\bigl[D_{f_p}(v,x_{n-1}) - D_{f_p}(v,x_n)\bigr]_+ \qquad (3.28)
$$

for all $n \ge n_0$. Using (2.2), we see that

$$
\begin{aligned}
\theta_n\bigl[D_{f_p}(v,x_{n-1}) - D_{f_p}(v,x_n)\bigr]_+ &= -\theta_n D_{f_p}(x_{n-1},x_n) + \theta_n\langle v - x_{n-1}, J_p^E(x_n) - J_p^E(x_{n-1})\rangle \\
&\le \theta_n\langle v - x_{n-1}, J_p^E(x_n) - J_p^E(x_{n-1})\rangle \\
&\le \theta_n\bigl\|J_p^E(x_n) - J_p^E(x_{n-1})\bigr\|M,
\end{aligned}
$$

where $M := \sup_{n\ge n_0}\{\|x_{n-1} - v\|\}$. From Remark 3.8, we can deduce that

$$
\sum_{n=n_0}^{\infty}\theta_n\bigl[D_{f_p}(v,x_{n-1}) - D_{f_p}(v,x_n)\bigr]_+ \le \sum_{n=n_0}^{\infty}\theta_n\bigl\|J_p^E(x_n) - J_p^E(x_{n-1})\bigr\|M < \infty. \qquad (3.29)
$$

Thus, from (3.28) with Lemma 2.6, we have the limit of $\{D_{f_p}(v,x_n)\}$ exists. Note that (3.29) implies that

$$
\lim_{n\to\infty}\theta_n\bigl[D_{f_p}(v,x_{n-1}) - D_{f_p}(v,x_n)\bigr]_+ = 0. \qquad (3.30)
$$

From (3.27), we have

$$\alpha_n\tau_n\left(1-\frac{\kappa_q\mu}{q}\left(\frac{\tau_n}{\tau_{n+1}}\right)^{q-1}\right)\|w_n\|^p+\alpha_n D_{f_p}(y_n,R_{\tau_n}y_n)$$
$$\leq D_{f_p}(v,x_n)-D_{f_p}(v,x_{n+1})+\theta_n\big[D_{f_p}(v,x_{n-1})-D_{f_p}(v,x_n)\big]_+.$$

This implies by (3.30) that

$$\lim_{n\to\infty}\|Tu_n-Q_{\mu_n}Tu_n\|=0\quad\text{and}\quad\lim_{n\to\infty}D_{f_p}(y_n,R_{\tau_n}y_n)=0.$$

Thus,

$$\lim_{n\to\infty}\big\|J_p^F(Tu_n-Q_{\mu_n}Tu_n)\big\|=0\quad\text{and}\quad\lim_{n\to\infty}\|y_n-R_{\tau_n}y_n\|=0.$$

By the definition of y_n, we can show that $\lim_{n\to\infty}\|J_p^E(y_n)-J_p^E(u_n)\|=0$. Also, by the definition of u_n, we have

$$\lim_{n\to\infty}\big\|J_p^E(u_n)-J_p^E(x_n)\big\|=\lim_{n\to\infty}\theta_n\big\|J_p^E(x_n)-J_p^E(x_{n-1})\big\|=0.$$

Consequently, $\lim_{n\to\infty}\|u_n-x_n\|=0$. Since the rest of the proof is the same as the proof of Theorem 3.6, we omit the details here. □

Next, we apply our algorithms to solve the split feasibility problem in Banach spaces.

Let C and Q be nonempty, closed, and convex subsets of E and F, respectively. Let T be a bounded linear operator with its adjoint operator T^* and $T\neq 0$. We consider the *split feasibility problem* (SFP):

$$\text{find}\quad w\in C\quad\text{such that}\quad Tw\in Q. \tag{3.31}$$

We denote the set of solutions of SFP by $\Gamma:=C\cap T^{-1}(Q)$. The SFP was first introduced in 1994 by Censor and Elfving [15] for inverse problems of intensity-modulated radiation therapy (IMRT) in the field of medical care (see [12, 14]).

Setting $A:=\partial\delta_C$ and $B:=\partial\delta_Q$ in Theorems 3.6 and 3.9, we obtain the following results.

Corollary 3.10 *Let E, F, C, Q, T, and T^* be the same as mentioned above. Let τ_1, β, μ, $\{s_n\}$, $\{\beta_n\}$, $\{\theta_n\}$, and $\{\bar{\theta}_n\}$ be the same as in Algorithm 1, where $\{\bar{\theta}_n\}$ is defined the same as in (3.1). Suppose that $\Gamma\neq\emptyset$. Let $x_0,x_1\in E$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n=J_q^{E^*}(J_p^E(x_n)+\theta_n(J_p^E(x_n)-J_p^E(x_{n-1}))),\\ y_n=J_q^{E^*}(J_p^E(u_n)-\tau_nT^*J_p^F(I-P_Q)Tu_n),\\ x_{n+1}=J_q^{E^*}((1-\alpha_n)J_p^E(u_n)+\alpha_nJ_p^E(\Pi_C^{f_p}y_n)),\\ \tau_{n+1}=\begin{cases}\min\{(\frac{\mu\|(I-P_Q)Tu_n)\|^p}{\|T^*J_p^F(I-P_Q)Tu_n\|^q})^{\frac{1}{q-1}},\tau_n+s_n\}, & \text{if } T^*J_p^F(I-P_Q)Tu_n\neq 0,\\ \tau_n+s_n, & \text{otherwise.}\end{cases}\end{cases}$$

Suppose, in addition, that J_p^E is weakly sequentially continuous on E. Then, the sequence $\{x_n\}$ converges weakly to a point in Γ.

Corollary 3.11 *Let E, F, C, Q, T, and T^* be the same as mentioned above. Let τ_1, β, μ, $\{s_n\}$, $\{\beta_n\}$, $\{\theta_n\}$, and $\{\bar{\theta}_n\}$ be the same as in Algorithm 2, where $\{\bar{\theta}_n\}$ is defined the same as in (3.22). Suppose that $\Gamma \neq \emptyset$. Let $x_0, x_1 \in E$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = J_q^{E^*}(J_p^E(x_n) + \theta_n(J_p^E(x_{n-1}) - J_p^E(x_n))), \\ y_n = J_q^{E^*}(J_p^E(u_n) - \tau_n T^* J_p^F(I - P_Q)Tu_n), \\ x_{n+1} = J_q^{E^*}((1-\alpha_n)J_p^E(u_n) + \alpha_n J_p^E(\Pi_C^{f_p} y_n)), \\ \tau_{n+1} = \begin{cases} \min\{(\frac{\mu\|(I-P_Q)Tu_n)\|^p}{\|T^*J_p^F(I-P_Q)Tu_n\|^q})^{\frac{1}{q-1}}, \tau_n + s_n\}, & \text{if } T^*J_p^F(I-P_Q)Tu_n \neq 0, \\ \tau_n + s_n, & \text{otherwise.} \end{cases} \end{cases}$$

Suppose, in addition, that J_p^E is weakly sequentially continuous on E. Then, the sequence $\{x_n\}$ converges weakly to a point in Γ.

4 Numerical and experiments results

In this section, we apply our Algorithms 1 and 2 to numerically solve some problems in science and engineering and we also compare the numerical performances with the iterative scheme (1.8) proposed by Tang [43] (namely, Tang Algorithm) and the iterative scheme (1.5) proposed by Suantai et al. [39] (namely, Suantai et al. Algorithm).

Problem 4.1 Split feasibility problem in infinite-dimensional Banach spaces

Let $E = F = \ell_p^0(\mathbb{R})$ $(1 < p < \infty, p \neq 2)$, where $\ell_p^0(\mathbb{R})$ is the subspace of $\ell_p(\mathbb{R})$, that is,

$$\ell_p^0(\mathbb{R}) = \left\{ x = (x_1, x_2, \dots, x_i, 0, 0, 0, \dots), x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

with norm $\|x\|_{\ell_p} = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ and duality pairing $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ for all $x = (x_1, x_2, \dots, x_i, \dots) \in l_p(\mathbb{R})$ and $y = (y_1, y_2, \dots, y_i, \dots) \in l_q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$. The generalized duality mapping $J_p^{\ell_p(\mathbb{R})}$ is computed by the following explicit formula (see [3]):

$$J_p^{\ell_p(\mathbb{R})}(x) = \left(|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \dots, |x_i|^{p-2}x_i, \dots\right), \quad \forall x \in \ell_p(\mathbb{R}).$$

In this example, let $p = 3$, we have $q = \frac{3}{2}$. Then, the smoothness constant $\kappa_q \approx 1.3065$. Let $C = \{x \in \ell_3^0(\mathbb{R}) : \|x\|_{\ell_3} \leq 1\}$ and $Q = \{x \in \ell_3^0(\mathbb{R}) : \langle x, a\rangle \leq 1\}$, where $a := (1, 1, \dots, 1, 0, 0, 0, \dots) \in \ell_{3/2}^0(\mathbb{R})$. Define an operator $Tx = \frac{x}{2}$ with its adjoint $T^* = T$ and $\|T\| = \frac{1}{2}$. In this experiment, we only perform the numerical tests of our Algorithms 1, 2, and Suantai et al. Algorithm [39] since Tang Algorithm [43] cannot be implemented in $\ell_3(\mathbb{R})$. For Algorithms 1 and 2, we set $\tau_1 = 1.99$, $\beta = 0.75$, $\mu = 10^{-5}$, $s_n = \frac{1}{(n+1)^4}$, $\alpha_n = 0.1$, and $\beta_n = \frac{1}{(n+10)^5}$. For Suantai et al. Algorithm [39], we set $\lambda_n = 10^{-5}$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{n+1}$, and $u_n = (\frac{1}{n^2}, \frac{1}{n^2}, \frac{1}{n^2}, 0, 0, 0, \dots)^{\mathsf{T}}$. The initial points x_0 and x_1 are generated randomly in $\ell_p^0(\mathbb{R})$. We use $E_n = \|x_{n+1} - x_n\|_{l_3} < 10^{-6}$ to terminate iterations for all algorithms. To test the robustness of each algorithm, we run the experiment several times and choose the best four tests of sequences generated by each algorithm. The numerical results are presented in Figs. 1 and 2.

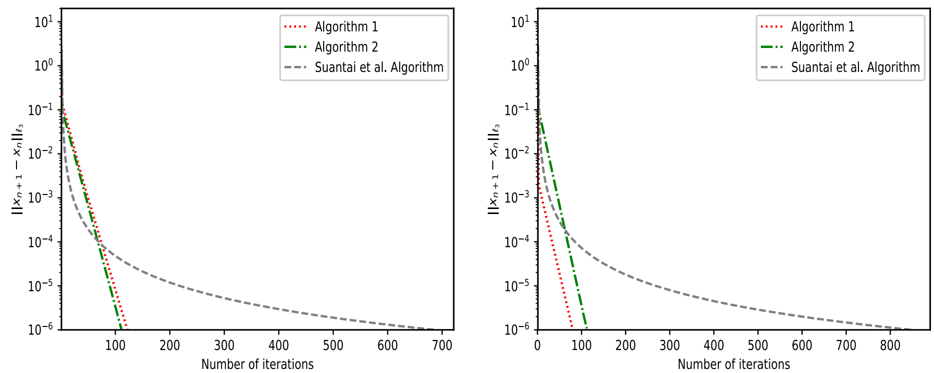

Figure 1 Numerical behavior of all algorithms for Tests 1 and 2

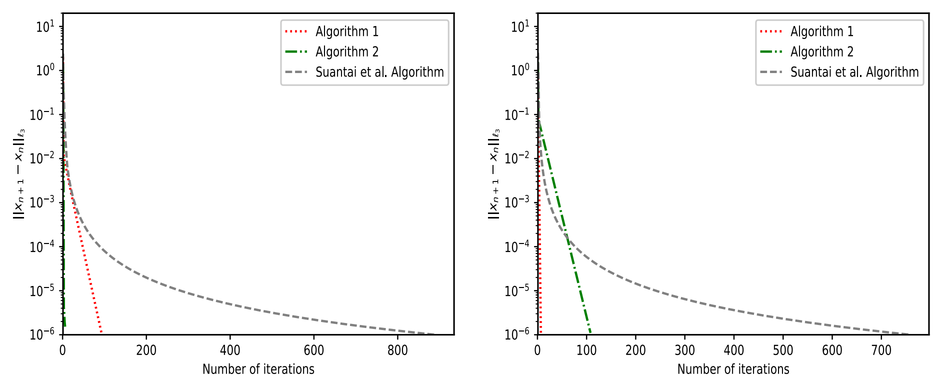

Figure 2 Numerical behavior of all algorithms for Tests 3 and 4

Problem 4.2 [39] Split minimization problem in finite-dimensional spaces

Let $E = F = \mathbb{R}^3$. For each $x \in \mathbb{R}^3$, let $f, g : \mathbb{R}^3 \to (-\infty, +\infty]$ be defined by

$$f(x) = 20\|x\|^2 + (33, 14, -95)x + 40$$

and

$$g(x) = \frac{1}{2}\|Lx - y\|,$$

where $L = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 2 & -2 & 4 \end{pmatrix}$ and $y = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$. Let $T = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & 2 \\ 2 & 1 & 0 \end{pmatrix}$. In this case, the split common null-point problem becomes the split minimization problem, that is, find $w \in (\partial f)^{-1}0 \cap T^{-1}(\partial g)^{-1}0$. Note that $T^* = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 5 & 1 \\ 3 & 2 & 0 \end{pmatrix}$ and $\|T\|^2$ is the largest eigenvalue of T^*T. In this experiment, we compare the numerical performances of our Algorithms 1 and 2 with Tang Algorithm [43] and Suantai et al. Algorithm [39]. For Algorithms 1 and 2, we set $\tau_1 = 1.99$, $\beta = 0.75$, $\mu = 0.1$, $\alpha_n = 0.1$, $s_n = \frac{1}{(n+1)^4}$, $\beta_n = \frac{1}{(n+1)^{1.1}}$, and $\tau_n = \mu_n = 0.01$. For Tang Algorithm [43], we set $\alpha = 0.75$, $\rho_n = 3 - \frac{1}{n+1}$, $\epsilon_n = \frac{1}{(n+1)^{1.1}}$, and $r = \mu = 10^{-5}$. For Suantai et al. Algorithm [39], we set $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.5$, $r_n = 1$, $\lambda_n = 0.01$, and $u_n = (\frac{1}{n^2}, \frac{1}{n^2}, \frac{1}{n^2})^\top$. The initial points x_0 and x_1 are generated randomly in $\mathbb{R}^3$. We use $E_n = \|x_{n+1} - x_n\| < 10^{-5}$ to terminate iterations for all

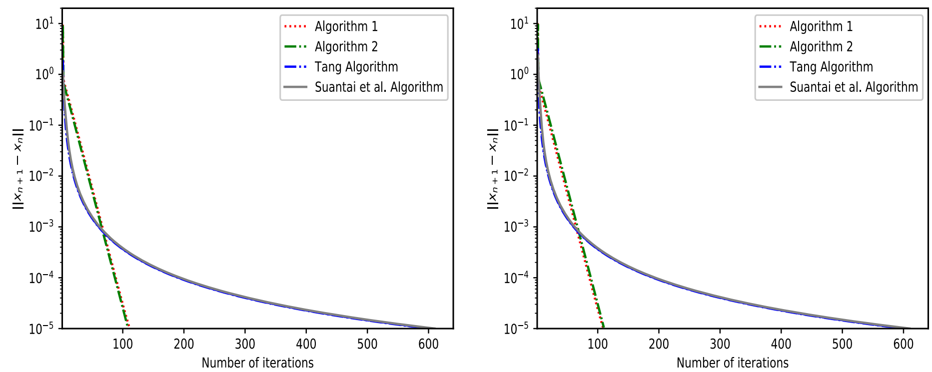

Figure 3 Numerical behavior of all algorithms for Tests 1 and 2

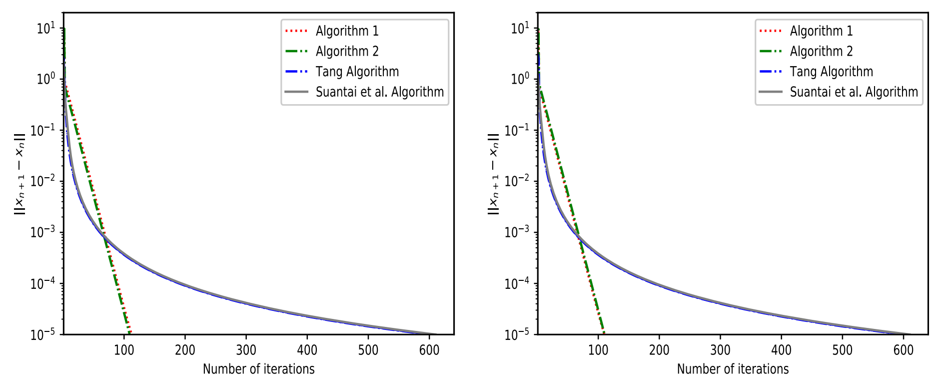

Figure 4 Numerical behavior of all algorithms for Tests 3 and 4

algorithms. To test the robustness of each algorithm, we run the experiment several times and choose the best four testes of sequences generated by each algorithm. The numerical results are presented in Figs. 3 and 4.

Problem 4.3 Signal-recovery problem

In signal processing, compressed sensing involves the recovery of a "sparse signal" from measured data, aiming to reconstruct the original signal using fewer measurements (see, e.g [27, 47]). In this context, we can model the compressed sensing as the following uncertain linear system:

$$y = Tx + b, \tag{4.1}$$

where $x \in \mathbb{R}^N$ is a K-sparse signal K ($K \ll N$), to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy b and $T : \mathbb{R}^N \to \mathbb{R}^M$ is a bounded linear operator. It is known that the above problem can be seen as the following LASSO problem [46]:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Tx - y\|^2 \quad \text{subject to} \quad \|x\|_1 \le t, \tag{4.2}$$

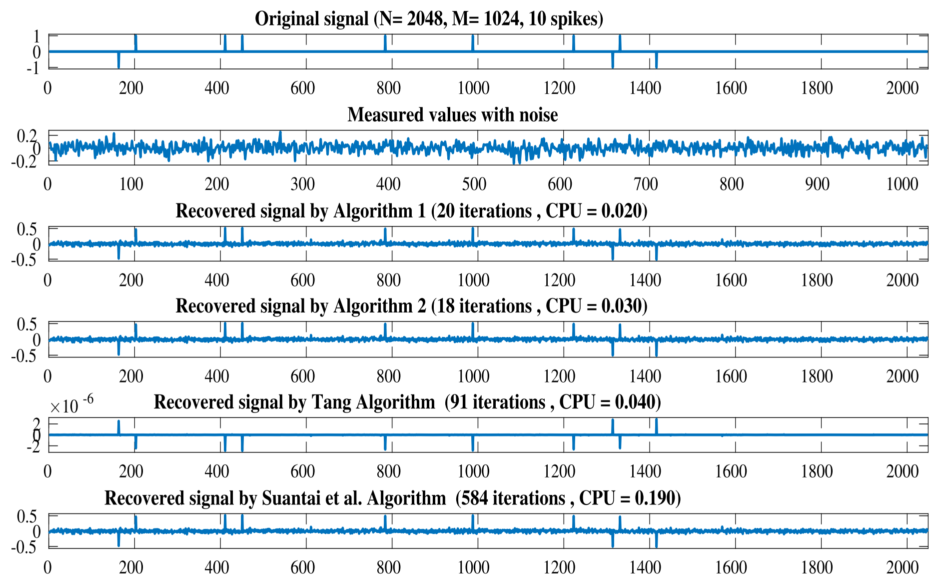

Figure 5 Restored signal via Algorithms 1, 2, Tang Algorithm, and Suantai et al. Algorithm for $K = 10$

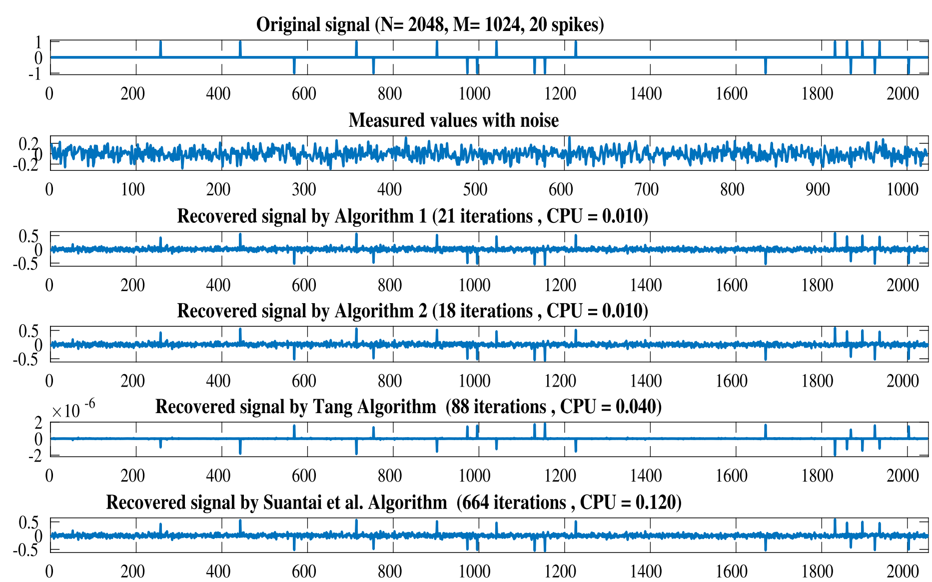

Figure 6 Restored signal via Algorithms 1, 2, Tang Algorithm, and Suantai et al. Algorithm for $K = 20$

where $t > 0$ is a given constant and $\|\cdot\|_1$ is the ℓ_1 norm. If $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q = \{y\}$, then (4.2) is a particular case of the SFP (3.31) in the finite-dimensional spaces.

We generated a sparse signal $x \in \mathbb{R}^N$ with K nonzero entries having a of length $N = 2048$ and made $M = 1024$ observations. The values of the sparse signal are sampled uniformly from the interval $[-1, 1]$. The observation y is generated from Gaussian noise of variance 10^{-4}. The matrix $T \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one variance. Additionally, the initial signals $x_0 = x_1 = T^*(Tx - y)$. In this experiment, we set $\tau_1 = 1.99$, $\beta = 0.75$, $\mu = 10^{-5}$, $\alpha_n = 0.1$, $s_n = \frac{1}{(n+1)^4}$, $\beta_n = \frac{1}{(n+1)^{1.1}}$, and $\tau_n = \mu_n = 0.01$ in

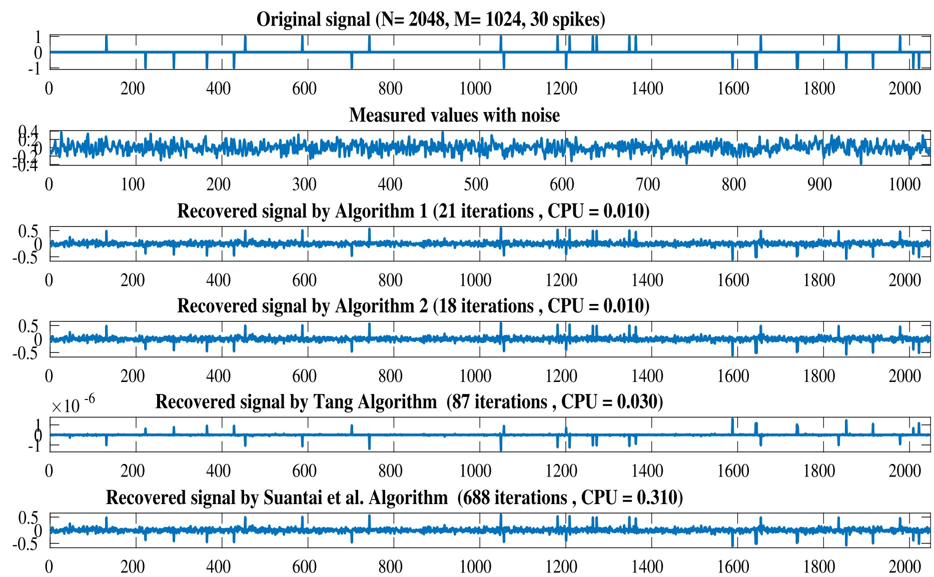

Figure 7 Restored signal via Algorithms 1, 2, Tang Algorithm, and Suantai et al. Algorithm for $K = 30$

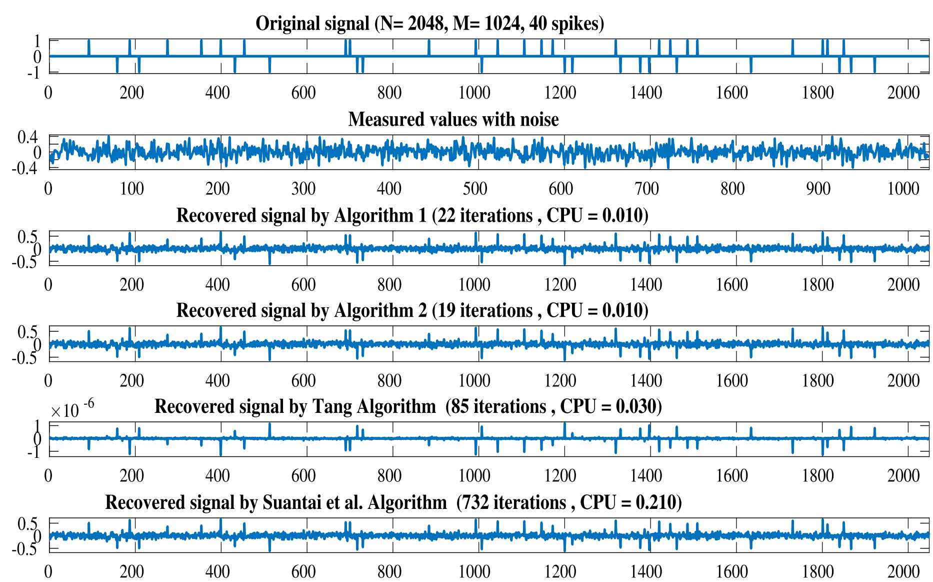

Figure 8 Restored signal via Algorithms 1, 2, Tang Algorithm, and Suantai et al. Algorithm for $K = 40$

Algorithms 1 and 2, we set $\alpha = 0.75$, $\rho_n = 0.5$, $\epsilon_n = \frac{1}{(n+1)^{1.1}}$, and $r = \mu = 0.001$ in Tang Algorithm [43] and we set $\alpha_n = \beta_n = \frac{1}{n+1}$, $u_n = Tx - y$, and $\lambda_n = 0.001$ in Suantai et al. Algorithm [39]. We consider five different tests for the spikes $K \in \{10, 20, 30, 40, 50\}$. Our stopping criterion is $E_n = \|x_{n+1} - x_n\| < 10^{-7}$. The results of the numerical simulations are presented in Figs. 5–9.

Remark 4.4 We observe from the numerical simulations presented in Figs. 5–9 that our proposed Algorithms 1 and 2 outperform Tang Algorithm [43] and Suantai et al. Algorithm [39] in the sense that they satisfy the stopping criteria in fewer iterations and less

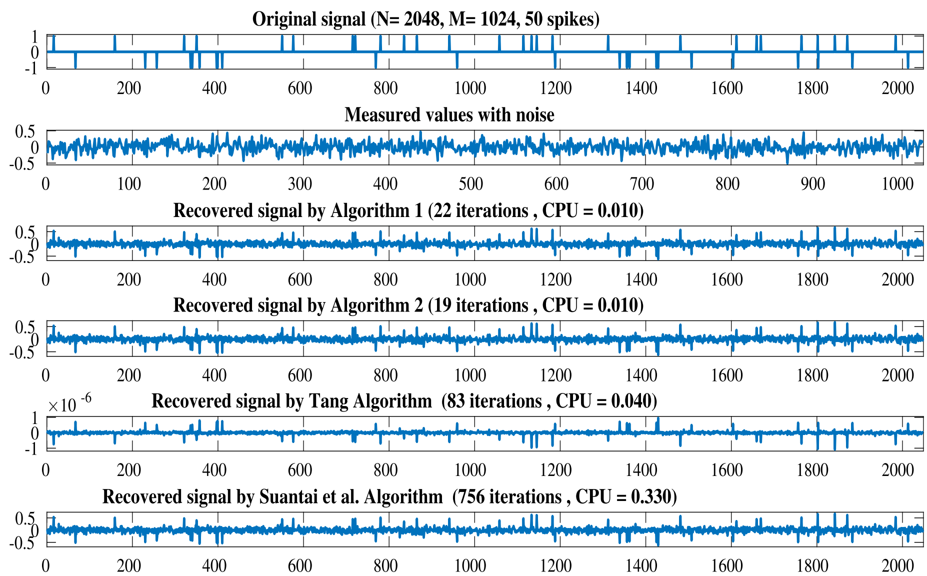

Figure 9 Restored signal via Algorithms 1, 2, Tang Algorithm, and Suantai et al. Algorithm for $K = 50$

computational time in the signal-recovery tests. Furthermore, we observe that while Tang Algorithm [43] requires fewer iterations to satisfy the stopping criteria compared to Suantai et al. Algorithm [39], the reconstructed signal by Tang Algorithm [43] is NOT very close to the original signal compared to that reconstructed via Suantai et al. Algorithm [39].

Problem 4.5 Data classifications

In this example, we apply our algorithms to data-classification problems, which are based on a learning technique called the extreme learning machine (ELM). Let $\mathcal{U} = \{(x_n, y_n) : x_n \in \mathbb{R}^N, y_n \in \mathbb{R}^M, n = 1, 2, 3, \dots, K\}$ be a training set of K distinct samples, x_n is an input training data and y_n is a training target. For the output of ELM with a single hidden layer at the ith hidden node is $h_i(x) = U(\langle a_i, x_n\rangle + b_i)$, where U is an activation function, a_i is the weight at the ith hidden node, and b_i is the bias at the ith hidden node. The output function with L hidden nodes is the single hidden-layer feedforward neural networks (SLFNs)

$$O_n = \sum_{i=1}^{L} \omega_i h_i(x_n),$$

where ω_i is the optimal output weight at the ith hidden node. The hidden-layer output matrix T is defined by

$$T = \begin{bmatrix} U(\langle a_1, x_1\rangle + b_1) & \cdots & U(\langle a_L, x_1\rangle + b_L) \\ \vdots & \ddots & \vdots \\ U(\langle a_1, x_K\rangle + b_1) & \cdots & U(\langle a_L, x_K\rangle + b_L) \end{bmatrix}.$$

The main aim of ELM is to calculate an optimal weight $\omega = (\omega_1, \omega_2, \dots, \omega_L)^{\mathsf{T}}$ such that $T\omega = b$, where $b = (t_1, t_2, \dots, t_K)^{\mathsf{T}}$ is the training target data. A successful model used to find the solution ω can be translated into the following convex constraint minimization problem:

$$\min_{\omega\in\mathbb{R}^L} \frac{1}{2}\|T\omega - b\|^2 \quad \text{subject to} \quad \|\omega\|_1 \le \xi, \tag{4.3}$$

where $\xi > 0$ is a given constant. If $C = \{\omega \in \mathbb{R}^L : \|\omega\|_1 \le \xi\}$ and $Q = \{b\}$, then (4.3) is a particular case of the SFP (3.31) in the finite-dimensional spaces.

The binary crossentropy loss function along with sigmoid activation function for binary classification calculates the loss of an example by computing the following average:

$$\text{Loss} = -\frac{1}{J}\sum_{j=1}^{J}\bigl(y_j \log \hat{y}_j + (1-y_j)\log(1-\hat{y}_j)\bigr), \tag{4.4}$$

where $\hat{y}_j$ is the jth scalar value in the model output, y_j is the corresponding target value, and J is the number of scalar values in the model output.

The performance evaluation in classification can be justified by precision and recall. The Recall/True Positive Rate can be defined as the level of accuracy of predictions in positive classes and the percentage of the number of predictions that are correct on the positive observations. Then, calculate the accuracy, prediction, and F1-score using the following standard criteria [22]:

(1) Precision = $\frac{\text{TP}}{\text{TP+FP}} \times 100\%$;
(2) Recall = $\frac{\text{TP}}{\text{TP+FN}} \times 100\%$;
(3) Accuracy = $\frac{\text{TP+TN}}{\text{TP+FP+TN+FN}} \times 100\%$;
(4) F1-score = $\frac{2\times\text{Precision}\times\text{Recall}}{\text{Precision+Recall}}$,

where a confusion matrix for original and predicted classes is shown in terms of TP := True Positive, TN := True Negative, FP := False Positive, and FN := False Negative.

Next, we consider the following two datasets:

Dataset 1 UCI Machine Learning Heart Disease dataset [20]. This dataset contains 14 attributes and 303 records. This dataset contains the attributes: Age, Gender, CP, Trestbps, Chol, Fbs, Restecg, Thalach, Exang, Oldpeak, Slope, Ca, Thal, and Num (the predicted attribute). The dataset consists of 138 normal instances versus 165 abnormal instances.

Dataset 2 PIMA Indians diabetes dataset [1]. The dataset contains 768 pregnant female patients of which 500 were nondiabetics and 268 that were diabetics. This dataset contains 9 attributes: Pregnancies, Glucose, Blood Pressure, Skin Thickness, Insulin, BMI, Diabetes Pedigree Function, Age, and Outcome (the predicted attribute).

In particular, we apply our algorithms to the optimized weight parameter in training data for machine learning by using 5-fold crossvalidation [25] in the extreme learning machine (ELM).

For Dataset 1, we start computation by setting the activation function as a sigmoid, hidden nodes $L = 80$, regularization parameter $\xi = 10$, $x_0 = \mathbf{1} := \underbrace{(1,1,\dots,1)}_{N} \in \mathbb{R}^N$, $x_1 = \mathbf{0} := \underbrace{(0,0,\dots,0)}_{N} \in \mathbb{R}^N$, $\tau_1 = 1$, $s_n = 0$, $\beta_n = \frac{1}{(n+1)^{10}}$, and $\alpha_n = \frac{n+2}{2n+1}$. The stopping criteria is the

Table 1 Numerical results of β for Algorithm 1 when $\mu = 0.9$ for Dataset 1

β	Training	Loss	
		Train	Test
0.1	0.0808	0.246457	0.224908
0.2	0.0816	0.246016	0.224521
0.3	0.0841	0.245528	0.224100
0.4	0.0824	0.244935	0.223599
0.5	0.0882	0.244198	0.222994
0.6	0.0839	0.243251	0.222248
0.7	0.0794	0.241977	0.221302
0.8	0.0833	0.240139	0.220054
0.9	0.0875	0.237138	0.218174

Table 2 Numerical results of β for Algorithm 2 when $\mu = 0.9$ for Dataset 1

β	Training	Loss	
		Train	Test
0.1	0.0775	0.247571	0.225920
0.2	0.0774	0.247902	0.226225
0.3	0.0771	0.248156	0.226459
0.4	0.0783	0.248382	0.226669
0.5	0.0755	0.248584	0.226858
0.6	0.0815	0.248767	0.227030
0.7	0.1114	0.248933	0.227185
0.8	0.0784	0.249083	0.227327
0.9	0.0800	0.249221	0.227457

Table 3 Numerical results of μ for Algorithm 1 when $\beta = 0.9$ for Dataset 1

μ	Training	Loss	
		Train	Test
0.1	0.0870	0.245923	0.224461
0.2	0.0846	0.243328	0.222218
0.3	0.0829	0.241784	0.221118
0.4	0.0830	0.240556	0.220319
0.5	0.0838	0.239611	0.219739
0.6	0.0872	0.238817	0.219257
0.7	0.0872	0.238143	0.218838
0.8	0.0870	0.237596	0.218483
0.9	0.0848	0.237138	0.218174

number of iterations 300. We compare the performance of Algorithms 1 and 2 with different parameters β for Dataset 1, as seen in Tables 1 and 2.

From Table 1, we see that β increases from 0.1 to 0.9. The training loss and test loss decrease, it appears that $\beta = 0.9$ performs better for Algorithm 1.

From Table 2, we see that the training loss and test loss increase when β increases, it appears that $\beta = 0.1$ performs better for Algorithm 2. Also, we compare the performance of Algorithms 1 and 2 with different parameters μ for Dataset 1, as seen in Tables 3 and 4.

From Tables 3 and 4, we see that the training loss and test loss decrease when μ increases, it show that $\mu = 0.9$ gives the highly improved the performance of Algorithm 1 and Algorithm 2 for Dataset 1.

Next, we compare the performance of our algorithms with Tang Algorithm [43] and Suantai et al. Algorithm [39] for Dataset 1. For Algorithms 1 and 2, we set $\tau_1 = 1$, $\mu = 0.9$, $s_n = 0$, and $\beta_n = \frac{1}{(n+1)^{10}}$. Moreover, we set $\beta = 0.9$ and $\beta = 0.1$ for Algorithms 1 and 2,

Table 4 Numerical results of μ for Algorithm 2 when $\beta = 0.1$ for Dataset 1

μ	Training	Loss	
		Train	Test
0.1	0.0775	0.250309	0.227553
0.2	0.0802	0.250964	0.229119
0.3	0.0778	0.250372	0.228554
0.4	0.1137	0.249819	0.228024
0.5	0.0883	0.249305	0.227536
0.6	0.0808	0.248827	0.227085
0.7	0.0787	0.248381	0.226668
0.8	0.1336	0.247963	0.226281
0.9	0.0805	0.247571	0.225920

Table 5 The performance of each algorithm for Dataset 1

Algorithms	Iterations	Training time	Precision	Recall	F1-score	Accuracy
Algorithm 1	49	0.0680	87.96	100	93.59	88.21
Algorithm 2	524	0.3763	87.96	100	93.59	88.21
Tang Algorithm [43]	2299	1.4604	87.96	100	93.59	88.21
Suantai et al. Algorithm [39]	1695	0.3028	87.96	100	93.59	88.21

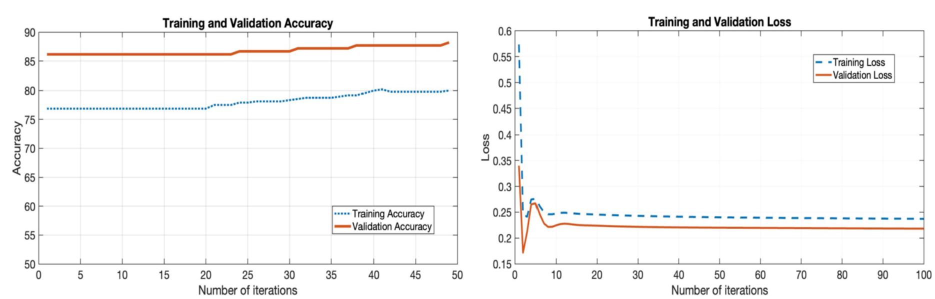

Figure 10 Accuracy and loss plots of Algorithm 1 for Dataset 1

respectively. For Tang Algorithm [43], we set $\alpha = 0.6$, $\rho_n = 3.5$, and $\epsilon_n = \frac{1}{n+1}$. For Suantai et al. Algorithm [39], we set $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n-1}{2n}$, $u_n = \mathbf{1} \in \mathbb{R}^N$, and $\lambda_n = \frac{1}{\|T\|^2}$.

The comparison of all algorithms is presented in Table 5.

From Table 5, we observe that our Algorithms 1 and 2 have fewer iterations than Tang Algorithm [43] and Suantai et al. Algorithm [39] with the same precision, recall, F1-score, and accuracy. This shows that our algorithms have the highest probability of correctly classifying heart disease compared to other algorithms.

Next, we present graphs of the accuracy and loss of training data and testing data for overfitting of Algorithms 1 and 2 to show that our algorithms have no overfitting in the training Dataset 1.

From Figs. 10 and 11, we see that our Algorithms 1 and 2 have suitably learned the training dataset for Dataset 1.

For Dataset 2, we present the comparison of our algorithms with Tang Algorithm [43] and Suantai et al. Algorithm [39]. We start computation by setting the activation function as sigmoid, hidden nodes $L = 160$. For Algorithms 1 and 2, we set $\tau_1 = 1$, $\mu = 0.9$, $s_n = 0$, and $\beta_n = \frac{1}{(n+1)^{10}}$. Moreover, we set $\beta = 0.9$ and $\beta = 0.1$ for Algorithms 1 and 2, respectively. For

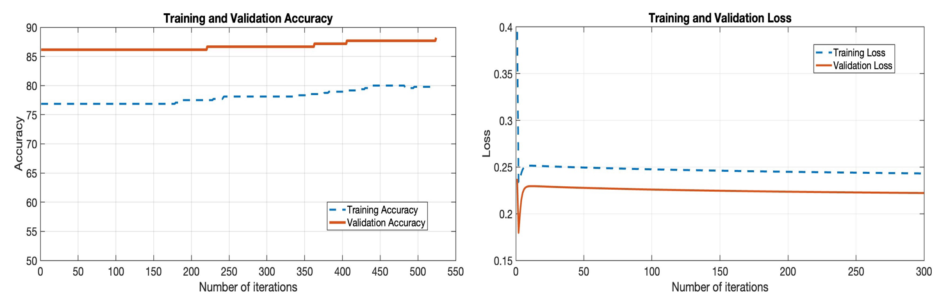

Figure 11 Accuracy and loss plots of Algorithm 2 for Dataset 1

Table 6 The performance of each algorithm for Dataset 2

Algorithms	Iterations	Training time	Precision	Recall	F1-score	Accuracy
Algorithm 1	14	0.0728	80.97	97.50	88.47	80.03
Algorithm 2	14	0.0612	80.97	97.50	88.47	80.03
Tang Algorithm [43]	33	0.0967	80.97	97.50	88.47	80.03
Suantai et al. Algorithm [39]	44	0.0620	80.97	97.50	88.47	80.03

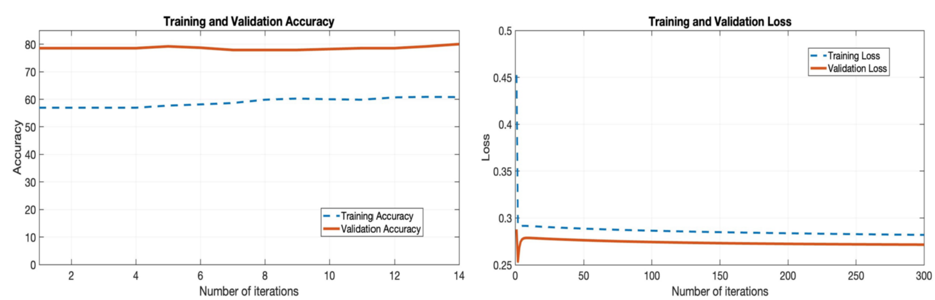

Figure 12 Accuracy and loss plots of Algorithm 1 for Dataset 2

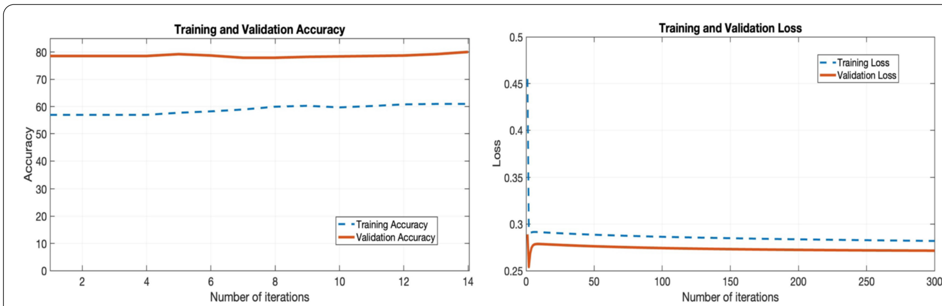

Figure 13 Accuracy and loss plots of Algorithm 2 for Dataset 2

Tang Algorithm [43], we set $\alpha = 0.6$, $\rho_n = 3.5$, and $\epsilon_n = \frac{1}{(n+1)^{10}}$. For Suantai et al. Algorithm [39], we set $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n-1}{2n}$, $u_n = \mathbf{1} \in \mathbb{R}^N$, and $\lambda_n = \frac{1}{\|T\|^2}$.

The comparison of all algorithms is presented in Table 6.

From Table 6, we see that Algorithms 1 and 2 have the most efficiency in precision, recall, F1-score, and accuracy for Dataset 2. This mean that our algorithms have the highest probability of correctly classifying the PIMA Indians diabetes dataset (Dataset 2) compared to Tang Algorithm [43] and Suantai et al. Algorithm [39].

Next, we present graphs of the accuracy and loss of training data and testing data for overfitting of Algorithms 1 and 2 to show that our algorithms have no overfitting in the training Dataset 2.

From Figs. 12 and 13, we see that Algorithms 1 and 2 have suitably learned the training dataset for Dataset 2.

5 Conclusions

In this paper, we have proposed two inertial self-adaptive algorithms to solve the split common null-point problem for two set-valued mappings in Banach spaces. The step sizes used in our proposed algorithms are adaptively updated without the prior knowledge of the operator norm of the bounded linear operator. We have proved the weak-convergence theorems of the proposed algorithms under suitable conditions in p-uniformly convex, real Banach spaces that are also uniformly smooth. Finally, we have performed experiments to numerically solve some problems in science and engineering, such as, the split feasibility problem, the split minimization problem, signal recovery, and data classifications, and also have compared them with some existing methods to demonstrate the implementability and efficiency of our methods.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization: P.S.; Writing-original draft: R.P., P.S.; Formal analysis: E.K., S.K.; Investigation: R.P., P.S.; Software: S.K., R.P.; Review and editing: P.S., R.P.; Project administration: P.S.; Funding: R.P., E.T. All authors have read and approved final version of the manuscript.

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References
1. https://www.kaggle.com/datasets/whenamancodes/predict-diabities
2. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Fixed Point Theory for Lipschitzian-Type Mappings with Applications. Springer, Berlin (2009)
3. Alber, Y., Ryazantseva, I.: Nonlinear Ill-Posed Problems of Monotone Type. Springer, Dordrecht (2006)
4. Alber, Y.I.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, pp. 15–50. Dekker, New York (1996)
5. Alofi, A.S., Alsulami, S.M., Takahashi, W.: Strongly convergent iterative method for the split common null point problem in Banach spaces. J. Nonlinear Convex Anal. **17**, 311–324 (2016)

6. Alvarez, F.: Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces. SIAM J. Optim. **14**, 773–782 (2004)
7. Alvarez, F., Attouch, H.: An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Var. Anal. **9**, 3–11 (2001)
8. Anh, P.K., Thong, D.V., Dung, V.T.: A strongly convergent Mann-type inertial algorithm for solving split variational inclusion problems. Optim. Eng. **22**, 159–185 (2021)
9. Aoyama, K., Kohsaka, F., Takahashi, W.: Three generalizations of firmly nonexpansive mappings: their relations and continuity properties. J. Nonlinear Convex Anal. **10**, 131–147 (2009)
10. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Bregman monotone optimization algorithms. SIAM J. Control Optim. **42**, 596–636 (2003)
11. Butnariu, D., Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. **2006**, Article ID 084919 (2006)
12. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**, 441–453 (2002)
13. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal. **13**, 759–775 (2012)
14. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
15. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in product space. Numer. Algorithms **8**, 221–239 (1994)
16. Chen, H.Y.: Weak and strong convergence of inertial algorithms for solving split common fixed point problems. J. Inequal. Appl. **2021**, 26 (2021)
17. Chuang, C.S.: Hybrid inertial proximal algorithm for the split variational inclusion problem in Hilbert spaces with applications. Optimization **66**, 777–792 (2017)
18. Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic, Dordrecht (1990)
19. Combette, P.L.: The convex feasibility problem in image recovery. Adv. Imaging Electron Phys. **95**, 155–270 (1996)
20. Dua, D., Graff, C.: UCI Machine Learning Repository. University of California, School of Information and Computer Science, Irvine (2019). http://archive.ics.uci.edu/ml
21. Duan, P., Zhang, Y., Bu, Q.: New inertial proximal gradient methods for unconstrained convex optimization problems. J. Inequal. Appl. **2020**, 255 (2020)
22. Han, J., Kamber, M., Pei, J.: Data Mining: Concepts and Techniques, vol. 10, 978-1. Kaufman, Waltham (2012)
23. Hanner, O.: On the uniform convexity of L_p and l_p. Ark. Mat. **3**, 239–244 (1956)
24. Kesornprom, S., Cholamjiak, K.: Proximal type algorithms involving linesearch and inertial technique for split variational inclusion problem in Hilbert spaces with applications. Optimization **68**, 2369–2395 (2019)
25. Kumari, V.A., Chitra, R.: Classification of diabetes disease using support vector machine. Int. J. Eng. Res. Appl. **3**, 1797–1801 (2013)
26. Kuo, L.W., Sahu, D.R.: Bregman distance and strong convergence of proximal-type algorithms. Abstr. Appl. Anal. **2013**, Article ID 590519 (2006)
27. Kutyniok, G.: Theory and applications of compressed sensing. GAMM-Mitt. **36**, 79–101 (2013)
28. López, G., Martin-Marquez, V., Wang, F., Xu, H.K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Probl. **28**, 085004 (2012)
29. Moudafi, M.: Split monotone variational inclusions. J. Optim. Theory Appl. **150**, 275–283 (2011)
30. Ogbuisi, F.U., Shehu, Y., Yao, J.C.: Convergence analysis of new inertial method for the split common null point problem. Optimization **71**, 3767–3795 (2022)
31. Reich, S.: A weak convergence theorem for the alternating method with Bregman distances. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 313–318. Dekker, New York (1996)
32. Reich, S., Tuyen, T.M., Sunthrayuth, P., Cholamjiak, P.: Two new inertial algorithms for solving variational inequalities in reflexive Banach spaces. Numer. Funct. Anal. Optim. **42**, 1954–1984 (2021)
33. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209–216 (1970)
34. Schöpfer, F.: Iterative regularization method for the solution of the split feasibility problem in Banach spaces. Ph.D. thesis, Saabrücken (2007)
35. Schöpfer, F., Louis, A.K., Schuster, T.: Nonlinear iterative methods for linear ill-posed problems in Banach spaces. Inverse Probl. **22**, 311–329 (2006)
36. Schöpfer, F., Schuster, T., Louis, A.K.: An iterative regularization method for the solution of the split feasibility problem in Banach spaces. Inverse Probl. **24**, 055008 (2008)
37. Shehu, Y., Iyiola, O.S., Ogbuisi, F.U.: Iterative method with inertial terms for nonexpansive mappings: applications to compressed sensing. Numer. Algorithms **83**, 1321–1347 (2020)
38. Suantai, S., Pholasa, N., Cholamjiak, P.: Relaxed CQ algorithms involving the inertial technique for multiple-sets split feasibility problems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**, 1081–1099 (2019)
39. Suantai, S., Shehu, Y., Cholamjiak, P.: Nonlinear iterative methods for solving the split common null point problem in Banach spaces. Optim. Methods Softw. **34**, 853–874 (2019)
40. Takahashi, E.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
41. Tan, B., Sunthrayuth, P., Cholamjiak, P., Cho, Y.J.: Modified inertial extragradient methods for finding minimum-norm solution of the variational inequality problem with applications to optimal control problem. Int. J. Comput. Math. **100**, 525–545 (2023)
42. Tan, K.K., Xu, H.K.: Approximating fixed points of nonexpensive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301–308 (1993)
43. Tang, Y.: New inertial algorithm for solving split common null point problem in Banach spaces. J. Inequal. Appl. **2019**, 17 (2019)
44. Tang, Y., Sunthrayuth, P.: An iterative algorithm with inertial technique for solving the split common null point problem in Banach spaces. Asian-Eur. J. Math. **15**, 2250120 (2022)

45. Tian, M., Jiang, B.N.: Inertial hybrid algorithm for variational inequality problems in Hilbert spaces. J. Inequal. Appl. **2020**, 12 (2020)
46. Tibshirani, R.: Regression shrinkage and selection via the LASSO. J. R. Stat. Soc. Ser. B **58**, 267–288 (1996)
47. Tropp, J.A.: A mathematical introduction to compressive sensing. Bull. Am. Math. Soc. **54**, 151–165 (2017)
48. Tuyen, T.M., Cholamjiak, P., Sunthrayuth, P.: A new self-adaptive method for the multiple-sets split common null point problem in Banach spaces. Vietnam J. Math. (2022). https://doi.org/10.1007/s10013-022-00574-3
49. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal., Theory Methods Appl. **16**, 1127–1138 (1991)
50. Xu, Z.B., Roach, G.F.: Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces. J. Math. Anal. Appl. **157**, 189–210 (1991)
51. Zhou, Z., Tan, B., Li, S.: Inertial algorithms with adaptive stepsizes for split variational inclusion problems and their applications to signal recovery problem. Math. Methods Appl. Sci. (2023). https://doi.org/10.1002/mma.9436

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A TECHNIQUE WITH DIMINISHING AND NON-SUMMABLE STEP-SIZE FOR MONOTONE INCLUSION PROBLEMS IN BANACH SPACES

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Abstract. In this paper, an algorithm for approximating zeros of sum of three monotone operators is introduced and its convergence properties are studied in the setting of 2-uniformly convex and uniformly smooth Banach spaces. Unlike the existing algorithms whose step-sizes usually depend on the knowledge of the operator norm or Lipschitz constant, a nice feature of the proposed algorithm is the fact that it requires only a diminishing and non-summable step-size to obtain strong convergence of the iterates to a solution of the problem. Finally, the proposed algorithm is implemented in the setting of a classical Banach space to support the theory established.

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1. Introduction

The following notions will appear frequently in this section. We will first introduce them for familiarity before we introduce the problem. Let E be a real Banach space with dual E^*. A mapping $A : E \to 2^{E^*}$ is called monotone if

$$\langle x - y, u - v\rangle \geq 0,\ u \in Ax,\ v \in Ay,\ \forall x, y \in E$$

and maximal monotone if its has no monotone extension. A mapping $A : E \to E^*$ is called β-strongly monotone if there exists $\beta > 0$ such that for all $x, y \in E$,

$$\langle x - y, Ax - Ay\rangle \geq \beta\|x - y\|^2.$$

It is also called β-cocoercive (or β-inverse strongly monotone) if

$$\langle x - y, Ax - Ay\rangle \geq \beta\|Ax - Ay\|^2.$$

It is called α-Lipschitz if there exists $\alpha > 0$ such that

$$\|Ax - Ay\| \leq \alpha\|x - y\|.$$

Monotone maps were studied by Minty [31], Zarantonello [42], Deepho et al. [23], Chidume et al. [15, 12], Muangchoo et al. [32] and many other authors in Hilbert spaces and more general Banach spaces. These mappings have caught the interest of many authors largely because they are useful in real-world applications, especially when it comes to solving convex optimization problems (see, e.g., [3, 4, 7, 18, 27, 28, 40, 41]).

Now, let $L : E \to 2^{E^*}$ be a set-valued map and $M, N : E \to E^*$ be single-valued maps. Consider the following inclusion problem:

$$\text{find} \;\; x \in E \;\; \text{such that} \;\; 0 \in (L + M + N)x. \tag{1.1}$$

The variational inclusion problem (1.1) popularly known as monotone inclusion problem, when the operators involved are monotone was first studied by Davis and Yin [21] in the setting of real Hilbert spaces, to the best of our knowledge (see [1, 2]). In theory, one may wonder why the interest in problem (1.1) since it can be redefined as $A := L + M + N$ and thus the problem is equivalent to finding a zero of A which the *proximal point algorithm (PPA)* and its variants have been used to solve such cases (see, e.g., [11, 14, 16, 17, 25, 33, 34]). We recall that the PPA of Martinet [30] involving a maximal monotone operator A generates its iterates by solving the recursive equation:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (I + \lambda_n A)^{-1}x_n, \end{cases}$$

where $\lambda_n > 0$, H is a real Hilbert space and I is the identity mapping on H. However, evaluating the resolvent of A, $(I + \lambda A)^{-1}$ can be challenging in practice especially when A is nonlinear. This challenge is what led to the introduction of problem (1.1). In the literature, several splitting methods have been proposed by many authors to overcome this challenge. The idea is to split A as sum of operators with so that the linear part of A can be used to compute the resolvent easily and other properties of the remaining operators can be exploited independently (see, e.g., [3, 5, 13, 22, 24, 26, 36, 43, 44]).

Davis and Yin [21] studied problem (1.1) due to its numerous applications in solving problems arising from optimization. In particular, they studied the following 3-objective optimization problem which is to find $x \in H$ such that

$$\min_x f(x) + g(x) + h(Bx), \tag{1.2}$$

where f, g and h are proper closed and convex functions, h is $\frac{1}{\beta}$-Lipschitz differentiable and B is a linear mapping. Davis and Yin [21] gave several interesting applications of problem (1.1). In fact, models arising from image inpainting which has to do with reconstructing missing regions in an image appear naturally as the 3-objective minimization problem (see, e.g., [20, 37]). In [21], Davis and Yin recast problem (1.2) to fit in the setting of problem (1.1) by setting $L = \partial f,\ M = \partial g$ and $N = \nabla(hB)$. Then, they introduced the following algorithm for solving problem (1.1) and established a weak convergence result:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases} \tag{1.3}$$

where $T = J_\lambda^L\big(2J_\lambda^M - I - \lambda_n N J_\lambda^M\big) + I - J_\lambda^M$, $J_{\lambda_n}^L = (I-\lambda L)^{-1}$, L and M are maximal monotone, N is β-cocoercive, $\{\lambda\} \subset (0, 2\beta)$, $\{\alpha_n\} \subset \Big(0, \frac{4\beta-\lambda}{2\beta}\Big)$.

Recently, using the idea of Tseng [38], Malistky and Tam [29] proposed a simple algorithm for solving problem (1.1) and established weak convergence result. Their algorithm is the following:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = J_\lambda^L\big(x_n - 2\lambda M x_n + \lambda M x_{n-1} + \lambda N x_n\big), \end{cases} \tag{1.4}$$

where L is maximal monotone, M is monotone and l_1-Lipschitz, N is monotone and l_2-cocoercive and $\lambda \in \big(0, \frac{2}{4l_1+l_2}\big)$.

Remark 1.1. The algorithm of Malistky and Tam [29] requires only one computation of the resolvent operator J_λ per iteration, which reduces the computational cost of implementing the algorithm. On the other hand, the method of Davis and Yin [21] requires the computation of the resolvent twice per iteration. In addition to this, one of the shortcomings of these methods is that the control parameters depend on the knowledge of the Lipschitz constant, which is difficult to compute. In most cases, estimations of the constants are used to implement the algorithm, which affects their performance.

Question A. Can an iteration process be developed that will address the shortcomings of algorithms (1.3) and (1.4) mentioned in Remark 1.1.

This question was answered in the affirmative by Hieu et al. [39] in the setting of Hilbert spaces. They introduced the following algorithm:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = J^L_{\lambda_n}\big(x_n - \lambda_n M x_n - \lambda_{n-1}(Mx_n - Mx_{n-1}) - \lambda_n N x_n\big), \end{cases} \tag{1.5}$$

where L is maximal monotone, M is α-strongly monotone and l_1-Lipschitz continuous and N is β-cocoercive, $\{\lambda_n\} \subset (0,\infty)$ such that $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. Then $\{x_n\}$ converges strongly to a solution of problem (1.1).

Motivated by Question A and the results of Hieu et al. [39], it is our purpose in this paper to provide an affirmative answer to Question A in the setting of real Banach spaces. Furthermore, we will provide some numerical illustrations to compare the performance of the algorithms of Davis and Yin [21], Malistky and Tam [29] and our proposed algorithm in the setting of Hilbert spaces. Furthermore, we will give a numerical illustration in the setting of the classical Banach space $\ell_{1.5}$ to support the theory we established. Finally, our proposed method extend and generalize many iterative techniques for approximating zeros of sum of two monotone operators in the setting of real Banach spaces.

2. Preliminaries

The following definitions and lemmas will be needed in the proof of our main theorem.

Definition 2.1. Let E be a strictly convex and smooth real Banach space. For $p > 1$, define $J_p : E \to 2^{E^*}$ by

$$J_p(x) := \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|^{p-1}\}.$$

J_p is called the *generalized duality map on E.* If $p = 2$, then $J_2 := J$ is called the *normalized duality map* and is denoted by J.

In a real Hilbert space H, J is the identity map on H. It is easy to see from the definition that

$$J_p(x) = \|x\|^{p-2}Jx \quad \text{and} \quad \langle x, J_p(x)\rangle = \|x\|^p,\ \forall x \in E.$$

It is well known that if E is smooth, then J is single-valued and if E is strictly convex, J is one-to-one, and J is surjective if E is reflexive.

The next definition is for the Lyapunov functional ϕ introduced by Alber and Ryazantseva [8]. It is useful for estimations involving J and its inverse J^{-1} on smooth Banach space.

Definition 2.2. Let E be a real Banach space that is smooth and $\phi : E \times E \to \mathbb{R}$ be a map. The Lyapunov functional ϕ is defined by

$$\phi(x,y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall\, x, y \in X.$$

Observe that if E is a real Hilbert space, (2.1) reduces to

$$\phi(x,y) = \|x-y\|^2, \forall x, y \in E.$$

The next definition is for the resolvent operator in the setting of a real Banach space.

Definition 2.3. Let E be a reflexive, strictly convex and smooth real Banach space and let $B : E \to 2^{E^*}$ be a maximal monotone operator. Then for any $\lambda > 0$ and $u \in E$, there exists a unique element $u_\lambda \in E$ such that $Ju \in Ju_\lambda + \lambda B u_\lambda$. The element u_λ is called the resolvent of B and it is denoted by $J_\lambda^B u$. Alternatively, $J_\lambda^B = \big(J + \lambda B\big)^{-1} J$ for all $\lambda > 0$.

It is easy to verify that $B^{-1}0 = F(J_\lambda^B)$ for all $\lambda > 0$, where $F(J_\lambda^B)$ denotes the set of fixed points of J_λ^B.

The next two lemmas will play a central role in establishing the strong convergence of the sequence generated by our proposed algorithm.

Lemma 2.4. ([9]) *Let E be a uniformly convex and smooth real Banach space. Then the following inequalities holds:*

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \ \forall x, y, z \in E.$$

Lemma 2.5. ([9]) *Let E be a 2-uniformly convex real Banach space. Then there exists $\kappa > 0$ such that*

$$\frac{1}{\kappa}\|x-y\|^2 \le \phi(x,y), \ \forall x, y \in E.$$

3. Main result

Algorithm 3.1. (Three Operator Splitting Algorithm)

Step 0. Choose $x_0, x_1 \in E$ and $\{\lambda_n\} \subset (0,1)$ such that $\lim\limits_{n\to\infty} \lambda_n = 0$ and $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$.

Step 1. Having $x_{n-1},\ x_n$, compute

$$x_{n+1}=\big(J+\lambda_n L\big)^{-1}\big(Jx_n-\lambda_n Mx_n-\lambda_{n-1}(Mx_n-Mx_{n-1})-\lambda_n Nx_n\big). \quad (3.1)$$

Step 2. If $\max\{\|x_{n+1}-x_n\|,\ \|x_n-x_{n-1}\|\}<\epsilon$ for any $\epsilon>0$, STOP else set $n=n+1$ and return to Step 1.

Theorem 3.2. *Let E be a 2-uniformly convex and uniformly smooth real Banach space. Let $L: E\to 2^{E^*}$ be a maximal monotone operator, $M: E\to E^*$ be an η-strongly monotone and γ-Lipschitz operator and $N: E\to E^*$ be μ-inverse strongly monotone. Let $\{x_n\}$ be a sequence in E generated by Algorithm* 3.1. *Then $\{x_n\}$ converges strongly to a solution of problem* (1.1).

Proof. Let x^* be a solution of problem (1.1). Observe that from (3.1), we have

$$Jx_n-\lambda_n Mx_n-\lambda_{n-1}(Mx_n-Mx_{n-1})-\lambda_n Nx_n\in Jx_{n+1}+\lambda_n Lx_{n+1}.$$

Set

$$\begin{aligned}w_n&=Jx_n-Jx_{n+1}-\lambda_n Mx_n-\lambda_{n-1}(Mx_n-Mx_{n-1})-\lambda_n Nx_n\\&\in\lambda_n Lx_{n+1}.\end{aligned}$$

Furthermore, since x^* is a solution, we get

$$w^*=-\lambda_n Mx^*-\lambda_n Nx^*\in\lambda_n Lx^*.$$

Therefore, by the monotonicity of L, we have that

$$\langle w_n-w^*,x_{n+1}-x^*\rangle\ge 0.$$

That is

$$\begin{aligned}&\langle Jx_n-Jx_{n+1},x_{n+1}-x^*\rangle-\lambda_n\langle Mx_n-Mx^*,x_{n+1}-x^*\rangle\\&-\lambda_{n-1}\langle Mx_n-Mx_{n-1},x_{n+1}-x^*\rangle-\lambda_n\langle Nx_n-Nx^*,x_{n+1}-x^*\rangle\ge 0.\end{aligned}\quad (3.2)$$

We estimate the first three terms of inequality (3.2) above as follows. By Lemma 2.4,

$$2\langle Jx_{n+1}-Jx_n,x^*-x_{n+1}\rangle=\phi(x^*,x_n)-\phi(x^*,x_{n+1})-\phi(x_{n+1},x_n). \quad (3.3)$$

Also, using the η-strong monotonicity of M, we get

$$\begin{aligned}\langle Mx_n-Mx^*,x_{n+1}-x^*\rangle&=\langle Mx_{n+1}-Mx^*,x_{n+1}-x^*\rangle\\&\quad+\langle Mx_n-Mx_{n+1},x_{n+1}-x^*\rangle\\&\ge\eta\|x_{n+1}-x^*\|^2+\langle Mx_n-Mx_{n+1},x_{n+1}-x^*\rangle.\end{aligned}\quad (3.4)$$

Furthermore,

$$\begin{aligned}\langle Mx_n-Mx_{n-1},x^*-x_{n+1}\rangle&=\langle Mx_n-Mx_{n-1},x^*-x_n\rangle\\&\quad+\langle Mx_n-Mx_{n-1},x_n-x_{n+1}\rangle.\end{aligned}\quad (3.5)$$

Substituting, equations (3.3) and (3.5), inequality (3.4) in inequality (3.2), we get

$$
\begin{aligned}
0 \leq\, & \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n) - 2\lambda_n \eta \|x_{n+1} - x^*\|^2 \\
& - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle + 2\lambda_{n-1} \langle Mx_n - Mx_{n-1}, x^* - x_n \rangle \\
& + 2\lambda_{n-1} \langle Mx_n - Mx_{n-1}, x_n - x_{n+1} \rangle - 2\lambda_n \langle Nx_n - Nx^*, x_{n+1} - x^* \rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
2\lambda_n \eta \|x_{n+1} - x^*\|^2 \leq\, & \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n) \\
& - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle \\
& + 2\lambda_{n-1} \langle Mx_{n-1} - Mx_n, x_n - x^* \rangle \\
& + 2\gamma \lambda_{n-1} \|x_n - x_{n-1}\| \|x_{n+1} - x_n\| \\
& - 2\lambda_n \langle Nx_n - Nx^*, x_{n+1} - x^* \rangle. \qquad (3.6)
\end{aligned}
$$

Next, we estimate the last term in inequality (3.6). Now,

$$
\begin{aligned}
2\langle Nx_n - Nx^*, x_{n+1} - x^* \rangle &= 2\langle Nx_n - Nx^*, x_{n+1} - x_n \rangle \\
&\quad + 2\langle Nx_n - Nx^*, x_n - x^* \rangle \\
&\geq -2\|Nx_n - Nx^*\| \|x_{n+1} - x_n\| \\
&\quad + 2\mu \|Nx_n - Nx^*\|^2 \\
&\geq -2\mu \|Nx_n - Nx^*\|^2 - \frac{1}{2\mu} \|x_{n+1} - x_n\|^2 \\
&\quad + 2\mu \|Nx_n - Nx^*\|^2 \\
&= -\frac{1}{2\mu} \|x_{n+1} - x_n\|^2.
\end{aligned}
$$

Substituting this inequality in (3.6) and using Lemma 2.5, we get

$$
\begin{aligned}
2\lambda_n \eta \|x_{n+1} - x^*\|^2 \leq\, & \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n) \\
& - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle \\
& + 2\lambda_{n-1} \langle Mx_{n-1} - Mx_n, x_n - x^* \rangle + \gamma \lambda_{n-1} \|x_n - x_{n-1}\|^2 \\
& + \gamma \lambda_{n-1} \|x_{n+1} - x_n\|^2 + \frac{\lambda_n}{2\mu} \|x_{n+1} - x_n\|^2 \\
\leq\, & \phi(x^*, x_n) + \gamma \lambda_{n-1} \|x_n - x_{n-1}\|^2 \\
& + 2\lambda_{n-1} \langle Mx_{n-1} - Mx_n, x_n - x^* \rangle - \phi(x^*, x_{n+1}) \\
& - \left(\frac{1}{\kappa} - \gamma \lambda_{n-1} - \frac{\lambda_n}{2\mu} \right) \|x_{n+1} - x_n\|^2 \\
& - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle. \qquad (3.7)
\end{aligned}
$$

Let $\Theta_n = \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 + 2\lambda_{n-1}\langle Mx_{n-1} - Mx_n, x_n - x^*\rangle$. Then inequality (3.7) can rewritten as

$$2\lambda_n\eta\|x_{n+1} - x^*\|^2 + \Big(\frac{1}{\kappa} - \gamma\lambda_{n-1} - \frac{\lambda_n}{2\mu} - \lambda_n\gamma\Big)\|x_{n+1} - x_n\|^2 \leq \Theta_n - \Theta_{n+1}. \tag{3.8}$$

Observe that

$$\begin{aligned}\Theta_n &= \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 + 2\lambda_{n-1}\langle Mx_{n-1} - Mx_n, x_n - x^*\rangle \\ &\geq \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 - 2\gamma\lambda_{n-1}\|x_{n-1} - x_n\|\|x_n - x^*\| \\ &\geq \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 \\ &\quad - \gamma\lambda_{n-1}\|x_{n-1} - x_n\|^2 - \gamma\lambda_{n-1}\|x_n - x^*\|^2 \\ &\geq \Big(\frac{1}{\kappa} - \gamma\lambda_{n-1}\Big)\|x_n - x^*\|^2. \end{aligned} \tag{3.9}$$

Let $\theta \in (0, \frac{1}{\kappa})$ be fixed. Since $\lambda_n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty}\Big(\frac{1}{\kappa} - \gamma\lambda_{n-1} - \frac{\lambda_n}{2\mu} - \gamma\lambda_n\Big) = \frac{1}{\kappa} > \theta.$$

Thus, there exists $n_0 \geq 1$ such that

$$\Big(\frac{1}{\kappa} - \gamma\lambda_{n-1} - \frac{\lambda_n}{2\mu} - \gamma\lambda_n\Big) \geq \theta,\ \forall n \geq n_0.$$

In addition,

$$\frac{1}{\kappa} - \gamma\lambda_{n-1} \geq \theta,\ \forall n \geq n_0.$$

Thus, $\{\Theta_n\}$ is nonnegative. Hence,

$$2\lambda_n\eta\|x_{n+1} - x^*\|^2 + \theta\|x_{n+1} - x_n\|^2 \leq \Theta_n - \Theta_{n+1},\ \forall\ n \geq n_0.$$

Thus, the sequence $\{\Theta_n\}$ is non-increasing. Consequently, the limit of $\{\Theta_n\}$ exists. Thus, from (3.9), we can conclude that $\{x_n\}$ is bounded.

Next, we show that $\{x_n\}$ converges strongly to a solution of problem (1.1). From (3.8), taking the finite sum of both sides, we have

$$\sum_{k=n_0}^{N}\big(2\lambda_k\gamma\|x_{k+1} - x^*\|^2 + \epsilon\|x_{k+1} - x_k\|^2\big) \leq \Theta_{n_0} - \Theta_{k+1} \leq \Theta_{n_0}.$$

Thus,

$$\sum_{k=n_0}^{\infty}\big(2\lambda_k\gamma\|x_{k+1} - x^*\|^2 + \epsilon\|x_{k+1} - x_k\|^2\big) \leq \Theta_{n_0} - \lim_{k\to\infty}\Theta_{k+1} \leq \Theta_{n_0}.$$

This implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\|^2 = 0 \quad \text{and} \quad \sum_{n=n_0}^{\infty} \lambda_n \|x_{n+1} - x^*\|^2 < \infty. \tag{3.10}$$

It follows from inequality (3.10) and the fact that $\sum_{n=n_0}^{\infty} \lambda_n = \infty$ that

$$\liminf_{n\to\infty} \|x_{n+1} - x^*\|^2 = 0$$

and thus

$$\liminf_{n\to\infty} \phi(x^*, x_{n+1}) = 0.$$

We recall that

$$\begin{aligned}\Theta_{n+1} = \phi(x^*, x_{n+1}) &+ \lambda_n \gamma \|x_{n+1} - x_n\|^2 \\ &+ 2\lambda_n \langle Mx_n - Mx_{n+1} - x_{n+1} - x^* \rangle.\end{aligned}$$

Using equation (3.10), the boundedness of $\{x_n\}$, the Lipschitz continuity of M and the fact that $\lambda_n \to 0$, we have

$$\lim_{n\to\infty} \big(\lambda_n \gamma \|x_{n+1} - x_n\|^2 + 2\lambda_n \langle Mx_n - Mx_{n+1} - x_{n+1} - x^* \rangle\big) = 0.$$

This implies that

$$\lim_{n\to\infty} \Theta_{n+1} = \lim_{n\to\infty} \phi(x^*, x_{n+1}).$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0.$$

This completes the proof. □

Remark 3.3. Theorem 3.2 extends and generalizes many established results in a Banach space that are 2-uniformly convex and uniformly smooth in the literature. It extends:

(1) Theorem 3.1 of Bello et al. [10]. In the sense that their *weak* convergence result can be modified to obtain strong convergence by just setting $N \equiv 0$ in Algorithm 3.1. Furthermore, the dependency of their step-size on the Lipschitz constant of one of the operators can be dispensed with by using the non-summable and diminishing step size we used in Algorithm 3.1 and using our method of proof in Theorem 3.2.

(2) Algorithm 3.3 of Shehu [35], Algorithm 1 of Cholamjiak et al. [19], Algorithm 3.12 of Adamu et al. [6] and other Tseng-type algorithms in the literature. In the sense that the number of function evaluation in the algorithm can be reduced and by just setting $N \equiv 0$ in our proposed Algorithm 3.1 and using our idea of proof the dependency of the step-size on the Lipschitz constant can be dispensed with.

4. Numerical illustrations and applications

In this section, we will give two numerical examples to compare the performance of our proposed algorithm and that of Davis and Yin [21], Malistky and Tam [29] in solving problem (1.1).

Example 4.1. Let A be an $n \times n$ symmetric and positive definite matrix (spdm). Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Lx := Ax$. Then L is maximal monotone. Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Mx := Ax + b$, $b \in \mathbb{R}^n$. Then, M is γ-strongly monotone. Let $N : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Nx = Ax$. Then N is μ-cocoercive. To implement Algorithms (1.3), (1.4) and (3.1), we will use a particular spdm to define L, M and N on $\mathbb{R}^3$. In algorithms (1.3), (1.4) and (3.1), set

$$
\begin{aligned}
Lx &= \begin{pmatrix} 3 & -2 & 0 \\ -1 & 4 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\
Mx &= \begin{pmatrix} 3 & -2 & 0 \\ -1 & 4 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \\
Nx &= \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\end{aligned}
$$

It is not difficult to verify that the coefficient matrices are symmetric and positive definite. Furthermore, it is easy to verify that L is maximal monotone, M is 1-strongly monotone and 5-Lipschitz continuous (where 5 is an estimated value of the Lipschitz constant) and thus, M is maximal monotone. In addition, N is 2-cocoercive and therefore, it is $\frac{1}{2}$-Lipschitz continuous. Moreover, the solution is $x^* = (-0.14, -0.05, -0.35)^T$.

In Algorithm (1.3), we set $\lambda \in (0, 4)$ to be the sequence $\lambda_n = \frac{1}{n+1}$, because we observe using this choice the algorithm gives a better approximation and $\{\alpha_n\} \subset \left(0, \frac{8-\lambda}{4}\right)$ to be $\frac{2n}{n+1}$. In Algorithm (1.4), we set $\lambda \in (0, 0.17)$; to be $\lambda = 0.01$ finally, in our proposed algorithm, we choose $\lambda_n = \frac{1}{n+1}$. Clearly, these parameters satisfy the hypothesis of Algorithms (1.3), (1.4) and (3.1), respectively. To test the robustness of the algorithms, we vary the starting points as follows:

Test 1: $x_0 = (1, 1, 0)^T$ and $x_1 = (-2, 0.5, 1)^T$;

Test 2: $x_0 = (0, 0, 0)^T$ and $x_1 = (0.5, 0.6, -0.7)^T$;

Test 3: $x_0 = (-1, 3, -5)^T$ and $x_1 = (0, -2, 4)^T$;

Test 4: $x_0 = (\frac{2}{3}, \frac{3}{5}, \frac{5}{7})^T$ and $x_1 = (1, 2, 3)^T$.

Setting maximum number of iterations $n = 300$, the results obtained from the simulations are reported in Table 1 and Figures 1 and 2.

TABLE 1. Numerical results with different starting points for Example 4.1

	Numerical Results of $\|x_n - x^*\|$ for Example 4.1											
	Algorithm (1.3)				Algorithm (1.4)				Algorithm (3.1)			
n	Test 1	Test 2	Test 3	Test 4	Test 1	Test 2	Test 3	Test 4	Test 1	Test 2	Test 3	Test 4
1	2.36	0.97	4.77	4.09	2.36	0.97	4.77	4.09	2.36	0.97	4.77	4.09
2	0.95	0.57	2.05	1.81	2.34	0.96	4.73	4.07	5.01	1.79	1.61	2.86
3	0.51	0.51	0.67	0.74	2.31	0.95	4.66	4.01	6.97	2.79	2.62	3.26
4	0.41	0.41	0.41	0.45	2.27	0.94	4.58	4.04	8.25	3.61	4.03	1.91
5	0.32	0.32	0.32	0.33	2.24	0.93	4.51	3.98	8.81	4.28	5.22	1.56
10	0.16	0.16	0.16	0.16	2.10	0.87	4.17	3.84	4.08	2.19	2.87	0.51
30	0.057	0.057	0.057	0.057	1.61	0.69	3.13	3.31	3.9E-4	1.4E-4	8.7E-5	6.2E-4
50	0.034	0.034	0.034	0.034	1.23	0.54	2.41	2.85	1.1E-5	7.7E-5	7.3E-5	1.3E-4
100	0.017	0.017	0.017	0.017	0.63	0.32	1.37	1.94	7.2E-5	7.1E-5	7.1E-5	7.3E-5
150	0.011	0.011	0.011	0.011	0.31	0.23	0.82	1.31	7.1E-5	7.1E-5	7.1E-5	7.2E-5
200	8.7E-3	8.7E-3	8.7E-3	8.7E-3	0.17	0.21	0.48	0.86	7.1E-5	7.1E-5	7.1E-5	7.1E-5
300	5.8E-3	5.8E-3	5.8E-3	5.8E-3	0.18	0.24	0.14	0.32	7.1E-5	7.1E-5	7.1E-5	7.1E-5

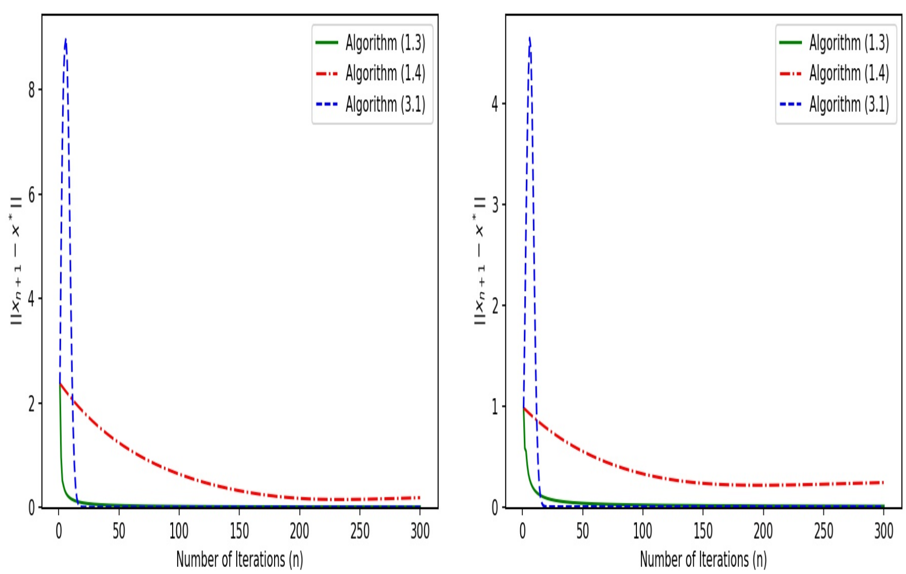

FIGURE 1. Graphical Simulations of Tests 1 and 2 for Algorithms (1.3), (1.4) and (3.1)

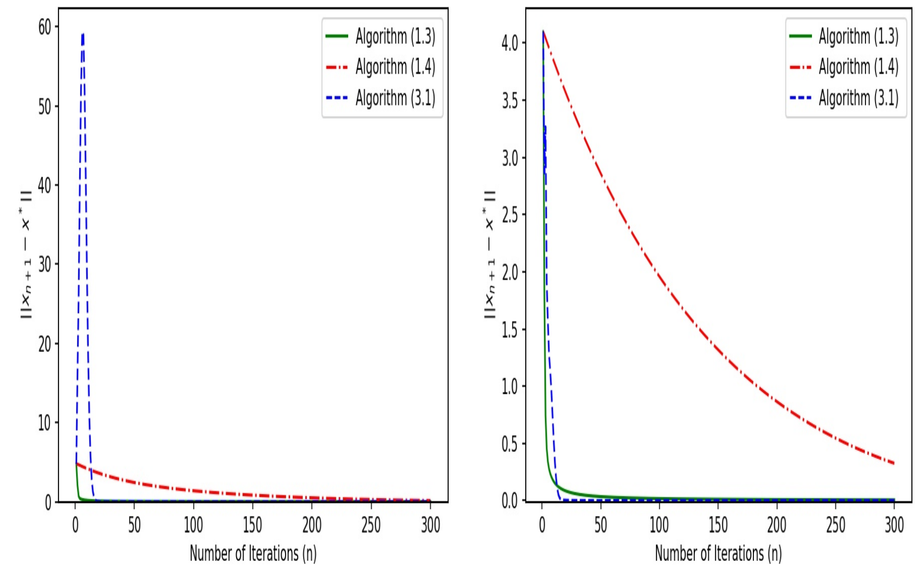

FIGURE 2. Graphical Simulations of Tests 3 and 4 for Algorithms (1.3), (1.4) and (3.1)

In the next example we are going to implement Algorithm 3.1 in the setting of a classical Banach space. Since the theorems of Davis and Yin [21], Malistky and Tam [29] we established in Hilbert spaces, we will not compare the performance of our algorithm with their algorithm in this example.

Example 4.2. In this example, we are going to implement Algorithm (3.1) on the subspace ℓ_p^0 of ℓ_p which consist of finitely many nonzero elements. We recall that

$$\ell_p = \big\{\{x_n\} \subset \mathbb{R} : \sum_{i=1}^{\infty} |x_i| < \infty\big\},$$

$$\ell_p^0 = \big\{\{x_n\} \in \ell_p : \{x_n\} = \{x_1, x_2, \cdots, x_k, 0, 0, 0, \cdots\}\big\}.$$

We also recall that for $1 < p \leq 2$, ℓ_p spaces are 2-uniformly convex and uniformly smooth. Let $p = 1.5$, $k = 4$. Consider $\ell_{1.5}^o$ with dual space ℓ_3^0. Its is well known that if $1 < p < q < \infty$, $\ell_p \subset \ell_q$. Thus, $\ell_{1.5}^0 \subset \ell_3^0$. Following Alber [8] the duality $J_{1.5}$ map and its inverse J_3 on these subspaces are

$$\begin{aligned}
&J_{1.5}(x) = \|x\|_{\ell_{1.5}}^{0.5} y \in \ell_3^0,\\
&y = \{|x_1|^{-0.5}x_1, |x_2|^{-0.5}x_2, |x_3|^{-0.5}x_3, |x_4|^{-0.5}x_4, 0, 0, \cdots\},\\
&x = \{x_1, x_2, x_3, x_4, 0, 0, \cdots\}
\end{aligned}$$

and

$$J_3(x) = \|x\|_{\ell_3}^{-1} y \in \ell_{1.5}^0,$$
$$y = \{|x_1|x_1, |x_2|x_2, |x_3|x_3, |x_4|x_4, 0, 0, \cdots\},$$
$$x = \{x_1, x_2, x_3, x_4, 0, 0, \cdots\},$$

where

$$\|x\|_{\ell_{1.5}} = \Big(\sum_{i=1}^{4} |x_i|^{1.5}\Big)^{\frac{1}{1.5}}$$

and

$$\|x\|_{\ell_3} = \Big(\sum_{i=1}^{4} |x_i|^{3}\Big)^{\frac{1}{3}}.$$

Remark 4.3. Observe that if $x_i = 0$, $i \in \{1, 2, 3, 4\}$, $J_{1.5}$ is NOT well-defined.

In MATLAB, we constructed a function that returns 0 when $x = \{0, 0, \cdots\}$ and it returns 1 when $x_i = 0$ in computing $|x_i|^{-0.5}$. The following is obtained for testing the function:

$$J_{1.5}(\{1, 0, 3, -0.5, 0, 0, \cdots\}) = \{1.8710, 0, 3.2407, -1.3230, 0, 0, \cdots\}.$$

This new function we constructed took care of the problem raised in Remark 4.3. Now, we are ready to implement Algorithm (3.1) on $\ell_{1.5}^0$.

In Algorithm (3.1), let $L, M, N : \ell_{1.5}^0 \to \ell_3^0$ be defined by

$$Lx = 2J_{1.5}(x),$$

$$Mx = 2J_{1.5}(x)$$

and

$$Nx = \{2|x_1|, 2^2|x_2|, 2^3|x_3|, 2^4|x_4|, 0, 0, \cdots\}.$$

Setting $\lambda_n = \frac{1}{n+1}$ and maximum number of iterations $n = 500$. To test the robustness of the algorithms, we vary the starting points as follows:

Test 1: $x_0 = \{1, 0, 3, -0.5, 0, 0, \cdots\}$ and $x_1 = \{2, 3, 0, 1, 0, 0, \cdots\}$;

Test 2: $x_0 = \{-0.1, -0.2, 0.3, 0.4, 0, 0, \cdots\}$ and $x_1 = \{2, 4, 6, 8, 0, 0, \cdots\}$.

The results of the numerical simulations are presented in Table 2 and Figure 3 below:

TABLE 2. Numerical results with different starting points for Example 4.2

Numerical Results of $\|x_{n+1} - x_n\|_{\ell_3}$ for Example 4.2		
	Test 1	Test 2
n	$\|x_{n+1} - x_n\|_{\ell_3}$	$\|x_{n+1} - x_n\|_{\ell_3}$
1	5.5936	12.8774
2	12.9783	56.0761
50	1.85E-3	2.07E-3
100	2.43E-4	2.73E-4
150	7.32E-5	8.17E-5
200	3.15E-5	3.46E-5
250	1.71E-5	1.79E-5
300	1.09E-5	1.06E-5
350	7.75E-6	7.09E-6
400	5.89E-6	5.09E-6
450	4.72E-6	3.87E-6
500	3.93E-6	3.08E-6

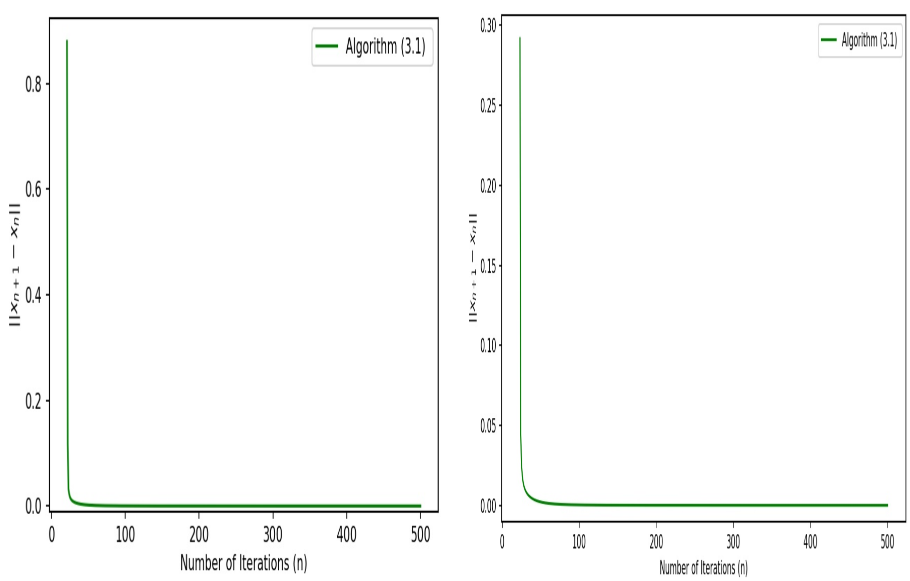

FIGURE 3. Graphical Simulations of Tests 1 and 2 for Algorithm (1.3)

Conclusion: This paper presents an algorithm with diminishing and non-summable step-size for approximating zeros of sum of three monotone operators in the setting of a real Banach space. A nice and interesting feature of the proposed algorithm is the fact that the step-size does not depend on the knowledge of Lipschitz or cocoercive constant of any of the operators involved. The fact that the approach used in dispensing this dependency does not follow the well-known approaches in the literature made the method of proof of convergence new, technical and innovative. To the best of our knowledge, this is the first paper that considered the inclusion problem (1.1) in the setting of Banach spaces.

Furthermore, numerical illustrations are presented to support the theory established in the paper. Finally, the proposed method extends and generalizes several methods established in the literature for approximating zeros of sum of two monotone operators as seen in Remark 3.3.

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References

[1] H. A. Abass, O. K. Narain and O. M. Onifade, *Inertial extrapolation method for solving systems of monotone variational inclusion and fixed point problems using Bregman distance approach,* Nonlinear Funct. Anal. Appl., **28**(2) (2023), 497-520.

[2] J. A. Abuchu, G. C. Ugunnadi and O. K. Narain, *Inertial proximal and contraction methods for solving monotone variational inclusion and fixed point problems,* Nonlinear Funct. Anal. Appl., **28**(1) (2023), 175-203.

[3] A. Adamu, J. Deepho, A.H. Ibrahim and A.B. Abubakar, *Approximation of zeros of sum of monotone mappings with applications to variational inequality and image restoration problems*, Nonlinear Funct. Anal. Appl., **26**(2) (2021), 411–432.

[4] A. Adamu, D. Kitkuan, P. Kumam, A. Padcharoen and T. Seangwattana, *Approximation method for monotone inclusion problems in real Banach spaces with applications*, J. Inequal. Appl., **2022**(70) (2022), 1–22, https://doi.org/10.1186/s13660-022-02805-0.

[5] A. Adamu, D. Kitkuan, A. Padcharoen, C.E. Chidume and P. Kumam, *Inertial viscosity-type iterative method for solving inclusion problems with applications*, Math. Comput. Simul., **194** (2022), 445–459.

[6] A. Adamu, P. Kumam, D. Kitkuan and A. Padcharoen, *Relaxed modified Tseng algorithm for solving variational inclusion problems in real Banach spaces with applications*, Carpathian J. Math., **39**(1) (2023), 1–26.

[7] A. Adamu, P. Kumam, D. Kitkuan and A. Padcharoen, *A Tseng-type algorithm for approximating zeros of monotone inclusion and J-fixed point problems with applications*, Fixed Point Theory Algo. Sci. Eng., **2023**(1) (2023), 1–23.

[8] Y. Alber and I. Ryazantseva, *Nonlinear ill-posed problems of monotone type*, Dordrecht: Springer, 2006.

[9] K. Aoyama and F. Kohsaka, *Strongly relatively nonexpansive sequences generated by firmly nonexpansive-like mappings*, Fixed Point Theory Appl., **2014**(1) (2014), 1–13.

[10] A.U. Bello, C.C. Okeke, M. Isyaku and M.T. Omojola, *Forward-reflected-backward splitting method without cocoercivity for the sum of maximal monotone operators in Banach spaces*, Optimization, (2022), 1–22.

[11] C.E. Chidume, *Strong convergence theorems for bounded accretive operators in uniformly smooth Banach spaces*, Contemp. Math., **659** (2016), 31–41.

[12] C.E. Chidume, A. Adamu and L.O. Chinwendu, *Approximation of solutions of Hammerstein equations with monotone mappings in real Banach spaces,* Carpathian J. Math., **35**(3) (2019), 305–316.

[13] C.E. Chidume, A. Adamu, P. Kumam and D. Kitkuan, *Generalized hybrid viscosity-type forward-backward splitting method with application to convex minimization and image restoration problems*, Numer. Funct. Anal. Optim., (2021), 1–23, https://doi.org/10.1080/01630563.2021.1933525.

[14] C.E Chidume, A. Adamu, M.S. Minjibir and U.V. Nnyaba, *On the strong convergence of the proximal point algorithm with an application to Hammerstein equations*, J. Fixed Point Theory Appl., (2020), 22:61 https://doi.org/10.1007/s11784-020-00793-6.

[15] C.E. Chidume, A. Adamu and M.O. Nnakwe, *Strong convergence of an inertial algorithm for maximal monotone inclusions with applications*, Fixed Point Theory Appl., (2020), https://doi.org/10.1186/s13663-020-00680-2.

[16] C.E. Chidume, M.O. Nnakwe and A. Adamu, *A strong convergence theorem for generalized-ϕ-strongly monotone maps, with applications*, Fixed Point Theory Appl., **2019**(1) (2019), 1–19.

[17] C.E. Chidume, G.S De Souza, U.V. Nnyaba, O.M. Romanus and A. Adamu, *Approximation of zeros of m-accretive mappings, with applications to Hammerstein integral equations*, Carpathian J. Math., **36**(1) (2020), 45–55.

[18] P. Cholamjiak, N. Pholasa, S. Suantai and P. Sunthrayuth, *The generalized viscosity explicit rules for solving variational inclusion problems in Banach spaces*, Optimization, **70**(12) (2021), 2607–2633.

[19] P. Cholamjiak, P. Sunthrayuth, A. Singta and K. Muangchoo, *Iterative methods for solving the monotone inclusion problem and the fixed point problem in Banach spaces*, Thai J. Math., **18**(3) (2020), 1225–1246.

[20] F. Cui, Y. Tang and Y. Yang, *An inertial three-operator splitting algorithm with applications to image inpainting*, Applied Set-Valued Anal. Optimization, **1**(2) (2019), 113–134.

[21] D. Davis and W. Yin, *A three-operator splitting scheme and its optimization applications*, Set-valued Var. Anal., **25** (2017), 829–858.

[22] P. Dechboon A. Adamu and P. Kumam, *A generalized Halpern-type forward-backward splitting algorithm for solving variational inclusion problems,* AIMS Mathematics, **8**(5) (2023), 11037–11056.

[23] J. Deepho, A. Adamu, A.H. Ibrahim and A.B. Abubakar, *Relaxed viscosity-type iterative methods with application to compressed sensing*, J. Analysis, (2023), https://doi.org/10.1007/s41478-022-00547-2.

[24] Z. Hu and Q.L. Dong, *A three-operator splitting algorithm with deviations for generalized DC programming*, Appl. Numer. Math., **191** (2023), 62–74.

[25] N. Lehdili and A. Moudafi, *Combining the proximal algorithm and Tikhonov regularization*, Optimization, **37**(3) (1996), 239–252.

[26] G. Lopez, V. Martin-Marquez, F. Wang and H.K. Xu, *Forward-backward splitting methods for accretive operators in Banach spaces*, Abstr. Appl. Anal., (2012), Article ID 109236, Hindawi, https://doi.org/10.1155/2012/109236.

[27] K. Kankam, W. Cholamjiak and P. Cholamjiak, *New inertial forward-backward algorithm for convex minimization with applications*, Demon. Math., **56**(1) (2023), https://doi.org/10.1515/dema-2022-0188.

[28] K. Kankam, W. Cholamjiak and P. Cholamjiak, *Convergence analysis of a modified forward-backward splitting algorithm for minimization and application to image recovery*, Comput. Math. Methods, (2022), Article ID 3455998, https://doi.org/10.1155/2022/3455998.

[29] Y. Malitsky and M. K. Tam, *A forward-backward splitting method for monotone inclusions without cocoercivity*, SIAM J. Optimization, **30**(2) (2020), 1451–1472.

[30] B. Martinet, *Regularisation dinequations variationnelles par approximations successives* Rev. Franaise Informat. Recherche Operationnelle, **4** (1970), 154–158.

[31] G.J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J., **29**(4) (1962), 341–346.

[32] K. Muangchoo, A. Adamu, A.H. Ibrahim and A.B. Abubakar, *An inertial Halpern-type algorithm involving monotone operators on real Banach spaces with application to image recovery problems*, Comput. Applied Math., (2022), 41:364, https://doi.org/10.1007/s40314-022-02064-1.

[33] S. Reich and S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optimization, **31**(1) (2010), 22–44.

[34] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimization, **14**(5) (1976), 877–898.

[35] Y. Shehu, *Convergence results of forward-backward algorithms for sum of monotone operators in Banach spaces*, Results Math., **74**(4) (2019), 74:138.

[36] S. Suantai, P. Cholamjiak and P. Sunthrayuth, *Iterative methods with perturbations for the sum of two accretive operators in q-uniformly smooth Banach spaces*, RACSAM, **113** (2019), 203–223.

[37] S. Suantai, K. Kankam and P. Cholamjiak, *A projected forward-backward algorithm for constrained minimization with applications to image inpainting*, Mathematics, **9**(8) (2021), 890, https://doi.org/10.3390/math9080890.

[38] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optimization, **38**(2) (2000), 431–446.

[39] D. Van Hieu, L. Van Vy and P.K. Quy, *Three-operator splitting algorithm for a class of variational inclusion problems*, Bulletin Iranian Math. Soc., **46**(4) (2020), 1055–1071.

[40] Z.B. Wang, P. Sunthrayuth, A. Adamu and P. Cholamjiak, *Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities*, Optimization, (2023), 1–35, https://doi.org/10.1080/02331934.2023.2187663.

[41] J. Yang, P. Cholamjiak and P. Sunthrayuth, *Modified Tsengs splitting algorithms for the sum of two monotone operators in Banach spaces*, AIMS Math., **6**(5) (2021), 4873–4900.

[42] E.H. Zarantonello, *Solving functional equations by contractive averaging,* U.S. Army Math. Research Center, Madison, Wisconsin, 1960 Technical Report 160.

[43] C. Zong, Y. Tang and Y.J. Cho, *Convergence analysis of an inexact three-operator splitting algorithm*, Symmetry, **10**(11) (2018), 563, https://doi.org/10.3390/sym10110563.

[44] C. Zong, Y. Tang and G. Zhang, *An accelerated forward-backward-half forward splitting algorithm for monotone inclusion with applications to image restoration*, Optimization, (2022), 1–28, https://doi.org/10.1080/02331934.2022.2107926.

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Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities

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ABSTRACT

In this paper, we introduce three new inertial-like Bregman projection methods with a nonmonotone adaptive step-size for solving quasi-monotone variational inequalities in real Hilbert spaces. Under some suitable conditions, the weak convergence of these methods is proved without the prior knowledge of the Lipschitz constant of the operator and the strong convergence of some proposed methods under a strong quasi-monotonicity assumption of the mapping is also provided. Finally, several numerical experiments and applications in image restoration problems are provided to illustrate the performance of the proposed methods.

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Bregman projection; Hilbert space; weak convergence; variational inequality problem; quasi-monotone mapping

1. Introduction

Throughout this paper, let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, and let C be a nonempty, closed and convex subset of H. Recall that a mapping $F : C \to H$ is said to be *monotone* if

$$\langle Fx - Fy, x - y \rangle \geq 0, \quad \forall\, x, y \in C.$$

A mapping $F : C \to H$ is said to be *pseudo-monotone* if

$$\langle Fx, y - x \rangle \geq 0 \Rightarrow \langle Fy, y - x \rangle \geq 0, \quad \forall\, x, y \in C.$$

A mapping $F : C \to H$ is said to be *quasi-monotone* if

$$\langle Fx, y - x \rangle > 0 \Rightarrow \langle Fy, y - x \rangle \geq 0, \quad \forall\, x, y \in C$$

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and it is said to be *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall\, x, y \in C.$$

Remark 1.1: From the above definition, the following implication holds: monotone $\Rightarrow$ pseudo-monotone $\Rightarrow$ quasi-monotone. However, the converse is not necessarily valid.

Let $F : C \to H$ be a nonlinear operator. The *variational inequality problem* (VIP) is formulated as follows: find a point $z \in C$ such that

$$\langle Fz, x - z\rangle \geq 0, \quad \forall\, x \in C. \tag{1}$$

We denote by S the solution set of problem (1). It is well-known that $z \in S$ if and only if $z = P_C(z - \lambda Fz)$, where λ is any positive real number and P_C is the metric projection from H onto C, that is,

$$P_C(x) := \arg\min\{\|x - y\| : y \in C\}, \quad \forall\, x \in H.$$

The dual variational inequality problem, which is called *Minty variational inequality problem* is formulated as follows: find a point $z \in C$ such that

$$\langle Fx, x - z\rangle \geq 0, \quad \forall\, x \in C. \tag{2}$$

We also denote S_D by the solution set of the dual variational inequality problem. It is obvious that S_D is a closed and convex subset of C. In general, it is well-known that if F is continuous and C is convex, then $S_D \subset S$ and if F is a pseudo-monotone and continuous mapping, then $S = S_D$ (see [1,Lemma 2.1]). However, the inclusion $S \subset S_D$ is false, if F is a quasi-monotone and continuous mapping (see Ref. [2]).

Variational inequalities were first introduced by Fichera [3,4] for solving the Signorini problem in 1963 and later were studied by Stampacchia [5] for solving partial differential equations with unilateral boundary conditions and free boundary value problems of elliptic type in mechanics. Variational inequalities have become one of the significant tools for studying a wide class of unrelated linear and nonlinear problems arising in elasticity, economics, transportation, optimization, network analysis, control theory and engineering sciences (see, for example, Refs [6–12]).

One well-known method to solve the VIP is the *projected gradient method*. In fact, this method may not converge when the cost operator is monotone (see Ref. [13]). To overcome the drawback of the projected gradient method, Korpelevich [14] established the following so-called *extragradient method* (EGM) for solving VIP in $\mathbb{R}^m$ when F is monotone and L-Lipschitz continuous:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda F x_n), \\ x_{n+1} = P_C(x_n - \lambda F y_n), \end{cases} \tag{3}$$

where P_C is the metric projection from H onto the closed and convex set C and $\lambda \in (0, 1/L)$. Note that this method requires to calculate two projections onto C in each iteration step. In the case of the set C is not explicit or complicated, it increases the computational burden of the EGM to calculate such projections. Later in 2011, Censor et al. [15] proposed the so-called *subgradient extragradient method* (SEGM) for solving monotone VIP in a real Hilbert space H. In this method, they replaced the second projection onto C of the EGM by a projection onto a specific, constructible half-space. Note that the projection onto the half-space has an explicit expression and thus essentially reduces the complexity in computing of projections. The SEGM is of the following form:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda F y_n), \end{cases} \tag{4}$$

where $T_n := \{x \in H : \langle x_n - \lambda F x_n - y_n, x - y_n \rangle \leq 0\}$ and $\lambda \in (0, 1/L)$.

Another improvement of the EGM is the Tseng's extragradient method (TEGM) which was studied by Tseng [16]. The TEGM is of the following form:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ x_{n+1} = y_n - \lambda(F y_n - F x_n), \end{cases} \tag{5}$$

where $\lambda \in (0, 1/L)$. This method also requires only one projection onto C in each iteration. Accordingly, it is easier to implement than the EGM. Recently, the TEGM has received great attention from many authors who modify in different approaches to solve the monotone VIP and even the pseudo-monotone VIP (see, for example, Refs [17–20]).

Implementing of the original TEGM, EGM, SEGM and some of their variant forms requires to know in advance the Lipschitz constant of the potential operator. However, the Lipschitz constant is sometimes unknown or difficult to estimate, see Ref. [21]. Some algorithms without knowledge of prior information of Lipschitz constants have been established for solving VIP, see Refs [22–25]. In the field of applied sciences, it would be interesting to broaden the ways for solving the VIP to more general class of the quasi-monotone operator. In 2015, Ye and He [2] proposed a double projection algorithm for solving quasi-monotone (or without monotonicity) variational inequalities in the finite-dimensional Euclidean space $\mathbb{R}^m$. Salahuddin [26] extended the original EGM to solve the VIP with quasi-monotone and Lipschitz continuous operators in infinite dimensional Hilbert spaces. Liu and Yang [25] proposed modifications of the EGM, SEGM and TEGM with a new adaptive step-size for solving the quasi-monotone VIP in real Hilbert spaces. Note that this new adaptive step-size is explicitly updated and is allowed to increase slightly at each iteration of algorithm which can be more easily implemented in practice. Very recently, Alakoya et al. [22] also proposed two modifications of TEGM with this new step-size for approximating the solution of the quasi-monotone VIP in real Hilbert spaces.

In addition, the concept of inertial techniques was first introduced by Polyak [27] as an acceleration process in solving a smooth convex minimization problem. It derived from an implicit time discretization of a second-order dynamical system, so-called *heavy ball with friction*. Let $\gamma, \delta : [0, \infty) \to [0, \infty)$ be Lebesgue measurable functions. Following the works in Refs [28,29], we consider the following second-order dynamical system for variational inequalities:

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \delta(t)(x(t) - \varphi_\lambda(x(t))) = 0, \\ x(0) = \alpha_0, \quad \dot{x}(0) = \beta_0, \end{cases} \tag{6}$$

where $\alpha_0, \beta_0 \in H$ and $\ddot{x}(t)$, and $\dot{x}(t)$ denote the first and second derivatives of x at t. In this case, $\varphi_\lambda(\cdot) = (I - \lambda F)P_C(I - \lambda F) + \lambda F$, where $\lambda > 0$. It is known that $x \in S$ if and only if $x = \varphi_\lambda(x)$. As $h_n > 0$ step-size, we discretize (6) about time variable t and get the following form:

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h_n^2} + \gamma_n \frac{x_n - x_{n-1}}{h_n} + \delta_n(w_n - \varphi_\lambda(w_n)) = 0, \quad \forall\, n \geq 1, \tag{7}$$

where w_n is an extrapolated point from x_n and x_{n-1} that will be defined later (see Ref. [30]). Now, if we set $h_n = 1$ and $\theta_n = 1 - \gamma_n$ for all $n \geq 1$, then (7) can be rewritten as follows:

$$x_{n+1} = x_n + \theta_n(x_n - x_{n-1}) - \delta_n(w_n - \varphi_\lambda(w_n)). \tag{8}$$

Take $w_n = x_n + \theta_n(x_n - x_{n-1})$, which is the Nesterove choice for the extrapolation term, we obtain

$$x_{n+1} = (1 - \delta_n)w_n + \delta_n \varphi_\lambda(w_n), \tag{9}$$

where $\varphi_\lambda(w_n) = (I - \lambda F)P_C(w_n - \lambda F w_n) + \lambda F w_n$. Now, let $y_n = P_C(w_n - \lambda F w_n)$, then we obtain the following iterative algorithm for variational inequalities:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ x_{n+1} = (1 - \delta_n)w_n + \delta_n(y_n - \lambda(F y_n - F w_n)). \end{cases} \tag{10}$$

Such a term $\theta_n(x_n - x_{n-1})$ is called an *inertial extrapolation term* which is intended to speed up the convergence rate of the sequence generated by (10).

In recent years, the inertial technique is wildly used to accelerate the convergence rate of algorithms for many kinds of optimization problem (see, for example, Refs [31–38]). Quite recently, Chbani and Riahi [39] proposed an inertial extrapolation term is a convex combination of x_{n-1} and x_n, namely,

$$(1-\theta_n)x_n + \theta_n x_{n-1} = x_n + \theta_n(x_{n-1} - x_n). \tag{11}$$

Also, they proposed algorithms with an inertial-like (11) for solving the Ky Fan minimax inequalities.

In 1967, Bregman [40] introduced a dissimilarity measure that describes the difference between two points on a differentiable convex function *f*. This flexible approach is known as the Bregman distance or the Bregman divergence which is useful in various applications in many areas of modern physical sciences, such as machine learning [41], speech processing [42], statistical learning [43] and data mining [44,45]. The main role of the Bregman distances is to replace the Squared Euclidean distance with a non-Euclidean distance. Many well-known concepts of distance, such as Squared Euclidean distance, Kullback–Leibler distance, Itakura–Saito distance and Squared Mahalanobis distance are special cases of the Bregman distances generated by various types of function. In recent years, using technique of Bregman distances has become an attractive approach for solving VIP and related optimization problems (see, for example, Refs [23,24,35,46–49]). In the context of VIP, Bregman distances have a nice property of projection which generalized effect rather than metric projection on a closed and convex set (see Refs [23,50]). This may allow more flexibility in computations when chosen projections are appropriate in some cases.

Motivated and inspired by the above works, in this paper, we introduce three new inertial-like algorithms with Bregman distance for solving quasi-monotone VIP in real Hilbert spaces. Our three methods are based on Tseng's extragradient method, extragradient method and subgradient extragradient method. The proposed methods use a nonmonotone adaptive step-size which does not need to know the Lipschitz constant of any mapping. Then the weak convergence of these methods is proved. Also, we provide the strong convergence of the proposed methods under strong quasi-monotonicity assumption of the mapping. Finally, several numerical experiments are performed to illustrate the effectiveness of the proposed algorithms. Our results obtained in this paper improve and generalize the corresponding results in Refs [2,22,23,25,26]

The remainder of the paper is organized as follows. In Section 2, we recall some definitions and preliminary results that need to be used in the sequel. In Section 3, we analyse and prove the convergence of the proposed algorithms. Finally in Section 4, we provide several numerical experiments to illustrate the effectiveness of the proposed algorithms.

2. Preliminaries

Let H be a real Hilbert space. For a sequence $\{x_n\} \subset H$, the strong convergence and the weak convergence of $\{x_n\}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let $f : H \to (-\infty, \infty]$ be a function. We denote by $\operatorname{dom} f$, the domain of f, that is, $\operatorname{dom} f := \{x \in H : f(x) < \infty\}$ and also denote by $\operatorname{int}(\operatorname{dom} f)$, the interior of the domain of f.

Definition 2.1: Recall that the function $f : H \to (-\infty, \infty]$ is said to be:

(1) *proper* if $\operatorname{dom} f \neq \emptyset$;
(2) *lower semi-continuous* if the set $\{x \in \operatorname{dom} f : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$;
(3) *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in \operatorname{dom} f$ and $t \in [0, 1]$, and it is said to be *strictly convex* if the strict inequality holds in a convex function for all $x, y \in \operatorname{dom} f$ with $x \neq y$ and $t \in (0, 1)$;
(4) *uniformly convex* if there exists a continuous and increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)\psi(\|x - y\|)$ for all $x, y \in \operatorname{dom} f$ and $t \in [0, 1]$, and it is said to be *strongly convex* if f is uniformly convex with $\psi(s) = \kappa s^2$, $\kappa > 0$. It is known that a strongly convex function is also strictly convex;
(5) *uniformly smooth* if there exists a continuous and increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y) - t(1-t)\phi(\|x - y\|)$ for all $x, y \in \operatorname{dom} f$ and $t \in [0, 1]$;
(6) *bounded on bounded sets* if $f(U)$ is bounded for each bounded subset U of H.

Let $f : H \to (-\infty, \infty]$ be proper, lower semi-continuous and convex function. The *Fenchel conjugate* of f is the functional $f^* : H \to (-\infty, \infty]$ defined by $f^*(x^*) := \sup_{x \in H}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in H$.

Next, we give some examples of the Fenchel conjugate of f.

Example 2.2: Let $f : H \to \mathbb{R}$ be defined by $f(x) = \frac{\|x\|^2}{2}$. Then for each $a \in H$, we have

$$f^*(a) = \sup_{x \in H}\{\langle a, x\rangle - f(x)\} = \sup_{x \in H}\left\{\langle a, x\rangle - \frac{\|x\|^2}{2}\right\}$$
$$= \sup_{x \in H}\left\{\frac{\|a\|^2}{2} - \frac{\|x - a\|^2}{2}\right\} = \frac{\|a\|^2}{2}.$$

The *directional derivative* of f at $x \in \operatorname{int}(\operatorname{dom} f)$ in the direction $y \in H$ is defined by

$$f'(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{12}$$

If the limit (12) exists for each y, then f is said to be *Gâteaux differentiable at* x and $f'(x, y)$ coincides with $\nabla f(x)$ which is the value of the gradient ∇f at x. The function f is said to be *Gâteaux differentiable* if it Gâteaux differentiable at each $x \in \mathrm{int}(\mathrm{dom} f)$. When the limit (12) is attained uniformly for $y \in H$ with $\|y\| = 1$, we say that f is *Fréchet differentiable* at x and it is said to be *uniformly Fréchet differentiable* on a subset C of H if the limit (12) is attained uniformly for $x \in C$ and $\|y\| = 1$. We know that if f is Fréchet differentiable function, then it is Gâteaux differentiable and if f is Fréchet differentiable, then it is also continuous (see [51,p. 142]). It is also known that if $f : H \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of H, then ∇f is uniformly continuous on bounded subsets of H (see [52,Proposition 1]) and it is also uniformly smooth (see [53,p. 207])

Let $f : H \to (-\infty, \infty]$ be a function. We know the following facts:

(i) f is strongly convex if and only if it satisfies the following inequality (see [54,Theorem 5.24]):

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle + \frac{\kappa}{2}\|x - y\|^2 \tag{13}$$

for all $x \in \mathrm{dom} f$ and $y \in \mathrm{int}(\mathrm{dom} f)$;

(ii) f is uniformly convex if and only if f^* is Fréchet differentiable and ∇f^* is uniformly continuous on bounded sets of H (see [53,Theorem 3.5.10]).

Definition 2.3: A proper, lower semi-continuous and convex function $f : H \to (-\infty, \infty]$ is said to be *Legendre* if it satisfies the following two conditions:

(L1) $\mathrm{int}(\mathrm{dom} f) \neq \emptyset$, f is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f)$ and $\mathrm{dom} \nabla f = \mathrm{int}(\mathrm{dom} f)$;

(L2) $\mathrm{int}(\mathrm{dom} f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f^*)$ and $\mathrm{dom} \nabla f^* = \mathrm{int}(\mathrm{dom} f^*)$.

For a Legendre function f, it is known that ∇f is a bijection from $\mathrm{int}(\mathrm{dom} f)$ into $\mathrm{int}(\mathrm{dom} f^*)$ satisfying $\nabla f = (\nabla f^*)^{-1}$ (see [55,Theorem 5.10]). Next, we give some examples of a Legendre function as below:

- The Boltzmann–Shannon entropy function $f(x) = \sum_{i=1}^m x_i \ln(x_i)$ for $x \in \mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x_i \geq 0, i = 1, 2, \ldots, m\}$ with $0 \ln 0 := 0$. This function is useful to handle simplex constraints (see [56]).
- The Burg's entropy function $f(x) = -\sum_{i=1}^m \ln(x_i)$ for $x \in \mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : x_i > 0, i = 1, 2, \ldots, m\}$. This function is useful in Poisson linear inverse problems (see Ref. [57]).

More examples of Legendre function can be found in Refs [55,58]. Next, we give the definition of Bregman distance.

Definition 2.4: Let $f : H \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function. The *Bregman distance* [40] with respect to f is the function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle$$

for all $x \in \operatorname{dom} f$ and $y \in \operatorname{int}(\operatorname{dom} f)$. The geometric of Bregman distance with respect to f is shown in Figure 1.

It is worth noting that D_f is not a metric because the symmetry and the triangle inequality fail to hold. Note that $D_f(x, x) = 0$, but $D_f(x, y) = 0$ does not imply $x = y$. In this case, when f is Legendre, this indeed holds (see [55,Lemma 7.3 (vi)]). Some Bregman distances can be seen as follows:

- The Bregman distance with respect to the Legendre function $f : \mathbb{R}^m \to \mathbb{R}$ given by $f(x) = 0.25\|x\|_2^4 + 0.5\|x\|_2^2$ is used in phase retrieval problems (see Ref. [59]).
- The Bregman distance with respect to the Legendre function $f : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ given by $f(x, y) = c_1(\|x\|^2 + \|y\|^2)^2 + c_2(\|x\|^2 + \|y\|^2)$ with $c_1, c_2 > 0$ and $m_1, m_2 \in \mathbb{N}$ is used in matrix factorization problems (see Refs [60,61]).
- The *Kullback–Leibler distance* $D_f(x, y) = \sum_{i=1}^m (x_i \ln(\frac{x_i}{y_i}) + y_i - x_i)$ generated from the Boltzmann–Shannon entropy function $f(x) = \sum_{i=1}^m x_i \ln(x_i)$. Note that the Kullback–Leibler distance is used to measure the difference between two probability distributions in statistics.
- The *Squared Euclidean distance* $D_f(x, y) = \frac{1}{2}\|x - y\|^2$ generated from the squared norm $f(x) = \frac{1}{2}\|x\|^2$.

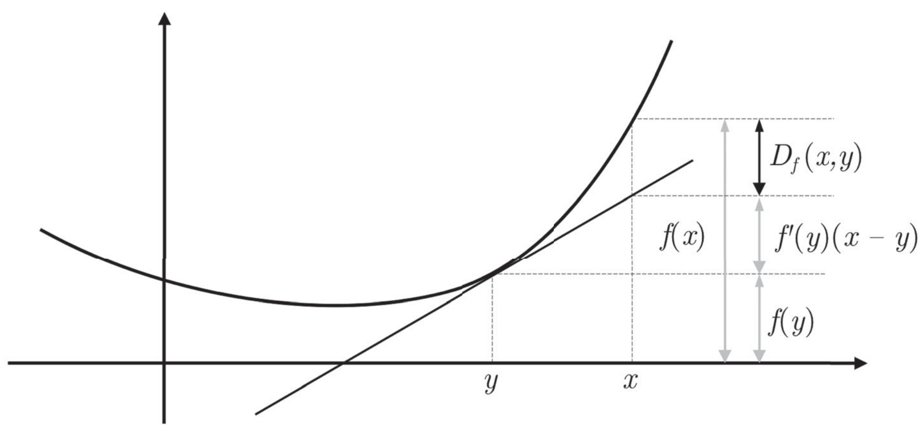


Figure 1. The Bregman distance with respect to *f*.

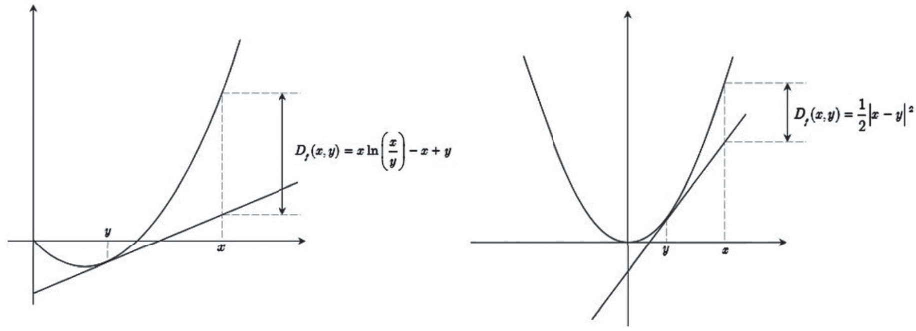


Figure 2. Left: Kullback–Leibler distance; Right: Squared Euclidean distance.

The geometric of Kullback–Leibler distance and Squared Euclidean distance are shown in Figure 2.

The Bregman distance has the following useful property called the *three-point identity*: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$, it holds that (Figure 2)

$$D_f(x,y) = D_f(x,z) - D_f(y,z) + \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{14}$$

Moreover, if f is additionally assumed to be strongly convex with constant κ, then it follows from (13) and Definition 2.4 that

$$D_f(x,y) \geq \frac{\kappa}{2}\|x-y\|^2 \tag{15}$$

for all $x \in \operatorname{dom} f$ and $y \in \operatorname{int}(\operatorname{dom} f)$.

Let $f : H \to (-\infty, \infty]$ be a strongly convex and Gâteaux differentiable function. Let C be a nonempty, closed and convex subset of $\operatorname{dom} f$. The *Bregman projection* with respect to f of $x \in \operatorname{int}(\operatorname{dom} f)$ is the unique point in C, denoted by Π^f_C, defined by

$$\Pi^f_C(x) := \arg\min\{D_f(y,x) : y \in C\}.$$

If $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in H$, then Π^f_C coincides the metric projection P_C. It is known that, $\Pi^f_C(x)$ has the following properties (see [62,Corollary 4.4]): for each $x \in H$,

$$z = \Pi^f_C(x) \iff \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \quad \forall\, y \in C \tag{16}$$

and

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \quad \forall\, y \in C. \tag{17}$$

Let $f : H \to \mathbb{R}$ be a Legendre function. We define $V_f : H \times H \to [0,\infty)$ associated with f by

$$V_f(x,x^*) := f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall\, x, x^* \in H.$$

It is known that V_f is convex in the second variable and $V_f(x,x^*) = D_f(x, \nabla f^*(x^*))$ for all $x, x^* \in H$ (see [63,Proposition 1]). Then for each $x, y, z \in$

H and $t \in [0,1]$, we get

$$\begin{aligned} &D_f(x, \nabla f^*(t\nabla f(y) + (1-t)\nabla f(z))) \\ &\quad = V_f(x, t\nabla f(y) + (1-t)\nabla f(z)) \\ &\quad \le tV_f(x, \nabla f(y)) + (1-t)V_f(x, \nabla f(z)) \\ &\quad = tD_f(x, \nabla f^* \nabla f(y)) + (1-t)D_f(x, \nabla f^* \nabla f(z)) \\ &\quad = tD_f(x, y) + (1-t)D_f(x, z). \end{aligned}$$

The following lemma is an Opial-like property for the Bregman distance, which can be found in [64,Lemma 5.1].

Lemma 2.5: *Let $f : H \to (-\infty, \infty]$ be a proper strictly convex function so that it is Gâteaux differentiable. Suppose that $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ for some $x \in H$. Then*

$$\liminf_{n\to\infty} D_f(x, x_n) < \liminf_{n\to\infty} D_f(y, x_n)$$

for all $y \in H$ with $y \neq x$.

Proposition 2.6 ([2,Proposition 2.1]): *If one of the following conditions hold:*

- (i) *F is pseudo-monotone on C and $S \neq \emptyset$;*
- (ii) *F is the gradient of G, where G is a differentiable quasi-convex function on an open set $K \supset C$ and attains its global minimum on C;*
- (iii) *F is quasi-monotone on C, $F \neq 0$ on C and C is bounded;*
- (iv) *F is quasi-monotone on C, $F \neq 0$ on C and there exists a positive number r such that, for every $x \in C$ with $\|x\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle Fx, y - x\rangle \leq 0$;*
- (v) *F is quasi-monotone on C, $\operatorname{int} C$ is nonempty and there exists $x^* \in S$ such that $Fx^* \neq 0$.*

Then S_D is nonempty.

Lemma 2.7 ([65,Lemma 1]): *Assume that $\{s_n\}$ and $\{t_n\}$ are two nonnegative real sequences such that $s_{n+1} \leq s_n + t_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.*

3. Main results

In this section, we present modified inertial algorithms for solving quasi-monotone VIP and their convergence analyses. In what follows, we adopt the convention $[a]_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. In order to establish our main result, we assume the following conditions are satisfied.

Condition 1 The feasible set C is nonempty, closed and convex subset of H.
Condition 2 The function $f : H \to \mathbb{R}$ satisfies the following conditions:

(A1) f is strongly convex with constant $\kappa > 0$;
(A2) f is Legendre which is uniformly Fréchet differentiable;
(A3) f is bounded on bounded subsets of H.

Condition 3 The mapping $F : H \to H$ satisfies the following conditions:

(B1) F is quasi-monotone on H;
(B2) F is Lipschitz continuous with constant $L > 0$;
(B3) F satisfies the following property:

$$\text{whenever } \{q_n\} \subset C,\ q_n \rightharpoonup x, \quad \text{one has } \|Fx\| \leq \liminf_{n\to\infty} \|Fq_n\|.$$

(B4) $S_D \neq \emptyset$.

First algorithm is proposed as below.

Remark 3.1: (1) If $z_n = w_n$ or $Fz_n = 0$, then Algorithm 1 stops in finite iterations and $z_n \in S$. Indeed, let $z_n = w_n$, we see that $z_n = \Pi_C^f \nabla f^*(\nabla f(z_n) - \lambda_n F z_n)$ for $\lambda_n > 0$. In view of the property of Bregman projection (16), we have $\langle Fz_n, y - z_n\rangle \geq 0$ for all $y \in C$, that is, z_n is a solution of VIP. In the rest of this paper, we assume that the Algorithm 1 does not stop in any finite iterations and generates an infinite sequence $\{x_n\}$. This implies that $z_n \neq w_n$ or $Fz_n \neq 0$.

(2) The condition $(B3)$ is weaker than the sequential weak continuity of the mapping F, which is often assumed in many recent works related quasi-monotone VIP (see, e.g. Refs [22,25,26,34,38]). Indeed, let $F : \ell_2 \to \ell_2$ be a mapping define by $Fx = x\|x\|$ for all $x \in \ell_2$. Let $\{q_n\} \subset \ell_2$ such that $q_n \rightharpoonup x$ and $\|Fx\| \leq \liminf_{n\to\infty} \|Fq_n\|$. By the weak lower semi-continuity of the norm, we have $\|x\| \leq \liminf_{n\to\infty} \|q_n\|$. It follows that $\|Fx\| = \|x\|^2 \leq (\liminf_{n\to\infty} \|q_n\|)^2 \leq \liminf_{n\to\infty} \|q_n\|^2 = \liminf_{n\to\infty} \|Fq_n\|$. To show that F is not sequentially weakly continuous, choose $q_n = e_n + e_1$, where $\{e_n\}$ is a standard basis of ℓ_2, that is, $e_n = (0, 0, \ldots, 1, \ldots)$ with 1 at the nth position. It is clear that $q_n \rightharpoonup e_1$ and $Fq_n = F(e_n + e_1) = (e_n + e_1)\|e_n + e_1\| \rightharpoonup \sqrt{2}e_1$, but $Fe_1 = e_1\|e_1\| = e_1$. Hence F is not sequentially weakly continuous. Moreover, the condition $(B3)$ is not strictly necessary to assume when F is monotone or H is a finite-dimensional Hilbert space.

(3) The step-size λ_n generated by (20) is a nonmonotone step-size, this is allowed to increase when the iteration increases. Hence it reduces the dependence on the initial step-size λ_1.

(4) Some special cases of Algorithm 1 are shown as below.

Algorithm 1: Modified inertial Tseng-type extragradient method for VIP

Initialization: Choose $\beta \in [0,1)$, $\delta \in (0,1]$, $\lambda_1 > 0$ and $\mu \in (0,\kappa)$.
Choose sequences $\{\xi_n\}$ and $\{p_n\}$ satisfy the following conditions:
(1) $\{\xi_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \xi_n < \infty$;
(2) $\{p_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} p_n < \infty$.

Let $x_0, x_1 \in H$ be arbitrary.
Iterative steps: Given the iterates x_{n-1} and x_n for each $n \geq 1$, calculate x_{n+1} as follows:
Step 1. Choose β_n such that $\beta_n \in [0, \bar{\beta}_n]$, where

$$\bar{\beta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|\nabla f(x_{n-1}) - \nabla f(x_n)\|}, \beta\right\} & \text{if } \nabla f(x_{n-1}) \neq \nabla f(x_n), \\ \beta & \text{otherwise.} \end{cases} \tag{18}$$

Step 2. Compute

$$\begin{cases} w_n = \nabla f^*(\nabla f(x_n) + \beta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ z_n = \Pi_C^f \nabla f^*(\nabla f(w_n) - \lambda_n F w_n), \\ y_n = \nabla f^*(\nabla f(z_n) - \lambda_n(F z_n - F w_n)), \\ x_{n+1} = \nabla f^*((1-\delta)\nabla f(x_n) + \delta \nabla f(y_n)), \quad \forall\, n \geq 1, \end{cases} \tag{19}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - z_n\|}{\|F w_n - F z_n\|}, \lambda_n + p_n\right\} & \text{if } F w_n - F z_n \neq 0, \\ \lambda_n + p_n & \text{otherwise.} \end{cases} \tag{20}$$

Set $n := n+1$ and go to the iterative step.

- If $\beta_n = 0$, then Algorithm 1 reduces to the modified Tseng-type iteration without the relaxed inertial term for VIP.
- If $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in H$, then Algorithm 1 reduces to the following relaxed inertial Tseng-type method for VIP:

$$\begin{cases} w_n = x_n + \beta_n(x_{n-1} - x_n), \\ z_n = P_C(w_n - \lambda_n F w_n), \\ y_n = z_n - \lambda_n(F z_n - F w_n), \\ x_{n+1} = (1-\delta)x_n + \delta y_n, \quad \forall\, n \geq 1, \end{cases} \tag{21}$$

where $\beta_n \in [0, \bar{\beta}_n]$ and

$$\bar{\beta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|x_{n-1} - x_n\|}, \beta\right\} & \text{if } x_{n-1} \neq x_n, \\ \beta & \text{otherwise.} \end{cases} \tag{22}$$

- If $\delta = 1$, $\beta_n = 0$ and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in H$, then Algorithm 1 reduces to Algorithm 3.1 of Liu and Yang [25].

(5) From (18), it is easy to see that $\beta_n\|\nabla f(x_{n-1}) - \nabla f(x_n)\| \leq \xi_n$. Since $\sum_{n=1}^{\infty} \xi_n < \infty$, it follows that

$$\sum_{n=1}^{\infty} \beta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| < \infty \tag{23}$$

and

$$\lim_{n\to\infty} \beta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| = 0. \tag{24}$$

We next give the following lemmas, which are necessary for proving our main result.

Lemma 3.1: *Let* $\{\lambda_n\}$ *be the sequence generated by* (20). *Then we have* $\lim_{n\to\infty} \lambda_n = \lambda$, *where* $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + p]$ *and* $p = \sum_{n=1}^{\infty} p_n$.

Proof: The proof of lemma is similar to the proof of [25,Lemma 3.1]. ■

Lemma 3.2: *Let* $\{x_n\}$ *be a sequence generated by Algorithm 1. Then, for each* $p \in S_D$, *the following inequality holds:*

$$D_f(p, y_n) \leq (1 - \beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}) - \sigma_n,$$

where $\sigma_n := (1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}) D_f(z_n, w_n) + (1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}) D_f(y_n, z_n)$.

Proof: Let $p \in S_D$. By the definition of Bregman distance, we have

$$\begin{aligned} &D_f(p, y_n) \\ &\quad = D_f(p, \nabla f^*(\nabla f(z_n) - \lambda_n(Fz_n - Fw_n))) \\ &\quad = f(p) - f(y_n) - \langle \nabla f(z_n) - \lambda_n(Fz_n - Fw_n), p - y_n\rangle \end{aligned}$$

$$
\begin{aligned}
&= f(p) - f(y_n) - \langle \nabla f(z_n), p - y_n\rangle + \lambda_n \langle Fz_n - Fw_n, p - y_n\rangle \\
&= f(p) - f(z_n) - \langle \nabla f(y_n), p - z_n\rangle + \langle \nabla f(z_n), p - z_n\rangle \\
&\quad + f(z_n) - f(y_n) - \langle \nabla f(z_n), p - y_n\rangle + \lambda_n \langle Fz_n - Fw_n, p - y_n\rangle \\
&= f(p) - f(z_n) - \langle \nabla f(z_n), p - z_n\rangle - f(y_n) + f(z_n) \\
&\quad + \langle \nabla f(z_n), y_n - z_n\rangle + \lambda_n \langle Fz_n - Fw_n, p - y_n\rangle \\
&= D_f(p, z_n) - D_f(y_n, z_n) + \lambda_n \langle Fz_n - Fw_n, p - y_n\rangle. \qquad (25)
\end{aligned}
$$

By the three-point identity, we have

$$D_f(p, z_n) = D_f(p, w_n) - D_f(z_n, w_n) + \langle \nabla f(w_n) - \nabla f(z_n), p - z_n\rangle. \qquad (26)$$

Substituting (26) into (25), we have

$$
\begin{aligned}
D_f(p, y_n) = D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) \\
+ \langle \nabla f(w_n) - \nabla f(z_n), p - z_n\rangle + \lambda_n \langle Fz_n - Fw_n, p - y_n\rangle. \qquad (27)
\end{aligned}
$$

Since $z_n = \Pi_C^f \nabla f^*(\nabla f(w_n) - \lambda_n Fw_n)$ and $p \in S_D \subset S \subset C$, we have

$$\langle \nabla f(w_n) - \lambda_n Fw_n - \nabla f(z_n), p - z_n\rangle \le 0.$$

Hence

$$\langle \nabla f(w_n) - \nabla f(z_n), p - z_n\rangle \le \lambda_n \langle Fw_n, p - z_n\rangle. \qquad (28)$$

Substituting (28) and (27), we have

$$
\begin{aligned}
&D_f(p, y_n) \\
&\quad \le D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n \langle Fw_n, p - z_n\rangle \\
&\qquad + \lambda_n \langle Fz_n - Fw_n, p - y_n\rangle \\
&\quad = D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n \langle Fw_n, p - z_n\rangle \\
&\qquad + \lambda_n \langle Fz_n, p - y_n\rangle - \lambda_n \langle Fw_n, p - y_n\rangle \\
&\quad = D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n \langle Fw_n, y_n - z_n\rangle \\
&\qquad + \lambda_n \langle Fz_n, p - y_n\rangle \\
&\quad = D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n \langle Fw_n, y_n - z_n\rangle \\
&\qquad - \lambda_n \langle Fz_n, z_n - p\rangle + \lambda_n \langle Fz_n, z_n - y_n\rangle \\
&\quad = D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) \\
&\qquad + \lambda_n \langle Fw_n - Fz_n, y_n - z_n\rangle - \lambda_n \langle Fz_n, z_n - p\rangle. \qquad (29)
\end{aligned}
$$

Since $z_n \in C$ and $p \in S_D$, it follows from the dual VIP that $\langle Fz_n, z_n - p\rangle \geq 0$. Then from (29), we have

$$\begin{aligned}
&D_f(p, y_n)\\
&\quad\leq D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n\langle Fw_n - Fz_n, y_n - z_n\rangle\\
&\quad\leq D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n\|Fw_n - Fz_n\|\|y_n - z_n\|\\
&\quad\leq D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \frac{\mu\lambda_n}{\lambda_{n+1}}\|w_n - z_n\|\|y_n - z_n\|\\
&\quad\leq D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n)\\
&\qquad+ \frac{\mu\lambda_n}{2\lambda_{n+1}}\|w_n - z_n\|^2 + \frac{\mu\lambda_n}{2\lambda_{n+1}}\|y_n - z_n\|^2\\
&\quad\leq D_f(p, w_n) - \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(z_n, w_n) - \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n, z_n).
\end{aligned} \tag{30}$$

Moreover, we see that

$$\begin{aligned}
D_f(p, w_n) &= D_f(p, \nabla f^*(\nabla f(x_n) + \beta_n(\nabla f(x_{n-1}) - \nabla f(x_n))))\\
&= D_f(p, \nabla f^*((1 - \beta_n)\nabla f(x_n) + \beta_n\nabla f(x_{n-1})))\\
&\leq (1 - \beta_n)D_f(p, x_n) + \beta_n D_f(p, x_{n-1}).
\end{aligned} \tag{31}$$

Substituting (31) into (30), we have

$$\begin{aligned}
D_f(p, y_n) \leq (1 - \beta_n)D_f(p, x_n) + \beta_n D_f(p, x_{n-1}) - \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(z_n, w_n)\\
- \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n, z_n).
\end{aligned} \tag{32}$$

From the definition of σ_n, we can write (32) in the following form:

$$D_f(p, y_n) \leq (1 - \beta_n)D_f(p, x_n) + \beta_n D_f(p, x_{n-1}) - \sigma_n.$$

Therefore, the proof of Lemma 3.2 is finished. ■

Lemma 3.3: *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then* $\lim_{n\to\infty} D_f(p, x_n)$ *exists for each* $p \in S_D$. *Moreover, we have* $\lim_{n\to\infty}\|z_n - w_n\| = \lim_{n\to\infty}\|y_n - z_n\| = 0$.

Proof: Since $\lim_{n\to\infty}\lambda_n = \lambda > 0$ and $\mu \in (0, \kappa)$, we have $\lim_{n\to\infty}(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \frac{\mu}{\kappa} = \frac{\kappa-\mu}{\kappa} > 0$. As a consequence, there exists $n_0 \in \mathbb{N}$ such that $1 -$

$\frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}} > 0$, $\forall n \geq n_0$. This implies that

$$\sigma_n := \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, w_n) + \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, z_n) \geq 0, \quad \forall n \geq n_0.$$

So, by Lemma 3.2, we have

$$D_f(p, y_n) \leq (1-\beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}), \quad \forall n \geq n_0. \tag{33}$$

It follows from (33) that

$$\begin{aligned} D_f(p, x_{n+1}) &= (1-\delta) D_f(p, x_n) + \delta D_f(p, y_n) \\ &\leq (1-\delta) D_f(p, x_n) + \delta[(1-\beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1})] \\ &= (1-\delta\beta_n) D_f(p, x_n) + \delta\beta_n D_f(p, x_{n-1}) \\ &\leq \max\{D_f(p, x_n), D_f(p, x_{n-1})\} \\ &\leq \ldots \leq \max\{D_f(p, x_1), D_f(p, x_0)\}. \end{aligned}$$

This implies that $\{D_f(p, x_n)\}$ is bounded and, in consequence $\{x_n\}$ is bounded by (15). Now by Lemma 3.2, we have

$$\begin{aligned} D_f(p, x_{n+1}) &\leq (1-\delta) D_f(p, x_n) + \delta D_f(p, y_n) \\ &\leq (1-\delta) D_f(p, x_n) + \delta[(1-\beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}) - \sigma_n] \\ &\leq D_f(p, x_n) + \delta\beta_n \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right]_+ - \delta\sigma_n. \end{aligned} \tag{34}$$

This implies that

$$D_f(p, x_{n+1}) \leq D_f(p, x_n) + \delta\beta_n \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right]_+. \tag{35}$$

By the three-point identity, we have

$$\begin{aligned} &\left[D_f(p, x_{n-1}) - D_f(p, x_n)\right]_+ \\ &\quad = -D_f(x_{n-1}, x_n) + \langle p - x_{n-1}, \nabla f(x_n) - \nabla f(x_{n-1})\rangle \\ &\quad \leq \langle p - x_{n-1}, \nabla f(x_n) - \nabla f(x_{n-1})\rangle \\ &\quad \leq \|\nabla f(x_n) - \nabla f(x_{n-1})\| M, \end{aligned}$$

where $M := \sup_{n\geq 1}\{\|x_{n-1} - p\|\}$. From (23), we get

$$\sum_{n=1}^{\infty} \beta_n \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right]_+ \leq \sum_{n=1}^{\infty} \beta_n \|\nabla f(x_n) - \nabla f(x_{n-1})\| M < \infty. \tag{36}$$

This, together with (35) and Lemma 2.7, yields that $\lim_{n\to\infty} D_f(p, x_n)$ exists. From (34), we see that

$$\delta\sigma_n \leq D_f(p, x_n) - D_f(p, x_{n+1}) + \delta\beta_n \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right]_+. \tag{37}$$

From (36), we also get $\lim_{n\to\infty} \beta_n [D_f(p, x_{n-1}) - D_f(p, x_n)]_+ = 0$. Again, since $\lim_{n\to\infty} D_f(p, x_n)$ exists and $\sigma_n \geq 0$ for all $n \geq n_0$, it follows from (37) that

$\lim_{n\to\infty}\sigma_n = 0$. From the definition of σ_n, we obtain

$$\lim_{n\to\infty} D_f(z_n, w_n) = \lim_{n\to\infty} D_f(y_n, z_n) = 0,$$

which imply by (15) that

$$\lim_{n\to\infty} \|z_n - w_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{38}$$

■

Lemma 3.4: *Let $\{w_n\}$ be a sequence generated by Algorithm 1. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to some point $z \in H$ and $\lim_{k\to\infty}\|w_{n_k} - z_{n_k}\| = 0$, then $z \in S_D$ or $Fz = 0$.*

Proof: Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ such that $w_{n_k} \rightharpoonup z \in H$. Since $\lim_{k\to\infty}\|w_{n_k} - z_{n_k}\| = 0$ and $z_{n_k} \in C$, we also get $z_{n_k} \rightharpoonup z \in C$. From the definition of z_{n_k} and (16), we have

$$\langle \nabla f(z_{n_k}) - \nabla f(w_{n_k}) + \lambda_{n_k} F w_{n_k}, x - z_{n_k}\rangle \geq 0, \quad \forall\, x \in C.$$

This implies that

$$\lambda_{n_k}\langle F w_{n_k}, x - z_{n_k}\rangle \geq \langle \nabla f(w_{n_k}) - \nabla f(z_{n_k}), x - z_{n_k}\rangle, \quad \forall\, x \in C.$$

Thus we have

$$\begin{aligned}&\langle F z_{n_k}, x - z_{n_k}\rangle \\ &\quad \geq \frac{1}{\lambda_{n_k}}\langle \nabla f(w_{n_k}) - \nabla f(z_{n_k}), x - z_{n_k}\rangle - \langle F w_{n_k} - F z_{n_k}, x - z_{n_k}\rangle, \quad \forall\, x \in C.\end{aligned} \tag{39}$$

Since $\lim_{k\to\infty}\|w_{n_k} - z_{n_k}\| = 0$, ∇f is uniformly continuous and F is Lipschitz continuous, it then follows from (39) that

$$0 \leq \liminf_{k\to\infty}\langle F z_{n_k}, x - z_{n_k}\rangle \leq \limsup_{k\to\infty}\langle F z_{n_k}, x - z_{n_k}\rangle, \quad \forall\, x \in C. \tag{40}$$

In order to prove $z \in S_D$, we consider two possible cases as follows:

Case 1. Suppose that $\limsup_{k\to\infty}\langle F z_{n_k}, x - z_{n_k}\rangle > 0$. Then there exists a subsequence $\{z_{n_{k_j}}\}$ of $\{z_{n_k}\}$ such that $\limsup_{j\to\infty}\langle F z_{n_{k_j}}, x - z_{n_{k_j}}\rangle > 0$. Thus there exists $N_0 \geq 1$ such that $\langle F z_{n_{k_j}}, x - z_{n_{k_j}}\rangle > 0$ for all $j \geq N_0$. By the quasi-monotonicity of F, we have

$$\langle Fx, x - z_{n_{k_j}}\rangle \geq 0, \quad \forall\, j \geq N_0. \tag{41}$$

Taking the limit as $j \to \infty$ in (41), we have $\langle Fx, x - z\rangle \geq 0$ for all $x \in C$, that is, $z \in S_D$.

Case 2. Suppose that $\limsup_{k\to\infty}\langle Fz_{n_k}, x - z_{n_k}\rangle = 0$. Then from (40), we get

$$\lim_{k\to\infty}\langle Fz_{n_k}, x - z_{n_k}\rangle = 0, \quad \forall\, x \in C. \tag{42}$$

Set $\epsilon_k := |\langle Fz_{n_k}, x - z_{n_k}\rangle| + \frac{1}{k}$ for $k \geq 1$. Thus we have

$$\langle Fz_{n_k}, x - z_{n_k}\rangle + \epsilon_k > 0, \quad \forall\, x \in C. \tag{43}$$

For each $k \geq 1$, we note that $\{z_{n_k}\} \subset C$ and $Fz_{n_k} \neq 0$. Let $v_{n_k} := \frac{Fz_{n_k}}{\|Fz_{n_k}\|^2}$, we have $\langle Fz_{n_k}, v_{n_k}\rangle = 1$. Thus from (43), we obtain $\langle Fz_{n_k}, x - z_{n_k}\rangle + \epsilon_k\langle Fz_{n_k}, v_{n_k}\rangle > 0$. That is, $\langle Fz_{n_k}, x + \epsilon_k v_{n_k} - z_{n_k}\rangle > 0$. The quasi-monotonicity of F implies that $\langle F(x + \epsilon_k v_{n_k}), x + \epsilon_k v_{n_k} - z_{n_k}\rangle \geq 0$. It then follows that

$$\begin{aligned}\langle Fx, x - z_{n_k}\rangle &\geq \langle Fx - F(x + \epsilon_k v_{n_k}), x + \epsilon_k v_{n_k} - z_{n_k}\rangle - \langle Fx, \epsilon_k v_{n_k}\rangle \\ &\geq -\|Fx - F(x + \epsilon_k v_{n_k})\|\,\|x + \epsilon_k v_{n_k} - z_{n_k}\| - \|Fx\|\,\|\epsilon_k v_{n_k}\| \\ &\geq -L\|\epsilon_k v_{n_k}\|\,\|x + \epsilon_k v_{n_k} - z_{n_k}\| - \|Fx\|\,\|\epsilon_k v_{n_k}\|. \end{aligned} \tag{44}$$

Now, we show that $\lim_{k\to\infty}\epsilon_k v_{n_k} = 0$. Since $z_{n_k} \rightharpoonup z \in C$ and F satisfies Condition $(B3)$, we have

$$\|Fz\| \leq \liminf_{k\to\infty}\|Fz_{n_k}\|.$$

We assume that $Fz \neq 0$ (otherwise, $z \in S$). It follows that

$$0 \leq \limsup_{k\to\infty}\|\epsilon_k v_{n_k}\| = \limsup_{k\to\infty}\left(\frac{\epsilon_k}{\|Fz_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty}\epsilon_k}{\liminf_{k\to\infty}\|Fz_{n_k}\|} \leq \frac{0}{\|Fz\|} = 0,$$

which implies that $\lim_{k\to\infty}\epsilon_k v_{n_k} = 0$. Taking limit as $k \to \infty$ in (44), by the boundedness of $\{\|x + \epsilon_k v_{n_k} - z_{n_k}\|\}$, we obtain $\langle Fx, x - z\rangle \geq 0$. Therefore, $z \in S_D$. ■

Now, we state and prove the weak convergence of Algorithm 1.

Theorem 3.5: *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then $\{x_n\}$ converges weakly to a point of $S_D \subset S$.*

Proof: Since f is Legendre, from (19) and (24), we see that

$$\lim_{n\to\infty}\|\nabla f(w_n) - \nabla f(x_n)\| = \lim_{n\to\infty}\beta_n\|\nabla f(x_{n-1}) - \nabla f(x_n)\| = 0.$$

By the uniform continuity of ∇f^*, we obtain

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|\nabla f^*(\nabla f(w_n)) - \nabla f^*(\nabla f(x_n))\| = 0. \tag{45}$$

As proved in Lemma 3.3, we have $\{x_n\}$ is bounded. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z \in H$. From (45), we also get $w_{n_k} \rightharpoonup z$.

This, together with (38) and Lemma 3.4, yields that $z \in S_D$. We now show that the sequence $\{x_n\}$ converges weakly to z. In order to do this, it is sufficient to show that $\{x_n\}$ has a unique weak cluster point in S_D. Let $\{x_{m_k}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_k} \rightharpoonup z'$ with $z' \neq z$. As prove in above, we have $z' \in S_D$. From Lemma 3.3, we have $\lim_{n\to\infty} D_f(p, x_n)$ exists for any $p \in S_D$, it follows from Lemma 2.5 that

$$\begin{aligned}\lim_{n\to\infty} D_f(z, x_n) &= \lim_{k\to\infty} D_f(z, x_{n_k}) = \liminf_{k\to\infty} D_f(z, x_{n_k}) \\ &< \liminf_{k\to\infty} D_f(z', x_{n_k}) = \lim_{k\to\infty} D_f(z', x_{m_k}) \\ &= \lim_{n\to\infty} D_f(z', x_n).\end{aligned}$$

Moreover, in a similar way, we have $\lim_{n\to\infty} D_f(z', x_n) < \lim_{n\to\infty} D_f(z, x_n)$. This is a contradiction. Therefore, $z = z'$ and the proof is complete. ■

Next, we also proposed an inertial-like extragradient-type method for solving VIP. The algorithm is shown as follows.

Algorithm 2: Modified inertial extragradient-type method for VIP

Initialization: Choose β, δ, λ_1, μ, $\{\xi_n\}$ and $\{p_n\}$ satisfy the same conditions as in Algorithm 1. Let $x_0, x_1 \in H$ be arbitrary.

Iterative steps: Given the iterates x_{n-1} and x_n for each $n \geq 1$, calculate x_{n+1} as follows:

Step 1. Choose β_n such that $\beta_n \in [0, \bar{\beta}_n]$, where $\bar{\beta}_n$ is defined as in (18).

Step 2. Compute

$$\begin{cases} w_n = \nabla f^*(\nabla f(x_n) + \beta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ z_n = \Pi_C^f \nabla f^*(\nabla f(w_n) - \lambda_n F w_n), \\ y_n = \Pi_C^f \nabla f^*(\nabla f(w_n) - \lambda_n F z_n), \\ x_{n+1} = \nabla f^*((1-\delta)\nabla f(x_n) + \delta \nabla f(y_n)), \quad \forall\, n \geq 1, \end{cases} \tag{46}$$

where λ_n is defined as in (20). Set $n := n + 1$ and go to the iterative step.

Lemma 3.6: *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then, for each $p \in S_D$, the following inequality holds:*

$$D_f(p, y_n) \leq (1 - \beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}) - \sigma_n',$$

where $\sigma_n' := (1 - \frac{\mu}{\kappa} \frac{\lambda_n}{\lambda_{n+1}}) D_f(z_n, w_n) + (1 - \frac{\mu}{\kappa} \frac{\lambda_n}{\lambda_{n+1}}) D_f(y_n, z_n)$.

Proof: Let $p \in S_D$. In view of (17), we have

$$
\begin{aligned}
&D_f(p, y_n)\\
&\quad = D_f(p, \Pi_C^f \nabla f^*(\nabla f(w_n) - \lambda_n F z_n))\\
&\quad \le D_f(p, \nabla f^*(\nabla f(w_n) - \lambda_n F z_n)) - D_f(y_n, \nabla f^*(\nabla f(w_n) - \lambda_n F z_n))\\
&\quad = V_f(p, \nabla f(w_n) - \lambda_n F z_n) - V_f(y_n, \nabla f(w_n) - \lambda_n F z_n)\\
&\quad = f(p) - \langle p, \nabla f(w_n) - \lambda_n F z_n\rangle + f^*(\nabla f(w_n) - \lambda_n F z_n) - f(y_n)\\
&\qquad + \langle y_n, \nabla f(w_n) - \lambda_n F z_n\rangle - f^*(\nabla f(w_n) - \lambda_n F z_n)\\
&\quad = f(p) - \langle p, \nabla f(w_n)\rangle + \lambda_n\langle p, F z_n\rangle - f(y_n) + \langle y_n, \nabla f(w_n)\rangle - \lambda_n\langle y_n, F z_n\rangle\\
&\quad = f(p) - \langle p, \nabla f(w_n)\rangle + f(w_n) - f(y_n) + \langle y_n \nabla f(w_n)\rangle\\
&\qquad - f(w_n) + \lambda_n\langle p, F z_n\rangle - \lambda_n\langle y_n, F z_n\rangle\\
&\quad = D_f(p, w_n) - D_f(y_n, w_n) - \lambda_n\langle F z_n, y_n - p\rangle\\
&\quad = D_f(p, w_n) - D_f(y_n, w_n) - \lambda_n\langle F z_n, z_n - p\rangle + \lambda_n\langle F z_n, z_n - y_n\rangle. \qquad (47)
\end{aligned}
$$

Since $z_n \in C$, it follows from the dual VIP that $\langle F z_n, z_n - p\rangle \ge 0$. Thus from (47), we have

$$D_f(p, y_n) \le D_f(p, w_n) - D_f(y_n, w_n) + \lambda_n\langle F z_n, z_n - y_n\rangle. \qquad (48)$$

By the three-point identity, we have

$$D_f(y_n, w_n) = D_f(y_n, z_n) + D_f(z_n, w_n) - \langle \nabla f(w_n) - \nabla f(z_n), y_n - z_n\rangle. \qquad (49)$$

Substituting (49) into (48), we have

$$
\begin{aligned}
D_f(p, y_n) &\le D_f(p, w_n) - D_f(y_n, z_n) - D_f(z_n, w_n)\\
&\quad + \langle \nabla f(w_n) - \nabla f(z_n), y_n - z_n\rangle + \lambda_n\langle F z_n, z_n - y_n\rangle\\
&= D_f(p, w_n) - D_f(y_n, z_n) - D_f(z_n, w_n) + \lambda_n\langle F w_n - F z_n, y_n - z_n\rangle\\
&\quad + \langle \nabla f(w_n) - \lambda_n F w_n - \nabla f(z_n), y_n - z_n\rangle. \qquad (50)
\end{aligned}
$$

By the definition of z_n and (16), we have

$$\langle \nabla f(w_n) - \lambda_n F w_n - \nabla f(z_n), y_n - z_n\rangle \le 0. \qquad (51)$$

Combining (50) and (51), we obtain

$$D_f(p, y_n) \le D_f(p, w_n) - D_f(y_n, z_n) - D_f(z_n, w_n) + \lambda_n\langle F w_n - F z_n, y_n - z_n\rangle. \qquad (52)$$

The rest of the proof follows similarly to the proof of Lemma 3.2, and here, is omitted. Therefore, this lemma is proved. ∎

Theorem 3.7: *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then $\{x_n\}$ converges weakly to a point of $S_D \subset S$.*

Proof: The proof is similar to Theorem 3.5, so we omit. ■

Finally, we propose the following inertial-like subgradient extragradient-type method, which is a slight modification of Algorithm 2.

Algorithm 3: Modified inertial subgradient extragradient-type method for VIP

Initialization: Choose β, δ, λ_1, μ, $\{\xi_n\}$ and $\{p_n\}$ satisfy the same conditions as in Algorithm 1. Let $x_0, x_1 \in H$ be arbitrary.

Iterative steps: Given the iterates x_{n-1} and x_n for each $n \geq 1$, calculate x_{n+1} as follows:

Step 1. Choose β_n such that $\beta_n \in [0, \bar{\beta}_n]$, where $\bar{\beta}_n$ is defined as in (18).

Step 2. Compute

$$\begin{cases} w_n = \nabla f^*(\nabla f(x_n) + \beta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ z_n = \Pi_C^f \nabla f^*(\nabla f(w_n) - \lambda_n F w_n), \\ y_n = \Pi_{T_n}^f \nabla f^*(\nabla f(w_n) - \lambda_n F z_n), \\ x_{n+1} = \nabla f^*((1-\delta)\nabla f(x_n) + \delta \nabla f(y_n)), \quad \forall\, n \geq 1, \end{cases} \tag{53}$$

where T_n is a half-space defined by

$$T_n := \{x \in H : \langle \nabla f(w_n) - \lambda_n F w_n - \nabla f(z_n), x - z_n \rangle \leq 0\}$$

and λ_n is defined as in (20). Set $n := n+1$ and go to the iterative step.

Lemma 3.8: *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then, for each $p \in S_D$, the following inequality holds:*

$$D_f(p, y_n) \leq (1-\beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}) - \sigma_n'',$$

where $\sigma_n'' := (1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}) D_f(z_n, w_n) + (1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}) D_f(y_n, z_n)$.

Proof: It is observed that a half-space T_n is closed and convex. Then we can replace one Bregman projection Π_C^f in Algorithm 2 by $\Pi_{T_n}^f$, that is, $y_n = \Pi_{T_n}^f \nabla f^*(\nabla f(w_n) - \lambda_n F z_n)$. Using the same arguments as in Lemma 3.6, we arrive at

$$\begin{aligned} D_f(p, y_n) \leq{}& D_f(p, w_n) - D_f(y_n, z_n) - D_f(z_n, w_n) + \lambda_n \langle F w_n - F z_n, y_n - z_n \rangle \\ &+ \langle \nabla f(w_n) - \lambda_n F w_n - \nabla f(z_n), y_n - z_n \rangle. \end{aligned} \tag{54}$$

From the definition of y_n, it is clear that $y_n \in T_n$ and hence

$$\langle \nabla f(w_n) - \lambda_n F w_n - \nabla f(z_n), y_n - z_n \rangle \leq 0. \tag{55}$$

Combining (54) and (55), we have

$$D_f(p, y_n) \leq D_f(p, w_n) - D_f(y_n, z_n) - D_f(z_n, w_n) + \lambda_n \langle F w_n - F z_n, y_n - z_n \rangle. \tag{56}$$

The rest of the proof follows similarly to the proof of Lemma 3.2, and here, is omitted. ■

Theorem 3.9: *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then $\{x_n\}$ converges weakly to the point of $S_D \subset S$.*

Proof: The proof is similar to Theorem 3.5, so we omit. ■

Next, we present the strong convergence theorem of Algorithm 1 when the mapping F is strongly quasi-monotone. Recall that $F : H \to H$ is said to be *strongly quasi-monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Fx, y - x \rangle > 0 \Rightarrow \langle Fy, y - x \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in C.$$

It is clear that strongly quasi-monotone mapping is a quasi-monotone mapping.

Theorem 3.10: *Suppose that $F : H \to H$ strongly quasi-monotone and Conditions 1, 2 and 3 $(B2) - (B4)$ are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to a point in $S_D \subset S$.*

Proof: Let $p \in S_D$. In the same way as in the proof of Lemma 3.2, we can show that

$$D_f(p, y_n) \leq D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n \langle F w_n - F z_n, y_n - z_n \rangle \\ - \lambda_n \langle F z_n, z_n - p \rangle. \tag{57}$$

Note that $z_n \in C$ and $p \in S_D$. The strong quasi-monotonicity of F ensures that $\langle Fz_n, z_n - p\rangle \geq \gamma \|z_n - p\|^2$ for $\gamma > 0$. Then from (57), we have

$$\begin{aligned}
&D_f(p, y_n)\\
&\quad \leq D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \lambda_n \|Fw_n - Fz_n\| \|y_n - z_n\|\\
&\qquad - \gamma\lambda_n \|z_n - p\|^2\\
&\quad \leq D_f(p, w_n) - D_f(z_n, w_n) - D_f(y_n, z_n) + \frac{\mu\lambda_n}{2\lambda_{n+1}} \|w_n - z_n\|^2\\
&\qquad + \frac{\mu\lambda_n}{2\lambda_{n+1}} \|y_n - z_n\|^2 - \gamma\lambda_n \|z_n - p\|^2\\
&\quad \leq D_f(p, w_n) - \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, w_n) - \left(1 - \frac{\mu}{\kappa}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, z_n)\\
&\qquad - \gamma\lambda_n \|z_n - p\|^2.
\end{aligned} \tag{58}$$

Note that

$$D_f(p, w_n) \leq (1 - \beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}). \tag{59}$$

It follows from (58) and (59) that

$$D_f(p, y_n) \leq (1 - \beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}) - \gamma\lambda_n \|z_n - p\|^2 - \sigma_n, \tag{60}$$

where σ_n is defined as in Lemma 3.2. As proved in Lemma 3.2, we have $\sigma_n \geq 0$ for all $n \geq n_0$. This implies that

$$D_f(p, y_n) \leq (1 - \beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1}).$$

Using the same arguments as in Lemma 3.3, we can show that $\{x_n\}$ is bounded and

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq (1 - \delta) D_f(p, x_n) + \delta D_f(p, y_n)\\
&\leq (1 - \delta) D_f(p, x_n) + \delta[(1 - \beta_n) D_f(p, x_n) + \beta_n D_f(p, x_{n-1})\\
&\quad - \gamma\lambda_n \|z_n - p\|^2 - \sigma_n]\\
&\leq D_f(p, x_n) + \delta\beta_n \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right]_+\\
&\quad - \delta\gamma\lambda_n \|z_n - p\|^2 - \delta\sigma_n.
\end{aligned} \tag{61}$$

Moreover, we can show that $\sum_{n=1}^{\infty} \beta_n [D_f(p, x_{n-1}) - D_f(p, x_n)]_+ < \infty$ and consequently $\lim_{n\to\infty} \beta_n [D_f(p, x_{n-1}) - D_f(p, x_n)]_+ = 0$. This gives $\lim_{n\to\infty} D_f$

(p, x_n) exists and so $\lim_{n\to\infty} D_f(z_n, w_n) = \lim_{n\to\infty} D_f(y_n, z_n) = 0$. Hence

$$\lim_{n\to\infty} \|z_n - w_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{62}$$

From (61), we see that

$$\delta\gamma\lambda_n\|z_n - p\|^2 \leq D_f(p, x_n) - D_f(p, x_{n+1}) + \delta\beta_n \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right]_+ . \tag{63}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda > 0$, it follows from (63) that

$$\lim_{n\to\infty} \|z_n - p\| = 0. \tag{64}$$

Also, from (24), we have

$$\lim_{n\to\infty} \|\nabla f(w_n) - \nabla f(x_n)\| = \lim_{n\to\infty} \beta_n\|\nabla f(x_{n-1}) - \nabla f(x_n)\| = 0$$

and so

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{65}$$

It then follows from (62), (64) and (65) that

$$\|x_n - p\| \leq \|x_n - w_n\| + \|w_n - z_n\| + \|z_n - p\| \to 0.$$

Therefore, $\{x_n\}$ converges strongly to $p \in S_D$. This completes the proof. ■

4. Numerical and image restoration experiments

In this section, we present some numerical experiments to illustrate the performance of our proposed methods, including a comparison with some existing methods. All the programs were implemented in MATLAB R2021b on Intel(R) Core(TM) i7-7700HQ CPU@2.80GHZ computer with RAM 8.00GB. We denote the number of iterations by 'Iter.'

Example 4.1: Let the operator $F : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$\begin{aligned}
F(x) &= F_1(x) + F_2(x),\\
F_1(x) &= (f_1(x), f_2(x), \ldots, f_m(x)),\\
F_2(x) &= Dx + d,\\
f_i(x) &= x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_ix_{i+1}, \quad i = 1, 2, \ldots, m,\\
x_0 &= x_{m+1} = 0,
\end{aligned}$$

where $d = (-1, -1, \ldots, -1)^T$ and D is a square matrix of order m, given by

$$d_{ij} = \begin{cases} 4 & \text{if } i = j,\\ 1 & \text{if } i - j = 1,\\ -2 & \text{if } i - j = -1,\\ 0 & \text{otherwise.} \end{cases}$$

Set $C = \{x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m : \|x\| \le 1 \text{ and } x_i \ge \frac{1}{2\sqrt{m}} > 0,\ i = 1, 2, \ldots, m\}$. The VIP defined by F and C that is considered by many authors (see Refs [23,24]). In this experiment, we perform the numerical tests of Algorithm 1, Algorithm 2 with the Boltzmann–Shannon entropy $f(x) = \sum_{i=1}^m x_i \ln(x_i)$, which generates the Kullback–Leibler distance $D_f(x, y) = \sum_{i=1}^m (x_i \ln(\frac{x_i}{y_i}) + y_i - x_i)$ and compare their performance of them with Algorithm 3.1 of Wang et al. [38] (namely, WAlg1), Liu and Yang [25] (namely, LYAlg3.1), Algorithm 3.1 of Hieu and Cholamjiak [23] (namely, HCAlg3.1), Algorithm 2.1 of Ye and He [2] (namely, YHAlg2.1) and Algorithm 3.12 of Alakoya et al. [22] (namely, AMAlg 3.12). The Bregman projection onto C is computed by using the function fmincon in Optimization Toolbox of MATLAB. The parameters for algorithms are chosen as follows:

- Alg1: $\beta = 0.9$, $\delta = 0.9$, $\mu = 0.9$, $p_n = 0$, $\xi_n = \frac{0.1}{n^2}$, $\lambda_1 = 1$ and $x_0 = x_1 = (0.1, \ldots, 0.1)^T$;
- Alg2: $\beta = 0.9$, $\delta = 0.9$, $\mu = 0.85$, $p_n = 0$, $\xi_n = \frac{0.1}{n^2}$, $\lambda_1 = 1$ and $x_0 = x_1 = (0.1, \ldots, 0.1)^T$;
- WAlg3.1: $\alpha = 0.6$, $\rho_1 = 0.7$, $\rho_2 = \frac{2}{3}$, $\nu = 0.1$, $\mu = 0.7$, $\mu' = 0.8$, $\gamma = 1.99$, $\gamma' = 1.993$, $M = 10^{15}$, $\lambda_1 = 1$, $\epsilon_n = \frac{10000}{n^{1.001}}$, $\theta_n = \frac{10}{n}$, $\mu_n = \frac{1}{n^{0.5}}$ and $x_0 = x_1 = (0.1, \ldots, 0.1)^T$;
- LYAlg3.1: $\mu = 0.5$, $p_n = \frac{100}{(n+1)^{1.1}}$, $\lambda_0 = 1$ and $x_1 = (0.1, \ldots, 0.1)^T$;
- HCAlg3.1: $\mu = 0.41$, $\beta = 1$, $\alpha_0 = 1$ and $x_0 = y_0 = y_{-1} = (0.1, \ldots, 0.1)^T$;
- YHAlg2.1: $\gamma = 0.4$, $\sigma = 0.99$ and $x_1 = (0.1, \ldots, 0.1)^T$;
- AMAlg3.12: $\lambda_0 = 0.9$, $\theta = 0.8$, $\mu = 0.7$, $\rho_n = \frac{1000}{(n+1)^2}$, $f(x) = \frac{x}{3}$, $\xi_n = \frac{1}{(2n+1)^3}$, $\alpha_n = \frac{1}{2n+1}$ and $x_0 = x_1 = (0.1, \ldots, 0.1)^T$.

To terminate the iterations, we use the condition $E_n = \|x_{n+1} - x_n\|^2 < 10^{-9}$ for all the algorithms. Two different cases of m are considered.

Case I: $m = 10$;

Case II: $m = 15$.

The numerical results are presented in Table 1 and Figure 3.

It can be seen from Table 1 and Figure 3 that our proposed algorithms in each case have a better convergence than the above other algorithms in number of iterations.

Table 1. Numerical results of Example 4.1.

	Alg1	Alg2	WAlg3.1	LYAlg3.1	HCAlg3.1	YHAlg2.1	AMAlg3.12
	Iter.	Iter.	Iter.	Iter.	Iter.	Iter.	Iter.
Case I	49	51	66	71	85	55	105
Case II	51	53	83	73	78	76	117

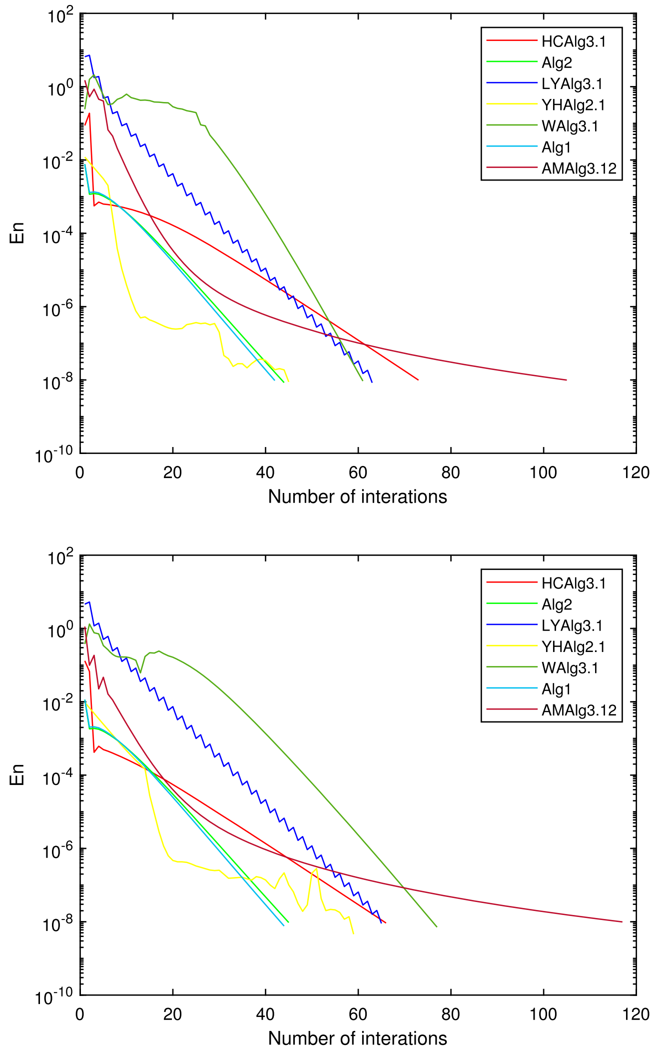


Figure 3. Numerical behaviour of E_n for Example 4.1, (up): Case I and (down): Case II.

Example 4.2: This example is taken from [25]. Let $C = [-1, 1]$ and

$$Fx = \begin{cases} 2x - 1, & x > 1, \\ x^2, & x \in [-1, 1], \\ -2x - 1, & x < -1. \end{cases}$$

Then F is a quasi-monotone and Lipschitz continuous mapping but neither pseudo-monotone nor monotone. Moreover, we have $S_D = \{-1\}$ and $S = \{-1, 0\}$. In this experiment, we perform the numerical tests of Algorithm 1, Algorithm 2 and Algorithm 3 with the squared norm $f(x) = \frac{1}{2}\|x\|^2$, which generates the squared Euclidean distance $D_f(x, y) = \frac{1}{2}\|x - y\|^2$ and compare their performance of them with Algorithm 3.1 of Liu and Yang [25] (LYAlg3.1), Algorithm 2.1 of Ye and He [2] (YHAlg2.1) and Algorithm 3.12 of Alakoya et al. [22] (AMAlg 3.12). The parameters of each algorithm are set as follows:

- Alg1: $\beta = 0.5, \delta = 0.5, \mu = 0.8, p_n = 0, \xi_n = \frac{1}{2n+10}$ and $\lambda_1 = 1$;
- Alg2: $\beta = 0.5, \delta = 0.7, \mu = 0.8, p_n = 0, \xi_n = \frac{1}{2n+10}$ and $\lambda_1 = 1$;
- Alg3: $\beta = 0.5, \delta = 0.8, \mu = 0.8, p_n = 0, \xi_n = \frac{1}{2n+10}$ and $\lambda_1 = 1$;
- LYAlg3.1: $\mu = 0.5, p_n = \frac{100}{(n+1)^{1.1}}$ and $\lambda_0 = 1$;
- AMAlg3.12: $\lambda_0 = 0.9, \theta = 0.8, \mu = 0.7, \rho_n = \frac{1000}{(n+1)^2}, f(x) = \frac{x}{3}, \xi_n = \frac{1}{(2n+1)^3}$ and $\alpha_n = \frac{1}{2n+1}$;
- YHAlg2.1: $\gamma = 0.4, \sigma = 0.99$.

To terminate the iterations, we use the condition $E_n = \min\{\|x_n - 0\|, \|x_n + 1\|\} < 10^{-5}$ for all the algorithms. The following two cases of x_0 and x_1 are considered.

Case I: Take $x_0 = -0.1, x_1 = -0.7$ in Alg1, Alg2, Alg3, LYAlg3.1, YHAlg2.1 and $x_0 = 0.7, x_1 = 0.1$ in AMAlg3.12;

Case II: Take $x_0 = -0.25, x_1 = -0.85$ in Alg1, Alg2, Alg3, LYAlg3.1, YHAlg2.1 and $x_0 = 0.85, x_1 = 0.25$ in AMAlg3.12.

The numerical results are presented in Table 2 and Figure 4.

Next, we consider a numerical example in infinite dimensional Hilbert spaces which is taken from Ref. [26].

Example 4.3: Let $H = \ell_2$, where ℓ_2 is a real Hilbert space whose elements are the square-summable sequence of real numbers, that is, $\ell_2 = \{x = (v_1, v_2, \dots, v_i, \dots), i = 1, 2, \dots, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$. Let $a, b \in \mathbb{R}$ be such that $b > a > \frac{b}{2} > 0$. Let $C = \{x \in \ell_2 : \|x\| \le a\}$ and $Fx = (b - \|x\|)x$. Then F is quasi-monotone and Lipschitz continuous with $S_D = \{0\}$. We set $a = 3$ and $b = 5$. In this experiment, we perform the numerical tests of Algorithm 1, Algorithm 2 and Algorithm 3 with the squared norm $f(x) = \frac{1}{2}\|x\|^2$ and compare the performance of them with Algorithm 3.12 of Alakoya et al. [22] (AMAlg 3.12). The parameters of each algorithm are set as follows:

- Alg1: $\beta = 0.9, \delta = 0.9, \mu = 0.9, p_n = 0$ and $\xi_n = \frac{0.1}{n^2}, \lambda_1 = 1$;
- Alg2: $\beta = 0.9, \delta = 0.9, \mu = 0.85, p_n = 0$ and $\xi_n = \frac{0.1}{n^2}, \lambda_1 = 1$;
- Alg3: $\beta = 0.4, \delta = 0.95, \mu = 0.6, p_n = 0$ and $\xi_n = \frac{0.1}{n^2}, \lambda_1 = 1$;

Table 2. Numerical results of Example 4.2.

	Alg1	Alg2	Alg3	LYAlg3.1	AMAlg3.12	YHAlg2.1
	Iter.	Iter.	Iter.	Iter.	Iter.	Iter.
Case I	6	10	9	19	48	17
Case II	14	9	7	17	50	15

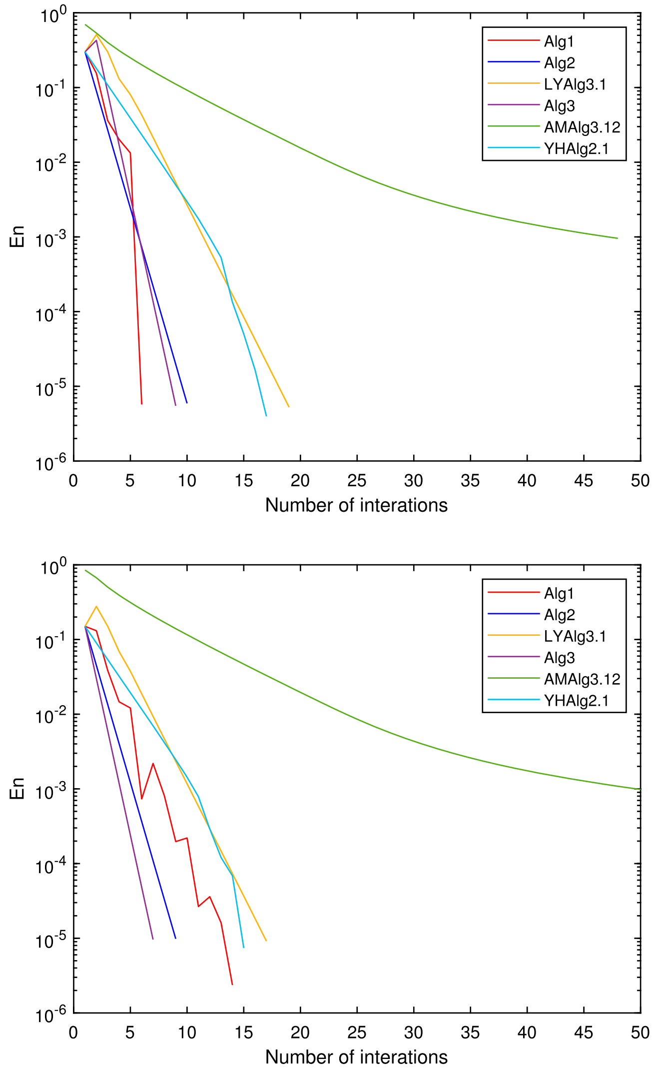


Figure 4. Numerical behaviour of E_n for Example 4.2, (up): Case I and (down): Case II.

- AMAlg3.12: $\lambda_0 = 0.9, \theta = 0.8, \mu = 0.7, \rho_n = \frac{1000}{(n+1)^2}, f(x) = \frac{x}{3}, \xi_n = \frac{1}{(2n+1)^3}$ and $\alpha_n = \frac{1}{2n+1}$.

We use the stopping condition $E_n = \|x_{n+1} - 0\|^2 < 10^{-4}$ for all the algorithms. The following two cases of x_0 and x_1 are considered.

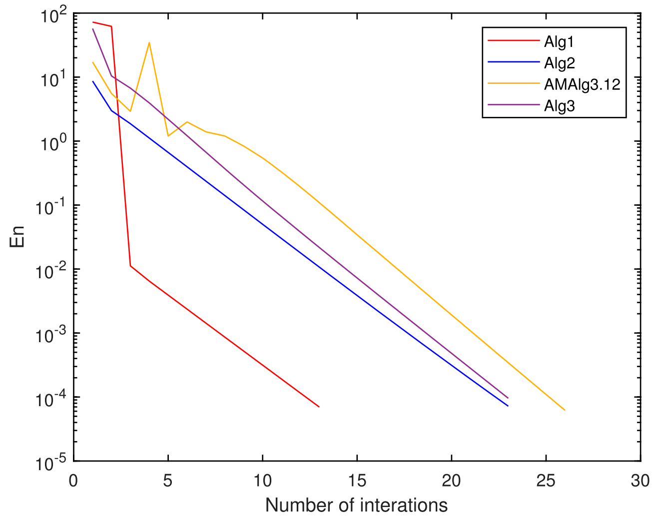
Case I: $x_0 = x_1 = (\underbrace{0.5, \ldots, 0.5}_{350}, 0, \ldots, 0, \ldots)$;

Case II: $x_0 = x_1 = (\underbrace{0.7, \ldots, 0.7}_{500}, 0, \ldots, 0, \ldots)$.

The numerical results are presented in Table 3 and Figure 5.

Table 3. Numerical results of Example 4.3.

	Alg1	Alg2	Alg3	AMAlg3.12
	Iter.	Iter.	Iter.	Iter.
Case I	13	23	23	26
Case II	23	23	24	29



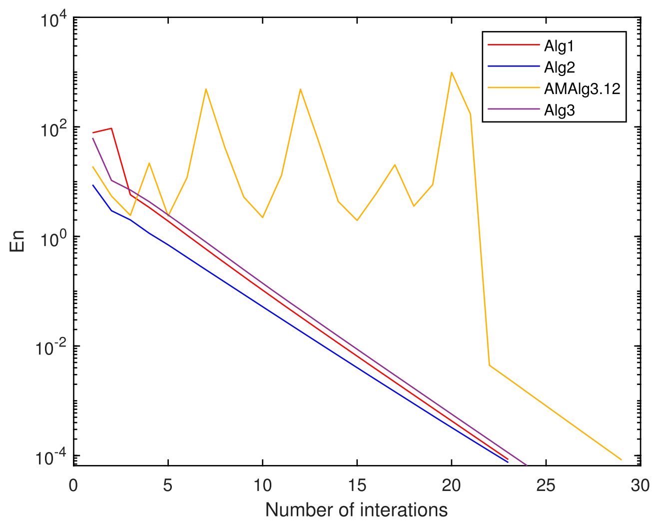


Figure 5. Numerical behaviour of E_n for Example 4.3, (up): Case I and (down): Case II.

From Table 3 and Figure 5, we see that our proposed algorithms have a less number of iterations than Algorithm 3.12 of Alakoya et al. [22]. This shows that our proposed algorithms have a good convergence behaviour in number of iterations.

Finally, we consider a numerical example in the image recovery problem.

Example 4.4: In this experiment, we assume that noisy images have a dimension $n \times n$ with missing pixels. The model for this problem is given by

$$y = Ax + b,$$

where y is an observed image, x is an original image, b is a noise term and A is a blurring matrix. Now, we consider a model originated from a restored image, that is, to find a solution to the following minimization problem:

$$\min_{x \in C} \frac{1}{2}\|Ax - y\|^2, \tag{66}$$

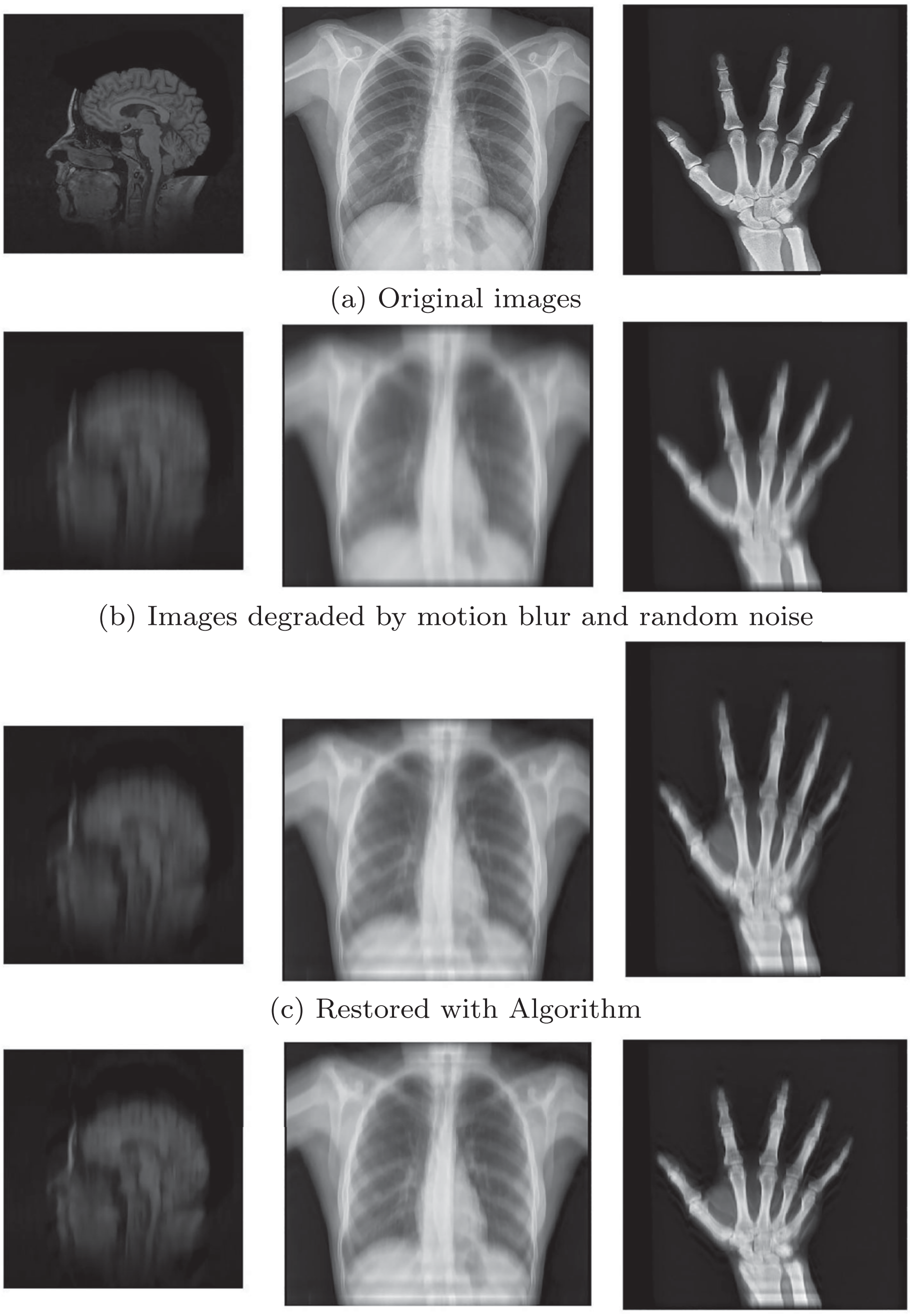
where $\|\cdot\|$ is the Euclidean norm. Then (66) can be expressed as a variational inequality problem by setting $Fx := A^T(Ax - y)$. In this case F is monotone (hence it is quasi-monotone) and Lipschitz continuous with $L = \|A^T A\|$. Now, we set $C = \mathbb{R}^n$. By doing so, the subspace T_n defined in the modified subgradient extragradient-type method (Algorithm 3) is equal to $\mathbb{R}^n$ and hence Algorithms 2 and 3 coincide for this problem. Therefore, we only present the comparison between our Algorithm 1 and Algorithm 2. In this numerical simulations, we use the same control parameters. We choose $\beta = 0.99$, $\delta = 0.5, \lambda_1 = 0.2, p_n = \frac{1}{(n+1)^3}$. Since $\beta_n \leq \bar{\beta}_n \leq \beta = 0.99$, we can set $\beta_n = 0.9$. We use digital X-ray of brain, chest and hand for the test images. The images are degraded using MATLAB blur function '$fspecial('motion', 30, 45)$' and we add random noise with scaling factor 0.001. We initialize the vectors $x_0 = zeros(x)$ and $x_1 = Ax + b$ (where x here is the test image). Finally, we use a tolerance of 10^{-4} and maximum number of iterations (n) to be 100, for all the algorithms. The results are presented in Figure 6.

Observe that it is difficult to tell which algorithm has the best restoration from Figure 6. Thus, we use a well-known metric called the SNR (Signal to Noise Ratio) to measure the quality of the restored image. Using this performance metric, the higher the SNR value for a restored image, the better the restoration process via the algorithm. The SNR is defined as follows:

$$\text{SNR} := 20 \log \frac{\|x\|}{\|x_n - x\|},$$

where x is an original image and x_n is an estimated image of x at nth iteration. The SNR for the restored test images are reported in Table 4.

We observe from the numerical simulations presented in Figure 6 and Table 4, our proposed Algorithm 1 appears to be competitive and promising as it outperforms Algorithm 2 and Algorithm 3 in the restoration process of degraded test images.



(a) Original images

(b) Images degraded by motion blur and random noise

(c) Restored with Algorithm

(d) Restored with our Algorithm

Figure 6. Degraded images and their restorations via Algorithms 1 and 2. (a) Original images; (b) Images degraded by motion blur and random noise; (c) Restored with Algorithm 2 and (d) Restored with our Algorithm 1.

Table 4. The SNR for the restored test images in Figure 6.

Test images	Algorithm 1	Algorithm 2
	SNR	SNR
Digital X-ray of brain	13	23
Digital X-ray of chest	23	22
Digital X-ray of hand	23	22

5. Conclusions

In this paper, we have presented three inertial-like Bregman projection methods for solving variational inequality problems involving quasi-monotone and Lipschitz continuous mappings in a real Hilbert space. The sequence of step-size of our methods is adaptively and explicitly updated without prior knowledge of the Lipschitz constant of the mapping. We have proved the weak convergence theorems of the proposed methods under some suitable conditions imposed on parameters and also, we have proposed the strong convergence of the proposed methods under a strong quasi-monotonicity assumption of the mapping. Finally, we have presented several numerical tests which illustrate the efficiency of our approaches.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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References

[1] Cottle RW, Yao JC. Pseudo-monotone complementarity problems in Hilbert space. J Optim Theory Appl. 1992;75:281–295.
[2] Ye M, He Y. A double projection method for solving variational inequalities without monotonicity. Comput Optim Appl. 2015;60:141–150.
[3] Fichera G. Sul problema elastostatico di Signorini con ambigue condizioni al contorno. Atti Accad Naz Lincci VIII Scr Rend Cl Sci Fis Mat Nat. 1963;34:138–142.

[4] Fichera G. Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Atti Accad Naz Lincci Mem Cl Sci Fis Mat Nat. 1964;7:91–140.
[5] Stampacchia G. Formes bilinearieres coercitivities sur les ensembles convexes. CR Acad Sci Paris. 1964;258:4413–4416.
[6] Baiocchi C, Capelo A. Variational and quasivariational inequalities: applications to free boundary problems. New York (NY): Wiley; 1984.
[7] Bensoussan A, Lions JL. Applications of variational inequalities to stochastic control. Amsterdam (AMS): North-Holland; 1982.
[8] Facchinei F, Pang JS. Finite-dimensional variational inequalities and complementarity problem. New York (NY): Springer; 2003.
[9] Grannessi F, Mangeri A. Variational inequalities and network equilibrium problems. New York (NY): Plenum Press; 1995.
[10] Khobotov EN. Modification of the extragradient method for solving variational inequalities and certain optimization problems. USSR Comput Math & Math Phys. 1989;27:120–127.
[11] Kinderlehrer D, Stampacchia G. An introduction to variational inequalities and their applications. New York (NY): Academic Press Inc; 1980.
[12] Nagurney A. Network economics: a variational inequality approach. Dordrecht: Kluwer Academic Publishers; 1989.
[13] He BS, Liao LZ. Improvements of some projection methods for monotone nonlinear variational inequalities. J Optim Theory Appl. 2002;112:111–128.
[14] Korpelevich GM. The extragradient method for finding saddle points and other problems. Ekon Mate Metody. 1976;12:747–756.
[15] Censor Y, Gibali A, Reich S. The subgradient extragradient method for solving variational inequalities in Hilbert space. J Optim Theory Appl. 2011;148:318–335.
[16] Tseng P. A modified forward-backward splitting method for maximal monotone mapping. SIAM J Control Optim. 2000;38:431–446.
[17] Shehu Y, Dong QL, Jiang D. Single projection method for pseudo-monotone variational inequality in Hilbert spaces. Optimization. 2019;68(1):385–409.
[18] Thong DV, Vinh NT, Cho YJ. Accelerated subgradient extragradient methods for variational inequality problems. J Sci Comput. 2019;80:1438–1462.
[19] Vuong PT. On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. J Optim Theory Appl. 2018;176:399–409.
[20] Zhao TY, Wang DQ, Ceng LC, et al. Quasi-inertial Tseng's extragradient algorithms for pseudomonotone variational inequalities and fixed point problems of quasi-nonexpansive operators. Numer Funct Anal Optim. 2020;42:69–90.
[21] Hieu DV, Anh PK, Muu LD. Modified extragradient-like algorithms with new stepsizes for variational inequalities. Comput Optim Appl. 2019;73:913–932.
[22] Alakoya TO, Mewomo OT, Shehu Y. Strong convergence results for quasimonotone variational inequalities. Math Methods Oper Res. 2021;95:249–279.
[23] Hieu DV, Cholamjiak P. Modified extragradient method with Bregman distance for variational inequalities. Appl Anal. 2022;101(2):655–670.
[24] Hieu DV, Reich S. Two Bregman projection methods for solving variational inequalities. Optimization. 2022;71(7):1777–1802.
[25] Liu H, Yang J. Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput Optim Appl. 2020;77:491–508.
[26] Salahuddin. The extragradient method for quasi-monotone variational inequalities. Optimization. 2020;71(9):2519–2528.

[27] Polyak BT. Some methods of speeding up the convergence of iterates methods. USSR Comput Math Phys. 1964;4(5):1–17.
[28] Attouch H, Maingé PE. Asymptotic behavior of second-order dissipative equations evolution combining potential with non-potential effects. ESAIM Contr Optim Calc. 2011;17:836–857.
[29] Boţ RI, Csetnek ER. Second order forward-backward dynamical systems for monotone inclusion problems. SIAM J Control Optim. 2016;54:1423–1443.
[30] Attouch H, Cabot A. Convergence rate of a relaxed inertial proximal algorithm for convex minimization. Optimization. 2020;69(6):1281–1312.
[31] Ceng LC, Petrusel A, Qin X, et al. A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems. Fixed Point Theory. 2020;21:93–108.
[32] Ceng LC, Petrusel A, Qin X, et al. Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints. Optimization. 2021;70:1337–1358.
[33] Ceng LC, Shang M. Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings. Optimization. 2021;70:715–740.
[34] Ogwo GN, Izuchukwu C, Shehu Y, et al. Convergence of relaxed inertial subgradient extragradient methods for quasimonotone variational inequality problems. J Sci Comput. 2022;90:10.
[35] Reich S, Tuyen TM, Sunthrayuth S, et al. Two new inertial algorithms for solving variational inequalities in reflexive Banach spaces. Numer Funct Anal Optim. 2021;42(16):1954–1984.
[36] Tang Y, Sunthrayuth P. An iterative algorithm with inertial technique for solving the split common null point problem in Banach spaces. Asian-Eur J Math. 2022;15(6):2250120.
[37] Tuyen TM, Sunthrayuth P, Trang NM. An inertial self-adaptive algorithm for the generalized split common null point problem in Hilbert spaces. Rend Circ Mat Palermo. 2022;71(2):537–557.
[38] Wang ZB, Chen X, Yi J, et al. Inertial projection and contraction algorithms with larger step sizes for solving quasimonotone variational inequalities. J Glob Optim. 2022;82:499–522.
[39] Chbani Z, Riahi H. Weak and strong convergence of an inertial proximal method for solving Ky Fan minimax inequalities. Optim Lett. 2013;7:185–206.
[40] Bregman LM. The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput Math Math Phys. 1967;7(3):200–217.
[41] Amid E, Warmuth MK, Anil R, et al. Robust bi-tempered logistic loss based on Bregman divergences. In: Conference on Neural Information Processing Systems. Vancouver; 2019. p. 14987–14996.
[42] Gray RM, Buzo A, Gray AH, et al. Distortion measures for speech processing. IEEE Trans Acoust Speech Signal Process. 1980;28:367–376.
[43] Cesa-Bianchi N, Lugosi G. Prediction, learning, and games. New York: Cambridge University Press; 2006.
[44] Banerjee A, Merugu S, Dhillon I, et al. Clustering with Bregman divergences. J Mach Learn Res. 2005;6:1705–1749.
[45] Fischer A. Quantization and clustering with Bregman divergences. J Multivar Anal. 2010;101:2207–2221.
[46] Chen JZ, Hu HY, Ceng LC. Strong convergence of hybrid Bregman projection algorithm for split feasibility and fixed point problems in Banach spaces. J Nonlinear Sci Appl. 2017;10:192–204.

[47] Gibali A. A new Bregman projection method for solving variational inequalities in Hilbert spaces. Pure Appl Funct Anal. 2018;3:403–415.
[48] Jolaoso LO, Oyewole OK, Aremu KO. A Bregman subgradient extragradient method with self-adaptive technique for solving variational inequalities in reflexive Banach spaces. Optimization. 2022;71:3835–3860. DOI: 10.1080/02331934.2021.1925669 .
[49] Sunthrayuth P, Jolaoso LO, Cholamjiak P. New Bregman projection methods for solving pseudo-monotone variational inequality problem. J Appl Math Comput. 2022;68:1565–1589.
[50] Alber Y, Butnariu D. Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces. J Optim Theory Appl. 1997;92:33–61.
[51] Pathak HK. An introduction to nonlinear analysis and fixed point theory. Singapore: Springer; 1980.
[52] Reich S, Sabach S. A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J Nonlinear Convex Anal. 2009;1:471–485.
[53] Zălinescu C. Convex analysis in general vector spaces. Singapore: World Scientific; 2002.
[54] Beck A. First-order methods in optimization. Philadelphia (PA): Society for Industrial and Applied Mathematics; 2017.
[55] Bauschke HH, Borwein JM, Combettes PL. Essential smoothness, essential strict convexity, and legendre functions in Banach spaces. Commun Contemp Math. 2001;3:615–647.
[56] Beck A, Teboulle M. Mirror descent and nonlinear projected subgradient methods for convex optimization. Oper Res Lett. 2003;31(3):167–175.
[57] Bauschke HH, Bolte J, Teboulle M. A descent lemma beyond Lipschitz gradient continuity: first-order methods revisited and applications. Math Oper Res. 2016;42(2):330–348.
[58] Bauschke HH, Borwein JM. Legendre functions and the method of random Bregman projections. J Convex Anal. 1997;4:27–67.
[59] Bolte J, Sabach S, Teboulle M, et al. First order methods beyond convexity and Lipschitz gradient continuity with applications to quadratic inverse problems. SIAM J Control Optim. 2018;28(3):2131–2151.
[60] Mukkamala MC, Ochs P. Beyond alternating updates for matrix factorization with inertial Bregman proximal gradient algorithms. Adv Neural Inf Process Syst. 2019;4266–4276.
[61] Teboulle M, Vaisbourd Y. Novel proximal gradient methods for nonnegative matrix factorization with sparsity constraints. SIAM J Imaging Sci. 2020;13(1):381–421.
[62] Butnairu D, Resmerita E. Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr Appl Anal. 2006;2006:1–39. Article ID 84919.
[63] Martín-Márquez V, Reich S, Sabach S. Iterative methods for approximating fixed points of Bregman nonexpansive operators. Discrete Contin Dyn Syst-S. 2013;6(4):1043–1063.
[64] Huang YY, Jeng JC, Kuo TY, et al. Fixed point and weak convergence theorems for point-dependent λ-hybrid mappings in Banach spaces. Fixed Point Theory Appl. 2011;2011:105.
[65] Tan KK, Xu HK. Approximating fixed points of nonexpensive mappings by the Ishikawa iteration process. J Math Anal Appl. 1993;178(2):301–308.

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On the Solution of Fractional Biswas–Milovic Model via Analytical Method

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Abstract: Through the use of a unique approach, we study the fractional Biswas–Milovic model with Kerr and parabolic law nonlinearities in this paper. The Caputo approach is used to take the fractional derivative. The method employed here is the homotopy perturbation transform method (HPTM), which combines the homotopy perturbation method (HPM) and Yang transform (YT). The HPTM combines the homotopy perturbation method, He's polynomials, and the Yang transform. He's polynomial is a wonderful tool for dealing with nonlinear terms. To confirm the validity of each result, the technique was substituted into the equation. The described techniques can be used to find the solutions to these kinds of equations as infinite series, and when these series are in closed form, they give a precise solution. Graphs are used to show the derived numerical results. The maple software package is used to carry out the numerical simulation work. The results of this research are highly positive and demonstrate how effective the suggested method is for mathematical modeling of natural occurrences.

Keywords: Yang transform; homotopy perturbation method; Caputo operator; time-fractional Biswas–Milovic model

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1. Introduction

Due to its numerous applications in numerous nonlinear phenomena, fractional calculus (FC) has gained the attention of academics. To describe the memory and heredity characteristics of many phenomena, FC is a reliable source. The expansion of integer to non-integer order of differentiation is known as fractional differentiation. Few phenomenons including quantum mechanics, viscoelasticity, diffusion processes, fluid mechanics, etc., are effectively described by fractional differential equations (FDEs). FC is connected to practical endeavours and is frequently used in human diseases, nanotechnology, chaos theory, optics, and other disciplines, as noted in Refs. [1–4]. A helpful tool for representing nonlinear events in scientific and engineering models is the fractional differential equation. In applied mathematics and engineering, partial differential equations, particularly nonlinear ones, have been utilised to simulate a wide range of scientific phenomena. Fractional-order partial differential equations (FPDEs) allowed researchers to recognise and model a wide range of significant and real-world physical issues in parallel with their work in the physical sciences. It has always been claimed how important it is to obtain approximations for scientists by using either numerical or analytical methods. Because of this, symmetry analysis is a fantastic tool for comprehending partial differential equations, especially when looking at equations generated from mathematical concepts connected to accounting. Despite the notion that symmetry is the foundation of nature, the bulk of observations in the natural world lack it. A clever technique for disguising symmetry is to provide unanticipated symmetry-breaking events. The two categories are finite and infinitesimal symmetry. There are two types of discrete and continuous finite symmetries. Natural symmetries such as parity and temporal inversion are discrete, while space is a continuous transformation. Mathematicians have always been fascinated by patterns.

Due to the numerous engineering and scientific applications of fractional differential equations, they have become more significant and well-liked. For example, these equations are more frequently used to explain phenomena in a wide range of physical processes [5–7], such as biology, acoustics, signal processing, electromagnetics, and many others. The main advantage of fractional differential equations in these and other applications is their non-locality [8–10].

The fractional order differential operator is non-local, whereas the integer order differential operator is commonly conceived of as a local operator. This demonstrates how a system's future state depends on both its current state and its previous state. This increases the utility of fractional calculus, which is one of the reasons it is gaining popularity [11–16]. Therefore, solving fractional differential equations has drawn a lot of attention. The exact solution of a fractional differential equation is often difficult. Numerical methods, such as the perturbation method, have attracted the interest of researchers. However, perturbation approaches have certain important limitations. It is challenging since most nonlinear problems do not have any smaller parameters at all, for example, the approximate solution generally requires a lot of small parameters. Although a proper choice of minor factors might occasionally yield the best outcome, unsuitable choices typically have adverse impact on the solutions [17–20].

This work presents the homotopy perturbation method (HPM) and the Yang transform (YT). Ji-Huan He of Shanghai University introduced the homotopy perturbation method (HPM) in 1998 as a potent tool for solving technical and scientific nonlinear issues [21,22]. Numerous mathematicians have handled the nonlinear equations that appear in engineering and research using the homotopy perturbation approach [23–26]. Refs. [27–31] address the application of the Adomian decomposition method, closely related to the homotopy perturbation method, to various diffusive and transport models (including fractional and nonlinear cases as well). Refs. [32–35] address time-fractional subdiffusion equations and inverse problems of determining their coefficients and fractional orders. Ref. [36] introduces a homotopy perturbation method for nonlinear transport equations. Ref. [37] proposes a perturbational approach to construct analytical approximations based on the double-parameter transformation perturbation expansion method. Ref [38] contains an exhaustive review of various modern fractional calculus applications. Ref [39] discusses some non-standard definitions of Caputo fractional derivatives. Ref. [40] provides an overview of the computational practices used in fractional calculus. Recently, a lot of authors have studied the solutions to partial differential equations, both linear and nonlinear, utilizing a variety of methodologies including the homotopy perturbation transform technique [41,42], the Elzaki transform decomposition method [43,44], the iterative Laplace transform method [45], the homotopy analysis transform method [46], the variational iteration method (VIM) [47,48], and many others.

Now, using HPTM, we will study the fractional model of the Biswas–Milovic equation (BME). The BME generalises the well-known nonlinear Schrodinger's equation to describe solitons transcontinental and transoceanic propagation across optical fibres. The BME is written as [49]

$$\imath\mathbb{F}^{\delta}_{\vartheta} + \lambda\mathbb{F}^{\delta}_{\varphi\varphi} + \chi\mathcal{H}(|\mathbb{F}|^2)\mathbb{F}^{\delta} = 0, \tag{1}$$

$\mathbb{F}(\varphi,\vartheta)$ denotes the wave profile, λ and χ are real-valued constants meeting the condition $\lambda\cdot\chi > 0$, and the parameter $\delta \geq 0$, which transforms the nonlinear Schrödinger equation to BME. The independent variables φ and ϑ denote the distance along the fibre and the time, respectively. The algebraic function $\mathcal{H}$ is real-valued and is assumed to be as smooth as the complex function $\mathcal{H}(|\mathbb{F}|^2) : C \to C$. Assuming that the complex plane C is a 2D linear space R^2 and that the function $\mathcal{H}(|\mathbb{F}|^2)$ is differentiable n times, so

$$\mathcal{H}(|\mathbb{F}|^2) \in U_{l,m=1}^{\infty} C^n((-m,m) \times (-l,l); R^2).$$

Here, we examine the following issue

$$\mathcal{H}(v) = v^m + \beta v^{2m}.$$

Here, parameter m denotes the power law nonlinearity, and β denotes the nonlinear term's coefficient. Researchers have used a variety of methodologies to study the BME. For $\delta = 1$, Ahmed et al. [50] analysed the BME using the Adomian decomposition approach, while Arnous and Mirzazadeh [51] used the HPM for solving the BME. For the first time, Ahmadian and Darvishi [52] examined the generalised version of the sine-cosine method of fractional BME. The (1 + 1) dimensional BME of fractional-order was then explored by Ahmadian and Darvishi [53] using the sec-csc, sech-csch, tan-cot, and tanh-coth approaches.

By using the homotopy perturbation approach, Darvishi and Zaidan [54] studied the nonlinear (1 + 1) dimensional BME of order fraction. Additionally, to examine the fractional BME with the Atangana–Baleanu derivative, Jagdev et al. [55] introduced the fractional homotopy analysis transform method (FHATM) and discussed several novel elements of the discovered solution. There are six sections throughout the entire paper. The introduction is in Section 1, and the definitions and attributes are explained in Section 2. An implementation of the suggested analytical technique is provided in Section 3. The suggested technique are put into practise on a few test examples in Section 4. The conclusion is covered in Section 5.

2. Preliminaries

In this part, we provide the basic definitions related to this study.

Definition 1. *The fractional Caputo derivative is given as [56,57]*

$$D_\vartheta^\varsigma \mathbb{F}(\varphi,\vartheta) = \frac{1}{\Gamma(k-\varsigma)} \int_0^\vartheta (\vartheta-\psi)^{k-\varsigma-1} \mathbb{F}^{(k)}(\varphi,\psi)d\psi, \quad k-1 < \varsigma \le k, \quad k \in \mathbf{N}. \tag{2}$$

Definition 2. *For the function $\mathbb{F}(\vartheta)$, the YT is given as [57]*

$$Y\{\mathbb{F}(\vartheta)\} = M(u) = \int_0^\infty e^{\frac{-\vartheta}{u}} \mathbb{F}(\vartheta)d\vartheta, \;\; \vartheta > 0, \;\; u \in (-\vartheta_1, \vartheta_2), \tag{3}$$

with inverse YT as

$$Y^{-1}\{M(u)\} = \mathbb{F}(\vartheta). \tag{4}$$

Definition 3. *The inverse YT is given by [57]*

$$Y^{-1}[Y(u)] = \mathbb{F}(\vartheta) = \frac{1}{2\pi\imath} \int_{\varsigma-\imath\infty}^{\varsigma+\imath\infty} \mathbb{F}\left(\frac{1}{u}\right) e^{u\vartheta} u du = \Sigma \;\; residues \;\; of \;\; \mathbb{F}\left(\frac{1}{u}\right) e^{u\vartheta} u.$$

Definition 4. *The fractional derivative YT is given as [57]*

$$Y\{\mathbb{F}^{(\varsigma)}(\vartheta)\} = \frac{M(u)}{u^\varsigma} - \sum_{k=0}^{n-1} \frac{\mathbb{F}^{(k)}(0)}{u^{\varsigma-(k+1)}}, \;\; n-1 < \varsigma \le n. \tag{5}$$

3. General Idea of HPTM

We consider the following differential equation to give the general implementation of HPTM.

$$D_\vartheta^\varsigma \mathbb{F}(\varphi,\vartheta) = \mathcal{P}_1[\varphi]\mathbb{F}(\varphi,\vartheta) + \mathcal{Q}_1[\varphi]\mathbb{F}(\varphi,\vartheta), \;\; 0 < \varsigma \le 1, \tag{6}$$

subject to initial conditions

$$\mathbb{F}(\varphi,0)=\xi(\varphi).$$

where $D_\vartheta^\varsigma=\frac{\partial^\varsigma}{\partial\vartheta^\varsigma}$ stand for the Caputo fractional derivative, $\mathcal{P}_1[\varphi]$, $\mathcal{Q}_1[\varphi]$ denote linear and nonlinear terms.

On operating YT, we get

$$Y[D_\vartheta^\varsigma\mathbb{F}(\varphi,\vartheta)]=Y[\mathcal{P}_1[\varphi]\mathbb{F}(\varphi,\vartheta)+\mathcal{Q}_1[\varphi]\mathbb{F}(\varphi,\vartheta)], \tag{7}$$

$$\frac{1}{u^\varsigma}\{M(u)-u\mathbb{F}(0)\}=Y[\mathcal{P}_1[\varphi]\mathbb{F}(\varphi,\vartheta)+\mathcal{Q}_1[\varphi]\mathbb{F}(\varphi,\vartheta)]. \tag{8}$$

After simplification, we get

$$M(\mathbb{F})=u\mathbb{F}(0)+u^\varsigma Y[\mathcal{P}_1[\varphi]\mathbb{F}(\varphi,\vartheta)+\mathcal{Q}_1[\varphi]\mathbb{F}(\varphi,\vartheta)]. \tag{9}$$

By implementing inverse YT, we get

$$\mathbb{F}(\varphi,\vartheta)=\mathbb{F}(\varphi,0)+Y^{-1}[u^\varsigma Y[\mathcal{P}_1[\varphi]\mathbb{F}(\varphi,\vartheta)+\mathcal{Q}_1[\varphi]\mathbb{F}(\varphi,\vartheta)]]. \tag{10}$$

By utilizing the HPM

$$\mathbb{F}(\varphi,\vartheta)=\sum_{k=0}^{\infty}\epsilon^k\mathbb{F}_k(\varphi,\vartheta). \tag{11}$$

having perturbation parameter $\epsilon\in[0,1]$.

The decomposition of nonlinear terms is stated as

$$\mathcal{Q}_1[\varphi]\mathbb{F}(\varphi,\vartheta)=\sum_{k=0}^{\infty}\epsilon^k H_n(\mathbb{F}), \tag{12}$$

and $H_n(\mathbb{F})$ represent He's polynomials as [58]

$$H_n(\mathbb{F}_0,\mathbb{F}_1,\ldots,\mathbb{F}_n)=\frac{1}{\Gamma(n+1)}D_\epsilon^n\left[\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\epsilon^i\mathbb{F}_i\right)\right]_{\epsilon=0}, \tag{13}$$

where $D_\epsilon^n=\frac{\partial^n}{\partial\epsilon^n}$.

By putting (11) and (12) in (10), we obtain

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{F}_k(\varphi,\vartheta)=\mathbb{F}(\varphi,0)+\epsilon\times\left(Y^{-1}\left[u^\varsigma Y\{\mathcal{P}_1\sum_{k=0}^{\infty}\epsilon^k\mathbb{F}_k(\varphi,\vartheta)+\sum_{k=0}^{\infty}\epsilon^k H_k(\mathbb{F})\}\right]\right). \tag{14}$$

Comparing the coefficient of ϵ, we obtain

$$\begin{aligned}
&\epsilon^0:\mathbb{F}_0(\varphi,\vartheta)=\mathbb{F}(\varphi,0),\\
&\epsilon^1:\mathbb{F}_1(\varphi,\vartheta)=Y^{-1}[u^\varsigma Y(\mathcal{P}_1[\varphi]\mathbb{F}_0(\varphi,\vartheta)+H_0(\mathbb{F}))],\\
&\epsilon^2:\mathbb{F}_2(\varphi,\vartheta)=Y^{-1}[u^\varsigma Y(\mathcal{P}_1[\varphi]\mathbb{F}_1(\varphi,\vartheta)+H_1(\mathbb{F}))],\\
&.\\
&.\\
&.\\
&\epsilon^k:\mathbb{F}_k(\varphi,\vartheta)=Y^{-1}[u^\varsigma Y(\mathcal{P}_1[\varphi]\mathbb{F}_{k-1}(\varphi,\vartheta)+H_{k-1}(\mathbb{F}))],
\end{aligned} \tag{15}$$

$k>0, k\in\mathbf{N}$.

Thus, the analytical solution $\mathbb{F}_k(\varphi, \vartheta)$ is obtained using the truncated series

$$\mathbb{F}(\varphi, \vartheta) = \lim_{M \to \infty} \sum_{k=1}^{M} \mathbb{F}_k(\varphi, \vartheta). \tag{16}$$

4. Applications

In this section, we implement HPTM to obtain the solution of time-fractional Biswas–Milovic model. Let us assume nonlinear fractional BME

$$\iota \frac{\partial^{\varsigma} \mathbb{F}}{\partial \vartheta^{\varsigma}} + \lambda \frac{\partial^2 \mathbb{F}}{\partial \varphi^2} + \chi |\mathbb{F}(\varphi, \vartheta)|^2 \mathbb{F}(\varphi, \vartheta) = 0, \quad 0 < \varsigma \leq 1, \tag{17}$$

subject to initial source

$$\mathbb{F}(\varphi, 0) = \exp(\iota \varphi).$$

On operating YT, we get

$$Y\left(\iota \frac{\partial^{\varsigma} \mathbb{F}}{\partial \vartheta^{\varsigma}}\right) = -Y\left[\lambda \frac{\partial^2 \mathbb{F}}{\partial \varphi^2} + \chi |\mathbb{F}(\varphi, \vartheta)|^2 \mathbb{F}(\varphi, \vartheta)\right], \tag{18}$$

After simplification, we get

$$\frac{1}{u^{\varsigma}}\{M(u) - u\mathbb{F}(0)\} = Y\left[\lambda \iota \frac{\partial^2 \mathbb{F}}{\partial \varphi^2} + \iota \chi |\mathbb{F}(\varphi, \vartheta)|^2 \mathbb{F}(\varphi, \vartheta)\right], \tag{19}$$

$$M(u) = u\mathbb{F}(0) + u^{\varsigma} Y\left[\lambda \iota \frac{\partial^2 \mathbb{F}}{\partial \varphi^2} + \iota \chi |\mathbb{F}(\varphi, \vartheta)|^2 \mathbb{F}(\varphi, \vartheta)\right]. \tag{20}$$

By implementing inverse YT, we get

$$\begin{aligned} \mathbb{F}(\varphi, \vartheta) &= \mathbb{F}(\varphi, 0) + Y^{-1}\left[u^{\varsigma}\left\{Y\left[\lambda \iota \frac{\partial^2 \mathbb{F}}{\partial \varphi^2} + \iota \chi |\mathbb{F}(\varphi, \vartheta)|^2 \mathbb{F}(\varphi, \vartheta)\right]\right\}\right], \\ \mathbb{F}(\varphi, \vartheta) &= \exp(\iota \varphi) + Y^{-1}\left[u^{\varsigma}\left\{Y\left[\lambda \iota \frac{\partial^2 \mathbb{F}}{\partial \varphi^2} + \iota \chi |\mathbb{F}(\varphi, \vartheta)|^2 \mathbb{F}(\varphi, \vartheta)\right]\right\}\right]. \end{aligned} \tag{21}$$

On utilizing the HPM

$$\sum_{k=0}^{\infty} \epsilon^k \mathbb{F}_k(\varphi, \vartheta) = \exp(\iota \varphi) + \epsilon\left(Y^{-1}\left[u^{\varsigma} Y\left[\lambda \iota \left(\sum_{k=0}^{\infty} \epsilon^k \mathbb{F}_k(\varphi, \vartheta)\right)_{\varphi\varphi} + \iota \chi \left(\sum_{k=0}^{\infty} \epsilon^k H_k(\mathbb{F})\right)\right]\right]\right). \tag{22}$$

The non-linear terms by means of He's polynomial $H_k(\mathbb{F})$ is given as

$$\sum_{k=0}^{\infty} \epsilon^k H_k(\mathbb{F}) = |\mathbb{F}(\varphi, \vartheta)|^2 \mathbb{F}(\varphi, \vartheta) \tag{23}$$

Some He's polynomial terms are determined as

$$\begin{aligned} H_0(\mathbb{F}) &= |\mathbb{F}_0(\varphi, \vartheta)|^2 \mathbb{F}_0(\varphi, \vartheta), \\ H_1(\mathbb{F}) &= \frac{1}{1!} \frac{\partial}{\partial \epsilon}[(|\mathbb{F}_0(\varphi, \vartheta) + \epsilon \mathbb{F}_1(\varphi, \vartheta))|^2 (\mathbb{F}_0(\varphi, \vartheta) + \epsilon \mathbb{F}_0(\varphi, \vartheta))]_{\epsilon=0} \\ H_2(\mathbb{F}) &= \frac{1}{2!} \frac{\partial^2}{\partial \epsilon^2}[(|\mathbb{F}_0(\varphi, \vartheta) + \epsilon \mathbb{F}_1(\varphi, \vartheta) + \epsilon^2 \mathbb{F}_2(\varphi, \vartheta)|^2)(\mathbb{F}_0(\varphi, \vartheta) + \epsilon \mathbb{F}_1(\varphi, \vartheta) + \epsilon^2 \mathbb{F}_2(\varphi, \vartheta))]_{\epsilon=0} \end{aligned}$$

Comparing the coefficient of ϵ, we have

$$
\begin{aligned}
&\epsilon^0 : \mathbb{F}_0(\varphi,\vartheta) = \exp(\iota\varphi),\\
&\epsilon^1 : \mathbb{F}_1(\varphi,\vartheta) = Y^{-1}\left(u^{\varsigma} Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \iota\chi H_0(\mathbb{F})\right]\right) = \iota(\chi-\lambda)\exp(\iota\varphi)\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)},\\
&\epsilon^2 : \mathbb{F}_2(\varphi,\vartheta) = Y^{-1}\left(u^{\varsigma} Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \iota\chi H_1(\mathbb{F})\right]\right) = \frac{1}{2}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^2(\chi-\lambda)(\lambda-\chi)\exp(\iota\varphi),\\
&\epsilon^3 : \mathbb{F}_3(\varphi,\vartheta) = Y^{-1}\left(u^{\varsigma} Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \iota\chi H_2(\mathbb{F})\right]\right) = \frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^3\left\{\frac{-\iota\lambda}{2}(\chi-\lambda)(\lambda-\chi)\exp(\iota\varphi)+\right.\\
&\left.\frac{3}{2}\chi\iota(\chi-\lambda)(\lambda-\chi)\exp(\iota\varphi) + \chi\iota(\chi-\lambda)^2\exp(\iota\varphi)\right\},\\
&\vdots
\end{aligned}
$$

Thus the analytical solution is obtained using the truncated series as

$$
\begin{aligned}
\mathbb{F}(\varphi,\vartheta) &= \exp(\iota\varphi) + \iota(\chi-\lambda)\exp(\iota\varphi)\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{1}{2}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^2(\chi-\lambda)(\lambda-\chi)\exp(\iota\varphi)+\\
&\frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^3\left\{\frac{-\iota\lambda}{2}(\chi-\lambda)(\lambda-\chi)\exp(\iota\varphi) + \frac{3}{2}\chi\iota(\chi-\lambda)(\lambda-\chi)\exp(\iota\varphi) + \chi\iota(\chi-\lambda)^2\exp(\iota\varphi)\right\} + \cdots
\end{aligned}
$$

Example 1. *Let us assume nonlinear fractional BME*

$$
\iota\frac{\partial^{\varsigma}\mathbb{F}}{\partial\vartheta^{\varsigma}} + \lambda\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \chi|\mathbb{F}(\varphi,\vartheta)|^4\mathbb{F}(\varphi,\vartheta) = 0, \quad 0<\varsigma\le 1, \tag{24}
$$

subject to initial source

$$
\mathbb{F}(\varphi,0) = \exp(\iota\varphi).
$$

On operating YT, we get

$$
Y\left(\iota\frac{\partial^{\varsigma}\mathbb{F}}{\partial\vartheta^{\varsigma}}\right) = -Y\left[\lambda\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \chi|\mathbb{F}(\varphi,\vartheta)|^4\mathbb{F}(\varphi,\vartheta)\right], \tag{25}
$$

After simplification, we get

$$
\frac{1}{u^{\varsigma}}\{M(u) - u\mathbb{F}(0)\} = Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \iota\chi|\mathbb{F}(\varphi,\vartheta)|^4\mathbb{F}(\varphi,\vartheta)\right], \tag{26}
$$

$$
M(u) = u\mathbb{F}(0) + u^{\varsigma}Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \iota\chi|\mathbb{F}(\varphi,\vartheta)|^4\mathbb{F}(\varphi,\vartheta)\right]. \tag{27}
$$

By implementing inverse YT, we get

$$
\begin{aligned}
\mathbb{F}(\varphi,\vartheta) &= \mathbb{F}(\varphi,0) + Y^{-1}\left[u^{\varsigma}\left\{Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \iota\chi|\mathbb{F}(\varphi,\vartheta)|^4\mathbb{F}(\varphi,\vartheta)\right]\right\}\right],\\
\mathbb{F}(\varphi,\vartheta) &= \exp(\iota\varphi) + Y^{-1}\left[u^{\varsigma}\left\{Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2} + \iota\chi|\mathbb{F}(\varphi,\vartheta)|^4\mathbb{F}(\varphi,\vartheta)\right]\right\}\right].
\end{aligned} \tag{28}
$$

On utilizing the HPM

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{F}_k(\varphi,\vartheta)=\exp(\iota\varphi)+\epsilon\left(Y^{-1}\left[u^{\varsigma}Y\left[\lambda\iota\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{F}_k(\varphi,\vartheta)\right)_{\varphi\varphi}+\iota\chi\left(\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{F})\right)\right]\right]\right). \quad (29)$$

The non-linear terms by means of He's polynomial $H_k(\mathbb{F})$ are given as

$$\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{F})=|\mathbb{F}(\varphi,\vartheta)|^2\mathbb{F}(\varphi,\vartheta) \quad (30)$$

Few He's polynomial terms are determined as

$$\begin{aligned}
H_0(\mathbb{F})&=|\mathbb{F}_0(\varphi,\vartheta)|^4\mathbb{F}_0(\varphi,\vartheta),\\
H_1(\mathbb{F})&=\frac{1}{1!}\frac{\partial}{\partial\epsilon}[(|\mathbb{F}_0(\varphi,\vartheta)+\epsilon\mathbb{F}_1(\varphi,\vartheta))|^4(\mathbb{F}_0(\varphi,\vartheta)+\epsilon\mathbb{F}_0(\varphi,\vartheta))]_{\epsilon=0}\\
H_2(\mathbb{F})&=\frac{1}{2!}\frac{\partial^2}{\partial\epsilon^2}[(|\mathbb{F}_0(\varphi,\vartheta)+\epsilon\mathbb{F}_1(\varphi,\vartheta)+\epsilon^2\mathbb{F}_2(\varphi,\vartheta)|^4)(\mathbb{F}_0(\varphi,\vartheta)+\epsilon\mathbb{F}_1(\varphi,\vartheta)+\epsilon^2\mathbb{F}_2(\varphi,\vartheta))]_{\epsilon=0}
\end{aligned}$$

Comparing the coefficient of ϵ, we have

$$\begin{aligned}
&\epsilon^0:\mathbb{F}_0(\varphi,\vartheta)=\exp(\iota\varphi),\\
&\epsilon^1:\mathbb{F}_1(\varphi,\vartheta)=Y^{-1}\left(u^{\varsigma}Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2}+\iota\chi H_0(\mathbb{F})\right]\right)=\iota(\chi-\lambda)\exp(\iota\varphi)\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)},\\
&\epsilon^2:\mathbb{F}_2(\varphi,\vartheta)=Y^{-1}\left(u^{\varsigma}Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2}+\iota\chi H_1(\mathbb{F})\right]\right)=\frac{1}{2}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^2(\chi-\lambda)(\chi-\lambda-4\chi\exp(2\iota\varphi))\exp(\iota\varphi),\\
&\epsilon^3:\mathbb{F}_3(\varphi,\vartheta)=Y^{-1}\left(u^{\varsigma}Y\left[\lambda\iota\frac{\partial^2\mathbb{F}}{\partial\varphi^2}+\iota\chi H_2(\mathbb{F})\right]\right)=\frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^3(\chi-\lambda)\Bigg\{\iota\lambda(-\lambda\exp(\iota\varphi)+16\chi\exp(5\iota\varphi))+\\
&\chi\iota[\exp(5\iota\varphi)(2+\lambda-\chi-\exp(4\iota\varphi))]\Bigg\}+\frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^3(\chi-\lambda)\chi\iota\left\{\frac{3}{2}\exp(3\iota\varphi)[(\lambda-\chi)\exp(\iota\varphi)+\lambda-\chi\exp(4\iota\varphi)]\right\},\\
&\vdots
\end{aligned}$$

Thus the analytical solution is obtained using the truncated series as

$$\begin{aligned}
\mathbb{F}(\varphi,\vartheta)&=\exp(\iota\varphi)+\iota(\chi-\lambda)\exp(\iota\varphi)\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}+\frac{1}{2}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^2(\chi-\lambda)(\chi-\lambda-4\chi\exp(2\iota\varphi))\exp(\iota\varphi)+\\
&\frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^3(\chi-\lambda)\left\{\iota\lambda(-\lambda\exp(\iota\varphi)+16\chi\exp(5\iota\varphi))+\chi\iota[\exp(5\iota\varphi)(2+\lambda-\chi-\exp(4\iota\varphi))]\right\}+\\
&\frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^3(\chi-\lambda)\chi\iota\left\{\frac{3}{2}\exp(3\iota\varphi)[(\lambda-\chi)\exp(\iota\varphi)+\lambda-\chi\exp(4\iota\varphi)]\right\}+\cdots
\end{aligned}$$

Example 2. *Let us assume nonlinear fractional BME*

$$\iota\frac{\partial^{\varsigma}\mathbb{F}}{\partial\vartheta^{\varsigma}}+\lambda\frac{\partial^2\mathbb{F}}{\partial\varphi^2}+\chi(|\mathbb{F}(\varphi,\vartheta)|^2+|\mathbb{F}(\varphi,\vartheta)|^4)\mathbb{F}(\varphi,\vartheta)=0,\quad 0<\varsigma\leq 1, \quad (31)$$

subject to initial source

$$\mathbb{F}(\varphi,0)=\exp^{-\frac{\iota\varphi}{2}}.$$

On operating YT, we get

$$Y\left(\iota\frac{\partial^{\varsigma}\mathbb{F}}{\partial\vartheta^{\varsigma}}\right)=-Y\left[\lambda\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\chi(|\mathbb{F}(\varphi,\vartheta)|^{2}+|\mathbb{F}(\varphi,\vartheta)|^{4})\mathbb{F}(\varphi,\vartheta)\right], \tag{32}$$

After simplification, we get

$$\frac{1}{u^{\varsigma}}\{M(u)-u\mathbb{F}(0)\}=Y\left[\lambda\iota\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\iota\chi(|\mathbb{F}(\varphi,\vartheta)|^{2}+|\mathbb{F}(\varphi,\vartheta)|^{4})\mathbb{F}(\varphi,\vartheta)\right], \tag{33}$$

$$M(u)=u\mathbb{F}(0)+u^{\varsigma}Y\left[\lambda\iota\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\iota\chi(|\mathbb{F}(\varphi,\vartheta)|^{2}+|\mathbb{F}(\varphi,\vartheta)|^{4})\mathbb{F}(\varphi,\vartheta)\right]. \tag{34}$$

By implementing inverse YT, we get

$$\begin{aligned}\mathbb{F}(\varphi,\vartheta)&=\mathbb{F}(\varphi,0)+Y^{-1}\left[u^{\varsigma}\left\{Y\left[\lambda\iota\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\iota\chi(|\mathbb{F}(\varphi,\vartheta)|^{2}+|\mathbb{F}(\varphi,\vartheta)|^{4})\mathbb{F}(\varphi,\vartheta)\right]\right\}\right],\\ \mathbb{F}(\varphi,\vartheta)&=\exp^{-\frac{\iota\varphi}{2}}+Y^{-1}\left[u^{\varsigma}\left\{Y\left[\lambda\iota\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\iota\chi(|\mathbb{F}(\varphi,\vartheta)|^{2}+|\mathbb{F}(\varphi,\vartheta)|^{4})\mathbb{F}(\varphi,\vartheta)\right]\right\}\right].\end{aligned} \tag{35}$$

On utilizing the HPM

$$\sum_{k=0}^{\infty}\epsilon^{k}\mathbb{F}_{k}(\varphi,\vartheta)=\exp^{-\frac{\iota\varphi}{2}}+\epsilon\left(Y^{-1}\left[u^{\varsigma}Y\left[\lambda\iota\left(\sum_{k=0}^{\infty}\epsilon^{k}\mathbb{F}_{k}(\varphi,\vartheta)\right)_{\varphi\varphi}+\iota\chi\left(\sum_{k=0}^{\infty}\epsilon^{k}H_{k}(\mathbb{F})\right)\right]\right]\right). \tag{36}$$

The non-linear terms by means of He's polynomial $H_k(\mathbb{F})$ are given as

$$\sum_{k=0}^{\infty}\epsilon^{k}H_{k}(\mathbb{F})=(|\mathbb{F}(\varphi,\vartheta)|^{2}+|\mathbb{F}(\varphi,\vartheta)|^{4})\mathbb{F}(\varphi,\vartheta) \tag{37}$$

Some He's polynomial terms are determined as

$$\begin{aligned}&H_{0}(\mathbb{F})=|\mathbb{F}_{0}(\varphi,\vartheta)|^{2}\mathbb{F}_{0}(\varphi,\vartheta)+|\mathbb{F}_{0}(\varphi,\vartheta)|^{4}\mathbb{F}_{0}(\varphi,\vartheta),\\ &H_{1}(\mathbb{F})=\frac{1}{1!}\frac{\partial}{\partial\epsilon}[(|\mathbb{F}_{0}(\varphi,\vartheta)+\epsilon\mathbb{F}_{1}(\varphi,\vartheta)|^{2}+|\mathbb{F}_{0}(\varphi,\vartheta)+\epsilon\mathbb{F}_{1}(\varphi,\vartheta)|^{4})(\mathbb{F}_{0}(\varphi,\vartheta)+\epsilon\mathbb{F}_{1}(\varphi,\vartheta))]_{\epsilon=0}\\ &H_{2}(\mathbb{F})=\frac{1}{2!}\frac{\partial^{2}}{\partial\epsilon^{2}}[(|\mathbb{F}_{0}(\varphi,\vartheta)+\epsilon\mathbb{F}_{1}(\varphi,\vartheta)+\epsilon^{2}\mathbb{F}_{2}(\varphi,\vartheta)|^{2}+|\mathbb{F}_{0}(\varphi,\vartheta)+\epsilon\mathbb{F}_{1}(\varphi,\vartheta)+\epsilon^{2}\mathbb{F}_{2}(\varphi,\vartheta)|^{4})(\mathbb{F}_{0}(\varphi,\vartheta)+\\ &\epsilon\mathbb{F}_{1}(\varphi,\vartheta)+\epsilon^{2}\mathbb{F}_{2}(\varphi,\vartheta))]_{\epsilon=0}\end{aligned}$$

Comparing the coefficient of ϵ, we have

$$\begin{aligned}&\epsilon^{0}:\mathbb{F}_{0}(\varphi,\vartheta)=\exp^{-\frac{\iota\varphi}{2}},\\ &\epsilon^{1}:\mathbb{F}_{1}(\varphi,\vartheta)=Y^{-1}\left(u^{\varsigma}Y\left[\lambda\iota\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\iota\chi H_{0}(\mathbb{F})\right]\right)=\iota\left(2\chi-\frac{\lambda}{4}\right)\exp^{-\frac{\iota\varphi}{2}}\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)},\\ &\epsilon^{2}:\mathbb{F}_{2}(\varphi,\vartheta)=Y^{-1}\left(u^{\varsigma}Y\left[\lambda\iota\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\iota\chi H_{1}(\mathbb{F})\right]\right)=\frac{1}{2}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^{2}\left(2\chi-\frac{\lambda}{4}\right)\left(\frac{\lambda}{4}-2\chi-3\iota+3\iota\exp(-2\iota\varphi)\right)\exp^{-\frac{\iota\varphi}{2}},\\ &\epsilon^{3}:\mathbb{F}_{3}(\varphi,\vartheta)=Y^{-1}\left(u^{\varsigma}Y\left[\lambda\iota\frac{\partial^{2}\mathbb{F}}{\partial\varphi^{2}}+\iota\chi H_{2}(\mathbb{F})\right]\right)=\frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^{3}\left(2\chi-\frac{\lambda}{4}\right)\Big\{\iota\lambda(-\lambda\exp(\iota\varphi)+16\chi\exp(5\iota\varphi))+\\ &\chi\iota\exp(\iota\varphi)(\chi-\lambda)^{2}(2+\lambda-\chi-\exp(4\iota\varphi))\Big\},\\ &\vdots\end{aligned}$$

Thus the analytical solution is obtained using the truncated series as

$$\mathbb{F}(\varphi,\vartheta)=\exp^{-\frac{\iota\varphi}{2}}+\iota\left(2\chi-\frac{\lambda}{4}\right)\exp^{-\frac{\iota\varphi}{2}}\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}+\frac{1}{2}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^{2}\left(2\chi-\frac{\lambda}{4}\right)\left(\frac{\lambda}{4}-2\chi-3\iota+3\iota\exp(-2\iota\varphi)\right)\exp^{-\frac{\iota\varphi}{2}}+$$
$$\frac{1}{3}\left(\frac{\vartheta^{\varsigma}}{\Gamma(\varsigma+1)}\right)^{3}\left(2\chi-\frac{\lambda}{4}\right)\left\{\iota\lambda(-\lambda\exp(\iota\varphi)+16\chi\exp(5\iota\varphi))+\chi\iota\exp(\iota\varphi)(\chi-\lambda)^{2}(2+\lambda-\chi-\exp(4\iota\varphi))\right\}+\cdots$$

Numerical Simulation Studies

To verify the suggested strategy, numerical simulation studies for the nonlinear time-fractional Biswas–Milovic equations are conducted. With the help of the 3D plots of the real and imaginary divisions of the wave profile $\mathbb{F}(\varphi,\vartheta)$ and their corresponding contours, once can clearly see how the wave solution behaves for various numeric values. Figure 1 displays the 3D plots of the numerical solution for Ex. 4.1 real and imaginary division when $\chi=2,\lambda=1$, and $\varsigma=1$ within the domain $-5\le\varphi\le 5$ and $\vartheta\in[0,0.1]$. Figure 2 displays the 3D plots of the numerical solution for Ex. 4.2 real and imaginary division when $\chi=2,\lambda=1$, and $\varsigma=1$ within the domain $-10\le\varphi\le 10$ and $\vartheta\in[0,0.1]$. Similarly, Figure 3 displays the 3D plots of the numerical solution for Ex. 4.3 real and imaginary division when $\chi=2,\lambda=4$, and $\varsigma=1$ within the domain $-20\le\varphi\le 20$ and $\vartheta\in[0,0.1]$. The numerical solution's contour plots, which express the three-dimensional data in a two-dimensional plane, are also provided. The third iteration provided all of the results, and other iterations can be found to produce more precise results.

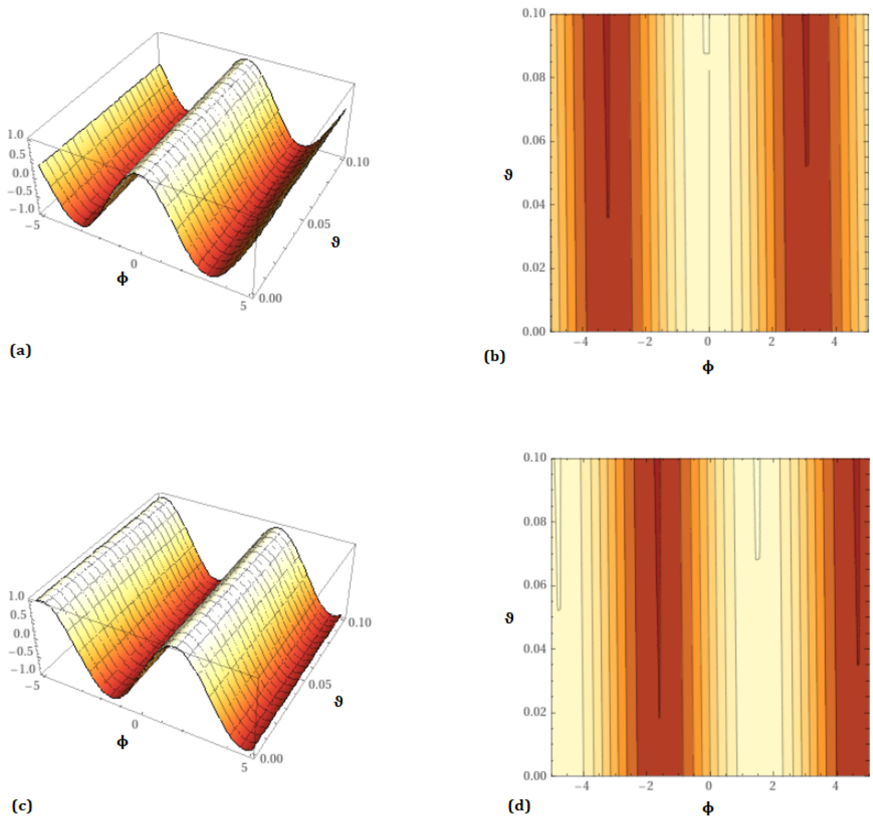

Figure 1. Aspects of the analytical result of problem 1 in 3D and its contour for $\chi=2,\lambda=1$, and $\varsigma=1$. (**a**) Real part, (**b**) Real part contour, (**c**) Imaginary part, and (**d**) Imaginary part contour.

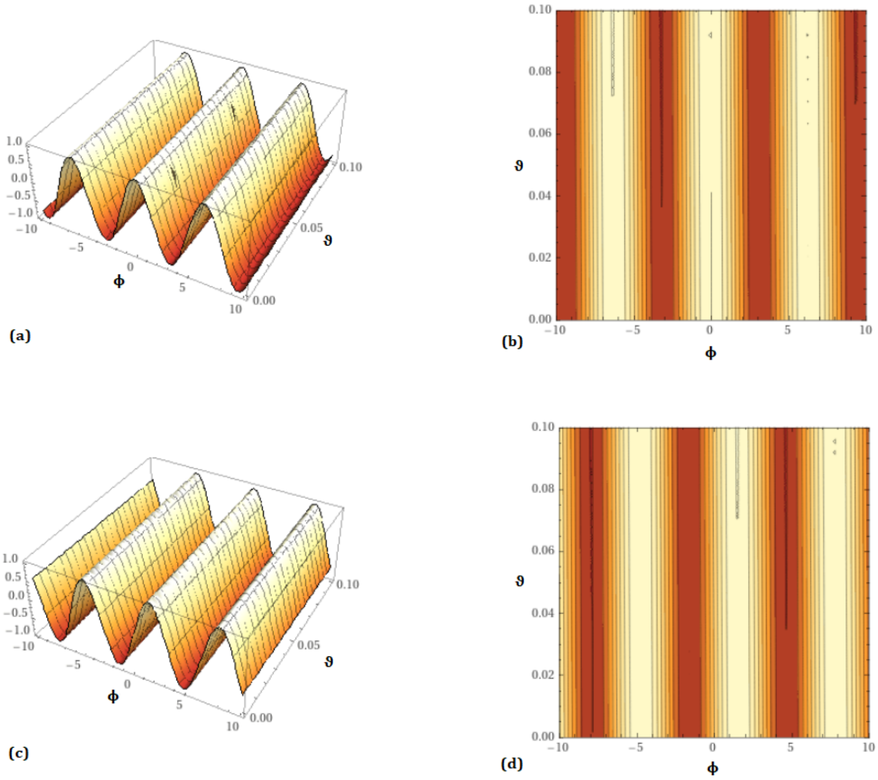

Figure 2. Aspects of the analytical result of problem 2 in 3D and its contour for $\chi = 2, \lambda = 1$, and $\varsigma = 1$. (**a**) Real part, (**b**) Real part contour, (**c**) Imaginary part, and (**d**) Imaginary part contour.

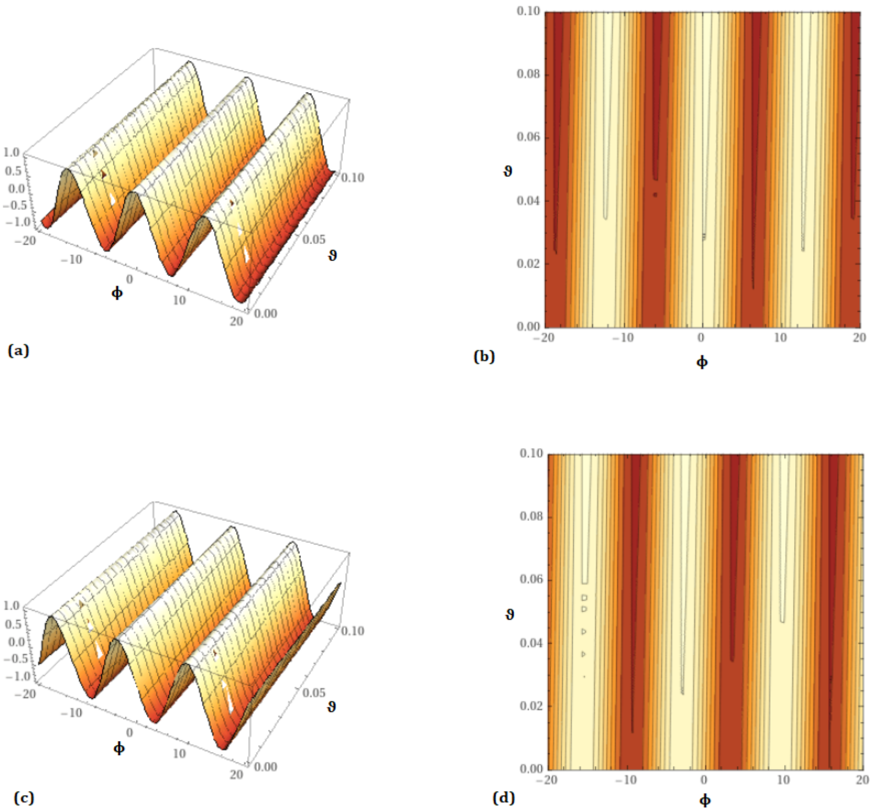

Figure 3. Aspects of the analytical result of problem 3 in 3D and its contour for $\chi = 2, \lambda = 1$, and $\varsigma = 1$. (**a**) Real part, (**b**) Real part contour, (**c**) Imaginary part, and (**d**) Imaginary part contour.

5. Conclusions

With the use of the HPTM, the approximate and analytical solutions to the fractional Biswas–Milovic equations are successfully achieved in this study. Numerous domains, including communications, all-optical rapid switching devices, nonlinear fibre optics, and others, analyse the Biswas–Milovic equation. Many phenomena in biology, fluid flow, economics, control theory, chemistry, the life sciences, and other branches of research and engineering may now be well described using fractional calculus. An accurate simulation of a physical phenomenon depends on both the current time and the past time history. Fractional calculus can be used in this regard. Therefore, science and engineering may benefit from any new solutions to fractional equations. This paper's main contribution is to offer a straightforward, trustworthy, and effective solution method for challenging fractional partial differential equations. The results obtained with this innovative approaches have greater accuracy in the numerical results and take less time and computational effort.

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References

1. Goyal, M.; Baskonus, H.M.; Prakash, A. An efficient technique for a time fractional model of lassa hemorrhagic fever spreading in pregnant women. *Eur. Phys. J. Plus* **2019**, *134*, 482. [CrossRef]
2. Prakash, A.; Kumar, M.; Baleanu, D. A new iterative technique for a fractional model of nonlinear Zakharov-Kuznetsov equations via Sumudu transform. *Appl. Math. Comput.* **2018**, *334*, 30–40. [CrossRef]
3. Prakash, A.; Kaur, H. A reliable numerical algorithm for fractional model of Fitzhugh-Nagumo equation arising in the transmission of nerve impulses. *Nonlinear Eng.-Model. Appl.* **2019**, *8*, 719–727. [CrossRef]
4. Prakash, A.; Veeresha, P.; Prakasha, D.G.; Goyal, M. A homotopy technique for a fractional order multi-dimensional telegraph equation via the Laplace transform. *Eur. Phys. J. Plus* **2019**, *134*, 1–18. [CrossRef]
5. Yang, D.; Zhu, T.; Wang, S.; Wang, S.; Xiong, Z. LFRSNet: A Robust Light Field Semantic Segmentation Network Combining Contextual and Geometric Features. *Front. Environ. Sci.* **2022**, 1443. [CrossRef]
6. Lv, Z.; Chen, D.; Feng, H.; Wei, W.; Lv, H. Artificial Intelligence in Underwater Digital Twins Sensor Networks. *ACM Trans. Sen. Netw.* **2022**, *18*, 39. [CrossRef]
7. Lv, Z.; Chen, D.; Lv, H. Smart City Construction and Management by Digital Twins and BIM Big Data in COVID-19 Scenario. *ACM Trans. Multimed. Comput. Commun. Appl.* **2022**, *18*, 117. [CrossRef]
8. Kovalnogov, V.N.; Fedorov, R.V.; Chukalin, A.V.; Simos, T.E.; Tsitouras, C. Eighth Order Two-Step Methods Trained to Perform Better onKeplerian-Type Orbits. *Mathematics* **2021**, *9*, 3071. [CrossRef]
9. Kovalnogov, V.N.; Fedorov, R.V.; Generalov, D.A.; Chukalin, A.V.; Katsikis, V.N.; Mourtas, S.D.; Simos, T.E. Portfolio Insurance through Error-Correction Neural Networks. *Mathematics* **2022**, *10*, 3335. [CrossRef]
10. Sun, L.; Hou, J.; Xing, C.; Fang, Z. A Robust Hammerstein-Wiener Model Identification Method for Highly Nonlinear Systems. *Processes* **2022**, *10*, 2664. [CrossRef]
11. Young, G.O. Definition of physical consistent damping laws with fractional derivatives. *Z. Angew. Math. Mech.* **1995**, *75*, 623–635.
12. He, J.H. Some applications of nonlinear fractional differential equations and their approximations. *Bull. Sci. Technol.* **1999**, *15*, 86–90.

13. He, J.H. Approximate analytic solution for seepage flow with fractional derivatives in porous media. *Comput. Methods Appl. Mech. Eng.* **1998**, *167*, 57–68. [CrossRef]
14. Hilfer, R. (Ed.) *Applications of Fractional Calculus in Physics*; World Scientific Publishing Company: Singapore, 2000; pp. 87–130.
15. Mainardi, F.; Luchko, Y.; Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. *Frac. Calcul. Appl. Anal.* **2001**, *4*, 153–192.
16. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
17. Lu, S.; Guo, J.; Liu, S.; Yang, B.; Liu, M.; Yin, L.; Zheng, W. An Improved Algorithm of Drift Compensation for Olfactory Sensors. *Appl. Sci.* **2022**, *12*, 9529. [CrossRef]
18. Dang, W.; Guo, J.; Liu, M.; Liu, S.; Yang, B.; Yin, L.; Zheng, W. A Semi-Supervised Extreme Learning Machine Algorithm Based on the New Weighted Kernel for Machine Smell. *Appl. Sci.* **2022**, *12*, 9213. [CrossRef]
19. Lu, S.; Yin, Z.; Liao, S.; Yang, B.; Liu, S.; Liu, M.; Zheng, W. An asymmetric encoder-decoder model for Zn-ion battery lifetime prediction. *Energy Rep.* **2022**, *8*, 33–50. [CrossRef]
20. Ban, Y.; Liu, M.; Wu, P.; Yang, B.; Liu, S.; Yin, L.; Zheng, W. Depth Estimation Method for Monocular Camera Defocus Images in Microscopic Scenes. *Electronics* **2022**, *11*, 2012. [CrossRef]
21. He, J.H. Homotopy perturbation technique. *Compt. Meth. Appl. Mech. Eng.* **1999**, *178*, 257–262. [CrossRef]
22. He, J.H. A new perturbation technique which is also valid for large parameters. *J. Sou. Vib.* **2000**, *229*, 1257–1263. [CrossRef]
23. Rashidi, M.M.; Ganji, D.D.; Dinarvand, S. Explicit analytical solutions of the generalized Burger and Burger-Fisher equations by homotopy perturbation method. *Numer. Meth.* **2009**, *25*, 409–417. [CrossRef]
24. Rashidi, M.M.; Ganji, D.D. Homotopy Perturbation Combined with Padé Approximation for Solving Two Dimensional Viscous Flow in the Extrusion Process. *Inter. J. Nonlinear Sci.* **2009**, *7*, 387–394.
25. Yildirim, A. An algorithm for solving the fractional nonlinear Schr¨ndinger equation by means of the homotopy perturbation method. *Int. J. Nonlinear Sci. Numer. Simul.* **2009**, *10*, 445–451. [CrossRef]
26. Kumar, S.; Singh, O.P. Numerical Inversion of the Abel Integral Equation using Homotopy Perturbation Method. *Z. Naturforschung* **2010**, *65a*, 677–682. [CrossRef]
27. Adomian, G. Solutions of Nonlinear P.D.E. *Appl. Math. Lett.* **1998**, *11*, 121–123. [CrossRef]
28. Adomian, G. Analytical solution of Navier-Stokes flow of a viscous compressible fluid. *Found. Phys. Lett.* **1995**, *8*, 389–400. [CrossRef]
29. Inc, M.; Cherruault, Y. A new approach to solve a diffusion-convection problem. *Kybernetes* **2002**, *31*, 536–549. [CrossRef]
30. Basto, M.; Semiao, V.; Calheiros, F.L. Numerical study of modified Adomian's method applied to Burgers equation. *J. Comput. Appl. Math.* **2007**, *206*, 927–949. [CrossRef]
31. Krasnoschok, M.; Pata, V.; Siryk, S.V.; Vasylyeva, N. A subdiffusive Navier-Stokes-Voigt system. *Phys. D Nonlinear Phenom.* **2020**, *409*, 132503. [CrossRef]
32. Gu, W.; Wei, F.; Li, M. Parameter Estimation for a Type of Fractional Diffusion Equation Based on Compact Difference Scheme. *Symmetry* **2022**, *14*, 560. [CrossRef]
33. Krasnoschok, M.; Pereverzyev, S.; Siryk, S.V.; Vasylyeva, N. Regularized reconstruction of the order in semilinear subdiffusion with memory. *Springer Proc. Math. Stat.* **2020**, *310*, 205–236.
34. Jin, B.; Kian, Y.; Zhou, Z. Reconstruction of a space-time-dependent source in subdiffusion models via a perturbation approach. *SIAM J. Math. Anal.* **2021**, *53*, 4445–4473. [CrossRef]
35. Krasnoschok, M.; Pereverzyev, S.; Siryk, S.V.; Vasylyeva, N. Determination of the fractional order in semilinear subdiffusion equations. *Fract. Calc. Appl. Anal.* **2020**, *23*, 694–722. [CrossRef]
36. Ahmad, S.; Ullah, A.; Akgul, A.; De la Sen, M. A Novel Homotopy Perturbation Method with Applications to Nonlinear Fractional Order KdV and Burger Equation with Exponential-Decay Kernel. *J. Funct. Spaces* **2021**, *2021*, 8770488. [CrossRef]
37. Xu, Y. Similarity solution and heat transfer characteristics for a class of nonlinear convection-diffusion equation with initial value conditions. *Math. Probl. Eng.* **2019**, *2019*, 3467276. [CrossRef]
38. Sun, H.; Zhang, Y.; Baleanu, D.; Chen, W.; Chen, Y. A new collection of real world applications of fractional calculus in science and engineering. *Commun. Nonlinear Sci. Numer. Simulat.* **2018**, *64*, 213–231. [CrossRef]
39. Krasnoschok, M.; Pata, V.; Siryk, S.V.; Vasylyeva, N. Equivalent definitions of Caputo derivatives and applications to subdiffusion equations. *Dyn. Partial. Differ. Equ.* **2020**, *17*, 383–402. [CrossRef]
40. Diethelm, K.; Garrappa, R.; Stynes, M. Good (and Not So Good) practices in computational methods for fractional calculus. *Mathematics* **2020**, *8*, 324. [CrossRef]
41. Alaoui, M.K.; Fayyaz, R.; Khan, A.; Shah, R.; Abdo, M.S. Analytical investigation of Noyes-Field model for time-fractional Belousov-Zhabotinsky reaction. *Complexity* **2021**, *2021*, 3248376. [CrossRef]
42. Qin, Y.; Khan, A.; Ali, I.; Al Qurashi, M.; Khan, H.; Shah, R.; Baleanu, D. An efficient analytical approach for the solution of certain fractional-order dynamical systems. *Energies* **2020**, *13*, 2725. [CrossRef]
43. Nonlaopon, K.; Alsharif, A.M.; Zidan, A.M.; Khan, A.; Hamed, Y.S.; Shah, R. Numerical investigation of fractional-order Swift-Hohenberg equations via a Novel transform. *Symmetry* **2021**, *13*, 1263. [CrossRef]
44. Khan, H.; Khan, A.; Kumam, P.; Baleanu, D.; Arif, M. An approximate analytical solution of the Navier-Stokes equations within Caputo operator and Elzaki transform decomposition method. *Adv. Differ. Equ.* **2020**, *2020*, 1–23.

45. Khan, H.; Khan, A.; Al-Qurashi, M.; Shah, R.; Baleanu, D. Modified modelling for heat like equations within Caputo operator. *Energies* **2020**, *13*, 2002. [CrossRef]
46. Dehghan, M.; Manafian, J.; Saadatmandi, A. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numer. Methods Partial. Differ. Equ. Int. J.* **2010**, *26*, 448–479. [CrossRef]
47. Alderremy, A.A.; Aly, S.; Fayyaz, R.; Khan, A.; Shah, R.; Wyal, N. The Analysis of Fractional-Order Nonlinear Systems of Third Order KdV and Burgers Equations via a Novel Transform. *Complexity* **2022**, *2022*, 4935809. [CrossRef]
48. Areshi, M.; Khan, A.; Shah, R.; Nonlaopon, K. Analytical investigation of fractional-order Newell-Whitehead-Segel equations via a novel transform. *Aims Math.* **2022**, *7*, 6936–6958. [CrossRef]
49. Biswas, A.; Milovic, D. Bright and dark solitons of the generalized nonlinear Schrodinger's equation. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 1473–1484. [CrossRef]
50. Ahmed, I.; Mu, C.; Zhang, F. Exact solution of the Biswas-Milovic equation by Adomian decomposition method. *Int. J. Appl. Math. Res.* **2013**, *2*, 418–422. [CrossRef]
51. Mirzazadeh, M.; Arnous, A.H. Exact solution of Biswas- Milovic equation using new efficient method. *Elec. J. Math. Anal. Appl.* **2015**, *3*, 139–146.
52. Ahmadian, S.; Darvishi, M.T. A new fractional Biswas-Milovic model with its periodic soliton solutions. *Optik* **2016**, *127*, 7694–7703. [CrossRef]
53. Ahmadian, S.; Darvishi, M.T. fractional version of (1+1) dimensional Biswas-Milovic equation and its solutions. *Optik* **2016**, *127*, 10135–10147. [CrossRef]
54. Zaidan, L.I.; Darvishi, M.T. Semi-analytical solutions of different kinds of fractional Biswas-Milovic equation. *Optik* **2017**, *136*, 403–410. [CrossRef]
55. Singh, J.; Kumar, D.; Baleanu, D. New aspects of fractional Biswas-Milovic model with Mittag-Leffler law. *Math. Model. Nat. Phenom.* **2019**, *14*, 303. [CrossRef]
56. Saad Alshehry, A.; Imran, M.; Khan, A.; Shah, R.; Weera, W. Fractional View Analysis of Kuramoto-Sivashinsky Equations with Non-Singular Kernel Operators. *Symmetry* **2022**, *14*, 1463. [CrossRef]
57. Sunthrayuth, P.; Alyousef, H.A.; El-Tantawy, S.A.; Khan, A.; Wyal, N. Solving Fractional-Order Diffusion Equations in a Plasma and Fluids via a Novel Transform. *J. Funct. Spaces* **2022**, *2022*, 1899130. [CrossRef]
58. Ghorbani, A. Beyond Adomian polynomials: He polynomials. *Chaos Solitons Fractals* **2009**, *39*, 1486–1492. [CrossRef]

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A Stochastic Mathematical Model for Understanding the COVID-19 Infection Using Real Data

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Abstract: Natural symmetry exists in several phenomena in physics, chemistry, and biology. Incorporating these symmetries in the differential equations used to characterize these processes is thus a valid modeling assumption. The present study investigates COVID-19 infection through the stochastic model. We consider the real infection data of COVID-19 in Saudi Arabia and present its detailed mathematical results. We first present the existence and uniqueness of the deterministic model and later study the dynamical properties of the deterministic model and determine the global asymptotic stability of the system for $\mathcal{R}_0 \leq 1$. We then study the dynamic properties of the stochastic model and present its global unique solution for the model. We further study the extinction of the stochastic model. Further, we use the nonlinear least-square fitting technique to fit the data to the model for the deterministic and stochastic case and the estimated basic reproduction number is $\mathcal{R}_0 \approx 1.1367$. We show that the stochastic model provides a good fitting to the real data. We use the numerical approach to solve the stochastic system by presenting the results graphically. The sensitive parameters that significantly impact the model dynamics and reduce the number of infected cases in the future are shown graphically.

Keywords: stochastic COVID-19 mathematical model; real data; stability results; parameters estimations; numerical results

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1. Introduction

The study of differential equations with symmetry (also known as equivariant differential equations) entails studying the structure of differential equation solutions that obey the limitations imposed by the symmetry group. Symmetries can be spatial or temporal in nature. Aside from their application in modeling, differential equations with symmetry have fascinating dynamical properties that are not seen in non-symmetric systems, such as durable heteroclinic cycles. In ordinary differential equations (including Hamiltonian systems) and delay-differential equations, I have looked at symmetry's importance in bifurcations from equilibria and periodic orbits. Mathematical models to study infectious diseases, in particular COVID-19, are gaining attention from researchers day by day, see [1–3]. The COVID-19 infection which brings negative effects by increasing infection and deaths is still active in many countries of the world. As per the follow-up of the World Health Organization (WHO) instructions, many countries are overcoming this infection while some of them are still facing it. Saudi Arabia also suffered from this infection but their policies and restrictions on their citizens to follow the WHO suggestions have succeeded in curtailing the COVID-19 infection. At present, the total number of coronavirus cases since the disease onset in Saudi Arabia is 819,083, with 9383 deaths. The number of individuals recovered so far is 805,670. The percentage of recovery is 99% while the death percentage is 1%. The proper management from the government and the response of citizens in following the guidelines of the World Health Organization can minimize future cases.

Mathematical models that studied COVID-19 infections are enormous in the literature. We highlight some of them in the present study. For example, the authors utilized the infected cases of coronavirus in UAE and presented a deterministic stochastic model in [4]. A mathematical study that investigated the coronavirus infection using asymptomatic and symptomatic classes is presented in [5]. In [6], the authors considered a mathematical model to emphasize the dynamics of healthcare workers and the community. A numerical study to study the coronavirus infection with environmental contact rates was investigated in [7]. The authors in [8] studied the COVID-19 infection with treatment. The COVID-19 infection with vaccination strategies using reinforcement learning is utilized in [9]. The authors in [10] considered an algorithm to investigate coronavirus infections for undetected individuals. Using the reported cases of coronavirus in Bangkok Thailand through the mathematical study was analyzed in [11]. A mathematical study was given on the COVID-19 disease with fractional derivative modeling for the coronavirus in [12]. A mathematical study to determine the upper bound for new COVID-19 infection in Germany is explored in [13]. Some recent literature related to COVID-19 infection can be seen in [14–19]. For example, the discovery of a new variant of the coronavirus called omicron has been modeled by the authors mathematically and the results were established [14]. The stability and the analysis of the second wave of the coronavirus infection were investigated in [15]. A non-integer system to study the COVID-19 infection in India was considered in [16]. The control theory was used in [17] by the authors to show some strategies for the elimination of the virus. The COVID-19 pandemic with the implementation of the control strategies was investigated in [18]. In [19], a fractional order was considered to investigate the COVID-19 infection, and the results regarding disease elimination were presented.

The concept of stochastic differential equations (SDEs) and their applications has been reported in the literature for various physical and biological problems. Some of the recent literature related to COVID-19 infection is presented here, where the concept of stochastic differential equations is utilized. For instance, the authors in [20] used stochastic modeling to understand the coronavirus infection dynamics. The authors in [21] consider the stochastic modeling with time delay and obtained results for the COVID-19 infection. Stochastic differential equations and their application to the study of COVID-19 real data have been obtained in [22]. Analysis of the COVID-19 infection using stochastic modeling with Jump was analyzed in [23]. The coronavirus infection and its detection in the workplace using the concept of SDEs was considered in [24]. The important theory about stochastic modeling, formulation, analysis, and their numerical simulations have been discussed in detail for stochastic differential equations [25]. We followed this work [25] with care and present our model results.

This paper focuses on the transmission dynamics of COVID-19 disease in stochastic differential equations. We study the basic properties related to the deterministic model and then present in detail the analysis for the stochastic case. We utilize the real infection data of COVID-19 in Saudi Arabia and fit well the data into the model. We consider the data fitting using the deterministic and stochastic differential equations model and show that the stochastic case provides better fitting. Some important graphical results for the model are presented for the sensitive parameters that can be useful for disease elimination. The rest of the work in this paper is organized as follows: The formulation of the new COVID-19 model in deterministic and stochastic differential equations is discussed in Section 2. In Section 3, we discuss the analysis of the deterministic model. In Section 4, we carry out the mathematical results for the stochastic model. Section 5 discusses the parameter estimations for the model. Sections 6 and 7, respectively, present the numerical results and their discussion, and the conclusions of the present work.

2. Model Formulation

The COVID-19 infection that caused many infections and deaths throughout the world still needs proper attention from researchers around the world to create some useful controls, treatments, and effective vaccinations for its elimination. Mathematical modeling in this regard provides interesting information about the disease's current propagation and its future possible controls. In this section, we consider a mathematical model for understanding the coronavirus infection using the stochastic ordinary nonlinear differential equation model. We consider the total humans denoted by $N(t)$ into five components: the healthy or the susceptible population $S(t)$ (the individuals in this compartment are not yet infected or immune, but capable of attracting infection while exposed to symptomatic, asymptomatic or exposed individuals); the healthy individuals that have close contact with infected or become exposed to the disease are shown by $E(t)$; individuals in the exposed class who complete their incubation and show clear disease symptoms are symptomatically infected and are given by $I(t)$; while those who do not show clear disease symptoms are known as asymptomatic infected and are described by $A(t)$; the people who are recovered either from the symptomatic, asymptomatic or exposed stage are recovered and join the recovered class, $R(t)$, so that $N(t) = S(t) + E(t) + I(t) + A(t) + R(t)$. It should be noted that the exposed individuals have the ability to infect other healthy individuals, and this route of transmission has been included in this paper [26]. It is obvious that asymptomatic people that do not have obvious disease symptoms play a significant role in disease transmission, and it is very difficult to control the disease until we can increase the number of testing for COVID-19 people. The spread due to exposed individuals has been documented in several clinical studies [27–30]. The above discussion can be shown mathematically in the form of a nonlinear model governed by the evolutionary differential equations given by:

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - dS, \\
\frac{dE}{dt} &= (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - (\delta + d)E, \\
\frac{dI}{dt} &= (1-\tau)\delta E - (d + d_1 + \gamma_1)I, \\
\frac{dA}{dt} &= \tau\delta E - (d + \gamma_2)A, \\
\frac{dR}{dt} &= \gamma_1 I + \gamma_2 A - dR, \qquad (1)
\end{aligned}
$$

where the non-negative initial values of the model variables are

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, A(0) = A_0 \geq 0, R(0) = R_0 \geq 0. \qquad (2)$$

The population of healthy people is increased through the birth rate Λ while it is decreased through the natural mortality rate d. In each compartment of the model, the natural death rate is given by d. The healthy individuals after interacting with exposed individuals can become infective after completing their incubation period through the effective contact rate β_1, and the route of the transmission for this is given by $\beta_1 SE/N$. The healthy individuals that interact with infected people (that show clinical symptoms) and become infected through the contact rate are given by β_2, and the route of the transmission for this rate is given by $\beta_2 SI/N$. Asymptomatic individuals with no clinical symptoms are considered more dangerous than those showing clinical symptoms who come in contact with healthy people and become infected with a rate given by β_3, so the route of the transmission for this is given by $\beta_3 SA/N$. The parameter δ defines the successful incubation period of the individuals after coming in contact with exposed, symptomatic, or asymptomatic individuals and after completing the incubation period become infected and are either asymptomatic or symptomatic. At the rate $(1-\tau)\delta$, the exposed individuals become symptomatically infected while $\tau\delta$ are the individuals that join the asymptomatic individuals class. The recovery from the infection in the symptomatic or asymptomatic classes is shown, respectively, by γ_1 and γ_2. The natural death due to infection of the individuals in the symptomatic class is given by d_1.

Using the concept of stochastic differential equations, one can represent the ordinary differential equations system, given by (1), in the form given by:

$$\begin{aligned}
dS &= \Big(\Lambda - (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - dS\Big)dt + \sigma_1 S dB_1(t),\\
dE &= \Big((\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - (\delta + d)E\Big)dt + \sigma_2 E dB_2(t),\\
dI &= \Big((1-\tau)\delta E - (d + d_1 + \gamma_1)I\Big)dt + \sigma_3 I dB_3(t),\\
dA &= \Big(\tau\delta E - (d + \gamma_2)A\Big)dt + \sigma_4 A dB_4(t),\\
dR &= \Big(\gamma_1 I + \gamma_2 A - dR\Big)dt + \sigma_5 R dB_5(t),
\end{aligned} \tag{3}$$

where σ_i for $i = 1, \ldots 5$ denote the real constant that describes the intensity of the stochastic differential equations while $B_i(t)$ for $i = 1, \ldots, 5$ are refereed to be the stochastic Brownian motion. Due to the biological meaning of the classes $(S(t), E(t), I(t), A(t), R(t))$, our focus is to study the model in the first quadrant:

$$\mathbb{R}^5_+ = \Big\{(S, E, I, A, R) \in \mathbb{R}^5 : S \geq 0,\ E \geq 0,\ I \geq 0,\ A \geq 0,\ R \geq 0\Big\}.$$

In the following section, we first investigate the dynamics of the model (3) in the absence of stochastic noises.

3. Dynamics of the Deterministic Model

This section studies the dynamics of the deterministic system when $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 0$, so the model (1) can be easily obtained. We now prove the existence of the solution of the system (1).

3.1. Existence and Uniqueness

We present here the existence and uniqueness of the system (1). We assume that for every $t \in [0,\ T]$ (T is refereed "final time"), the functions f_S, f_E, f_I, f_A and f_R are bounded, i.e., $\|S\|_\infty < M_S$, $\|E\|_\infty < M_E$, $\|I\|_\infty < M_I$, $\|A\|_\infty < M_A$, and $\|R\|_\infty < M_R$. We write the model (1) in the form given by:

$$\begin{aligned}
f_S(t, \Psi) &= \Lambda - (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - dS,\\
f_E(t, \Psi) &= (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - (\delta + d)E,\\
f_I(t, \Psi) &= (1-\tau)\delta E - (d + d_1 + \gamma_1)I,\\
f_A(t, \Psi) &= \tau\delta E - (d + \gamma_2)A,\\
f_R(t, \Psi) &= \gamma_1 I + \gamma_2 A - dR,
\end{aligned}$$

where $\Psi = (S, E, I, A, R)$. The parameters of the system (1) positive because it is dealing with the human population. We now show that these functions $f_S, \ldots f_R$ hold the linear growth property and the Lipschitz condition. Let us start with linear growth property,

$$
\begin{aligned}
|f_S(t,\Psi)| &= |\Lambda - (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - dS|, \\
&< |\Lambda - (\beta_1 E + \beta_2 I + \beta_3 A)S - dS|, \\
&\leq \Lambda + (\beta_1 |E| + \beta_2 |I| + \beta_3 |A|)|S| + d|S|, \\
&\leq \Lambda + (\beta_1 \sup_{t\in D_E} |E(t)| + \beta_2 \sup_{t\in D_I} |I(t)| + \beta_3 \sup_{t\in D_A} |A(t)|) \sup_{t\in D_S} |S(t)| \\
&\quad + d \sup_{t\in D_S} |S(t)|, \\
&\leq \Lambda + (\beta_1 M_E + \beta_2 M_I + \beta_3 M_A) M_S + d M_S, \\
&= M_{SS}, \\
&< \infty,
\end{aligned}
$$

$$
\begin{aligned}
|f_E(t,\Psi)| &\leq |(\beta_1 E + \beta_2 I + \beta_3 A)S - (\delta + d)E|, \\
&\leq (\beta_1 |E| + \beta_2 |I| + \beta_3 |A|)|S| + (\delta + d)|E|, \\
&\leq (\beta_1 \sup_{t\in D_E} |E(t)| + \beta_2 \sup_{t\in D_I} |I(t)| + \beta_3 \sup_{t\in D_A} |A(t)|) \sup_{t\in D_S} |S(t)| \\
&\quad + (\delta + d) \sup_{t\in D_E} |E(t)|, \\
&\leq (\beta_1 M_E + \beta_2 M_I + \beta_3 M_A) M_S + (\delta + d) M_E, \\
&= M_{EE}, \\
&< \infty,
\end{aligned}
$$

$$
\begin{aligned}
|f_I(t,\Psi)| &= |(1-\tau)\delta E - (d + d_1 + \gamma_1) I|, \\
&\leq (1-\tau)\delta |E| + (d + d_1 + \gamma_1)|I|, \\
&\leq (1-\tau)\delta \sup_{t\in D_E} |E(t)| + (d + d_1 + \gamma_1) \sup_{t\in D_I} |I(t)|, \\
&\leq (1-\tau)\delta M_E + (d + d_1 + \gamma_1) M_I, \\
&= M_{II}, \\
&< \infty.
\end{aligned}
$$

$$
\begin{aligned}
|f_A(t,\Psi)| &= |\tau\delta E - (d + \gamma_2) A|, \\
&\leq \tau\delta |E| + (d + \gamma_2)|A|, \\
&\leq \tau\delta \sup_{t\in D_E} |E(t)| + (d + \gamma_2) \sup_{t\in D_A} |A(t)|, \\
&\leq \tau\delta M_E + (d + \gamma_2) M_A,
\end{aligned}
$$

$$
\begin{aligned}
&= M_{AA}, \\
&< \infty.
\end{aligned}
$$

$$
\begin{aligned}
|f_R(t,\Psi)| &= |\gamma_1 I + \gamma_2 A - dR|, \\
&\leq \gamma_1|I| + \gamma_2|A| + d|R|, \\
&\leq \gamma_1 \sup_{t\in D_I}|I(t)| + \gamma_2 \sup_{t\in D_A}|A(t)| + d \sup_{t\in D_R}|R(t)|, \\
&\leq \gamma_1 M_I + \gamma_2 M_A + dM_R, \\
&= M_{RR}, \\
&< \infty.
\end{aligned}
$$

Now, we prove the Lipschitz conditions. Let us start with the first equation of the system (1):

$$
\begin{aligned}
|f_S(t,\Psi_{S_1}) - f_S(t,\Psi_{S_2})| &= |(\beta_1 E + \beta_2 I + \beta_3 A)(S_2 - S_1) + d(S_2 - S_1)|, \\
&\leq |(\beta_1 E + \beta_2 I + \beta_3 A)(S_2 - S_1)| + d|S_2 - S_1|, \\
&\leq |(\beta_1|E| + \beta_2|I| + \beta_3|A|)||S_1 - S_2| + d|S_1 - S_2|, \\
&\leq (\beta_1 M_E + \beta_2 M_I + \beta_3 M_A + d)|S_1 - S_2|, \\
&\leq K_S|S_1 - S_2|,
\end{aligned}
$$

where $K_S = (\beta_1 M_E + \beta_2 M_I + \beta_3 M_A + d)$.

$$
\begin{aligned}
|f_E(t,\Psi_{E_1}) - f_E(t,\Psi_{E_2})| &= |\beta_1 S(E_1 - E_2) + (\delta + d)(E_2 - E_1)|, \\
&\leq \beta_1|S||E_1 - E_2| + (\delta + d)|E_1 - E_2|, \\
&\leq (\beta_1 \sup_{t\in D_S}|S(t)| + \delta + d)|E_1 - E_2|, \\
&\leq (\beta_1 M_S + \delta + d)|E_1 - E_2|, \\
&\leq K_E|E_1 - E_2|,
\end{aligned}
$$

where $K_E = (\beta_1 M_S + \delta + d)$.

$$
\begin{aligned}
|f_I(t,\Psi_{I_1}) - f_I(t,\Psi_{I_2})| &\leq |(d + d_1 + \gamma_1)|I_1 - I_2|, \\
&\leq K_I|I_1 - I_2|,
\end{aligned}
$$

where $K_I = (\beta_1 M_S + \delta + d)$.

$$
\begin{aligned}
|f_A(t,\Psi_{A_1}) - f_I(t,\Psi_{A_2})| &\leq (d + \gamma_2)|A_1 - A_2|, \\
&\leq K_A|A_1 - A_2|,
\end{aligned}
$$

where $K_A = (d + \gamma_2)$.

$$
|f_R(t,\Psi_{R_1}) - f_R(t,\Psi_{R_2})| \leq d|R_1 - R_2|,
$$

$$\leq \quad K_R|R_1 - R_2|,$$

where $K_R = d$. Therefore, the given system (1) satisfies the linear growth as well as the Lipschitz conditions. Hence, the model (1) admits a unique system of solutions.

3.2. Equilibrium Points Analysis

The dynamics of the model (1) can be studied while investigating their possible equilibrium points. Usually, the disease epidemic models associated with human populations, often involve the disease free and the endemic equilibrium. We find the disease-free equilibrium of the model (1) denoted by $E_0 = (S^0, 0, 0, 0, 0)$, and it is given by

$$E_0 = (\frac{\Lambda}{d}, 0, 0, 0, 0).$$

The local asymptotic stability of an epidemic model at its equilibrium points is determined with the available threshold number. For disease elimination, the value of the threshold should be less than 1, while for the disease to exist in the population permanently, it should be greater than unity. To study the asymptotic stability analysis of the model, we first investigate the basic reproduction number. The basic reproduction number can be stated as "the number of people infected by a single infected person when introduced into a healthy population by generating secondary infection". Usually, the basic reproduction number is denoted by $\mathcal{R}_0$, and for our proposed model (1), we use the next-generation approach given in [31] to obtain the basic reproduction number. According to this approach [31], we obtain the following matrices,

$$F = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} d+\delta & 0 & 0 \\ \delta(\tau - 1) & d + d_1 + \gamma_1 & 0 \\ -\delta\tau & 0 & d + \gamma_2 \end{pmatrix}.$$

The spectral radius that gives the basic reproduction number $\mathcal{R}_0$ is given by $\rho(FV^{-1})$:

$$\mathcal{R}_0 = \underbrace{\frac{\beta_1}{d+\delta}}_{\mathcal{R}_1} + \underbrace{\frac{\beta_2\delta(1-\tau)}{(\gamma_1 + d + d_1)(d+\delta)}}_{\mathcal{R}_2} + \underbrace{\frac{\beta_3\delta\tau}{(\gamma_2 + d)(d+\delta)}}_{\mathcal{R}_3}.$$

Here, $\mathcal{R}_1$ accounts for the infection generated through the exposed, $\mathcal{R}_2$ through the symptomatic, and $\mathcal{R}_3$ through the asymptomatic individuals.

3.3. Endemic Equilibria

We find the expression for the endemic equilibria and their existence. In the endemic case, the disease persists in the population for $\mathcal{R}_0 > 1$. For the existence of unique positive endemic equilibria of the model proposed, the requirement is that the basic reproduction number $\mathcal{R}_0 > 1$. Here, we find the model (1) by denoting it E_1 and obtain the following,

$$E_1 = \left(S^*, E^*, I^*, A^*, R^*\right)$$

where

$$\begin{cases} S^* = \frac{\Lambda}{d+\lambda^*} \\ E^* = \frac{\lambda S^*}{d+\delta} \\ I^* = \frac{\delta(1-\tau)E^*}{\gamma_1+d+d_1} \\ A^* = \frac{\delta\tau E^*}{\gamma_2+d} \\ R^* = \frac{\gamma_2 A^*+\gamma_1 I^*}{d}. \end{cases}$$

where

$$\lambda^* = \frac{\beta_1 E + \beta_2 I + \beta_3 A}{N}.$$

Using the above into

$$\lambda^* = \frac{\beta_1 E + \beta_2 I + \beta_3 A}{N},$$

we obtain the following,

$$b_0\lambda^* + b_1 = 0,$$

where

$$\begin{aligned} b_0 &= (\gamma_2+d)((\gamma_1+d)(d+\delta)+d_1(\delta\tau+d)), \\ b_1 &= d(\gamma_1+d+d_1)(\gamma_2+d)(d+\delta)(1-\mathcal{R}_0). \end{aligned}$$

The endemic positive equilibrium exists only if $\mathcal{R}_0 > 1$. The disease will spread in the community if $\mathcal{R}_0 > 1$ otherwise it will vanish if $\mathcal{R}_0 \leq 1$.

Lemma 1. *The plane given by $S+E+I+A+R = \frac{\Lambda}{d}$ is a manifold of model (1), and it is attracting in the first octant.*

Proof. Adding all five equations in system (1), we obtain

$$\frac{dN}{dt} = \Lambda - dN - d_1 I \leq \Lambda - dN. \quad (4)$$

It is obvious that $N(t) = \Lambda/d$ is the solution of (4) and for any $N(t) \geq 0$, we may obtain the general solution for (4) given by

$$N(t) = \frac{\Lambda}{d} + \left(N(0) - \frac{\Lambda}{d}\right)e^{-dt}. \quad (5)$$

So,

$$\lim_{t\to\infty} N(t) = \frac{\Lambda}{d}, \quad (6)$$

that summarizes the result. □

3.4. Global Stability Disease Free Case

The following theorem shows the stability of the model (1) in the disease-free case. The global stability of epidemic models is important in the sense that there are no possibilities of backward bifurcation if the model is globally asymptotically stable.

Theorem 1. *The deterministic model (1) is globally asymptotically stable when $\mathcal{R}_0 \leq 1$.*

Proof. We define the Lyapunov function given by

$$L = (\gamma_1 + d + d_1)E + \beta_2 I + \frac{(\gamma_1 + d + d_1)(\beta_3)}{(\gamma_2 + d)}A.$$

Time differentiation along the solution of the model (1) gives the following:

$$\begin{aligned}
\dot{L} &= (\gamma_1 + d + d_1)\dot{E} + \beta_2\dot{I} + \frac{(\gamma_1 + d + d_1)(\beta_3)}{(\gamma_2 + d)}\dot{A}, \\
&= (\gamma_1 + d + d_1)[(\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - (\delta + d)E] \\
&\quad + \beta_2[(1-\tau)\delta E - (d + d_1 + \gamma_1)I] \\
&\quad + \frac{(\gamma_1 + d + d_1)(\beta_3)}{(\gamma_2 + d)}[\tau\delta E - (d + \gamma_2)A], \\
&\leq \Big(\frac{\beta_1}{\delta + d} + \frac{\beta_2\delta(1-\tau)}{(\delta + d)(\gamma_1 + d + d_1)} + \frac{\beta_3\tau\delta}{(\gamma_2 + d)(\delta + d)} - 1\Big)(\gamma_1 + d + d_1)(\delta + d)E, \\
&= (\mathcal{R}_0 - 1)(\gamma_1 + d + d_1)(\delta + d)E.
\end{aligned}$$

So,

$$\dot{L} \leq (\mathcal{R}_0 - 1)(\gamma_1 + d + d_1)(\delta + d)E.$$

Here, $\dot{L} \leq 0$ if and only if $\mathcal{R}_0 \leq 1$. Thus, the deterministic model given by (1) is globally asymptotically stable. □

4. Dynamics of the Stochastic Model

To study the dynamic behavior of stochastic models (3), we first discuss whether the solution of the proposed model is non-negative and whether it concerns global existence. We need to follow [32]. Now let us take a closer look at the dynamics of the probabilistic model (3). First, we present some important results that will help us in later results.

4.1. Preliminaries

This subsection provides a preliminary concept regarding stochastic calculus and some other useful theorem that is based on the onward results in the present paper. We assumed that $(\Xi, \mathcal{Q}, \{\mathcal{Q}_t\}_{t\geq 0}, \mathbb{P})$ defined to be a complete probability space with the filtration $\{\mathcal{Q}_t\}_{t\geq 0}$. Then, we introduce the following notations:

$$\mathbb{R}^m_+ = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m; u_k > 0, 1 \leq k \leq m\},$$
$$\bar{\mathbb{R}}^m_+ = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m; u_k \geq 0, 1 \leq k \leq m\}.$$

The d-dimensional stochastic ordinary differential equation [32] can be written as:

$$du(t) = \Phi(u(t))dt + v(u(t))dG(t) \ \ for\ t \geq t_0, \tag{7}$$

with the initial condition $u(0) = u_0 \in \mathbb{R}^m$ where, $G(t)$ is the d-dimensional standard Brownian motion that is defined on the given probability space. Suppose $C^2(\mathbb{R}^m; \bar{\mathbb{R}}_+)$ describes the family of all functions that are non-negative $\mathcal{V}$ and defined on $\mathbb{R}^m$, which has the property to be continuously differentiable twice in u. Follows [32], we can present for model (7), the differential operator $\mathcal{L}$ shown by:

$$\mathcal{L} = \sum_{k=1}^{m} \Phi_k(u,t)\frac{\partial}{\partial u_k} + \frac{1}{2}\sum_{k,l=1}^{m}[v^T(u,t)v(u,t)]_{k,l}\frac{\partial^2}{\partial u_k \partial u_l}.$$

When $\mathcal{L}$ acts on a function $\mathcal{V} \in C^2(\mathbb{R}^m; \bar{\mathbb{R}}_+)$, then

$$\mathcal{L}\mathcal{V} = V_u(u)\Phi(u) + \frac{1}{2}trace[v^T(u)V_{uu}(u)g(u)],$$

where, $V_u = \left(\frac{\partial V}{\partial u_1}, \ldots, \frac{\partial V}{\partial u_m}\right), \quad V_{uu} = \left(\frac{\partial^2 V}{\partial u_k \partial u_l}\right)_{m\times m}$.
By Ito's lemma [32], if $u \in \mathbb{R}^m$, we have

$$d\mathcal{V}(u(t)) = \mathcal{L}\mathcal{V}(u(t))dt + V_u(u(t))v(u(t))dG(t).$$

Consider that $B(t)$ defines the Brownian motion while $Z(t)$ be an Ito drift-diffusion process that satisfies the following stochastic differential equation:

$$dZ(t) = \mu(Z(t), t)dt + \sigma(Z(t), t)dG(t).$$

If $f(z,t) \in C^2(\mathbb{R}^2, \mathbb{R})$ then $\Phi(Z(t), t)$ is also an Ito drift-diffusion process, with their differential, given by:

$$d(\Phi(Z(t), t)) = \frac{\partial \Phi}{\partial t}(Z(t), t)dt + \Phi'(Z(t), t)dZ + \frac{1}{2}\Phi''(Z(t), t)dZ(t)^2,$$

with $dZ(t)^2$ given by: $dt^2 = 0, dtdG(t) = 0$ and $dG(t)^2 = dt$.

Now, we give the following result:

4.2. Existence of the Positive Unique Global Solution

We provide the results for the unique positive global solution existence of the system (3) which are shown in the following:

Theorem 2. *For the initial values of the model variables,* $X(0) = (S(0), E(0), I(0), A(0), R(0)) \in R^5_+$*, a solution (non-negative) exists, say,* $X(t) = (S(t), E(t), I(t), A(t), R(t))$ *for the system (3) on* $t \geq 0$ *and it is remains in* R^5_+ *with the unit probability.*

Proof. The coefficients given in (3) of all the equations are locally continuous in Lipschitz sense for the initial data $(S(0), E(0), I(0), A(0), R(0)) \in \mathbb{R}^5_+$, so there should be a unique solution or, in other words, a local solution, say $(S(0), E(0), I(0), A(0), R(0))$ on $t \in [0, \chi_m)$, where χ_m defines to be the explosion time, the readers can see more details in the work [22–24,33,34]. To obtain a global solution, one has to show $\chi_m = \infty$. We need to choose a positive real number, say, j_0, to be large enough so that all the initial values associated with the states should lie in $[\frac{1}{j_0}, j_0]$. Further, the stopping time can be defined as,

$$\begin{aligned} \chi_j &= \{t \in [0, \chi_m) : \frac{1}{j} \geq \min\{S(t), E(t), I(t), A(t), R(t)\} \\ &\quad \text{or } \max\{S(t), E(t), I(t), A(t), R(t)\} \geq j\} \end{aligned} \tag{8}$$

for every non-negative integer j ($j \geq j_0$). Considering $\inf \phi = \infty$ for the case ϕ when it shows the null set. The definition of stopping time ensures that χ_j has the monotonically increasing behavior for $j \to \infty$. Consider, $\lim_{j\to\infty} = \chi_\infty$ with $\chi_m \geq \chi_\infty$ a.s.

For the non-negative value of t, we consider the case and claim that $\chi_\infty = \infty$ a.s., then it can be said that $\chi_m = \infty$ and a.s. $(S(t), E(t), I(t), A(t), R(t)) \in R^5_+$. So, we have to show that $\chi_m = \infty$ a.s. if such a result is wrong then there may exists two constants such as $0 < T$ and $\epsilon \in (0,1)$ such that

$$P\{T \geq \chi_\infty\} > \epsilon. \tag{9}$$

Define the following Lyapunov function

$$H(S, E, I, A, R) = S + I + E + A + R - 5 - (\log S + \log E + \log I + \log A + \log R). \tag{10}$$

Applying *Itô* formula on the Equation (10), we obtain,

$$
\begin{aligned}
dH(S,E,I,A,R) = {} & \left(1-\frac{1}{S}\right)dS + \sigma_1(S-1)dB_1(t) + \left(1-\frac{1}{E}\right)dE + \sigma_2(E-1)dB_2(t) \\
& + \left(1-\frac{1}{I}\right)I + \sigma_3(I-1)dB_3(t) + \left(1-\frac{1}{A}\right)A + \sigma_3(A-1)dB_4(t) \\
& + \left(1-\frac{1}{R}\right)dR + \sigma_5(R-1)dB_5(t), \\
& = LH(S,E,I,A,R)dt + \sigma_1(S-1)dB_1(t) + \sigma_2(E-1)dB_2(t) \\
& + \sigma_3(I-1)dB_3(t) + \sigma_4(A-1)dB_4(t) + \sigma_5(R-1)dB_5(t).
\end{aligned} \tag{11}
$$

In Equation (11), the relation $LH : R^5_+ \to R_+$ can be shown to the form given by,

$$
\begin{aligned}
LH(S,E,I,A,R) = {} & \left(1-\frac{1}{S}\right)\left(\Lambda - (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - dS\right) \\
& + \left(1-\frac{1}{E}\right)\left((\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - (\delta+d)E\right) \\
& + \left(1-\frac{1}{I}\right)\left((1-\tau)\delta E - (d+d_1+\gamma_1)I\right) \\
& + \left(1-\frac{1}{A}\right)\left(\tau\delta E - (d+\gamma_2)A\right) + \left(1-\frac{1}{R}\right)\left(\gamma_1 I + \gamma_2 A - dR\right) \\
& + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2}{2}, \\
& = \Lambda - (\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - dS - \frac{\Lambda}{S} + (\beta_1 E + \beta_2 I + \beta_3 A)\frac{1}{N} + d \\
& + ((\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - (\delta+d)E - ((\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{NE} + (\delta+d) \\
& + (1-\tau)\delta E - (d+d_1+\gamma_1)I - \frac{(1-\tau)\delta E}{I} + (d+d_1+\gamma_1) \\
& + \tau\delta E - (d+\gamma_2)A - \frac{\tau\delta E}{A} + (d+\gamma_2) + \gamma_1 I + \gamma_2 A - dR - \frac{\gamma_1 I}{R} + \frac{\gamma_2 A}{R} + d \\
& + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2}{2}, \\
& \leq \Lambda + \beta_1 + \beta_2 + \beta_3 + d + (\delta+d) + (d+d_1+\gamma_1) + (d+\gamma_2) + d \\
& + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2}{2} \\
& \leq \Lambda + \beta_1 + \beta_2 + \beta_3 + 5d + \delta + d_1 + \gamma_1 + \gamma_2 \\
& + \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2}{2} := K.
\end{aligned} \tag{12}
$$

The constant K is obviously positive, which is free from independent state variables. Therefore,

$$
\begin{aligned}
dH(S,E,I,A,R) \leq {} & Kdt + \sigma_1(S-1)dB_1(t) + \sigma_2 E dB_2(t) \\
& + \sigma_3(I-1)dB_3(t) + \sigma_4(A-1)dB_3(t) + \eta_5(R-1)dB_5(t).
\end{aligned} \tag{13}
$$

Integrating both sides of Equation (13) from 0 to $\chi_j \wedge T$

$$\begin{aligned}
&E\left[H(S(\chi_j \wedge T), E(\chi_j \wedge T), I(\chi_j \wedge T), A(\chi_j \wedge T), R(\chi_j \wedge T))\right],\\
&\leq H(S(0), E(0), I(0), A(0), R(0)) + E\left[\int_0^{\chi_j \wedge T} K\right],\\
&\leq H(S(0), E(0), I(0), A(0), R(0)) + TK.
\end{aligned} \tag{14}$$

Setting $\Omega_j = \{T \geq \chi_j\}$ for $j_1 \leq j$ and thus by Equation (9), $P(\Omega_j) \geq \epsilon$. It should be kept in mind that for every ω in Ω_j, there should be at least one $S(\chi_j, \omega)$, $E(\chi_j, \omega)$, $I(\chi_j, \omega)$,$A(\chi_j, \omega)$,$R(\chi_j, \omega)$, that equals $\frac{1}{j}$ or j. Hence, $(S(\chi_j), E(\chi_j), I(\chi_j), I(\chi_j), R(\chi_j))$ is not less than $j - \log j - 1$ or $\log j - 1 + \frac{1}{j}$. As a result

$$\left(\log j - 1 + \frac{1}{j}\right) \wedge E\big(j - \log j - 1\big) \leq H(S(\chi_j), E(\chi_j), I(\chi_j), A(\chi_j), R(\chi_j)). \tag{15}$$

Referring to Equations (9) and (14), we can write

$$\begin{aligned}
H(S(0), E(0), I(0), A(0), R(0)) + TK &\geq E\left[1_{\Omega(\omega)} H\big(S(\chi_j), E(\chi_j), I(\chi_j), A(\chi_j), R(\chi_j)\big)\right],\\
&\geq \epsilon\left[(-1 + j - \log j) \wedge \left(-1 + \frac{1}{j} + \log j\right)\right].
\end{aligned} \tag{16}$$

Here, the notion $1_{\Omega(\omega)}$ describes the indicator function of Ω. Considering $j \to \infty$ leads to dichotomy $\infty > H\big(S(0), E(0), I(0), A(0), R(0)\big) + TK = \infty$, that implies $\chi_\infty = \infty$ a.s. $\square$

4.3. Extinction for the Proposed Model

Here, we find the conditions for the extinction of the system (3). Some definitions and notations are given here, which shall be utilized later in the proof of the result. Consider,

$$\langle x(t) \rangle = \frac{1}{t}\int_0^t x(r)dr.$$

For the stochastic model (3), we give the following threshold $\mathbf{R_0}$:

$$\mathbf{R_0} = \frac{\beta}{(3d + d_1 + \gamma_1 + \gamma_2 + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2})}. \tag{17}$$

Theorem 3. *Let us assume that (S, E, I, A, R) shows the solution of the system (3) with the given initial conditions $(S(0), E(0), I(0), A(0), R(0)) \in R_+^5$, then*

$$\lim_{t\to\infty} \frac{A(t) + E(t) + I(t) + A(t) + R(t)}{t} = 0, \text{ a.s.} \tag{18}$$

then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)dB_1(u)=0,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t EdB_2(u)=0,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t I(u)dB_3(u)=0, \tag{19}$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t A(u)dB_4(u)=0,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(u)dB_5(u)=0.$$

To obtain the proof of Theorem 3, the readers can see the work of Zhao and Jiang [35], which follows the same procedure to obtain the proof of Lemmas (2.1) *and* (2.2)*. Moreover, one can see in the work [36] the same steps to obtain the result for the Lemma 3. So, the proof of this is omitted.*

We give the following result for disease extinction.

Theorem 4. *Let* (S, E, I, A, R) *denote the solution to the system (3) with the initial conditions given by*
$(S(0), E(0), I(0), A(0), R(0)) \in R^5$*. If* $\mathbf{R_0} < 1$*, then*

$$\lim_{t\to\infty}\frac{\langle \log E(t)\rangle}{t}<0,\ \lim_{t\to\infty}\frac{\langle \log I(t)\rangle}{t}<0 \quad \textit{and} \quad \lim_{t\to\infty}\frac{\langle \log A(t)\rangle}{t}<0, \quad \text{a.s.}$$

Which means that disease will die out with probability one. In addition

$$\lim_{t\to\infty}\langle S(t)\rangle=\frac{\Lambda}{d},$$
$$\lim_{t\to\infty}\langle E(t)\rangle=0,$$
$$\lim_{t\to\infty}\langle I(t)\rangle=0, \tag{20}$$
$$\lim_{t\to\infty}\langle A(t)\rangle=0,$$
$$\lim_{t\to\infty}\langle R(t)\rangle=0,$$

Proof. We directly integrate the model (3), and obtain the following equations:

$$\frac{S(t)-S(0)}{t}=\Lambda-\big(\beta_1\langle E\rangle+\beta_2\langle I\rangle+\beta_3\langle A\rangle\big)\frac{\langle S\rangle}{\langle N\rangle}-d\langle S\rangle+\frac{\sigma_1}{t}\int_0^t S(r)dB_1(r),$$
$$\frac{E(t)-E(0)}{t}=\big(\beta_1\langle E\rangle+\beta_2\langle I\rangle+\beta_3\langle A\rangle\big)\frac{\langle S\rangle}{\langle N\rangle}-(\delta+d)\langle E\rangle+\frac{\sigma_2}{t}\int_0^t E(r)dB_2(r),$$
$$\frac{I(t)-I(0)}{t}=(1-\tau)\delta\langle E\rangle-(d+d_1+\gamma_1)\langle I\rangle+\frac{\sigma_3}{t}\int_0^t I(r)dB_3(r), \tag{21}$$
$$\frac{A(t)-I_A(0)}{t}=\tau\delta\langle E\rangle-(d+\gamma_2)\langle A\rangle+\frac{\sigma_4}{t}\int_0^t A(r)dB_4(r),$$
$$\frac{R(t)-R(0)}{t}=\gamma_1\langle I\rangle+\gamma_2\langle A\rangle-d\langle R\rangle+\frac{\sigma_5}{t}\int_0^t R(r)dB_5(r).$$

Define $\Phi = E + I + A$, and using the *Itô* formula to the second equation of model (3), we have

$$
\begin{aligned}
d\log\Phi(t) = & \frac{1}{\Phi}\left[(\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - (d + d_1 + \gamma_1)I - (d + \gamma_2)A\right]dt \\
& - \frac{\sigma_2^2 E^2 dt}{2\Phi^2} - \frac{\sigma_3^2 I^2 dt}{2\Phi^2} - \frac{\sigma_4^2 A^2 dt}{2\Phi^2} + \frac{\sigma_2 E dB_2(t)}{\Phi} + \frac{\sigma_3 I dB_3(t)}{\Phi} \\
& + \frac{\sigma_4 A dB_4(t)}{\Phi}.
\end{aligned} \tag{22}
$$

We integrate the Equation (22) in the interval given by $[0, t]$ and then divide with t, which gives

$$
\begin{aligned}
\frac{\log\Phi(t) - \log\Phi(0)}{t} = & \frac{1}{\Phi}\left[(\beta_1 E + \beta_2 I + \beta_3 A)\frac{S}{N} - dE - (d + d_1 + \gamma_1)I\right. \\
& \left. -(d + \gamma_2)A - \frac{\sigma_2^2 E^2}{2\Phi^2} - \frac{\sigma_3^2 I^2}{2\Phi^2} - \frac{\sigma_4^2 A^2}{2\Phi^2}\right] \\
& + \frac{\sigma_2}{t}\int_0^t \frac{E dB_2(t)}{\Phi} + \frac{\sigma_3}{t}\int_0^t \frac{I dB_3(t)}{\Phi} + \frac{\sigma_4}{t}\int_0^t \frac{A dB_4(t)}{\Phi}.
\end{aligned} \tag{23}
$$

Moreover, $M(t) = \frac{\sigma_2}{t}\int_0^t \frac{EdB_2(t)}{\Phi} + \frac{\sigma_3}{t}\int_0^t \frac{IdB_3(t)}{\Phi} + \frac{\sigma_4}{t}\int_0^t \frac{AdB_4(t)}{\Phi}$, which is a local continuous martingale and $M(0) = 0$. Applying the result given in (3), we have

$$
\lim_{t\to\infty} \sup \frac{M(t)}{t} = 0. \tag{24}
$$

Then

$$
\begin{aligned}
\frac{\log\Phi(t)}{t} & \leq (\beta - (d + (d + d_1 + \gamma_1) + (d + \gamma_2) + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2}) + \frac{\log\Phi(0)}{t}, \\
& \leq (\beta - (3d + d_1 + \gamma_1 + \gamma_2 + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2}) + \frac{\log\Phi(0)}{t}.
\end{aligned} \tag{25}
$$

Where $\beta = Max\{\beta_1, \beta_2, \beta_3\}$, and we know that $S \leq N$, and $E + I + A \leq (\beta_1 E + \beta_2 I + \beta_3 A)$, and $\frac{\log\Phi(0)}{t} = 0$.
If $\mathbf{R_0} < 1$ is satisfied, then Equation (25) becomes

$$
\lim_{t\to\infty} \sup \frac{\log\Phi(t)}{t} \leq \left(3d + d_1 + \gamma_1 + \gamma_2 + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2}\right)\left(\mathbf{R_0} - 1\right) < 0, \text{ a.s.} \tag{26}
$$

The above Equation (26) implies that

$$
\lim_{t\to\infty} \langle\Phi(t)\rangle = 0, \text{ a.s.} \tag{27}
$$

Then

$$
\lim_{t\to\infty} \langle\Phi(t)\rangle = \lim_{t\to\infty} (E + I + A) = 0, \text{ a.s.} \tag{28}
$$

Therefore, as a result,

$$
\begin{aligned}
\lim_{t\to\infty} E &= 0, \\
\lim_{t\to\infty} I &= 0, \\
\lim_{t\to\infty} A &= 0.
\end{aligned} \tag{29}
$$

Furthermore, considering the first equation of the model (21) and integrating with the limit ranging from 0 to t, after that dividing by t, and using Equation (29), then, we have

$$\begin{aligned}
\frac{S(t)-S(0)}{t} &= \Lambda - \big(\beta_1\langle E\rangle + \beta_2\langle I\rangle + \beta_3\langle A\rangle\big)\frac{\langle S\rangle}{\langle N\rangle} - d\langle S\rangle + \frac{\sigma_1}{t}\int_0^t S(r)dB_1(r), \\
\langle S\rangle &= \frac{1}{d}\left[\Lambda - \big(\beta_1\langle E\rangle + \beta_2\langle I\rangle + \beta_3\langle A\rangle\big)\frac{\langle S\rangle}{\langle N\rangle} + \frac{S(0)-S(t)}{t} + \frac{\sigma_1}{t}\int_0^t S(r)dB_1(r)\right].
\end{aligned} \tag{30}$$

Thus implying that

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{\Lambda}{d}, \ \text{ a.s.} \tag{31}$$

Similarly, the fourth equation of system (21) by integrating within the range from 0 to t and later dividing it by t, and using Equation (29), then we have

$$\begin{aligned}
\frac{R(t)-R(0)}{t} &= \gamma_1\langle I\rangle + \gamma_2\langle A\rangle - d\langle R\rangle + \frac{\sigma_5}{t}\int_0^t R(r)dB_5(r), \\
\langle R(t)\rangle &= \frac{1}{(d+\sigma+\gamma_3)}\left[\gamma_1\langle I\rangle + \gamma_2\langle A\rangle + \frac{R(0)-R(t)}{t} + \frac{\sigma_5}{t}\int_0^t R(r)dB_5(r)\right].
\end{aligned} \tag{32}$$

Thus implying that

$$\lim_{t\to\infty}\langle R(t)\rangle = 0, \ \text{ a.s.} \tag{33}$$

The prove the result. □

5. Parameters Estimation

This section studies the estimation of the parameters for the model using the nonlinear least square curve fitting method. The data is considered for the infected cases of COVID-19 in Saudi Arabia. The cases are available in worldometers [37]. We consider the cases per day, thus, we consider the time unit given by *per-day*. Among these model parameters, we consider the demographic parameters of birth and the natural death rate that are obtained using estimations while the rest of the parameter values have been obtained through model fitting to the data. The stochastic model (3) is used to obtain the parameter estimations. We consider the infected cases of the Kingdom of Saudi Arabia of coronavirus infection for the period starting March–July 2020 [38] is considered. The total population is considered to be $N(0) = 34,813,871$ in Saudi Arabia in 2020 [39]. The average lifespan in the Kingdom of Saudi Arabia is given by $1/74.87$ [40]. The birth rate is determined from the equation $\lambda = d * N(0)$ which is $\Lambda = 1273.94$ *per day*. The initial conditions are $S(0) = 34,811,870$, $E(0) = 2000$, $I(0) = 1$, $A(0) = R(0) = 0$. The basic reproduction number obtained through the data fitting is approximate $\mathcal{R}_0 = 1.1367$. The graphics for the data fitting are given in Figures 1 and 2, while the parameters definitions and their realistic values are given in Table 1. Moreover, in Figures 1 and 2, the legend "Data" referred to the infected cases while the "Model simulation" is the solution of the model". Figure 1 is the model versus data fitting in the absence of $\sigma_i = 0$, for $i = 1, 2, \ldots, 5$ while Figure 2 is the data fitting versus model using $\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = 0.002$ and $\sigma_5 = 0.004$. Figure 3 describes the comparison of stochastic and deterministic models versus infected data. It is clear that the stochastic case provides good results for data compared to the deterministic case. It should be noted that in Figure 1 and 2, the legend "Data" referred to the infected cases while the "Model simulation" is the solution of the model. The fitting with the stochastic case is better than the deterministic case.

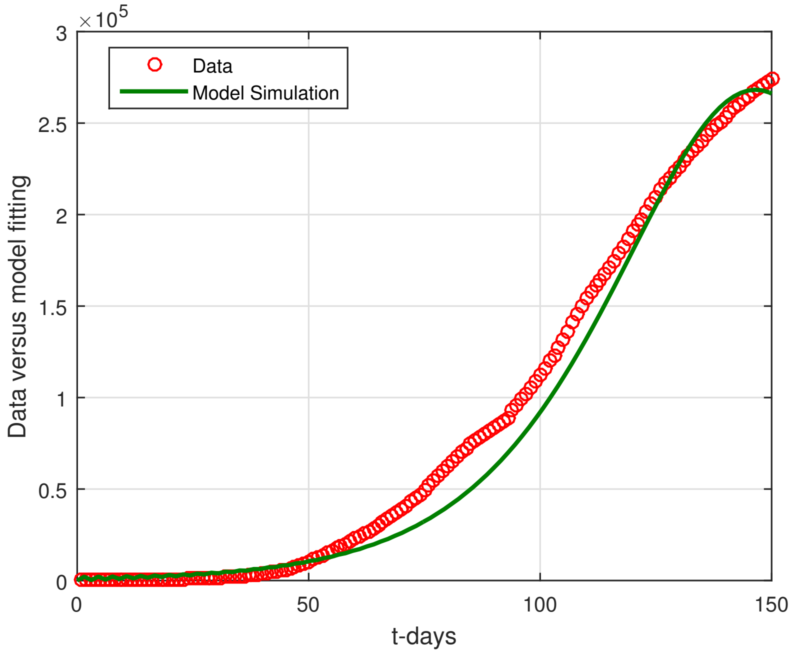


Figure 1. Model fitting with deterministic cases when $\mathcal{R}_0 = 1.1367 > 1$. The circle denotes the infected cases while the bold line referred to the model solutions.

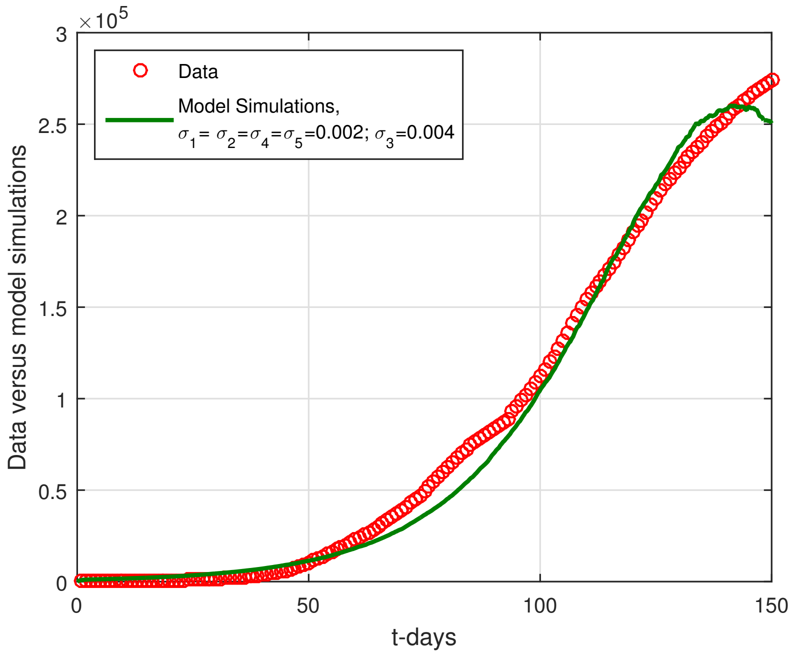


Figure 2. Model fitting for the stochastic case. The circle denotes the infected cases while the bold line referred to the model solutions.

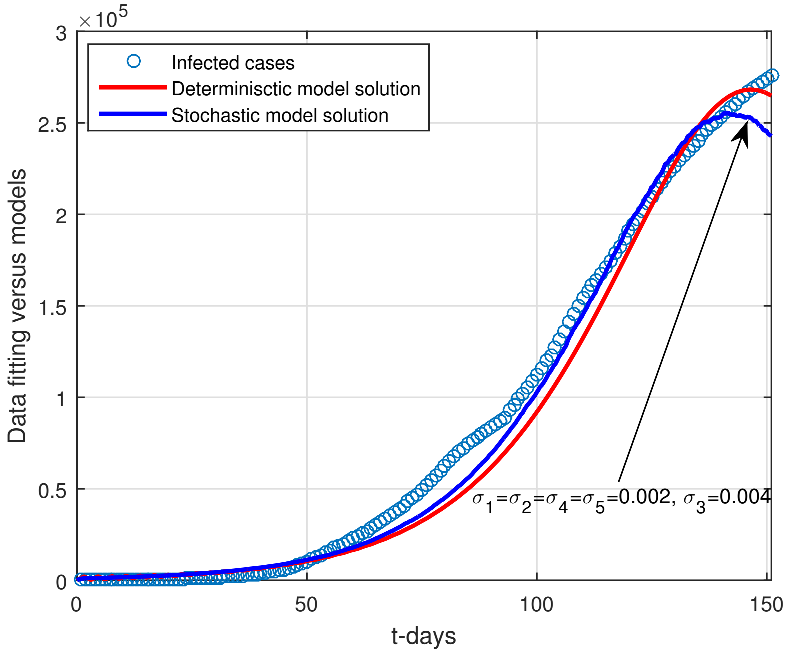


Figure 3. Data fitting using stochastic and deterministic models, blue line "stochastic case", red line "deterministic case" while the circle shows infected cases.

Table 1. Parameters used in data fitting.

Parameter	Description	Value	Source
Λ	Birth rate	1273.94	Estimated
d	Natural death rate	$\frac{1}{74.87\times365}$	Estimated
β_1	Contact rate among E and S	0.1321	Fitted
β_2	Contact rate among I and S	0.5926	Fitted
β_3	Contact rate among A and S	0.5465	Fitted
δ	Incubation period	0.7526	Fitted
τ	Incubation period	0.0746	Fitted
d_1	Natural mortality due to disease	0.3691	Fitted
γ_1	Recovery from I	0.2588	Fitted
γ_2	Recovery from A	0.4639	Fitted

6. Numerical Results

This section is divided into three subsections where we study the numerical scheme, the results, and finally in the last subsection, the discussion on the results achieved for Model (3). We first present a numerical algorithm for solving the model, then obtain graphical results with a detailed discussion.

6.1. Numerical Scheme

We give the following numerical scheme for the model (3) using the scheme adopted from [41] called the Milstein method to obtain the numerical results. It follows from [41] by writing our model (3) in the following form:

$$
\begin{aligned}
S_{i+1} &= S_i + \left[\Lambda - (\beta_1 E_i + \beta_2 I_i + \beta_3 A_i)\frac{S_i}{N} - dS_i\right] \triangle t + \sigma_1 S_i \sqrt{\triangle t}\varsigma_{1,i} + \frac{\sigma_1^2}{2} S_i(\varsigma_{1,i}^2 - 1) \triangle t, \\
E_{i+1} &= E_i + \left[(\beta_1 E_i + \beta_2 I_i + \beta_3 A_i)\frac{S_i}{N} - (\delta + d)E_i\right] \triangle t + \sigma_2 E_i \sqrt{\triangle t}\varsigma_{2,i} + \frac{\sigma_2^2}{2} E_i(\varsigma_{2,i}^2 - 1) \triangle t, \\
I_{i+1} &= I_i + \left[(1-\tau)\delta E_i - (d + d_1 + \gamma_1)I_i\right] \triangle t + \xi_3 I_i \sqrt{\triangle t}\varsigma_{3,i} + \frac{\sigma_3^2}{2} I_i(\varsigma_{3,i}^2 - 1) \triangle t, \\
A_{i+1} &= A_i + \left[\tau\delta E_i - (d + \gamma_2)A_i\right] \triangle t + \sigma_4 A_i \sqrt{\triangle t}\varsigma_{4,i} + \frac{\sigma_4^2}{2} A_i(\varsigma_{4,i}^2 - 1) \triangle t, \\
R_{i+1} &= R_i + \left[\gamma_1 I_i + \gamma_2 A_i - dR_i\right] \triangle t + \sigma_5 R_i \sqrt{\triangle t}\varsigma_{5,i} + \frac{\sigma_5^2}{2} R_i(\varsigma_{5,i}^2 - 1) \triangle t.
\end{aligned}
\tag{34}
$$

In the above scheme, $\varsigma_{i,j}$, when $i = 1, \ldots, 5$, displays the independent Gaussian random variables with the given distribution $N(0,1)$, and where the step size is shown by Δt, and $\sigma_i > 0$, for $i = 1, \ldots, 5$ represents the numerical values of the noise.

6.2. Results

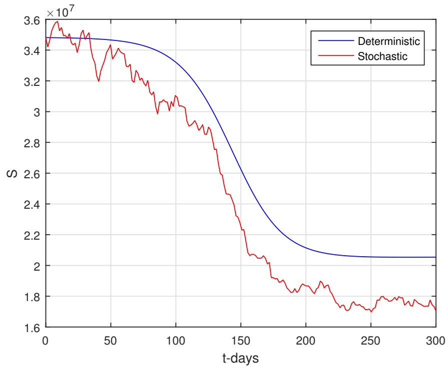
We solve the stochastic model (3) using the values of the parameters given in Table 1 and obtain the required graphical results, see Figures 4–9. We use in these graphical results the values of the stochastic noises are $\sigma_1 = 0.01$, $\sigma_2 = 0.02$, $\sigma_3 = 0.04$, $\sigma_4 = 0.02$, $\sigma_5 = 0.02$. The subgraphs in Figures 4–9 represent the population of susceptible, exposed, infected with disease symptoms, infected with no disease symptoms, and the recovery individuals, respectively, in (a–e). Figure 4 represents the simulation of the stochastic model versus the deterministic model.

Figure 5 represents the simulation of the model for different values of β_1. Reducing the contact between exposed and susceptible decreases the number of infective cases, also, one can see a better reduction in future cases if we restrict healthy individuals by following WHO recommendations.

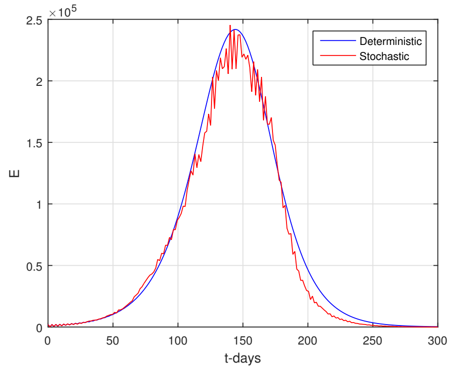
Figure 6 gives the simulation of the stochastic model for many values of the parameter β_2. We can obtain a better decrease if the contact among healthy and symptomatic individuals is decreased, so β_2 can decrease best the number of infected cases caused by the symptomatic infected individuals.

Similarly, we can see from Figure 7 the decrease in the parameter β_3 decreases the number of infected people due to the asymptomatic infection.

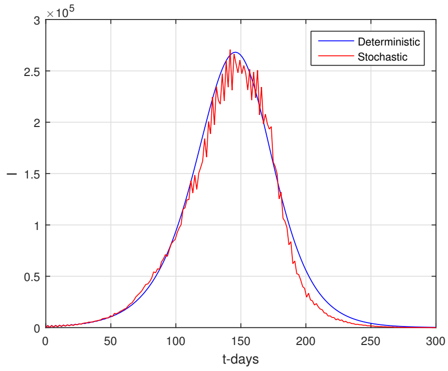
The impact of the parameters δ and τ is shown in Figures 8 and 9.



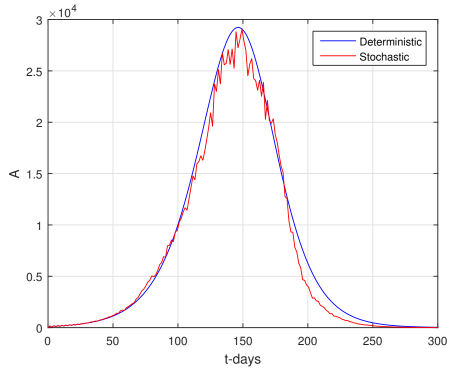
(**a**)



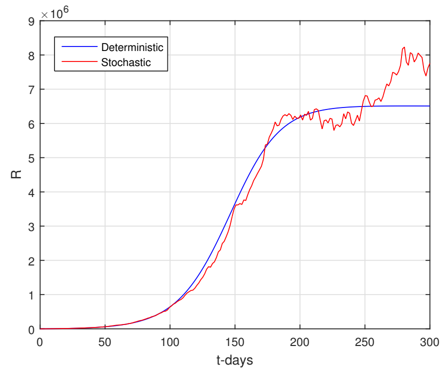
(**b**)



(**c**)



(**d**)



(**e**)

Figure 4. The comparison of stochastic and deterministic model. For the stochastic model, we consider the values $\sigma_1 = 0.01$, $\sigma_3 = 0.04$, $\sigma_2 = \sigma_4 = \sigma_5 = 0.02$. Subfigures (**a**–**e**) represent, respectively, the comparison of healthy, exposed, symptomatic infected, asymptomatic infected and the recovered with and without stochastic noise.

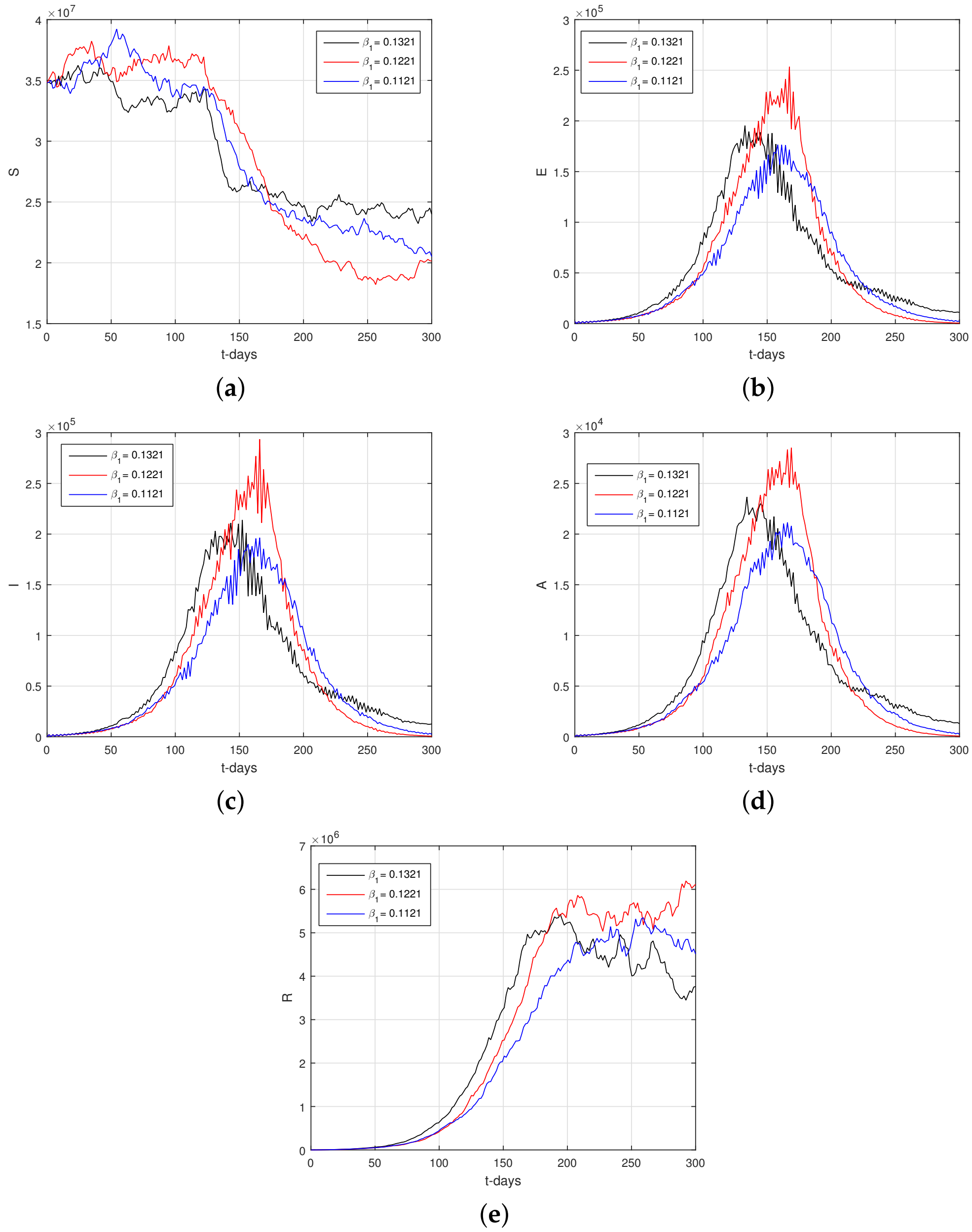


Figure 5. The impact of the parameter β_1 on the stochastic model. Subfigures (**a**–**e**) represent, respectively, the dynamics of susceptible, exposed, symptomatic, asymptomatic and recovered population under different values of the parameters β_1.

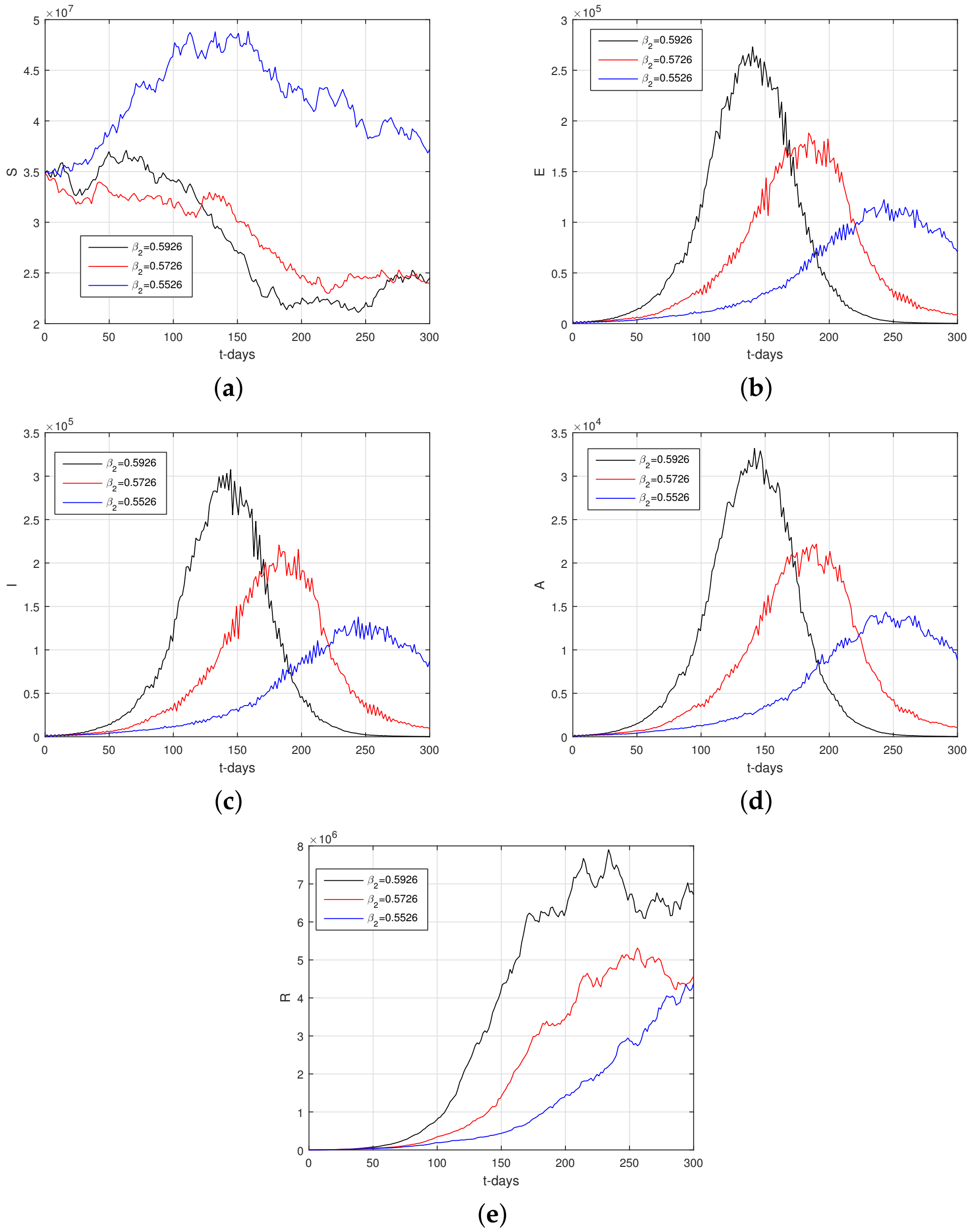


Figure 6. The impact of the parameter β_2 on the stochastic model. Subfigures (**a**–**e**) represent, respectively, the dynamics of susceptible, exposed, symptomatic, asymptomatic and recovered population under different values of the parameters β_2.

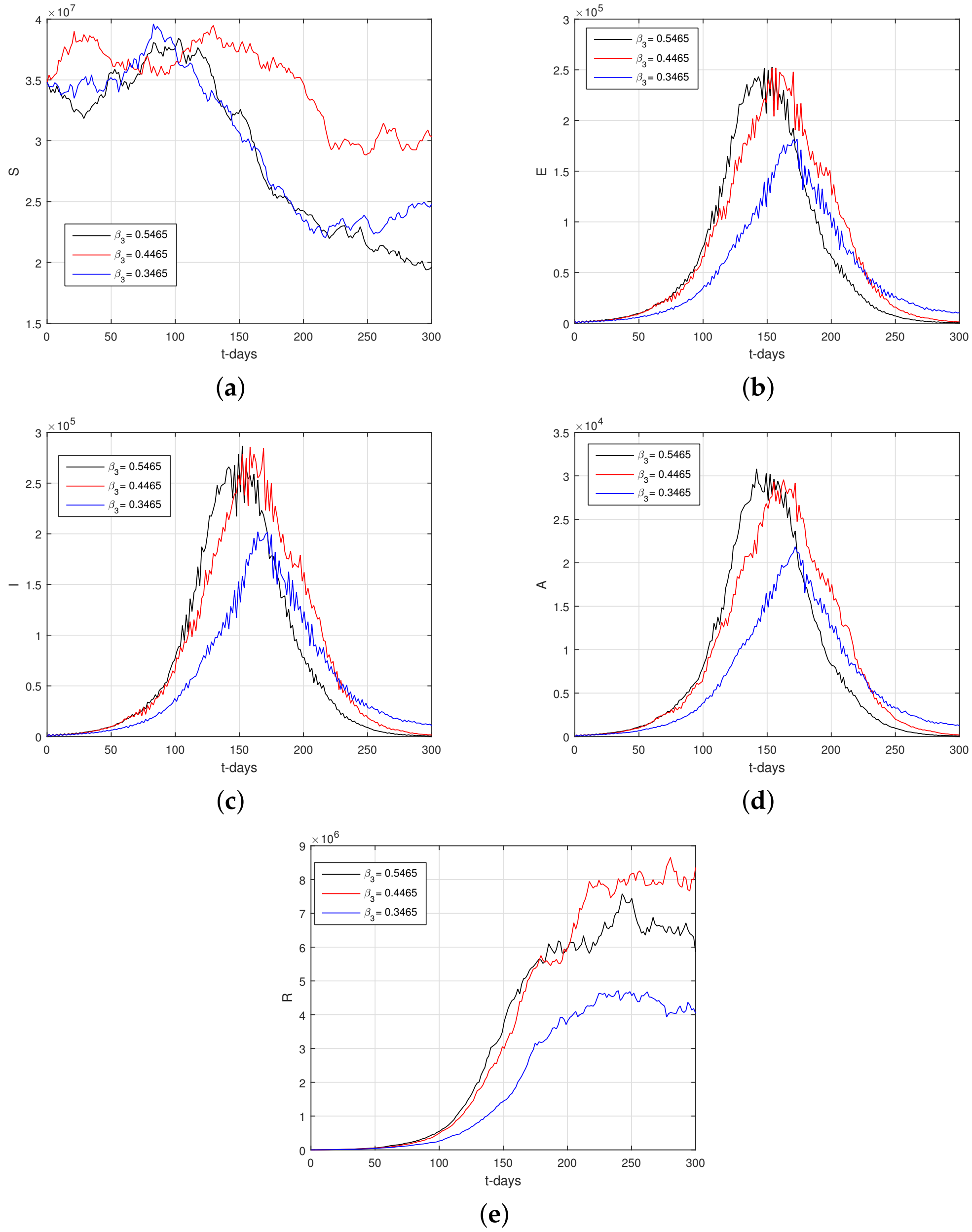


Figure 7. The impact of the parameter β_3 on the stochastic model. Subfigures (**a**–**e**) represent, respectively, the dynamics of susceptible, exposed, symptomatic, asymptomatic and recovered population under different values of the parameters β_3.

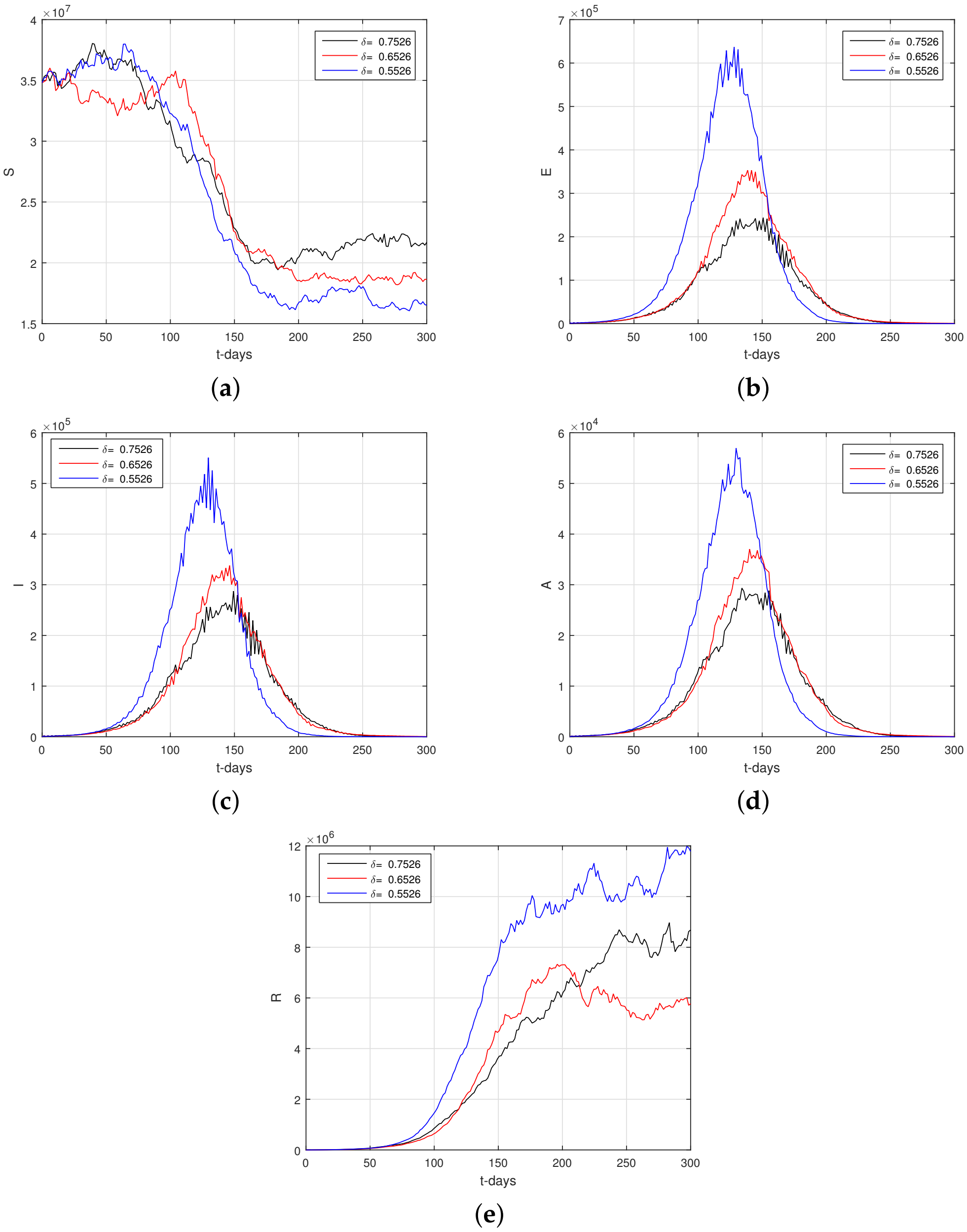


Figure 8. The plot shows the simulation with δ for the stochastic model. Subfigures (**a**–**e**) represent, respectively, the dynamics of susceptible, exposed, symptomatic, asymptomatic and recovered population under different values of the parameters δ.

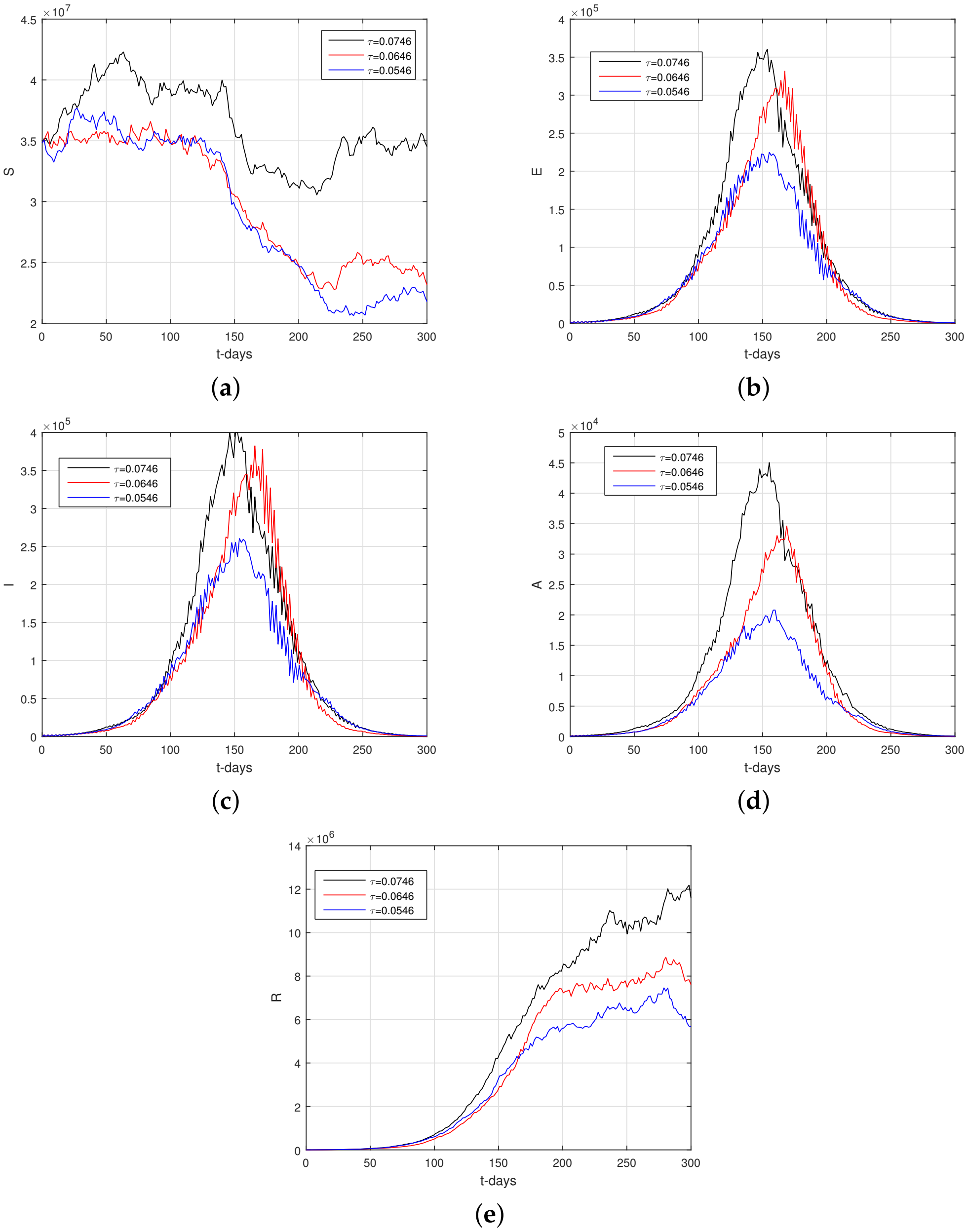


Figure 9. The impact of the parameter τ on the stochastic model. Subfigures (**a**–**e**) represent, respectively, the dynamics of susceptible, exposed, symptomatic, asymptomatic and recovered population under different values of the parameters τ.

6.3. Discussion

The graphical results obtained in this paper can be helpful regarding disease elimination in the country. The comparison of the stochastic and the deterministic model is shown in Figure 4. In particular, in Figure 4e, it can be seen that the number of recovered cases increases in the stochastic case as compared to the deterministic case. Increasing the testing of people and identifications of the symptomatic and asymptomatic individuals by quarantine and educating them about the disease can best minimize the future infected cases in the population. It is useful if individuals in a certain country increase COVID-19 testing upon identifying the asymptomatic and symptomatic individuals. The asymptomatic individuals that do not show visible disease symptoms should be isolated in order to decrease the future risk of infected cases. The symptomatic individuals should be quarantined, possibly at home, and can also be restricted from visiting other places which can possibly increase the population of infected cases in society. It can be seen from the graphical results that the model behaves well and leads to the equilibrium point.

7. Conclusions

In this work, we presented the analysis of the COVID-19 model using stochastic differential equations. We first formulated the model in ordinary differential equations and then we extended it to stochastic differential equations. We studied with care the stochastic COVID-19 mathematical model and presented their mathematical as well as numerical results in detail. Initially, we formulated the model by taking into account the assumptions of the transmission coefficients, such as exposed, symptomatic, and asymptomatic with their interaction with healthy people. It is well-known that these interactions are possible methods of increasing the cases of COVID-19 in a population. With these assumptions, we extended the model into stochastic differential equations. We studied the analysis of the model for the deterministic case and presented the related mathematical results for it. We proved the existence and uniqueness of the deterministic model and found the existence and uniqueness of the model. We studied the existence of the endemic equilibria and found that the model has a unique endemic equilibrium. We found that the deterministic model is globally asymptotically stable when the basic reproduction number $\mathcal{R}_0 \leq 1$.

Further, we studied the stochastic model and presented their unique global positive solution. We carried out the results of extinction for the stochastic model. The extinction results for the stochastic model have been provided.

We used the infected cases of the Kingdom of Saudi Arabia for the period March–July 2020 and parameterized the model. Using the nonlinear square curve fitting method, we obtained the data fitting to the model and presented it graphically. The results of the stochastic case for different σ_i values have good fitting as compared to the deterministic case.

The obtained parameters from the least square curve fit have been used to obtain the numerical results. The basic reproduction number computed for the given set of parameters is approximately, $\mathcal{R}_0 \approx 1.1367$.

We also compared the data fitting to the model using deterministic and stochastic and showed that stochastic fitting is better than the deterministic case for this work, and present a comparative graph of both solutions. We gave a comparison of graphical results for the case of deterministic and stochastic solutions. Some important parameters that can possibly best decrease future cases have been shown graphically. Among these parameters, the contact between susceptible and exposed, susceptible and symptomatic, and susceptible and asymptomatic, can best decrease future cases, if the recommendations of the World Health Organization (WHO) are properly followed.

According to the results of our simulations, one can see that our results are in line with the WHO recommendations and can be useful for disease elimination in the country of Saudi Arabia, see for more details [42,43].

Author Contributions: Conceptualization, M.A.K., A.D. and P.S.; Data curation, F.S.A., F.T.A. and A.D.; Formal analysis, F.T.A., M.A.K. and P.S.; Investigation, F.S.A., M.A.K., A.D. and P.S.; Methodology, F.S.A., F.T.A., M.A.K. and A.D.; Project administration, F.S.A., F.T.A. and P.S.; Resources, F.S.A. and M.A.K.; Software, F.T.A. and P.S.; Supervision, M.A.K. and P.S.; Visualization, F.S.A., F.T.A. and M.A.K.; Writing—original draft, F.S.A., F.T.A., M.A.K., A.D. and P.S. All authors have read and agreed to the published version of the manuscript.

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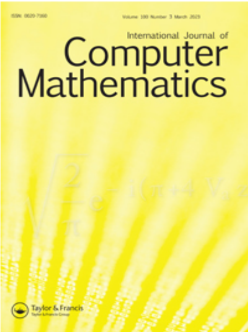
References

1. Khan, M.A.; Atangana, A. Modeling the dynamics of novel coronavirus (2019-nCov) with fractional derivative. *Alex. Eng. J.* **2020**, *59*, 2379–2389. [CrossRef]
2. Khan, M.A.; Atangana, A.; Alzahrani, E.; Fatmawati. The dynamics of COVID-19 with quarantined and isolation. *Adv. Differ. Equ.* **2020**, *2020*, 425. [CrossRef]
3. Oud, M.A.A.; Ali, A.; Alrabaiah, H.; Ullah, S.; Khan, M.A.; Islam, S. A fractional order mathematical model for COVID-19 dynamics with quarantine, isolation, and environmental viral load. *Adv. Differ. Equ.* **2021**, *2021*, 106. [CrossRef]
4. Rihan, F.; Alsakaji, H. Dynamics of a stochastic delay differential model for COVID-19 infection with asymptomatic infected and interacting people: Case study in the UAE. *Results Phys.* **2021**, *28*, 104658. [CrossRef] [PubMed]
5. Anggriani, N.; Ndii, M.Z.; Amelia, R.; Suryaningrat, W.; Pratama, M.A.A. A mathematical COVID-19 model considering asymptomatic and symptomatic classes with waning immunity. *Alex. Eng. J.* **2022**, *61*, 113–124. [CrossRef]
6. Masandawa, L.; Mirau, S.S.; Mbalawata, I.S. Mathematical modeling of COVID-19 transmission dynamics between healthcare workers and community. *Results Phys.* **2021**, *29*, 104731. [CrossRef] [PubMed]
7. Rwezaura, H.; Tchoumi, S.; Tchuenche, J. Impact of environmental transmission and contact rates on Covid-19 dynamics: A simulation study. *Informatics Med. Unlocked* **2021**, *27*, 100807. [CrossRef]
8. Liu, X.; Ullah, S.; Alshehri, A.; Altanji, M. Mathematical assessment of the dynamics of novel coronavirus infection with treatment: A fractional study. *Chaos Solitons Fractals* **2021**, *153*, 111534. [CrossRef]
9. Beigi, A.; Yousefpour, A.; Yasami, A.; Gómez-Aguilar, J.; Bekiros, S.; Jahanshahi, H. Application of reinforcement learning for effective vaccination strategies of coronavirus disease 2019 (COVID-19). *Eur. Phys. J. Plus* **2021**, *136*, 609. [CrossRef] [PubMed]
10. Martinez-Guerra, R.; Flores-Flores, J.P. An algorithm for the robust estimation of the COVID-19 pandemics population by considering undetected individuals. *Appl. Math. Comput.* **2021**, *405*, 126273. [PubMed]
11. Riyapan, P.; Shuaib, S.E.; Intarasit, A. A Mathematical model of COVID-19 Pandemic: A case study of Bangkok, Thailand. *Comput. Math. Methods Med.* **2021**, *2021*, 6664483. [CrossRef] [PubMed]
12. Khan, M.A.; Ullah, S.; Kumar, S. A robust study on 2019-nCOV outbreaks through non-singular derivative. *Eur. Phys. J. Plus* **2021**, *136*, 168. [CrossRef]
13. Bock, W.; Jayathunga, Y.; Götz, T.; Rockenfeller, R. Are the upper bounds for new SARS-CoV-2 infections in Germany useful? *Comput. Math. Biophys.* **2021**, *9*, 242–260. [CrossRef]
14. Khan, M.A.; Atangana, A. Mathematical modeling and analysis of COVID-19: A study of new variant Omicron. *Phys. A Stat. Mech. Its Appl.* **2022**, *599*, 127452. [CrossRef] [PubMed]
15. Muniyappan, A.; Sundarappan, B.; Manoharan, P.; Hamdi, M.; Raahemifar, K.; Bourouis, S.; Varadarajan, V. Stability and numerical solutions of second wave mathematical modeling on covid-19 and omicron outbreak strategy of pandemic: Analytical and error analysis of approximate series solutions by using hpm. *Mathematics* **2022**, *10*, 343. [CrossRef]
16. Pandey, P.; Gómez-Aguilar, J.; Kaabar, M.K.; Siri, Z.; Abd Allah, A.M. Mathematical modeling of COVID-19 pandemic in India using Caputo-Fabrizio fractional derivative. *Comput. Biol. Med.* **2022**, *145*, 105518. [CrossRef] [PubMed]
17. Asamoah, J.K.K.; Okyere, E.; Abidemi, A.; Moore, S.E.; Sun, G.Q.; Jin, Z.; Acheampong, E.; Gordon, J.F. Optimal control and comprehensive cost-effectiveness analysis for COVID-19. *Results Phys.* **2022**, *33*, 105177. [CrossRef]
18. Khajanchi, S.; Sarkar, K.; Banerjee, S. Modeling the dynamics of COVID-19 pandemic with implementation of intervention strategies. *Eur. Phys. J. Plus* **2022**, *137*, 129. [CrossRef]
19. Ullah, M.S.; Higazy, M.; Kabir, K.A. Modeling the epidemic control measures in overcoming COVID-19 outbreaks: A fractional-order derivative approach. *Chaos Solitons Fractals* **2022**, *155*, 111636. [CrossRef] [PubMed]
20. Tesfaye, A.W.; Satana, T.S. Stochastic model of the transmission dynamics of COVID-19 pandemic. *Adv. Differ. Equ.* **2021**, *2021*, 457. [CrossRef] [PubMed]
21. Rihan, F.A.; Alsakaji, H.J.; Rajivganthi, C. Stochastic SIRC epidemic model with time-delay for COVID-19. *Adv. Differ. Equ.* **2020**, *2020*, 502. [CrossRef] [PubMed]
22. Niño-Torres, D.; Ríos-Gutiérrez, A.; Arunachalam, V.; Ohajunwa, C.; Seshaiyer, P. Stochastic modeling, analysis, and simulation of the COVID-19 pandemic with explicit behavioral changes in Bogotá: A case study. *Infect. Dis. Model.* **2022**, *7*, 199–211. [CrossRef] [PubMed]
23. Tesfay, A.; Saeed, T.; Zeb, A.; Tesfay, D.; Khalaf, A.; Brannan, J. Dynamics of a stochastic COVID-19 epidemic model with jump-diffusion. *Adv. Differ. Equ.* **2021**, *2021*, 228. [CrossRef] [PubMed]
24. Guo, X.; Gupta, A.; Sampat, A.; Zhai, C. A stochastic contact network model for assessing outbreak risk of COVID-19 in workplaces. *PLoS ONE* **2022**, *17*, e0262316. [CrossRef] [PubMed]
25. Allen, L.J. A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis. *Infect. Dis. Model.* **2017**, *2*, 128–142. [CrossRef] [PubMed]
26. Zhong, H.; Wang, W. Mathematical analysis for COVID-19 resurgence in the contaminated environment. *Math. Biosci. Eng.* **2020**, *17*, 6909–6927. [CrossRef]
27. Li, S.; Ying, S. Latest Research Advances on Novel Coronavirus Pneumonia. 2020, SciFinder; 2020. Preprint | SciFinder | ID: Ppcovidwho-4385. Available online: https://pesquisa.bvsalud.org/global-literature-on-novel-coronavirus-2019-ncov/resource/pt/ppcovidwho-4385 (accessed on 20 September 2022).

28. Chen, Y.; Wang, A.; Yi, B.; Ding, K.; Wang, H.; Wang, J.; Shi, H.; Wang, S.; Xu, G. Epidemiological characteristics of infection in COVID-19 close contacts in Ningbo city. *Zhonghua Liu Xing Bing Xue Zhi Zhonghua Liuxingbingxue Zazhi* **2020**, *41*, 667–671.
29. Heilongjiang Provincial Health Committee. Available online: http://wsjkw.hlj.gov.cn/ (accessed on 10 July 2022.).
30. Xiong, C. A detailed explanation of survival time for COVID-19 virus in the environment. *China Food Safty Mag.* **2020**, *5*, 22–25.
31. Van den Driessche, P.; Watmough, J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Math. Biosci.* **2002**, *180*, 29–48. [CrossRef]
32. Mao, X. *Stochastic Differential Equations and their Applications*; Horwood Publ.: Chichester, UK, 1997.
33. Zhang, Q.; Zhou, K. Stationary distribution and extinction of a stochastic SIQR model with saturated incidence rate. *Math. Probl. Eng.* **2019**, *2019*, 3575410. [CrossRef]
34. Din, A.; Liang, J.; Zhou, T. Detecting critical transitions in the case of moderate or strong noise by binomial moments. *Phys. Rev. E* **2018**, *98*, 012114. [CrossRef] [PubMed]
35. Zhao, Y.; Jiang, D. The threshold of a stochastic SIRS epidemic model with saturated incidence. *Appl. Math. Lett.* **2014**, *34*, 90–93. [CrossRef]
36. Zhao, Y.; Jiang, D. The threshold of a stochastic SIS epidemic model with vaccination. *Appl. Math. Comput.* **2014**, *243*, 718–727. [CrossRef]
37. World/Countries/Saudi Arabia. Available online: https://www.worldometers.info/coronavirus/country/saudi-arabia/ (accessed on 20 September 2022).
38. Daily New Cases in Saudi Arabia. Available online: https://www.worldometers.info/coronavirus/country/saudi-arabia/ (accessed on 20 September 2022).
39. Saudi Arabia Population. Available online: https://www.worldometers.info/world-population/saudi-arabia-population/ (accessed on 20 September 2022).
40. Life Expectancy at Birth, Total (Years)—Saudi Arabia. Available online: https://data.worldbank.org/indicator/SP.DYN.LE00.IN?locations=SA (accessed on 20 September 2022).
41. Higham, D.J. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Rev.* **2001**, *43*, 525–546. [CrossRef]
42. Yezli, S.; Khan, A. COVID-19 social distancing in the Kingdom of Saudi Arabia: Bold measures in the face of political, economic, social and religious challenges. *Travel Med. Infect. Dis.* **2020**, *37*, 101692. [CrossRef] [PubMed]
43. Al-Harbi, S.K.; Al-Tuwairqi, S.M. Modeling the effect of lockdown and social distancing on the spread of COVID-19 in Saudi Arabia. *PLoS ONE* **2022**, *17*, e0265779. [CrossRef] [PubMed]



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Modified inertial extragradient methods for finding minimum-norm solution of the variational inequality problem with applications to optimal control problem

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RESEARCH ARTICLE

Modified inertial extragradient methods for finding minimum-norm solution of the variational inequality problem with applications to optimal control problem

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ABSTRACT

In order to discover the minimum-norm solution of the pseudomonotone variational inequality problem in a real Hilbert space, we provide two variants of the inertial extragradient approach with a novel generalized adaptive step size. Two of the suggested algorithms make use of the projection and contraction methods. We demonstrate several strong convergence findings without requiring the prior knowledge of the Lipschitz constant of the mapping. Finally, we give a number of numerical examples that highlight the benefits and effectiveness of the suggested algorithms and how they may be used to solve the optimal control problem.

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1. Introduction

The primary goal of this study is to construct several accelerated iterative methods with adaptive step sizes for finding the solutions of variational inequality problems in infinite-dimensional Hilbert spaces. Let $A : \mathcal{H} \to \mathcal{H}$ be an operator and let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Take $C \subset \mathcal{H}$ is a nonempty, closed, and convex subset of $\mathcal{H}$. The *variational inequality problem* (shortly, VIP) is find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C. \tag{VIP}$$

Variational inequality theory provides a fundamental model for many areas; for example engineering, economics, traffic management, operations optimization, and mathematical programming, and it constructs a unified framework for many optimization problems (see, e.g. [1,6,22,28,42]). Therefore, the theory and solution methods of variational inequalities have received more and more attention from scholars.

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A vast variety of numerical approaches for solving variational inequality problems have been presented throughout the last few decades. Next, we review some known methods in the literature for solving variational inequalities in finite- and infinite-dimensional spaces, which motivate us to propose new iterative algorithms. The Korpelevich extragradient method [15], which calls for computing the projection on the feasible set twice in each iteration, is the oldest and simplest method for dealing with the variational inequality problem. It is well known that computing projections may be challenging, particularly when the structure of the feasible set is intricate. Some approaches that only need computing the projection on the feasible set once per iteration have been developed to solve this problem; see, e.g. [3,11,40]. The main idea of these methods is to replace the iterative process of the second step in the extragradient method with a display calculation. Numerous variations based on these techniques [3,11,40] have recently been presented (see, e.g. [14,24,29,30,34,36,39,43]). Their numerical tests demonstrate the computational effectiveness and benefits of the suggested algorithms.

Recently, inspired by the work of Dong, Jiang and Gibali [8], Thong and Gibali [32] proposed the following Algorithm 1.1 to solve VIP in Hilbert spaces. On the other hand, Gibali, Thong and Tuan [10] also proposed the following Algorithm 1.2 for solving the monotone variational inequality problem based on the projection and contraction method [11].

Algorithm 1.1

Initialization: Given $\lambda > 0$, $l \in (0,1)$, $\mu \in (0,1)$, and $\gamma \in (0,2)$.
Iterative Steps: Let $x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Compute $v_n = P_C(x_n - \lambda_n A x_n)$, where λ_n is chosen to be the largest $\kappa \in \{\lambda, \lambda l, \lambda l^2, \dots\}$ satisfying

$$\kappa \|Ax_n - Av_n\| \le \mu \|x_n - v_n\| \tag{1}$$

If $x_n = v_n$ then stop and v_n is a solution of (VIP). Otherwise, go to **Step 2**.
Step 2. Compute $z_n = P_{T_n}(x_n - \gamma \lambda_n \rho_n A v_n)$, where $T_n := \{x \in \mathcal{H} : \langle x_n - \lambda_n A x_n - v_n, x - v_n\rangle \le 0\}$, and

$$\rho_n := (1-\mu)\frac{\|x_n - v_n\|^2}{\|g_n\|^2}, \quad g_n := x_n - v_n - \lambda_n (Ax_n - Av_n). \tag{2}$$

Step 3. Compute $x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n z_n$.
Set $n := n + 1$ go to **Step 1**.

Algorithm 1.2

Initialization: Given $\lambda > 0$, $l \in (0,1)$, $\mu \in (0,1)$, and $\gamma \in (0,2)$.
Iterative Steps: Let $x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Compute $v_n = P_C(x_n - \lambda_n A x_n)$, where λ_n is generated by (1).
Step 2. Compute $z_n = x_n - \gamma \rho_n g_n$, where ρ_n and g_n are defined in (2).
Step 3. Compute $x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n z_n$.
Set $n := n + 1$ go to **Step 1**.

The strong convergence theorems for the suggested iterative techniques in infinite-dimensional Hilbert spaces were obtained by Thong and Gibali [32] and Gibali et al. [10], respectively, under some reasonable restrictions imposed on the mapping and parameters. It is important to keep in mind that the Algorithms 1.1 and 1.2 only need to perform the projection on the feasible set once throughout each iteration. Their numerical tests demonstrate that the suggested algorithms outperform the existing approaches [3,8,24] in terms of computational efficiency and accuracy. Furthermore, we note that the Algorithms 1.1 and 1.2 employ an Armijo-type line search step size criterion enabling them to operate without requiring prior knowledge of the Lipschitz constant of the mapping. However, using Armijo-type step sizes may require the proposed algorithm to calculate the projection values on the feasible set multiple times per iteration. To overcome this drawback, Yang and Liu [46] introduced a new adaptive step size criterion which only needs to use some previously known information to complete the calculation of the step size. Recently, many scholars have used the idea of this criterion to construct numerous algorithms for finding the solutions of variational inequalities and equilibrium problems; see, e.g. [9,16,31,33,36,45,47].

Many scholars have focussed a lot of their attention and study on the concept of inertial as one of the ways of acceleration. The primary characteristic of inertial-type approaches is that the combination of the previous two (or more) iterations determines the outcome of the subsequent iteration. It has been observed that this minor adjustment might accelerate the convergence of inertial-free algorithms. Numerous inertial-type methods have been developed to handle variational inequalities, equilibrium problems, split feasibility problems, fixed point problems, inclusion problems, and others (see, e.g. [4,7,12,23,25,26,29,31,35,36,43]). Numerous numerical simulations show the benefits and effectiveness of their inertial methods compared to the version without inertial terms.

In this paper, we suggest two adaptive algorithms with inertial terms to handle variational inequality problems in real Hilbert spaces, inspired and motivated by the aforementioned findings. We made the following contributions to this research.

- Our two algorithms use a new step size without any line search procedure, which generalizes the step size suggested by Liu and Yang [16]. In addition, our two adaptive algorithms are preferable to the fixed-step algorithms suggested in [4,35]. Numerical experimental results show that our step size is useful and efficient, and that our two algorithms require less execution time than the algorithms in [10,32] that use the Armijo step size.
- Our two algorithms are designed to solve pseudo-monotone variational inequality problems, which improves the results used in [8,10,24,32,45,46] for finding the solutions of monotone variational inequalities.
- To accelerate the convergence speed of the proposed algorithms, the inertial term is also embedded in our algorithms. Numerical experimental results demonstrate that the proposed algorithms converge faster than the methods without inertial in [10,32].
- The strong convergence theorems of the proposed algorithms are proved under some suitable conditions. This improves the weak convergence results obtained in [3,8,16,25].
- To demonstrate the benefits and computational effectiveness of the suggested methods in comparison to those that were previously known in [10,32], several numerical experiments and applications in optimal control problems are provided.

The rest of this paper is structured as follows. Basic definitions and lemmas that should be utilized are gathered in Section 2. In Section 3, we describe two new non-monotonic inertial extragradient algorithms and examine their convergence. In Section 4, a few numerical tests are provided to demonstrate the benefits and effectiveness of the suggested algorithms. In Section 5, we solve the optimal control problem utilizing the suggested methods. Finally, Section 6 provides a succinct review of the research.

2. Preliminaries

The following equality and inequality are useful for our proofs.

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \quad \forall\, x, y \in \mathcal{H}, \tag{3}$$

and

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall\, x, y \in \mathcal{H}. \tag{4}$$

Let $C \subset \mathcal{H}$ be a nonempty, closed, and convex. Recall that the *metric projection* of $\mathcal{H}$ onto C, denoted by P_C, which is defined as for any $x \in \mathcal{H}$, there exists a unique nearest point in C, given as $P_C(x)$ such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall\, y \in C.$$

Note that P_C has following properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall\, x \in \mathcal{H},\ y \in C, \tag{5}$$

and

$$\|x-y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \quad \forall\, x \in \mathcal{H},\ y \in C. \tag{6}$$

Let $\mathrm{VIP}(C, A)$ denote the solution set of the variational inequality problem (VIP). It is easy to check the following relation according to (5).

$$z \in \mathrm{VI}(C, A) \ \Leftrightarrow\ z = P_C(z - \lambda Az), \quad \forall\, \lambda > 0. \tag{7}$$

Definition 2.1: A mapping $A : C \to \mathcal{H}$ is said to be:

(1) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$;
(2) *pseudomonotone* if $\langle Ax, y - x\rangle \geq 0$, we have $\langle Ay, y - x\rangle \geq 0$ for all $x, y \in C$;
(3) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$;
(4) *sequentially weakly continuous* on C if, for each sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$.

Remark 2.1: From the above definitions, we see that (1) $\Rightarrow$ (2), but the converse is not true in general (see, e.g. [27, Example 4.2]).

Lemma 2.1 ([5]): *Let $C \subset \mathcal{H}$ be a nonempty closed and convex set and $A : C \to \mathcal{H}$ be a pseudomonotone and continuous mapping. Then z is a solution of the problem* (VIP) *if and only if*

$$\langle Ax, x - z\rangle \geq 0, \quad \forall\, x \in C.$$

Lemma 2.2 ([17]): *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\delta_n)a_n + b_n + c_n, \quad \forall\, n \geq 1,$$

where $\{\delta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(1) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(2) *If $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.3 ([18]): *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each $i \in \mathbb{N}$. Define the sequence $\{\kappa(n)\}_{n\geq n_0}$ of integers as follows:*

$$\kappa(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then the following results hold:

(1) $\kappa(n_0) \le \kappa(n_0+1) \le \cdots$ *and* $\kappa(n) \to \infty$.
(2) $\Gamma_{\kappa(n)} \le \Gamma_{\kappa(n)+1}$ *and* $\Gamma_n \le \Gamma_{\kappa(n)+1}$ *for each* $n \ge n_0$.

3. Main results

We make the following assumptions about our algorithms in order to prove some strong convergence theorems for them:

(A1) The feasible set C is a closed and convex subset of a real Hilbert space $\mathcal{H}$;
(A2) The mapping $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and pseudomonotone on $\mathcal{H}$;
(A3) The mapping $A : \mathcal{H} \to \mathcal{H}$ satisfies the following condition: for each $\{t_n\} \subset C$ such that $t_n \rightharpoonup x$,

$$\|Ax\| \le \liminf_{n\to\infty} \|At_n\|; \tag{8}$$

(A4) The solution set of the problem (VIP) is nonempty, that is, $\Omega := \mathrm{VIP}(C, A) \neq \emptyset$, where $\mathrm{VIP}(C, A)$ denotes the solution set of the problem (VIP);
(A5) The positive sequence $\{\xi_n\}$ satisfies $\lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Remark 3.1: (1) For Assumption (A2), it suffices to assume that the mapping A is continuous pseudomonotone if $\mathcal{H}$ is a finite-dimensional Hilbert space and it is not necessary to assume A satisfies (8).
(2) Note that Assumption (A3) is weaker than the sequential weak continuity of the mapping A, which often assumed in many recent works related to the pseudomonotone problem (VIP) (see, for example, [4,14,29,34,36,39,43]). Indeed, let $A : \mathcal{H} \to \mathcal{H}$ be a mapping define by $Ax = x\|x\|$ for all $x \in \mathcal{H}$. It can be shown that A satisfies Assumption (A3), but not sequentially weakly continuous (see [21,38]). However, if A is monotone, then Assumption (A3) can be removed.

Now, we are in a position to describe the proposed Algorithm 3.1.
The following lemma is crucial for proving the convergence results.

Lemma 3.1: *Let* $\{\lambda_n\}$ *be a sequence generated by* (11). *Then there exists* $\lambda \in [\min\{\frac{\mu}{L}, \lambda_0\}, \lambda_0 + \sum_{n=1}^{\infty} p_n]$ *such that* $\lambda = \lim_{n\to\infty} \lambda_n$.

Proof: The proof of this lemma follows as that of Lemma 3.1 in [44], so we omit it here. ∎

Remark 3.2: The adaptive step size in this work is different from the studied adaptive step size as in many works. In particular, if $p_n = 0$ and $q_n = 1$ for all $n \ge 0$, then the step size reduces to the step size of many methods (see, e.g. [9,33,36,45–47]). In addition, if $p_n \neq 0$ and $q_n = 1$ for all $n \ge 0$, then the step size becomes the step size in [16].

Lemma 3.2: *Let* $\{r_n\}$, $\{v_n\}$ *and* $\{g_n\}$ *be the sequences generated by Algorithm* 3.1. *If* $r_n = v_n$ *or* $g_n = 0$, *then* $v_n \in \Omega$.

Proof: By the definition of g_n, we have

$$\begin{aligned}
\|g_n\| &= \|r_n - v_n - \lambda_n(Ar_n - Av_n)\| \\
&\ge \|r_n - v_n\| - \lambda_n\|Ar_n - Av_n\| \\
&\ge \|r_n - v_n\| - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\|r_n - v_n\| \\
&= \left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|.
\end{aligned}$$

Algorithm 3.1 Modified inertial subgradient extragradient method

Initialization: Given $\lambda_0 > 0$, $\phi > 0$, $\sigma > 1$, $\gamma \in (0, \frac{2}{\sigma})$ and $\mu \in (0,1)$. Choose $\{p_n\} \subset [0,\infty)$ such that $\sum_{n=0}^{\infty} p_n < \infty$ and $\{q_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} q_n = 1$.
Iterative Steps: Let $x_{-1}, x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n ($n \geq 0$). Set

$$r_n = (1-\alpha_n)(x_n + \phi_n(x_n - x_{n-1})),$$

where

$$\phi_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|x_n - x_{n-1}\|}, \phi\right\}, & \text{if } x_n \neq x_{n-1}, \\ \phi, & \text{otherwise.} \end{cases} \tag{9}$$

Step 2. Compute

$$v_n = P_C(r_n - \lambda_n A r_n).$$

If $r_n = v_n$ or $Av_n = 0$, then stop and v_n is a solution of the problem (VIP). Otherwise, go to **Step 3**.
Step 3. Compute

$$x_{n+1} = P_{T_n}(r_n - \gamma\lambda_n\rho_n A v_n),$$

where $T_n := \{x \in \mathcal{H} : \langle r_n - \lambda_n A r_n - v_n, x - v_n\rangle \leq 0\}$ and ρ_n is defined as follows:

$$\rho_n := (1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|^2}, \quad g_n := r_n - v_n - \lambda_n(Ar_n - Av_n), \tag{10}$$

and update the step size by

$$\lambda_{n+1} = \min\left\{\lambda_n + p_n, \frac{q_n\mu\|r_n - v_n\|}{\|Ar_n - Av_n\|}\right\}. \tag{11}$$

Set $n := n+1$ go to **Step 1**.

We can also show that

$$\|g_n\| \leq \left(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|.$$

Therefore, we conclude that

$$\left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\| \leq \|g_n\| \leq \left(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|. \tag{12}$$

By Lemma 3.1, one sees that $\lim_{n\to\infty}\lambda_n$ exists, which together with $\lim_{n\to\infty} q_n = 1$ gives

$$\lim_{n\to\infty}\frac{q_n\lambda_n}{\lambda_{n+1}} = 1.$$

Therefore, there exists a constant n_0 such that $1 - \frac{q_n\mu\lambda_n}{\lambda_{n+1}} > 0$ for all $n \geq n_0$. Hence we have that $r_n = v_n$ if and only if $g_n = 0$ by means of (12). If $r_n = v_n$, then $v_n = P_C(v_n - \lambda_n A v_n)$. This means that $v_n \in \Omega$ by means of (5). ∎

Lemma 3.3: *Suppose that Assumptions* (A1)–(A4) *hold. Let* $\{x_n\}$ *be formed by Algorithm* 3.1. *Then, for each* $p \in \Omega$ *and* $n \geq n_0$, *we have*

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2,$$

where $\chi_n := (\dfrac{1-\mu}{1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}}})^2$.

Proof: Let $p \in \Omega$. Then it follows from (6) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|r_n - \gamma\lambda_n\rho_n Av_n - p\|^2 - \|r_n - \gamma\lambda_n\rho_n Av_n - x_{n+1}\|^2 \\
&= \|r_n - p\|^2 - 2\gamma\lambda_n\rho_n\langle r_n - p, Av_n\rangle + \gamma^2\lambda_n^2\rho_n^2\|Av_n\|^2 - \|r_n - x_{n+1}\|^2 \\
&\quad + 2\gamma\lambda_n\rho_n\langle r_n - x_{n+1}, Av_n\rangle - \gamma^2\lambda_n^2\rho_n^2\|Av_n\|^2 \\
&= \|r_n - p\|^2 - \|r_n - x_{n+1}\|^2 - 2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - p\rangle \\
&= \|r_n - p\|^2 - \|r_n - x_{n+1}\|^2 - 2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle - 2\gamma\lambda_n\rho_n\langle Av_n, v_n - p\rangle.
\end{aligned}$$

Since $p \in \Omega$ and $v_n \in C$, one has $\langle Ap, v_n - p\rangle \geq 0$. Then, by the pseudomonotonicity of A, we have $\langle Av_n, v_n - p\rangle \geq 0$. Hence we have

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \|r_n - x_{n+1}\|^2 - 2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle. \tag{13}$$

It is clear that $x_{n+1} \in T_n$ and hence

$$\begin{aligned}
&- 2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle \\
&= 2\gamma\rho_n \underbrace{\langle r_n - \lambda_n Ar_n - v_n, x_{n+1} - v_n\rangle}_{\leq 0} - 2\gamma\rho_n\langle r_n - v_n - \lambda_n(Ar_n - Av_n), x_{n+1} - v_n\rangle \\
&\leq -2\gamma\rho_n\langle r_n - v_n - \lambda_n(Ar_n - Av_n), x_{n+1} - v_n\rangle \\
&= -2\gamma\rho_n\langle g_n, x_{n+1} - v_n\rangle \\
&= -2\gamma\rho_n\langle g_n, r_n - v_n\rangle + 2\gamma\rho_n\langle g_n, r_n - x_{n+1}\rangle.
\end{aligned} \tag{14}$$

Now, we estimate $-2\gamma\rho_n\langle g_n, r_n - v_n\rangle$ and $2\gamma\rho_n\langle g_n, r_n - x_{n+1}\rangle$. By the definition of g_n and (11), we have

$$\begin{aligned}
\langle g_n, r_n - v_n\rangle &= \langle r_n - v_n - \lambda_n(Ar_n - Av_n), r_n - v_n\rangle \\
&\geq \|r_n - v_n\|^2 - \lambda_n\|(Ar_n - Av_n)\|\|r_n - v_n\| \\
&\geq \left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \mu > \frac{1-\mu}{\sigma} > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0, \quad \forall\, n \geq n_0.$$

Thus we deduce

$$\langle g_n, r_n - v_n\rangle \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|^2, \quad \forall\, n \geq n_0.$$

Since $\rho_n = (1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|^2}$, we have $\rho_n\|g_n\|^2 = (1-\mu)\|r_n - v_n\|^2$. Therefore we obtain

$$-2\gamma\rho_n\langle g_n, r_n - v_n\rangle \le \frac{-2\gamma\rho_n^2}{\sigma}\|g_n\|^2, \quad \forall\, n \ge n_0. \tag{15}$$

On the other hand, it follows from the equality $2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$ that

$$2\gamma\rho_n\langle g_n, r_n - x_{n+1}\rangle = \|r_n - x_{n+1}\|^2 + \gamma^2\rho_n^2\|g_n\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2. \tag{16}$$

Substituting (15) and (16) into (14), we obtain

$$-2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle \le \|r_n - x_{n+1}\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\rho_n^2\|g_n\|^2. \tag{17}$$

By the definition of g_n, we see that

$$\begin{aligned}\|g_n\| &\le \|r_n - v_n\| + \lambda_n\|Ar_n - Av_n\| \\ &\le \|r_n - v_n\| + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\|r_n - v_n\| \\ &= \left(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|.\end{aligned}$$

This implies that

$$\frac{1}{\|g_n\|^2} \ge \frac{1}{(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|r_n - v_n\|^2}.$$

Hence we have

$$\rho_n^2\|g_n\|^2 = (1-\mu)^2\frac{\|r_n - v_n\|^4}{\|g_n\|^2} \ge \frac{(1-\mu)^2}{(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2}\|r_n - v_n\|^2. \tag{18}$$

Combining (13), (17), and (18), we obtain

$$\|x_{n+1} - p\|^2 \le \|r_n - p\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2, \quad \forall\, n \ge n_0, \tag{19}$$

where $\chi_n := \left(\dfrac{1-\mu}{1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}}\right)^2$. ■

Lemma 3.4 ([37]): *Suppose that Assumptions (A1)–(A4) hold. Let $\{r_n\}$ be generated by Algorithm* 3.1. *If there exists a subsequence $\{r_{n_k}\} \subset \{r_n\}$ such that $\{r_{n_k}\}$ converges weakly to $v \in \mathcal{H}$ and* $\lim_{k\to\infty}\|r_{n_k} - v_{n_k}\| = 0$, *then $v \in \Omega$.*

Now, we prove the strong convergence of Algorithm 3.1.

Theorem 3.1: *Suppose that Assumptions (A1)–(A5) hold. Then the sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges strongly to* $x^* = P_\Omega(0)$, *where* $\|x^*\| = \min\{\|x\| : x \in \Omega\}$.

Proof: First, we show that $\{x_n\}$ is bounded. From Lemma 3.3 and $\gamma \in (0, \frac{2}{\sigma})$, one has

$$\begin{aligned}
\|x_{n+1} - p\| &\le \|r_n - p\| \\
&= \|(1-\alpha_n)(x_n - p + \phi_n(x_n - x_{n-1})) - \alpha_n p\| \\
&\le (1-\alpha_n)\|x_n - p + \phi_n(x_n - x_{n-1})\| + \alpha_n\| - p\| \\
&\le (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\phi_n\|x_n - x_{n-1}\| + \alpha_n\|p\|, \quad \forall\, n \ge n_0.
\end{aligned}$$

Putting $\iota_n := (1-\alpha_n)\frac{\phi_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|$ for all $n \ge n_0$. It is easy to see that $\lim_{n\to\infty} \iota_n$ exists, which implies that $\{\iota_n\}$ is bounded. Then by Lemma 2.2, one has $\{\|x_n - p\|\}$ is bounded. Note that

$$\|x_n\| \le \|x_n - p + p\| \le \|x_n - p\| + \|p\|.$$

Hence $\{x_n\}$ is bounded and consequently so are $\{r_n\}$ and $\{v_n\}$. Let x^* be the minimum-norm solution of Ω, that is, $x^* = P_\Omega(0)$. From (4), we have

$$\begin{aligned}
\|r_n - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^* + \phi_n(x_n - x_{n-1})) - \alpha_n x^*\|^2 \\
&\le (1-\alpha_n)^2\|x_n - x^* + \phi_n(x_n - x_{n-1})\|^2 + 2\alpha_n\langle x^*, x^* - r_n\rangle \\
&\le (1-\alpha_n)^2\left(\|x_n - x^*\|^2 + 2\phi_n\langle x_n - x_{n-1}, x_n - x^* + \phi_n(x_n - x_{n-1})\rangle\right) \\
&\quad + 2\alpha_n\langle x^*, x^* - r_n\rangle \\
&\le (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x^* - r_n\rangle,
\end{aligned} \tag{20}$$

where $K_1 := \sup_{n\ge 0}\{\|x_n - x^* + \phi_n(x_n - x_{n-1})\|\}$. Putting (20) into (19), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x^* - r_n\rangle \\
&\quad - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2,
\end{aligned} \tag{21}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x_{n+1} - r_n\rangle \\
&\quad + 2\alpha_n\langle x^*, x^* - x_{n+1}\rangle
\end{aligned} \tag{22}$$

for all $n \ge n_0$. From (21), we have

$$\|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 + \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n K_2 \tag{23}$$

for all $n \ge n_0$, where $K_2 := \sup_{n\ge n_0}\{(1-\alpha_n)\frac{\phi_n}{\alpha_n}\|x_n - x_{n-1}\|K_1, \|x^*\|\|r_n - x^*\|\}$.

Now, we prove the strong convergence of $\{\|x_n - x^*\|^2\}$ converges to zero by consider the following two cases.

Case 1. Suppose there exists $N \in \mathbb{N}$ such that $\{\|x_n - x^*\|^2\}$ is monotonically nonincreasing for $n \geq N$. Since $\{\|x_n - x^*\|^2\}$ is bounded, we have $\{\|x_n - x^*\|^2\}$ converges and hence

$$\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0.$$

Since $\gamma \in (0, \frac{2}{\sigma})$ and $\lim_{n\to\infty} \chi_n > 0$, it follows from (23) that

$$\lim_{n\to\infty} \|r_n - x_{n+1} - \gamma \rho_n g_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|r_n - v_n\| = 0. \tag{24}$$

For all $n \geq n_0$, we note that $\|g_n\| \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|$, which gives $\frac{1}{\|g_n\|} \leq \frac{\sigma}{(1-\mu)\|r_n - v_n\|}$. Hence we have

$$\begin{aligned}
\|r_n - x_{n+1}\| &\leq \|r_n - x_{n+1} - \gamma \rho_n g_n\| + \gamma \rho_n \|g_n\| \\
&= \|r_n - x_{n+1} - \gamma \rho_n g_n\| + \gamma(1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|} \\
&\leq \|r_n - x_{n+1} - \gamma \rho_n g_n\| + \gamma\sigma \|r_n - v_n\|.
\end{aligned}$$

Then it follows from (24) that

$$\lim_{n\to\infty} \|r_n - x_{n+1}\| = 0. \tag{25}$$

Moreover, we see that

$$\begin{aligned}
\|x_n - r_n\| &= \|(1-\alpha_n)\phi_n(x_n - x_{n-1}) - \alpha_n x_n\| \\
&\leq (1-\alpha_n)\phi_n\|x_n - x_{n-1}\| + \alpha_n\|x_n\| \\
&= \alpha_n\left((1-\alpha_n)\frac{\phi_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x_n\|\right).
\end{aligned}$$

Thus we have

$$\lim_{n\to\infty} \|x_n - r_n\| = 0. \tag{26}$$

It follows from (25) and (26) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - r_n\| + \|r_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{27}$$

Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some point $v \in \mathcal{H}$ such that

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle = \lim_{k\to\infty}\langle x^*, x^* - x_{n_k}\rangle = \langle x^*, x^* - v\rangle.$$

From (26), we also get $\{r_{n_k}\}$ converges weakly to $v \in \mathcal{H}$, which together with Lemma 3.4 and (24) implies that $v \in \Omega := \mathrm{VIP}(C, A)$. From (5), we obtain

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle = \langle x^*, x^* - v\rangle \leq 0. \tag{28}$$

Moreover, from (27) and (28), we also get

$$\limsup_{n\to\infty}\langle x^*, x^* - x_{n+1}\rangle = \limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle \leq 0. \tag{29}$$

This together with (22) and Lemma 2.2 yields that $\lim_{n\to\infty} \|x_n - x^*\|^2 \to 0$, that is, $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define an integer sequence $\kappa(n)$ by $\kappa(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}$ for all $n \geq n_0$ (for some n_0 large enough). By Lemma 2.3, $\{\kappa(n)\}$ is a nondecreasing sequence such that $\kappa(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\kappa(n)} \leq \Gamma_{\kappa(n)+1}$ for all $n \geq n_0$. Put $\Gamma_n := \|x_n - x^*\|^2$ for all $n \in \mathbb{N}$. By (23), one has

$$\begin{aligned}
&\|r_{\kappa(n)} - x_{\kappa(n)+1} - \gamma\rho_{\kappa(n)}g_{\kappa(n)}\|^2 + \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_{\kappa(n)}\|r_{\kappa(n)} - v_{\kappa(n)}\|^2 \\
&\quad \leq \|x_{\kappa(n)} - x^*\|^2 - \|x_{\kappa(n)+1} - x^*\|^2 + 2\alpha_{\kappa(n)}K_2 \\
&\quad \leq 2\alpha_{\kappa(n)}K_2,
\end{aligned}$$

where $K_2 > 0$. Following similar argument as in Case 1, one has

$$\lim_{n\to\infty}\|r_{\kappa(n)} - x_{\kappa(n)+1} - \gamma\rho_{\kappa(n)}g_{\kappa(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|r_{\kappa(n)} - v_{\kappa(n)}\| = 0.$$

Moreover, we have

$$\lim_{n\to\infty}\|x_{\kappa(n)+1} - r_{\kappa(n)}\| = 0 \tag{30}$$

and

$$\limsup_{n\to\infty}\langle x^*, x^* - x_{\kappa(n)+1}\rangle \leq 0. \tag{31}$$

From (22) and $\Gamma_{\kappa(n)} \leq \Gamma_{\kappa(n)+1}$, one gets

$$\begin{aligned}
\|x_{\kappa(n)+1} - x^*\|^2 &\leq (1 - \alpha_{\kappa(n)})\|x_{\kappa(n)} - x^*\|^2 + 2(1 - \alpha_{\kappa(n)})\phi_{\kappa(n)}\|x_{\kappa(n)} - x_{\kappa(n)-1}\|K_1 \\
&\quad + 2\alpha_{\kappa(n)}\langle x^*, x_{\kappa(n)+1} - r_n\rangle + 2\alpha_{\kappa(n)}\langle x^*, x^* - x_{\kappa(n)+1}\rangle \\
&\leq (1 - \alpha_{\kappa(n)})\|x_{\kappa(n)+1} - x^*\|^2 + 2(1 - \alpha_{\kappa(n)})\phi_{\kappa(n)}\|x_{\kappa(n)} - x_{\kappa(n)-1}\|K_1 \\
&\quad + 2\alpha_{\kappa(n)}\langle x^*, x_{\kappa(n)+1} - r_{\kappa(n)}\rangle + 2\alpha_{\kappa(n)}\langle x^*, x^* - x_{\kappa(n)+1}\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{\kappa(n)+1} - x^*\|^2 &\leq 2(1 - \alpha_{\kappa(n)})\frac{\phi_{\kappa(n)}}{\alpha_{\kappa(n)}}\|x_{\kappa(n)} - x_{\kappa(n)-1}\|K_1 \\
&\quad + 2\|x_{\kappa(n)+1} - r_{\kappa(n)}\|\|x^*\| + 2\langle x^*, x^* - x_{\kappa(n)+1}\rangle,
\end{aligned}$$

where $K_1 > 0$. Combining (30) and (31), we obtain

$$\lim_{n\to\infty}\|x_{\kappa(n)+1} - x^*\|^2 = 0.$$

By Lemma 2.3, we have

$$\|x_n - x^*\|^2 \leq \|x_{\kappa(n)+1} - x^*\|^2 \to 0 \text{ as } n \to \infty.$$

Hence $x_n \to x^*$. Therefore we can conclude that $\{x_n\}$ converges strongly to the minimum-norm solution of (VIP) from the above two cases. ■

Next, we introduce the second modification of inertial extragradient method (see Algorithm 3.2 below) for solving pseudomonotone VIPs. This method motivated by the projection and contraction method [11] with a generalized adaptive step size.

Lemma 3.5: *Suppose that Assumptions* (A1)–(A4) *hold. Let* $\{x_n\}$ *be created by Algorithm* 3.2. *We have*

Algorithm 3.2 Modified inertial projection and contraction method

Initialization: Given $\lambda_0 > 0$, $\phi > 0$, $\sigma > 1$, $\gamma \in (0, \frac{2}{\sigma})$ and $\mu \in (0,1)$. Choose $\{p_n\} \subset [0,\infty)$ such that $\sum_{n=0}^{\infty} p_n < \infty$ and $\{q_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} q_n = 1$.
Iterative Steps: Let $x_{-1}, x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 0)$. Set

$$r_n = (1-\alpha_n)(x_n + \phi_n(x_n - x_{n-1})),$$

where ϕ_n is defined in (9).
Step 2. Compute

$$v_n = P_C(r_n - \lambda_n A r_n).$$

If $r_n = v_n$ or $Av_n = 0$, then stop and v_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute

$$x_{n+1} = r_n - \gamma \rho_n g_n,$$

where ρ_n and g_n are defined in (10), and update the step size by (11).
Set $n := n+1$ go to **Step 1**.

(1) $\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \frac{1}{\gamma}(\frac{2}{\sigma} - \gamma)\|x_{n+1} - r_n\|^2$ *for each* $n \geq n_0$ *and* $p \in \Omega$;

(2) $\|r_n - v_n\|^2 \leq \chi_n' \|x_{n+1} - r_n\|^2$, *where* $\chi_n' := \left(\frac{1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\right)^2$.

Proof: (1) Let $p \in \Omega$, one sees that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|r_n - \gamma\rho_n g_n - p\|^2 \\ &= \|r_n - p\|^2 - 2\gamma\rho_n\langle r_n - p, g_n\rangle + \gamma^2\rho_n^2\|g_n\|^2.\end{aligned} \tag{32}$$

From the definition of g_n, we see that

$$\begin{aligned}\langle r_n - p, g_n\rangle &= \|r_n - v_n\|^2 - \lambda_n\langle r_n - v_n, Ar_n - Av_n\rangle + \langle v_n - p, r_n - v_n - \lambda_n(Ar_n - Av_n)\rangle \\ &\geq \|r_n - v_n\|^2 - \lambda_n\|r_n - v_n\|\|Ar_n - Av_n\| + \langle v_n - p, r_n - v_n - \lambda_n(Ar_n - Av_n)\rangle \\ &\geq \left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|^2 + \langle v_n - p, r_n - v_n - \lambda_n(Ar_n - Av_n)\rangle.\end{aligned}$$

According to $\lim_{n\to\infty}(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \mu > \frac{1-\mu}{\sigma} > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0, \quad \forall\, n \geq n_0.$$

Thus we have

$$\langle r_n - p, g_n\rangle \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|^2 + \langle r_n - v_n - \lambda_n(Ar_n - Av_n), v_n - p\rangle, \quad \forall\, n \geq n_0. \tag{33}$$

Since $v_n = P_C(r_n - \lambda_n A r_n)$ and from (5), one has

$$\langle r_n - \lambda_n A r_n - v_n, v_n - p\rangle \geq 0.$$

Moreover, using $\langle Ap, v_n - p\rangle \geq 0$ and the pseudomonotonicity of A, one gets

$$\langle Av_n, v_n - p\rangle \geq 0.$$

Hence

$$\langle r_n - v_n - \lambda_n(Ar_n - Av_n), v_n - p\rangle = \underbrace{\langle r_n - \lambda_n Ar_n - v_n, v_n - p\rangle}_{\geq 0} + \lambda_n \underbrace{\langle Av_n, v_n - p\rangle}_{\geq 0} \geq 0. \tag{34}$$

Combining (33) and (34), we obtain

$$\langle r_n - p, g_n\rangle \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|^2, \quad \forall\, n \geq n_0.$$

It follows from the definition of ρ_n that

$$\langle r_n - p, g_n\rangle \geq \frac{1}{\sigma}\rho_n\|g_n\|^2, \quad \forall\, n \geq n_0. \tag{35}$$

By using (33) and (36), one has

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\rho_n^2\|g_n\|^2, \quad \forall\, n \geq n_0.$$

Since $x_{n+1} = r_n - \gamma\rho_n g_n$, we have $\rho_n^2\|g_n\|^2 = \frac{1}{\gamma^2}\|x_{n+1} - r_n\|^2$. Hence we have

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - r_n\|^2, \quad \forall\, n \geq n_0. \tag{36}$$

(2) By the definition of ρ_n, we have

$$\begin{aligned}\|r_n - v_n\|^2 &= \frac{1}{1-\mu}\cdot\rho_n\|g_n\|^2 = \frac{1}{1-\mu}\cdot\frac{1}{\gamma^2\rho_n}(\gamma^2\rho_n^2\|g_n\|^2)\\ &= \frac{1}{1-\mu}\cdot\frac{1}{\gamma^2\rho_n}\|x_{n+1} - r_n\|^2.\end{aligned} \tag{37}$$

From $\|g_n\| \leq (1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})\|r_n - v_n\|$, we have $\frac{1}{\|g_n\|^2} \geq \frac{1}{(1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|r_n - v_n\|^2}$. So

$$\rho_n = (1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|^2} \geq \frac{1-\mu}{(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2}. \tag{38}$$

Combining (37) and (38), one has

$$\|r_n - v_n\|^2 \leq \chi_n'\|x_{n+1} - r_n\|^2, \tag{39}$$

where $\chi_n' := \left(\frac{1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\right)^2$. ∎

Theorem 3.2: *Suppose that Assumptions* (A1)–(A5) *hold. Then the sequence* $\{x_n\}$ *created by Algorithm* 3.2 *converges strongly to* $x^* = P_\Omega(0)$*, where* $\|x^*\| = \min\{\|x\| : x \in \Omega\}$.

Proof: From Lemma 3.5 and $\gamma \in (0, \frac{2}{\sigma})$, by using the same argument as in Theorem 3.1, we have that $\{x_n\}$ is bounded. Moreover, we can show that

$$\|r_n - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x^* - r_n\rangle, \tag{40}$$

where $x^* = P_\Omega(0)$ and $K_1 > 0$. Putting (40) into (36), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 \\
&\quad + 2\alpha_n\langle x^*, x^* - r_n\rangle - \frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - r_n\|^2 \\
&\le (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 \\
&\quad + 2\alpha_n\langle x^*, x_{n+1} - r_n\rangle + 2\alpha_n\langle x^*, x^* - x_{n+1}\rangle
\end{aligned} \tag{41}$$

for all $n \ge n_0$. From (41), we have

$$\frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - r_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n K_2, \quad \forall\, n \ge n_0, \tag{42}$$

where $K_2 > 0$. Finally, we prove the strong convergence of $\{x_n\}$ converges to $x^* = P_\Omega(0)$ by consider the two cases, which are the same as in Theorem 3.1. Thus it follows from (42) that $\lim_{n\to\infty}\|x_{n+1} - r_n\| = 0$. This together with (39) gives that $\lim_{n\to\infty}\|r_n - v_n\| = 0$. The rest of the proof can be easily proved by similar arguments to that of Theorem 3.1 and so we omit it. ■

4. Numerical experiments

The purpose of this part is to illustrate the benefits and computing effectiveness of the suggested algorithms in comparison to several strongly convergent schemes in the literature [10,32]. The numerical examples take place in both finite- and infinite-dimensional spaces. The programmes are all executed in MATLAB 2018a using a PC with an Intel(R) Core(TM) i5-8250U CPU running at 1.60 GHz and 8.00 GB of RAM.

Example 4.1: Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be given as $Ax := Gx + g$, where $g \in \mathbb{R}^m$ and $G := BB^\mathsf{T} + S + E$, matrix $B \in \mathbb{R}^{m\times m}$, matrix $S \in \mathbb{R}^{m\times m}$ is skew-symmetric, and matrix $E \in \mathbb{R}^{m\times m}$ is diagonal matrix whose diagonal terms are nonnegative (hence G is positive symmetric definite). The feasible set C is given by $C := \{x \in \mathbb{R}^m : -2 \le x_i \le 5, i = 1, 2, \ldots, m\}$. It is easy to see that A is monotone (hence it is pseudomonotone) L-Lipschitz continuous with $L = \|G\|$. In this example, all entries of B, E are produced randomly in $[0, 2]$ and S is produced randomly in $[-2, 2]$. Let $g = \mathbf{0}$. Then the solution set is $x^* = \{\mathbf{0}\}$.

We compare the proposed algorithms with the following.

- Algorithm 3.1 in Thong and Gibali [32] (shortly, TG Alg. 3.1).
- Algorithm 3.1 in Gibali et al. [10] (shortly, GTT Alg. 3.1).

The parameters of our algorithms and the compared ones are set as follows.

- Taking $\lambda_0 = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 1/(n+1)$, $p_n = 1/(n+1)^{1.1}$, $q_n = (n+1)/n$, $\phi = 0.4$ and $\xi_n = 100/(n+1)^2$ for our Algorithms 3.1 and 3.2.
- Choosing $\lambda = 0.5$, $l = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 1/(n+1)$ and $\beta_n = 0.5(1-\alpha_n)$ for TG Alg. 3.1 and GTT Alg. 3.1.

The starting values $x_0 = x_1$ are produced at random using $5rand(m,1)$ in MATLAB, and the maximum number of iterations 200 serves as a common stopping condition for all methods. At the nth

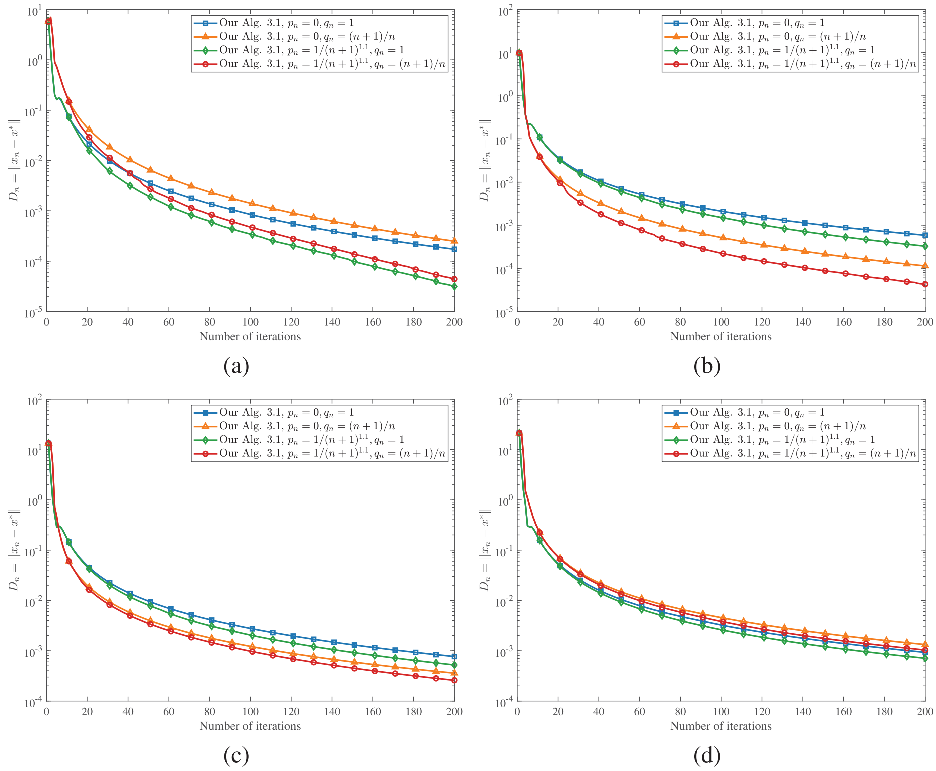


Figure 1. The behaviour of our Algorithm 3.1 for different p_n and q_n in Example 4.1. (a) $m = 20$. (b) $m = 50$. (c) $m = 100$ and (d) $m = 200$.

Table 1. Numerical results for all algorithms under different dimensions in Example 4.1.

	$m = 20$		$m = 50$		$m = 100$		$m = 200$	
Algorithms	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 3.1	2.09E−05	0.0349	4.42E−05	0.0273	3.74E−04	0.0337	1.09E−03	0.0419
Our Alg. 3.2	2.34E−05	0.0239	4.58E−05	0.0228	3.78E−04	0.0267	1.08E−03	0.0370
TG Alg. 3.1	1.11E−02	0.0430	3.49E−02	0.0412	5.77E−02	0.1538	8.88E−02	0.1683
GTT Alg. 3.1	1.11E−02	0.0370	3.49E−02	0.0364	5.77E−02	0.0709	8.88E−02	0.1286

step, we utilize $D_n := \|x_n - x^*\|$ to calculate the iteration error. First, we test the effect of different parameters p_n and q_n on the proposed algorithms with different dimensions, as shown in Figures 1 and 2. Next, Table 1 shows the results of the proposed methods compared to some known ones in different dimensions, where 'CPU' denotes the execution time in seconds.

Example 4.2: We consider an example in the Hilbert space $\mathcal{H} := L^2([0,1])$ associated with the inner product

$$\langle p, q\rangle := \int_0^1 p(t)q(t)\,\mathrm{d}t, \quad \forall\, p, q \in \mathcal{H},$$

and the induced norm

$$\|p\| := \left(\int_0^1 |p(t)|^2\,\mathrm{d}t\right)^{1/2}, \quad \forall\, p \in \mathcal{H}.$$

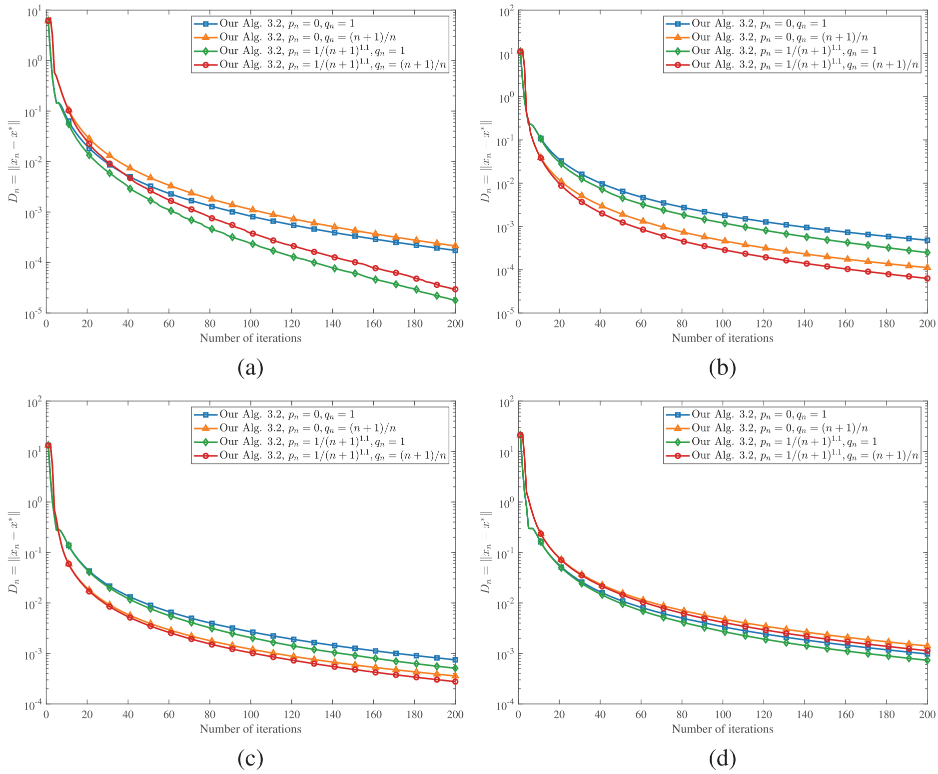


Figure 2. The behaviour of our Algorithm 3.2 for different p_n and q_n in Example 4.1. (a) $m = 20$. (b) $m = 50$. (c) $m = 100$ and (d) $m = 200$.

The feasible set is given by $C := \{x \in \mathcal{H} : \|x\| \le 1\}$. Let $A : C \to \mathcal{H}$ be as follows.

$$(Ax)(t) := \int_0^1 \big(x(t) - Q(t,v)g(x(v))\big)\,\mathrm{d}v + h(t), \quad \forall\, t \in [0,1],\ x \in C,$$

where

$$Q(t,v) := \frac{2tv\,\mathrm{e}^{t+v}}{e\sqrt{\mathrm{e}^2 - 1}}, \quad g(x) := \cos x, \quad h(t) := \frac{2t\,\mathrm{e}^t}{e\sqrt{\mathrm{e}^2 - 1}}.$$

Note that A is monotone (hence it is pseudomonotone) and L-Lipschitz continuous with $L = 2$ (see [13] for more details) and $x^*(t) = \{\mathbf{0}\}$ is the solution of the (VIP).

The parameters of all algorithms are maintained the same as in Example 4.1. We utilize $D_n := \|x_n(t) - x^*(t)\|$ to calculate the iteration error of the nth step and set the maximum number of iterations for all algorithms to 50. The numerical behaviours of all algorithms with four starting points $x_0(t) = x_1(t)$ are reported in Table 2.

From Examples 4.1 and 4.2, we have the following observations.

(1) It can be seen from Figures 1 and 2 that the suggested methods have different impacts with different parameters p_n and q_n. Note that when $m = 50, 100$, the proposed algorithms on $q_n \neq 1$ has a higher accuracy than $q_n = 1$ when the values of p_n are the same. In addition, the proposed algorithms on $p_n \neq 0$ has a better performance than $p_n = 0$ when the values of q_n are the same. Thus, the iteration step sizes of the proposed algorithms are useful and efficient.

Table 2. Numerical results for all algorithms at different initial values in Example 4.2.

	$x_1 = 5t^3$		$x_1 = 4\sin(2t)$		$x_1 = 8\log(t)$		$x_1 = 3\exp(t)$	
Algorithms	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 3.1	8.44E−21	28.0391	8.80E−21	28.6204	1.83E−21	29.3688	3.27E−17	33.3884
Our Alg. 3.2	3.95E−21	26.4142	5.39E−22	27.1204	6.45E−18	27.3436	2.94E−13	34.7676
TG Alg. 3.1	7.47E−06	35.4475	1.02E−05	35.3399	2.68E−05	37.8135	1.50E−05	44.1810
GTT Alg. 3.1	6.70E−06	34.3776	8.30E−06	34.3631	2.05E−05	36.7857	1.25E−05	43.5128

(2) From Tables 1 and 2, we can obtain that our two algorithms have a better accuracy and less execution time than the algorithms presented in the literature [10,32]. These findings are independent of the size of the dimension and the choice of starting values. On the other hand, it is worth noting that the algorithms presented in [10,32] use an Armijo-type step size, which may lead them to require more execution time than our suggested adaptive algorithms.

5. Applications to optimal control problems

In this section, we use the proposed algorithms to solve the optimal control problem (see [20,29,41] for a description of this problem). Next, we run two tests in optimal control problems to illustrate the performance of our algorithms and compare them with the ones in [10,32]. The parameters of the algorithms are set as follows.

- Taking $\lambda_0 = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 10^{-4}/(n+1)$, $p_n = 10^{-1}/(n+1)^{1.1}$, $q_n = (n+1)/n$, $\phi = 0.01$ and $\xi_n = 10^{-4}/(n+1)^2$ for our Algorithms 3.1 and 3.2.
- Choosing $\lambda = 1$, $l = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 10^{-4}/(n+1)$ and $\beta_n = 0.5(1-\alpha_n)$ for TG Alg. 3.1 and GTT Alg. 3.1.

Example 5.1 (See [19]): Consider the following problem:

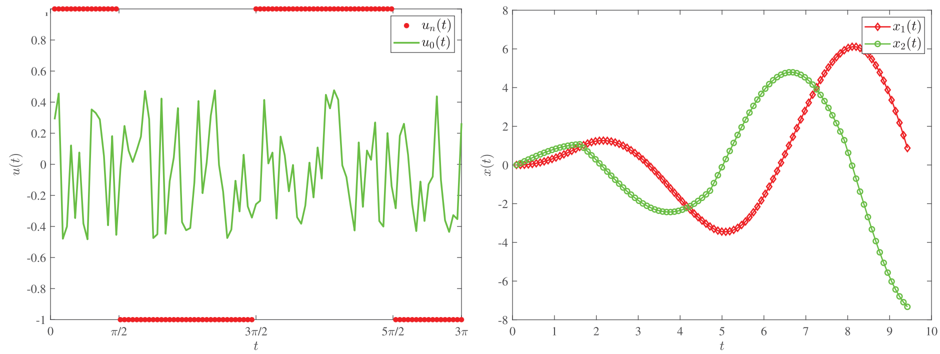
$$\begin{aligned} &\text{minimize} && x_2(3\pi) \\ &\text{subject to} && \dot{x}_1(t) = x_2(t), \\ &&& \dot{x}_2(t) = -x_1(t) + u(t), \quad \forall\, t \in [0, 3\pi], \\ &&& x(0) = 0, \\ &&& u(t) \in [-1, 1]. \end{aligned}$$

The exact optimal control of Example 5.1 is known:

$$u^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2); \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

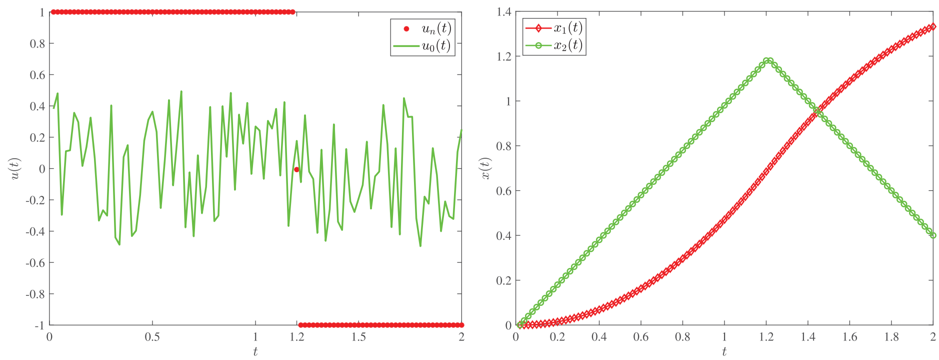
The initial controls $u_0(t) = u_1(t)$ are randomly generated in $[-1, 1]$ and the stopping criterion is either $D_n := \|u_{n+1} - u_n\| \le 10^{-4}$ or the maximum number of iterations is reached 1000. Figure 3 gives the approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.1.

We now consider an example in which the terminal function is not linear.



(a) (b)

Figure 3. Numerical results of the proposed Algorithm 3.1 for Example 5.1. (a) Initial and optimal controls and (b) Optimal trajectories.



(a) (b)

Figure 4. Numerical results of the proposed Algorithm 3.2 for Example 5.2. (a) Initial and optimal controls and (b) Optimal trajectories.

Example 5.2 (See [2]): Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & -x_1(2) + (x_2(2))^2, \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = u(t), \quad \forall\, t \in [0,2], \\ & x_1(0) = 0, \quad x_2(0) = 0, \\ & u(t) \in [-1,1]. \end{aligned}$$

The exact optimal control of Example 5.2 is known:

$$u^*(t) = \begin{cases} 1, & \text{if } t \in [0, 1.2); \\ -1, & \text{if } t \in (1.2, 2]. \end{cases}$$

The approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.2 are shown in Figure 4.

The results of our methods as well as the compared algorithms in Examples 5.1 and 5.2 are given in Table 3, where 'Iter.' represent the number of iterations.

Table 3. Numerical results for all algorithms in Examples 5.1 and 5.2.

	Example 5.1			Example 5.2		
Algorithms	Iter.	CPU	D_n	Iter.	CPU	D_n
Our Alg. 3.1	100	0.0468	9.9010E−05	175	0.0680	6.4170E−05
Our Alg. 3.2	111	0.0507	9.9305E−05	273	0.0823	8.7029E−05
TG Alg. 3.1	202	0.1245	9.9507E−05	417	0.1623	9.9175E−05
GTT Alg. 3.1	224	0.0856	9.9756E−05	1000	0.6143	2.4875E−04

From Figures 3, 4 and Table 3, it is clear that whether the terminal function is linear or nonlinear, the suggested techniques for solving optimal control problems can still produce satisfactory results. Additionally, compared to the algorithms described in the literature [10,32], they take fewer iterations and less time.

6. Conclusions

In this paper, two iterative approaches with a novel adaptive step size rule are suggested for locating the minimum-norm solution of a pseudomonotone variational inequality problem in a real Hilbert space. Without previous knowledge of the operator's Lipschitz constant, the strong convergence of the sequences produced by these methods has been demonstrated. To confirm the effectiveness and benefits of the suggested algorithms and to compare them with some related approaches in the literature, several numerical experiments have been carried out. Additionally, the optimum control problem has been investigated as an application of our main results.

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Disclosure statement

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References

[1] Q.H. Ansari, M. Islam, and J.C. Yao, *Nonsmooth variational inequalities on Hadamard manifolds*, Appl. Anal. 99 (2020), pp. 340–358.
[2] B. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control*, American Institute of Mathematical Sciences, San Francisco, 2007.
[3] Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. 148 (2011), pp. 318–335.
[4] P. Cholamjiak, D.V. Thong, and Y.J. Cho, *A novel inertial projection and contraction method for solving pseudomonotone variational inequality problems*, Acta Appl. Math. 169 (2020), pp. 217–245.

[5] R.W. Cottle and J.C. Yao, *Pseudo-monotone complementarity problems in Hilbert space*, J. Optim. Theory Appl. 75 (1992), pp. 281–295.

[6] P. Cubiotti and J.C. Yao, *On the Cauchy problem for a class of differential inclusions with applications*, Appl. Anal. 99 (2020), pp. 2543–2554.

[7] Q.L. Dong, D. Jiang, P. Cholamjiak, and Y. Shehu, *A strong convergence result involving an inertial forward-backward algorithm for monotone inclusions*, J. Fixed Point Theory Appl. 19 (2017), pp. 3097–3118.

[8] Q.L. Dong, D. Jiang, and A. Gibali, *A modified subgradient extragradient method for solving the variational inequality problem*, Numer. Algorithms 79 (2018), pp. 927–940.

[9] A. Gibali and D.V. Thong, *Tseng typemethods for solving inclusion problems and its applications*, Calcolo 55 (2018), Article ID 49.

[10] A. Gibali, D.V. Thong, and P.A. Tuan, *Two simple projection-type methods for solving variational inequalities*, Anal. Math. Phys. 9 (2019), pp. 2203–2225.

[11] B.S. He, *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim. 35 (1997), pp. 69–76.

[12] D.V. Hieu and A. Gibali, *Strong convergence of inertial algorithms for solving equilibrium problems*, Optim. Lett. 14 (2020), pp. 1817–1843.

[13] D.V. Hieu, P.K. Anh, and L.D. Muu, *Modified hybrid projection methods for finding common solutions to variational inequality problems*, Comput. Optim. Appl. 66 (2017), pp. 75–96.

[14] P.Q. Khanh, D.V. Thong, and N.T. Vinh, *Versions of the subgradient extragradient method for pseudomonotone variational inequalities*, Acta Appl. Math. 170 (2020), pp. 319–345.

[15] G.M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Èkonom. i Mat. Metody 12 (1976), pp. 747–756.

[16] H. Liu and J. Yang, *Weak convergence of iterative methods for solving quasimonotone variational inequalities*, Comput. Optim. Appl. 77 (2020), pp. 491–508.

[17] P.E. Maingé, *Inertial iterative process for fixed points of certain quasi-nonexpansive mappings*, Set-Valued Anal. 15 (2007), pp. 67–79.

[18] P.E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. 16 (2008), pp. 899–912.

[19] A. Pietrus, T. Scarinci, and V.M. Veliov, *High order discrete approximations to Mayer's problems for linear systems*, SIAM J. Control Optim. 56 (2018), pp. 102–119.

[20] J. Preininger and P.T. Vuong, *On the convergence of the gradient projection method for convex optimal control problems with bang-bang solutions*, Comput. Optim. Appl. 70 (2018), pp. 221–238.

[21] S. Reich, D.V. Thong, Q.L. Dong, X.H. Li, and V.T. Dung, *New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings*, Numer. Algorithms 87 (2021), pp. 527–549.

[22] D.R. Sahu, J.C. Yao, M. Verma, and K.K. Shukla, *Convergence rate analysis of proximal gradient methods with applications to composite minimization problems*, Optimization 70 (2021), pp. 75–100.

[23] Y. Shehu and A. Gibali, *New inertial relaxed method for solving split feasibilities*, Optim. Lett. 15 (2021), pp. 2109–2126.

[24] Y. Shehu and O.S. Iyiola, *Strong convergence result for monotone variational inequalities*, Numer. Algorithms 76 (2017), pp. 259–282.

[25] Y. Shehu and O.S. Iyiola, *Projection methods with alternating inertial steps for variational inequalities: Weak and linear convergence*, Appl. Numer. Math. 157 (2020), pp. 315–337.

[26] B. Tan and S. Li, *Strong convergence of inertial Mann algorithms for solving hierarchical fixed point problems*, J. Nonlinear Var. Anal. 4 (2020), pp. 337–355.

[27] B. Tan, L. Liu, and X. Qin, *Self adaptive inertial extragradient algorithms for solving bilevel pseudomonotone variational inequality problems*, Jpn. J. Ind. Appl. Math. 38 (2021), pp. 519–543.

[28] B. Tan, X. Qin, and J.C. Yao, *Strong convergence of self-adaptive inertial algorithms for solving split variational inclusion problems with applications*, J. Sci. Comput. 87 (2021), Article ID 20.

[29] B. Tan, X. Qin, and J.C. Yao, *Two modified inertial projection algorithms for bilevel pseudomonotone variational inequalities with applications to optimal control problems*, Numer. Algorithms 88 (2021), pp. 1757–1786.

[30] B. Tan, S.Y. Cho, and J.C. Yao, *Accelerated inertial subgradient extragradient algorithms with non-monotonic step sizes for equilibrium problems and fixed point problems*, J. Nonlinear Var. Anal.6 (2022), pp. 89–122.

[31] B. Tan, X. Qin, and S.Y. Cho, *Revisiting subgradient extragradient methods for solving variational inequalities*, Numer. Algorithms 90 (2022), pp. 1593–1615.

[32] D.V. Thong and A. Gibali, *Two strong convergence subgradient extragradient methods for solving variational inequalities in Hilbert spaces*, Jpn. J. Ind. Appl. Math. 36 (2019), pp. 299–321.

[33] D.V. Thong and D.V. Hieu, *Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems*, Numer. Algorithms 82 (2019), pp. 761–789.

[34] D.V. Thong and P.T. Vuong, *Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities*, Optimization 68 (2019), pp. 2207–2226.

[35] D.V. Thong, N.T. Vinh, and Y.J. Cho, *New strong convergence theorem of the inertial projection and contraction method for variational inequality problems*, Numer. Algorithms 84 (2020), pp. 285–305.
[36] D.V. Thong, D.V. Hieu, and T.M. Rassias, *Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems*, Optim. Lett. 14 (2020), pp. 115–144.
[37] D.V. Thong, L.V. Long, X.H. Li, Q.L. Dong, Y.J. Cho, and P.A. Tuan, *A new self-adaptive algorithm for solving pseudomonotone variational inequality problems in Hilbert spaces*, Optimization (2021). https://doi.org/10.1080/02331934.2021.1909584.
[38] D.V. Thong, J. Yang, Y.J. Cho, and T.M. Rassias, *Explicit extragradient-like method with adaptive stepsizes for pseudomonotone variational inequalities*, Optim. Lett. 15 (2021), pp. 2181–2199.
[39] D.V. Thong, Y. Shehu, O.S. Iyiola, and H.V. Thang, *New hybrid projection methods for variational inequalities involving pseudomonotone mappings*, Optim. Eng. 22 (2021), pp. 363–386.
[40] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim. 38 (2000), pp. 431–446.
[41] P.T. Vuong and Y. Shehu, *Convergence of an extragradient-type method for variational inequality with applications to optimal control problems*, Numer. Algorithms 81 (2019), pp. 269–291.
[42] J. Wang and Y. Wang, *Strong convergence of a cyclic iterative algorithm for split common fixed-point problems of demicontractive mappings*, J. Nonlinear Var. Anal. 2 (2018), pp. 295–303.
[43] J. Yang, *Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities*, Appl. Anal. 100 (2021), pp. 1067–1078.
[44] J. Yang, *Projection and contraction methods for solving bilevel pseudomonotone variational inequalities*, Acta Appl. Math. 177 (2022), Article ID 7.
[45] J. Yang and H. Liu, *A modified projected gradient method for monotone variational inequalities*, J. Optim. Theory Appl. 179 (2018), pp. 197–211.
[46] J. Yang and H. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numer. Algorithms 80 (2019), pp. 741–752.
[47] J. Yang, P. Cholamjiak, and P. Sunthrayuth, *Modified Tseng's splitting algorithms for the sum of two monotone operators in Banach spaces*, AIMS Math. 6 (2021), pp. 4873–4900.

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A FRACTIONAL SARS-COV-2 MODEL WITH ATANGANA–BALEANU DERIVATIVE: APPLICATION TO FOURTH WAVE

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Abstract

A dynamical model of SARS-CoV-2 in fractional derivative using the cases of coronavirus of the fourth wave is presented. We construct basically the model in an integer case, and later it is extended to a fractional-order system by applying the Atangana–Baleanu operator definition. We give some background definitions and results for the fractional-order model. We present for the disease-free case that the model is locally asymptotically stable when $\mathcal{R}_0 < 1$. The global dynamics of the fractional model are given when $\mathcal{R}_0 \leq 1$ for the disease-free case. The model is further extended to fractional stochastic piecewise equations in the Atangana–Baleanu case. The reported cases from the fourth wave in Pakistan starting from July 1 up to November 16, 2021 are considered for the estimation of the parameters. We fitted our model to the suggested data and obtained the numerical value of the basic reproduction number $\mathcal{R}_0 \approx 0.9775$ for fractional order. We give the data fitting to both the fractional and piecewise stochastic differential equations, and show them both as having a good fitting to the data. We use further the numerical values of the model parameters and present its numerical results graphically using the effective numerical approaches. Some sensitive parameters that are reasonable for disease eliminations are used to obtain the graphical results.

Keywords: Fourth Wave Cases; COVID-19 Fractional Model; Estimations of Parameters; Results and Discussions.

1. INTRODUCTION

The COVID-19 infection is still producing large numbers of deaths and infected cases worldwide. Less developed countries that lack facilities are suffering from this infection. Due to the COVID-19 infection, the less developed and the developing countries are affected a lot. Among these countries, Pakistan has also been suffering. The COVID-19 infection produced multiple waves in different parts of the world with rising number of cases. Pakistan earlier faced three waves of COVID-19 infection and now, the fourth wave is ending. The fourth wave in the country began in July and will end in November. At present, the total cases reported are 1,281,872, whereas the death cases are 28,659. It is reasonable that the recovered cases are 1,230,734 which is 98% of the total infected cases. The death cases are 2% of the total infected cases.[1] The current cases of coronavirus in the country are decreasing day by day and life is returning back to normalcy. In order to avoid another wave in the country, the governments and the people should follow the prevention and other precautions that are mentioned by the World Health Organization (WHO). About 50% of the population in the country are vaccinated with one dose, while it is needed to make an action to vaccinate possibly the rest of the individuals to avoid the future fifth wave.

Scientists and biologists worked day and night to determine how the infection spreads. WHO and other health authorities are offering suggestions for possible prevention measures. We should mention that every country that faces this infection implements the necessary strategies in order to reduce the infection and its spread. Mathematical modeling was used first to estimate the basic reproduction number for the early cases reported in China and to predict the disease's current burden in the country and its possible propagation to the rest of the world in the future. Mathematical models are also necessary for understanding the COVID-19 infection and for determining when the infection peak occurs. A tremendous amount of literature has been produced on the modeling of COVID-19 infection with researchers from around the world using the simulated results achieved in specific countries and making recommendations based on the results.

According to our previous discussion, mathematical modeling of COVID-19 infection is vital in estimating the peak of the infection curve and basic reproduction numbers. In this paper, we highlight some mathematical models that were used for COVID-19 infection. The spread of COVID-19 infection in Saudi Arabia and its possible control mechanisms have been examined in Ref. 2. A SARS-CoV-2 model with different prevention mechanisms is discussed in Ref. 3. Dynamical analysis and optimal control of coronavirus infection have been considered in Ref. 4. A mathematical model to determine the control strategies for COVID-19 infection in Ethiopia is studied in Ref. 5. The COVID-19 infection with multiple strains and its control have been considered in Ref. 6. The authors in Ref. 7 considered the Indian coronavirus infected cases and presented the forecasting using a mathematical model. The study of infection trends in Italy and France through a mathematical model is explained in Ref. 8. The infected cases in Pakistan with optimal control analysis are discussed in Ref. 9. The dynamics and control of COVID-19 infection using the reported cases from Nigeria have been considered by the authors in Ref. 10. The authors in Ref. 11 considered a mathematical model to study the infected cases of coronavirus in China. A mathematical approach to study and predict the coronavirus infection in Ethiopia is explored in Ref. 12. An approach of graphical modeling to explore the dynamics of COVID-19 infection has been investigated in Ref. 13. A mathematical study designed for the understanding of infected cases of COVID-19 in India, is explored in Ref. 14. The authors in Ref. 15 explored the COVID-19 infection with cost-effective analysis and its optimal control theory. A fuzzy approach to investigate the solution of COVID-19 infection model is discussed in Ref. 16. The COVID-19 infection has hit many countries of the world and hence, there is a lot of related literature; therefore, we suggest the readers to check the references inside the above-mentioned studies.

In the recent era, with the advancement in fractional differential equations, many researches have been focused on studying the scientific and engineering problems, see the applications to integro-differential equations,[17] the new advancement and development in fractional operators,[18,19] application to epidemiology,[20–22] applications to wave dynamics equations,[23,24] etc. Due to its applications to scientific and engineering problems and particularly to disease epidemiology, the field is growing day by day. The fractional differential equations have so many properties and the memory effect is considered to be one of their important properties,[25] while this property cannot be observed in integer-order studies. With so many properties involved in fractional differential equations, authors around the world formulated interesting works regarding their applications to engineering and scientific areas. The COVID-19 infection considered a deadly infection, has been studied by many authors through fractional-order mathematical models. A mathematical model in fractional derivative using power-law kernel is discussed in Ref. 26. The transmission dynamics of COVID-19 infection through fractional-order derivative is discussed in Ref. 27. A mathematical model for COVID-19 with Mittag-Leffler law is considered in Ref. 28. The modeling for coronavirus infection using the infected cases from Turkey and Africa is made in Ref. 29. An approach of fractional-order modeling to investigate and analyze the infected cases in Saudi Arabia, is discussed by the authors in Ref. 30. The mathematical modeling of COVID-19 infection and its numerical investigations have been done in Ref. 31. SARS-CoV-2 with immune response through mathematical modeling and its optimal control study have been discussed in Ref. 32. The early cases in China using fractional-order model in the sense of Atangana–Baleanu derivative are considered in Ref. 33.

We study in this paper the mathematical modeling of COVID-19 infection with fractional derivatives using the Atangana–Baleanu operator. We first discuss briefly the modeling of the problem with integer-order derivative, while we extend it to fractional-order one using the Atangana–Baleanu derivative with recent cases of COVID-19 from Pakistan of the fourth wave. Further, we explored many possible interactions of the susceptible humans that make a healthy person infected by COVID-19. We explored the very vast literature on COVID-19 infection using fractional-order models. The rest of the paper is categorized section-wise as follows: The COVID-19 mathematical modeling with integer and noninteger orders is given in Sec. 2. We study the mathematical results involved in the fractional model in Sec. 3. Section 4 presents the analysis of the second-order differential model. Section 5 extends the model to piecewise stochastic differential equations and an analysis is also provided. We consider the reported cases of the coronavirus infection and estimate the parameters of the model in Sec. 6. We present the numerical algorithm for the Atangana–Baleanu derivative model in Sec. 7, while the discussion on the numerical results is also given in this section. We summarize the main achievements in this paper in Sec. 8.

2. MODEL FRAMEWORK

The coronavirus infection also knows as SARS-CoV-2 is still causing alarming situations in many countries including Pakistan. This study is concerned with studying this infection via a new mathematical model. In order to do this, we split the total population of humans into six different compartments. The first compartment is the healthy people $S(t)$ (the individuals who are not yet infected but have the ability to attract the disease, such people are not immune). Second one comprises the healthy individuals who after coming into contact with the asymptomatic, symptomatic and hospitalized individuals, become exposed, denoted by $E(t)$. The exposed individuals after completing their incubation periods become infected, either asymptomatically or symptomatically (asymptomatic people usually are strong and they do not show the disease symptoms, while the symptomatic people are those who have visible disease symptoms), shown, respectively, by $A(t)$ and $I(t)$. The infected individuals who show symptoms and are with severe infections are hospitalized, shown by $H(t)$. The individuals under the asymptomatic, symptomatic and hospitalized cases after recovery or removal join the recovered class given by $R(t)$. We also added the death class to the model, denoted by $D(t)$, which is generated by the deaths of the symptomatic and hospitalized individuals. So, we denote the total human population by $N(t)$, such that: $N(t) = S(t)+E(t)+A(t)+I(t)+H(t)+R(t)$. It is obvious from the literature that about 80% of the cases who are infected, fall under the asymptomatic infected people. The asymptomatic infected cases are considered more dangerous than the symptomatic ones due to not showing any visible symptoms and making the disease control more complex. The exposed individuals after completing their incubation period become infected and transmit the infection to the other healthy individuals and increasing the burden of disease on the population. The important characteristics of the exposed and asymptomatic individuals coming into contact with susceptible people, are regarded to be one of the important features of COVID-19 disease transmission. The above discussion on the modeling of the new COVID-19 infection is formulated through the following evolutionary differential equations:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \Pi - \frac{\rho_1 ES}{N-D} - \frac{\rho_2 AS}{N-D} - \frac{\rho_3 IS}{N-D} \\
&\quad - \frac{\rho_4 HS}{N-D} - \mu S, \\
\frac{dE(t)}{dt} &= \frac{\rho_1 ES}{N-D} + \frac{\rho_2 AS}{N-D} + \frac{\rho_3 IS}{N-D} \\
&\quad + \frac{\rho_4 HS}{N-D} - (\sigma+\mu)E, \\
\frac{dA(t)}{dt} &= \sigma\varepsilon E - (\xi_1+\mu)A, \\
\frac{dI(t)}{dt} &= (1-\varepsilon)\sigma E - (\xi_2+\upsilon+\varpi+\mu)I, \\
\frac{dH(t)}{dt} &= \upsilon I - (\phi+\xi_3+\mu)H, \\
\frac{dR(t)}{dt} &= \xi_1 A + \xi_2 I + \xi_3 H - \mu R, \\
\frac{dD(t)}{dt} &= \varpi I + \phi H,
\end{aligned} \tag{1}
$$

with nonnegative initial conditions. In model (1), the population of healthy people is increased by the birth rate given by Π, while the natural death rate given by μ decreases the population of every compartment. The exposed individuals that are infected after coming into close contact with the exposed individuals, are shown by the route $\frac{\rho_1 ES}{N-D}$, where ρ_1 is the disease contact rate. The healthy individuals who become infected after interacting with asymptomatic individuals, are given by the transmission route $\frac{\rho_2 AS}{N-D}$, where ρ_2 defines the disease contact rate. The susceptible ones after coming into close contact with the symptomatic people, suffer from infection, which is shown by the route $\frac{\rho_3 IS}{N-D}$, where ρ_3 is the disease contact rate. Similarly, the healthy individuals who become inflected after contact with hospitalized individuals, are given by $\frac{\rho_4 HS}{N-D}$, where the disease transmission coefficient is ρ_4. The hospitalized individuals usually infect those working in the hospitals, i.e. usually the doctors, nurses and other staff. It is evident that many health workers, nurses and doctors have died through this. It is assumed further that $\rho_1 \neq \rho_2 \neq \rho_3 \neq \rho_4$. The healthy individuals after interacting with all these routes of transmission become exposed to the disease with the rate given by σ, and a proportion $\varepsilon \in (0,1]$ of people in E join A without disease symptoms, while those showing disease symptoms join the class I with the rate given by $1-\varepsilon$. The parameters ξ_i for $i = 1,2,3$ denote the recovery of the individuals from the asymptomatic, symptomatic and hospitalized classes, respectively. The symptomatic individuals are hospitalized at a rate given by ν. Here, we consider the hospitalization of the symptomatic individuals only because they show visible clinical symptoms, and it is shown by υ. The infected individuals dying due to disease in the compartments I and H are given by ϖ and ϕ, respectively. The total deaths reported due to coronavirus infection can be estimated through the death equation given by: $\frac{dD}{dt} = \varpi I + \phi H$.

2.1. A Noninteger-Order Model

We give some definitions before constructing the noninteger-order model. For the construction of the noninteger-order model, we apply the Atangana–Baleanu derivative in the sense of Caputo. The definition and other results are taken from the work given in Ref. 34.

Definition 1. The following defines the Atangana–Baleanu derivative:

$$
\begin{aligned}
&{}^{\mathrm{ABC}}D_t^{\tau}[g(t)]\\
&\quad = \frac{K(\tau)}{1-\tau}\int_b^t g'(z)E_{\tau}\left[\frac{-\tau}{1-\tau}(t-z)^{\tau}\right]dz, \quad (2)
\end{aligned}
$$

where $\tau \in [0,1]$, $b > a$, $g \in H^1(a,b)$ and $K(\tau) = 1-\tau+\tau/\Gamma(\tau)$.

Definition 2. The fractional integral for the above definition is

$$
\begin{aligned}
{}_a^{\mathrm{ABC}}I_t^{\tau}\{g(t)\} &= \frac{1-\tau}{K(\tau)}g(t) + \frac{\tau}{K(\tau)\Gamma(\tau)}\\
&\quad \times \int_a^t g(z)(t-z)^{\tau-1}dz. \quad (3)
\end{aligned}
$$

The normalization function satisfies $K(0) = K(1) = 1$, and τ is the fractional order.

Definition 3. The Laplace transform of the Atangana–Baleanu fractional derivative of order τ in Caputo sense is given by

$$
L\{{}_0^{\mathrm{ABC}}D_t^{\tau}g(t)\} = \frac{K(\tau)(s^{\tau}\bar{g}(s) - s^{\tau-1}g(0))}{s^{\tau}(1-\tau)+\tau}, \quad (4)
$$

where L defines the Laplace operator.

For more details and interesting discussion about Atangana–Baleanu derivative, we refer the reader to see Ref. 34. Keeping in mind the above definition of Atangana–Baleanu derivative in Caputo sense, the model in integer order given in (1) can be generalized as follows:

$$
\begin{cases}
{}_0^{\mathrm{ABC}}D_t^{\tau}S(t) = \Pi - \Phi(t)S - \mu S,\\
{}_0^{\mathrm{ABC}}D_t^{\tau}E(t) = \Phi(t)S - (\sigma+\mu)E,\\
{}_0^{\mathrm{ABC}}D_t^{\tau}A(t) = \sigma\varepsilon E - (\xi_1+\mu)A,\\
{}_0^{\mathrm{ABC}}D_t^{\tau}I(t) = (1-\varepsilon)\sigma E - (\xi_2+\upsilon+\varpi+\mu)I,\\
{}_0^{\mathrm{ABC}}D_t^{\tau}H(t) = \upsilon I - (\phi+\xi_3+\mu)H,\\
{}_0^{\mathrm{ABC}}D_t^{\tau}R(t) = \xi_1 A + \xi_2 I + \xi_3 H - \mu R,
\end{cases}
\quad (5)
$$

where

$$
\Phi(t) = \frac{\rho_1 E}{N} + \frac{\rho_2 A}{N} + \frac{\rho_3 I}{N} + \frac{\rho_4 H}{N},
$$

and the nonnegative initial conditions are

$$
\begin{aligned}
&S(0) = S_0 \geq 0, \quad E(0) = E_0 \geq 0, \quad A(0) = A_0 \geq 0,\\
&I(0) = I_0 \geq 0, \quad H(0) = H_0 \geq 0, \quad R(0) = R_0 \geq 0.
\end{aligned}
$$

For the noninteger-order model (5), we can get the total dynamics by summing all their equations, and it is given by

$$
{}_0^{\mathrm{ABC}}D_t^{\tau}N(t) = \Pi - \mu N - \varpi I - \phi H \leq \Pi - \mu N,
$$

which can be rewritten further as

$$
{}_0^{\mathrm{ABC}}D_t^{\tau}N(t) \leq \Pi - \mu N. \quad (6)
$$

Applying the Laplace transform on (6), we get

$$
\begin{aligned}
N(t) &\leq \left(\frac{K(\tau)}{K(\tau)+(1-\tau)\mu}N(0) + \frac{(1-\tau)\Pi}{K(\tau)+(1-\tau)\mu}\right)\\
&\quad \times E_{\tau,1}\left(-\frac{\tau\mu}{K(\tau)+(1-\tau)\mu}t^{\tau}\right)
\end{aligned}
$$

$$+\frac{\tau\Pi}{K(\tau)+(1-\tau)\mu}E_{\tau,\tau+1}$$
$$\times\left(-\frac{\tau\mu}{K(\tau)+(1-\tau)\mu}t^{\tau}\right).$$

Due to the asymptotic nature of the Mittag-Leffler function, we have for $t\to\infty$

$$\lim_{t\to\infty}N(t)\leq\frac{\Pi}{\mu}.$$

So, the solution of Eq. (5) will lie in Θ, which is defined in the following. The feasibility of the model solution for system (5) can be studied as follows:

$$\Theta=\left\{Z\in\mathbb{R}_{+}^{6}:S+E+A+I+H+R\leq\frac{\Pi}{\mu}\right\}, \tag{7}$$

$Z=(S,E,A,I,H,R)$, where all the related solutions, such as the positive invariant, boundedness, etc., lie.

3. EQUILIBRIA AND THEIR STABILITY

There always exist two equilibrium points for the noninteger-order model (5), i.e. the disease-free and endemic equilibria. The disease-free equilibrium, denoted by P_0, can be easily obtained by equating to zero the time rate of change of the noninteger-order model (5),

$$P_0=(S_0,E_0,A_0,I_0,H_0,R_0)=\left(\frac{\Pi}{\mu},0,0,0,0,0\right). \tag{8}$$

The stability of the equilibrium points for the model can be discussed by using the basic reproduction number $\mathcal{R}_0$. This important threshold parameter can decide on the disease elimination or its persistence in the population when it is lesser or greater than unity. We compute this threshold for our noninteger-order model (5) by using the well-known approach of the next-generation matrix given in Ref. 35. This method involves the computation of some matrices, which are given in the following for our model (5):

$$F=\begin{pmatrix}\rho_1&\rho_2&\rho_3&\rho_4\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$V=\begin{pmatrix}\chi_1&0&0&0\\-\sigma\varepsilon&\chi_2&0&0\\-\sigma(1-\varepsilon)&0&\chi_3&0\\0&0&-\upsilon&\chi_4\end{pmatrix}.$$

The spectral radius $\gamma(FV^{-1})$ of the above matrices gives the basic reproduction number $\mathcal{R}_0$ for the noninteger-order system (5) given by

$$\mathcal{R}_0=\underbrace{\frac{\rho_1}{\chi_1}}_{\mathcal{R}_E}+\underbrace{\frac{\rho_2\sigma\varepsilon}{\chi_1\chi_2}}_{\mathcal{R}_A}+\underbrace{\frac{\rho_3\sigma(1-\varepsilon)}{\chi_1\chi_3}}_{\mathcal{R}_I}+\underbrace{\frac{\rho_4\sigma\upsilon(1-\varepsilon)}{\chi_1\chi_3\chi_4}}_{\mathcal{R}_H},$$

where $\chi_1=(\sigma+\mu)$, $\chi_2=(\xi_1+\mu)$, $\chi_3=(\xi_2+\upsilon+\mu+\varpi)$ and $\chi_4=(\phi+\xi_3+\mu)$. For the model (5), we claim that the disease-free equilibrium is locally asymptotically stable if $\mathcal{R}_0<1$. The condition $\mathcal{R}_0<1$ means that the disease will not spread in the community and can be eliminated by following the recommendations of the health department. Mathematically, we show the result in the following theorem.

Theorem 1. *The noninteger-order model* (5) *at the equilibrium point P_0 is locally asymptotically stable if $\mathcal{R}_0<1$.*

Proof. Compute the Jacobian at P_0 that is given by

$$J(P_0)=\begin{pmatrix}-\mu&-\rho_1&-\rho_2&-\rho_3&-\rho_4&0\\0&\rho_1-\chi_1&\rho_2&\rho_3&\rho_4&0\\0&\sigma\varepsilon&-\chi_2&0&0&0\\0&\sigma(1-\varepsilon)&0&-\chi_3&0&0\\0&0&0&\upsilon&-\chi_4&0\\0&0&\xi_1&\xi_2&\xi_3&-\mu\end{pmatrix}. \tag{9}$$

The Jacobian matrix $J(P_0)$ gives the eigenvalues $-\mu$(twice) which obviously contain negative real parts, while to get the remaining eigenvalues, we consider the following fourth-order polynomial:

$$\lambda^4+\pi_1\lambda^3+\pi_2\lambda^2+\lambda_3\pi+\pi_4=0, \tag{10}$$

where

$$\pi_1=\chi_2+\chi_3+\chi_4+\chi_1(1-\mathcal{R}_E),$$
$$\pi_2=\chi_3\chi_4+\chi_2(\chi_3+\chi_4)+\chi_1\chi_4(1-\mathcal{R}_E)$$
$$+\chi_1\chi_3[1-(\mathcal{R}_E+\mathcal{R}_I)]$$
$$+\chi_1\chi_2[1-(\mathcal{R}_E+\mathcal{R}_A)],$$
$$\pi_3=\chi_2\chi_3\chi_4+\chi_1\chi_2\chi_4(1-\mathcal{R}_E-\mathcal{R}_A)$$

$$+\chi_1\chi_2\chi_3(1-\mathcal{R}_E-\mathcal{R}_A-\mathcal{R}_I)$$
$$+\chi_1\chi_3\chi_4(1-\mathcal{R}_E-\mathcal{R}_I-\mathcal{R}_H),$$
$$\pi_4=\chi_1\chi_2\chi_3\chi_4(1-\mathcal{R}_0).$$

The coefficients $\pi_j>0$ for $j=1,\ldots,4$ are positive whenever $\mathcal{R}_0<1$ and further to ensure that the fourth-order polynomial will give four eigenvalues with negative real parts, they must satisfy the Routh–Hurwitz condition given by $\pi_1\pi_2\pi_3>\pi_3^2+\pi_1^2\pi_4$. This condition can be easily determined through the algebraic software and one can get four eigenvalues with negative real parts. These criteria ensure us that the noninteger-order model (5) gives all the eigenvalues containing negative real parts and thus the noninteger-order system (5) is locally asymptotically stable at P_0 whenever $\mathcal{R}_0<1$. □

Further, we claim that the model (5) is globally asymptotically stable at P_0. The statement and proof of the claim are given in the following theorem.

Theorem 2. *The model* (5) *at* P_0 *is globally asymptotically stable if* $\mathcal{R}_0\le 1$.

Proof. We need to define the Lyapunov function given by

$$L(t)=\varrho_1E(t)+\varrho_2A(t)+\varrho_3I(t)+\varrho_4H(t), \quad (11)$$

where $\varrho_n>0$, for $n=1,\ldots,4$. Time differentiating (11) in line with (5), we have

$$\begin{aligned}
&{}^{\mathrm{ABC}}D^{\tau}L(t)\\
&\quad=\varrho_1{}^{\mathrm{ABC}}D^{\tau}E(t)+\varrho_2{}^{\mathrm{ABC}}D^{\varpi}A(t)\\
&\qquad+\varrho_3{}^{\mathrm{ABC}}D^{\tau}I(t)+\varrho_4{}^{\mathrm{ABC}}D^{\tau}H(t)\\
&\quad=\varrho_1\left[\frac{(\rho_1E+\rho_2A+\rho_3I+\rho_4H)}{N}S-\chi_1E\right]\\
&\qquad+\varrho_2[\sigma\varepsilon E-\chi_2A]+\varrho_3[(1-\varepsilon)\sigma E-\chi_3I]\\
&\qquad+\varrho_4[\upsilon I-\chi_4H].
\end{aligned}$$

Making the assumption $S(t)\le N(t)$ and then rearranging, we arrive at

$$\begin{aligned}
&{}^{\mathrm{ABC}}D^{\tau}L(t)\\
&\quad=[\varrho_1\rho_1+\varrho_2\sigma\varepsilon+\varrho_3(1-\varepsilon)\sigma-\chi_1\varrho_1]E\\
&\qquad+[\rho_2\varrho_1-\varrho_2\chi_2]A+[\varrho_1\rho_3+\varrho_4\upsilon-\varrho_3\chi_3]I\\
&\qquad+[\varrho_1\rho_4-\varrho_4\chi_4]H.
\end{aligned}$$

Choosing the constants $\varrho_1=\chi_2$, $\varrho_2=\rho_2$, $\varrho_3=\frac{\chi_2}{\chi_3}(\rho_3+\frac{\upsilon\rho_4}{\chi_4})$ and $\varrho_4=\frac{\chi_2\rho_4}{\chi_4}$, we have

$${}^{\mathrm{ABC}}D^{\tau}L(t)=\chi_1\chi_2(\mathcal{R}_0-1).$$

So, ${}^{\mathrm{ABC}}D^{\tau}L(t)\le 0$ if $\mathcal{R}_0\le 1$. Also, ${}^{\mathrm{ABC}}D^{\varpi}L(t)=0$ if and only if $E=A=I=H=0$. The result in Ref. 36 ensures that the noninteger-order model (5) is globally asymptotically stable if $\mathcal{R}_0\le 1$. □

3.1. Existence of Endemic Equilibria

Here, we determine the expressions for the endemic equilibrium and denote them by $P_1=(S^*,E^*,A^*,I^*,H^*,R^*)$ of system (5). We give the details of the computations of the endemic equilibria in the following:

$$\begin{cases}
S^*=\dfrac{\Pi}{\Phi^*+\mu},\\
E^*=\dfrac{\Phi^*S^*}{\chi_1},\\
A^*=\dfrac{\sigma\varepsilon E^*}{\chi_2},\\
I^*=\dfrac{\sigma(1-\varepsilon)E^*}{\chi_3},\\
H^*=\dfrac{\upsilon I^*}{\chi_4},\\
R^*=\dfrac{\xi_1A^*+\xi_3H^*+\xi_2I^*}{\mu}.
\end{cases}$$

The above is used in the following expression:

$$\Phi(t)=\frac{\rho_1E^*}{N^*}+\frac{\rho_2A^*}{N^*}+\frac{\rho_3I^*}{N^*}+\frac{\rho_4H^*}{N^*},$$

and we get the following:

$$\begin{aligned}
&l_1\Phi^*+l_2=0,\\
&\qquad l_1=\sigma\upsilon(1-\varepsilon)\chi_2(\mu+\xi_3)\\
&\qquad\quad+\chi_4\sigma(1-\varepsilon)\chi_2(\mu+\xi_2)\\
&\qquad\quad+\chi_3\chi_4(\sigma\varepsilon(\mu+\xi_1)+\mu\chi_2),\\
&\qquad l_2=\mu\chi_1\chi_2\chi_3\chi_4(1-\mathcal{R}_0).
\end{aligned}\quad (12)$$

It is obvious that $l_1>0$ and l_2 is positive whenever $\mathcal{R}_0<1$. The existence of (12) to give positive equilibria depends on the value of $\mathcal{R}_0>1$ and hence, there exists a positive and unique solution for Eq. (12). Thus, the system (5) when $\mathcal{R}_0>1$ gives a unique endemic equilibrium.

4. A MODIFIED COVID-19 MODEL

The prediction of next waves of any infectious disease based on mathematical modeling tools is an interesting area for the health authorities. When there is any signal for the next wave, the government, the health agencies and the people can make themselves ready to fight or protect against the infection. For determining the second wave or subsequent ones using any mathematical model related to infectious diseases, there needs to be a second-order differential equation, and then utilizing the procedure to find the strength, and on the basis of the equilibria, one can have the possibilities of the wave from their proposed model. We will explore all detailed steps for our proposed model (1). Before starting our analysis, first, we modify the model (1) by taking its second time derivative, which is given by

$$\begin{cases} \dfrac{d^2S(t)}{dt^2} = -(\rho_1 A_1 + \rho_2 A_2 + \rho_3 A_3 \\ \qquad\qquad + \rho_4 A_4) - \mu\dot{S}, \\ \dfrac{d^2E(t)}{dt^2} = (\rho_1 A_1 + \rho_2 A_2 + \rho_3 A_3 + \rho_4 A_4) \\ \qquad\qquad - (\sigma+\mu)\dot{E}, \\ \dfrac{d^2A(t)}{dt^2} = \sigma\varepsilon\dot{E} - (\xi_1+\mu)\dot{A}, \\ \dfrac{d^2I(t)}{dt^2} = (1-\varepsilon)\sigma\dot{E} - (\xi_2+\upsilon+\varpi+\mu)\dot{I}, \\ \dfrac{d^2H(t)}{dt^2} = \upsilon\dot{I} - (\phi+\xi_3+\mu)\dot{H}, \\ \dfrac{d^2R(t)}{dt^2} = \xi_1\dot{A} + \xi_2\dot{I} + \xi_3\dot{H} - \mu\dot{R}, \end{cases} \tag{13}$$

where

$$A_1 = \frac{[(E\dot{S} + S\dot{E}) - ES\dot{N}]}{N^2},$$

$$A_2 = \frac{[(A\dot{S} + S\dot{A}) - AS\dot{N}]}{N^2},$$

$$A_3 = \frac{[(I\dot{S} + S\dot{I}) - IS\dot{N}]}{N^2},$$

$$A_4 = \frac{\rho_4[(H\dot{S} + S\dot{H}) - HS\dot{N}]}{N^2},$$

and $\dot{()}$ defines the time derivative. Next, we compute the strength number for model (13), by taking again the time derivative of the matrix F, and then find the spectral radius of $\gamma_1(GV^{-1})$, with the following matrices:

$$G = \begin{pmatrix} -\dfrac{2\mu\rho_1}{\Pi} & -\dfrac{2\mu\rho_2}{\Pi} & -\dfrac{2\mu\rho_3}{\Pi} & -\dfrac{2\mu\rho_4}{\Pi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \chi_1 & 0 & 0 & 0 \\ -\sigma\varepsilon & \chi_2 & 0 & 0 \\ -\sigma(1-\varepsilon) & 0 & \chi_3 & 0 \\ 0 & 0 & -\upsilon & \chi_4 \end{pmatrix}.$$

Thus, $\gamma_2(GV^{-1})$ gives

$$\mathcal{R}_{\text{st}} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4,$$

where

$$\mathcal{R}_1 = -\frac{2\mu\rho_1}{\Pi\chi_1}, \quad \mathcal{R}_2 = \frac{2\mu\rho_2\sigma\varepsilon}{\Pi\chi_1\chi_2},$$

$$\mathcal{R}_3 = \frac{2\mu\rho_3\sigma(1-\varepsilon)}{\Pi\chi_1\chi_3}, \quad \mathcal{R}_4 = -\frac{2\mu\upsilon\rho_4\sigma(1-\varepsilon)}{\Pi\chi_1\chi_3\chi_4}.$$

4.1. Analysis of Model (13)

In order to check the possibility of concavity in model (13), we need to insert the equations of model (1) into model (13). After that, we check the conditions of concavity for each equation of model (13) at the disease-free and endemic equilibria. We obtained that each of the equations of model (13) at the disease-free and endemic cases gives zero, i.e. ($d^2S/dt^2 = 0, \ldots, d^2R/dt^2 = 0$). This shows that there is no possibility of concavity up or down in model (13).

5. MODEL WITH PIECEWISE STOCHASTIC DIFFERENTIAL EQUATION

Here, we formulate model (1) further using the concept of piecewise differential equation.[37] To do this, we write model (1) in the extended form, as follows:

$$\begin{cases} {}_0^{\text{ABC}}D_t^{\tau} S(t) = \Pi - \Phi(t)S - \mu S, \\ {}_0^{\text{ABC}}D_t^{\tau} E(t) = \Phi(t)S - (\sigma+\mu)E, \\ {}_0^{\text{ABC}}D_t^{\tau} A(t) = \sigma\varepsilon E - (\xi_1+\mu)A, \\ {}_0^{\text{ABC}}D_t^{\tau} I(t) = (1-\varepsilon)\sigma E - (\xi_2+\upsilon+\varpi+\mu)I, \\ {}_0^{\text{ABC}}D_t^{\tau} H(t) = \upsilon I - (\phi+\xi_3+\mu)H, \\ {}_0^{\text{ABC}}D_t^{\tau} R(t) = \xi_1 A + \xi_2 I + \xi_3 H - \mu R, \end{cases} \tag{14}$$

where

$$\Phi(t) = \frac{\rho_1 E}{N} + \frac{\rho_2 A}{N} + \frac{\rho_3 I}{N} + \frac{\rho_4 H}{N}.$$

We use $0 \leq t < T_1$ for the case in (14). When $t \in [T_1, T_2]$, we consider the following case:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Pi - \Phi(t)S - \mu S, \\ \dfrac{dE(t)}{dt} = \Phi(t)S - (\sigma + \mu)E, \\ \dfrac{dA(t)}{dt} = \sigma\varepsilon E - (\xi_1 + \mu)A, \\ \dfrac{dI(t)}{dt} = (1-\varepsilon)\sigma E - (\xi_2 + \upsilon + \varpi + \mu)I, \\ \dfrac{dH(t)}{dt} = \upsilon I - (\phi + \xi_3 + \mu)H, \\ \dfrac{dR(t)}{dt} = \xi_1 A + \xi_2 I + \xi_3 H - \mu R, \end{cases} \tag{15}$$

and for $t \in [T_2, T]$, we present

$$\begin{cases} dS(t) = [\Pi - \Phi(t)S - \mu S]dt + \sigma_1 S dB_1(t), \\ dE(t) = [\Phi(t)S - (\sigma + \mu)E] + \sigma_2 E dB_2(t), \\ dA(t) = [\sigma\varepsilon E - (\xi_1 + \mu)A] + \sigma_3 A dB_3(t), \\ dI(t) = [(1-\varepsilon)\sigma E - (\xi_2 + \upsilon + \varpi + \mu)I] \\ \qquad\qquad + \sigma_4 I dB_4(t), \\ dH(t) = [\upsilon I - (\phi + \xi_3 + \mu)H] + \sigma_5 H dB_5(t), \\ dR(t) = [\xi_1 A + \xi_2 I + \xi_3 H - \mu R] + \sigma_6 R dB_6(t), \end{cases} \tag{16}$$

where σ_i for $i = 1, \ldots, 6$ define the positive constants (i.e. the intensity of the stochastic environment) and B_k for $k = 1, \ldots, 6$ are the standard Brownian motions. Numerically, the extended stochastic model with piecewise differential equation can be solved through the following scheme: Let us consider the case with the Atangana–Baleanu derivative, we write in general the equation

$$\begin{cases} \dfrac{dY_i(t)}{dt} = g(t, Y_i) \quad \text{if } 0 \leq t \leq T_1, \\ Y_i(0) = Y_{i,0}, \quad i = 1, 2, \ldots, n, \\ {}^{\text{ABC}}_{T}D_t^{\tau} Y_i(t) = g(t, Y_i) \quad \text{if } T_1 \leq t \leq T_2, \\ Y_i(T_1) = Y_{i,1}, \\ dY_i(t) = g(t, Y_i)dt + \sigma_i Y_i dB_i(t) \quad \text{if } T_2 \leq t \leq T, \\ Y_i(T_2) = Y_{i,2}. \end{cases} \tag{17}$$

The simplified scheme for the above Eq. (17) is given as follows:

$$\begin{aligned} Y_i^{n_1} &= Y_i(0) + \sum_{j_1=2}^{n_1} \left[\frac{23}{12} g(t_{j_1}, Y_{j_1}) - \frac{4}{3} g(t_{j_1-1}, Y_{j_1-1}) \right. \\ &\quad \left. + \frac{5}{12} g(t_{j_1-2}, Y_{j_1-2})\right] \Delta t, \quad 0 \leq t \leq T_1, \end{aligned}$$

$$\begin{aligned} Y_i^{n_2} &= Y_i(T_1) + \frac{1-\tau}{\text{AB}(\tau)} g(t_{n_2}, Y_{n_2}) \\ &\quad + \frac{\tau(\Delta t)^{\tau}}{\text{AB}(\tau)\Gamma(\tau+1)} \sum_{j_2=n_1+3}^{n_2} g(t_{j_2-2}, Y_{j_2-2})\Delta_1 \\ &\quad + \frac{\tau(\Delta t)^{\tau}}{\text{AB}(\tau)\Gamma(\tau+2)} \sum_{j_2=n_1+3}^{n_2} \\ &\quad \times [g(t_{j_2-1}, Y_{j_2-1}) - g(t_{j_2-2}, Y_{j_2-2})]\Delta_2 \\ &\quad + \frac{\tau(\Delta t)^{\tau}}{2\text{AB}(\tau)\Gamma(\tau+3)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2}, Y_{j_2}) \\ &\quad - 2g(t_{j_2-1}, Y_{j_2-1}) + g(t_{j_2-2}, \\ &\quad Y_{j_2-2})]\Delta_3, \quad T_1 \leq t \leq T_2, \end{aligned} \tag{18}$$

$$\begin{aligned} Y_i^{n_3} &= Y_i(T_2) + \sum_{j_3=n_2+3}^{n_3} \left[\frac{23}{12} g(t_{j_3}, Y_{j_3}) \right. \\ &\quad \left. - \frac{4}{3} G(t_{j_3-1}, Y_{j_3-1}) + \frac{5}{12} g(t_{j_3-2}, Y_{j_3-2})\right] \Delta t \\ &\quad + \sigma_i \sum_{j=n_2+3}^{n_3} Y_i^{j_3} (B_i^{j_3} - B_i^{j_3-1}), \quad T_2 \leq t \leq T, \end{aligned} \tag{19}$$

where

$$\Delta_1 = \{(n_2 - j_2 + 1)^{\tau} - (n_2 - j_2)^{\tau}\},$$

$$\Delta_2 = \begin{Bmatrix} (n_2 - j_2 + 1)^{\tau}(n_2 - j_2 + 3 + 2\tau) \\ -(n_2 - j_2)^{\tau}(n_2 - j_2 + 3 + 3\tau) \end{Bmatrix},$$

$$\Delta_3 = \begin{Bmatrix} (n_2 - j_2 + 1)^{\tau} \\ \times \begin{bmatrix} 2(n_2 - j_2)^2 + (3\tau + 10)(n_2 - j_2) \\ +2\tau^2 + 9\tau + 12 \end{bmatrix} \\ -(n_2 - j_2)^{\tau} \\ \times \begin{bmatrix} 2(n_2 - j_2)^2 + (5\tau + 10)(n_2 - j_2) \\ +6\tau^2 + 18\tau + 12 \end{bmatrix} \end{Bmatrix}.$$

For systems (14)–(16), the following scheme is used:

$$\begin{aligned} S_i^{n_1} &= S_i(0) + \sum_{j_1=2}^{n_1} \\ &\quad \times \left[\frac{23}{12} g(t_{j_1}, S_{j_1}, E_{j_1}, A_{j_1}, I_{j_1}, H_{j_1}, R_{j_1})\right] \Delta t \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{j_1=2}^{n_1}\left[\frac{4}{3}g(t_{j_1-1}, S_{j_1-1}, E_{j_1-1},\right.\\
&\qquad \left. A_{j_1-1}, I_{j_1-1}, H_{j_1-1}, R_{j_1-1})\right]\Delta t\\
&\quad + \sum_{j_1=2}^{n_1}\left[\frac{5}{12}g(t_{j_1-2}, S_{j_1-2}, E_{j_1-2},\right.\\
&\qquad \left. A_{j_1-2}, I_{j_1-2}, H_{j_1-2}, R_{j_1-2})\right]\Delta t,\\
&\quad 0 \le t \le T_1,
\end{aligned}
$$

$$
\begin{aligned}
E_i^{n_1} &= E_i(0) + \sum_{j_1=2}^{n_1}\\
&\quad \times \left[\frac{23}{12}g(t_{j_1}, S_{j_1}, E_{j_1}, A_{j_1}, I_{j_1}, H_{j_1}, R_{j_1})\right]\Delta t\\
&\quad - \sum_{j_1=2}^{n_1}\left[\frac{4}{3}g(t_{j_1-1}, S_{j_1-1}, E_{j_1-1},\right.\\
&\qquad \left. A_{j_1-1}, I_{j_1-1}, H_{j_1-1}, R_{j_1-1})\right]\Delta t\\
&\quad + \sum_{j_1=2}^{n_1}\left[\frac{5}{12}g(t_{j_1-2}, S_{j_1-2}, E_{j_1-2}, A_{j_1-2}, I_{j_1-2},\right.\\
&\qquad \left. H_{j_1-2}, R_{j_1-2})\right]\Delta t, \quad 0 \le t \le T_1,
\end{aligned}
$$

$$
\begin{aligned}
A_i^{n_1} &= A_i(0) + \sum_{j_1=2}^{n_1}\\
&\quad \times \left[\frac{23}{12}g(t_{j_1}, S_{j_1}, E_{j_1}, A_{j_1}, I_{j_1}, H_{j_1}, R_{j_1})\right]\Delta t\\
&\quad - \sum_{j_1=2}^{n_1}\left[\frac{4}{3}g(t_{j_1-1}, S_{j_1-1}, E_{j_1-1},\right.\\
&\qquad \left. A_{j_1-1}, I_{j_1-1}, H_{j_1-1}, R_{j_1-1})\right]\Delta t\\
&\quad + \sum_{j_1=2}^{n_1}\left[\frac{5}{12}g(t_{j_1-2}, S_{j_1-2}, E_{j_1-2}, A_{j_1-2}, I_{j_1-2},\right.\\
&\qquad \left. H_{j_1-2}, R_{j_1-2})\right]\Delta t, \quad 0 \le t \le T_1,
\end{aligned}
$$

$$
\begin{aligned}
I_i^{n_1} &= I_i(0) + \sum_{j_1=2}^{n_1}\\
&\quad \times \left[\frac{23}{12}g(t_{j_1}, S_{j_1}, E_{j_1}, A_{j_1}, I_{j_1}, H_{j_1}, R_{j_1})\right]\Delta t\\
&\quad - \sum_{j_1=2}^{n_1}\left[\frac{4}{3}g(t_{j_1-1}, S_{j_1-1}, E_{j_1-1}, A_{j_1-1},\right.\\
&\qquad \left. I_{j_1-1}, H_{j_1-1}, R_{j_1-1})\right]\Delta t\\
&\quad + \sum_{j_1=2}^{n_1}\left[\frac{5}{12}g(t_{j_1-2}, S_{j_1-2}, E_{j_1-2}, A_{j_1-2}, I_{j_1-2},\right.\\
&\qquad \left. H_{j_1-2}, R_{j_1-2})\right]\Delta t, \quad 0 \le t \le T_1,
\end{aligned}
$$

$$
\begin{aligned}
H_i^{n_1} &= H_i(0) + \sum_{j_1=2}^{n_1}\\
&\quad \times \left[\frac{23}{12}g(t_{j_1}, S_{j_1}, E_{j_1}, A_{j_1}, I_{j_1}, H_{j_1}, R_{j_1})\right]\Delta t\\
&\quad - \sum_{j_1=2}^{n_1}\left[\frac{4}{3}g(t_{j_1-1}, S_{j_1-1}, E_{j_1-1}, A_{j_1-1},\right.\\
&\qquad \left. I_{j_1-1}, H_{j_1-1}, R_{j_1-1})\right]\Delta t\\
&\quad + \sum_{j_1=2}^{n_1}\left[\frac{5}{12}g(t_{j_1-2}, S_{j_1-2}, E_{j_1-2}, A_{j_1-2}, I_{j_1-2},\right.\\
&\qquad \left. H_{j_1-2}, R_{j_1-2})\right]\Delta t, \quad 0 \le t \le T_1,
\end{aligned}
$$

$$
\begin{aligned}
R_i^{n_1} &= R_i(0) + \sum_{j_1=2}^{n_1}\\
&\quad \times \left[\frac{23}{12}g(t_{j_1}, S_{j_1}, E_{j_1}, A_{j_1}, I_{j_1}, H_{j_1}, R_{j_1})\right]\Delta t\\
&\quad - \sum_{j_1=2}^{n_1}\left[\frac{4}{3}g(t_{j_1-1}, S_{j_1-1}, E_{j_1-1}, A_{j_1-1},\right.\\
&\qquad \left. I_{j_1-1}, H_{j_1-1}, R_{j_1-1})\right]\Delta t\\
&\quad + \sum_{j_1=2}^{n_1}\left[\frac{5}{12}g(t_{j_1-2}, S_{j_1-2}, E_{j_1-2}, A_{j_1-2}, I_{j_1-2},\right.\\
&\qquad \left. H_{j_1-2}, R_{j_1-2})\right]\Delta t, \quad 0 \le t \le T_1,
\end{aligned}
$$

$$
S_i^{n_2} = S_i(T_1) + \frac{1-\tau}{\mathrm{AB}(\tau)}g(t_{n_2}, S_{n_2}, E_{n_2}, A_{n_2}, I_{n_2},
$$

$$H_{n_2}, R_{n_2}) + \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+1)} \times \sum_{j_2=n_1+3}^{n_2} g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})\Delta_1$$
$$+ \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+2)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-1}, S_{j_2-1}, E_{j_2-1}, A_{j_2-1}, I_{j_2-1}, H_{j_2-1}, R_{j_2-1})]\Delta_2$$
$$- \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+2)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})]\Delta_2$$
$$+ \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2}, S_{j_2}, E_{j_2}, A_{j_2}, I_{j_2}, H_{j_2}, R_{j_2})]\Delta_3 - \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}$$
$$\times \sum_{j_2=n_1+3}^{n_2} [2g(t_{j_2-1}, S_{j_2-1}, E_{j_2-1}, A_{j_2-1}, I_{j_2-1}, H_{j_2-1}, R_{j_2-1})]\Delta_3 + \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}$$
$$\times \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})]\Delta_3, \quad T_1 \le t \le T_2,$$

$$E_i^{n_2} = E_i(T_1) + \frac{1-\tau}{\mathrm{AB}(\tau)} g(t_{n_2}, S_{n_2}, E_{n_2}, A_{n_2}, I_{n_2}, H_{n_2}, R_{n_2}) + \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+1)}$$
$$\times \sum_{j_2=n_1+3}^{n_2} g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})\Delta_1$$
$$+ \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+2)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-1}, S_{j_2-1}, E_{j_2-1}, A_{j_2-1}, I_{j_2-1}, H_{j_2-1}, R_{j_2-1})]\Delta_2$$
$$- \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+2)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})]\Delta_2$$
$$+ \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2}, S_{j_2}, E_{j_2}, A_{j_2}, I_{j_2}, H_{j_2}, R_{j_2})]\Delta_3 - \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}$$
$$\times \sum_{j_2=n_1+3}^{n_2} [2g(t_{j_2-1}, S_{j_2-1}, E_{j_2-1}, A_{j_2-1}, I_{j_2-1}, H_{j_2-1}, R_{j_2-1})]\Delta_3 + \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}$$
$$\times \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})]\Delta_3, \quad T_1 \le t \le T_2,$$

$$A_i^{n_2} = A_i(T_1) + \frac{1-\tau}{\mathrm{AB}(\tau)} g(t_{n_2}, S_{n_2}, E_{n_2}, A_{n_2}, I_{n_2}, H_{n_2}, R_{n_2})$$
$$+ \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+1)} \sum_{j_2=n_1+3}^{n_2} g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})\Delta_1$$
$$+ \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+2)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-1}, S_{j_2-1}, E_{j_2-1}, A_{j_2-1}, I_{j_2-1}, H_{j_2-1}, R_{j_2-1})]\Delta_2$$
$$- \frac{\tau(\Delta t)^\tau}{\mathrm{AB}(\tau)\Gamma(\tau+2)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})]\Delta_2$$
$$+ \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)} \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2}, S_{j_2}, E_{j_2}, A_{j_2}, I_{j_2}, H_{j_2}, R_{j_2})]\Delta_3 - \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}$$
$$\times \sum_{j_2=n_1+3}^{n_2} [2g(t_{j_2-1}, S_{j_2-1}, E_{j_2-1}, A_{j_2-1}, I_{j_2-1}, H_{j_2-1}, R_{j_2-1})]\Delta_3 + \frac{\tau(\Delta t)^\tau}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}$$
$$\times \sum_{j_2=n_1+3}^{n_2} [g(t_{j_2-2}, S_{j_2-2}, E_{j_2-2}, A_{j_2-2}, I_{j_2-2}, H_{j_2-2}, R_{j_2-2})]\Delta_3, \quad T_1 \le t \le T_2,$$

$$I_i^{n_2} = I_i(T_1)$$

$$+\frac{1-\tau}{\mathrm{AB}(\tau)}g(t_{n_2},S_{n_2},E_{n_2},A_{n_2},I_{n_2},H_{n_2},R_{n_2})$$
$$+\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+1)}\sum_{j_2=n_1+3}^{n_2}g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})\Delta_1$$
$$+\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+2)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-1},S_{j_2-1},E_{j_2-1},A_{j_2-1},I_{j_2-1},H_{j_2-1},R_{j_2-1})]\Delta_2$$
$$-\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+2)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})]\Delta_2$$
$$+\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2},S_{j_2},E_{j_2},A_{j_2},I_{j_2},H_{j_2},R_{j_2})]\Delta_3$$
$$-\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[2g(t_{j_2-1},S_{j_2-1},E_{j_2-1},A_{j_2-1},I_{j_2-1},H_{j_2-1},R_{j_2-1})]\Delta_3$$
$$+\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}6[g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})]\Delta_3,\quad T_1\le t\le T_2,$$

$$H_i^{n_2}=H_i(T_1)+\frac{1-\tau}{\mathrm{AB}(\tau)}g(t_{n_2},S_{n_2},E_{n_2},A_{n_2},I_{n_2},H_{n_2},R_{n_2})$$
$$+\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+1)}\sum_{j_2=n_1+3}^{n_2}g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})\Delta_1$$
$$+\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+2)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-1},S_{j_2-1},E_{j_2-1},A_{j_2-1},I_{j_2-1},H_{j_2-1},R_{j_2-1})]\Delta_2$$
$$-\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+2)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})]\Delta_2$$
$$+\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2},S_{j_2},E_{j_2},A_{j_2},I_{j_2},H_{j_2},R_{j_2})]\Delta_3$$
$$-\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[2g(t_{j_2-1},S_{j_2-1},E_{j_2-1},A_{j_2-1},I_{j_2-1},H_{j_2-1},R_{j_2-1})]\Delta_3$$
$$+\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})]\Delta_3,\quad T_1\le t\le T_2,$$

$$R_i^{n_2}=R_i(T_1)+\frac{1-\tau}{\mathrm{AB}(\tau)}g(t_{n_2},S_{n_2},E_{n_2},A_{n_2},I_{n_2},H_{n_2},R_{n_2})+\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+1)}$$
$$\times\sum_{j_2=n_1+3}^{n_2}g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})\Delta_1$$
$$+\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+2)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-1},S_{j_2-1},E_{j_2-1},A_{j_2-1},I_{j_2-1},H_{j_2-1},R_{j_2-1})]\Delta_2$$
$$-\frac{\tau(\Delta t)^{\tau}}{\mathrm{AB}(\tau)\Gamma(\tau+2)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})]\Delta_2$$
$$+\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2},S_{j_2},E_{j_2},A_{j_2},I_{j_2},H_{j_2},R_{j_2})]\Delta_3$$
$$-\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[2g(t_{j_2-1},S_{j_2-1},E_{j_2-1},A_{j_2-1},I_{j_2-1},H_{j_2-1},R_{j_2-1})]\Delta_3$$
$$+\frac{\tau(\Delta t)^{\tau}}{2\mathrm{AB}(\tau)\Gamma(\tau+3)}\sum_{j_2=n_1+3}^{n_2}[g(t_{j_2-2},S_{j_2-2},E_{j_2-2},A_{j_2-2},I_{j_2-2},H_{j_2-2},R_{j_2-2})]\Delta_3,\quad T_1\le t\le T_2,$$

$$
\begin{aligned}
S_i^{n_3} &= S_i(T_2) + \sum_{j_3=n_2+3}^{n_3}\left[\frac{23}{12}g(t_{j_3}, S_{j_3}, E_{j_3}, A_{j_3},\right.\\
&\quad \left. I_{j_3}, H_{j_3}, R_{j_3})\right]\Delta t - \sum_{j_3=n_2+3}^{n_3}\left[\frac{4}{3}G(t_{j_3-1},\right.\\
&\quad \left. S_{j_3-1}, E_{j_3-1}, A_{j_3-1}, I_{j_3-1}, H_{j_3-1}, R_{j_3-1})\right]\Delta t\\
&\quad + \sum_{j_3=n_2+3}^{n_3}\left[\frac{5}{12}g(t_{j_3-2}, S_{j_3-2}, E_{j_3-2},\right.\\
&\quad \left. A_{j_3-2}, I_{j_3-2}, H_{j_3-2}, R_{j_3-2})\right]\Delta t\\
&\quad + \sigma_1 \sum_{j=n_2+3}^{n_3} S_i^{j_3}(B_i^{j_3} - B_i^{j_3-1}), \quad T_2 \leq t \leq T,
\end{aligned}
$$

$$
\begin{aligned}
E_i^{n_3} &= E_i(T_2) + \sum_{j_3=n_2+3}^{n_3}\\
&\quad \times\left[\frac{23}{12}g(t_{j_3}, S_{j_3}, E_{j_3}, A_{j_3}, I_{j_3}, H_{j_3}, R_{j_3})\right]\Delta t\\
&\quad - \sum_{j_3=n_2+3}^{n_3}\left[\frac{4}{3}G(t_{j_3-1}, S_{j_3-1}, E_{j_3-1},\right.\\
&\quad \left. A_{j_3-1}, I_{j_3-1}, H_{j_3-1}, R_{j_3-1})\right]\Delta t\\
&\quad + \sum_{j_3=n_2+3}^{n_3}\left[\frac{5}{12}g(t_{j_3-2}, S_{j_3-2}, E_{j_3-2},\right.\\
&\quad \left. A_{j_3-2}, I_{j_3-2}, H_{j_3-2}, R_{j_3-2})\right]\Delta t\\
&\quad + \sigma_2 \sum_{j=n_2+3}^{n_3} E_i^{j_3}(B_i^{j_3} - B_i^{j_3-1}), \quad T_2 \leq t \leq T,
\end{aligned}
$$

$$
\begin{aligned}
A_i^{n_3} &= A_i(T_2) + \sum_{j_3=n_2+3}^{n_3}\\
&\quad \times\left[\frac{23}{12}g(t_{j_3}, S_{j_3}, E_{j_3}, A_{j_3}, I_{j_3}, H_{j_3}, R_{j_3})\right]\Delta t\\
&\quad - \sum_{j_3=n_2+3}^{n_3}\left[\frac{4}{3}g(t_{j_3-1}, S_{j_3-1}, E_{j_3-1}, A_{j_3-1},\right.\\
&\quad \left. I_{j_3-1}, H_{j_3-1}, R_{j_3-1})\right]\Delta t\\
&\quad + \sum_{j_3=n_2+3}^{n_3}\left[\frac{5}{12}g(t_{j_3-2}, S_{j_3-2}, E_{j_3-2},\right.\\
&\quad \left. A_{j_3-2}, I_{j_3-2}, H_{j_3-2}, R_{j_3-2})\right]\Delta t\\
&\quad + \sigma_3 \sum_{j=n_2+3}^{n_3} A_i^{j_3}(B_i^{j_3} - B_i^{j_3-1}), \quad T_2 \leq t \leq T,
\end{aligned}
$$

$$
\begin{aligned}
I_i^{n_3} &= I_i(T_2) + \sum_{j_3=n_2+3}^{n_3}\\
&\quad \times\left[\frac{23}{12}g(t_{j_3}, S_{j_3}, E_{j_3}, A_{j_3}, I_{j_3}, H_{j_3}, R_{j_3})\right]\Delta t\\
&\quad - \sum_{j_3=n_2+3}^{n_3}\left[\frac{4}{3}g(t_{j_3-1}, S_{j_3-1}, E_{j_3-1}, A_{j_3-1},\right.\\
&\quad \left. I_{j_3-1}, H_{j_3-1}, R_{j_3-1})\right]\Delta t\\
&\quad + \sum_{j_3=n_2+3}^{n_3}\left[\frac{5}{12}g(t_{j_3-2}, S_{j_3-2}, E_{j_3-2},\right.\\
&\quad \left. A_{j_3-2}, I_{j_3-2}, H_{j_3-2}, R_{j_3-2})\right]\Delta t\\
&\quad + \sigma_4 \sum_{j=n_2+3}^{n_3} I_i^{j_3}(B_i^{j_3} - B_i^{j_3-1}), \quad T_2 \leq t \leq T,
\end{aligned}
$$

$$
\begin{aligned}
H_i^{n_3} &= H_i(T_2) + \sum_{j_3=n_2+3}^{n_3}\\
&\quad \times\left[\frac{23}{12}g(t_{j_3}, S_{j_3}, E_{j_3}, A_{j_3}, I_{j_3}, H_{j_3}, R_{j_3})\right]\Delta t\\
&\quad - \sum_{j_3=n_2+3}^{n_3}\left[\frac{4}{3}g(t_{j_3-1}, S_{j_3-1}, E_{j_3-1}, A_{j_3-1},\right.\\
&\quad \left. I_{j_3-1}, H_{j_3-1}, R_{j_3-1})\right]\Delta t\\
&\quad + \sum_{j_3=n_2+3}^{n_3}\left[\frac{5}{12}g(t_{j_3-2}, S_{j_3-2}, E_{j_3-2}, A_{j_3-2},\right.\\
&\quad \left. I_{j_3-2}, H_{j_3-2}, R_{j_3-2})\right]\Delta t\\
&\quad + \sigma_5 \sum_{j=n_2+3}^{n_3} H_i^{j_3}(B_i^{j_3} - B_i^{j_3-1}), \quad T_2 \leq t \leq T,
\end{aligned}
$$

$$R_i^{n_3} = R_i(T_2) + \sum_{j_3=n_2+3}^{n_3} \times \left[\frac{23}{12} g(t_{j_3}, S_{j_3}, E_{j_3}, A_{j_3}, I_{j_3}, H_{j_3}, R_{j_3})\right] \Delta t - \sum_{j_3=n_2+3}^{n_3} \left[\frac{4}{3} g(t_{j_3-1}, S_{j_3-1}, E_{j_3-1}, A_{j_3-1}, I_{j_3-1}, H_{j_3-1}, R_{j_3-1})\right] \Delta t + \sum_{j_3=n_2+3}^{n_3} \left[\frac{5}{12} g(t_{j_3-2}, S_{j_3-2}, E_{j_3-2}, A_{j_3-2}, I_{j_3-2}, H_{j_3-2}, R_{j_3-2})\right] \Delta t + \sigma_6 \sum_{j=n_2+3}^{n_3} R_i^{j_3}(B_i^{j_3} - B_i^{j_3-1}), \quad T_2 \le t \le T.$$

6. ESTIMATION OF PARAMETERS

Here, we estimate the model parameters using the technique of nonlinear least-square method. This technique is so effective for a system of nonlinear differential equations epidemic model. It has been used by researchers for data fitting for the epidemic models, etc. We consider this technique to parameterize our model by considering the fractional-order parameter $\tau = 1$ and with the infected cases from July 1 to November 16, 2021, which are the fourth wave cases in Pakistan. The reported cases of the fourth wave of COVID-19 in Pakistan are taken from Worldometer website, which are the currently infected cases. There are 14 parameters involved in model (5); among these parameters, we take some of them from the already available literature including: $\xi_1 = 1/5.1$, $\xi_2 = 1/10$, $\xi_3 = 1/8$ and $\varpi = 0.015$. The demographic parameters, i.e. the birth rate Π and death rate μ, can be estimated using the expression: $N(0) = \Pi/\mu$. The population at the time of epidemic is considered to be $N(0) = 220{,}000{,}000$, and the average life expectancy for Pakistani population is $\mu = 1/(67.7)$, so, following the aforementioned expressions, we get $\Pi \approx 8903$ per day. The other parameters have been obtained by data fitting of the model and are listed in Table 1.

In the data fitting, we assumed the initial conditions of the model variables as follows: The total population of Pakistan in 2021 was $N(0) = 220{,}000{,}000$, and to obtain the initial condition for susceptible individuals, we consider in disease absence the value of $S(0) = 219{,}768{,}233$. Subject to data fitting, the exposed individuals initial condition is $E(0) = 200{,}000$, and we consider $A(0) = 0$, $H(0) = 0$ and $R(0) = 0$ due to nonavailability of the asymptomatic, hospitalized and recovered individuals data, while the currently infected cases on July 1, 2021 are $I(0) = 31{,}767$. After fitting the data to the model, the basic reproduction number calculated with the real parameters given in Table 1 is $\mathcal{R}_0 \approx 0.9775$. The data versus model fitting and its predictions are given in Fig. 1, which is a reasonable fitting and the parameters are considered to be effective to study the further dynamics of COVID-19 in Pakistan and to draw much useful recommendations regarding the current and future infection predictions. Figures 1c and 1d are the model fits for the stochastic case with different noises.

7. NUMERICAL SIMULATIONS AND DISCUSSION

In this section, the numerical simulations of the noninteger-order model (5) using the parameter values given in Table 1 are carried out. Initially, a numerical procedure based on the Adams–Bashforth scheme[40] for the solution of model (5) in the sense of Atangana–Baleanu derivative is presented. After the implementation of the scheme, we present a fruitful discussion on the numerical results.

7.1. Numerical Scheme for Model (5)

Here, we implement the numerical scheme based on the Adams–Bashforth method for fractional differential equations with nonsingular and nonlocal kernel. First, we consider a general system and then particularize it to our proposed model,

$$\begin{cases} {}_0^{\mathrm{ABC}}D^\tau w(t) = g(t, w(t)), \\ w(0) = w_0. \end{cases} \tag{20}$$

It follows from the fundamental theorem of fractional calculus that Eq. (20) can be written as

$$w(t) - w(0) = \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g(t, w(t)) + \frac{\tau}{\Gamma(\tau) \times \mathrm{ABC}(\tau)} \times \int_0^t g(\psi, w(\psi))(t-\psi)^{\tau-1} d\psi. \tag{21}$$

Table 1 Fitting of the Parameters to the Model.

Notation	Description	Value/per day	Source
Π	Requirement rate	$\mu \times N(0)$	Estimated
μ	Natural death rate	$\frac{1}{67.7\times 365}$	Ref. 38
σ	Incubation period	0.7	Fitted
ε	Individuals' progress to A	0.8	Fitted
ξ_1	Recovery from asymptomatic class	1/5.1	Ref. 39
ξ_2	Recovery from symptomatic class	0.1	Ref. 39
ξ_3	Recovery of hospitalized individuals	1/8	Ref. 39
ϕ	Death due to infection at H	0.01	Fitted
ϖ	Death due to infection at I	0.015	Ref. 39
υ	The rate at which the individuals in I are hospitalized	0.7000	Fitted
ρ_1	Coefficient of disease transmission due to E	0.6715	Fitted
ρ_2	Coefficient of disease transmission due to A	0.001	Fitted
ρ_3	Coefficient of disease transmission due to I	0.05	Fitted
ρ_4	Coefficient of disease transmission due to H	0.100	Fitted

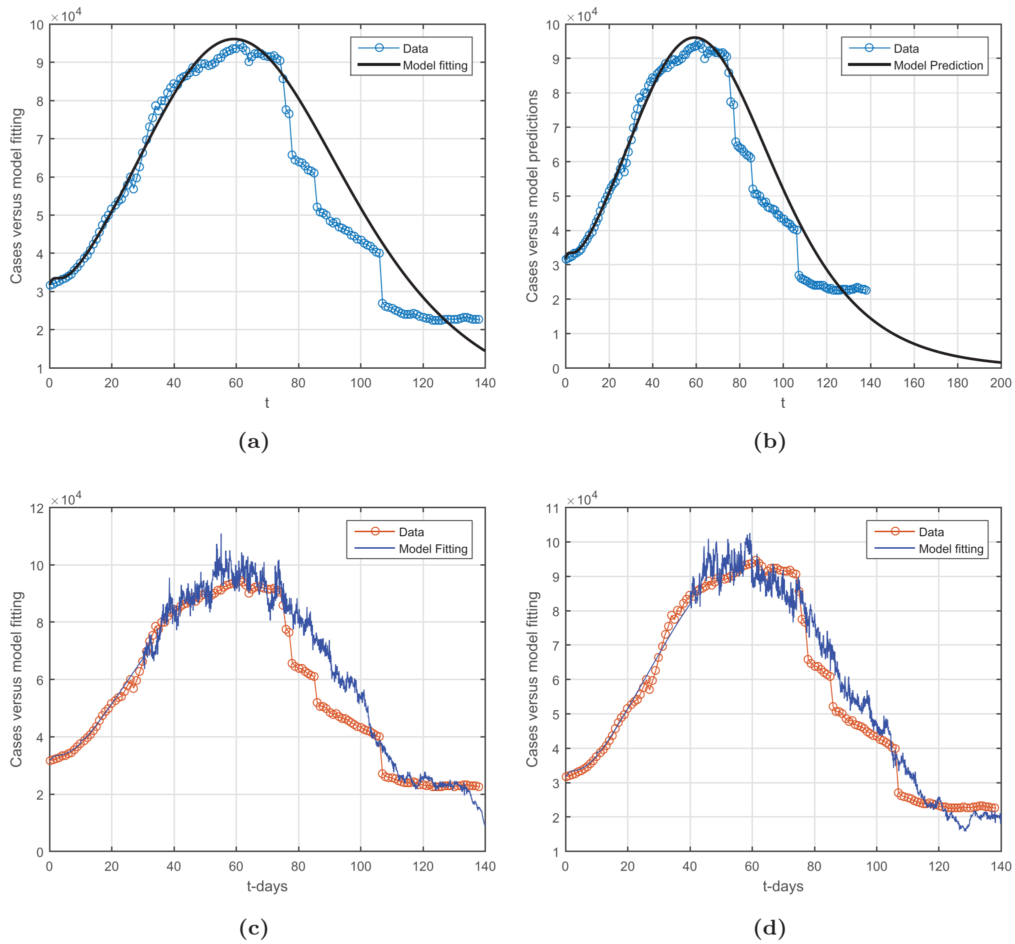


Fig. 1 Data versus model fitting for July 1–November 16, 2021: **(a)** model versus data fitting, **(b)** model predictions, **(c)** stochastic model fitting to data, parameters used from Table 1 except $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.04$, $\sigma_4 = 0.001$, $\sigma_5 = 0.1$, $\sigma_6 = 0.6$, $T_1 = 10$, $T_2 = 30$ and $T_3 = 140$, and **(d)** stochastic model fitting, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.04$, $\sigma_4 = 0.001$, $\sigma_5 = 0.1$, $\sigma_6 = 0.4$, $T_1 = 10$, $T_2 = 40$ and $T_3 = 140$.

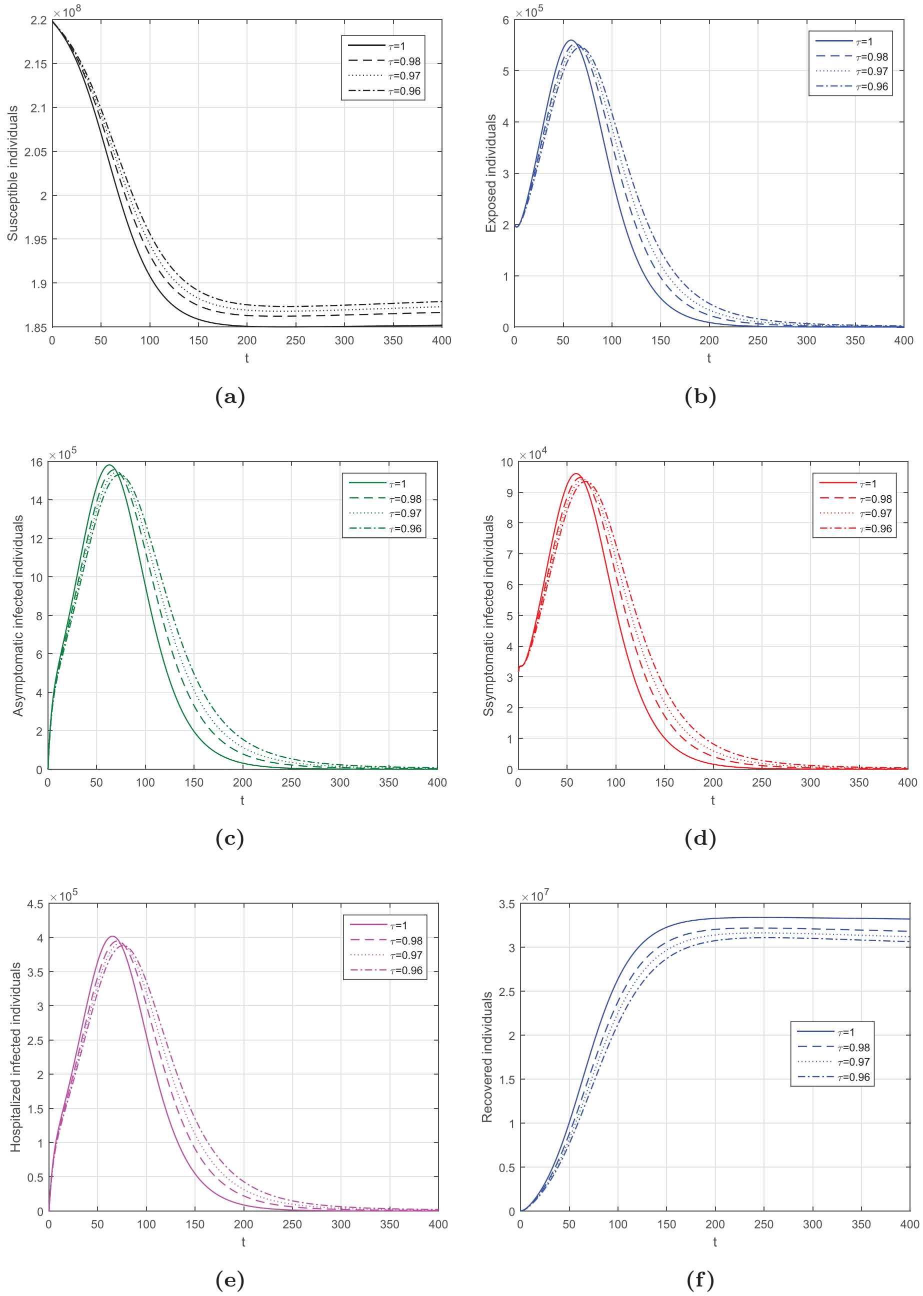


Fig. 2 Simulations of the fractional model using different values of the fractional order τ.

Re-formulating Eq. (21) at $t = t_{m+1}$, $m = 0, 1, 2, \ldots$,

$$\begin{aligned} w(t_{m+1}) - w(0) &= \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g(t_m, w(t_n)) \\ &\quad + \frac{\tau}{\mathrm{ABC}(\tau)\times\Gamma(\tau)} \int_0^{t_{m+1}} \\ &\quad \times g(\psi, w(\psi))(t_{m+1}-\psi)^{\tau-1} d\psi \\ &= \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g(t_m, w(t_m)) \\ &\quad + \frac{\tau}{\mathrm{ABC}(\tau)\times\Gamma(\tau)} \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \\ &\quad \times g(\psi, w(\psi))(t_{m+1}-\psi)^{\tau-1} d\psi. \end{aligned} \tag{22}$$

Approximating the function $g(\psi, w(\psi))$ within the interval $[t_l, t_{l+1}]$ by Lagrange interpolation polynomial of two steps, we have

$$Z_l(\psi) = \frac{\psi - t_{l-1}}{t_l - t_{l-1}} g(t_l, w(t_l))$$

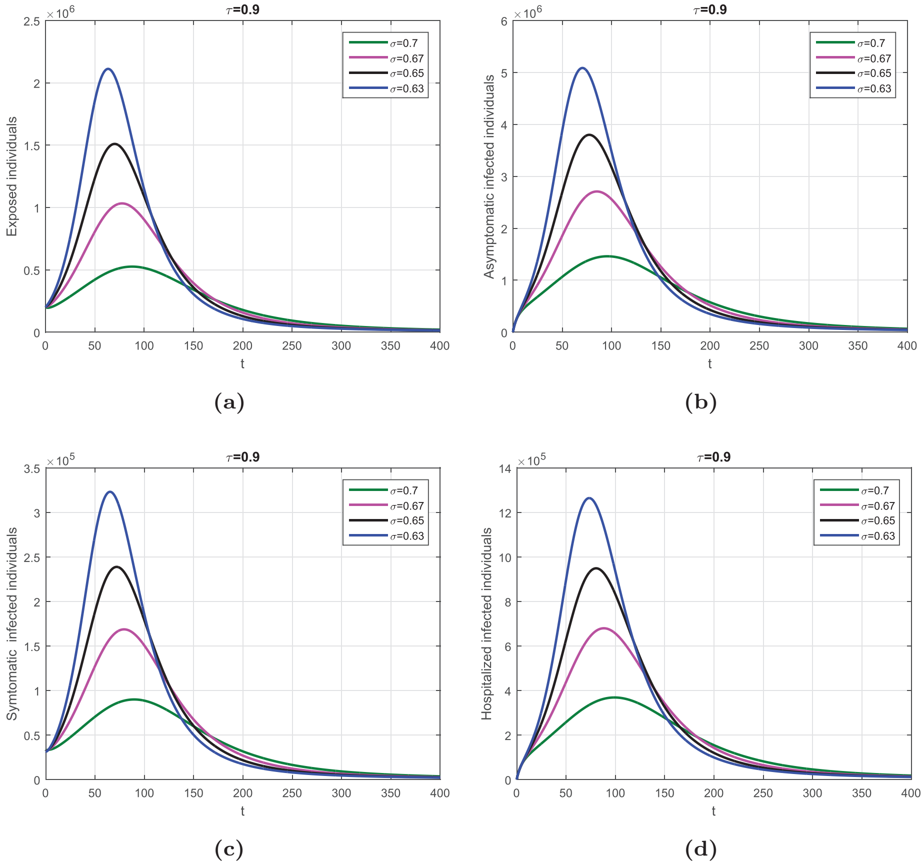


Fig. 3 The impacts of various values of σ on the infected compartments.

$$
\begin{aligned}
&\quad - \frac{\psi - t_l}{t_l - t_{l-1}} g(t_{l-1}, w(t_{l-1})) \\
&= \frac{g(t_l, w(t_l))}{h}(\psi - t_{l-1}) \\
&\quad - \frac{g(t_{l-1}, w(t_{l-1}))}{h}(\psi - t_l) \\
&\simeq \frac{g(t_l, w_l)}{h}(\psi - t_{l-1}) \\
&\quad - \frac{g(t_{l-1}, w_{l-1})}{h}(\psi - t_l). \qquad (23)
\end{aligned}
$$

Inserting Eq. (23) into (22), we get

$$
\begin{aligned}
w_{m+1} &= w_0 + \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g(t_m, w(t_m)) \\
&\quad + \frac{\tau}{\mathrm{ABC}(\tau) \times \Gamma(\tau)} \sum_{l=0}^{m} \\
&\quad \times \left(\frac{g(t_l, w_l)}{h} \int_{t_l}^{t_{l+1}} (\psi - t_{l-1})(t_{m+1} \right. \\
&\quad \left. - \psi)^{\tau-1} d\psi \right) - \frac{\tau}{\mathrm{ABC}(\tau) \times \Gamma(\tau)} \sum_{l=0}^{m} \\
&\quad \times \left(\frac{g(t_{l-1}, w_{l-1})}{h} \int_{t_l}^{t_{l+1}} \right. \\
&\quad \left. \times (\psi - t_l)(t_{m+1} - \psi)^{\tau-1} d\psi \right). \qquad (24)
\end{aligned}
$$

Let for simplicity

$$
B_{\tau,l,1} = \int_{t_l}^{t_{l+1}} (\psi - t_{l-1})(t_{m+1} - \tau)^{\tau-1} d\psi, \qquad (25)
$$

and also

$$
B_{\tau,l,2} = \int_{t_l}^{t_{l+1}} (\psi - t_l)(t_{m+1} - \psi)^{\tau-1} d\psi,
$$

$$
B_{\tau,l,1} = h^{\tau+1} \frac{\begin{array}{c}(m+1-l)^{\tau}(m-l+2+\tau) \\ -(m-l)^{\tau}(m-l+2+2\tau)\end{array}}{\tau(\tau+1)},
$$

$$
B_{\tau,l,2} = h^{\tau+1} \frac{\begin{array}{c}(m+1-l)^{\tau+1} \\ -(m-l)^{\tau}(m-l+1+\tau)\end{array}}{\tau(\tau+1)}. \qquad (26)
$$

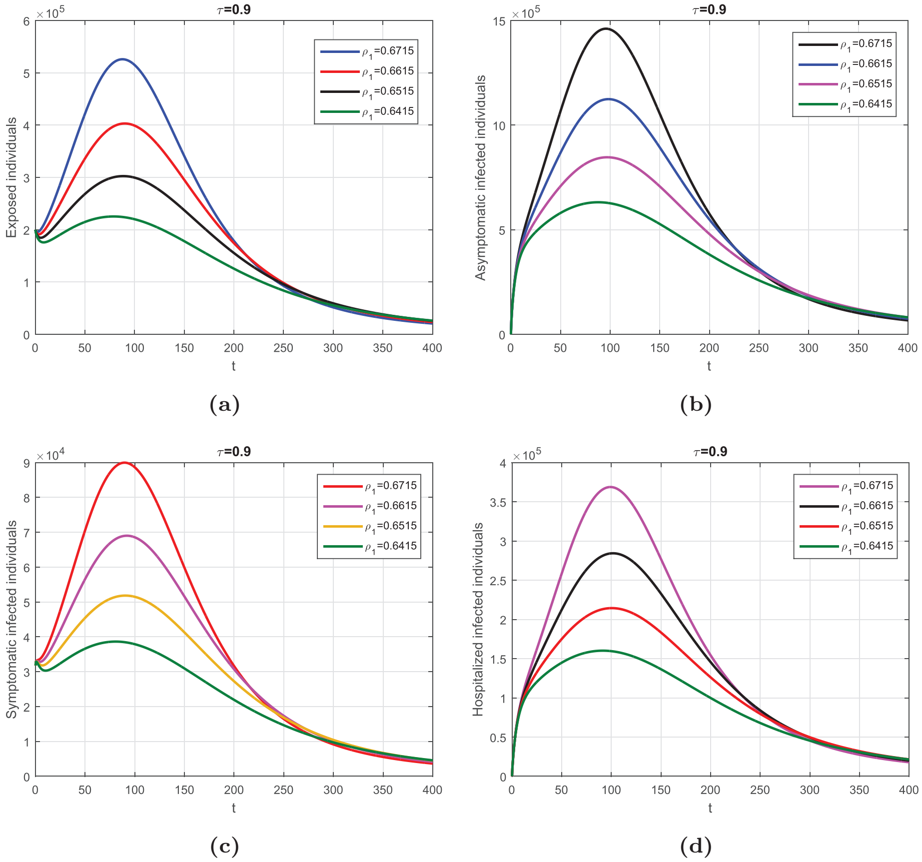


Fig. 4 The impacts of various values of ρ_1 on the infected compartments.

Let us integrate Eqs. (25) and (26) and insert it into (24), then we have

$$\begin{aligned} w_{m+1} = w_0 &+ \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g(t_m, w(t_m)) \\ &+ \frac{\tau}{\mathrm{ABC}(\tau)} \sum_{l=0}^{m} \left(\frac{h^\tau g(t_l, w_l)}{\Gamma(\tau+2)} \right. \\ &\times ((m+1-l)^\tau (m-l+2+\tau) \\ &- (m-l)^\tau (m-l+2+2\tau)) \\ &- \frac{h^\tau g(t_{l-1}, w_{l-1})}{\Gamma(\tau+2)} ((m+1-l)^{\tau+1} \\ &\left. -(m-l)^\tau (m-l+1+\tau)) \right). \end{aligned} \tag{27}$$

The final general scheme given in (27) is used further to present a numerical scheme for the noninteger-order model (5), as given in the following:

$$\begin{aligned} S_{m+1} = S_0 &+ \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g_1[t_m, S(t_m), E(t_m), \\ &A(t_m), I(t_m), H(t_m), R(t_m)] \\ &+ \frac{\tau}{\mathrm{ABC}(\tau)} \sum_{l=0}^{m} \{G_1[(m+1-l)^\tau (m-l+2 \\ &+\tau) - (m-l)^\tau (m-l+2+2\tau)]\} \\ &- \frac{\tau}{\mathrm{ABC}(\tau)} \sum_{l=0}^{m} \{G_2[(m+1-l)^{\tau+1} \\ &- (m-l)^\tau (m-l+1+\tau)]\}, \\ E_{m+1} = E_0 &+ \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g_2[t_m, S(t_m), E(t_m), \\ &A(t_m), I(t_m), H(t_m), R(t_m)] \\ &+ \frac{\tau}{\mathrm{ABC}(\tau)} \sum_{l=0}^{m} \{G_3[(m+1-l)^\tau (m-l \\ &+2+\tau) - (m-l)^\tau (m-l+2+2\tau)]\} \\ &- \frac{\tau}{\mathrm{ABC}(\tau)} \sum_{l=0}^{m} \{G_4[(m+1-l)^{\tau+1} \\ &- (m-l)^\tau (m-l+1+\tau)]\}, \\ A_{m+1} = A_0 &+ \frac{(1-\tau)}{\mathrm{ABC}(\tau)} g_3[t_m, S(t_m), E(t_m), \end{aligned}$$

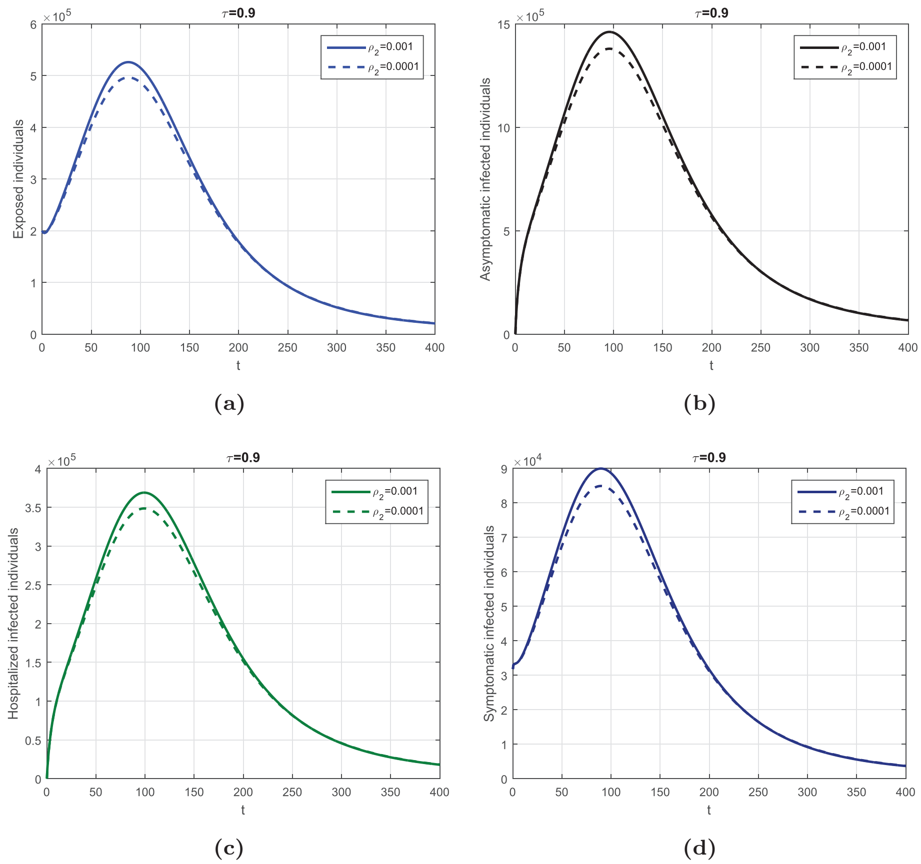


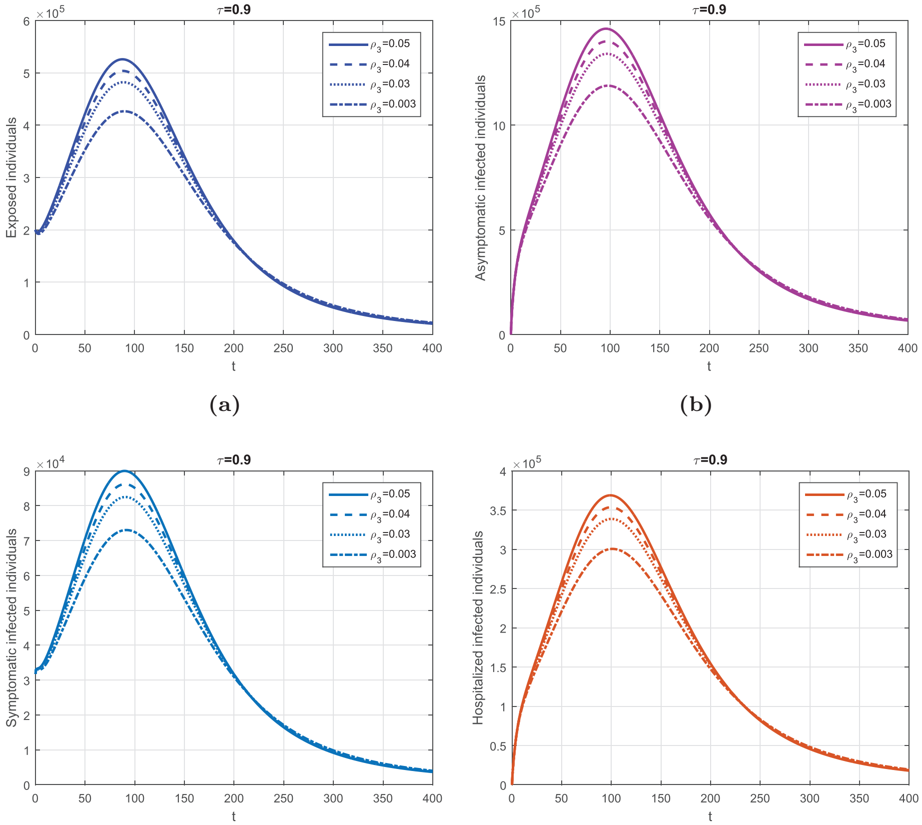
Fig. 5 The impacts of various values of ρ_2 on the infected compartments.

$$
\begin{aligned}
&A(t_m), I(t_m), H(t_m), R(t_m)]\\
&+\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_5[(m+1-l)^{\tau}\\
&\times(m-l+2+\tau)-(m-l)^{\tau}\\
&\times(m-l+2+2\tau)]\}\\
&-\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_6[(m+1-l)^{\tau+1}\\
&-(m-l)^{\tau}(m-l+1+\tau)]\},
\end{aligned}
$$

$$
\begin{aligned}
I_{m+1} = I_0 + \frac{(1-\tau)}{\mathrm{ABC}(\tau)}g_4[&t_m, S(t_m), E(t_m),\\
&A(t_m), I(t_m), H(t_m), R(t_m)]\\
&+\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_7[(m+1-l)^{\tau}\\
&\times(m-l+2+\tau)-(m-l)^{\tau}\\
&\times(m-l+2+2\tau)]\}\\
&-\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_8[(m+1-l)^{\tau+1}\\
&-(m-l)^{\tau}(m-l+1+\tau)]\},
\end{aligned}
$$

$$
\begin{aligned}
H_{m+1} = H_0 + \frac{(1-\tau)}{\mathrm{ABC}(\tau)}g_5[&t_m, S(t_m), E(t_m),\\
&A(t_m), I(t_m), H(t_m), R(t_m)]\\
&+\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_9[(m+1-l)^{\tau}\\
&\times(m-l+2+\tau)-(m-l)^{\tau}\\
&\times(m-l+2+2\tau)]\}\\
&-\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_{10}[(m+1-l)^{\tau+1}\\
&-(m-l)^{\tau}(m-l+1+\tau)]\},
\end{aligned}
$$

$$
\begin{aligned}
R_{m+1} = R_0 + \frac{(1-\tau)}{\mathrm{ABC}(\tau)}g_6[&t_m, S(t_m), E(t_m),\\
&A(t_m), I(t_m), H(t_m), R(t_m)]
\end{aligned}
$$



(a) (b)

(c) (d)

Fig. 6 The impacts of various values of ρ_3 on the infected compartments.

$$
\begin{aligned}
&+\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_{11}[(m+1-l)^{\tau}\\
&\times(m-l+2+\tau)-(m-l)^{\tau}\\
&\times(m-l+2+2\tau)]\}\\
&-\frac{\tau}{\mathrm{ABC}(\tau)}\sum_{l=0}^{m}\{G_{12}[(m+1-l)^{\tau+1}\\
&-(m-l)^{\tau}(m-l+1+\tau)]\},
\end{aligned}
\tag{28}
$$

where

$$G_1=\frac{h^{\tau}g_1(t_l,S_l,E_l,A_l,I_l,H_l,R_l)}{\Gamma(\tau+2)},$$

$$G_2=\frac{h^{\tau}g_1(t_{l-1},S_{l-1},E_{l-1},A_{l-1},I_{l-1},H_{l-1},R_{l-1})}{\Gamma(\tau+2)},$$

$$G_3=\frac{h^{\tau}g_2(t_l,S_l,E_l,A_l,I_l,H_l,R_l)}{\Gamma(\tau+2)},$$

$$G_4=\frac{h^{\tau}g_2(t_{l-1},S_{l-1},E_{l-1},A_{l-1},I_{l-1},H_{l-1},R_{l-1})}{\Gamma(\tau+2)},$$

$$G_5=\frac{h^{\tau}g_3(t_l,S_l,E_l,A_l,I_l,H_l,R_l)}{\Gamma(\tau+2)},$$

$$G_6=\frac{h^{\tau}g_3(t_{l-1},S_{l-1},E_{l-1},A_{l-1},I_{l-1},H_{l-1},R_{l-1})}{\Gamma(\tau+2)},$$

$$G_7=\frac{h^{\tau}g_4(t_l,S_l,E_l,A_l,I_l,H_l,R_l)}{\Gamma(\tau+2)},$$

$$G_8=\frac{h^{\tau}g_4(t_{l-1},S_{l-1},E_{l-1},A_{l-1},I_{l-1},H_{l-1},R_{l-1})}{\Gamma(\tau+2)},$$

$$G_9=\frac{h^{\tau}g_5(t_l,S_l,E_l,A_l,I_l,H_l,R_l)}{\Gamma(\tau+2)},$$

$$G_{10}=\frac{h^{\tau}g_5(t_{l-1},S_{l-1},E_{l-1},A_{l-1},I_{l-1},H_{l-1},R_{l-1})}{\Gamma(\tau+2)},$$

$$G_{11}=\frac{h^{\tau}g_6(t_l,S_l,E_l,A_l,I_l,H_l,R_l)}{\Gamma(\tau+2)},$$

$$G_{12}=\frac{h^{\tau}g_6(t_{l-1},S_{l-1},E_{l-1},A_{l-1},I_{l-1},H_{l-1},R_{l-1})}{\Gamma(\tau+2)}.$$

Using the effective numerical scheme shown in (28), we obtained the graphical solution of model (5) formulated with Atangana–Baleanu derivative. For these simulations, the initial conditions of the variables are mentioned in Sec. 6, while the parameter values are given in Table 1. We show

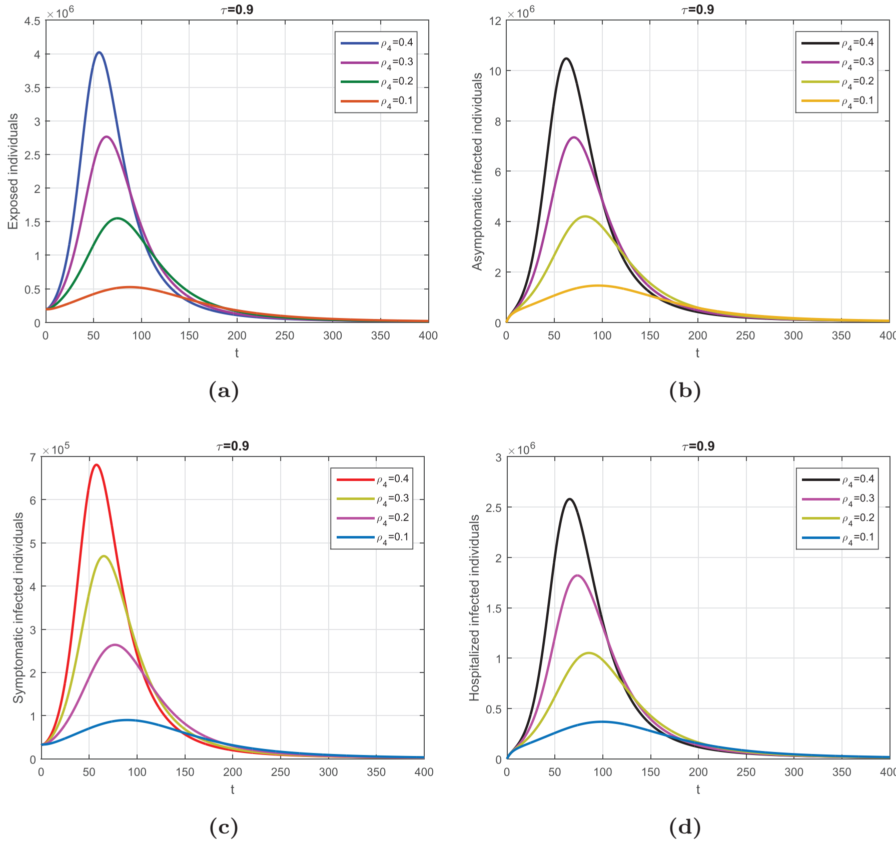


Fig. 7 The impacts of various values of ρ_4 on the infected compartments.

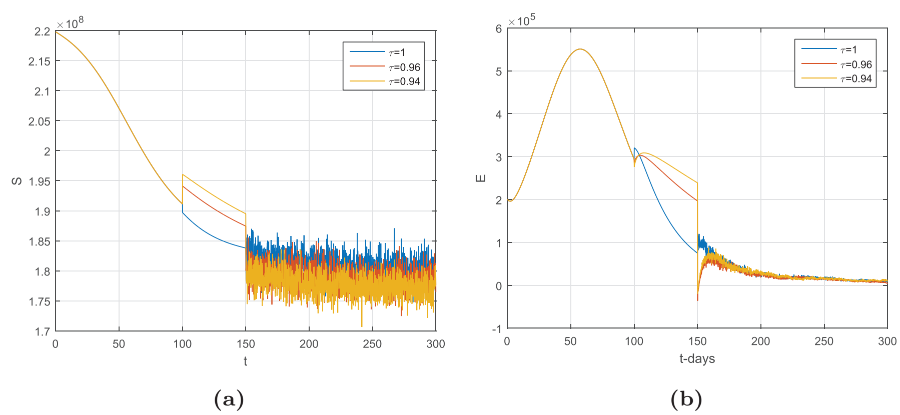
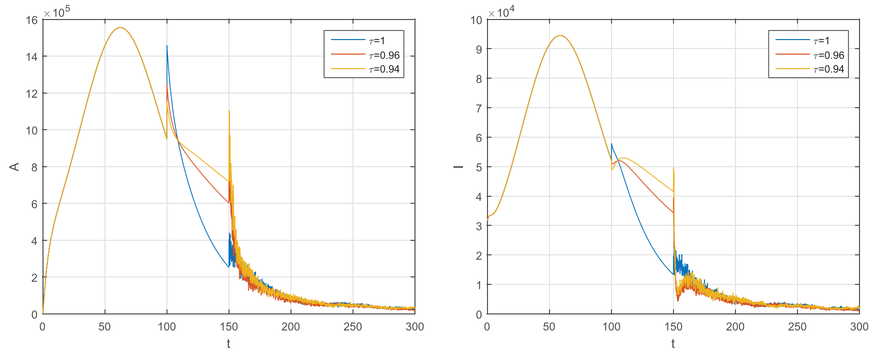


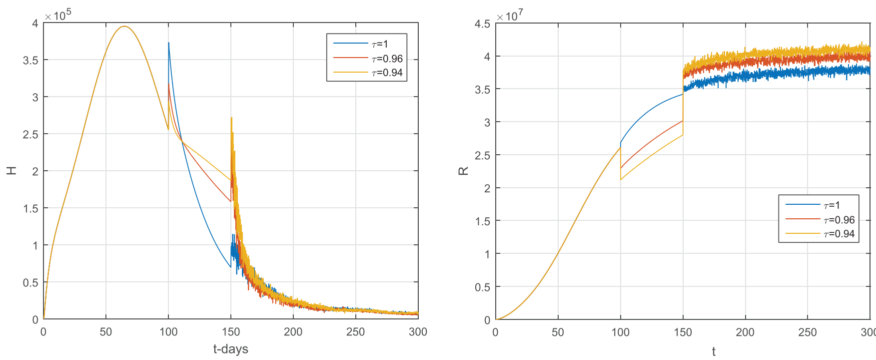
Fig. 8 Simulations of the stochastic model when $\tau = 1$, 0.96 and 0.94.

the simulation results for our fractional model (5) in Figs. 2–7. In Fig. 2, we simulate the model using arbitrary values of the fractional order τ to obtain the model dynamics. The solutions of the model using arbitrary values of the fractional order parameter converge to the equilibrium. Using $\tau = 0.9$ as fixed, we obtain the graphical results shown in Figs. 3–7. In Fig. 3, we use the infected compartments for various values of the parameter σ. We can observe that by decreasing the value of σ, the individuals in the exposed, asymptomatic, infected and hospitalized compartments decrease. The graphical result for the contact parameter ρ_1 and its impacts on the infected compartments have been obtained and shown in Fig. 4. We see that decreasing the contact among the exposed and the healthy individuals can decrease efficiently the number of COVID-19 infected cases. In Fig. 5, we present the impact of the contact parameter ρ_2 which is responsible for asymptomatic infected



(a) **(b)**

Fig. 9 Simulations of the stochastic model when $\tau = 1$, 0.96 and 0.94.



(a) **(b)**

Fig. 10 Simulations of the stochastic model when $\tau = 1$, 0.96 and 0.94.

people. It can be observed that decreasing the contact among the symptomatic and the healthy individuals can better decrease the disease infection. In Figs. 6 and 7, the impacts of contact among infected and healthy individuals and contact among hospitalized, infected and healthy individuals are shown, respectively. It is observed that if we can minimize these contacts, then the number of infected cases will decline faster.

The graphical results for the fractional piecewise stochastic differential equations are shown in Figs. 8–10. The values of the intensity parameters are: $\sigma_1 = 0.01$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$, $\sigma_4 = 0.1$, $\sigma_5 = 0.1$, $\sigma_6 = 0.01$, $T_1 = 100$, $T_2 = 150$ and $T_3 = 300$, and the rest of the parameters are the same as given in Table 1.

8. CONCLUSION

A model with fractional-order derivative using the definition of Atangana–Baleanu derivative has been considered to study the coronavirus infection using the fourth wave cases in Pakistan. One of the important features of this novel model is considering the contact among the exposed, asymptomatic, symptomatic and hospitalized healthy individuals, which are the possible ones that can generate infection in the society. The model is first formulated in integer order and then extended to the fractional-order case. We presented in detail the mathematical analysis of the model and found that the model is locally asymptotically stable when the basic reproduction number $\mathcal{R}_0 < 1$. We studied the global asymptotical stability of the model and found that the model when $\mathcal{R}_0 \leq 1$ is globally asymptotically stable in the disease-free case. The model is then extended to a stochastic fractional-order one with the concept of piecewise differential equations. Further, we considered the infected cases from the fourth wave in Pakistan starting from July 1 to November 16, 2021 and obtained the fitted parameters. For the curve fitting, we used the nonlinear least-square curve fitting and the results of the data versus model are excellent. The estimated basic reproduction number for the fourth wave of COVID-19 cases in Pakistan is $\mathcal{R}_0 \approx 0.9775$.

Further, we considered a numerical algorithm for the simulations of the fractional differential equations in the Atangana–Baleanu case, and also, we provided the numerical scheme in the sense of piecewise fractional stochastic differential equations. The model fitting to the data for both the cases has been given and was found to be useful. The estimated parameters of the model have been used and the graphical results for the key parameters have been presented. We observed from the numerical results of the model that the COVID-19 infection cases can be quickly minimized if the transmissions or contacts are minimized among susceptible individuals, those who are exposed to infections, asymptomatic cases, those who show no clinical symptoms, and the individuals who are hospitalized. Using proper kits to protect themselves during the treatment of the infected hospitalized people can better reduce the infection spread and the life of the health workers.

ACKNOWLEDGMENTS

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REFERENCES

1. Worldometer, Active cases in Pakistan (2022), https://www.worldometers.info/coronavirus/country/pakistan.
2. M. Bachar, M. A. Khamsi and M. Bounkhel, A mathematical model for the spread of COVID-19 and control mechanisms in Saudi Arabia, *Adv. Differ. Equ.* **2021** (2021) 253.
3. H. M. Yang, L. P. L. Junior, F. F. M. Castro and A. C. Yang, Mathematical modeling of the transmission of SARS-CoV-2: Evaluating the impact of isolation in São Paulo State (Brazil) and lockdown in Spain associated with protective measures on the epidemic of CoViD-19, *PLoS ONE* **16** (2021) e0252271.
4. Y. Li and Z. Wei, Dynamics and optimal control of a stochastic coronavirus (COVID-19) epidemic model with diffusion, *Nonlinear Dyn.* **109** (2022) 91–120.
5. H. T. Alemneh and G. T. Telahun, Mathematical modeling and optimal control analysis of COVID-19 in Ethiopia, medRxiv (2020), doi:10.1101/2020.07.23.20160473.
6. E. F. Arruda, D. H. Pastore, C. M. Dias and S. S. Das, Modelling and optimal control of multi strain epidemics, with application to COVID-19, preprint (2021), arXiv:2101.08137 [eess.SY].
7. K. Sarkar, S. Khajanchi and J. J. Nieto, Modeling and forecasting the COVID-19 pandemic in India, *Chaos Solitons Fractals* **139** (2020) 110049.
8. L. López and X. Rodo, A modified SEIR model to predict the COVID-19 outbreak in Spain and Italy: Simulating control scenarios and multi-scale epidemics, *Results Phys.* **21** (2021) 103746.
9. S. Ullah and M. A. Khan, Modeling the impact of non-pharmaceutical interventions on the dynamics of novel coronavirus with optimal control analysis with a case study, *Chaos Solitons Fractals* **139** (2020) 110075.
10. D. Okuonghae and A. Omame, Analysis of a mathematical model for COVID-19 population dynamics in Lagos, Nigeria, *Chaos Solitons Fractals* **139** (2020) 110032.
11. T. Sun and Y. Wang, Modeling COVID-19 epidemic in Heilongjiang province, China, *Chaos Solitons Fractals* **138** (2020) 109949.
12. C. T. Deressa and G. F. Duressa, Modeling and optimal control analysis of transmission dynamics of COVID-19: The case of Ethiopia, *Alex. Eng. J.* **60** (2021) 719–732.
13. R. Alguliyev, R. Aliguliyev and F. Yusifov, Graph modelling for tracking the COVID-19 pandemic spread, *Infect. Dis. Model.* **6** (2021) 112–122.
14. A. Tripathi, R. N. Tripathi and D. Sharma, A mathematical model to study the COVID-19 pandemic in India, *Model. Earth Syst. Environ.* (2021), doi:10.1007/s40808-021-01280-8.
15. J. K. K. Asamoah, E. Okyere, A. Abidemi, S. E. Moore, G. Q. Sun, Z. Jin, E. Acheampong and J. F. Gordon, Optimal control and comprehensive cost-effectiveness analysis for COVID-19, preprint (2021), arXiv:2107.09595 [math.OC].
16. T. Zha, O. Castillo, H. Jahanshahi, A. Yusuf, M. O. Alassafi, F. Alsaadi and Y. Chu, A fuzzy-based strategy to suppress the novel coronavirus (2019-nCoV) massive outbreak, *Appl. Comput. Math.* **20** (2021) 160–176.
17. K. Karthikeyan, P. Karthikeyan, H. Baskonus, K. Venkatachalam and Y. Chu, Almost sectorial operators on ψ-Hilfer derivative fractional impulsive integro-differential equations, *Math. Methods Appl. Sci.* **45** (2021) 8045–8059.
18. S. Rashid, S. Sultana, Y. Karaca, A. Khalid and Y.-M. Chu, Some further extensions considering discrete proportional fractional operators, *Fractals* **30** (2022) 2240026.
19. S. Rashid, E. Abouelmagd, S. Sultana and Y. Chu, New developments in weighted n-fold type inequalities via discrete generalized-proportional fractional operators, *Fractals* **30** (2022) 2240056.

20. Z. He, A. Abbes, H. Jahanshahi, N. Alotaibi and Y. Wang, Fractional-order discrete-time SIR epidemic model with vaccination: Chaos and complexity, *Mathematics* **10** (2022) 165.
21. F. Jin, Z. Qian, Y. Chu and M. Rahman, On nonlinear evolution model for drinking behavior under Caputo–Fabrizio derivative, *J. Appl. Anal. Comput.* **12** (2022) 790–806.
22. Y. Gu, M. A. Khan, Y. S. Hamed and B. F. Felemban, A comprehensive mathematical model for SARS-CoV-2 in Caputo derivative, *Fractal Fract.* **5** (2021) 271.
23. M. Iqbal, Y. Wang, M. Miah and M. S. Osman, Study on Date–Jimbo–Kashiwara–Miwa equation with conformable derivative dependent on time parameter to find the exact dynamic wave solutions, *Fractal Fract.* **6** (2021) 4.
24. F. Wang, M. N. Khan, I. Ahmad, H. Ahmad, H. Abu-Zinadah and Y. Chu, Numerical solution of traveling waves in chemical kinetics: Time-fractional Fishers equations, *Fractals* **30** (2022) 2240051.
25. M. Du, Z. Wang and H. Hu, Measuring memory with the order of fractional derivative, *Sci. Rep.* **3** (2013) 3431.
26. N. H. Tuan, H. Mohammadi and S. Rezapour, A mathematical model for COVID-19 transmission by using the Caputo fractional derivative, *Chaos Solitons Fractals* **140** (2020) 110107.
27. D. Baleanu, H. Mohammadi and S. Rezapour, A fractional differential equation model for the COVID-19 transmission by using the Caputo–Fabrizio derivative, *Adv. Differ. Equ.* **2020** (2020) 299.
28. N. Sene, SIR epidemic model with Mittag-Leffler fractional derivative, *Chaos Solitons Fractals* **137** (2020) 109833.
29. A. Atangana and S. Araz, Mathematical model of COVID-19 spread in Turkey and South Africa: Theory, methods, and applications, *Adv. Differ. Equ.* **2020** (2022) 659.
30. Y. Chu, A. Ali, M. A. Khan, S. Islam and S. Ullah, Dynamics of fractional order COVID-19 model with a case study of Saudi Arabia, *Results Phys.* **21** (2021) 103787.
31. M. Odaka and K. Inoue, Modeling viral dynamics in SARS-CoV-2 infection based on differential equations and numerical analysis, *Heliyon* **7** (2021) e08207.
32. A. Chatterjee, F. Basir, M. Almuqrin, J. Mondal and I. Khan, SARS-CoV-2 infection with lytic and non-lytic immune responses: A fractional order optimal control theoretical study, *Results Phys.* **26** (2021) 104260.
33. M. A. Khan and A. Atangana, Modeling the dynamics of novel coronavirus (2019-nCoV) with fractional derivative, *Alex. Eng. J.* **59** (2020) 2379–2389.
34. A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, preprint (2016), arXiv:1602.03408 [math.GM].
35. P. V. Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.* **180** (2002) 29–48.
36. J. P. LaSalle, An invariance principle in the theory of stability, Technical Report No. 66-1, Center for Dynamical Systems, Brown University (1966).
37. A. Atangana and S. Araz, Modeling third waves of COVID-19 spread with piecewise differential and integral operators: Turkey, Spain and Czechia, *Results Phys.* **29** (2021) 104694.
38. Worldometer, Pakistan population, 1950–2020 (2022), https://www.worldometers.info/world-population/pakistan-population/.
39. E. A. Iboi, C. N. Ngonghala and A. B. Gumel, Will an imperfect vaccine curtail the COVID-19 pandemic in the U.S.? *Infect. Dis. Model.* **5** (2020) 510–524.
40. M. Toufik and A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models, *Eur. Phys. J. Plus* **132** (2017) 444.

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ORIGINAL ARTICLE

On nonlinear dynamics of COVID-19 disease model corresponding to nonsingular fractional order derivative

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Abstract

This manuscript is devoted to investigate the mathematical model of fractional-order dynamical system of the recent disease caused by Corona virus. The said disease is known as Corona virus infectious disease (COVID-19). Here we analyze the modified SEIR pandemic fractional order model under nonsingular kernel type derivative introduced by Atangana, Baleanu and Caputo ($\mathcal{ABC}$) to investigate the transmission dynamics. For the validity of the proposed model, we establish some qualitative results about existence and uniqueness of solution by using fixed point approach. Further for numerical interpretation and simulations, we utilize Adams-Bashforth method. For numerical investigations, we use some available clinical data of the Wuhan city of China, where the infection initially had been identified. The disease free and pandemic equilibrium points are computed to verify the stability analysis. Also we testify the proposed model through the available data of Pakistan. We also compare the simulated data with the reported real data to demonstrate validity of the numerical scheme and our analysis.

Keywords Non-integer order Adams-Bashforth technique · Approximate solution · COVID-19 model

1 Introduction

COVID-19 which is a threatful and terrible disease has been identified initially in Wuhan city of China in December 2019. The said infection transmitted in all over the world in coming few months. The spreading of this little and quickly transmissible virus in the recent time is due to corona virus [1, 2]. In 2020, the disease of COVID-19 is the world big threat that affected nearly all countries and continents around the globe. By the data given by the worldo-meter [3] and WHO [4, 5] shows that nearly 150 million cases of infection occurred while more than five million of population has been died. In the past history of the said virus, it is started in 1965 by the Tyrrell and Bynoe for identification of a virus

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due to B814 [6]. Such types of viruses had been identified in human organs of embryonic tracheal organ moves taken from the respiration vain of an aged person [7].

Most of scientists, scholars and politicians are trying to stabilize and control the transmission of the pandemic, because the aforesaid disease has killed millions of people around the world in last 2 years. One of the factors of transmitting the said pandemic so rapidly is the migration of infectious population from one place to another. Therefore, locally and globally, some precautionary measures have been implemented. Most of the countries have stopped their air traffic and avoid unnecessary traveling [8]. They also banned crowds and lockdown in the cities to minimize the loss of human lives. In this scenario, the researchers and policies makers are searching to discover a cure or vaccine for the mention disease to stabilize and control it in the coming days. In preparation of vaccine some countries have got succession and now vaccines are available.

To properly controlled this pandemic, it is important to know much about the transmission, symptoms and features of this disease. Implementation of a proper method against the disease out breaking which is the big type of challenge faced by the human population in past history. Therefore, scientists and researchers are trying continuously to model this disease mathematically. In the past, different mathematical models have been developed for infectious diseases, for instance (see the references [9–15]). Recently, a lot of research work has been published in the form of mathematical models. For reference, we give for instance some published work as [1, 2, 16–22].

Most of mathematical models which have been investigated in the past were either the system of differential, difference and integration equation having natural or discrete-order. But after fractional calculus has got attention in last few decades, fractional order differential equations (FDEs) applied in excessive numbers to model various real-world problems. FDEs have many applications in various fields of engineering and medical laboratories like physics, business, controlling phenomena, accounting and in biological problems. Therefore, the scientists and researchers increasingly have used FDEs to formulate the real-globe phenomena. Because of the extra degree of choices in fractional derivative which is not present in traditional order operator. Further traditional order derivatives of integer order are not generalized as compared to fractional order which is generalized. Hence fractional order derivative is non-locale in nature and preserves the memory properties which makes it better. Further fractional order derivative of a function produces accumulation of the function which include the corresponding integer order counter part as a special case. Further geometrically it gives spectrum of the function and hence produce the whole density of the function on whom it applies. This is consider the best one, in the conditions where the coming states of models not only related to the present state but may also depend on the past timing of each quantity. For some significance applications see [23–26]. Due to these properties FDEs not only formulate the problems containing the non-Gaussian nature but can also describe the dynamics for the non-Markovian conditions also. As the natural order derivative and its constituting equations do not give knowledge lying between any of the two consecutive different natural numbers. Therefore, FDEs have been introduced to overcome these limitations. Fractional differential operators have many applications in different areas of mathematical and physical sciences. Liouville, Euler, Reimann and Fourier established some definitions for fractional order derivative during. After that the area has given much more attention. Modern calculus has a lot of applications in the area of mathematical modeling where hereditary characteristic and memorization properties have been studied very easily. Integer order derivative is rarely used to study such behaviors. Non-integer order derivative is the generalization of the natural order derivative having extra degree of freedom as compared the natural order derivative (see [23–30]). Keeping these properties scholars and researchers have taken much interest to study FDEs from different aspects. In the definition of arbitrary order operators, theirs lie a definite integration which predicts physically the area under the function curve or spectrum to generalize it. Integer order differentiation is a specific class of the non-integer order derivative. Although, sufficient contributions have been made by the researchers to analyze the solution of various problems (see [31–37]). Remarkably, arbitrary order operators have been formulated in different mathematical forms. Fractional differential operators can be classified in two major classes. One is devoted to singular kernel type fractional order differential operators like Reimann-Liouville, Caputo, etc. While the other class is devoted to non-singular type operator where exponential or Mittag-Leffler function play the role of kernel. One of the famous operator of fractional derivative with Mittag-Leffler type kernel is known as $\mathscr{ABC}$ introduced by Atangana, Baleanu and Caputo [38] in 2016. This operator replaced the singular kernel by non-singular one [39–41]. But this classification has own merits and de-merits. But researchers increasingly used these operators to investigate various real-world problems.

To treat FDEs under various operators for their numerical solution, optimization and numerical analysis, the traditional techniques have been extended for these purposes. For instance decomposition and homotopy perturbation techniques have been previously used to investigate various problems of FDEs (see [42–44]). For numerical solution mostly, RKM methods have been applied to various fractional order models. Here, in our work, the fractional Adams Bash-forth method is used for numerical simulation as applied in [45, 46]. This technique is an easy bi-step method which is more powerful than Taylor series, Euler method, and RKM techniques. Moreover, it is rapidly convergent and stable.

The investigation of epidemiological models of infected disease have gained great attention from research point of view. Several scholars have investigated the solution existence and uniqueness of many fractional order models [47–50]. For the

Table 1 Parameters description given in the model (1)

Notation	Parameters description
n_P	Birth rate
m_p	Infection death rate
b_p	rate of transmission
b_w	Disease transmission coefficient
ω_p, ω'_p	signified incubation period
γ_p, γ'_p	rate of recovered of I_p, A_p
ε, σ	Influence rate of virus from I_p and A_p to M
δ_p,	density of Asymptotic infectious population
κ	Multiplicity of transmissibility
ϑ	Eliminating rate of virus from M

description of the mathematical formulation of COVID-19, and to observe that how this disease impacts the susceptible, infected and quarantined people have been investigated. Some of the researchers have focused on the mathematical perspective of COVID-19. For knowing the dynamics structure and physical behavior of the outbreak of COVID-19, Mathematical models are playing important roles. In the problem presented in [51, 52] contains the susceptible people $S_p(t)$, exposed population $E_p(t)$, infectious density $I_p(t)$, asymptotically infectious people $A_p(t)$, humans recovery population $R_p(t)$, reservoir $M(t)$ and the their interactions have been modeled as

$$\begin{cases} \mathcal{D}_t(S_p(t)) = n_p - S_p m_p - b_p S_p(I_p + \kappa A_p) - b_w M S_p, \\ \mathcal{D}_t(E_p(t)) = (I_p + \kappa A_p) b_p S_p + b_w S_p M - \omega_p E_p(1-\delta_p) - \delta_p E_p \omega'_p - E_p m_p, \\ \mathcal{D}_t(I_p(t)) = \omega_p E_p(1-\delta_p) - I_p(\gamma_p + m_p), \\ \mathcal{D}_t(A_p(t)) = \delta_p \omega'_p E_p - (\gamma'_p + m_p) A_p, \\ \mathcal{D}_t(R_p(t)) = \gamma_p I_p + \gamma'_p A_p - m_p \gamma_p, \\ \mathcal{D}_t(M(t)) = \varepsilon I_p + \sigma A_p - \vartheta M, \\ S_p(0) = S_0,\ E_p(0) = E_0,\ I_p(0) = I_0,\ A_p(0) = A_0,\ R_p(0) = A_0,\ M(0) = M_0. \end{cases} \tag{1}$$

The detail of parameters applied in the problem (1), with full explanation is provided in Table 1.

Here authors have established some global, local stability by computing basic reproductive numbers. Also using simple integral transform method, they have presented some numerical results.

Motivated from the aforesaid literature and work published in the corresponding area, we consider Model (1) under the $\mathcal{ABC}$ fractional order derivative as

$$\begin{cases} {}^{ABC}\mathcal{D}_t^r(S_p(t)) = n_p - m_p S_p - b_p S_p(I_p + \kappa A_p) - b_w S_p M, \\ {}^{ABC}\mathcal{D}_t^r(E_p(t)) = (I_p + \kappa A_p) b_p S_p + b_w S_p M - \omega_p E_p(1-\delta_p) - \delta_p \omega'_p E_p - m_p E_p, \\ {}^{ABC}\mathcal{D}_t^r(I_p(t)) = \omega_p E_p(1-\delta_p) - I_p(\gamma_p + m_p), \\ {}^{ABC}\mathcal{D}_t^r(A_p(t)) = \delta_p \omega'_p E_p - (\gamma'_p + m_p) A_p, \\ {}^{ABC}\mathcal{D}_t^r(R_p(t)) = \gamma_p I_p + \gamma'_p A_p - m_p \gamma_p, \\ {}^{ABC}\mathcal{D}_t^r(M(t)) = \varepsilon I_p + \sigma A_p - \vartheta M, \\ S_p(0) = S_0,\ E_p(0) = E_0,\ I_p(0) = I_0,\ A_p(0) = A_0,\ R_p(0) = A_0, \\ M(0) = M_0, \quad 0 < r \le 1. \end{cases} \tag{2}$$

We establish some appropriate results for existence theory of solution via fixed point approach. Further, we attempt on stability results for the suggested model. Some sensitivity results about the parameters of the model are also discussed. Further, numerical technique of Adams-Bashforth method is used to handle this model (2) for the approximate solution and numerical simulations. Further we testify the numerical interpretation by two sets of data one of Wuhan city reported and other one is reported in about Pakistan. Further we also compared our simulated data and real data in case of infected cases to see the validity of the numerical scheme.

Here we remark some limitations of using mathematical models to understand the mechanism of infections disease or other real-world problems. For instance, models that establish for addressing forecasts are usually designed to produce either short-term or long-term forecasts. Some times models designed for long-term forecasting often do not produce good short-term forecasts and vice versa. Also, the associated factors, assumptions and structure, required for the one purpose often make the model less suitable for the other. To construct an appropriate model is a crucial job, because it is often the only link between the model and the model user. Also in majority cases to verify the model by real data, we often have no access to the afore data or information. In short we say that models are abstractions of reality, because, real-world systems are complex and composed of many interrelated components. For a modeler it is impossible or tedious to include all the comments (see detail in [57]). On the other hand for simulations, different numerical schemes are using to deal mathematical models. The concerned schemes have some short comings. For instance often numerical scheme is stable that we are using but on the other hands it will suffer from convergency. In same fashion it is not necessary that a scheme we use is convergent then it must also be stable. Here we use Adams-Bashforth method to simulate our results. The advantage of the proposed method is that it uses only one additional function evaluation per step and produces preserve high-order accuracy. But the limitation of the said method is the necessity of using another method to start.

2 Method, feasibility and stability analysis

Here in this part, we have to find feasibility and stability analysis of the proposed model. We first here re-collect some required results, definitions from [39, 40].

Definition 2.1 $\mathcal{ABC}$ fractional operator for a function $\Psi(t)$ and $\Psi(t) \in \mathscr{H}^1(0, \tau)$ is formulated as:

$${}^{ABC}\mathbf{D}_0^r\Psi(t) = \frac{ABC(r)}{1-r}\int_0^t \frac{d}{dz}\Psi(z)\kappa_r\left[\frac{-r}{1-r}\left(t-z\right)^r\right]dz. \tag{3}$$

If we change $\kappa_r\left[\frac{-r}{1-r}\left(t-z\right)^r\right]$ to $\kappa_1 = \exp\left[\frac{-r}{1-r}\left(t-z\right)\right]$, in (3), then we will obtain the Caputo-Fabrizo (CF) operator of fractional orders. Further, it is to be noted that

$${}^{ABC}\mathbf{D}^r[constant] = 0.$$

$\mathcal{ABC}(r)$ is called normalized mapping as $\mathcal{ABC}(0) = \mathcal{ABC}(1) = 1$. Also κ_r represents specific mapping known as Mittag-Leffler which is the general form of the exponential mapping [28–30].

Definition 2.2 Consider $\Psi \in L[0,T]$, then the fractional order integration in the sense of $\mathcal{ABC}$ is as follows:

$${}^{ABC}\mathbf{I}_0^r\Psi(t) = \frac{1-r}{ABC(r)}\Psi(t) + \frac{r}{ABC(r)\Gamma(r)}\int_0^t (t-z)^{r-1}\Psi(z)dz. \quad (4)$$

Lemma 2.3 ([54]) If $Y(t) \to 0$ as $t=0$, then the solution for $0 < r < 1$ of the problem

$$\begin{aligned} {}^{ABC}\mathbf{D}_0^r\Psi(t) &= Y(t),\ t \in [0,T], \\ \Psi(0) &= \Psi_0 \end{aligned}$$

is given by

$$\Psi(t) = \Psi_0 + \frac{(1-r)}{ABC(r)}Y(t) + \frac{r}{\Gamma(r)ABC(r)}\int_0^t (t-z)^{r-1}Y(z)dz.$$

Note: For existence of solution, closed norm space is defined by:

$$\mathbf{Y} = \mathbf{Z} = C([0,T] \times R^6, R),$$

where $\mathbf{Z} = C[0,T]$ under the norm:

$$\|\mathbf{Y}\| = \|\Psi\| = \sup_{t\in[0,T]}\left[|S_p(t)| + |E_p(t)| + |I_p(t)| + |A_p(t)| + |R_p(t)| + |M(t)|\right].$$

The Krasnosilkii's theorem of fixed point theory is applied for the main result.

Theorem 2.4 [55] Consider **A** be any convexed subset of **Y** and consider that **F**,**G** are two different operators in the integral equations with

1. $\mathbf{G}w + \mathbf{F}w \in \mathbf{A}$ for all $w \in \mathbf{A}$;
2. **F** is contracted operator;
3. **G** is compact and continuous operator.

Then equation $\mathbf{F}w + \mathbf{G}w = w$ in operator form, has one or more than one solution.

Lemma 2.5 The solution of the proposed problem (2) is bounded in the region of feasibility given by

$$\Psi = \left\{(S_p(t), E_p(t), I_p(t), A_p(t), R_p(t), M(t)) \in \mathbf{R}_+^6\ :\ 0 \le N(t) \le \frac{n_p}{m_p}\right\}.$$

Proof Let consider

$$N_p = S_p(t) + E_p(t) + I_p(t) + A_p(t) + R_p(t) + M(t).$$

By adding all the equations of (2) we get as

$$ll\frac{{}^{ABC}d^r(N_p)}{dt^r} \le n_p - m_pN_p. \quad (5)$$

Solving Eq. (5), we get

$$N_p \le \frac{n_p}{m_p} - C\exp(-m_pt),$$

or

$$N_p(t) \le \frac{n_p}{m_p},$$

hence proved.

Next we find the disease free and the pandemic equilibrium points by setting all the equation of system (2) equal zero as

$$\begin{aligned} {}^{ABC}\mathcal{D}_t^r(S_p(t)) &= 0, \\ {}^{ABC}\mathcal{D}_t^r(E_p(t)) &= 0, \\ {}^{ABC}\mathcal{D}_t^r(I_p(t)) &= 0, \\ {}^{ABC}\mathcal{D}_t^r(A_p(t)) &= 0, \\ {}^{ABC}\mathcal{D}_t^r(R_p(t)) &= 0, \\ {}^{ABC}\mathcal{D}_t^r(M(t)) &= 0, \end{aligned}$$

or

$$E_0(\frac{n_p}{m_p}, 0, 0, 0, 0, 0).$$

Theorem 2.6 The basic reproductive number is computed as

$$R_0 = \frac{\delta_p\omega_p'(m_p+\gamma_p)(\vartheta\kappa b_pn_p + n_p\delta b_w) + (1-\delta_p)\omega_p(\gamma_p'+m_p)(\vartheta b_pn_p + n_p\varepsilon b_w)}{\vartheta m_p(m_p+\gamma_p)(\gamma_p'+m_p)(\delta_p(\omega_p'-\omega_p)+m_p+w_p)}$$

Proof For this we take the four equations of model (2) as

$$\frac{{}^{ABC}d^r(N_p)}{dt^r} = \begin{pmatrix} (I_p+\kappa A_p)b_pS_p + b_wS_pM - \omega_pE_p(1-\delta_p) - \delta_p\omega_p'E_p - m_pE_p \\ \omega_pE_p(1-\delta_p) - I_p(\gamma_p+m_p) \\ \delta_p\omega_p'E_p - (\gamma_p'+m_p)A_p \\ \varepsilon I_p + \sigma A_p - \vartheta M \end{pmatrix}.$$

We define F and V as follows

$$F = \begin{pmatrix} (I_p+\kappa A_p)b_pS_p + b_wS_pM \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} \omega_p E_p(1-\delta_p) + \delta_p \omega'_p E_p + m_p E_p \\ \omega_p E_p(1-\delta_p) - I_p(\gamma_p + m_p) \\ \delta_p \omega'_p E_p - (\gamma'_p + m_p) A_p \\ \varepsilon I_p + \sigma A_p - \vartheta M \end{pmatrix}.$$

Next, taking the Jacobian of F and V w.r.t, and putting the value of E_0, we get

$$\mathcal{F} = \begin{pmatrix} 0 & \frac{b_p n_p}{m_p} & \frac{\kappa b_p n_p}{m_p} & \frac{b_w n_p}{m_p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} \delta_p \omega'_p + \omega_p(1-\delta_p) + m_p & 0 & 0 & 0 \\ \omega_p(\delta_p - 1) & m_p + \gamma_p & 0 & 0 \\ -\delta_p \omega'_p & 0 & \gamma'_p + m_p & 0 \\ 0 & -\varepsilon_p & -\delta & \vartheta \end{pmatrix}.$$

Then the dominant eigen value of $\mathcal{F}\mathcal{V}^{-1} = \rho(\mathcal{F}\mathcal{V}^{-1})$ is called basic reproduction number R_0 and hence equal to

$$R_0 = \frac{\delta_p \omega'_p (m_p+\gamma_p)(\vartheta \kappa b_p n_p + n_p \delta b_w) + (1-\delta_p)\omega_p(\gamma'_p + m_p)(\vartheta b_p n_p + n_p \varepsilon b_w)}{\vartheta m_p (m_p+\gamma_p)(\gamma'_p+m_p)(\delta_p(\omega'_p - \omega_p) + m_p + w_p)}. \quad (6)$$

Hence proved

Theorem 2.7 E_0 is locally asymptotically stable if $R_0 < 1$.

Proof The derivation of the theorem can be obtained by taking Jacobian of the system (2) and putting $E_0 = (\frac{n_p}{m_p}, 0, 0, 0, 0, 0)$, one has

$$\mathcal{J} = \begin{pmatrix} -m_p & 0 & \frac{-b_p n_p}{m_p} & \frac{-\kappa b_p n_p}{m_p} & 0 & \frac{-b_w n_p}{m_p} \\ 0 & -m_p - \delta_p \omega'_p - (1-\delta_p)\omega_p & \frac{b_p n_p}{m_p} & \frac{\kappa b_p n_p}{m_p} & 0 & \frac{b_w n_p}{m_p} \\ 0 & (1-\delta_p)\omega_p & -m_p - \gamma_p & 0 & 0 & 0 \\ 0 & \delta_p \omega'_p & 0 & -m_p - \gamma'_p & 0 & 0 \\ 0 & 0 & \gamma_p & \gamma'_p & -m_p & 0 \\ 0 & 0 & \varepsilon & \delta & 0 & -\vartheta \end{pmatrix}.$$

In the above matrix, two of the eigen values on the main diagonal are negative, while the rest of the eigen values can be computed by characteristic equation as

$$\Lambda^4 + b_1 \Lambda^3 + b_2 \Lambda^2 + b_3 \Lambda + b_4 = 0. \quad (7)$$

Here

$$\begin{aligned} b_1 &= \gamma'_p + \delta + \delta_p \omega'_p + (1-\delta_p)\omega_p + 3m_p + \gamma_p, \\ b_2 &= \underbrace{(m_p+\gamma_p)(\delta_p \omega'_p + (1-\delta_p)\omega_p + m_p) - b_p(1-\delta_p)\omega_p} \\ &+ \underbrace{(m_p+\gamma_p)(\delta_p(\omega'_p - \omega_p) + m_p - \omega_p) - \kappa b_p \delta_p \omega'_p} \\ &+ \vartheta(\gamma'_p + m_p) + (m_p+\gamma_p)(m_p+\gamma'_p) + \vartheta(\delta_p(\omega'_p - \omega_p) + m_p + \omega_p) + \vartheta(m_p + \gamma_p), \\ b_3 &= \vartheta(\delta_p \omega'_p + (1-\delta_p)\omega_p + m_p)[(\gamma'_p + m_p)(1-R_1) + (m_p+\gamma_p)(1-R_2)] \\ &+ (m_p+\gamma_p)\underbrace{(\delta(\gamma'_p + m_p) - (\kappa b_p \delta_p \omega'_p))} + b_p \delta_p \omega_p (\gamma'_p + m_p) \\ &+ (\gamma'_p + m_p)((\gamma_p + m_p)\delta_p(\omega'_p - \omega_p) + m_p + \omega_p) - b_p \omega_p, \\ b_4 &= \vartheta(\gamma_p + m_p)(\gamma'_p + m_p)(\delta + \delta_p \omega'_p + (1-\delta_p)\omega_p + m_p)(1 - R_0), \end{aligned}$$

where $R_0 = R_1 + R_2$ as follows

$$R_1 = \frac{\delta_p \omega'_p (\vartheta \kappa b_p n_p + n_p \delta b_w)}{\vartheta m_p (\gamma'_p + m_p)(\delta_p(\omega'_p - \omega_p) + m_p + w_p)}$$

and

$$R_2 = \frac{(1-\delta_p)\omega_p(\gamma_p + m_p)(\vartheta b_p n_p + n_p \varepsilon b_w)}{\vartheta m_p (\gamma_p + m_p)(\delta_p(\omega'_p - \omega_p) + m_p + w_p)}.$$

In the above characteristic equation, the terms which are underlines are less than R_0, also b_4 is positive if $R_0 < 1$. Further if $R_1 < 1$ and $R_2 < 1$, then b_3 will be positive. Hence all the coefficients are positive being the conditions for Routh-Hurwitz criteria [53]. Hence E_0 is locally asymptotically stable.

Next we have to find the pandemic equilibrium point as

$$E^* = (S^*_P, E^*_P, I^*_P, A^*_P, R^*_P, M^*)$$

and

$$\begin{aligned} S^*_p &= \frac{n_p}{\Lambda + m_p}, \\ E^*_p &= \frac{\Lambda S^*_p}{\delta_p \omega'_p - \delta_p \omega_p + m_p + \omega_p}, \\ I^*_p &= \frac{E^*_p (1-\delta_p)\omega_p}{\gamma_p + m_p}, \\ A^*_P &= \frac{E^*_p \delta_p \omega'_p}{\gamma'_p + m_p}, \\ R^*_P &= \frac{A^*_p \gamma'_p + I^*_p \gamma_p}{+m_p}, \\ M^* &= \frac{A^*_p \delta + I^*_p \varepsilon}{\vartheta}. \end{aligned}$$

Here

$$\Lambda = \frac{b_p(\kappa n_p A^*_p + m_p I^*_p)}{m_p(S^*_p + E^*_p + I^*_p + A^*_p + R^*_p)} + \frac{b_p M^*}{m_p},$$

satisfying the given equation

$$P(\Lambda^*) = a_1(\Lambda^*)2 + a_2(\Lambda^*) = 0.$$

Here

$$a_1 = \vartheta(m_p + \gamma_p)(m_p + \gamma_p')(\delta_p(\omega_p' - \omega_p) + m_p + \omega_p)$$

$$a_2 = \vartheta m_p(m_p + \gamma_p)(m_p + \gamma_p')(\delta_p(\omega_p' - \omega_p) + m_p + \omega_p)(1 - R_0)$$

As $a1 > 0, a_2 \geq 0$ if $R_0 < 1$, then $\Lambda^* = \frac{-a_2}{a_1} \leq 0$. Hence no pandemic equilibrium will lie if $R_0 \leq 1$. This implies that the endemic equilibrium exists and stable if $R_0 > 1$.

3 Existence, uniqueness of solution and numerical simulations

It is natural to ask whether a dynamical system that we are investigating exists or not in reality. Fixed point theory answer this question. We examine our considered problem (2) for existence results about the solution. Regarding this, we write the right sides of our problem (2) as:

$$\begin{cases} F_1(t, S_p, E_p, I_p, A_p, R_p, M) = n_p - S_p m_p - b_p S_p (I_p + \kappa A_p) - S_p M b_w, \\ F_2(t, S_p, E_p, I_p, A_p, R_p, M) = (I_p + \kappa A_p) b_p S_p + b_w S_p M - (1 - \delta_p)\omega_p E_p - E_p \delta_p \omega_p' - E_p m_p, \\ F_3(t, S_p, E_p, I_p, A_p, R_p, M) = (1 - \delta_p)\omega_p E_p - (\gamma_p + m_p) I_p, \\ F_4(t, S_p, E_p, I_p, A_p, R_p, M) = \delta_p \omega_p' E_p - (\gamma_p' + m_p) A_p, \\ F_5(t, S_p, E_p, I_p, A_p, R_p, M) = \gamma_p I_p + \gamma_p' A_p - m_p \gamma_p, \\ F_6(t, S_p, E_p, I_p, A_p, R_p, M) = \varepsilon I_p + \sigma A_p - \vartheta M, \\ S_p(0) = S_0,\ E_p(0) = E_0,\ I_p(0) = I_0,\ A_p(0) = A_0,\ R_p(0) = A_0,\ M(0) = M_0. \end{cases} \tag{8}$$

To symbolize the system (2) by using (8) as follows

$$\begin{aligned} {}^{ABC}\mathbf{D}^r_{+0}\mathcal{Y}(t) &= \Omega(t, \mathcal{Y}(t)),\ t \in [0, \tau],\ 0 < r \leq 1, \\ \mathcal{Y}(0) &= \mathcal{Y}_0. \end{aligned} \tag{9}$$

By applying integral in sense of $\mathcal{ABC}$ and by using lemma 2.3 we get

$$\mathcal{Y}(t) = \mathcal{Y}_0(t) + \frac{(1-r)}{ABC(r)}\Big[\Omega(t, \mathcal{Y}(t))\Big] + \frac{r}{ABC(r)\Gamma(r)}\int_0^t (t-z)^{r-1}\Omega(z, \mathcal{Y}(z))dz, \tag{10}$$

where,

$$\mathcal{Y}(t) = \begin{cases} S_p(t) \\ E_p(t) \\ I_p(t) \\ A_p(t) \\ R_p(t) \\ M(t) \end{cases},\quad \mathcal{Y}_0(t) = \begin{cases} S_0 \\ E_0 \\ I_0 \\ A_0 \\ R_0 \\ M_0 \end{cases},\quad \Omega(t, \mathcal{Y}(t)) = \begin{cases} F_1(S_p, E_p, I_p, A_p, R_p, M, t) \\ F_2(t, S_p, E_p, I_p, A_p, R_p, M) \\ F_3(t, S_p, E_p, I_p, A_p, R_p, M) \\ F_4(t, S_p, E_p, I_p, A_p, R_p, M) \\ F_5(t, S_p, E_p, I_p, A_p, R_p, M) \\ F_6(t, S_p, E_p, I_p, A_p, R_p, M). \end{cases} \tag{11}$$

Using (9) and define operators $\mathbf{F}$,$\mathbf{G}$ by using (10) as

$$\begin{aligned} \mathbf{F}(\mathcal{Y}) &= \mathcal{Y}_0(t) + \tfrac{(1-r)}{ABC(r)}\Big[\Omega(t, \mathcal{W}(t))\Big], \\ \mathbf{G}(\mathcal{Y}) &= \tfrac{r}{\Gamma(r)ABC(r)}\int_0^t (t-z)^{r-1}\Omega(Z, \mathcal{Y}(z))dz. \end{aligned} \tag{12}$$

Witting the growth condition and Lipschitz condition for solution's existence and uniqueness as given below.

(B1) Let we have a constants A_Ω,E_Ω, as:

$|\Omega(t, \mathcal{Y}(t))| \leq A_\Omega|\mathcal{Y}| + E_\Omega$.

(B2) Let we have a constants $L_\Omega > 0$, as for all $\mathcal{Y},\ \bar{Y} \in \mathbf{Y}$ as:

$|\Omega(t, \mathcal{Y}) - \Omega(t, \bar{Y})| \leq L_\Omega[|\mathcal{Y}| - \bar{Y}|]$.

Theorem 3.1 Under hypothesis $(B1,B2)$, the problem (10) has at least one solution which implies that the proposed model (2) has at least one solution if $\frac{(1-r)}{\mathcal{ABC}(r)}L_\Omega < 1$.

Proof The theorem can be proved by using the following two steps.

Step I: Consider $\bar{Y} \in \mathbf{B}$ and $\mathbf{B} = \{\mathcal{Y} \in \mathbf{Y} :\ \|\mathcal{Y}\| \leq \sigma, \sigma > 0\}$ is convex and close set. Then by $\mathbf{F}$ in (12), we obtain

$$\begin{aligned} \|\mathbf{F}(\mathcal{Y}) - \mathbf{F}(\bar{Y})\| &= \tfrac{(1-r)}{ABC(r)}\max_{t\in[0,\tau]}\Big|\Omega(t, \mathcal{Y}(t)) - \Omega(t, \bar{Y}(t))\Big| \\ &\leq \tfrac{(1-r)}{ABC(r)}L_\Omega\|\mathcal{Y} - \bar{Y}\|. \end{aligned} \tag{13}$$

Hence, $\mathbf{F}$ is contracted.

Step-II: To show that $\mathbf{G}$ is compact relative, it is enough to show that $\mathbf{G}$ is bounded and equi-continuous. Clearly, $\mathbf{G}$ is defined on their domain as Ω is defined on domain and for any $\mathcal{Y} \in \mathbf{B}$, we follow

$$\begin{aligned} \|\mathbf{G}(\mathcal{Y})\| &= \max_{t\in[0,\tau]}\|\tfrac{r}{ABC(r)\Gamma(r)}\textstyle\int_0^t (t-z)^{r-1}\Omega(z, \mathcal{Y}(z))dz\| \\ &\leq \tfrac{r}{ABC(r)\Gamma(r)}\textstyle\int_0^\tau (\tau-z)^{r-1}|\Omega(z, \mathcal{Y}(z))|dz \\ &\leq \tfrac{\tau^r}{ABC(r)\Gamma(r)}[A_\Omega\sigma + E_\Omega]. \end{aligned} \tag{14}$$

So, from (14) it is clear that $\mathbf{G}$ have bounds. Further, for equi-continuous let $t_1 > t_2 \in [0,\tau]$, we continue as

$$\begin{aligned} |\mathbf{G}(\mathcal{Y}(t_2) - \mathbf{G}(\mathcal{Y}(t_1)| &= \tfrac{r}{ABC(r)\Gamma(r)}\Big|\textstyle\int_0^{t_2}(t_2-z)^{r-1}\Omega(z, \mathcal{Y}(z))dz - \int_0^{t_1}(t_1-z)^{r-1}\Omega(z, \mathcal{Y}(z))dz\Big| \\ &\leq \tfrac{[A_\Omega\sigma + E_\Omega]}{ABC(r)\Gamma(r)}[t_2^r - t_1^r]. \end{aligned} \tag{15}$$

Equation (15) implies that as $t_2 \to t_1$ then the right side will approaches to zero. As, $\mathbf{G}$ is continuous and hence

$|\mathbf{G}(\mathcal{Y}(t_2) - \mathbf{G}(\mathcal{Y}(t_1)| \to 0,$ as $t_2 \to t_1$.

Hence as $\mathbf{G}$ have bounds and are continuous so

$\|\mathbf{G}(\mathcal{Y}(t_2) - \mathbf{G}(\mathcal{Y}(t_1)\| \to 0,$ as $t_2 \to t_1$.

Thus, $\mathbf{G}$ have bounds and equi-continuous operator. Also, from theorem of Arzelá-Ascoli, the operator $\mathbf{G}$ is relative compact and hence continuous completely. Thus, from Theorem 3.1, the integration Equation (10) has Als at least one solution and therefore the proposed problem has at least one solution.

For unique solution we give the given theorem.

Theorem 3.2 By hypothesis ($B2$), the integral Equation (10) has unique solution which yields that the system under consideration (2) has unique solution if:

$$\left[\frac{(1-r)L_\Omega}{\mathcal{ABC}(r)} + \frac{\tau^r L_\Omega}{\mathcal{ABC}(r)\Gamma(r)}\right] < 1.$$

Proof Consider the mapping $\mathbf{T} : \mathbf{Y} \to \mathbf{Y}$ defined by

$$\begin{aligned}\mathbf{T}\mathcal{Y}(t) &= \mathcal{Y}_0(t) + \left[\Omega(t,\mathcal{Y}(t)) - \Omega_0(t)\right]\tfrac{(1-r)}{\mathcal{ABC}(r)}\\ &+ \tfrac{r}{\mathcal{ABC}(r)\Gamma(r)}\int_0^t (t-z)^{r-1}\Omega(z,\mathcal{Y}(z))dz,\ t\in[0,\tau].\end{aligned} \tag{16}$$

Let $\mathcal{Y}, \bar{Y} \in \mathbf{Y}$, then one can take

$$\begin{aligned}\|\mathbf{T}\mathcal{Y} - \mathbf{T}\bar{Y}\| &\le \tfrac{(1-r)}{\mathcal{ABC}(r)}\max_{t\in[0,\tau]}\left|\Omega(t,\mathcal{Y}(t)) - \Omega(t,\bar{Y}(t))\right|\\ &+ \tfrac{r}{\mathcal{ABC}(r)\Gamma(r)}\max_{t\in[0,\tau]}\left|\int_0^t (t-z)^{r-1}\Omega(z,\mathcal{Y}(z))dz - \int_0^t (t-z)^{r-1}\Omega(z,\bar{Y}(z))dz\right|\\ &\le \Upsilon\|\mathcal{Y} - \bar{Y}\|,\end{aligned} \tag{17}$$

where

$$\Upsilon = \left[\tfrac{(1-r)L_\Omega}{\mathcal{ABC}(r)} + \tfrac{\tau^r L_\Omega}{\mathcal{ABC}(r)\Gamma(r)}\right]. \tag{18}$$

Hence, $\mathbf{T}$ is contracted from (17). So, the integration Equation (10) has one root. This implies that the problem (2) has one solution.

For approximate solution, we continue this part of manuscript to the proposed fractional order (2) model in sense of ABC operator. The iterative technique are then simulated on different fractional orders. For this, we use the arbitrary order AB iterative technique [56] to find the numerical scheme for the graphical representation of the problem (2). For model (8) we develop a numerical scheme as

$$\begin{cases} {}^{ABC}\mathcal{D}_t^r(S_p(t)) = \mathrm{F}_1(S_p(t),t) = n_p - m_pS_p - b_pS_p(I_p+\kappa A_p) - b_wS_pM,\\ {}^{ABC}\mathcal{D}_t^r(E_p(t)) = \mathrm{F}_2(E_p(t),t) = (I_p+\kappa A_p)b_pS_p + b_wS_pM - \omega_pE_p(1-\delta_p) - \delta_p\omega_p'E_p - E_pm_p,\\ {}^{ABC}\mathcal{D}_t^r(I_p(t)) = \mathrm{F}_3(I_p(t),t) = \omega_pE_p(1-\delta_p) - I_p(\gamma_p+m_p),\\ {}^{ABC}\mathcal{D}_t^r(A_p(t)) = \mathrm{F}_4(A_p(t),t) = \delta_p\omega_p'E_p - (\gamma_p'+m_p)A_p,\\ {}^{ABC}\mathcal{D}_t^r(R_p(t)) = \mathrm{F}_5(R_p(t),t) = \gamma_pI_p + \gamma_p'A_p - m_p\gamma_p,\\ {}^{ABC}\mathcal{D}_t^r(M(t)) = \mathrm{F}_6(M(t),t) = \varepsilon I_p + \sigma A_p - \vartheta M,\\ S_p(0)=S_0,\ E_p(0)=E_0,\ I_p(0)=I_0,\ A_p(0)=A_0,\ R_p(0)=A_0,\ M(0)=M_0. \end{cases} \tag{19}$$

Integrating first equation of (19) in $\mathcal{ABC}$ approach, we get

$$S_p(t) - S_p(0) = \tfrac{(1-r)}{\mathcal{ABC}(r)}\left[\mathrm{F}_1(S_p(t),t)\right] + \tfrac{r}{\mathcal{ABC}(r)\Gamma(r)}\int_0^t (t-z)^{r-1}\mathrm{F}_1(S_p(z),z)dz.$$

Consider $t = t_{i+1}$ for $i = 0,1,2\cdots$, it follows that

$$\begin{aligned}S_p(t_{i+1}) - S_p(0) &= \tfrac{(1-r)}{\mathcal{ABC}(r)}\left[\mathrm{F}_1(S_p(t_i),t_i)\right]\\ &+ \tfrac{r}{\mathcal{ABC}(r)\Gamma(r)}\int_0^{t_{i+1}}(t_{i+1}-z)^{r-1}\mathrm{F}_1(S_p(z),z)dz,\\ &= \tfrac{(1-r)}{\mathcal{ABC}(r)}\left[\mathrm{F}_1(S_p(t_i),t_i)\right]\\ &+ \tfrac{r}{\mathcal{ABC}(r)\Gamma(r)}\sum_{q=0}^{i}\int_q^{t_{q+1}}(t_{i+1}-z)^{r-1}\mathrm{F}_1(S_p(z),z)dz.\end{aligned}$$

Next, approximating the mapping $F_1(S_p(t),t)$ on time interval $[t_q,t_{q+1}]$, by the interpolating expression as follows:

$$F_1(S_p(t),t) \cong \frac{F_1(S_p(t_q),t_q)}{\Delta}(t-t_{q-1}) + \frac{F_1(S_p(t_{q-1}),t_{q-1})}{\Delta}(t-t_q)$$

or

$$\begin{aligned}S_p(t_{i+1}) &= S_p(0) + \tfrac{(1-r)}{\mathcal{ABC}(r)}\left[\mathrm{F}_1(S_p(t_i),t_i)\right]\\ &+ \tfrac{r}{\mathcal{ABC}(r)\Gamma(r)}\textstyle\sum_{q=0}^{i}\Big(\tfrac{F_1(S_p(t_q),t_q)}{\Delta}\int_q^{t_{q+1}}(t-t_{q-1})(t_{i+1}-t)^{r-1}dt\\ &- \tfrac{F_1(S_p(t_{q-1}),t_{q-1})}{\Delta}\int_q^{t_{q+1}}(t-t_q)(t_{i+1}-t)^{r-1}dt\Big)\\ &= S_p(0) + \tfrac{(1-r)}{\mathcal{ABC}(r)}\left[\mathrm{F}_1(S_p(t_i),t_i)\right]\\ &+ \tfrac{r}{\mathcal{ABC}(r)\Gamma(r)}\textstyle\sum_{q=0}^{i}\Big(\tfrac{F_1(S_p(t_q),t_q)}{\Delta}I_{q-1,r} - \tfrac{F_1(S_p(t_{q-1}),t_{q-1})}{\Delta}I_{q,r}\Big).\end{aligned} \tag{20}$$

Now the integrals $I_{q-1,r}$ and $I_{q,r}$ can be calculated as follow:

$$\begin{aligned}I_{q-1,r} &= \int_q^{t_{q+1}}(t-t_{q-1})(t_{i+1}-t)^{r-1}dt\\ &= -\tfrac{1}{r}\left[(t_{q+1}-t_{q-1})(t_{i+1}-t_{q+1})^r - (t_q-t_{q-1})(t_{i+1}-t_q)^r\right]\\ &- \tfrac{1}{r(r-1)}\left[(t_{i+1}-t_{q+1})^{r+1} - (t_{i+1}-t_q)^{r+1}\right],\end{aligned}$$

and

$$\begin{aligned}I_{q,r} &= \int_q^{t_{q+1}}(t-t_q)(t_{i+1}-t)^{r-1}dt\\ &= -\tfrac{1}{r}\left[(t_{q+1}-t_q)(t_{i+1}-t_{q+1})^r\right]\\ &- \tfrac{1}{r(r-1)}\left[(t_{i+1}-t_{q+1})^{r+1} - (t_{i+1}-t_q)^{r+1}\right],\end{aligned}$$

put $t_q = q\Delta$, we get

Table 2 Description of the parameters given in model (1) for Wuhan

Notation	Numerical value
n_P	0.073
m_p	0.00408
b_p	0.05
b_w	0.000001231
$\omega_p,\ \omega'_p$	0.1243, 0.005
γ_p, γ'_p	0.09871, 0.854302
$\varepsilon,\ \sigma$	0.1243, 0.01
ϑ	0.398
$\delta_p,$	0.1243
κ	0.02

$$\begin{aligned}
I_{q-1,r} &= -\tfrac{\Delta^{r+1}}{r}\Big[(q+1-(q-1))(i+1-(q+1))^r-(q-(q-1))(i+1-q)^r\Big]\\
&\quad-\tfrac{\Delta^{r+1}}{r(r-1)}\Big[(i+1-(q+1))^{r+1}-(i+1-q)^{r+1}\Big],\\
&= \tfrac{\Delta^{r+1}}{r(r-1)}\Big[-2(r+1)(i-q)^r+(r+1)(i+1-q)^r-(i-q)^{r+1}+(i+1-q)^{r+1}\Big],\\
&= \tfrac{\Delta^{r+1}}{r(r-1)}\Big[(i-q)^r(-2(r+1)-(i-q))+(i+1-q)^r(r+1+i+1-q)\Big],\\
&= \tfrac{\Delta^{r+1}}{r(r-1)}\Big[(i+1-q)^r(i-q+2+r)-(i-q)^r(i-q+2+2r)\Big],
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
I_{q,r} &= -\tfrac{\Delta^{r+1}}{r}\Big[(q+1-q)(i+1-(q+1))^r\Big]-\tfrac{\Delta^{r+1}}{r(r-1)}\Big[(i+1-(q+1))^{r+1}-(i+1-q)^{r+1}\Big],\\
&= \tfrac{\Delta^{r+1}}{r(r-1)}\Big[-(r+1)(i-q)^r-(i-q)^{r+1}+(i+1-q)^{r+1}\Big],\\
&= \tfrac{\Delta^{r+1}}{r(r-1)}\Big[(i-q)^r(-(r+1)-(i-q))+(i+1-q)^{r+1}\Big],\\
&= \tfrac{\Delta^{r+1}}{r(r-1)}\Big[(i+1-q)^{r+1}-(i-q)^r(i-q+1+r)\Big],
\end{aligned} \tag{22}$$

substitute (21) and (22) in (20), we get

$$\begin{aligned}
S_p(t_{i+1}) &= S_p(0)+\tfrac{(1-r)}{ABC(r)}\Big[\mathrm{F}_1(S_p(t_i),t_i)\Big]\\
&\quad+\tfrac{r}{ABC(r)}\sum_{q=0}^{i}\Bigg(\tfrac{F_1(S_p(t_q),t_q)}{\Gamma(r+2)}\Delta^r\Big[(i+1-q)^r(i-q+2+r)\\
&\quad-(i-q)^r(i-q+2+2r)\Big]\\
&\quad-\tfrac{F_1(S_p(t_{q-1}),t_{q-1})}{\Gamma(r+2)}\Delta^r[(i+1-q)^{r+1}-(i-q)^r(i-q+1+r)]\Bigg).
\end{aligned}$$

Similarly, the proposed method can be used for the remaining five equations of (19) to form general algorithms as

$$\begin{aligned}
E_p(t_{i+1}) &= E_p(0)+\tfrac{(1-r)}{ABC(r)}\Big[\mathrm{F}_2(E_p(t_i),t_i)\Big]\\
&\quad+\tfrac{r}{ABC(r)}\sum_{q=0}^{i}\Bigg(\tfrac{F_2(E_p(t_q),t_q)}{\Gamma(r+2)}\Delta^r\Big[(i+1-q)^r(i-q+2+r)\\
&\quad-(i-q)^r(i-q+2+2r)\Big]\\
&\quad-\tfrac{F_2(E_p(t_{q-1}),t_{q-1})}{\Gamma(r+2)}\Delta^r[(i+1-q)^{r+1}-(i-q)^r(i-q+1+r)]\Bigg).
\end{aligned}$$

$$\begin{aligned}
I_p(t_{i+1}) &= I_p(0)+\tfrac{(1-r)}{ABC(r)}\Big[\mathrm{F}_3(I_p(t_i),t_i)\Big]\\
&\quad+\tfrac{r}{ABC(r)}\sum_{q=0}^{i}\Bigg(\tfrac{F_3(I_p(t_q),t_q)}{\Gamma(r+2)}\Delta^r\Big[(i+1-q)^r(i-q+2+r)\\
&\quad-(i-q)^r(i-q+2+2r)\Big]\\
&\quad-\tfrac{F_3(I_p(t_{q-1}),t_{q-1})}{\Gamma(r+2)}\Delta^r[(i+1-q)^{r+1}-(i-q)^r(i-q+1+r)]\Bigg).
\end{aligned}$$

$$\begin{aligned}
A_p(t_{i+1}) &= A_p(0)+\tfrac{(1-r)}{ABC(r)}\Big[\mathrm{F}_4(A_p(t_i),t_i)\Big]\\
&\quad+\tfrac{r}{ABC(r)}\sum_{q=0}^{i}\Bigg(\tfrac{F_4(A_p(t_q),t_q)}{\Gamma(r+2)}\Delta^r\Big[(i+1-q)^r(i-q+2+r)\\
&\quad-(i-q)^r(i-q+2+2r)\Big]\\
&\quad-\tfrac{F_4(A_p(t_{q-1}),t_{q-1})}{\Gamma(r+2)}\Delta^r[(i+1-q)^{r+1}-(i-q)^r(i-q+1+r)]\Bigg).
\end{aligned}$$

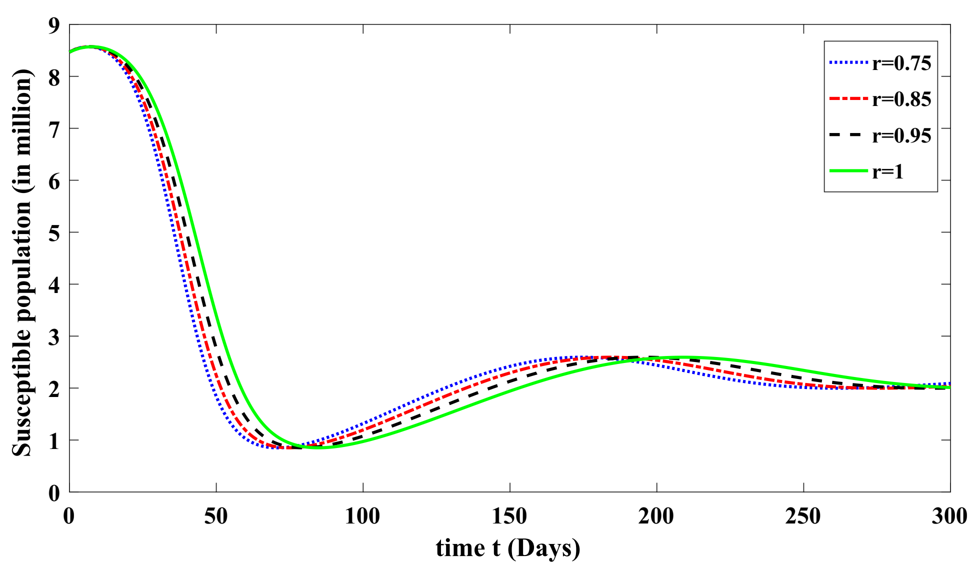

Fig. 1 Behavior of Susceptible population $S_p(t)$ at various arbitrary order r of the proposed system (2) for $h = 0.1, b_p = 0.05$

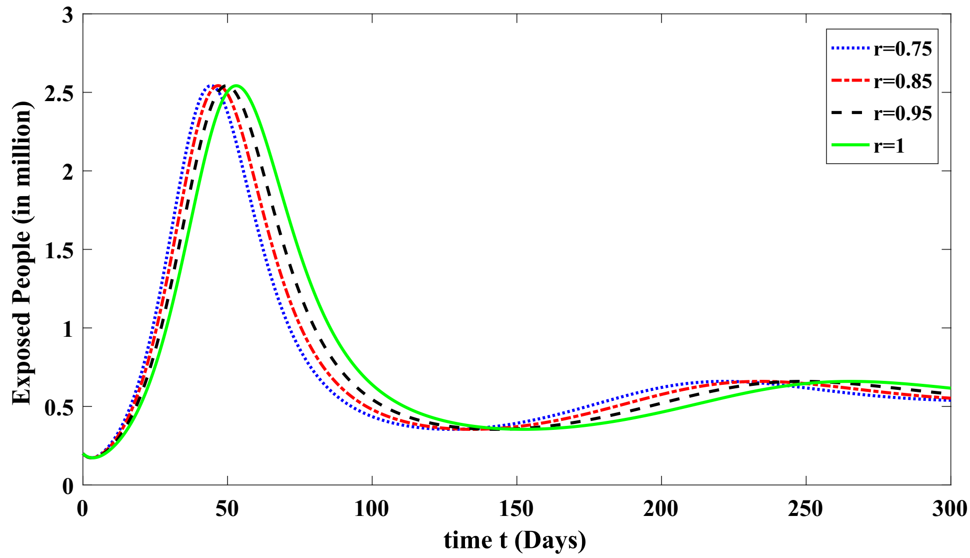

Fig. 2 Behavior of Exposed individuals $E_p(t)$ at various arbitrary order r of the proposed system (2) for $h = 0.1, b_p = 0.05$

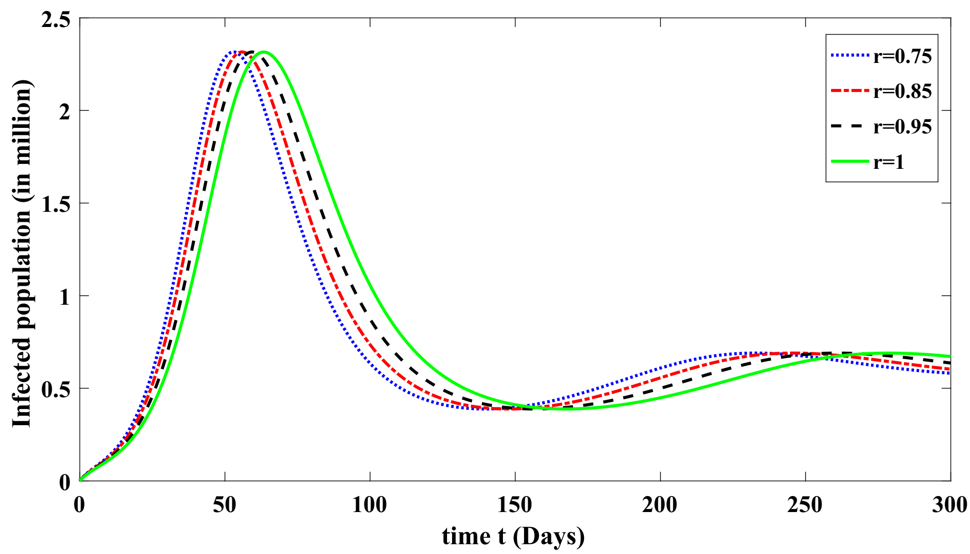

Fig. 3 Behavior of total infected population $I_p(t)$ at various arbitrary order r of the proposed system (2) for $h = 0.1, b_p = 0.05$

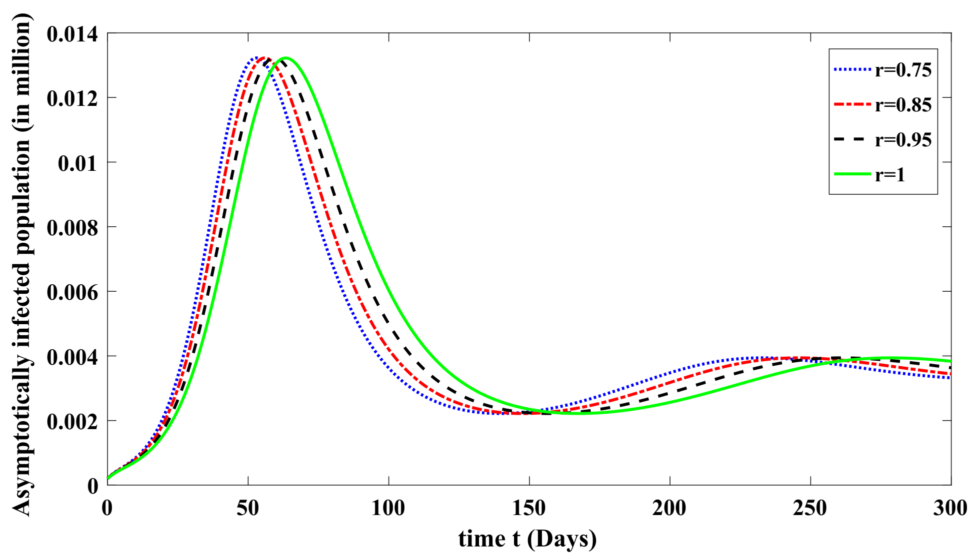

Fig. 4 Behavior of asymptotically infectious population $A_p(t)$ at various arbitrary order r of the proposed system (2) for $h = 0.1, b_p = 0.05$

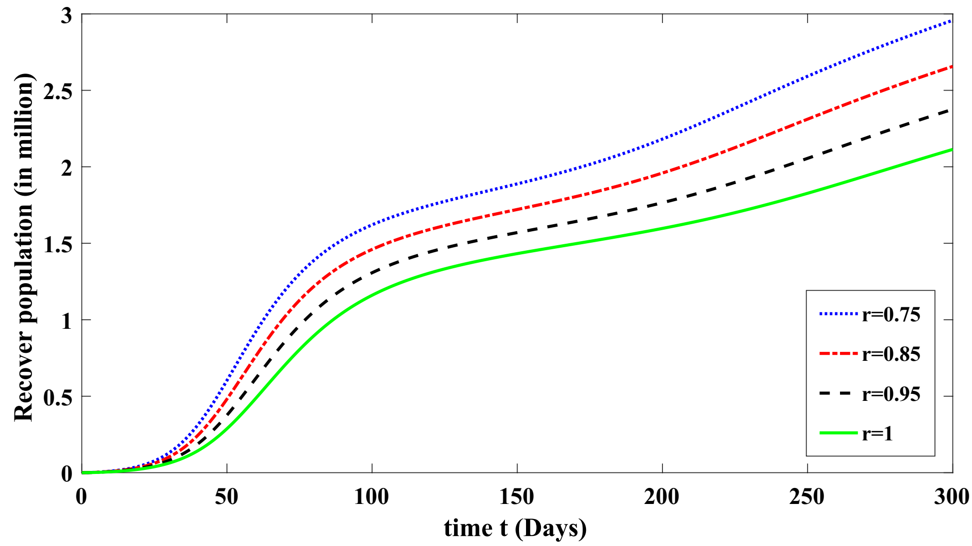

Fig. 5 Behavior of recovered population $R_p(t)$ at various arbitrary order r of the proposed system (2) for $h = 0.1, b_p = 0.05$

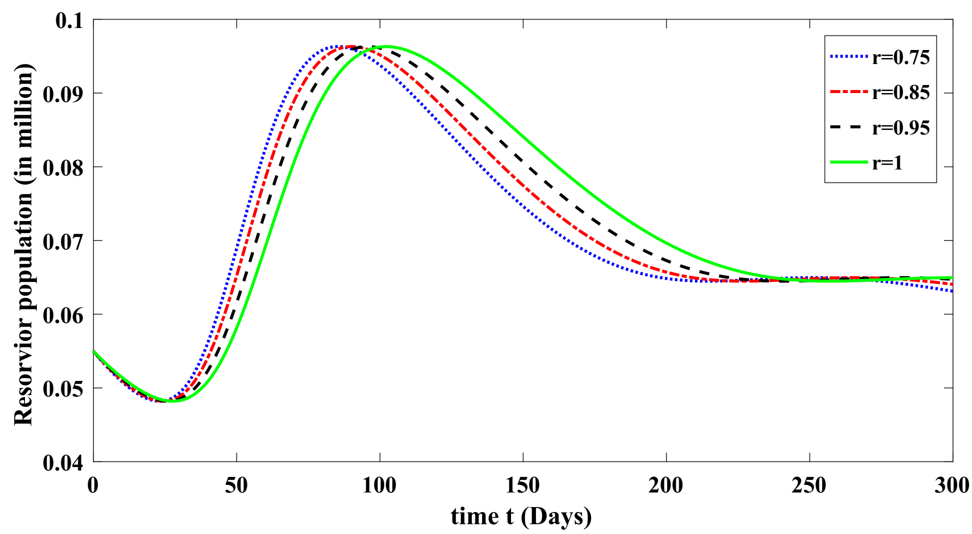

Fig. 6 Behavior of reservoir population $M(t)$ at various arbitrary order r of the proposed system (2) for $h = 0.1, b_p = 0.05$

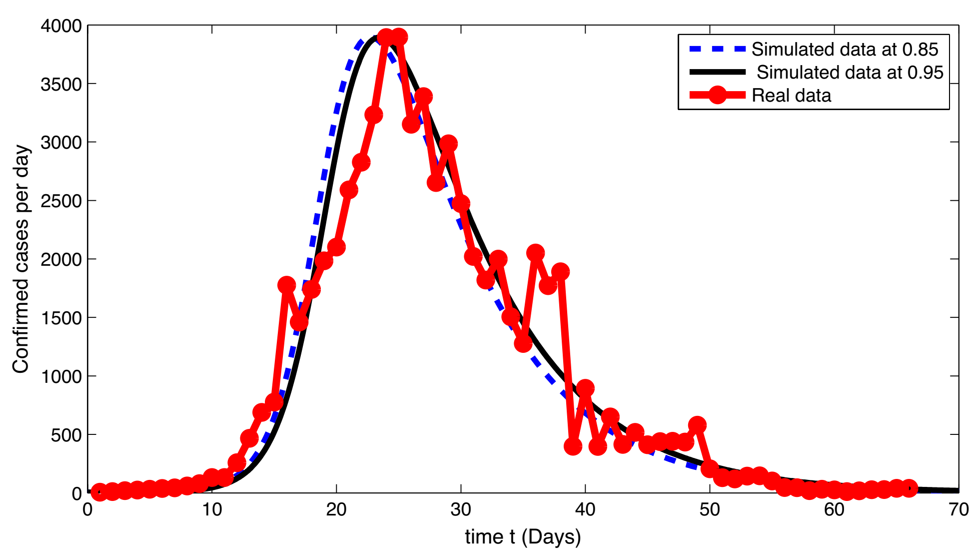

Fig. 7 Behavior of all populations at various arbitrary order r of the proposed system (2) for $h = 0.05, b_p = 0.5$

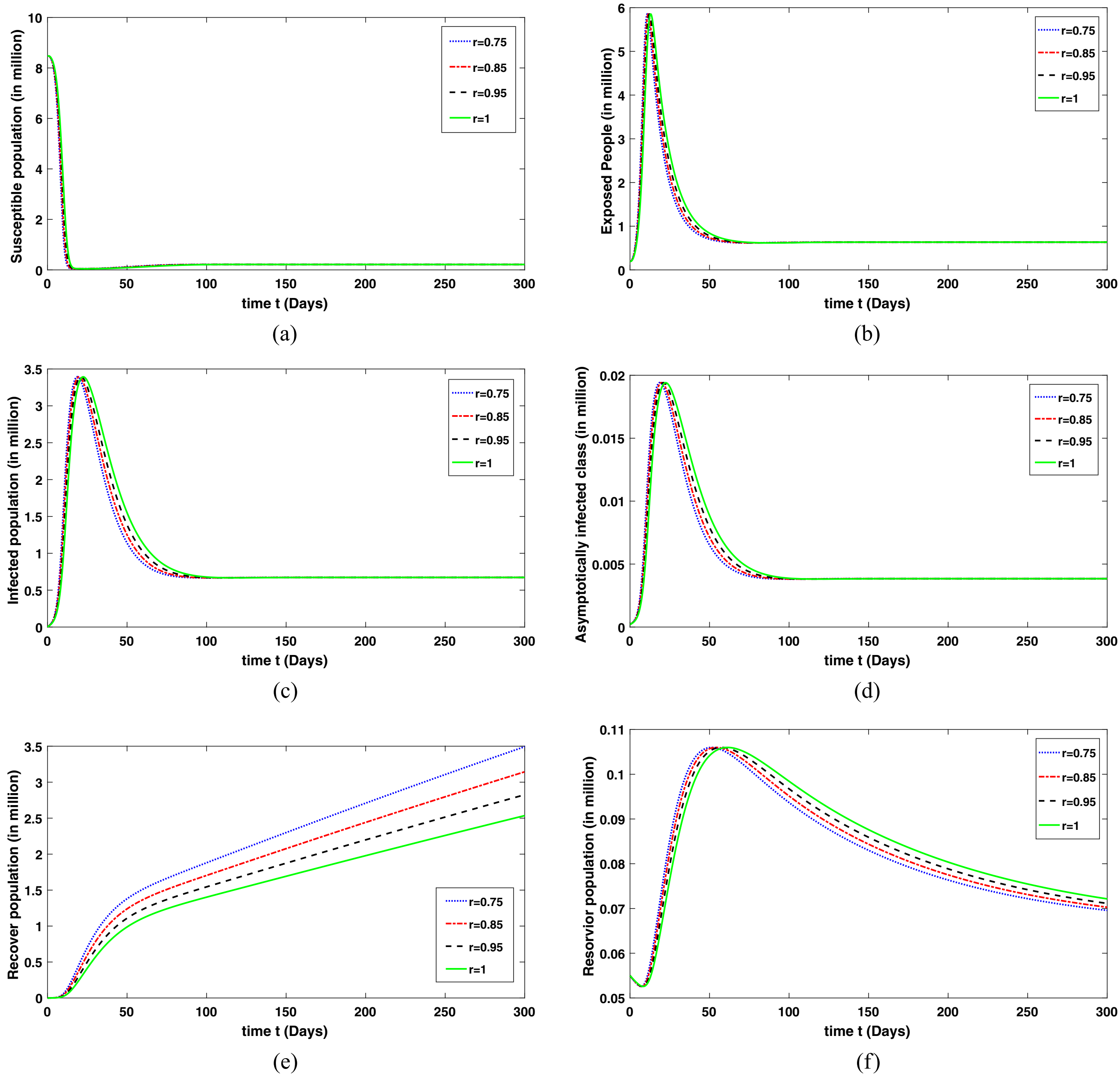

Fig. 8 Behavior of all populations at various arbitrary order r of the proposed system (2) for $h = 0.01, b_p = 0.1$

$$\begin{aligned} R_p(t_{i+1}) = {} & R_p(0) + \frac{(1-r)}{\mathcal{ABC}(r)}\Big[\mathrm{F}_5(R_p(t_i), t_i)\Big] \\ & + \frac{r}{\mathcal{ABC}(r)} \sum_{q=0}^{i} \Bigg(\frac{\mathrm{F}_5(R_p(t_q), t_q)}{\Gamma(r+2)} \Delta^r \Big[(i+1-q)^r (i-q+2+r) \\ & -(i-q)^r (i-q+2+2r)\Big] \\ & - \frac{\mathrm{F}_5(R_p(t_{q-1}), t_{q-1})}{\Gamma(r+2)} \Delta^r [(i+1-q)^{r+1} - (i-q)^r (i-q+1+r)] \Bigg). \end{aligned}$$

$$\begin{aligned} M(t_{i+1}) = {} & M(0) + \frac{(1-r)}{\mathcal{ABC}(r)}\Big[\mathrm{F}_6(M(t_i), t_i)\Big] \\ & + \frac{r}{\mathcal{ABC}(r)} \sum_{q=0}^{i} \Bigg(\frac{\mathrm{F}_6(M(t_q), t_q)}{\Gamma(r+2)} \Delta^r \Big[(i+1-q)^r (i-q+2+r) \\ & -(i-q)^r (i-q+2+2r)\Big] \\ & - \frac{\mathrm{F}_6(M(t_{q-1}), t_{q-1})}{\Gamma(r+2)} \Delta^r [(i+1-q)^{r+1} - (i-q)^r (i-q+1+r)] \Bigg). \end{aligned}$$

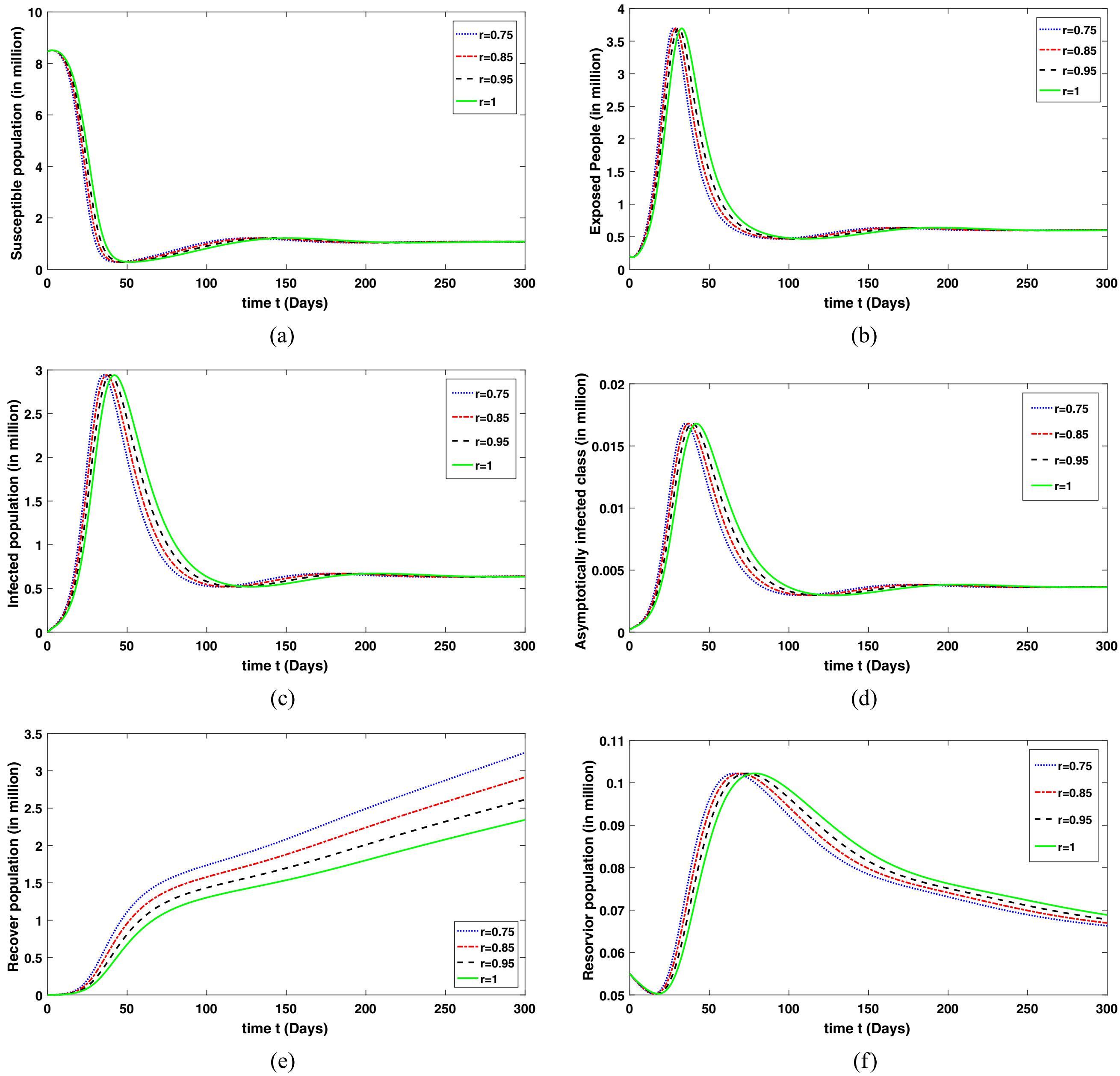

Fig. 9 Comparison between real and simulated data at given different fractional orders for our proposed model

We apply the above procedure to simulate the proposed model in the next section.

4 Experimental results and discussion

The numerical simulation is obtained from the results using data given in Table 2, from [51] with some updating assuming data. Further, the initial data are in million, we have taken in percentage as $S_p(0) = 8.465518$, $E_p(0) = 0.2$, $I_p(0) = 0.0002$, $A_p(0) = 0.0002$, $R_p(0) = 0.0002$, $M(0) = 0.055$, and the values of parameters are given in Table 2.

From Fig. 1, it is observed that increasing rate of transmission will decrease the number of susceptible individuals, and in return, it will increase number of infected population. In other words isolation and keeping social distance will greatly help in controlling the current outbreak for further spreading. The decrease of susceptible has been shown on different fractional order. The order of derivatives has also produced certain impact on the process, initially at smaller order the process of decay is faster than the higher order

Table 3 Description of the parameters given in model (1) for Pakistan

Notation	Numerical value
n_P	0.4673
m_p	0.018
b_p	0.083
b_w	0.000001231
ω_p, ω'_p	0.1243, 0.005
HCode γ_p, γ'_p	0.09871, 0.854302
ε, σ	0.1243, 0.01
ϑ	0.398
δ_p,	0.1243
κ	0.02

and vice versa. After 60 days the susceptible population decreases and then going towards convergency and stability.

From Fig. 2, we conclude that initially the number of exposed individuals is growing up to 50 days for about all orders of derivatives. After that the number decreases gradually, but at this time the decrease occurs differently at different order of derivative. It means that exposed population increases as the symptoms is recognized initially, when the outbreak of pandemic starts. After 40 or 45 days the exposed class begins to decay and then become constant or stable

From Fig. 3, we see that at the given data the number of infected cases are less than that of susceptible and exposed ones. Here up to 50 days the rate of increase is very high and same for all about orders of derivatives, but after that as the transmission of people from place to place decreases the infection is decreasing. The concerned increase and decrease in population are different due to different fractional orders. The infection reached to the peak value on 50^{th} day and then decays to some certain value of 10 infected cases per day, showing stability or convergency.

From Fig. 4, we conclude that initially the asymptotic infected population slightly increases up to 50 days, as the number of infected individuals at this stage is on peak. After that the number decreases gradually or very slowly at different orders of derivative. The number of this class after the 100^{th} day shows stability and became constant. As decrease and increase in this class are very very low or small as compared to other classes, therefore it is known as asymptotic class.

Figure 5 shows that the recovery from disease at the beginning is low as the numbers of infected and exposed are very low. But after some protective actions and precautionary measures, the number of recovered population increases. Here, the recovered population is different for different order of derivatives up to 300 days. After that the increases that occur in the recovered case are different at different fractional order.

Figure 6 demonstrates that the number of reservoir population decreases up to 35 days at different fractional order. After that the number of this class increases as compared to other classes with the passage of time. This means that as no precautionary actions are taken in the society more people will be infected but they will be unaware of their infection which will become cause or reservoir for infection in the future. It means that large number of population will be reserved (infection lies in their bodies) but with the passage of time it is also controlled.

Furthermore, to check the sensitivity of the fractional order model by changing the values of contact rate b_p and step size h in Fig. 7a–f.

Another set has been given from Fig. 8a–f by taking $h = 0.01, b_p = 0.1$.

Here in Fig. 9, we have compared the reported real data [58] of infected cases in Wuhan city from 4th January 2020 to 8th March 2020 for 67 days as [6,12,19,25,31,38,44,60,80,131,131,259,467,688,776,1776,1460,1739,1984,2101,2590,2827,3233,3892,3697,3151,3387,2653,2984,2473,2022,1820,1998,1506,1278,2051,1772,1891,399,894,397,650,415,518,412,439,441,435,579,206,130,120,143,146,102,46,45,20,31,26,11,18,27,29,39,39]

We see that the simulated data has close agreement with the plot of the real data. This phenomenon demonstrates the efficiency of our numerical results (Fig. 10).

Next we use the second data for Pakistan as in [59]. The initial population is $S_p(0) = 220$, $E_p(0) = 120, I_p(0) = 1.30$, $A_p(0) = 0.3$, $R_p(0) = 1.02$, $M(0) = 1.055$. The parameters values are given in Table 2 [59] (Table 3).

Hence present comparison between real data and simulated data Fig. 11. The confirmed cases in Pakistan per day reported in [60] from the 1 March 2021 to 15th of September 2021 for 200 days as [4,4,5,5,5,5,5,6,15,17,18,19,19,31,51,182,245,331,439,485,629,758,856,962,1034,1171,1139,1454,1554,1836,19972262,2520,2646,2899,3058,3549,3735,3852,3902,4162,4150,4307,4362,4824,5143,5122,5660,6043,6742,7286,7703,8479,8925,9438,10103,10586,11058,11747,11996,12380,12900,13818,14498,14814,15716,16370,17574,18003,20267,21587,22037,23268,25609,26003,26230,27054,27904,29266,30503,31775,32578,34386,34642,36228,37657,38150,38900,39690,40358,40880,42687,44777,47607,50234,53300,56144,59394,63400,57170,60470,75053,78699,83182,79700,84762,85321,89583,93233,97690,100324,104648,105087,106142,107733,107270,107607,107460,107784,106023,106775,106361,108100,108466,103543,95388,95241,95219,94522,91408,90358,89250,87345,86770,84234,77418,77360,73536,60234,57668,53431,53333,52203,51057,50080,40242,29274,29626,27189,26191,25279,24983,24941,24912,24908,24935,24827,20597,19230,18253,17573,17548,17555,17588,17103,16229,16685,16014,16001,13706,13385,12464,11697,11542,10378,10446,9940,9356,8739,8555,8585,8500,8553,8623,8633,8564,8512,8660,8883,6020,6234,6477,6545,5291,5546,5979,5786,5582,5525]

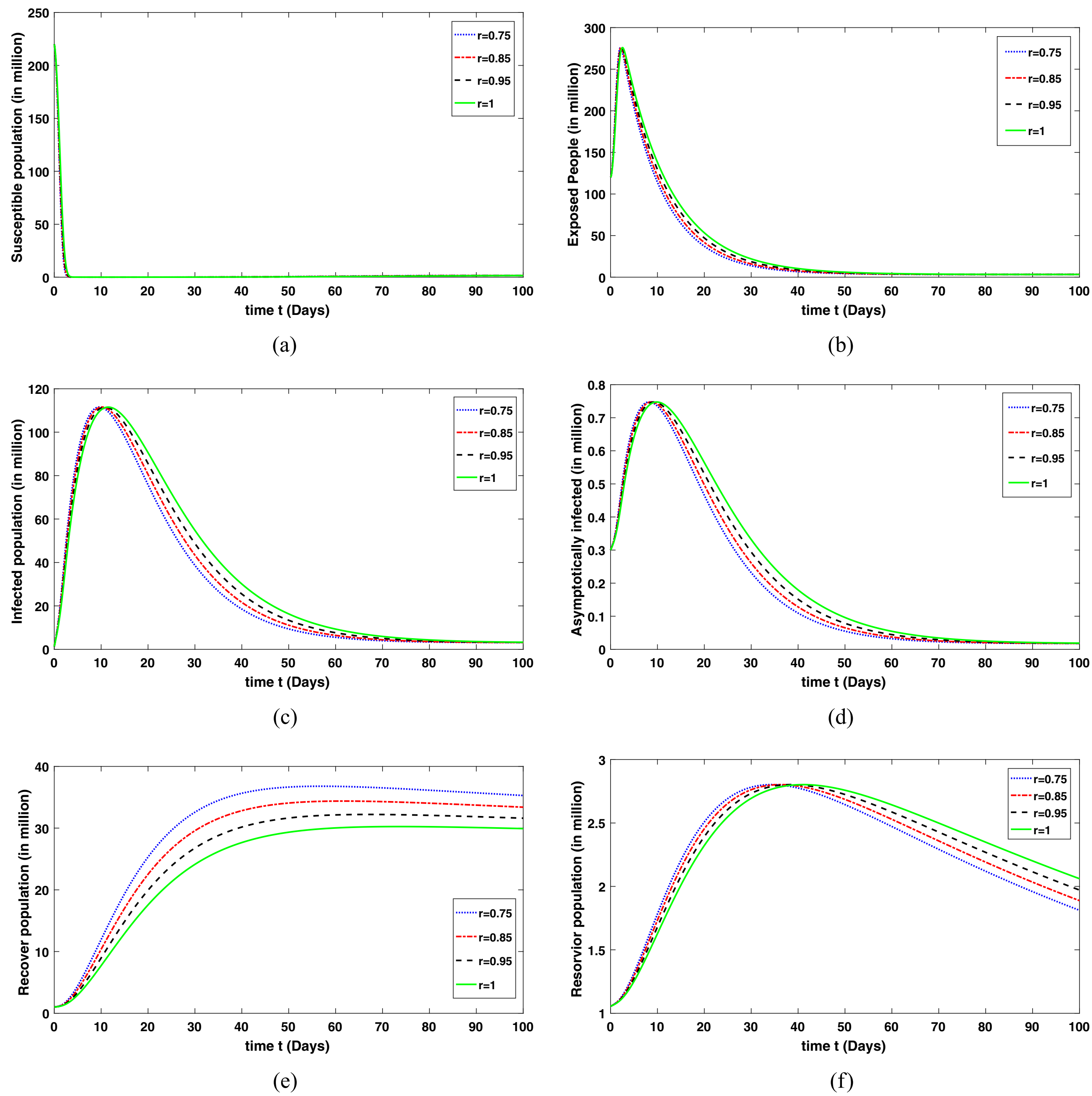

Fig. 10 Behavior of all populations at various arbitrary order r of the proposed system (2) for $h = 0.01$

5 Concluding remarks

Assigning different experimental values taken from [51, 59] to the parameters of (2), we have performed the required simulations as compared to integer order simulation of the system (1). We noticed that by increasing rate of protection, cure and decrease rate of transmission, the minimization and stabling in the numbers of infected individuals can be achieved. By studying such dynamical system, one can know how to control the population from being infected and isolation of infected ones from transmission (immigration). This will be very easy for policy makers and health sector to implement precautionary measures. We can predict for future on the basis of basic reproductive number. From epidemiological point of view it will be very interesting for medical science researchers to know about the history (past), present and future of infection by investigating such type of fractional mathematical model for the pandemic. This model can be applied to the population where social gathering occurs locally or globally. Further, by using fixed point theory the solution of the considered fractional dynamical system has been proved for the existence and uniqueness, while the rate of decaying and growth has been shown through global ways.

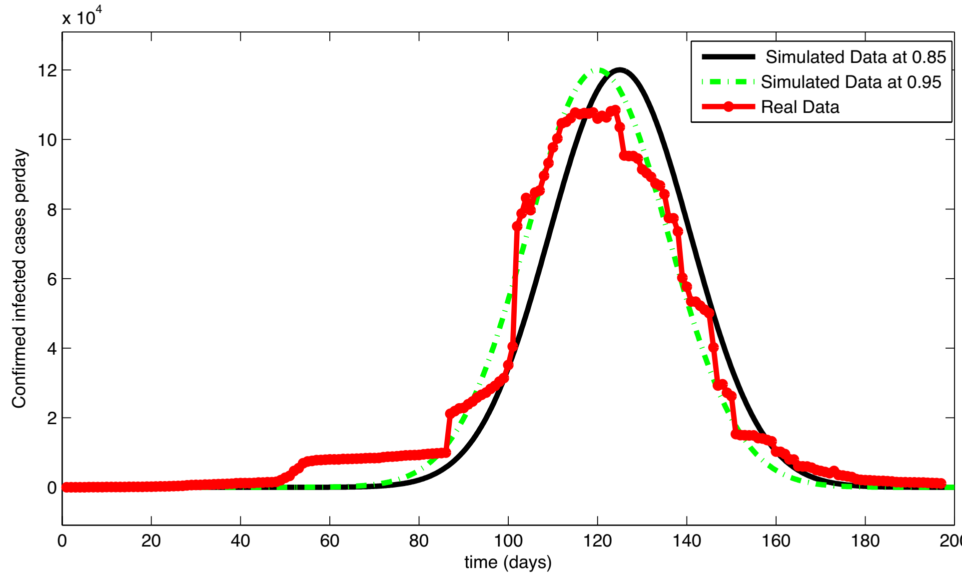

Fig. 11 Comparison between real and simulated data at given different fractional orders for our proposed model

Hence, fractional calculus can be used for comprehensive explanation of various dynamical models. Further, we observe that increasing precautionary measures will increase the recovered population. The data of Wuhan and Pakistan have been used to demonstrate the model. Also we have compared our simulated data with some reported real data of Wuhan and Pakistan for infected population. We have observed that both simulated and real data plots closely agreed. This shows that the established results are true and applicable. These types of models usually provide interesting indications for future planning and understanding the transmission dynamics of the disease.

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Declarations

Conflict of interest The authors declare no competing interests.

References

1. Is the World Ready for the Coronavirus. Editorial, The New York Times, 29 January 2020, Archived from the original on 30 January 2020
2. China virus death toll rises to 41, more than 1,300 infected worldwide, CNBC, 24 January 2020. Archived from the original on 26 January 2020. Retrieved 26 January 2020. Retrieved 30 January 2020
3. Zhai P, Ding Y, Wu X, Long J, Zhong Y, Li Y (2020) The epidemiology, diagnosis and treatment of COVID-19. Int J Antimicrob Agents 55(5):105955
4. Chowdhury SD, Oommen AM (2020) Epidemiology of COVID-19. J Dig Endosc 11(01):03–07
5. Lipsitch M, Swerdlow DL, Finelli L (2020) Defining the epidemiology of Covid-19-studies needed. N Engl J Med 382(13):1194–1196
6. Tyrrell DA, Bynoe ML (1966) Cultivation of viruses from a high proportion ofpatients with colds. Lancet 1:76–77
7. Kahn JS, McIntosh K (2005) History and recent advances in coronavirus discovery. Pediatr Infect Dis J 24(11):223–227
8. Lu H, Stratton CW, Tang YW (2020) Outbreak of pneumonia of unknown etiology in Wuhan, China: The mystery and the miracle. J Med Virol 92(4):401
9. Goyal M, Baskonus HM, Prakash A (2019) An efficient technique for a time fractional model of lassa hemorrhagic fever spreading in pregnant women. Eur Phys J Plus 134(481):1–10
10. Gao W, Veeresha P, Prakasha DG, Baskonus HM, Yel G (2020) New approach for the model describing the deathly disease in pregnant women using Mittag-Leffler function. Chaos Solit Fractals 134:109696
11. Kumar D, Singh J, Al-Qurashi M, Baleanu D (2019) A new fractional SIRS-SI malaria disease model with application of vaccines, anti-malarial drugs, and spraying. Adv Differ Equ 278:1–10
12. Shah K, Alqudah MA, Jarad F, Abdeljawad T (2020) Semi-analytical study of Pine Wilt Disease model with convex rate under Caputo-Febrizio fractional order derivative. Chaos Solitons Fractals 135:109754
13. Shahid A, Huang HL, Khalique CM, Bhatti MM (2021) Numerical analysis of activation energy on mhd nanofluid flow with exponential temperature-dependent viscosity past a porous plate. J Therm Anal Calorim 143(3):2585–2596
14. Zhang L, Bhatti MM, Marin M, Mekheimer KhS (2020) Entropy analysis on the blood flow through anisotropically tapered arteries filled with magnetic zinc-oxide (zno) nanoparticles. Entropy 22(10):1070
15. Bhatti MM, Alamri SZ, Ellahi R, Abdelsalam S (2020) Intra-uterine particle-fluid motion through a compliant asymmetric tapered channel with heat transfer. J Ther Anal Calor 21:1–9

16. Shi Z et al (2020) Estimating the serial interval of the novel coronavirus disease (COVID-19): a statistical analysis using the public data in Hong Kong from January 16 to February 15, 2020. Frontiers in Physics 8:347
17. Zhao SS et al (2020) Preliminary estimation of the basic reproduction number of novel coronavirus (2019-nCoV) in China, from 2019 to 2020: a data-driven analysis in the early phase of the outbreak. Int J Infect Dis 92:214–217
18. Riou J, Althaus CL (2020) Pattern of early human-to-human transmission of Wuhan 2019 novel coronavirus (2019-nCoV), December 2019 to January 2020. Eurosurveillance 25(4):2000058
19. Li X et al (2020) Transmission dynamics and evolutionary history of 2019-nCoV. J Med Virology 92(5):501–511
20. Mahase E (2020) Coronavirus: UK screens direct flights from Wuhan after US case, B. Med J 368:m265
21. Li QQ et al (2020) Early transmission dynamics in Wuhan, China, of novel Coronavirus-infected pneumonia. N Engl J Med 382:1199–1207
22. Zhang X (2020) Epidemiology of COVID-19. N Engl J Med 382(19):1869
23. Podlubny I (1999) Fractional Differential Equations, Mathematics in Science and Engineering. Academic Press, New York
24. Lakshmikantham V, Leela S, Vasundhara J (2009) Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge, UK
25. Hilfer R (2000) Applications of Fractional Calculus in Physics. World Scientific, Singapore
26. Rossikhin YA, Shitikova MV (1997) Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl Mech Rev 50:15–67
27. Kilbas AA, Marichev OI, Samko SG (1993) Fractional Integrals and Derivatives (Theory and Applications). Gordon and Breach, Switzerland
28. Toledo-Hernandez R, Rico-Ramirez V, Iglesias-Silva GA, Diwekar UM (2014) A fractional calculus approach to the dynamic optimization of biological reactive systems. Part I: Fractional models for biological reactions. Chemecal Engineering Science 117:217–228
29. Miller KS, Ross B (1993) An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York
30. Kilbas AA, Srivastava H, Trujillo J (1996) Theory and Application of Fractional Differential Equations North Holland Mathematics Studies, vol 204. Elseveir, Amsterdam
31. Shah K, Jarad F, Abdeljawad T (2020) On a nonlinear fractional order model of dengue fever disease under Caputo-Fabrizio derivative. Alex Eng J 59(4):2305–2313
32. Biazar J (2006) Solution of the epidemic model by Adomian decomposition method. Appl Math Comput 173:1101–1106
33. Rafei M, Ganji DD, Daniali H (2007) Solution of the epidemic model by homotopy perturbation method. Appl Math Comput 187:1056–1062
34. Rafei M, Daniali H, Ganji DD (2007) Variational iteration method for solving the epidemic model and the prey and predator problem. Appl Math Comput 186:1701–1709
35. Lin Q et al (2020) A conceptual model for the coronavirus disease 2019 (COVID-19) outbreak in Wuhan, China with individual reaction and governmental action. Int J Infec Diseases 93:211–216
36. Richard K, Wilhelm HD (2016) Numerical Methods and Modelling for Engineering. Springer, New York
37. Naz R, Naeem I (2019) The approximate Noether symmetries and approximate first integrals for the approximate Hamiltonian systems. Nonlinear Dyn 96:2225–2239
38. Al-Refai M, Abdeljawad T (2017) Analysis of the fractional diffusion equations with fractional derivative of non-singular kernel. Advances in Difference Equations, (2017): 315
39. Abdeljawad T (2017) Fractional operators with exponential kernels and a Lyapunov type inequality. Advances in Difference Equations, (2017): 313
40. Shatha H et al (2020) Atangana-Baleanu fractional framework of reproducing kernel technique in solving fractional population dynamics system. Chaos, Solitons Fractals 133:109624
41. Khan SA, Shah K, Jarad F, Zaman G (2019) Existence theory and numerical solutions to smoking model under Caputo-Fabrizio fractional derivative. Chaos 29(1):013128
42. Shah K, Alqudah MA, Jarad F, Abdeljawad T (2020) Semi-analytical study of Pine Wilt Disease model with convex rate under Caputo-Fabrizio fractional order derivative. Chaos, Solitons Fractals 135:109754
43. Haq F, Shah K, Rahman G, Shahzad M (2018) Numerical solution of fractional order smoking model via Laplace Adomian decomposition method. Alex Eng J 57(2):1061–1069
44. Haq F, Shah K, Rahman G, Shahzad M (2017) Numerical analysis of fractional order model of HIV-1 infection of CD4+ T-cells. Comput Method Diff Equ 5(1):1–11
45. Iserles A (1996) A First Course in the Numerical Analysis of Differential Equations. Cambridge University Press
46. Butcher JC (2003) Numerical Methods for Ordinary Differential Equations. John Wiley, New York
47. Naghipour A, Manafian J (2015) Application of the Laplace adomian decomposition method and implicit methods for solving Burger's equation. J Pure Apple Math 6(1):68–77
48. Rida SZ, Abdel Rady AS, Arafa AAM, Khalil M (2012) Approximate analytical solution of the fractional epidemic model. IJMR 1:17–19
49. Brailsford SC et al (2009) An analysis of the academic literature on simulation and modelling in health care. J Simul 3(3):130–140
50. Rappaz J, Touzani R (1992) On a two-dimensional magnetohydrodynamic problem: modelling and analysis. Math Model Numer Anal 26(2):347–364
51. Veeresha P et al (2020) Novel dynamical structures of COVID-19 with nonlocal operator via new computational technique. Biology 9(5):107
52. Khan MA, Atangana A (2020) Modeling the dynamics of novel coronavirus (2019-nCov) with fractional derivative. Alexandria Eng J 59(4):2379–2389
53. Porwal P, Shrivastava P, Tiwari SK (2015) Study of simple SIR epidemic model. Adv Appl Sci Res 6(4):1–4
54. Abdeljawad T, Baleanu D (2016) Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels. Adv Difference Equ 1:1–18
55. Burton TA (2019) Krasnoselskii N-tupled fixed point theorem with applications to fractional nonlinear dynamical system, Advances in Mathematical Physics 2019, Article ID 6763842 9 pages
56. Gómez JF et al (2019) Fractional Derivatives with Mittag-Leffler Kernel. Springer
57. Richardson BC (1979) Limitations on the use of mathematical models in transportation policy analysis, University Microfilms International. Ann Arbor, Mich
58. COVID-19 Coronavirus Pandemic, https://www.worldometers.info/coronavirus/reprot, Accessed March 26, 2020
59. https://www.worldometers.info/world-population/pakistan-population/ 25 January 2022
60. https://covid.gov.pk/ Situation of covid-19 in Pakistan from March 2021 to September 2021

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Analysis of two versions of relaxed inertial algorithms with Bregman divergences for solving variational inequalities

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Abstract
In this paper, we introduce and analyze two new inertial-like algorithms with the Bregman divergences for solving the pseudomonotone variational inequality problem in a real Hilbert space. The first algorithm is inspired by the Halpern-type iteration and the subgradient extragradient method and the second algorithm is inspired by the Halpern-type iteration and Tseng's extragradient method. Under suitable conditions, we prove some strong convergence theorems of the proposed algorithms without assuming the Lipschitz continuity and the sequential weak continuity of the given mapping. Finally, we give some numerical experiments with various types of Bregman divergence to illustrate the main results. In fact, the results presented in this paper improve and generalize the related works in the literature.

Keywords Bregman divergence · Hilbert space · Strong convergence · Variational inequality problem · Pseudomonotone mapping

Mathematics Subject Classification 47H09 · 47H10 · 47J25 · 47J05

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1 Introduction

Hartman and Stampacchia (1966) first introduced the variational inequality problem (VIP) for used in the study of partial differential equations with unilateral boundary conditions and free boundary value problems of elliptic type from mechanics. The VIP has been intensively and wildly studied and it has been found that it also can be applied to real-world problems such as equilibrium problems, optimal control problems, machine learning, signal processing and linear inverse problems (see, for example, Cholamjiak et al. 2020; Combettes and Pesquet 2020; Jolaoso 2021; Juditsky and Nemirovski 2016; Kinderlehrer and Stampacchia 1980; Luo and Zhang 2017; Tan et al. 2021).

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty, closed and convex subset of H and $A : C \to H$ be a given mapping. The *variational inequality problem* (shortly, (VIP)) is formulated as follows:

$$\text{Find a point} \quad z \in C \quad \text{such that} \quad \langle Az, x - z \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

We denote by $VI(C, A)$ the solution set of the problem (VIP). A concrete example of the problem (VIP) is the problem of solving a system of some equations. Clearly, if $C = H = \mathbb{R}^m$, then

$$z \in VI(C, A) \Longleftrightarrow Az = 0.$$

Another example of VIP is the constrained optimization problem. In fact, if we set $A := \nabla f$, where ∇f is the gradient of a continuously differentiable convex function f, then $z \in VI(C, A)$ if and only if z solves the following *minimization problem*:

$$\min_{x \in C} f(x), \tag{1.2}$$

where C is a closed and convex subset of $\mathbb{R}^m$. It is also known that the problem (VIP) can equivalently be rewritten as the following *fixed point equation* involving the metric projection P_C of H onto C:

$$z = P_C(z - \lambda Az), \tag{1.3}$$

where $\lambda > 0$.

There are various methods for solving the problem (VIP). One well-known method to solve the problem (VIP) is the *extragradient method* (EGM), which was originally introduced by Antipin (1976) for solving the saddle point problem, and was later extended by Korpelevich (1976) to the problem (VIP) in the finite-dimensional Euclidean space. The method (EGM) is of the following form: for each $n \geq 1$,

$$\begin{cases} x_1 \in \mathbb{R}^m, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{1.4}$$

where A is a monotone and L-Lipschitz continuous mapping and $\lambda \in \left(0, \frac{1}{L}\right)$.

The algorithm converges to a point of $VI(C, A)$ provided that $VI(C, A)$ is nonempty. In recent years, the method (EGM) was widely extended to infinite dimensional Hilbert spaces by many authors (see, for example, Censor et al. 2011; Iusem and Svaiter 1997; Popov 1980; Tseng 2000). It is remarked that this method requires calculating two projections onto C and two evaluations of A in each iteration. However, this may be difficult when the feasible set C has complicated structures.

In order to overcome some disadvantages of the method (EGM), Censor et al. (2011) replaced the second projection onto C of the method EGM by a projection onto a half-space, which significantly reduces the difficulty of calculating projection onto the whole feasible set twice. This method is called the *subgradient extragradient method* (SEGM), which is of the following form: for each $n \geq 1$,

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \\ T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}, \end{cases} \tag{1.5}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$. The weak convergence of SEGM was established provided that $VI(C, A)$ is nonempty.

On the other hand, Tseng (2000) proposed a single projection method known as *Tseng's extragradient method* (TEGM) and is of the following form: for each $n \geq 1$,

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n), \end{cases} \tag{1.6}$$

where A is monotone and L-Lipschitz continuous and $\lambda \in \left(0, \frac{1}{L}\right)$. He proved that this method converges weakly to a point of $VI(C, A)$. Note that this method only requires calculating one projection onto the feasible set C in each iteration, which is simple than the original method (EGM).

Another important method which overcomes the challenges in the method (EGM) is *Popov's subgradient extragradient method* (PSEGM), which was introduced by Malitsky and Semenov (2014). They improved the method (EGM) by combining the advantages of the method (SEGM) and Popov's extragradient method introduced by Popov (1980), which is of the following form: for each $n \geq 1$,

$$\begin{cases} y_0, x_1, y_1 \in H, \\ y_{n+1} = P_C(x_{n+1} - \lambda A y_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \\ T_n = \{x \in H : \langle x_n - \lambda A y_{n-1} - y_n, x - y_n \rangle \leq 0\}. \end{cases} \tag{1.7}$$

It was proved that the method (PSEM) converges weakly to point of $VI(C, A)$ provided $\lambda \in \left(0, \frac{1}{3L}\right)$. The advantages of the method (PSEGM) are computing one projection onto the feasible set C and one evaluation of the mapping A in each iteration.

Unfortunately, most of these methods mentioned above obtained only weak convergence results which are not enough to make it efficient from the numerical point of view. More so, in many applied disciplines, strong (or norm) convergence results are often more desirable than weak convergence. For instance, it translates the physically tangible property that the energy $\|x_n - p\|^2$ of the error between the iterate x_n and a solution p eventually become small (see Bauschke and Combettes 2001). More importance of strong convergence was also underlined in Güler (1991). Furthermore, the stepsizes of all methods mentioned above required a prior knowledge of the Lipschitz constant of the cost operators, which is very difficult to estimate.

Even when it could be estimated, it is often too small which affects the rate of convergence of the methods.

On the other hand, the *inertial technique* was introduced to speed up the convergence rate of algorithms by Polyak (1964). This technique originates from an implicit discretization method of the second-order dynamical systems (heavy ball with friction) in solving the smooth convex minimization problem. For approximating the null point of a maximal monotone operator A, Alvarez and Attouch (2001) introduced the following *inertial proximal point algorithm* (IPPA): for each $n \geq 1$,

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J^A_{\lambda_n}(y_n), \end{cases} \tag{1.8}$$

where $J^A_{\lambda_n}$ is the resolvent operator of A for any $\lambda_n > 0$ and $\theta_n(x_n - x_{n-1})$ is called the *inertial extrapolation* with $\theta_n \in [0, 1)$. In recent years, the inertial technique has been applied to improves the performance of the algorithms for solving the problem (VIP) and related optimization problems (see, for example, Cholamjiak et al. 2020; Gibali et al. 2020; Jolaoso 2021; Tan et al. 2021; Thong et al. 2019, 2021; Yang 2021). Chbani and Riahi (2013) proposed a new type of inertial term, which is known as *relaxed inertial algorithm* (RIA), whose structure is a convex combination of two iterates x_{n-1} and x_n, that is, for each $n \geq 1$,

$$y_n = (1 - \theta_n)x_n + \theta_n x_{n-1} = x_n + \theta_n(x_{n-1} - x_n). \tag{1.9}$$

They also proposed two modifications of the algorithm (IPPA) with relaxed inertial (1.9) for solving the equilibrium problem. Under suitable conditions, they obtained both weak and strong convergence of the algorithms to a solution of the equilibrium problem.

In general, many algorithms based on the Halpern-type algorithm (Halpern 1967), the viscosity approximation algorithm (Moudafi 2000), the hybrid projection algorithm (Nakajo and Takahashi 2003) and the shrinking projection algorithm (Kimura et al. 2009) have been usually constructed to provide the strong convergence.

Thong et al. (2019) applied the inertial technique in (1.8) with the method (SEGM) (1.5) for solving the monotone problem (VIP) in a real Hilbert space. They proposed two algorithms, that is, the first algorithm is based on the hybrid projection method, which is of the following form: for each $n \geq 1$,

$$\begin{cases} x_0, x_1 \in C, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda A u_n), \\ z_n = \alpha_n u_n + (1 - \alpha_n)(y_n - \lambda(Ay_n - Au_n)), \\ C_n = \{w \in H : \|z_n - w\| \leq \|u_n - w\|\}, \\ Q_n = \{w \in H : \langle w - x_n, x_1 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1). \end{cases} \tag{1.10}$$

The second algorithm is based on the shrinking projection method, which is of the following form: for each $n \geq 1$,

$$
\begin{cases}
C_1 = C, \\
x_0, x_1 \in C, \\
u_n = x_n + \theta_n(x_n - x_{n-1}), \\
y_n = P_C(u_n - \lambda A u_n), \\
z_n = \alpha_n u_n + (1 - \alpha_n)(y_n - \lambda(A y_n - A u_n)), \\
C_{n+1} = \{w \in C_n : \|z_n - w\| \le \|u_n - w\|\}, \\
x_{n+1} = P_{C_{n+1}}(x_1),
\end{cases}
\tag{1.11}
$$

where $\{\theta_n\}$ is a bounded real sequence and $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $0 \le \alpha_n \le \alpha < 1$. They proved that the sequences $\{x_n\}$ generated by (1.10) and (1.11) converge strongly to a point in $VI(C, A)$ provided $\lambda \in \left(0, \frac{1}{L}\right)$. However, the hybrid (shrinking) projection method requires constructing the sets C_n and Q_n (C_{n+1}) and computing a projection of x_1 onto the set $C_n \cap Q_n$ (C_{n+1}), which make calculating at each iteration even more complicated.

It would be interesting to extend the methods to solve the problem (VIP) in a more general class of monotone mappings. In this regards, Thong and Vuong (2019) proposed a modification of the method (TEGM) with Armijo-type linesearch procedure for solving the problem (VIP) involving a pseudomonotone mapping. To be more precise, they proposed the following algorithm:

Algorithm A. (The method (TEGM) for the pseudomonotone problem (VIP))

Step 0: Given $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, 1)$. Let $x_1 \in H$ be arbitrary.

Step 1: Compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \|A x_n - A y\| \le \mu \|x_n - y_n\|.$$

Step 2: Compute

$$x_{n+1} = y_n - \lambda_n (A y_n - A x_n).$$

Update $n := n + 1$ go to Step 1.

They proved that, if $A : H \to H$ is a pseudomonotone mapping satisfying the following additional assumptions:

(A1) A is L-Lipschitz continuous;
(A2) A is sequentially weakly continuous,

then the sequence $\{x_n\}$ generated by **Algorithm A** converges weakly to a point of $VI(C, A)$.

Very recently, Khanh et al. (2020) also proposed the following modified method (SEGM) with the Armijo-type linesearch procedure for solving the pseudomonotone problem (VIP) in a Hilbert space:

Algorithm B. The method (SEGM) for the pseudomonotone problem (VIP)

Step 0. Given $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, 1)$. Let $x_1 \in H$ be arbitrary.

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \|Ax_n - Ay\| \leq \mu \|x_n - y_n\|.$$

Step 2. Construct the half-space

$$T_n = \{x \in H : \langle x_n - \lambda_n Ax_n - y_n, x - y_n \rangle \leq 0\}$$

and compute

$$x_{n+1} = P_{T_n}(x_n - \lambda_n Ay_n).$$

Update $n := n + 1$ go to Step 1.

The weak convergence of the sequence $\{x_n\}$ generated by **Algorithm B** was also established under the assumptions (A1) and (A2). Note that theses assumptions are standard assumption, which often assumed in many recent works. However, these assumptions may be stringent in practice (see, for example, Bot et al. 2020; Cholamjiak et al. 2020; Khanh et al. 2020; Liu and Qin 2020; Tan et al. 2021; Thong and Vuong 2019; Thong et al. 2021).

It is worth noticing that most of the methods use the Euclidean squared norm. The use of the Bregman divergence instead of the Euclidean squared norm is an elegant and effective technique for solving problem in many areas of applied sciences, such as in machine learning (Amid et al. 2019), clustering (Banerjee et al. 2005) and optimization (Bregman 1967).

Gibali (2018) (see also Hieu and Cholamjiak 2020) proposed a nice extension of the method (PSEGM) with Bregman divergence technique for approximating a solution of the problem (VIP) for a class of monotone mapping in a real Hilbert space. Recently, Gibali et al. (2020) proposed two inertial Bregman method (SEGM) with Armijo-type linesearch procedure for solving the monotone problem (VIP) which such algorithms are based on the hybrid projection and shrinking projection methods. However, most of inertial algorithms with Bregman divergences for solving both monotone and pseudomonotone problems (VIP) have not considered the Halpern-type method due to the structure of Bregman divergence and the inertial term in such algorithms.

Motivated and inspired by the above works, in this paper, we propose two new *relaxed inertial algorithms* with the Bregman divergences for solving the pseudomonotone problem (VIP), which provide strong convergence in Hilbert spaces. For the first one, we combine the method (SEGM) and Halpern-type iteration and, for the second one, we combine the method (TEGM) and Halpern-type iteration. Finally, we give some numerical experiments with various types of the Bregman divergence to show the effectiveness of the algorithms and some numerical experiments to the image deblurring problem.

The main contributions of this paper are highlighted as follows:

(1) It is known that any inertial algorithm with the Bregman divergences requires to use the hybrid projection method or the shrinking projection method, which ensures to obtain the strong convergence. In this situation, we prove some strong convergence theorems of the proposed algorithms without using two mentioned methods.
(2) The inertial parameter of the proposed algorithms contain a computation procedure of the gradient of f at two iterates x_{n-1} and x_n. This approach is quite new and different from many recent works related to inertial algorithms for solving the problem (VIP) (see, for example, Anh et al. 2020; Cholamjiak et al. 2020; Tan et al. 2021; Thong et al. 2021).
(3) We prove some strong convergence theorems of the proposed algorithms without assuming standard the assumptions (A1) and (A2), which are more relaxed than many recent works related to the pseudomonotone problem (VIP) (see, for example, Bot et al. 2020;

Cholamjiak et al. 2020; Khanh et al. 2020; Liu and Qin 2020; Tan et al. 2021; Thong and Vuong 2019).

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The following notations are adopted throughout the paper:

- $\mathbb{R}$ denotes the set of all real numbers;
- $\mathbb{N}$ denotes the set of all positive integers;
- $x_n \rightharpoonup x$ denotes the weak convergence of the sequence $\{x_n\}$ to x;
- $x_n \to x$ denotes the strong convergence of the sequence $\{x_n\}$ to x.

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be the extend real-valued function. We denote the *domain* of f by $\mathrm{dom} f$, that is,

$$\mathrm{dom} f := \{x \in H : f(x) < +\infty\}.$$

A function f is said to be *proper* if $\mathrm{dom} f \neq \emptyset$ and it is said to be *lower semi-continuous* if the set $\{x \in H : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$. A function f is said to be *convex* if, for any $x, y \in \mathrm{dom} f$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) \tag{2.1}$$

and it is also said to be *strictly convex* if the strict inequality holds in (2.1) for all $x, y \in \mathrm{dom} f$ with $x \neq y$ and $t \in (0, 1)$. Throughout this paper, we assume that $f : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, semi-continuous and convex function. The *subdifferential* of f at x defined by

$$\partial f(x) := \{u \in H : f(y) - f(x) \geq \langle u, y - x \rangle, \ \forall y \in H\}.$$

The *conjugate function* of f is the function f^* on H defined by

$$f^*(x^*) := \sup_{x \in H} \{\langle x^*, x \rangle - f(x)\}, \quad \forall (x, x^*) \in H \times H.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x^*, x \rangle$ (see (Takahashi 2009, Theorem 7.4.5). We also know that, if f is a proper, lower seimi-continuous and convex function, then $f^* : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semi-continuous and convex function (see Takahashi 2009, Theorem 7.4.2).

A function f is said to be Gâteaux *differentiable* at $x \in \mathrm{int}(\mathrm{dom} f)$ if there is $\nabla f \in H$ such that

$$\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = \langle \nabla f(x), y \rangle, \quad \forall y \in H. \tag{2.2}$$

When the limit (2.2) is attained uniformly for $\|y\| = 1$, we say that f is *Fréchet differentiable* at x. A function f is said to be *Gâteaux differentiable* (*Fréchet differentiable*) if it is Gâteaux differentiable everywhere (Fréchet differentiable everywhere) and f is said to be *uniformly Fréchet differentiable* (or, equivalently, f is uniformly smooth) on a subset C of H if the limit (2.2) is attained uniformly for $x \in C$ and $\|y\| = 1$. We also know that, if f is uniformly Fréchet differentiable and bounded on bounded subsets of H, then ∇f is uniformly continuous on bounded subsets of H (see Reich and Sabach 2009, Proposition 2).

Definition 2.1 A function $f : H \to \mathbb{R}$ is said to be:

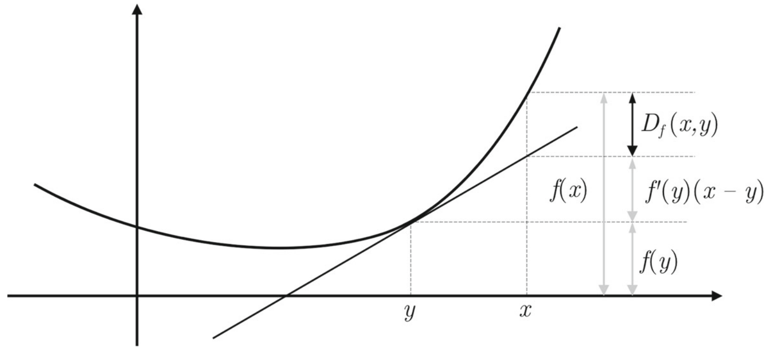

Fig. 1 The Bregman divergence with respect to f

(1) *uniformly convex* with modulus ϕ if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-t(1-t)\phi(\|x-y\|)$$

for all $x, y \in \operatorname{dom} f$ and $t \in [0,1]$, where ϕ is an increasing function vanishing only at 0;

(2) *strongly convex* with a constant $\sigma > 0$ if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-\frac{\sigma}{2}t(1-t)\|x-y\|^2 \tag{2.3}$$

for all $x, y \in \operatorname{dom} f$ and $t \in [0,1]$.

We know that f is uniformly convex if and only if f^* is Fr*é*chet differentiable and ∇f^* is uniformly continuous (see Zălinescu 2002, Theorem 3.5.10). Obviously, f is strongly convex with a constant σ if and only if it is uniformly convex with modulus $\phi(s) = \frac{\sigma}{2}s^2$ and it is also equivalent to the following inequality (see Beck 2017, Theorem 5.24):

$$f(x) \geq f(y) + \langle \nabla f(y), x-y\rangle + \frac{\sigma}{2}\|x-y\|^2$$

for all $x \in \operatorname{dom} f$ and $y \in \operatorname{int}(\operatorname{dom} f)$. A function f is said to be *Legendre* if f is essentially smooth and essentially strictly convex in the sense of Rockafellar (1970), Section 26. If f is additionally assumed to be G$\hat{a}$teaux differentiable, then the bifunction $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0,\infty)$ defined by

$$D_f(x,y) := f(x) - f(y) - \langle \nabla f(y), x-y\rangle$$

is called the *Bregman divergence* (distance) with respect to f (Bregman 1967) is shown in the Fig. 1.

In fact, the Bregman divergence is one kind of measurement of the difference between two points (or distribution in statistics) on a differentiable convex function of Legendre type. Note that the Bregman divergence is not a usual metric because it is asymmetric and does not satisfy the triangle inequality. The Bregman divergence with respect to various types of f can be seen as follows (Bauschke et al. 2009; Hieu and Cholamjiak 2020):

Example 2.2 Let $x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$ be two points in $\mathbb{R}^m$.

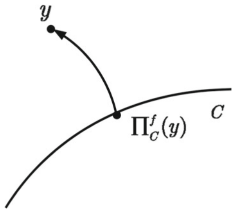

Fig. 2 The Bregman projection with respect to f

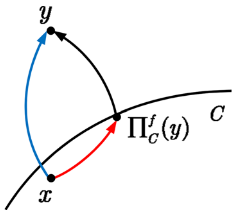

Fig. 3 The generalized Pythagorean theorem

(1) The *Kullback–Leibler divergence*

$$D_f^{\mathrm{KL}}(x, y) = \sum_{i=1}^{m} \left(x_i \ln \left(\frac{x_i}{y_i} \right) + y_i - x_i \right)$$

generated by the function $f^{\mathrm{KL}}(x) = \sum_{i=1}^{m} x_i \ln x_i$ with its domain $\mathrm{dom} f^{\mathrm{KL}} = \{x \in \mathbb{R}^m : x_i > 0,\ i = 1, 2, \ldots, m\}$ and its gradient

$$\nabla f^{\mathrm{KL}}(x) = (1 + \ln(x_1), 1 + \ln(x_2), \ldots, 1 + \ln(x_m))^T.$$

In statistics, the Kullback–Leibler divergence is used to measure the difference between two probability distributions.

(2) The *Itakura–Saito divergence*

$$D_f^{\mathrm{IS}}(x, y) = \sum_{i=1}^{m} \left(\frac{x_i}{y_i} - \ln \left(\frac{x_i}{y_i} \right) - 1 \right)$$

generated by the function $f^{\mathrm{IS}}(x) = -\sum_{i=1}^{m} \ln x_i$ with its domain $\mathrm{dom} f^{\mathrm{IS}} = \{x \in \mathbb{R}^m : x_i > 0,\ i = 1, 2, \ldots, m\}$ and its gradient $\nabla f^{\mathrm{IS}}(x) = -\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_m} \right)^T$. In signal processing, the Itakura–Saito divergence is used to measure the difference between original spectrum and approximation of that spectrum.

(3) The *Bregman divergence*

$$D_f^{\mathrm{FD}}(x, y) = \sum_{i=1}^{m} \left(x_i \ln \left(\frac{x_i}{y_i} \right) + (1 - x_i) \ln \left(\frac{1 - x_i}{1 - y_i} \right) \right)$$

generated by the Fermi–Dirac entropy function

$$f^{\mathrm{FD}}(x) = \sum_{i=1}^{m} \left(x_i \ln x_i + (1 - x_i) \ln(1 - x_i) \right)$$

with its domain $\mathrm{dom} f^{\mathrm{FD}} = \{x \in \mathbb{R}^m : 0 < x_i < 1,\ i = 1, 2, \ldots, m\}$ and its gradient

$$\nabla f^{\mathrm{FD}}(x) = \left(\ln\left(\frac{x_1}{1-x_1}\right), \ln\left(\frac{x_2}{1-x_2}\right), \ldots, \ln\left(\frac{x_m}{1-x_m}\right)\right)^{\mathrm{T}}.$$

(4) The *squared Mahalanobis divergence*

$$D_f^{\mathrm{SM}}(x, y) = \frac{1}{2}(x-y)^T Q(x-y)$$

generated by the function $f^{\mathrm{SM}}(x) = \frac{1}{2}x^{\mathrm{T}}Qx$ with its domain $\mathrm{dom} f^{\mathrm{SM}} = \mathbb{R}^m$ and its gradient $\nabla f^{\mathrm{SM}}(x) = Qx$, where $Q = \mathrm{diag}(1, 2, \ldots, m)$. The Squared Mahalanobis divergence is used to measure the difference between standard deviation and mean in a normal distribution.

(5) The *squared Euclidean divergence*

$$D_f^{\mathrm{SE}}(x, y) = \frac{1}{2}\|x-y\|^2$$

generated by the function $f^{\mathrm{SE}}(x) = \frac{1}{2}\|x\|^2$ with its domain $\mathrm{dom} f^{\mathrm{SE}} = \mathbb{R}^m$ and its gradient $\nabla f^{\mathrm{SE}}(x) = x$.

Note that, if f is strongly convex, then, for any $x \in \mathrm{dom} f$ and $y \in \mathrm{int}(\mathrm{dom} f)$,

$$D_f(x, y) \geq \frac{\sigma}{2}\|x-y\|^2. \tag{2.4}$$

Moreover, it is known that if f is twice continuously differentiable, then it is strongly convex if and only if $\nabla^2 f(x) \succeq \sigma I$ for all $x \in \mathrm{dom} f$, where $\nabla^2 f(x)$ is the Hessian matrix at x, I is the identity matrix and the notation $\succeq$ means that $\nabla^2 f(x) - \sigma I$ is positive semi-definite (see Beck 2017).

The following important properties follow from the definition of the Bregman divergence:

(1) (The two-point identity) for any $x, y \in \mathrm{int}(\mathrm{dom} f)$,

$$D_f(x, y) + D_f(y, x) = \langle \nabla f(x) - \nabla f(y), x - y \rangle; \tag{2.5}$$

(2) (The three-point identity) for any $x \in \mathrm{dom} f$ and $y, z \in \mathrm{int}(\mathrm{dom} f)$,

$$D_f(x, y) = D_f(x, z) - D_f(y, z) + \langle \nabla f(z) - \nabla f(y), x - y \rangle. \tag{2.6}$$

Recall that the *Bregman projection* with respect to f (Bregman 1967) of $y \in \mathrm{int}(\mathrm{dom} f)$ onto the nonempty, closed and convex set $C \subset \mathrm{int}(\mathrm{dom} f)$ is the unique point in C, denoted by Π_C^f, defined by,

$$\Pi_C^f(y) := \arg\min\{D_f(x, y) : x \in C\}.$$

The Bregman projection with respect to f is shown in the Fig. 2. In particular, if $f(x) = \frac{1}{2}\|x\|^2$, then Π_C^f reduces to the metric projection P_C. It is known that Π_C^f is continuous (see Bauschke et al. 2009, Theorem 4.3).

Moreover, Π_C^f has the following properties (see Butnariu and Resmerita 2006): for each $y \in H$,

$$\langle \nabla f(\Pi_C^f(y)) - \nabla f(y), x - \Pi_C^f(y) \rangle \geq 0, \quad \forall x \in C \tag{2.7}$$

and

$$D_f(x, \Pi_C^f(y)) + D_f(\Pi_C^f(y), y) \leq D_f(x, y), \quad \forall x \in C. \tag{2.8}$$

The property (2.8) is also called the *generalized Pythagorean theorem* and it is shown in the Fig. 3.

Let $f : H \to \mathbb{R}$ be a Legendre function. Let $V_f : H \times H \to [0, +\infty)$ associated with f be defined by

$$V_f(x, x^*) := f(x) - \langle x^*, x\rangle + f^*(x^*), \quad \forall (x, x^*) \in H \times H.$$

We know the following properties (Martín-Márquez et al. 2013, Proposition 1):

(1) V_f is nonnegative and convex in the second variable;
(2) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $(x, x^*) \in H \times H$;
(3) $V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \le V_f(x^* + y^*, x)$ for all $(x, x^*) \in H \times H$ and $y^* \in H$.

Since V_f is convex in the second variable, it follows that, for all $z \in H$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.9}$$

where $\{x_i\}_{i=1}^N \subset H$ and $\{t_i\}_{i=1}^N \subset [0, 1]$ with $\sum_{i=1}^N t_i = 1$.

Definition 2.3 A mapping $A : H \to H$ is said to be:

(1) *monotone* if $\langle Ax - Ay, x - y\rangle \ge 0$ for all $x, y \in H$;
(2) *pseudomonotone* if $\langle Ax, y - x\rangle \ge 0$, we have $\langle Ay, y - x\rangle \ge 0$ for all $x, y \in H$;
(3) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \le L\|x - y\|$ for all $x, y \in H$;
(4) *sequentially weakly continuous* if for each sequence $\{x_n\}$ such that $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$.

Remark 2.4 It is observe that every monotone mapping is a pseudomonotone mapping, but converse is not true. The example of a pseudomonotone mapping but not necessarily monotone can be found in Khanh and Vuong (2014).

Lemma 2.5 (Cottle and Yao 1992) *Let C be a nonempty, closed and convex subset of H and A be a pseudomonotone and continuous mapping of C into H. Then, z is a solution of the problem* (VIP) *if and only if*

$$\langle Ax, x - z\rangle \ge 0, \quad \forall x \in C.$$

Lemma 2.6 *For any $a, b \in \mathbb{R}$ and $\epsilon > 0$. Then, the following inequality holds:*

$$2ab \le \frac{a^2}{\epsilon} + \epsilon b^2.$$

Proof Since $0 \le \left(\frac{1}{\sqrt{\epsilon}}a - \sqrt{\epsilon}b\right)^2 = \frac{a^2}{\epsilon} - 2ab + \epsilon b^2$, we have $2ab \le \frac{a^2}{\epsilon} + \epsilon b^2$. This completes the proof. □

Following the proof line as in Lemma 2 of Denisov et al. (2015), we obtain the following result:

Lemma 2.7 *For all $x \in H$ and $\alpha \ge \beta > 0$, the following inequalities hold:*

$$\left\| \frac{x - \Pi_C^f \nabla f^*(\nabla f(x) - \alpha Ax)}{\alpha} \right\| \le \left\| \frac{x - \Pi_C^f \nabla f^*(\nabla f(x) - \beta Ax)}{\beta} \right\|$$

and

$$\|x - \Pi_C^f \nabla f^*(\nabla f(x) - \beta Ax)\| \leq \|x - \Pi_C^f \nabla f^*(\nabla f(x) - \alpha Ax)\|.$$

Lemma 2.8 (Iusem and Gárciga Otero 2001) *Let H_1 and H_2 be two real Hilbert spaces. Suppose that $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

The following lemmas are useful in our proofs.

Lemma 2.9 (Maingé 2008) *Let $\{a_n\}$ be a nonnegative real sequence such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k := \max\{j \leq k : a_j \leq a_{j+1}\}$.

Lemma 2.10 (Chbani and Riahi 2013) *Let $\{a_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{t_n\}$ be nonnegative real sequences such that $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup_{n\to\infty} s_n \leq 0$, $\sum_{n=n_0}^{\infty} \delta_n < \infty$, $\sum_{n=n_0}^{\infty} t_n = \infty$ and, for each $n \geq n_0$ (where n_0 is a positive integer),*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n a_{n-1} + t_n s_n + \delta_n.$$

Then, $\lim_{n\to\infty} a_n = 0$.

3 Algorithms and their convergence

In this section, we propose two relaxed inertial algorithms for solving pseudomonotone variational inequalities. In order to establish the convergence of the algorithms, the following assumptions are needed:

Assumption A1. The feasible set C is a nonempty, closed and convex subset of a real Hilbert space H.
Assumption A2. The function $f : H \to \mathbb{R}$ is σ-strongly convex, Legendre which is bounded and uniformly Fréchet differentiable on bounded subsets of H.
Assumption A3. The mapping $A : H \to H$ is pseudomonotone and uniformly continuous on H, which satisfies the following condition:

$$\text{whenever } \{q_n\} \subset C,\ q_n \rightharpoonup q, \text{ one has } \|Aq\| \leq \liminf_{n\to\infty} \|Aq_n\|. \tag{3.1}$$

Assumption A4. The solution set of VIP is nonempty, that is, $VI(C, A) \neq \emptyset$.
Assumption A5. The positive sequence $\{\xi_n\}$ satisfies $\lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Remark 3.1 From Assumption A3, we consider the following aspects:

(1) When H is a finite-dimensional Hilbert space, it suffices to assume that the mapping A is continuous pseudomonotone and it is not necessary to assume (3.1).

(2) The uniform continuity is weaker than the Lipschitz continuity. Clearly, if A is Lipschitz continuous, then A is uniformly continuous, but the converse is not true. For example, let $A : [0, \infty) \to [0, \infty)$ be a mapping defined by

$$Ax = \sqrt{x}, \quad \forall x \in [0, \infty).$$

For each $\epsilon > 0$, let $\delta = \epsilon^2$ and $|x - y| < \delta$, where $x, y \geq 0$. To estimate $|Ax - Ay|$, we consider possible two cases of x, y. In the case $x, y \in [0, \delta)$. Using the fact that A is strictly increasing, we have

$$|Ax - Ay| < A(\delta) - A(0) < \sqrt{\delta} = \epsilon.$$

Otherwise, in the case $x \notin [0, \delta)$ or $y \notin [0, \delta)$, we have $\max\{x, y\} \geq \delta$. It follows that

$$|Ax - Ay| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| \leq \frac{|x - y|}{\sqrt{\max\{x, y\}}} < \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} = \epsilon.$$

Thus, A is uniformly continuous and, for each $n \in \mathbb{N}$, we have

$$\left|A\left(\frac{1}{n}\right) - A(0)\right| = \sqrt{\frac{1}{n}} = \sqrt{n}\left|\frac{1}{n} - 0\right|.$$

Thus, A is not Lipschitz continuous.

(3) Note that the sequential weak continuity of the mapping A is satisfies (3.1), but the converse is not necessary true. Indeed, let $A : \ell_2 \to \ell_2$ be a mapping defined by

$$Ax = x\|x\|, \quad \forall x \in \ell_2.$$

Let $\{q_n\} \subset \ell_2$ such that $q_n \rightharpoonup q$. By the weak lower semi-continuity of the norm, we have

$$\|q\| \leq \liminf_{n\to\infty} \|q_n\|.$$

It follows that

$$\|Aq\| = \|q\|^2 \leq (\liminf_{n\to\infty} \|q_n\|)^2 \leq \liminf_{n\to\infty} \|q_n\|^2 = \liminf_{n\to\infty} \|Aq_n\|.$$

Hence, A satisfies (3.1). To show that A is not sequentially weakly continuous, choose $q_n = e_n + e_1$, where $\{e_n\}$ is a standard basis of ℓ_2, that is, $e_n = (0, 0, \ldots, 1, \ldots)$ with 1 at the n-th position. It is clear that $q_n \rightharpoonup e_1$ and $Aq_n = A(e_n + e_1) = (e_n + e_1)\|e_n + e_1\| \rightharpoonup \sqrt{2}e_1$, but $Ae_1 = e_1\|e_1\| = e_1$. Hence, A is not sequentially weakly continuous.

(4) If A is monotone, then (3.1) can be removed.

Now, we propose the first algorithm, which combines the Halpern-type iteration and the subgradient extragradient method. The algorithm is shown as follows.

Algorithm 1: The relaxed inertial subgradient extragradient algorithm for the problem (VIP)

Step 0. Given $\theta \in [0, 1/2]$, $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.4). Let $x_0, x_1 \in H$ be arbitrary.

Step 1. Given the current iterates x_{n-1} and x_n $(n \geq 1)$. Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|\nabla f(x_{n-1}) - \nabla f(x_n)\|}, \theta\right\}, & \text{if } \nabla f(x_{n-1}) \neq \nabla f(x_n), \\ \theta, & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n)))$ and compute

$$y_n = \Pi_C^f \nabla f^*(\nabla f(u_n) - \lambda_n A u_n),$$

where $\lambda_n := \gamma l^{m_n}$, with m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \|Au_n - Ay_n\| \leq \mu \|u_n - y_n\|. \tag{3.3}$$

If $u_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of the problem (VIP). Otherwise, go to Step 2.

Step 2. Construct the half-space

$$T_n = \{x \in H : \langle \nabla f(u_n) - \lambda_n A u_n - \nabla f(y_n), x - y_n \rangle \leq 0\}$$

and compute

$$z_n = \Pi_{T_n}^f \nabla f^*(\nabla f(u_n) - \lambda_n A y_n).$$

Step 3. Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1 - \alpha_n)\nabla f(z_n)).$$

Update $n := n + 1$ go to Step 1.

Remark 3.2 (1) If $f(x) = \frac{1}{2}\|x\|^2$ and $\theta_n = 0$, then Algorithm 1 reduces to the following one: for each $n \geq 1$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = P_{T_n}(x_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) z_n, \end{cases} \tag{3.4}$$

where $\lambda_n = \gamma l^{m_n}$, with m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\| \tag{3.5}$$

and

$$T_n = \{x \in H : \langle x_n - \lambda_n A x_n - y_n, x - y_n \rangle \leq 0\}.$$

Algorithm (3.4) is a modification of the method (SEGM) without the relaxed inertial term for pseudomonotone problem (VIP) with a non-Lipschitz mapping.

(2) From (3.2), it is easy to see that $\theta_n\|\nabla f(x_{n-1}) - \nabla f(x_n)\| \le \xi_n$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0$, it follows that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|\nabla f(x_{n-1}) - \nabla f(x_n)\| \le \lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0.$$

Lemma 3.3 *The Armijo-line search rule (3.3) is well defined.*

Proof If $u_n \in VI(C, A)$, then $u_n = \Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma A u_n)$ and $m_n = 0$. In this case, we consider $u_n \notin VI(C, A)$ and assume that the contrary for all $m \ge 1$. Thus, we have

$$\gamma l^m \|Au_n - A(\Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n))\| > \mu\|u_n - \Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n)\|,$$

which implies that

$$\|Au_n - A(\Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n))\| > \mu \frac{\|u_n - \Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n)\|}{\gamma l^m}. \tag{3.6}$$

Now, we consider two possible cases of u_n, that is, $u_n \in C$ and $u_n \notin C$. If $u_n \in C$, then $u_n = \Pi_C^f(u_n)$. By the continuity of Π_C^f, we have

$$\lim_{m\to\infty} \|u_n - \Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n)\| = 0$$

and, by the uniform continuity of A, we have

$$\lim_{m\to\infty} \|Au_n - A(\Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n))\| = 0. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\lim_{m\to\infty} \frac{\|u_n - \Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n)\|}{\gamma l^m} = 0.$$

Also, by the uniform continuity of ∇f, we have

$$\lim_{m\to\infty} \frac{\|\nabla f(u_n) - \nabla f(\Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n))\|}{\gamma l^m} = 0. \tag{3.8}$$

Let $v_n = \Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n)$. From (2.7), it follows that

$$\langle \nabla f(v_n) - \nabla f(u_n) + \gamma l^m Au_n, x - v_n\rangle \ge 0, \quad \forall x \in C,$$

which implies that

$$\left\langle \frac{\nabla f(v_n) - \nabla f(u_n)}{\gamma l^m}, x - v_n \right\rangle + \langle Au_n, x - v_n\rangle \ge 0, \quad \forall x \in C. \tag{3.9}$$

Letting $m \to \infty$ in (3.9), by (3.8), we have

$$\langle Au_n, x - u_n\rangle \ge 0, \quad \forall x \in C.$$

That is, $u_n \in VI(C, A)$, which is a contradiction.

On the other hand, if $u_n \notin C$, then we have

$$\lim_{m\to\infty} \|u_n - \Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n)\| = \lim_{m\to\infty} \|u_n - \Pi_C^f(u_n)\| > 0 \tag{3.10}$$

and

$$\lim_{m\to\infty} \gamma l^m \|Au_n - A(\Pi_C^f \nabla f^*(\nabla f(u_n) - \gamma l^m Au_n))\| = 0. \tag{3.11}$$

Combining (3.6), (3.10) and (3.11), we also get a contradiction. This completes the proof. □

Remark 3.4 (1) We note that the pseudomonotonicity of the mapping is not used in the proof of Lemma 3.3.

(2) It is obvious that $0 < \lambda_n \le \gamma$ for all $n \in \mathbb{N}$.

Lemma 3.5 *Suppose that Assumptions* A1–A4 *are satisfied. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 1 *satisfies the following inequality:*

$$D_f(p, z_n) \le D_f(p, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n), \quad \forall p \in VI(C, A).$$

In particular, if $\mu \in (0, \sigma)$, *then* $D_f(p, z_n) \le D_f(p, u_n)$.

Proof Let $p \in VI(C, A)$. By the definition of the Bregman divergence, we have

$$\begin{aligned}
D_f(p, z_n) &= D_f(p, \Pi_{T_n}^f \nabla f^*(\nabla f(u_n) - \lambda_n Ay_n)) \\
&\le D_f(p, \nabla f^*(\nabla f(u_n) - \lambda_n Ay_n)) - D_f(z_n, \nabla f^*(\nabla f(u_n) - \lambda_n Ay_n)) \\
&= V_f(p, \nabla f(u_n) - \lambda_n Ay_n) - V_f(z_n, \nabla f(u_n) - \lambda_n Ay_n) \\
&= f(p) - \langle \nabla f(u_n) - \lambda_n Ay_n, p\rangle + f^*(\nabla f(u_n) - \lambda_n Ay_n) - f(z_n) \\
&\quad + \langle \nabla f(u_n) - \lambda_n Ay_n, z_n\rangle - f^*(\nabla f(u_n) - \lambda_n Ay_n) \\
&= f(p) - \langle \nabla f(u_n), p\rangle + \lambda_n \langle Ay_n, p\rangle - f(z_n) + \langle \nabla f(u_n), z_n\rangle - \lambda_n \langle Ay_n, z_n\rangle \\
&= f(p) - \langle \nabla f(u_n), p\rangle + f(u_n) - f(z_n) + \langle \nabla f(u_n), z_n\rangle - f(u_n) \\
&\quad + \lambda_n \langle Ay_n, p\rangle - \lambda_n \langle Ay_n, z_n\rangle \\
&= D_f(p, u_n) - D_f(z_n, u_n) - \lambda_n \langle Ay_n, z_n - p\rangle.
\end{aligned} \tag{3.12}$$

Using the fact that $\langle Ap, y_n - p\rangle \ge 0$ and the pseudomonotonicity of A, we have $\langle Ay_n, y_n - p\rangle \ge 0$. It follows that

$$\langle Ay_n, z_n - p\rangle = \langle Ay_n, y_n - p\rangle + \langle Ay_n, z_n - y_n\rangle \ge \langle Ay_n, z_n - y_n\rangle. \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$D_f(p, z_n) \le D_f(p, u_n) - D_f(z_n, u_n) + \lambda_n \langle Ay_n, y_n - z_n\rangle. \tag{3.14}$$

Then, using (2.5) and (2.6), we get

$$\begin{aligned}
D_f(p, z_n) &\le D_f(p, u_n) - D_f(z_n, y_n) + D_f(u_n, y_n) - \langle \nabla f(y_n) - \nabla f(u_n), z_n - u_n\rangle \\
&\quad + \lambda_n \langle Ay_n, y_n - z_n\rangle \\
&= D_f(p, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \langle \nabla f(u_n) - \nabla f(y_n), u_n - y_n\rangle \\
&\quad - \langle \nabla f(y_n) - \nabla f(u_n), z_n - u_n\rangle + \lambda_n \langle Ay_n, y_n - z_n\rangle \\
&= D_f(p, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \langle \nabla f(u_n) - \nabla f(y_n), z_n - y_n\rangle \\
&\quad + \lambda_n \langle Ay_n, y_n - z_n\rangle \\
&= D_f(p, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \langle \nabla f(u_n) - \lambda_n Au_n - \nabla f(y_n), z_n - y_n\rangle \\
&\quad + \lambda_n \langle Au_n - Ay_n, z_n - y_n\rangle.
\end{aligned} \tag{3.15}$$

It is clear that $z_n \in T_n$ and hence

$$\langle \nabla f(u_n) - \lambda_n Au_n - \nabla f(y_n), z_n - y_n\rangle \le 0. \tag{3.16}$$

Combining (3.15) and (3.16), we have

$$\begin{aligned} D_f(p, z_n) &\le D_f(p, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \lambda_n \langle Au_n - Ay_n, z_n - y_n\rangle \\ &\le D_f(p, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \lambda_n \|Au_n - Ay_n\| \|z_n - y_n\| \\ &\le D_f(p, u_n) - D_f(y_n, u_n) - D_f(z_n, y_n) + \mu \|u_n - y_n\| \|z_n - y_n\| \\ &\le D_f(p, u_n) - D_f(y_n, u_n) - D_f(z_n, y_n) + \frac{\mu}{2}\|u_n - y_n\|^2 + \frac{\mu}{2}\|z_n - y_n\|^2 \\ &\le D_f(p, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n). \end{aligned} \tag{3.17}$$

Since $\mu \in (0, \sigma)$, we have $1 - \frac{\mu}{\sigma} = \frac{\sigma - \mu}{\sigma} > 0$. Consequently, we have

$$\left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, x_n) + \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n) \ge 0.$$

Then, from (3.17), it follows that

$$D_f(p, z_n) \le D_f(p, u_n). \tag{3.18}$$

This completes the proof. □

Lemma 3.6 *Suppose that Assumptions A1-A4 are satisfied. Let $\{u_n\}$ generated by Algorithm 1. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $v \in H$ and $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$, then $v \in VI(C, A)$.*

Proof Let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ such that $u_{n_k} \rightharpoonup v \in H$. Since $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$ and $\{y_{n_k}\} \subset C$, we have $y_{n_k} \rightharpoonup v \in C$. By the definition of y_{n_k} and (2.7), we have

$$\langle \nabla f(y_{n_k}) - \nabla f(u_{n_k}) + \lambda_{n_k} Au_{n_k}, x - y_{n_k}\rangle \ge 0, \quad \forall x \in C,$$

which implies that

$$\lambda_{n_k} \langle Au_{n_k}, x - y_{n_k}\rangle \ge \langle \nabla f(u_{n_k}) - \nabla f(y_{n_k}), x - y_{n_k}\rangle, \quad \forall x \in C.$$

Hence, we have

$$\langle Au_{n_k}, x - u_{n_k}\rangle \ge \left\langle \frac{\nabla f(u_{n_k}) - \nabla f(y_{n_k})}{\lambda_{n_k}}, x - y_{n_k}\right\rangle + \langle Au_{n_k}, y_{n_k} - u_{n_k}\rangle, \quad \forall x \in C. \tag{3.19}$$

Now, we consider two possible cases. In the first case, we assume that $\liminf_{k\to\infty} \lambda_{n_k} > 0$. By the weakly convergent of $\{u_{n_k}\}$, we have $\{u_{n_k}\}$ is bounded and since A is uniformly continuous, it follows from Lemma 2.8 that $\{Au_{n_k}\}$ is bounded. Moreover, since ∇f is uniformly continuous, we have

$$\lim_{k\to\infty} \|\nabla f(u_{n_k}) - \nabla f(y_{n_k})\| = 0.$$

Taking the limit inferior as $k \to \infty$ in (3.19), we have

$$\liminf_{k\to\infty} \langle Au_{n_k}, x - u_{n_k}\rangle \ge 0, \quad \forall x \in C.$$

In the second case, we assume that $\liminf_{k\to\infty} \lambda_{n_k} = 0$. Let

$$w_{n_k} = \Pi_C^f \nabla f^*(\nabla f(u_{n_k}) - \lambda_{n_k} l^{-1} Au_{n_k}).$$

Clearly, we have $\lambda_{n_k} l^{-1} > \lambda_{n_k}$. Then, from Lemma 2.7, it follows that

$$\|u_{n_k} - w_{n_k}\| \leq \frac{1}{l}\|u_{n_k} - y_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty.$$

Moreover, we have

$$\|Au_{n_k} - Aw_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{3.20}$$

By the Armijo linesearch rule (3.3), we have

$$\lambda_{n_k} l^{-1}\|Au_{n_k} - Aw_{n_k}\| > \mu\|u_{n_k} - w_{n_k}\|.$$

That is,

$$\frac{1}{\mu}\|Au_{n_k} - Aw_{n_k}\| > \frac{\|u_{n_k} - w_{n_k}\|}{\lambda_{n_k} l^{-1}}. \tag{3.21}$$

Combining (3.20) and (3.21), we get

$$\lim_{k\to\infty} \frac{\|u_{n_k} - w_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0$$

and hence

$$\lim_{k\to\infty} \frac{\|\nabla f(u_{n_k}) - \nabla f(w_{n_k})\|}{\lambda_{n_k} l^{-1}} = 0.$$

Moreover, we have

$$\langle \nabla f(w_{n_k}) - \nabla f(u_{n_k}) + \lambda_{n_k} l^{-1} Au_{n_k}, x - w_{n_k}\rangle \geq 0, \quad \forall x \in C.$$

It follows that

$$\langle Au_{n_k}, x - u_{n_k}\rangle \geq \left\langle \frac{\nabla f(u_{n_k}) - \nabla f(w_{n_k})}{\lambda_{n_k} l^{-1}}, x - w_{n_k}\right\rangle + \langle Au_{n_k}, w_{n_k} - u_{n_k}\rangle, \quad \forall x \in C. \tag{3.22}$$

Taking the limit inferior as $k \to \infty$ in (3.22), we have

$$\liminf_{k\to\infty}\langle Au_{n_k}, x - u_{n_k}\rangle \geq 0, \quad \forall x \in C.$$

On the other hand, we observe that

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Au_{n_k}, x - u_{n_k}\rangle + \langle Au_{n_k}, x - u_{n_k}\rangle + \langle Ay_{n_k}, u_{n_k} - y_{n_k}\rangle.$$

Again, since A is uniformly continuous, $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$ and $\liminf_{k\to\infty}\langle Au_{n_k}, x - u_{n_k}\rangle \geq 0$, we have

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0.$$

Let $\{\epsilon_k\}$ be a strictly decreasing and positive sequence such that $\epsilon_k \to 0$ as $k \to \infty$. For each ϵ_k, we denote by m_k the smallest positive integer such that $\langle Ay_{n_j}, x - y_{n_j}\rangle + \epsilon_k \geq 0, \quad \forall j \geq m_k$. In particular, let $j = m_k$, we have

$$\langle Ay_{n_{m_k}}, x - y_{n_{m_k}}\rangle + \epsilon_k \geq 0. \tag{3.23}$$

Since, for each $k \geq 1$, we can set $v_{n_{m_k}} = \frac{Ay_{n_{m_k}}}{\|Ay_{n_{m_k}}\|^2}$. It is easy to see that $\langle Ay_{n_{m_k}}, v_{n_{m_k}} \rangle = 1$. Thus, we can write (3.23) as

$$\langle Ay_{n_{m_k}}, x + \epsilon_k v_{n_{m_k}} - y_{n_{m_k}} \rangle \geq 0.$$

The pseudomonotonicity of A implies that $\langle A(x + \epsilon_k v_{n_{m_k}}), x + \epsilon_k v_{n_{m_k}} - y_{n_{m_k}} \rangle \geq 0$. It then follows that

$$\begin{aligned} \langle Ax, x - y_{n_{m_k}} \rangle &= \langle Ax - A(x + \epsilon_k v_{n_{m_k}}), x - y_{n_{m_k}} \rangle + \langle A(x + \epsilon_k v_{n_{m_k}}), x + \epsilon_k v_{n_{m_k}} - y_{n_{m_k}} \rangle \\ &\quad - \langle A(x + \epsilon_k v_{n_{m_k}}), \epsilon_k v_{n_{m_k}} \rangle \\ &\geq \langle Ax - A(x + \epsilon_k v_{n_{m_k}}), x - y_{n_{m_k}} \rangle - \langle A(x + \epsilon_k v_{n_{m_k}}), \epsilon_k v_{n_{m_k}} \rangle. \end{aligned} \tag{3.24}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k v_{n_{m_k}} = 0$. Since $y_{n_{m_k}} \rightharpoonup v \in C$ and A satisfies (3.1), we have $\|Av\| \leq \liminf_{k\to\infty} \|Ay_{n_{m_k}}\|$. We assume that $Av \neq 0$ (otherwise, v is a solution of problem (VIP)). It follows that

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{n_{m_k}}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ay_{n_{m_k}}\|} \right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ay_{n_{m_k}}\|} = 0.$$

Hence, $\lim_{k\to\infty} \epsilon_k v_{n_{m_k}} = 0$. Using the facts that A is uniformly continuous, then from (3.24), we have $\liminf_{k\to\infty} \langle Ax, x - y_{n_{m_k}} \rangle \geq 0$. Therefore,

$$\langle Ax, x - v \rangle = \lim_{k\to\infty} \langle Ax, x - y_{n_{m_k}} \rangle = \liminf_{k\to\infty} \langle Ax, x - y_{n_{m_k}} \rangle \geq 0, \quad \forall x \in C.$$

By Lemma 2.5, we get $v \in VI(C, A)$. This completes the proof. □

Now, we prove strong convergence theorem of Algorithm 1.

Theorem 3.7 *Suppose that Assumptions* A1–A5 *are satisfied. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 1 *converges strongly to* $z \in VI(C, A)$*, where* $z = \Pi^f_{VI(C,A)}(x_1)$.

Proof First, we show that $\{x_n\}$ is bounded. Let $p \in VI(C, A)$. From (2.9), it follows that

$$\begin{aligned} D_f(p, u_n) &= D_f(p, \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n)))) \\ &= D_f(p, \nabla f^*((1 - \theta_n)\nabla f(x_n) + \theta_n \nabla f(x_{n-1}))) \\ &\leq (1 - \theta_n) D_f(p, x_n) + \theta_n D_f(p, x_{n-1}) \end{aligned} \tag{3.25}$$

and so, from (3.18) and (3.25),

$$\begin{aligned} D_f(p, x_{n+1}) &\leq \alpha_n D_f(p, x_1) + (1 - \alpha_n) D_f(p, z_n) \\ &\leq \alpha_n D_f(p, x_1) + (1 - \alpha_n) D_f(p, u_n) \\ &\leq \alpha_n D_f(p, x_1) + (1 - \alpha_n)(1 - \theta_n) D_f(p, x_n) + (1 - \alpha_n)\theta_n D_f(p, x_{n-1}) \\ &\leq \alpha_n D_f(p, x_1) + (1 - \alpha_n) \max\{D_f(p, x_n), D_f(p, x_{n-1})\} \\ &\leq \max\{D_f(p, x_1), D_f(p, x_n), D_f(p, x_{n-1})\} \\ &\leq \cdots \leq \max\{D_f(p, x_1), D_f(p, x_0)\}. \end{aligned}$$

Hence, $\{D_f(p, x_n)\}$ is bounded. From the relation $D_f(x, y) \geq \frac{\sigma}{2}\|x - y\|^2$ for all $x, y \in H$, we can see that $\{x_n\}$ is bounded and consequently $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. Let $z = \Pi^f_{VI(C,A)}(x_1)$. From Lemma 3.5 and (3.25), we have

$$D_f(z, x_{n+1}) \leq \alpha_n D_f(z, x_1) + (1 - \alpha_n) D_f(z, z_n)$$

$$\leq \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, u_n) - (1-\alpha_n)\Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_n, u_n)$$
$$-(1-\alpha_n)\Big(1 - \frac{\mu}{\sigma}\Big) D_f(z_n, y_n)$$
$$\leq \alpha_n D_f(z, x_1) + (1-\alpha_n)(1-\theta_n) D_f(z, x_n) + (1-\alpha_n)\theta_n D_f(z, x_{n-1})$$
$$-(1-\alpha_n)\Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_n, u_n) - (1-\alpha_n)\Big(1 - \frac{\mu}{\sigma}\Big) D_f(z_n, y_n).$$

This implies that

$$(1-\alpha_n)\Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_n, u_n) + (1-\alpha_n)\Big(1 - \frac{\mu}{\sigma}\Big) D_f(z_n, y_n)$$
$$\leq D_f(z, x_n) - D_f(z, x_{n+1}) + (1-\alpha_n)\theta_n (D_f(z, x_{n-1}) - D_f(z, x_n)) + \alpha_n K, \tag{3.26}$$

where $K := \sup_{n \geq 1}\{|D_f(z, x_1) - D_f(z, x_n)|\}$.

Now, we consider the following two possible cases to prove $\lim_{n\to\infty} D_f(z, x_n) = 0$.

Case 1. There exists $N \in \mathbb{N}$ such that $D_f(z, x_{n+1}) \leq D_f(z, x_n)$ for all $n \geq N$. This gives $\{D_f(z, x_n)\}$ is convergent and consequently

$$\lim_{n\to\infty}(D_f(z, x_n) - D_f(z, x_{n+1})) = \lim_{n\to\infty}(D_f(z, x_{n-1}) - D_f(z, x_n)) = 0.$$

Then, (3.26) implies that $\lim_{n\to\infty} D_f(y_n, u_n) = \lim_{n\to\infty} D_f(z_n, y_n) = 0$. Hence, we have

$$\lim_{n\to\infty} \|\nabla f(y_n) - \nabla f(u_n)\| = \lim_{n\to\infty} \|\nabla f(z_n) - \nabla f(y_n)\| = 0.$$

Thus, we have

$$\|\nabla f(z_n) - \nabla f(u_n)\| \leq \|\nabla f(z_n) - \nabla f(y_n)\| + \|\nabla f(y_n) - \nabla f(u_n)\|$$
$$\to 0 \quad \text{as} \quad n \to \infty. \tag{3.27}$$

Note that

$$\|\nabla f(x_{n+1}) - \nabla f(u_n)\| \leq \|\nabla f(x_{n+1}) - \nabla f(z_n)\| + \|\nabla f(z_n) - \nabla f(u_n)\|$$
$$= \alpha_n \|\nabla f(x_1) - \nabla f(z_n)\| + \|\nabla f(z_n) - \nabla f(u_n)\|.$$

It follows from (3.27) that

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(u_n)\| = 0. \tag{3.28}$$

Since $\alpha_n \in (0, 1)$, we have $\theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \leq \frac{\theta_n}{\alpha_n} \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \to 0$ as $n \to \infty$. Thus, we have

$$\|\nabla f(u_n) - \nabla f(x_n)\| = \theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\| \to 0 \;\text{ as }\; n \to \infty. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$\|\nabla f(x_{n+1}) - \nabla f(x_n)\| \leq \|\nabla f(x_{n+1}) - \nabla f(u_n)\| + \|\nabla f(u_n) - \nabla f(x_n)\|$$
$$\to 0 \quad \text{as} \quad n \to \infty.$$

Hence, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.30}$$

In fact, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$ and

$$\limsup_{n\to\infty} \langle \nabla f(x_1) - \nabla f(z), x_n - z\rangle = \lim_{k\to\infty} \langle \nabla f(x_1) - \nabla f(z), x_{n_k} - z\rangle.$$

From (3.29), it follows that $\|u_n - x_n\| \to 0$ and hence $u_{n_k} \rightharpoonup v$. Since $\|\nabla f(y_{n_k}) - \nabla f(u_{n_k})\| \to 0$, we have $\|y_{n_k} - u_{n_k}\| \to 0$. By Lemma 3.6, we get $v \in VI(C, A)$. Then, from (2.7), we obtain

$$\limsup_{n\to\infty} \langle \nabla f(x_1) - \nabla f(z), x_n - z\rangle = \langle \nabla f(x_1) - \nabla f(z), v - z\rangle \le 0.$$

Also, from (3.30), we obtain

$$\limsup_{n\to\infty} \langle \nabla f(x_1) - \nabla f(z), x_{n+1} - z\rangle \le 0. \tag{3.31}$$

By the properties of V_f, we get

$$\begin{aligned}
D_f(z, x_{n+1}) &= V_f(z, \alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(z_n)) \\
&\le V_f(z, \alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(z_n) - \alpha_n(\nabla f(x_1) - \nabla f(z))) \\
&\quad + \alpha_n \langle \nabla f(x_1) - \nabla f(z), x_{n+1} - z\rangle \\
&= V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(z_n)) + \alpha_n \langle \nabla f(x_1) - \nabla f(z), x_{n+1} - z\rangle \\
&= D_f(z, \nabla f^*(\alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(z_n)) + \alpha_n \langle \nabla f(x_1) - \nabla f(z), x_{n+1} - z\rangle \\
&\le \alpha_n D_f(z, z) + (1-\alpha_n) D_f(z, z_n) + \alpha_n \langle \nabla f(x_1) - \nabla f(z), x_{n+1} - z\rangle \\
&\le (1-\alpha_n)((1-\theta_n) D_f(z, x_n) + \theta_n D_f(z, x_{n-1})) \\
&\quad + \alpha_n \langle \nabla f(x_1) - \nabla f(z), x_{n+1} - z\rangle \\
&= (1 - \alpha_n - (1-\alpha_n)\theta_n) D_f(z, x_n) + (1-\alpha_n)\theta_n D_f(z, x_{n-1}) \\
&\quad + \alpha_n \langle \nabla f(x_1) - \nabla f(z), x_{n+1} - z\rangle.
\end{aligned} \tag{3.32}$$

Using Lemma 2.10 and (3.31), we obtain $\lim_{n\to\infty} D_f(z, x_n) = 0$ and hence $x_n \to z$ as $n \to \infty$.

Case 2. There exists a subsequence $\{D_f(z, x_{n_i})\}$ of $\{D_f(z, x_n)\}$ such that

$$D_f(z, x_{n_i}) \le D_f(z, x_{n_i+1}), \quad \forall i \in \mathbb{N}.$$

It follows from Lemma 2.9 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$D_f(z, x_{m_k}) \le D_f(z, x_{m_k+1}) \tag{3.33}$$

and

$$D_f(z, x_k) \le D_f(z, x_{m_k+1}). \tag{3.34}$$

From (3.26), it follows that

$$\begin{aligned}
&(1-\alpha_{m_k})\Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_{m_k}, u_{m_k}) + (1-\alpha_{m_k})\Big(1 - \frac{\mu}{\sigma}\Big) D_f(z_{m_k}, y_{m_k}) \\
&\le D_f(z, x_{m_k}) - D_f(z, x_{m_k+1}) + (1-\alpha_{m_k})\theta_{m_k}(D_f(z, x_{m_k-1}) - D_f(z, x_{m_k})) + \alpha_{m_k} K \\
&\le \alpha_{m_k} K,
\end{aligned}$$

where $K > 0$. Then, we obtain

$$\lim_{k\to\infty} D_f(y_{m_k}, u_{m_k}) = \lim_{k\to\infty} D_f(z_{m_k}, y_{m_k}) = 0.$$

Hence, we have

$$\lim_{k\to\infty} \|\nabla f(y_{m_k}) - \nabla f(u_{m_k})\| = \lim_{k\to\infty} \|\nabla f(z_{m_k}) - \nabla f(y_{m_k})\| = 0.$$

Using the same arguments as in the proof of Case 1, we can show that

$$\lim_{k\to\infty} \|\nabla f(z_{m_k}) - \nabla f(u_{m_k})\| = 0, \quad \lim_{k\to\infty} \|\nabla f(x_{m_k+1}) - \nabla f(u_{m_k})\| = 0,$$
$$\lim_{k\to\infty} \|\nabla f(u_{m_k}) - \nabla f(x_{m_k})\| = 0, \quad \lim_{k\to\infty} \|\nabla f(x_{m_k+1}) - \nabla f(x_{m_k})\| = 0$$

and

$$\limsup_{k\to\infty} \langle \nabla f(x_1) - \nabla f(z), x_{m_k+1} - z\rangle \leq 0. \tag{3.35}$$

Also, from (3.32) and (3.33), we can show that

$$\begin{aligned} D_f(z, x_{m_k+1}) &\leq (1-\alpha_{m_k} - (1-\alpha_{m_k})\theta_{m_k}) D_f(z, x_{m_k}) + (1-\alpha_{m_k})\theta_{m_k} D_f(z, x_{m_k-1}) \\ &\quad + \alpha_{m_k} \langle \nabla f(x_1) - \nabla f(z), x_{m_k+1} - z\rangle \\ &\leq (1-\alpha_{m_k}) D_f(z, x_{m_k}) + \alpha_{m_k} \langle \nabla f(x_1) - \nabla f(z), x_{m_k+1} - z\rangle \\ &\leq (1-\alpha_{m_k}) D_f(z, x_{m_k+1}) + \alpha_{m_k} \langle \nabla f(x_1) - \nabla f(z), x_{m_k+1} - z\rangle. \end{aligned}$$

Since $\alpha_{m_k} > 0$, it follows from (3.34) that

$$D_f(z, x_k) \leq D_f(z, x_{m_k+1}) \leq \langle \nabla f(x_1) - \nabla f(z), x_{m_k+1} - z\rangle. \tag{3.36}$$

Combining (3.35) and (3.36), we get

$$\limsup_{k\to\infty} D_f(z, x_k) \leq 0.$$

This gives $\limsup_{k\to\infty} D_f(z, x_k) = 0$ and hence $x_k \to z$ as $k \to \infty$. From above Cases 1 and 2, we can conclude that the sequence $\{x_n\}$ converges strongly to $z = \Pi^f_{VI(C,A)}(x_1)$. This complete the proof. □

Next, we propose the second relaxed inertial algorithm, which combines the Halpern-type iteration and Tseng's extragradient method. The algorithm is of the following form:

Algorithm 2: Relaxed inertial Tseng's extragradient algorithm for the problem (VIP)

Step 0. Given $\theta \in [0, 1/2]$, $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.4). Let $x_0, x_1 \in H$ be arbitrary.

Step 1. Given the current iterates x_{n-1} and x_n for each $n \geq 1$. Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|\nabla f(x_{n-1}) - \nabla f(x_n)\|}, \theta\right\}, & \text{if } \nabla f(x_{n-1}) \neq \nabla f(x_n), \\ \theta, & \text{otherwise.} \end{cases}$$

Set $u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n)))$ and compute

$$y_n = \Pi_C^f \nabla f^*(\nabla f(u_n) - \lambda_n A u_n),$$

where $\lambda_n := \gamma l^{m_n}$, with m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \|Au_n - Ay_n\| \leq \mu \|u_n - y_n\|.$$

If $u_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of the problem (VIP). Otherwise, go to Step 2.

Step 2. Compute

$$z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Au_n)).$$

Step 3. Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1 - \alpha_n)\nabla f(z_n)).$$

Update $n := n + 1$ go to Step 1.

Remark 3.8 If $f(x) = \frac{1}{2}\|x\|^2$ and $\theta_n = 0$, then Algorithm 2 reduces to the following one: for each $n \geq 1$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = y_n - \lambda_n(Ay_n - Ax_n), \\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) z_n, \end{cases} \tag{3.37}$$

where λ_n is defined in (3.5). Algorithm (3.37) is a modification of the method (TEGM) without the relaxed inertial term for the pseudomonotone problem (VIP) with a non-Lipschitz mapping.

Lemma 3.9 *Suppose that Assumptions* A1–A4 *are satisfied. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 2 *satisfies the following inequality:*

$$D_f(p, z_n) \leq D_f(p, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n), \quad \forall p \in VI(C, A).$$

In particular, if $\mu \in (0, \sigma)$*, then* $D_f(p, z_n) \leq D_f(p, u_n)$.

Proof Let $p \in VI(C,A)$. By the definition of the Bregman divergence, we have

$$\begin{aligned}
D_f(p,z_n) &= D_f(p,\nabla f^*(\nabla f(y_n)-\lambda_n(Ay_n-Au_n)))\\
&= f(p)-f(z_n)-\langle \nabla f(y_n)-\lambda_n(Ay_n-Au_n), p-z_n\rangle\\
&= f(p)-f(z_n)-\langle \nabla f(y_n), p-z_n\rangle+\lambda_n\langle Ay_n-Au_n, p-z_n\rangle\\
&= f(p)-f(y_n)-\langle \nabla f(y_n), p-y_n\rangle+\langle \nabla f(y_n), p-y_n\rangle+f(y_n)-f(z_n)\\
&\quad -\langle \nabla f(y_n), p-z_n\rangle+\lambda_n\langle Ay_n-Au_n, p-z_n\rangle\\
&= f(p)-f(y_n)-\langle \nabla f(y_n), p-y_n\rangle-f(z_n)+f(y_n)+\langle \nabla f(y_n), z_n-y_n\rangle\\
&\quad +\lambda_n\langle Ay_n-Au_n, p-z_n\rangle\\
&= D_f(p,y_n)-D_f(z_n,y_n)+\lambda_n\langle Ay_n-Au_n, p-z_n\rangle. \qquad (3.38)
\end{aligned}$$

From (2.6), it follows that

$$D_f(p,y_n)=D_f(p,u_n)-D_f(y_n,u_n)+\langle \nabla f(u_n)-\nabla f(y_n), p-y_n\rangle. \qquad (3.39)$$

Substituting (3.39) into (3.38), we have

$$\begin{aligned}
D_f(p,z_n) = D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\langle \nabla f(u_n)-\nabla f(y_n), p-y_n\rangle\\
+\lambda_n\langle Ay_n-Au_n, p-z_n\rangle. \qquad (3.40)
\end{aligned}$$

By the definition of y_n, we have

$$\langle \nabla f(u_n)-\lambda_n Au_n-\nabla f(y_n), p-y_n\rangle \le 0,$$

which implies that

$$\langle \nabla f(u_n)-\nabla f(y_n), p-y_n\rangle \le \lambda_n\langle Au_n, p-y_n\rangle. \qquad (3.41)$$

Substituting (3.41) into (3.40), we have

$$\begin{aligned}
D_f(p,z_n) &\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle Au_n, p-y_n\rangle\\
&\quad +\lambda_n\langle Ay_n-Au_n, p-z_n\rangle\\
&= D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle Au_n, p-y_n\rangle\\
&\quad +\lambda_n\langle Ay_n, p-z_n\rangle-\lambda_n\langle Au_n, p-z_n\rangle\\
&= D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle Au_n, z_n-y_n\rangle\\
&\quad +\lambda_n\langle Ay_n, p-z_n\rangle\\
&= D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle Au_n, z_n-y_n\rangle\\
&\quad -\lambda_n\langle Ay_n, y_n-p\rangle+\lambda_n\langle Ay_n, y_n-z_n\rangle\\
&= D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle Au_n-Ay_n, z_n-y_n\rangle\\
&\quad -\lambda_n\langle Ay_n, y_n-p\rangle.
\end{aligned}$$

Since $p \in VI(C,A)$ and $y_n \in C$, we have $\langle Ap, y_n-p\rangle \ge 0$, which implies by the pseudomonotonicity of A that $\langle Ay_n, y_n-p\rangle \ge 0$. From (2.4), we have

$$\begin{aligned}
D_f(p,z_n) &\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle Au_n-Ay_n, z_n-y_n\rangle\\
&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\|Au_n-Ay_n\|\|z_n-y_n\|\\
&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\mu\|u_n-y_n\|\|z_n-y_n\|\\
&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\frac{\mu}{2}\|u_n-y_n\|^2+\frac{\mu}{2}\|z_n-y_n\|^2
\end{aligned}$$

$$\leq D_f(p, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n). \tag{3.42}$$

Since $\mu \in (0, \sigma)$, we have $1 - \frac{\mu}{\sigma} = \frac{\sigma - \mu}{\sigma} > 0$. This implies that

$$\left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, x_n) + \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n) \geq 0.$$

Then, from (3.42), we obtain

$$D_f(p, z_n) \leq D_f(p, u_n).$$

This completes the proof. □

Theorem 3.10 *Suppose that Assumptions* A1–A5 *are satisfied. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 2 *converges strongly to* $z \in VI(C, A)$*, where* $z = \Pi^f_{VI(C,A)}(x_1)$.

Proof The proof of theorem is quite similar to that of Theorem 3.7, so we omit it here. □

Next, we also utilize Algorithms 1 and 2 for solving the problem (VIP) with fixed point constraints.

Let C be a nonempty subset of H and $S : C \to C$ be a mapping with a fixed point set is nonempty, that is, $F(S) := \{x \in C : x = Sx\} \neq \emptyset$. A point $z \in C$ is called an *asymptotic fixed point* of S (Reich 1996) if C contains a sequence $\{x_n\}$, which converges weakly to z and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. We denote $\widehat{F}(S)$ by the set of asymptotic fixed points of S. A mapping S is said to be *Bregman quasi-nonexpansive* (Butnariu et al. 2001) if $F(S) \neq \emptyset$ and $D_f(v, Sx) \leq D_f(v, x)$ for all $v \in F(S)$ and $x \in C$. We know that $F(S)$ is closed and convex if f is a Legendre function (see Reich and Sabach 2011).

Algorithm 3: Relaxed inertial subgradient extragradient algorithm for the problem (VIP) with fixed point constraints

Step 0. Given $\theta \in [0, 1/2]$, $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.4). Let $x_0, x_1 \in H$ be arbitrary.

Step 1. Given the current iterates x_{n-1} and x_n for each $n \geq 1$. Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where $\bar{\theta}_n$ is defined by (3.2). Set

$$u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n)))$$

and compute

$$y_n = \Pi^f_C \nabla f^*(\nabla f(u_n) - \lambda_n A u_n),$$

where λ_n is defined in (3.3).

Step 2. Construct the half-space

$$T_n = \{x \in H : \langle \nabla f(u_n) - \lambda_n A u_n - \nabla f(y_n), x - y_n \rangle \leq 0\}$$

and compute

$$z_n = \Pi^f_{T_n} \nabla f^*(\nabla f(u_n) - \lambda_n A y_n).$$

(Step 3) Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1 - \alpha_n)(\beta_n \nabla f(z_n) + (1 - \beta_n) \nabla f(S z_n))).$$

Update $n := n + 1$ go to Step 1.

Theorem 3.11 *Suppose that Assumptions* A1–A5 *are satisfied. Let* $S : H \to H$ *be a Bregman quasi-nonexpansive mapping such that* $F(S) = \widehat{F}(S)$ *and* $\{\beta_n\} \subset (0, 1)$ *such that*

$$\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

If $\Omega := VI(C, A) \cap F(S) \neq \emptyset$*, then the sequence* $\{x_n\}$ *generated by Algorithm* 3 *converges strongly to* $z \in \Omega$*, where* $z = \Pi^f_\Omega(x_1)$.

Proof As proved in Theorem 3.7, it follows that $\{x_n\}$ is bounded and consequently $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. Let $z \in \Omega$ and $w_n = \nabla f^*(\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Sz_n))$ for all $n \geq 1$. Since f is uniformly Fr*é*chet differentiable, we have f is also uniformly smooth (see Zălinescu 2002, p. 207). This implies that f^* is uniformly convex (see Zălinescu 2002, Theorem 3.5.5). By the property of V_f and Lemma 3.5, we have

$$\begin{aligned}
D_f(z, w_n) &= V_f(z, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Sz_n)) \\
&= f(z) - \langle z, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Sz_n)\rangle + f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Sz_n)) \\
&\leq \beta_n f(z) + (1-\beta) f(z) - \beta_n \langle z, \nabla f(z_n)\rangle - (1-\beta_n)\langle z, \nabla f(Sz_n)\rangle + \beta_n f^*(\nabla f(z_n)) \\
&\quad + (1-\beta_n) f^*(\nabla f(Sz_n)) - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|) \\
&= \beta_n(f(z) - \langle z, \nabla f(z_n)\rangle + f^*(\nabla f(z_n))) \\
&\quad + (1-\beta_n)(f(z) - \langle z, \nabla f(Sz_n)\rangle + f^*(\nabla f(Sz_n))) \\
&\quad - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|) \\
&= \beta_n D_f(z, z_n) + (1-\beta_n) D_f(z, Sz_n) - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|) \\
&\leq \beta_n D_f(z, z_n) + (1-\beta_n) D_f(z, z_n) - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|) \\
&= D_f(z, z_n) - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|) \\
&\leq D_f(z, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n) \\
&\quad - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|).
\end{aligned}$$

From (3.25), it follows that

$$\begin{aligned}
D_f(z, w_n) \leq (1-\theta_n) D_f(z, x_n) + \theta_n D_f(p, x_{n-1}) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) \\
- \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n) - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|).
\end{aligned} \tag{3.43}$$

It follows that

$$\begin{aligned}
D_f(z, x_{n+1}) &\leq \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, w_n) \\
&\leq \alpha_n D_f(z, x_1) + (1-\alpha_n)(1-\theta_n) D_f(p, x_n) + \theta_n D_f(p, x_{n-1}) \\
&\quad - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n) \\
&\quad - \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|).
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\alpha_n)\left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, u_n) + (1-\alpha_n)\left(1 - \frac{\mu}{\sigma}\right) D_f(z_n, y_n) \\
&\quad + \beta_n(1-\beta_n)\phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|) \\
&\leq D_f(z, x_n) - D_f(z, x_{n+1}) + (1-\alpha_n)\theta_n(D_f(z, x_{n-1}) - D_f(z, x_n)) + \alpha_n K,
\end{aligned}$$

where $K := \sup_{n\geq 1}\{|D_f(z,x_1) - D_f(z,x_n)|\}$. Obviously, as in the proof of Theorem 3.7, we have

$$\lim_{n\to\infty} D_f(y_n,u_n) = \lim_{n\to\infty} D_f(z_n,y_n) = \lim_{n\to\infty} \phi^*(\|\nabla f(z_n) - \nabla f(Sz_n)\|) = 0.$$

Hence, we have

$$\lim_{n\to\infty} \|\nabla f(y_n) - \nabla f(u_n)\| = \lim_{n\to\infty} \|\nabla f(z_n) - \nabla f(y_n)\| = 0.$$

By the property of ϕ^*, we have $\lim_{n\to\infty} \|\nabla f(z_n) - \nabla f(Sz_n)\| = 0$ and hence $\lim_{n\to\infty} \|z_n - Sz_n\| = 0$. Moreover, we can show that

$$\lim_{n\to\infty} \|\nabla f(z_n) - \nabla f(u_n)\| = 0 \tag{3.44}$$

and

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(x_n)\| = 0. \tag{3.45}$$

It follows from (3.44) and (3.45) that

$$\|\nabla f(z_n) - \nabla f(x_n)\| \leq \|\nabla f(z_n) - \nabla f(u_n)\| + \|\nabla f(u_n) - \nabla f(x_n)\| \to 0 \quad \text{as } n \to \infty.$$

Hence, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.46}$$

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$. From (3.46), also it follows that $z_{n_k} \rightharpoonup v$ and, since $\|z_n - Sz_n\| \to 0$, we have $v \in \widehat{F}(S) = F(S)$. In the rest of the proof, we follow the lines of the proof of Theorem 3.7 and hence it is omitted. This completes the proof. □

Algorithm 4: Relaxed inertial Tseng's extragradient algorithm for the problem (VIP) with fixed point constraints

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Step 0. Given $\theta \in [0, 1/2]$, $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.4). Let $x_0, x_1 \in H$ be arbitrary.

Step 1. Given the current iterates x_{n-1} and x_n for each $n \geq 1$. Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where $\bar{\theta}_n$ is defined by (3.2). Set

$$u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n)))$$

and compute

$$y_n = \Pi_C^f \nabla f^*(\nabla f(u_n) - \lambda_n Au_n),$$

where λ_n is defined in (3.3).

Step 2. Compute

$$z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Au_n)).$$

Step 3. Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_1) + (1-\alpha_n)(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Sz_n))).$$

Update $n := n + 1$ go to Step 1.

Theorem 3.12 *Suppose that Assumptions* A1–A5 *are satisfied. Let $S : H \to H$ be a Bregman quasi-nonexpansive mapping such that $F(S) = \widehat{F}(S)$ and $\{\beta_n\} \subset (0, 1)$ such that*

$$\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

If $\Omega := VI(C, A) \cap F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 4 *converges strongly to $z \in \Omega$, where $z = \Pi_\Omega^f(x_1)$.*

Proof The proof of theorem is quite similar to that of Theorems 3.7 and 3.11, so we omit it here. □

4 Numerical experiments

In this section, we provide some numerical experiments with a non-Euclidean distance to illustrate the convergence behavior of the proposed algorithms.

Let $H = \mathbb{R}^m$, then $\nabla f^* = (\nabla f)^{-1}$. The following lists are values of $(\nabla f)^{-1}$ for various functions in Example 2.2:

(1) For $f^{\mathrm{KL}}(x)$, we have $\left(\nabla f^{\mathrm{KL}}\right)^{-1}(x) = (\exp(x_1-1), \exp(x_2-1), \ldots, \exp(x_m-1))^{\mathrm{T}}$.

(2) For $f^{\mathrm{IS}}(x)$, we have $\left(\nabla f^{\mathrm{IS}}\right)^{-1}(x) = -\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_m}\right)^T$.

(3) For $f^{\mathrm{FD}}(x)$, we have $\left(\nabla f^{\mathrm{FD}}\right)^{-1}(x) = \left(\frac{\exp(x_1)}{1+\exp(x_1)}, \frac{\exp(x_2)}{1+\exp(x_2)}, \ldots \frac{\exp(x_m)}{1+\exp(x_m)}\right)^{\mathrm{T}}$.

(4) For $f^{\mathrm{SM}}(x)$, we have $\left(\nabla f^{\mathrm{SM}}\right)^{-1}(x) = Q^{-1}x$.

(5) For $f^{\mathrm{SE}}(x)$, we have $\left(\nabla f^{\mathrm{SE}}\right)^{-1}(x) = x$.

Note that each f satisfies Assumption A2 (see Bauschke et al. 2009; Hieu and Cholamjiak 2020). Let C be the feasible set given by

$$C := \{x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}} \in \mathbb{R}^m : \|x\| \le 1,\ x_i \ge a > 0,\ i = 1, 2, \ldots, m\},$$

where $a < \frac{1}{\sqrt{m}}$. Also, we can calculate explicitly the Hessian matrix of each f. Then, it is easy to check that $\nabla^2 f^{\mathrm{KL}}(x) \succeq I$, $\nabla^2 f^{\mathrm{IS}}(x) \succeq I$, $\nabla^2 f^{\mathrm{FD}}(x) \succeq I$, $\nabla^2 f^{\mathrm{SM}}(x) \succeq I$ and $\nabla^2 f^{\mathrm{SE}}(x) \succeq I$ for all $x \in C$. This implies that all functions are strongly convex on C with $\sigma = 1$ (see Hieu and Cholamjiak 2020).

Example 4.1 Let $A : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 100)$ be an operator given by

$$Ax := \frac{1}{\|x\|^2 + 1} \arg\min_{y\in\mathbb{R}^m} \left\{ \frac{\|y\|^4}{4} + \frac{1}{2}\|x-y\|^2 \right\}.$$

Then, A is continuous pseudomonotone but not monotone. We choose $\theta = 0.333$, $\gamma = 2$, $l = 0.5$, $\mu = 0.38$, $\alpha_n = \frac{1}{n+1}$, $\xi_n = \alpha_n^2$ and two cases for θ_n, that is, $\theta_n = \theta_n^{\max} := \bar{\theta}_n$ and $\theta_n = \theta_n^{\min} := 0$.

Note that, when $\theta_n = \theta_n^{\min}$, Algorithms 1 and 2 are the modified method (SEGM) and the modified method (TEGM) without inertial terms, respectively. We use $E_n = \|u_n - y_n\| < 10^{-5}$ as the stopping criterion and the starting points x_0, x_1 are generated randomly in $\mathbb{R}^m$. In this experiments, we compare Algorithms 1 and 2 with Algorithms 1 and 2 without the inertial terms. The numerical results of our methods have been reported in the Table 1 and Fig. 4.

Table 1 Numerical results for Example 4.1

Bregman divergence	Algorithm 1 ($\theta_n = \theta_n^{\min}$)		Algorithm 1 ($\theta_n = \theta_n^{\max}$)		Algorithm 2 ($\theta_n = \theta_n^{\min}$)		Algorithm 2 ($\theta_n = \theta_n^{\max}$)	
	Iter.	Time	Iter.	Time	Iter.	Time	Iter.	Time
D_f^{KL}	25	0.0092	18	0.0041	16	0.0309	12	0.0124
D_f^{IS}	70	0.0163	65	0.0160	46	0.0395	38	0.0219
D_f^{FD}	15	0.0127	11	0.0097	8	0.0219	6	0.0068
D_f^{SM}	4	0.0082	2	0.0022	4	0.0156	3	0.0052

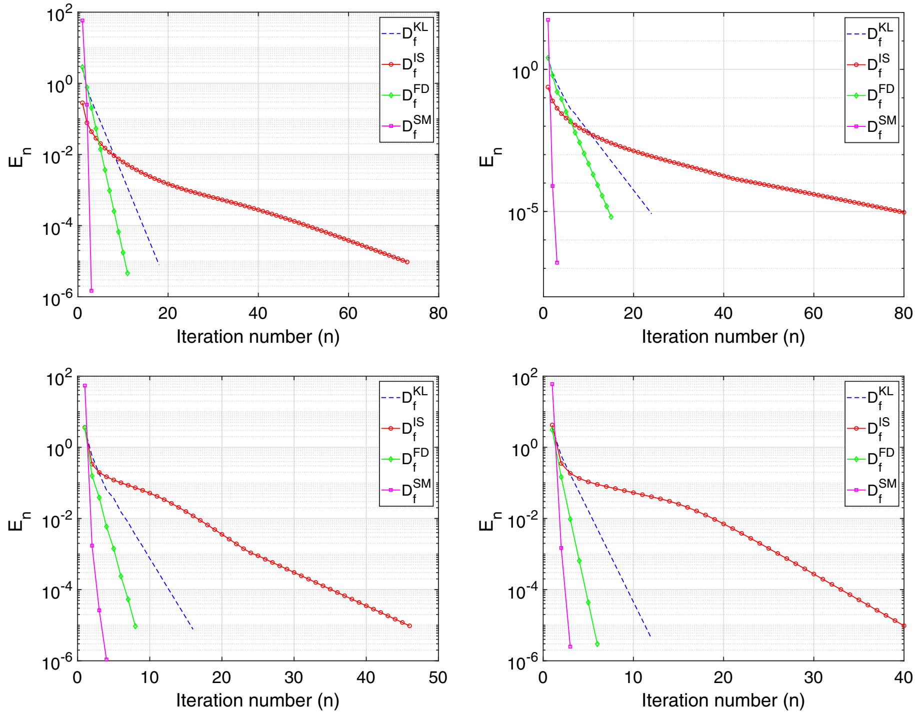

Fig. 4 Numerical result for Example 4.1. Top: Algorithm 1, left ($\theta_n = \theta_n^{\min}$), right ($\theta_n = \theta_n^{\max}$); bottom: Algorithm 2, left ($\theta_n = \theta_n^{\min}$), right ($\theta_n = \theta_n^{\max}$)

Remark 4.2 From aforementioned numerical results as above, we summarize the performance of our methods as follows:

(1) Algorithms 1 and 2 with relaxed inertial terms ($\theta_n = \theta_n^{\max}$) have a good running effect than the algorithms without relaxed inertial terms ($\theta_n = \theta_n^{\min}$) in each the Bregman divergence. This assured that adding the relaxed inertial term to algorithms has an effect similar to the classical inertial algorithms for solving the problem.

(2) Algorithms 1 and 2 with the Bregman divergence D_f^{SM} have a number of iterations and elapsed times less than the algorithms with the Bregman divergences D_f^{KL}, D_f^{IS} and D_f^{FD}. This is because the structure of D_f^{SM} is not complicated to perform.

In what follows, we let $f(x) = f^{\mathrm{SE}}(x) = \frac{1}{2}\|x\|^2$ for all $x \in H$. Then, D_f^{SE} is the Square Euclidean divergence, that is, $D_f^{\mathrm{SE}}(x, y) = \frac{1}{2}\|x - y\|^2$ for all $x, y \in H$. Next, we provide numerical experiments to illustrate the performance of our algorithms in solving the image deblurring problem and also compare them with Algorithm A proposed in Thong and Vuong (2019), Algorithm 1 and Algorithm B proposed in Khanh et al. (2020), Algorithm 3.1.

Example 4.3 The digital image restoration problem plays an important role in many applications of science and engineering such as film restoration, image and video coding, medical and astronomical imaging, etc. (Figueiredo et al. 2007; Shehu et al. 2020; Xiao and Zhu 2013). Restoring an image from a degraded one is typically an ill-posed inverse problem,

Table 2 Computational result for Example 4.3

Algorithms	Pout		Cameraman	
	Time (s)	SNR	Time (s)	SNR
Algorithm 1	28.6139	34.2679	26.0414	31.0415
Algorithm 2	26.9383	34.3372	24.6394	31.0582
Algorithm A	38.6904	34.0122	26.5851	33.3580
Algorithm B	45.6154	32.5071	36.8937	29.6873

which can be modeled by the following linear equation:

$$b = Bx + v, \tag{4.1}$$

where $x \in \mathbb{R}^N$ is the original image, $b \in \mathbb{R}^M$ is the degraded image, $B \in \mathbb{R}^{M\times N}$ is the blurring matrix and v is an additive noise. An efficient method for recovering the original image is the ℓ_1-norm regularized least square method given by

$$\min_{x\in\mathbb{R}^N}\left\{\frac{1}{2}\|Bx-b\|_2^2+\lambda\|x\|_1\right\}, \tag{4.2}$$

where $\|x\|_2$ is the Euclidean norm of x and $\|x\|_1 := \sum_{i=1}^N |a_i|$ is the l_1-norm of x. Our main task is to restore the original image x given the data of the blurred image b. Several iterative algorithms have been introduced for treating such problems with the earliest being the projection method by Figueiredo et al. (2007). More so, the least square problem (4.2) can be expressed as a variational inequality problem by setting $A = B^{\mathrm{T}}(Bx-b)$. It is known that the operator A in this case is monotone and $\|B^{\mathrm{T}}B\|$-Lipschitz continuous (hence, it is pseudomonotone and uniformly continuous).

We consider the grey scale image of M pixels wide and N pixel height, each value is known to be in the range $[0, 255]$. The quality of the restored image is measured by the signal-to-noise ratio defined by

$$\mathrm{SNR} := 20\log_{10}\left(\frac{\|x\|_2}{\|x-x^*\|_2}\right),$$

where x is the original image and x^* is the restored image. Note that the larger the value of SNR, the better the quality of the restored image.

In our experiments, we use the grey test image Pout (291×240) and Cameraman (256×256), each test image is degraded by Gaussian 7×7 blur kernel with standard deviation 4. We choose $\gamma = 2$, $l = 0.36$, $\mu = 0.64$, $x_0 = \mathbf{0} \in \mathbb{R}^{\mathbf{D}}$ and $x_1 = \mathbf{1} \in \mathbb{R}^{\mathbf{D}}$, where $\mathbf{D} = M \times N$. Also, we choose $\alpha_n = \frac{1}{200(n+1)}$, $\xi_n = \alpha_n^2$, $\theta = 0.0266$ and $\theta_n = \theta_n^{\max} := \bar{\theta}_n$.

Figures 5 and 6 show the original, blurred and restored image by using the Algorithms 1, 2, A and B. Also, Fig. 7 shows the graph of SNR against number of iterations for each test image using the algorithms. More so, we report the time (in seconds) for each algorithm in Table 2. The computational results shows that Algorithms 1 and 2 are more efficient for restoring the degraded image than Algorithms A and B.

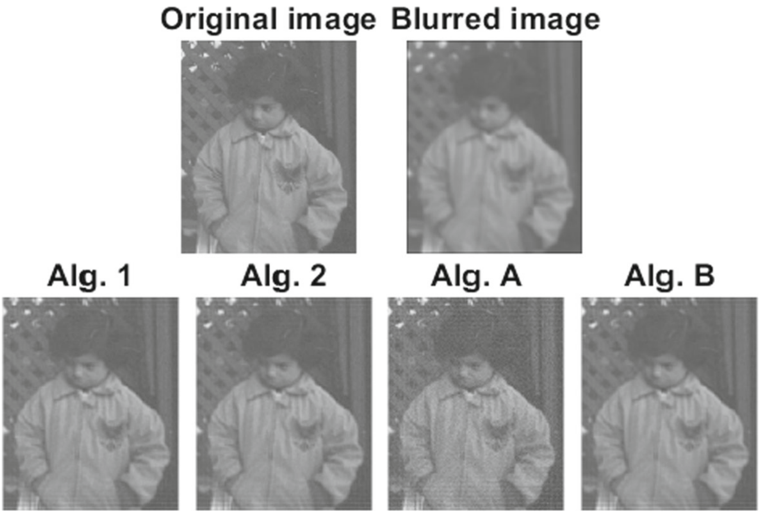

Fig. 5 Example 4.3, top shows original image of Pout (left) and degraded image of Pout (right); bottom shows recovered image by Algorithms 1, 2, Algorithm A and Algorithm B

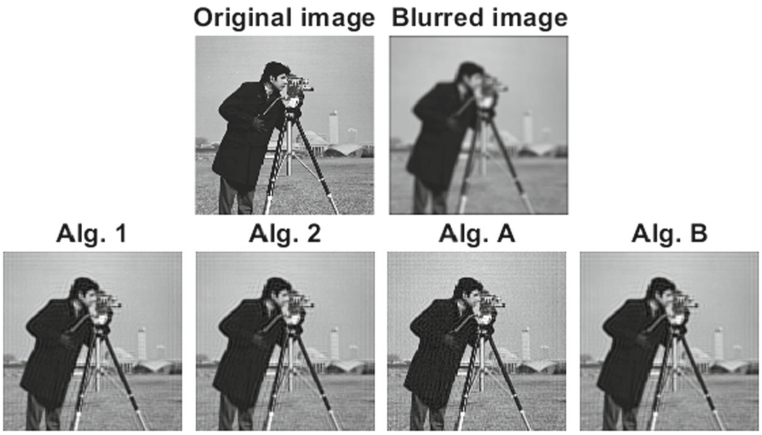

Fig. 6 Example 4.3, top shows original image of Cameraman (left) and degraded image of Cameraman (right); bottom shows recovered image by Algorithms 1, 2, Algorithm A and Algorithm B

5 Conclusions

In this paper, we have proposed and analyzed two Halpern relaxed inertial type algorithms with the Bregman divergence for approximating solutions of the pseudomonotone problem (VIP) in real Hilbert spaces. The strong convergence of the sequences generated by the proposed algorithms are established without assuming the Lipschitz continuity and the sequential weak continuity of the cost mapping. Finally, we give some numerical experiments to illustrate the performance and efficiency of the proposed methods in comparison with some existing methods.

In fact, we know that the following facts depend on the convergence rate of the proposed methods and the existence of a solution of the problem (VIP):

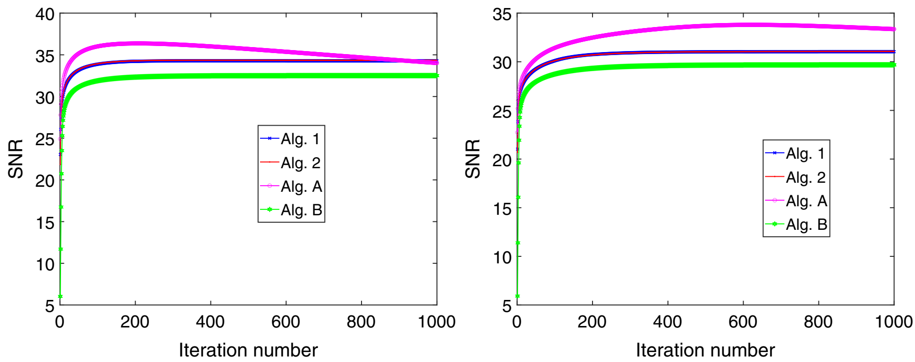

Fig. 7 Example 4.3: graphs of SNR values against number of iteration for Pout (left) and Cameraman (right)

(1) The inertial term;
(2) The stepsize;
(3) The Lipschitz constant:
(4) The Armijo linesearch rule:
(5) The pseudomonotonicity or the monotonicity of the given mapping;
(6) The norm of the given mapping.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

Alvarez F, Attouch H (2001) An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Anal 9:3–11

Amid E, Warmuth M K, Anil R, Koren T (2019) Robust bi-tempered logistic loss based on Bregman divergences. In: Conference on neural information processing systems, pp 14987–14996

Anh PK, Thong DV, Vinh NT (2020) Improved inertial extragradient methods for solving pseudomonotone variational inequalities. Optimization. https://doi.org/10.1080/02331934.2019.1616191

Antipin AS (1976) On a method for convex programs using a symmetrical modification of the Lagrange function. Ekonomika i Matematicheskie Metody 12:1164–1173

Banerjee A, Merugu S, Dhillon IS, Ghosh J (2005) Clustering with Bregman divergences. J Mach Learn Res 6:1705–1749

Bauschke HH, Combettes PL (2001) A weak-to-strong convergence principle for Féjer monotone methods in Hilbert spaces. Math Oper Res 26(2):248–264

Bauschke HH, Wang X, Ye J, Yuan X (2009) Bregman distances and Chebyshev sets. J Approx Theory 159:3–25

Beck A (2017) First-order methods in optimization. Society for Industrial and Applied Mathematics, Philadelphia

Bot RI, Csetnek ER, Vuong PT (2020) The forward–backward–forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces. Eur J Oper Res 287:49–60

Bregman LM (1967) The relaxation method of finding the common points of convex sets and its application to the solution of problems in convex programming. USSR Comput Math Math Phys 7(3):200–217
Butnariu D, Resmerita E (2006) Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr Appl Anal 2006:1–39 (**art. ID 84919**)
Butnariu D, Reich S, Zaslavski AJ (2001) Asymptotic behavior of relatively nonexpansive mappings in Banach spaces. J Appl Anal 7:151–174
Censor Y, Gibali A, Reich S (2011) The subgradient extragradient method for solving variational inequalities in Hilbert space. J Optim Theory Appl 148:318–335
Chbani Z, Riahi H (2013) Weak and strong convergence of an inertial proximal method for solving Ky Fan minimax inequalities. Optim Lett 7:185–206
Cholamjiak P, Thong DV, Cho YJ (2020) A novel inertial projection and contraction method for solving pseudomonotone variational inequality problems. Acta Appl Math 169:217–245
Combettes PL, Pesquet JC (2020) Deep neural structures solving variational inequalities. Set-Valued Var. Anal. 28:491–518
Cottle RW, Yao JC (1992) Pseudo-monotone complementarity problems in Hilbert space. J Optim Theory Appl 75:281–295
Denisov SV, Semenov VV, Chabak LM (2015) Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. Cybern Syst Anal 51:757–765
Figueiredo MAT, Nowak RD, Wright SJ (2007) Gradient projection for sparse reconstruction: application to compressed sensing and other inverse problems. IEEE J Sel Top Signal Process 1:586–597
Gibali A (2018) A new Bregman projection method for solving variational inequalities in Hilbert spaces. Pure Appl Funct Anal. 3:403–415
Gibali A, Jolaoso LO, Mewomo OT, Taiwo A (2020) Fast and simple Bregman projection methods for solving variational inequalities and related problems in Banach spaces. Results Math 75:179
Güler O (1991) On the convergence of the proximal point algorithms for convex minimization. SIAM J Control Optim 29:403–419
Halpern B (1967) Fixed points of nonexpanding maps. Bull Am Math Soc 73:95–961
Hartman P, Stampacchia G (1966) On some non-linear elliptic differential-functional equations. Acta Math 115:271–310
Hieu DV, Cholamjiak P (2020) Modified extragradient method with Bregman distance for variational inequalities. Appl Anal. https://doi.org/10.1080/00036811.2020.1757078
Hieu DV, Cho YJ, Xiao YB, Kumam P (2020) Relaxed extragradient algorithm for solving pseudomonotone variational inequalities in Hilbert spaces. Optimization 69:2279–2304
Iusem AN, Gárciga Otero R (2001) Inexact versions of proximal point and augmented Lagrangian algorithms in Banach spaces. Numer Funct Anal Optim 22:609–640
Iusem AN, Svaiter BF (1997) A variant of Korpelevich's method for variational inequalities with a new search strategy. Optimization 42:309–321
Jolaoso LO (2021) An inertial projection and contraction method with a line search technique for variational inequality and fixed point problems. Optimization. https://doi.org/10.1080/02331934.2021.1901289
Juditsky A, Nemirovski A (2016) Solving variational inequalities with monotone operators on domains given by linear minimization oracles. Math Progr 156:221–256
Khanh PD, Vuong PT (2014) Modified projection method for strongly pseudo-monotone variational inequalities. J Glob Optim 58:341–350
Khanh PQ, Thong DV, Vinh NT (2020) Versions of the subgradient extragradient method for pseudomonotone variational inequalities. Acta Appl Math 170:319–345
Kimura Y, Nakajo K, Takahashi W (2009) Convexity of the set of fixed points of a quasi-pseudocontractive type Lipschitz mapping and the shrinking projection method. Sci Math Jpn 70:213–220
Kinderlehrer D, Stampacchia G (1980) An introduction to variational inequalities and their applications. Academic Press Inc., New York
Korpelevich GM (1976) The extragradient method for finding saddle points and other problems. Matecon 12:747–756
Liu L, Qin X (2020) Strong convergence of an extragradient-like algorithm involving pseudo-monotone mappings. Numer Algorithms 83:1577–1590
Luo MJ, Zhang Y (2017) Robust solutions to box-contrained stochastic linear variational inequality problem. J Inequal Appl 2017:253
Maingé PE (2008) Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal 16:899–912
Malitsky YV, Semenov VV (2014) An extragradient algorithm for monotone variational inequalities. Cybern Syst Anal 50:271–277

Martín-Márquez V, Reich S, Sabach S (2013) Iterative Methods for approximating fixed points of Bregman nonexpansive operators. Discret Contin Dyn Syst Ser S 6(4):1043–1063
Moudafi A (2000) Viscosity approximation methods for fixed-points problems. J Math Anal Appl 241:46–55
Nakajo K, Takahashi W (2003) Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J Math Anal Appl 279:372–379
Polyak BT (1964) Some methods of speeding up the convergence of iterative methods. Zh Vychisl Mat Mat Fiz 4:1–17
Popov LD (1980) A modification of the Arrow–Hurwicz method for searching for saddle points. Mat Zametki 28:777–784
Reich S (1996) A weak convergence theorem for the alternating method with Bregman distances. In: Theory and applications of nonlinear operators of accretive and monotone type. Marcel Dekker, New York, pp 313–318
Reich S, Sabach S (2009) A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. 10:471–485
Reich S, Sabach S (2011) Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces. In: Fixed-point algorithms for inverse problems in science and engineering, Springer optimization and applications, vol 49. Springer, New York, pp 301–316
Rockafellar RT (1970) Convex analysis. Princeton University Press, Princeton
Shehu Y, Iyiola OS, Ogbuisi FU (2020) Iterative method with inertial terms for nonexpansive mappings, applications to compressed sensing. Numer Algorithms 83:1321–1347
Takahashi W (2009) Introduction to nonlinear and convex analysis. Yokohama Publishers, Yokohama
Tan B, Qin X, Yao JC (2021) Two modified inertial projection algorithms for bilevel pseudomonotone variational inequalities with applications to optimal control problems. Numer Algorithms. https://doi.org/10.1007/s11075-021-01093-x
Thong DV, Vuong PT (2019) Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities. Optimization 68:2207–2226
Thong DV, Vinh NT, Hieu DV (2019) Accelerated hybrid and shrinking projection methods for variational inequality problems. Optimization 68(5):981–998
Thong DV, Li XH, Dong QL, Cho YJ, Rassias TM (2021) An inertial Popov's method for solving pseudomonotone variational inequalities. Optimization Letters 15:757–777
Tseng P (2000) A modified forward-backward splitting method for maximal monotone mapping. SIAM J Control Optim 38:431–446
Xiao Y, Zhu H (2013) A conjugate gradient method to solve convex constrained monotone equations with applications in compressive sensing. J Math Anal Appl 405:310–319
Yang J (2021) Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities. Appl Anal 100:1067–1078
Zălinescu C (2002) Convex analysis in general vector spaces. World Scientific, Publishing Co., Inc, River Edge

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A Self-Adaptive Method for Split Common Null Point Problems and Fixed Point Problems for Multivalued Bregman Quasi-Nonexpansive Mappings in Banach Spaces

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Abstract. In this paper, we propose a self-adaptive algorithm for solving the split common null point problem and the fixed point problem for multivalued Bregman quasi-nonexpansive mappings in Banach spaces. We prove that the sequence generated by our iterative scheme converges strongly to a common solution of the above-mentioned problems under some suitable conditions. We also apply our main result to split feasibility problems in Banach spaces. Finally, numerical examples are given to support our main theorem. The results presented in this paper improve and extend many recent results in the literature.

1. Introduction

Let E_1 and E_2 be two real Banach spaces. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be two set-valued maximal monotone operators and $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$. The *split common null point problem* (SCNPP) is formulated as finding $x^* \in E_1$ such that

$$0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1}$$

This formalism is also at the core of the modeling of many inverse problems and other real life problems, for instance, in practice as a model in intensity-modulated radiation therapy treatment planning (see [15, 19]) and in sensor networks in computerized tomography and data compression (see [14]).

To solve the SCNPP in two Hilbert spaces H_1 and H_2, Byrne et al. [11] introduced the following algorithms: for $u, x_1 \in H_1$, compute the sequences $\{x_n\}$ generated iteratively by

$$x_{n+1} = J_\lambda(x_n - \gamma A^*(I - Q_\mu)Ax_n), \quad \forall n \geq 1 \tag{2}$$

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and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_\lambda(x_n - \gamma A^*(I - Q_\mu)Ax_n), \ \ \forall n \geq 1, \tag{3}$$

where J_λ and Q_μ are the resolvent operators of B_1 and B_2 for $\lambda, \mu > 0$, respectively, and the parameter γ satisfies $0 < \gamma < \frac{2}{\|A\|^2}$. They obtained weak and strong convergence results of (2) and (3), respectively under some control conditions.

Alofi et al. [4] introduced the modified Halpern's iteration for solving the SCNPP (1) in the case that E_1 is a Hilbert space and E_2 is a Banach space as follows:

$$\begin{cases} x_1 \in E_1, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A^* J_E(I - Q_{\mu_n})Ax_n)), \ \ \forall n \geq 1, \end{cases} \tag{4}$$

where J_E is the duality mapping on E_2, $\{u_n\}$ is a sequence in E_1 such that $u_n \to u$, and the stepsize λ_n satisfies $0 < a \leq \lambda_n \|A\|^2 \leq b < 2$ for some $a, b > 0$. Under some suitable assumptions, they proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a solution of the SCNPP.

Suantai et al. [49] also proposed the following algorithm for solving the SCNPP (1) between a Hilbert space and a Banach space:

$$\begin{cases} x_1 \in E_1, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n}(x_n - \lambda_n A^* J_E(I - Q_{\mu_n})Ax_n), \ \ \forall n \geq 1, \end{cases} \tag{5}$$

where $f : E_1 \to E_1$ is a contraction and the stepsize λ_n satisfies $0 < a \leq \lambda_n \|A\|^2 \leq b < 2$ for some $a, b > 0$. They proved a strong convergence result of $\{x_n\}$ generated by (5) under some suitable conditions. Recently, some iterative methods have been proposed and invented independently for solving such a problem in many different contexts (see for instance [24, 51, 53, 54, 56, 57]).

However, it is observed that the choice of the stepsize of the above results and other corresponding results depend on the operator norm or the matrix norm (in the finite-dimensional space). As a result, the implementation of such algorithms are usually difficult to handle (see [23]). To overcome this difficulty, López et al. [30] suggested an algorithm so-called a *self-adaptive method* for solving the split feasibility problem (SFP) in Hilbert spaces. We note that the SFP is an interest special case of SCNPP and it is very important in nonlinear analysis. To be more precise, they proposed the following method, which permits the stepsize λ_n being selected self-adaptively in such a way

$$\lambda_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \tag{6}$$

where $\{\rho_n\} \subset (0, 4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$ (P_C and P_Q denote the metric projections on C and Q, respectively). They proposed an iterative method for solving the SFP in two Hilbert spaces as follows:

$$\begin{cases} u, x_1 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)P_C(x_n - \lambda_n \nabla f(x_n)), \ \ \forall n \geq 1, \end{cases} \tag{7}$$

where the stepsize λ_n is chosen in (6), and also proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a solution of the SFP provided $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

On the other hand, let E be a real Banach space. We consider the fixed point problem which is the problem of finding a point

$$x^* \in E \ \text{ such that } \ x^* = Tx^*, \tag{8}$$

where T is a nonlinear mapping on E. In real life, many mathematical models have been formulated as this problem. Currently, many mathematicians are interested in finding solutions of some optimization problems with fixed point constraints (see for instance [18, 25–28, 40–43, 46, 47]).

In this paper, inspired and motivated by the above-mentioned works, we introduce a self-adaptive algorithm for finding a common solution of the SCNPP and the fixed point problem for multivalued Bregman quasi-nonexpansive mappings in the framework of Banach spaces. We prove a strong convergence theorem of the sequence generated by our proposed method under some suitable conditions as shown in Sec. 3. Furthermore, in Sec. 4, the result for solving the split feasibility problem and the fixed point problem in Banach spaces is a consequence of our main result. In the last, Sec. 5, we give some numerical examples to demonstrate the convergence behavior of our algorithm and support our main theorem. The results presented in this paper improve and extend many recent results in the literature.

2. Preliminaries

Let E be a real Banach spaces with its the dual space E^* of E. We write $\langle x, j\rangle$ for the value of a functional j in E^* at x in E. We shall use the notations $x_n \to x$ means that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$ which is defined by

$$\langle x, A^*\bar{y}\rangle := \langle Ax, \bar{y}\rangle, \ \ \forall x \in E_1, \ \ \bar{y} \in E_2^*$$

and the equalities $\|A^*\| = \|A\|$ and $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ are valid, where $\mathcal{R}(A)^\perp := \{x^* \in E_2^* : \langle u, x^*\rangle = 0, \ \forall u \in \mathcal{R}(A)\}$. For more details on bounded linear operators and their duals, please see ([21, 50]).

Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}.$$

A space E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and *p-uniformly convex* if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$.

The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

A space E is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$ and *q-uniformly smooth* if there exists a $c_q > 0$ such that $\rho_E(\tau) \leq c_q\tau^q$ for all $\tau > 0$. Note that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [5]). Furthermore, L_p (or ℓ_p) and the Sobolev spaces are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ while a Hilbert space is 2-uniformly smooth (see [58]).

Definition 2.1. A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.2. The mapping $J_\varphi : E \multimap E^*$ associated with a gauge function φ defined by

$$J_\varphi(x) := \{f \in E^* : \langle x, f\rangle = \|x\|\varphi(\|x\|), \ \|f\| = \varphi(\|x\|), \ \ \forall x \in E\},$$

is called the *duality mapping with gauge* φ, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called *normalized duality mapping*. In the case $\varphi(t) = t^{p-1}$, where $p > 1$, the duality mapping $J_\varphi = J_p$ is called the *generalized duality mapping* which is defined by

$$J_p(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}.$$

It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|}J(x)$ and $J_p(x) = \|x\|^{p-2}J(x)$, $p > 1$. It is well known that if E is uniformly smooth, the generalized duality mapping J_p is norm to norm uniformly continuous on bounded subsets of E (see [35]). Furthermore, J_p is one-to-one, single-valued and satisfies $J_p = J_q^{-1}$, where J_q is the generalized duality mapping of E^* (see [13, 36] for more details).

The following lemma can be found in [5, Theorem 2.8.17] (see also [29, Lemma 5]).

Lemma 2.3. *Let $p > 1$, $r > 0$ and let E be a uniformly convex Banach space. Then there exists a strictly, increasing and convex function $g_r : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^p \leq t\|x\|^p + (1-t)\|y\|^p - t(1-t)g_r(\|x-y\|),$$

for all $x, y \in B_r := \{z \in E : \|z\| \leq r\}$ and $t \in [0, 1]$.

Lemma 2.4. [58] *Let E be a q-uniformly smooth Banach space. Then there exists a constant $c_q > 0$ which is called the q-uniform smoothness coefficient of E such that*

$$\|x-y\|^q \leq \|x\|^q - q\langle y, J_q(x)\rangle + c_q\|y\|^q,$$

for all $x, y \in E$.

Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E. The *metric projection* of $x \in E$ onto C is the unique element $P_C(x) \in C$ such that

$$\|x - P_C(x)\| = \min_{y\in C}\|x-y\|.$$

The metric projection can be also characterized by the following variational inequality:

$$\langle y - P_C(x), J_\varphi(x - P_C(x))\rangle \leq 0, \quad \forall y \in C.$$

For a gauge φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Phi(t) := \int_0^t \varphi(s)ds$ is a continuous, convex and strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi'(t) = \varphi(t)$ and $\lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty$. Therefore, Φ has a continuous inverse function Φ^{-1}.

We next recall the Bregman distance, which was introduced and studied in [10].

Definition 2.5. Let E be a real smooth Banach space. The Bregman distance $D_\varphi(x, y)$ between x and y in E is defined by

$$D_\varphi(x, y) := \Phi(\|x\|) - \Phi(\|y\|) - \langle x - y, J_\varphi(y)\rangle.$$

We note that $D_\varphi(x, y) \geq 0$ and $D_\varphi(x, y) = 0$ if and only of $x = y$. Moreover, the Bregman distance has the following important properties:

$$D_\varphi(x, y) + D_\varphi(y, x) = \langle x - y, J_\varphi(x) - J_\varphi(y)\rangle, \quad \forall x, y \in E \tag{9}$$

and

$$D_\varphi(x, y) = D_\varphi(x, z) + D_\varphi(z, y) + \langle z - y, J_\varphi(x) - J_\varphi(z)\rangle, \quad \forall x, y, z \in E. \tag{10}$$

For a smooth and uniformly convex Banach space E, then there exists a strictly, increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$g(\|x-y\|) \leq D_\varphi(x, y) \tag{11}$$

for all $x, y \in E$ (see [29]).

In the case $\varphi(t) = t^{p-1}$, $p > 1$, we have $\Phi(t) = \int_0^t \varphi(s)ds = \frac{t^p}{p}$. So we have the distance $D_\varphi = D_p$ is called the *p-Lyapunov function* which was studied in [12] and it is given by

$$D_p(x, y) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle x - y, J_p(y)\rangle. \tag{12}$$

It is easy to show that (12) equivalent to the following:

$$D_p(x,y) = \frac{1}{p}\|x\|^p - \langle x, J_p(y)\rangle + \frac{1}{q}\|y\|^p, \tag{13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 2$, we have $D_2(x,y) = \frac{1}{2}\phi(x,y)$, where ϕ is called the *Lyapunov function* which was introduced by Alber [1].

Following [32], we make use of the function $V_p : E \times E^* \to \mathbb{R}^+$ which is defined by

$$V_p(x,\bar{x}) := \frac{1}{p}\|x\|^p - \langle x, \bar{x}\rangle + \frac{1}{q}\|\bar{x}\|^q, \ \ \forall x \in E, \ \bar{x} \in E^*. \tag{14}$$

Note that V_p is nonnegative and

$$V_p(x,\bar{x}) = D_p(x, J_q(\bar{x})), \ \ \forall x \in E, \ \bar{x} \in E^*. \tag{15}$$

By the subdifferential inequality, we have

$$V_p(x,\bar{x}) + \langle J_q(\bar{x}) - x, \bar{y}\rangle \le V_p(x, \bar{x} + \bar{y}), \ \ \forall x \in E, \ \bar{x}, \bar{y} \in E^*. \tag{16}$$

Moreover, V_p is convex in the second variable. Then, for all $z \in E$,

$$D_p\Big(z, J_q\Big(\sum_{i=1}^{M} t_i J_p(x_i)\Big)\Big) \le \sum_{i=1}^{M} t_i D_p(z, x_i), \tag{17}$$

where $\{x_i\}_{i=1}^M \subset E$ and $\{t_i\}_{i=1}^M \subset (0,1)$ with $\sum_{i=1}^M t_i = 1$.

Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E. The *Bregman projection*, denoted by Π_C^φ, is defined as the unique solution of the following minimization problem:

$$\Pi_C^\varphi(x) := \operatorname{argmin}_{y \in C} D_\varphi(x,y), \ \ x \in E. \tag{18}$$

When $\varphi(t) = t$, we have Π_C^φ coincides with the generalized projection which studied in [1]. When $\varphi(t) = t^{p-1}$, where $p > 1$, we have Π_C^φ becomes the Bregman projection with respect to p and denoted by Π_C.

Proposition 2.6. ([29]) *Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E and let $x \in E$. Then the following assertions are equivalent:*

(i) $z = \Pi_C^\varphi(x)$ *if and only if* $\langle y - z, J_\varphi(x) - J_\varphi(z)\rangle \le 0, \forall y \in C$.

(ii) $D_\varphi(y, \Pi_C^\varphi(x)) + D_\varphi(\Pi_C^\varphi(x), x) \le D_\varphi(y,x), \forall y \in C$.

Lemma 2.7. ([33]) *Let E be a smooth and uniformly convex real Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then,* $\lim_{n\to\infty} D_p(x_n, y_n) = 0$ *if and only if* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.8. ([38]) *Let E be a smooth and uniformly convex real Banach space. Suppose that $x \in E$, if $\{D_p(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Let C be a nonempty, closed and convex subset of a Banach space E. Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty, closed and bounded subsets of C, respectively. Let $\mathcal{H}$ be the Hausdorff metric on $CB(C)$ defined by

$$\mathcal{H}(A,B) := \max\Big\{\sup_{x\in A} d(x,B), \sup_{y \in B} d(y,A)\Big\},$$

for all $A, B \in CB(C)$, where $d(a,B) = \inf_{b\in B}\{\|a-b\|\}$ is the distance from the point a to the subset B.

Let C be a nonempty subset of E and $T : C \to CB(C)$ be a multi-valued mapping. We denote the set of fixed point of T by $F(T)$, *i.e.*, $F(T) := \{x \in C : x \in Tx\}$. A point $z \in C$ is called an *asymptotic fixed point* of T, if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup z$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$. We denote $\widehat{F}(T)$ by the set of asymptotic fixed points of T. The concept of an asymptotic fixed point was introduced by Reich [37].

We now give the definitions of some classes of Bregman multi-valued mappings.

Definition 2.9. A multivalued mapping $T : C \to CB(C)$ is said to be

(1) *φ-Bregman nonexpansive* if

$$D_\varphi(u, v) \leq D_\varphi(x, y), \ \ \forall u \in Tx, \ v \in Ty \text{ and } x, y \in C,$$

(2) *φ-Bregman relatively nonexpansive* if $\widehat{F}(T) = F(T) \neq \emptyset$ and

$$D_\varphi(z, u) \leq D_\varphi(z, x), \ \ \forall u \in Tx, \ x \in C \text{ and } z \in F(T),$$

(3) *φ-Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_\varphi(z, u) \leq D_\varphi(z, x), \ \ \forall u \in Tx, \ x \in C \text{ and } z \in F(T).$$

We remark that the class of φ-Bregman quasi-nonexpansive is more general than class of φ-Bregman relatively nonexpansive mappings and φ-Bregman nonexpansive mappings with nonempty fixed point set.

Remark 2.10. *(i) In the case $\varphi(t) = t^{p-1}$, where $p > 1$, we have φ-Bregman quasi-nonexpansive, φ-Bregman relatively nonexpansive and φ-Bregman nonexpansive mappings become Bregman quasi-nonexpansive, Bregman relatively nonexpansive and Bregman nonexpansive mappings, respectively.*

(ii) In a Hilbert space H and $\varphi(t) = t$, a Bregman quasi-nonexpansive mapping and quasi-nonexpansive mapping are equivalent, for $D_2(x, y) := \|x - y\|^2$ for all $x, y \in H$, i.e.,

$$D_2(z, u) \leq D_2(z, x) \iff \|z - u\| \leq \|z - x\|, \ \ \forall u \in Tx, x \in C \ \ and \ \ z \in F(T).$$

Let E be a Banach space and $B : E \multimap E^*$ be a mapping. The effective domain of B is denoted by $\mathcal{D}(B)$, *i.e.*, $\mathcal{D}(B) := \{x \in E : Bx \neq \emptyset\}$ and the range of B is also denoted by $\mathcal{R}(B) := \bigcup_{x \in \mathcal{D}(B)} Bx$. A multi-valued mapping B is said to be *monotone* if

$$\langle x - y, u - v \rangle \geq 0, \ \ \forall x, y \in \mathcal{D}(B), \ u \in Bx \text{ and } v \in By. \tag{19}$$

A monotone operator B on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.

Definition 2.11. Let E be a strictly convex, smooth and reflexive Banach space and let $B : E \multimap E^*$ be a maximal monotone operator. For $\lambda > 0$, the *φ-metric resolvent* of B is operator $Q_\lambda^\varphi : E \to \mathcal{D}(B)$ defined by

$$Q_\lambda^\varphi(x) := (I + \lambda J_\varphi^{-1} B)^{-1}(x) \ \text{ for all } \ x \in E. \tag{20}$$

The set of null points of B is defined by $B^{-1}0 := \{z \in E : 0 \in Bz\}$ and it is known that $B^{-1}0$ is closed and convex (see [50]). We see that

$$0 \in J_\varphi(Q_\lambda^\varphi(x) - x) + \lambda B Q_\lambda^\varphi(x) \tag{21}$$

and $F(Q_\lambda^\varphi) = B^{-1}0$ for $\lambda > 0$. By (21), we see that

$$\frac{J_\varphi(x - Q_\lambda^\varphi(x))}{\lambda} \in B Q_\lambda^\varphi(x) \tag{22}$$

and

$$\frac{J_\varphi(y - Q^\varphi_\lambda(y))}{\lambda} \in BQ^\varphi_\lambda(y) \tag{23}$$

for all $x, y \in E$. Adding up (22) with (23) and using the monotonicity of B, we obtain

$$\langle Q^\varphi_\lambda x - Q^\varphi_\lambda y, J_\varphi(x - Q^\varphi_\lambda x) - J_\varphi(y - Q^\varphi_\lambda y)\rangle \geq 0, \tag{24}$$

for all $x, y \in E$. It is also known that, if $B^{-1}0 \neq \emptyset$, then

$$\langle Q^\varphi_\lambda x - z, J_\varphi(x - Q^\varphi_\lambda x)\rangle \geq 0, \tag{25}$$

for all $x \in E$ and $z \in B^{-1}0$ (see [6]).

In fact, let $\{x_n\}$ be a bounded sequence in E. From (25), we have

$$\begin{aligned} \|x_n - z\|\varphi(\|x_n - Q^\varphi_\lambda x_n\|) &\geq \langle x_n - z, J_\varphi(x_n - Q^\varphi_\lambda x_n)\rangle \\ &\geq \langle x_n - Q^\varphi_\lambda x_n, J_\varphi(x_n - Q^\varphi_\lambda x_n)\rangle \\ &= \|x_n - Q^\varphi_\lambda x_n\|\varphi(\|x_n - Q^\varphi_\lambda x_n\|), \end{aligned}$$

which implies that

$$\|x_n - Q^\varphi_\lambda x_n\| \leq \|x_n - z\|,$$

for $z \in B^{-1}0$. Hence, $\{x_n - Q^\varphi_\lambda x_n\}$ is bounded. Moreover, let $x_n \to x$ as $n \to \infty$, then from (11) and (24), we have

$$\begin{aligned} &\langle x_n - x, J_\varphi(x_n - Q^\varphi_\lambda x_n) - J_\varphi(x - Q^\varphi_\lambda x)\rangle \\ \geq\ &\langle x_n - Q^\varphi_\lambda x_n - (x - Q^\varphi_\lambda x), J_\varphi(x_n - Q^\varphi_\lambda x_n) - J_\varphi(x - Q^\varphi_\lambda x)\rangle \\ =\ &D_\varphi(x_n - Q^\varphi_\lambda x_n, x - Q^\varphi_\lambda x) + D_\varphi(x - Q^\varphi_\lambda x, x_n - Q^\varphi_\lambda x_n) \\ \geq\ &g(\|x_n - Q^\varphi_\lambda x_n - (x - Q^\varphi_\lambda x)\|) + g(\|x - Q^\varphi_\lambda x - (x_n - Q^\varphi_\lambda x_n)\|) \\ =\ &2g(\|x_n - Q^\varphi_\lambda x_n - (x - Q^\varphi_\lambda x)\|). \end{aligned}$$

Since $x_n \to x$ and by the property of g, then $Q^\varphi_\lambda x_n \to Q^\varphi_\lambda x$. Hence, Q^φ_λ is continuous.

In the case $\varphi(t) = t^{p-1}$, where $p > 1$, we shall denote Q^φ_λ by $Q_\lambda := (I + \lambda J_p^{-1}B)^{-1}$.

Definition 2.12. ([29]) Let C be a nonempty, closed and convex subset of a smooth Banach space E and let $J_\varphi : E \to E^*$ be the duality mapping with gauge φ. Suppose that $B : E \multimap E^*$ is an operator satisfying the range condition

$$\mathcal{D}(B) \subset C \subset J_\varphi^{-1}\mathcal{R}(J_\varphi + rB), \tag{26}$$

where $r > 0$. For each $r > 0$, the φ-resolvent associated with operator B is the operator $R^\varphi_r : C \multimap E$ defined by

$$R^\varphi_r(x) := \{z \in E : J_\varphi(x) \in (J_\varphi + rB)z\}, \quad x \in C.$$

In addition, it is easy to show that $F(R^\varphi_r) = B^{-1}0$.

Proposition 2.13. ([29]) *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E and let $J_\varphi : E \to E^*$ be the duality mapping with gauge φ. Let $B : E \multimap E^*$ be a monotone operator satisfying* (26). *Let R^φ_r be a resolvent operator of B for $r > 0$, then $\widehat{F}(R^\varphi_r) = F(R^\varphi_r)$.*

Lemma 2.14. ([7]) *Let E be a uniformly convex and smooth Banach space. Let $B : E \multimap E^*$ be a monotone operator. Then, B is maximal if and only if for each $r > 0$,*

$$\mathcal{R}(J_\varphi + rB) = E^*,$$

where $\mathcal{R}(J_\varphi + rB)$ is the range of $J_\varphi + rB$.

Remark 2.15. *(i) If B is maximal monotone, then we see that the range condition holds for $C = \overline{\mathcal{D}(A)}$.*

(ii) By the smoothness and strict convexity of E, we obtain that $R_r^{\varphi,B}$ is single-valued. The range condition ensures that R_λ^φ is single-valued operator from C into $\overline{\mathcal{D}(A)}$. In other words,

$$R_r^\varphi(x) := (J_\varphi + rB)^{-1} J_\varphi(x), \quad \forall x \in C.$$

For a smooth Banach space E, when $\varphi(t) = t^{p-1}$, where $p > 1$, we denote R_r^φ by $R_r := (J_p + rB)^{-1} J_p$.

Lemma 2.16. ([29]) *Let $B : E \multimap E^*$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let R_r^φ be a resolvent operator of B for $r > 0$, then*

$$D_\varphi(z, R_r^\varphi x) + D_\varphi(R_r^\varphi x, x) \leq D_\varphi(z, x),$$

for all $x \in E$ and $z \in B^{-1}0$.

Lemma 2.17. ([59]) *Assume that $\{a_n\}$ is a nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.18. ([31]) *Let $\{\Gamma_n\}$ be a real sequence that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq ...$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Main Result

Throughout this paper, we denote by J_p^E and $J_q^{E^*}$ the duality mappings of a Banach space E and its dual space, respectively, where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We assume that E_1 is a p-uniformly convex and uniformly smooth Banach space, E_2 is a uniformly convex and smooth Banach space, $B_1 : E_1 \multimap E_1^*$, $B_2 : E_2 \multimap E_2^*$ are two maximal monotone operators, R_r is a resolvent operator of B_1 for $r > 0$, Q_λ is a metric resolvent operator of B_2 for $\lambda > 0$, $A : E_1 \to E_2$ is a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$, and $T : E_1 \to CB(E_1)$ is a multivalued Bregman quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. We introduce an iterative method (Algorithm 3.1) for solving the following problem:

$$\text{Find an element } x^* \in B_1^{-1}0 \cap F(T) \text{ such that } Ax^* \in B_2^{-1}0. \tag{27}$$

The solution set of the problem (27) is denoted by Ω.

Algorithm 3.1. *For $u \in E_1$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in E_1$ and*

$$\begin{cases} y_n = R_r(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))) \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(\beta_n J_p^{E_1}(y_n) + (1-\beta_n) J_p^{E_1}(u_n))\Big), \quad \forall n \geq 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

where $f(x_n) = \frac{1}{p}\|(I - Q_\lambda)Ax_n\|^p$ and $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$.

Remark 3.2. *Note that the choice in* (28) *of the stepsize λ_n is independent of the norm $\|A\|$.*

Lemma 3.3. *The stepsize λ_n defined by* (28) *is well-defined.*

Proof. Since $I - Q_\lambda$ is continuous, we have $\nabla f(x) = A^* J_p^{E_2}(I - Q_\lambda)Ax$ for all $x \in E_1$ (see [22, Proposition 5.7]). Let $z \in \Omega$, *i.e.*, $z \in B_1^{-1}0$ and $Az \in B_2^{-1}0$. Then, from (25), we have

$$\begin{aligned} \|x_n - z\|\|\nabla f(x_n)\| &\geq \langle x_n - z, \nabla f(x_n)\rangle \\ &= \langle x_n - z, A^* J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &= \langle Ax_n - Az, J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &\geq \langle Ax_n - Az, J_p^{E_2}(I - Q_\lambda)Ax_n\rangle + \langle Az - Q_\lambda(Ax_n), J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &= \langle Ax_n - Q_\lambda(Ax_n), J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &= \|(I - Q_\lambda)Ax_n\|^p = pf(x_n). \end{aligned} \tag{29}$$

We see that $\|\nabla f(x_n)\| > 0$, when $f(x_n) \neq 0$. This implies that $\|\nabla f(x_n)\| \neq 0$. That is λ_n is well-defined. □

The following proposition is needed before proving our main result.

Proposition 3.4. *Let E be a uniformly convex and uniformly smooth Banach space. Let $T : E \to CB(E)$ be a multivalued Bregman quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$, such that $x_n \to x$. Since T is a multivalued Bregman quasi-nonexpansive mapping, then for all $v \in Tx$ and by (9), we have

$$\begin{aligned} D_p(v, x_n) &\leq D_p(x, x_n) \\ &\leq \langle x - x_n, J_p^E(x) - J_p^E(x_n)\rangle \\ &\leq \|x - x_n\|\|J_p^E(x) - J_p^E(x_n)\| \to 0. \end{aligned}$$

This implies that $\lim_{n\to\infty} D_p(v, x_n) = 0$ and by Lemma 2.7, we have $\lim_{n\to\infty} \|x_n - v\| = 0$. We see that $x = v$. Hence, $x \in F(T)$, *i.e.*, $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $w = tx + (1-t)y$ for $t \in (0,1)$. Let $z \in Tw$, then we have

$$\begin{aligned}
D_p(w,z) &= \frac{1}{p}\|w\|^p - \frac{1}{p}\|z\|^p - \langle w - z, J_p^E(z)\rangle \\
&= \frac{1}{p}\|w\|^p - \frac{1}{p}\|z\|^p - \langle t(x-z) + (1-t)(y-z), J_p^E(z)\rangle \\
&= \frac{1}{p}\|w\|^p + tD_p(x,z) + (1-t)D_p(y,z) - t\frac{\|x\|^p}{p} - (1-t)\frac{\|y\|^p}{p} \\
&\le \frac{1}{p}\|w\|^p + tD_p(x,w) + (1-t)D_p(y,w) - t\frac{\|x\|^p}{p} - (1-t)\frac{\|y\|^p}{p} \\
&= \frac{1}{p}\|w\|^p + t\Big(\frac{1}{p}\|x\|^p - \frac{1}{p}\|w\|^p - \langle x - w, J_p^E(w)\rangle\Big) + (1-t)\Big(\frac{1}{p}\|y\|^p - \frac{1}{p}\|w\|^p - \langle y - w, J_p^E(w)\rangle\Big) \\
&\quad - t\frac{\|x\|^p}{p} - (1-t)\frac{\|y\|^p}{p} \\
&= -\langle tx + (1-t)y - w, J_p^E(w)\rangle = 0,
\end{aligned}$$

which implies that $z = w$. Hence, $w \in F(T)$, *i.e.*, $F(T)$ is convex. Therefore, $F(T)$ is closed and convex. □

We now prove a strong convergence theorem of Algorithm 3.1, which is the main result of this paper.

Theorem 3.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that $\Omega \neq \emptyset$ and the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < a \le \beta_n \le b < 1$ *for some* $a, b \in (0,1)$;

(C3) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0$.

Then, $\{x_n\}$ converges strongly to a common element $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from E_1 onto Ω.

Proof. Put $v_n := J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))$ for all $n \ge 1$. Since $(p-1)q = p$. Then, by (29) and Lemma 2.4, we have

$$\begin{aligned}
D_p(z, y_n) &\le D_p(z, v_n) \\
&= D_p\Big(z, J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))\Big) \\
&= \frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(x_n)\rangle + \lambda_n\langle z, \nabla f(x_n)\rangle + \frac{1}{q}\|J_p^{E_1}(x_n) - \lambda_n\nabla f(x_n))\|^q \\
&\le \frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(x_n)\rangle + \lambda_n\langle z, \nabla f(x_n)\rangle + \frac{1}{q}\|J_p^{E_1}(x_n)\|^q - \lambda_n\langle x_n, \nabla f(x_n)\rangle + \frac{c_q\lambda_n^q}{q}\|\nabla f(x_n)\|^q \\
&= \frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(x_n)\rangle + \frac{1}{q}\|x_n\|^p - \lambda_n\langle x_n - z, \nabla f(x_n)\rangle + \frac{c_q\lambda_n^q}{q}\|\nabla f(x_n)\|^q \\
&\le D_p(z, x_n) - \lambda_n p f(x_n) + \frac{c_q\lambda_n^q}{q}\|\nabla f(x_n)\|^q \\
&= D_p(z, x_n) - \frac{\rho_n p f^p(x_n)}{\|\nabla f(x_n)\|^p} + \frac{\rho_n^q c_q}{q}\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \\
&= D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}.
\end{aligned}$$

Put $z_n := J_q^{E_1^*}(\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n))$ for all $n \geq 1$. From Lemmas 2.3 and 2.16, we have

$$\begin{aligned}
D_p(z, z_n) &= D_p(z, J_q^{E_1^*}(\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n)) \\
&= \frac{1}{p}\|z\|^q - \beta_n\langle z, J_p^{E_1}(y_n)\rangle - (1-\beta_n)\langle z, J_p^{E_1}(u_n)\rangle + \frac{1}{q}\|\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n)\|^q \\
&\leq \frac{1}{p}\|z\|^q - \beta_n\langle z, J_p^{E_1}(y_n)\rangle - (1-\beta_n)\langle z, J_p^{E_1}(u_n)\rangle \\
&\quad + \frac{1}{q}\Big[\beta_n\|J_p^{E_1}(y_n)\|^q + (1-\beta_n)\|J_p^{E_1}(u_n)\|^q - \beta_n(1-\beta_n)g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|)\Big] \\
&= \beta_n\Big(\frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(y_n)\rangle + \frac{1}{q}\|y_n\|^p\Big) + (1-\beta_n)\Big(\frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(u_n)\rangle + \frac{1}{q}\|u_n\|^p\Big) \\
&\quad - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&= \beta_n D_p(z, y_n) + (1-\beta_n)D_p(z, u_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&\leq \beta_n D_p(z, y_n) + (1-\beta_n)D_p(z, y_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&= D_p(z, R_r v_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&\leq D_p(z, v_n) - D_p(R_r v_n, v_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&\leq D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - D_p(R_r v_n, v_n) \\
&\quad - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|),
\end{aligned} \tag{30}$$

which implies that

$$D_p(z, z_n) \leq D_p(z, x_n).$$

Then, it follows that

$$\begin{aligned}
D_p(z, x_{n+1}) &= D_p\Big(z, J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n))\Big) \\
&\leq \alpha_n D_p(z, u) + (1-\alpha_n)D_p(z, z_n) \\
&\leq \alpha_n D_p(z, u) + (1-\alpha_n)D_p(z, x_n) \\
&\leq \max\{D_p(z, u), D_p(z, x_n)\} \\
&\vdots \\
&\leq \max\{D_p(z, u), D_p(z, x_1)\}.
\end{aligned} \tag{31}$$

Hence, $\{D_p(z, x_n)\}$ is bounded and so $\{x_n\}$ is bounded by Lemma 2.8.

Let $x^* = \Pi_\Omega u$. Using (16) and (30), we have the following estimation:

$$\begin{aligned}
D_p(x^*, x_{n+1}) &= D_p\Big(x^*, J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n))\Big) \\
&= V_p(x^*, \alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n)) \\
&\leq V_p(x^*, \alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n) - \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(x^*))) \\
&\quad + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \\
&= V_p(x^*, \alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)J_p^{E_1}(z_n)) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \\
&= D_p(x^*, J_q^{E_1^*}(\alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)J_p^{E_1}(z_n))) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \\
&\leq \alpha_n D_p(x^*, x^*) + (1-\alpha_n)D_p(x^*, z_n) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \\
&\leq (1-\alpha_n)\Big[D_p(x^*, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - D_p(R_r v_n, v_n) \\
&\quad - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|)\Big] + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \\
&= (1-\alpha_n)D_p(x^*, x_n) - (1-\alpha_n)\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - (1-\alpha_n)D_p(R_r v_n, v_n) \\
&\quad - \frac{(1-\alpha_n)\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle. \qquad (32)
\end{aligned}$$

For each $n \geq 1$, we set

$$\begin{aligned}
\Gamma_n &:= D_p(x^*, x_n), \\
\eta_n &:= (1-\alpha_n)\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + (1-\alpha_n)D_p(R_r v_n, v_n) \\
&\quad + \frac{(1-\alpha_n)\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|), \\
\delta_n &:= \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle.
\end{aligned}$$

Then (32) reduces to the following formulae:

$$\Gamma_{n+1} \leq (1-\alpha_n)\Gamma_n - \eta_n + \delta_n, \quad \forall n \geq 1 \qquad (33)$$

and

$$\Gamma_{n+1} \leq (1-\alpha_n)\Gamma_n + \delta_n, \quad \forall n \geq 1. \qquad (34)$$

We now show that $\Gamma_n \to 0$ as $n \to \infty$ by considering two possible cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}_{n=n_0}^\infty$ is non-increasing. This implies that $\{\Gamma_n\}_{n=1}^\infty$ is convergent. From (33), we have

$$\eta_n \leq \Gamma_n - \Gamma_{n+1} + \delta_n - \alpha_n\Gamma_n. \qquad (35)$$

Since $\alpha_n \to 0$, $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big) > 0$ and $\beta_n \in (a, b)$. This implies that $\lim_{n\to\infty} \eta_n = 0$. Then, we have

$$\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \to 0, \;\; D_p(R_r v_n, v_n) \to 0 \;\text{ and }\; g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \to 0. \qquad (36)$$

Since $\{\|\nabla f(x_n)\|\}$ is bounded, there exists $M > 0$ such that $\|\nabla f(x_n)\| \leq M$. Thus we have

$$\frac{f^p(x_n)}{M^p} \leq \frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \to 0 \;\text{ as }\; n \to \infty.$$

This implies that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \|(I - Q_\lambda)Ax_n\| = 0. \tag{37}$$

Moreover, we have

$$\lim_{n\to\infty} \|R_r v_n - v_n\| = 0 \tag{38}$$

and

$$\lim_{n\to\infty} \|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\| = 0. \tag{39}$$

It follows from (39) that

$$\|J_p^{E_1}(z_n) - J_p^{E_1}(y_n)\| = (1-\beta_n)\|J_p^{E_1}(u_n) - J_p^{E_1}(y_n)\| \to 0 \text{ as } n \to \infty.$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0 \tag{40}$$

and

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{41}$$

From (37), we see that

$$\|J_p^{E_1}(v_n) - J_p^{E_1}(x_n)\| = \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\|\nabla f(x_n)\| = \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^{p-1}} \to 0 \text{ as } n \to \infty.$$

So we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{42}$$

From (38) and (42), we have

$$\|y_n - x_n\| \le \|y_n - v_n\| + \|v_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{43}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E_1$ as $k \to \infty$ and

$$\limsup_{n\to\infty} \langle x_n - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle = \lim_{k\to\infty} \langle x_{n_k} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle, \tag{44}$$

where $x^* = \Pi_\Omega u$. Since $\|y_n - x_n\| \to 0$ as $n \to \infty$, we also have $y_{n_k} \rightharpoonup \hat{x}$. So, by (40), we see that

$$d(y_n, Ty_n) \le \|y_n - u_n\| \to 0 \text{ as } n \to \infty \tag{45}$$

and by the demiclosedness of $I - T$ at zero, we get $\hat{x} \in F(T)$. Since $x_{n_k} \rightharpoonup \hat{x}$ and by (42), we also get $v_{n_k} \rightharpoonup \hat{x}$. Then from (38), we get $\hat{x} \in F(R_r)$. From (24) and (37), we see that

$$\begin{aligned} & \langle Q_\lambda Ax_n - Q_\lambda A\hat{x}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle \\ \le\ & \langle Q_\lambda Ax_n - Q_\lambda A\hat{x}, J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ \le\ & \|Q_\lambda Ax_n - Q_\lambda A\hat{x}\|\|(I - Q_\lambda)Ax_n\|^{p-1} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{46}$$

Moreover, we have

$$\begin{aligned} \|(I - Q_\lambda)A\hat{x}\|^p &= \langle (I - Q_\lambda)A\hat{x}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle \\ &= \langle A\hat{x} - Ax_{n_k}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle + \langle Ax_{n_k} - Q_\lambda Ax_{n_k}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle \\ &\quad + \langle Q_\lambda Ax_{n_k} - Q_\lambda A\hat{x}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle. \end{aligned} \tag{47}$$

Since A is continuous, we have $Ax_{n_k} \rightharpoonup A\hat{x}$ as $k \to \infty$, by (37) and (46), we have

$$\|A\hat{x} - Q_\lambda A\hat{x}\| = 0.$$

This shows that $A\hat{x} \in F(Q_\lambda)$. Hence $\hat{x} \in F(T) \cap B_1^{-1}0 \cap A^{-1}(B_2^{-1}0) = \Omega$. Note that

$$\begin{aligned} D_p(z_n, x_{n+1}) &= D_p(z_n, J_q^{E_1^*}\big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n)\big)) \\ &\leq \alpha_n D_p(z_n, u) + (1-\alpha_n) D_p(z_n, z_n) \to 0 \text{ as } n \to \infty \end{aligned}$$

and so

$$\|x_{n+1} - z_n\| \to 0 \text{ as } n \to \infty. \tag{48}$$

It follows from (41), (43) and (48) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{49}$$

Then by (44) and Proposition 2.6, we obtain

$$\limsup_{n\to\infty} \langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \leq 0. \tag{50}$$

This together with (34) and (50), we conclude by Lemma 2.17 that $\Gamma_n \to 0$ as $n \to \infty$. Hence, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Let us define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then by Lemma 2.18, we have

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Put $\Gamma_n := D_p(x_n, x^*)$ for all $n \in \mathbb{N}$. From (31), we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} (D_p(x^*, x_{\tau(n)+1}) - D_p(x^*, x_{\tau(n)})) \\ &\leq \lim_{n\to\infty} (D_p(x^*, u) + (1-\alpha_{\tau(n)}) D_p(x^*, x_{\tau(n)}) - D_p(x^*, x_{\tau(n)})) \\ &= \lim_{n\to\infty} \alpha_{\tau(n)}\big(D_p(x^*, u) - D_p(x^*, x_{\tau(n)})\big) = 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} (D_p(x^*, x_{\tau(n)+1}) - D_p(x^*, x_{\tau(n)})) = 0. \tag{51}$$

Following the proof line in **Case 1**, we can show that

$$\limsup_{n\to\infty} \langle x_{\tau(n)+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \leq 0.$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$, by (34), we have

$$D_p(x_n, x^*) \leq \langle x_{\tau(n)+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle.$$

Thus we have

$$\limsup_{n\to\infty} D_p(x^*, x_{\tau(n)}) \leq 0$$

and so

$$\lim_{n\to\infty} D_p(x^*, x_{\tau(n)}) = 0.$$

Since $\Gamma_n \leq \Gamma_{\tau(n)+1}$. Then from (51), we have

$$D_p(x^*, x_n) \leq D_p(x^*, x_{\tau(n)+1}) = D_p(x^*, x_{\tau(n)}) + (D_p(x^*, x_{\tau(n)+1}) - D_p(x^*, x_{\tau(n)})) \to 0 \text{ as } n \to \infty.$$

Therefore, $x_n \to x^*$ as $n \to \infty$. We thus complete the proof. □

Corollary 3.6. *Let E_1 be a p-uniformly convex and uniformly smooth Banach space and E_2 a uniformly convex and smooth Banach space. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be two maximal monotone operators such that R_r is a resolvent operator of B_1 for $r > 0$ and Q_λ is a metric resolvent operator of B_2 for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$ and let $T : E_1 \to CB(E_1)$ be a multivalued Bregman relatively nonexpansive mapping. Suppose that $\Omega \neq \emptyset$. For $u \in E_1$, let $\{x_n\}$ be the sequence generated by $x_1 \in E_1$ and*

$$\begin{cases} y_n = R_r(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))) \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n))\Big), \ \ \forall n \geq 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \textit{if } f(x_n) \neq 0; \\ 0, & \textit{otherwise}, \end{cases} \tag{52}$$

where $f(x_n) = \frac{1}{p}\|(I - Q_\lambda)Ax_n\|^p$ and $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0,1)$*;*

(C3) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0.$

Then, $\{x_n\}$ converges strongly to a common element $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from E_1 onto Ω.

If we take $T = I$ is a single-valued mapping in Theorem 3.5, then we obtain the following result.

Corollary 3.7. *Let E_1 be a p-uniformly convex and uniformly smooth Banach space and E_2 a uniformly convex and smooth Banach space. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be two maximal monotone operators such that R_r is a resolvent operator of B_1 for $r > 0$ and Q_λ is a metric resolvent operator of B_2 for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$. Suppose that $\Lambda := \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\} \neq \emptyset$. For $u \in E_1$, let $\{x_n\}$ be the sequence generated by $x_1 \in E_1$ and*

$$\begin{cases} y_n = J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n)) \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(R_r y_n)\Big), \ \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \textit{if } f(x_n) \neq 0; \\ 0, & \textit{otherwise}, \end{cases} \tag{53}$$

where $f(x_n) = \frac{1}{p}\|(I - Q_\lambda)Ax_n\|^p$ and $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0.$

Then, $\{x_n\}$ converges strongly to an element $x^* = \Pi_\Lambda u$, where Π_Λ is the Bregman projection from E_1 onto Λ.

In addition, we consequently obtain the following result in Hilbert spaces.

Corollary 3.8. *Let H_1 and H_2 be two Hilbert spaces. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be two maximal monotone operators such that R_r and Q_λ are resolvent operators of B_1 for $r > 0$ and B_2 for $\lambda > 0$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint $A^* : H_2 \to H_1$ and let $T : H_1 \to CB(H_1)$ be a multivalued quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Suppose that $\Omega \neq \emptyset$. For $u \in H_1$, let $\{x_n\}$ be the sequence generated by $x_1 \in H_1$ and*

$$\begin{cases} y_n = R_r(x_n - \lambda_n \nabla f(x_n)) \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(\beta_n y_n + (1-\beta_n)u_n), \ \ \forall n \geq 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases} \tag{54}$$

where $f(x_n) = \frac{1}{2}\|(I - Q_\lambda)Ax_n\|^2$ and $\{\rho_n\} \subset (0,4)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0,1)$;

(C3) $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$.

Then, $\{x_n\}$ converges strongly to a common element $x^* = P_\Omega u$, where P_Ω is the metric projection from H_1 onto Ω.

4. Application to Split Feasibility Problems

Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint A^*. The *split feasibility problem* (SFP) is formulated as finding an element

$$x^* \in C \ \text{ such that } \ Ax^* \in Q. \tag{55}$$

We denote by $\Gamma := \{x \in C : Ax \in Q\} = C \cap A^{-1}(Q)$ the set of solutions of the SFP. This problem was first introduced, in a finite dimensional Hilbert space, by Censor-Elfving [15] for modeling inverse problems which arise from phase retrieval and in medical image reconstruction. Moreover, the SFP has applications in signal processing, in image recovery, in radiation therapy, in data denoising and in data compression (see for instance [8, 9, 19, 20]).

In order to solve the SFP in Banach spaces, Schöpfer et al. [48] first introduced the following algorithm: for $x_1 \in E_1$ and

$$x_{n+1} = \Pi_C J_{E_1}^*(J_{E_1}(x_n) - \lambda_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))), \ \ \forall n \geq 1, \tag{56}$$

where $\{\lambda_n\}$ is a positive sequence, Π_C denotes the generalized projection on E_1, P_Q is the metric projection on E_2, J_{E_1} is the duality mapping on E_1 and $J_{E_1}^*$ is the duality mapping on E_1^*. It was proved that the sequence $\{x_n\}$ converges weakly to a solution of the SFP under some mild conditions.

To obtain a strong convergence theorem, Shehu [39] introduced the following iterative algorithm for solving the SFP in p-uniformly convex and uniformly smooth Banach spaces: for $u, x_1 \in E$ and

$$\begin{cases} y_n = J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Ax_n), \\ x_{n+1} = \Pi_C J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)), \ \ \forall n \geq 1, \end{cases} \tag{57}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n satisfies $0 < a \le \lambda_n \le b < \left(\frac{q}{\kappa_q\|A\|^q}\right)^{\frac{1}{q-1}}$ for some $a, b > 0$. Under suitable assumptions, he proved that the sequence $\{x_n\}$ generated by (57) converges strongly to a solution of the SFP.

Let C be a closed and convex subset of a strictly convex, smooth and reflexive Banach space E. Recall that the *indicator function* of C given by

$$i_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases} \tag{58}$$

It is known that i_C is proper convex, lower semicontinuous and convex function with its subdifferential ∂i_C is maximal monotone (see [34]). From [5], we know that

$$\partial i_C(z) = N_C(z) := \{u \in E^* : \langle y - z, u\rangle \le 0, \ \forall y \in C\}, \tag{59}$$

where N_C is the normal cone for C at a point $z \in C$. Thus, we can define the resolvent R_r of ∂i_C for $r > 0$ by

$$R_r(x) := (J_p + r\partial i_C)^{-1} J_p(x), \ \ \forall x \in E.$$

So we have for any $x \in E$ and $z \in C$,

$$\begin{aligned} z = R_r(x) \ &\Leftrightarrow \ J_p(x) \in J_p(z) + rN_C(z) \\ &\Leftrightarrow \ J_p(x) - J_p(z) \in rN_C(z) \\ &\Leftrightarrow \ \langle y - z, J_p(x) - J_p(z)\rangle \le 0, \ \ \forall y \in C \\ &\Leftrightarrow \ z = \Pi_C(x), \end{aligned}$$

where Π_C is the Bregman projection from E onto C. Moreover, we can define the metric resolvent Q_λ of ∂i_C for $\lambda > 0$ by

$$Q_\lambda(x) := (I + \lambda J_p^{-1}\partial i_C)^{-1}(x), \ \ \forall x \in E.$$

So we have for any $x \in E$ and $z \in C$,

$$\begin{aligned} z = Q_\lambda(x) \ &\Leftrightarrow \ x \in z + \lambda J_p^{-1} N_C(z) \\ &\Leftrightarrow \ x - z \in \lambda J_p^{-1} N_C(z) \\ &\Leftrightarrow \ J_\varphi(x - z) \in N_C(z) \\ &\Leftrightarrow \ \langle y - z, J_p(x - z)\rangle, \ \ \forall y \in C \\ &\Leftrightarrow \ z = P_C(x), \end{aligned}$$

where P_C is the metric projection from E onto C.

In fact, we set $B_1 := \partial i_C$ and $B_2 := \partial i_Q$, then $R_r = \Pi_C$ and $Q_\lambda = P_Q$ for $\lambda_1, \lambda_2 > 0$. We also have $F(R_r) = B_1^{-1}0 = C$ and $F(Q_\lambda) = B_2^{-1}0 = Q$. So we obtain the following result.

Theorem 4.1. *Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$ and let $T : C \to CB(C)$ be a multivalued Bregman quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Suppose that $\Theta := F(T) \cap \Gamma \neq \emptyset$. For $u \in C$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = \Pi_C(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))), \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(\beta_n J_p^{E_1}(y_n) + (1-\beta_n) J_p^{E_1}(u_n))\Big), \ \ \forall n \ge 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{60}$$

where $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$ and $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0,1)$*;*

(C3) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0.$

Then, $\{x_n\}$ *converges strongly to a common element* $x^* = \Pi_\Theta u$*, where* Π_Θ *is the Bregman projection from* E_1 *onto* Θ*.*

5. Numerical Results

In this section, we first give a numerical example to demonstrate the performance of Algorithm 3.1.

Example 5.1. Let $E_1 = \mathbb{R}$ and $E_2 = \mathbb{R}^3$ with the usual norms. Define a multi-valued mapping $T : \mathbb{R} \to CB(\mathbb{R})$ by

$$Tx := \begin{cases} \Big[0, \Big|\frac{5}{6} x \sin\Big(\frac{1}{x}\Big)\Big|\Big], & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

One can show that T is (Bregman) quasi-nonexpansive and it also satisfies the demiclosedness principle. Define a multi-valued mapping $B_1 : \mathbb{R} \multimap \mathbb{R}$ by

$$B_1(x) := \begin{cases} \Big\{y \in \mathbb{R} : z^2 + xz - 2x^2 \geq (z - x)y, \ \forall z \in [-9, 3]\Big\}, & x \in [-9, 3], \\ \emptyset, & \text{otherwise.} \end{cases}$$

By [55, Theorem 4.2], B_1 is a maximal monotone operator. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $g(z_1, z_2, z_3) = \frac{1}{2}|5z_1 - 3z_2 + 2z_3|^2$. Let $B_2 : \mathbb{R}^3 \multimap \mathbb{R}^3$ be a subdifferential of g, that is,

$$B_2(x) = \partial g(x) := \Big\{y \in \mathbb{R}^3 : \langle y, z - x\rangle \leq g(z) - g(x), \ \forall z \in \mathbb{R}^3\Big\}.$$

Since g is a proper, lower semicontinuous and convex function, then B_2 is a maximal monotone operator (see [34]). The explicit forms of the resolvent operators of B_1 and B_2 can be written by $R_r(x) = \frac{x}{4}$ and $Q_\lambda = M^{-1}$, where

$$M = \begin{pmatrix} 26 & -15 & 10 \\ -15 & 10 & -6 \\ 10 & -6 & 5 \end{pmatrix}$$

(see [17, 44, 55] for more details). Next, define a bounded linear operator $A : \mathbb{R} \to \mathbb{R}^3$ by $Ax := (-8x, -3x, x)$ and let $\Omega := F(T) \cap B_1^{-1}0 \cap A^{-1}(B_2^{-1}0)$.

Take $\alpha_n = \frac{1}{8500n}$, $\beta_n = \frac{n}{2n+1}$, $\rho_n = \frac{2n}{n+1}$, $r = \lambda = 1$ and $u = \frac{1}{2}$. If $y_n \neq 0$, then we choose $u_n = \Big|\frac{5}{12} y_n \sin\Big(\frac{1}{y_n}\Big)\Big|$; otherwise, $u_n = 0$. Now, Algorithm 3.1 becomes

$$\begin{cases} y_n = \frac{1}{4}\Big(x_n - \lambda_n A^\top \Big(I - M^{-1}\Big) A x_n\Big) \\ x_{n+1} = \frac{1}{2(8500n)} + \Big(1 - \frac{1}{8500n}\Big)\Big(\frac{n}{2n+1} y_n + \frac{n+1}{2n+1} u_n\Big), \ \forall n \geq 1, \end{cases} \tag{61}$$

where

$$\lambda_n = \begin{cases} \frac{n}{n+1} \frac{\|(I - M^{-1})Ax_n\|^2}{\|A^\top (I - M^{-1})Ax_n\|^2}, & \text{if } Ax_n \neq M^{-1}(Ax_n), \\ 0, & \text{otherwise.} \end{cases}$$

Let us start with the initial point $x_1 = 10$ and the stopping criterion for our testing method is set as: $E_n := |x_{n+1} - x_n| < 10^{-7}$. Now, we show the numerical experiment of the method (61) and plot the number of iterations n against E_n as seen in Table 1 and Figure 1. It is observed that our algorithm converges to a solution, i.e., $x_n \to 0 \in \Omega$.

n	y_n	x_{n+1}	E_n
1	1.2820513	0.6777851	9.3222149
2	0.0593786	0.0372240	0.6405611
3	0.0025055	0.0011798	0.0360442
4	0.0000650	0.0000475	0.0011323
5	0.0000022	0.0000131	0.0000344
6	0.0000005	0.0000101	0.0000029
7	0.0000004	0.0000087	0.0000015
8	0.0000003	0.0000075	0.0000012
9	0.0000002	0.0000067	0.0000008
$\vdots$	$\vdots$	$\vdots$	$\vdots$
25	3.909E-08	0.0000024	1.028E-07
26	3.665E-08	0.0000023	8.833E-08

Table 1: Numerical experiment of the iterative method (61)

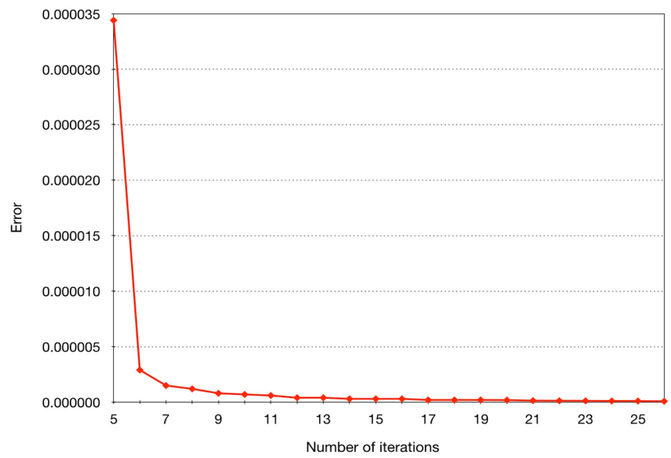

Figure 1: A gragh of error of the iterative method (61)

Finally, we give an example established in the infinite-dimensional space L_p but not a Hilbert space for supporting Theorem 3.5.

Example 5.2. For $p > 2$, let $E_1 = E_2 = L_p([\alpha, \beta])$. From [3], we have the duality mapping of E_1 is the function $J_p^{E_1} : L_p([\alpha, \beta]) \to L_q([\alpha, \beta])$ given by $J_p^{E_1}(x) = |x|^{p-2} \cdot x$ and the Bregman function $D(\cdot, \cdot)$ given by

$$D_p(x, y) = \frac{||x||^p}{p} + \frac{||y||^p}{q} - \langle x, |y|^{p-2} \cdot y \rangle.$$

Consider a hyperplane C of $L_p([\alpha, \beta])$

$$C := \{x \in L_p([\alpha, \beta]) : \langle a, x \rangle = b\},$$

where $a(t) \in L_q([\alpha, \beta])$, $b \in \mathbb{R}$ and $t \in [\alpha, \beta]$. Let $B_1 = \partial i_C$, where ∂i_C is the subdifferential of the indicator function of C. Then the resolvent operator R_r of B_1 becomes the Bregman projection operator Π_C given by [2]

$$\Pi_C(x) = \begin{cases} u_k, & \text{if } x \notin C; \\ x, & \text{if } x \in C, \end{cases}$$

where $u_k \in L_p([\alpha, \beta])$ is a solution of the problem: find $k \in \mathbb{R}$ such that $\langle a, u_k \rangle = b$ and

$$u_k := |k \cdot a + |x|^{p-2} \cdot x|^{q-2} \cdot (k \cdot a + |x|^{p-2} \cdot x).$$

Let a closed ball centered at $v \in L_p([\alpha, \beta])$ and radius $d > 0$ be defined by

$$Q := \{x \in L_p([\alpha, \beta]) : \|x - v\| \leq d\}.$$

Let $B_2 = \partial i_Q$, where ∂i_Q is the subdifferential of the indicator function of Q. Then the metric resolvent operator Q_λ of B_2 becomes the metric projection operator P_Q given by

$$P_Q(x) = \begin{cases} v + d\frac{x-v}{\|x-v\|}, & \text{if } x \notin Q; \\ x, & \text{if } x \in Q. \end{cases}$$

Let $\{\rho_n\}$ be a sequence in $\left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$ such that $\liminf_{n\to\infty} \rho_n\left(p - \frac{\rho_n^{q-1} c_q}{q}\right) > 0$, where $c_q = (1 + t_q^{q-1})(1 + t_q)^{1-q}$ and t_q is the unique solution of the equation $(q-2)t^{q-1} + (q-1)t^{q-2} - 1 = 0,\ 0 < t < 1$ (see [58]). In particular, we consider the following SFP and the fixed point problem:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q \text{ and } x^* \in Tx^*$$

with its solution set $\Theta := \Gamma \cap F(T)$. Let

$$C = \{x \in L_3([0, 1]) : \langle 1, x \rangle = 0\}$$

and

$$Q = \{x \in L_3([0, 1]) : \|x\| \leq 1\}.$$

Let $A : L_3([0, 1]) \to L_3([0, 1])$ be defined by $(Ax)(t) = \frac{x(t)}{2}, \forall x \in L_3([0, 1])$. We see that A is bounded and linear with $A^* = A$. Let $T : C \to CB(C)$ be defined by

$$Tx := \begin{cases} \{y \in C : x - \frac{1}{2} \leq y \leq x - \frac{1}{4}\}, & \text{if } x > 1; \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is shown in [45] that T is a multivalued Bregman quasi-nonexpansive mapping with $F(T) = \{0\}$ and T is demiclosed at zero. We see that $x^* = 0$ is solution in Γ and it is a fixed point of T. Hence, $x^* = 0 \in \Theta$. Suppose that $\alpha_n = \frac{n}{n^2+1}, \beta_n = \frac{n}{2n+1}$. So our Algorithm 3.1 has the following form:

$$\begin{cases} y_n = \Pi_C(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Ax_n)) \\ z_n \in J_q^{E_1^*}\left(\frac{n}{2n+1}J_p^{E_1}(y_n) + \frac{n+1}{2n+1}J_p^{E_1}(Ty_n)\right) \\ x_{n+1} = J_q^{E_1^*}\left(\frac{n}{n^2+1}J_p^{E_1}(u) + \frac{n^2-n+1}{n^2+1}J_p^{E_1}(z_n)\right), \ \forall n \geq 1, \end{cases} \tag{62}$$

where the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{63}$$

where $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$. By Theorem 3.5, the sequence $\{x_n\}$ generated by (62) converges strongly to $x^* = 0 \in \Theta$.

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References

[1] Y. I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, Lect. Notes Pure Appl. Math. (1996) 15–50.
[2] Y. I. Alber, D. Butnariu, Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces, J. Optim. Theory Appl. 92 (1997) 33–61.
[3] Y. I. Alber, I. Ryazantseva, Nonlinear Ill-Posed Problems of Monotone Type, Springer, Netherlands, 2006.
[4] A. S. Alofi, S. M. Alsulami, W. Takahashi, Strongly convergent iterative method for the split common null point problem in Banach spaces, J. Nonlinear Convex Anal. 17 (2016) 311–324.
[5] R. P. Agarwal, D. O'Regan, D. R. Sahu, Fixed Point Theory for Lipschitzian Type Mappings with Applications, Springer, 2009.
[6] K. Aoyama, F. Kohsaka, W. Takahashi, Three generalizations of firmly nonexpansive mappings: Their relations and continuity properties, J. Nonlinear Convex Anal. 10 (2009) 131–147.
[7] F. E. Browder, Nonlinear maximal monotone operators in Banach space, Math. Ann. 175 (1968) 89–113.
[8] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, Inverse Probl. 18(2) (2002) 441–453.
[9] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20(1) (2004) 103–120.
[10] H. H. Bauschke, J. M. Borwein, P. L.Combettes, Bregman monotone optimization algorithms, SIAM J. Control Optim. 42 (2003) 596–636.
[11] C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, J. Nonlinear Convex Anal. 13 (2012) 759–775.
[12] T. Bonesky, K. S. Kazimierski, P. Maass, F. Schöpfer, T. Schuster, Minimization of Tikhonov functionals in Banach spaces, Abstr. Appl. Anal. 2008 (2008) Article ID 192679.
[13] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer Academic, Dordrecht, 1990.
[14] P. L. Combette, The convex feasibility problem in image recovery, Adv. Imaging Electron Phys. 95 (1996) 155–270.
[15] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer Algorithms 8(2) (1994) 221–239.
[16] Y. Censor, A. Lent, An iterative row-action method for interval convex programming, J. Optim. Theory Appl. 34 (1981) 321–353.
[17] P. L. Combettes, J. C. Pesquet, Proximal splitting methods in signal processing, In: Fixed-Point Algor. Inv. Probl. Sci. Eng. 49 (2011) 185–212.
[18] P. Cholamjiak, P. Sunthrayuth, A Halpern-type iteration for solving the split feasibility problem and the fixed point problem of Bregman relatively nonexpansive semigroup in Banach spaces, Filomat 32(9) (2018) 3211–3227.
[19] Y. Censor, T. Bortfeld, B. Martin, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51(10) (2006) 2353–2365.
[20] Y. Censor, T. Elfving, N. Kopf, The multiple-sets split feasibility problem and its applications for inverse problems, Inverse Probl. 21(6) (2005) 2071–2084.
[21] N. Dunford, J. T. Schwartz, Linear Operators I, Wiley Interscience, New York, 1958.
[22] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, North–Holland, Amsterdam, 1976.
[23] J. M. Hendrickx, A. Olshevsky, Matrix P-norms are NP-hard to approximate if $P = 1, 2, \infty$, SIAM. J. Matrix Anal. Appl. 31 (2010) 2802–2812.
[24] M. Hojo, W. Takahashi, Strong convegence theorem by shrinking projection method for the split common null point problem in Banach spaces, Numerical Functional Analysis and Optimization 37(5) (2016) 541–553.
[25] L. O. Jolaoso, I. Karahan, A general alternative regularization method with line search technique for solving split equilibrium and fixed point problems in Hilbert spaces, Comp. Appl. Math. 39(3) (2020).
[26] P. Jailoka, S. Suantai, Split null point problems and fixed point problems for demicontractive multivalued mappings, Mediterr. J. Math. 15 (2018) Atricle no. 204.
[27] P. Jailoka, S. Suantai, Split common fixed point and null point problems for demicontractive operators in Hilbert spaces, Optim. Methods Softw. 34 (2019) 248–263.
[28] K. R. Kazmi, S. H. Rizvi, An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping, Optim. Lett. 8 (2014) 113–1124.
[29] L. W. Kuo, D. R. Sahu, Bregman distance and strong convergence of proximal-type algorithms, Abstract and Applied Analysis 2013 (2013) Article ID 590519.
[30] G. Lóṕez, V. Martin-Máŕquez, F. Wang, H. K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, Inverse Probl. 28 (2012) 085004.
[31] P. E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal 16 (2008) 899–912.
[32] V. Martín-Máŕquez, S. Reich, S. Sabach, Iterative Methods for approximating fixed points of Bregman nonexpansive operators, Discrete and Continuous Dynamical Systems Series S 6(4) (2013) 1043–1063.
[33] E. Naraghirad, J. C. Yao, Bregman weak relatively non expansive mappings in Banach space, Fixed Point Theory Appl. 2013 (2013) Article no. 141.
[34] R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, Pacific Journal of Mathematics 33 (1970) 209–216.
[35] S. Reich, On the asymptotic behavior of nonlinear semigroups and the range of accretive operators, J. Math. Anal. Appl. 79(1) (1981) 113–126.
[36] S. Reich, Review of geometry of Banach spaces, Duality Mappings and Nonlinear Problems by Ioana Cioranescu, Kluwer Academic Publishers, Dordrecht, 1990, Bull. Amer. Math. Soc. 26 (1992) 367–370.
[37] S. Reich, A weak convergence theorem for the alternating method with Bregman distance, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, Lecture Notes in Pure and Appl Math (1996) 313–318.

[38] S. Reich, S. Sabach, Two strong convergence theorems for a proximal method in reflexive Banach spaces, Numer. Funct. Anal. Optim. 31(1) (2010) 22–44.
[39] Y. Shehu, Iterative methods for split feasibility problems in certain Banach spaces, Journal of Nonlinear and Convex Analysis 16(12) (2015) 2351–2364.
[40] Y. Shehu, D. F. Agbebaku, On split inclusion problem and fixed point problem for multi-valued mappings, Comp. Appl. Math. 37 (2018) 1807–1824.
[41] S. Suantai, P. Jailoka, A self-adaptive algorithm for split null point problems and fixed point problems for demicontractive multivalued mappings, Acta Appl. Math. 170 (2020) 883–901.
[42] P. Sunthrayuth, P. Kumam, A system of generalized mixed equilibrium problems, maximal monotone operators, and fixed point problems with application to optimization problems, Abstract and Applied Analysis 2012 (2012) Article ID 316276.
[43] Y. Shehu, F. U. Ogbuisi, Convergence analysis for proximal split feasibility problems and fixed point problems, J. Appl. Math. Comput. 48 (2015) 221–239.
[44] U. Singthong, S. Suantai, Equilibrium problems and fixed point problems for nonspreading-type mappings in Hilbert space, Int. J. Nonlinear Anal. Appl. 2 (2011) 51–61.
[45] N. Shahzad, H. Zegeye, Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings, Fixed Point Theory and Applications 2014 (2014) Article no. 152.
[46] P. Sunthrayuth, Y. J. Cho, P. Kumam, General iterative algorithms approachto variational inequalities and minimum-norm fixed point for minimization and split feasibility problems, Opsearch 51(3) (2014) 400–415.
[47] S. Suantai, Y. Shehu, P. Cholamjiak, Nonlinear iterative methods for solving the split common null point problem in Banach spaces, Optimization Methods and Software 34(4) (2019) 853–874.
[48] F. Schöpfer, T. Schuster, A. K. Louis, An iterative regularization method for the solution of the split feasibility problem in Banach spaces, Inverse Probl. 24 (2008) 055008.
[49] S. Suantai, N. Srisap, N. Naprang, M. Mamat, V. Yundon, P. Cholamjiak, Convergence theorems for finding split common null point problem in Banach spaces, Appl. Gen. Topol. 18(2) (2017) 345–360.
[50] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
[51] W. Takahashi, The split common null point problem for generalized resolvents in two banach spaces, Numer. Algor. 75 (2017) 1065–1078.
[52] Y. Tang, New inertial algorithm for solving split common null point problem in Banach spaces, Journal of Inequalities and Applications 2019 (2019) Article no. 17.
[53] Y. Tang, P. Sunthrayuth, An iterative algorithm with inertial technique for solving the split common null point problem in Banach spaces, Asian-European Journal of Mathematics 15(61) (2022) Article ID 2250120.
[54] S. Takahashi, W. Takahashi, The split common null point problem and the shrinking projectionmethod in Banach spaces, Optimization 65(2) (2016) 281–287.
[55] S. Takahashi, W. Takahashi, M. T. Toyoda, Strong convergence theorem for maximal monotone operators with nonlinear mappings in Hilbert spaces, J. Optim. Theory Appl. 147 (2010) 27–41.
[56] W. Takahashi, J. C. Yao, Strong convergence theorems by hybrid methods for the split common null point problem in Banach spaces, Fixed Point Theory Appl. 2015 (2015) Article no. 87.
[57] T. M. Tuyen, P. Sunthrayuth, N. M. Trang, An inertial self-adaptive algorithm for the generalized split common null point problem in Hilbert spaces, Rendiconti del Circolo Matematico di Palermo 71 (2021) 537–557.
[58] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Analysis: Theory, Methods & Applications 16 (1991) 1127–1138.
[59] H. K. Xu, Iterative algorithms for nonlinear operators, Journal London Mathematical Society 66(1) (2002) 240–256.

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The dynamics of the HIV/AIDS infection in the framework of piecewise fractional differential equation

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ABSTRACT

We study the HIV/AIDS infection dynamics using the new concept of piecewise fractional differential equations in the sense of Atangana–Baleanu derivative. The cases of HIV/AIDS in Indonesia have been utilized to provide better fitting to the real data. We show that the piecewise modeling approach has the better fitting to the cases in the present work compared to the previous case of the model in the Caputo sense. We study that the model is locally asymptotically stable when $\mathcal{R}_0 < 1$. The fractional model is extended to the piecewise model and provided a numerical procedure for their graphical solution. We provide some graphical results for the model which indicates the usefulness of the piecewise modeling approach.

Introduction

There are many severe diseases in the human world that provide death cases and some becoming a global health issue for society. The Human Immunodeficiency Virus (HIV) is one of the big issues in the entire world for the human community which still needs proper treatment. Although the researchers and scientists working to discover some useful medicine to cure this infection which is underway. There are many cases of infection and deaths are reported everywhere in the world. The virus of HIV target mainly the cells of the immune system and make it infected and with the passage of time the immunity of the human decreases. This infection is continuously increasing and becoming a major health problem for the global community. Some reports show that more than 37.9 million people have been infected with HIV [1]. Blood, semen, cervical or vaginal secretions, breastfeeding of the infant by infected moms, and other bodily fluids are all ways that HIV can spread to other individuals. If the HIV infection is not treated well in its early stages, it progresses to the most severe form, known as the Acquired Immunodeficiency Syndrome (AIDS). Lymph nodes that are enlarged are one of the acute retroviral syndrome signs of HIV. For someone with HIV infection, the acute retroviral syndrome emerges in two to six weeks and becomes better with or without medication. Eight years on an average pass from the time that an HIV-positive person first experiences acute retroviral syndrome before they reach the AIDS stage. According to the literature, [2,3], Durham, the AIDS-infected person's stage lasts for 15 months on average until they pass away.

In our human society, there is a great impact due to infectious diseases, which are responsible for the 25% deaths in the world [4]. There are several factors such as migration of people, vaccinations, behavior changes, and mass media that have a large influence on the disease dynamics and their prevention policies [5]. HIV is one of these infectious diseases which is continuously rising and providing death and infection to humans. Presently, there is no vaccine or drug to properly cure HIV, but some treatments such as antiretroviral therapy (ART) treatments can enhance the conditions of the infected person better, and the risk of HIV transmission is reduced. Antiviral treatments for HIV reduce death cases around the world. The proper treatment of HIV may increase the life expectancy of those who not getting treatment [6]. In this social media era, people may receive more information about HIV/AIDS infection and can better prevent themselves. So, the role of media is vital in this regard by providing positive messages about HIV infection control and their prevention [7]. An individual can prevent

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himself from HIV by avoiding sexual relations with many partners and having safe sexual activities with a healthy individual [8].

Using ART, it has been documented that the new infection cases of HIV have been down to 37% and 45% deaths due to HIV and 13.6 million individuals benefited from 2002–2018. In 2018, approximately 23.3 million have access to ART treatment. If more countries of the world follow the guidelines suggested by World Health Organization (WHO) treating all the individuals who are diagnosed to be HIV infected can get health benefits [9]. In 1987, in the province of Bali, Indonesia, the first case has been reported. Until now the infection has been spread to more provinces in Indonesia [10].

The role of mathematical modeling regearing infectious diseases and controls cannot be ignored including the spread of HIV/AIDS. The authors and researchers studied the background mathematical results for the effective methods that are used in the modeling in science and engineering and its applications, see [11–16]. Mathematical modeling to study real-life and engineering problems, we can refer the readers to see [17–21]. Fractional calculus is the most interesting research area nowadays from researchers' point of view. There have been a lot of articles in literature focusing to study the problems in science and engineering and other social sciences areas, we can highlight some here, [22–29]. For example, the fractional calculus and its theory has been utilized in problems, see [22,23,26], multi-steps methods for the solution of q-Des [24], nonlinear models [25,27–29]. Moreover, the fractional calculus has been used in modeling chaotic systems, [30–32], to coronavirus infection, see [33,34].

A number of articles have been published in the literature highlighting the HIV/AIDS infection. For example, the authors in [35] studied the HIV/AIDS infection dynamics by studying the global stability of the model. The modeling of HIV/AIDS and highlighting of the role of media have been discussed in [36]. A fractional model to study HIV/AIDS infection is given in [37]. The authors in [38] discuss the modeling of HIV/AIDS in the Caputo fractional case. An Atangana–Baleanu approach to studying HIV infection has been suggested by the authors in [39]. The applications of the fractional models and their applications to daily life problems have been documented in [40–42]. The computational methods to solve fractional models have been given in [43,44]. The recent development in the field of fractional calculus increases the attention of the researcher to solve scientific and engineering problems. One of the reasons for fractional calculus is the memory and the heredity properties which is stronger than in the classical case. With the emergence of the new operators in the field of fractional calculus, the vast literature on their applications and numerical procedure are observed. Recently the authors proposed the new concept of piecewise fractional differential to solve scientific and engineering problems efficiently, see [45,46]. There are much recent literature on the fractional operators and their applications to the scientific problems, see [47–50] and especially in epidemiology, see [51–54].

The purpose of this study is to investigate the HIV/AIDS infection dynamics using the new concept of the piecewise fractional model in the sense of the Atangana–Baleanu derivative. We utilized the real data of HIV/AIDS for the parameters setting and show that the present model with a piecewise approach provides useful fitting to the previous case. The simulations of the model are shown graphically which illustrate the theoretical results. We divide section-wise the rest of the work in the present paper. Section "Model construction of HIV/AIDS" explores the detailed model formulation of the HIV/AIDS dynamics. Formulation of the model in the Atangana–Baleanu sense and their mathematical results have been investigated in Section "A fractional model in Atangana–Baleanu case". Section "Piecewise HIV/AIDS Model" studies the formulation of the piecewise model and its numerical procedure. In Section "Numerical solution" and Section "Conclusion" respectively we present the graphical solution of the model and concluded the work.

Model construction of HIV/AIDS

The authors in [55] formulated the dynamics of HIV using the assumptions of awareness. They divided the total human population denoted by N, into five classes, namely, the healthy or susceptible individuals that are unaware of the HIV infection, S_u, the healthy or susceptible individuals that know are aware of the HIV infection is denoted by S_a, individuals infected due to HIV infection are shown by I, individuals infected with HIV infection but are under ART treatment are denoted by C, and the individuals who are infected with AIDS given by (A). So, $N = S_a + S_u + I + C + A$. We assume that the individuals in the unaware class can join the aware class after getting information about HIV. The infected individuals under ART treatment are aware of the infection and it is assumed that they do not spread the infection. The AIDS-infected individuals are assumed to be not spreading the infection. HIV and AIDs infected people are considered to be able to access ART treatment. We considered the death rate of AIDs infected individuals. With these assumptions the model below is formulated using the nonlinear system of differential equations:

$$\begin{cases} \frac{dS_u}{dt} = \Pi - \frac{\beta S_u I}{N} - (\theta + \mu) S_u, \\ \frac{dS_a}{dt} = \theta S_u - (1-\kappa)\frac{\beta S_a I}{N} - \mu S_a, \\ \frac{dI}{dt} = \frac{\beta I}{N}(S_u + (1-\kappa)S_a) + \phi C + \nu A - (\psi + \zeta + \mu) I, \\ \frac{dC}{dt} = \psi I - (\phi + \mu) C, \\ \frac{dA}{dt} = \zeta I - (\nu + \delta + \mu) A, \end{cases} \tag{1}$$

subject to the initial conditions given by

$$S_u(0) = S_{u0} \geq 0, S_a(0) = S_{a0} \geq 0, I(0) = I_0 \geq 0, C(0) = C_0 \geq 0, A(0) = A_0 \geq 0. \tag{2}$$

The healthy population is recruited through the birth rate given by Π while its natural mortality rate is shown by μ. The contact rate that the healthy people become infected is shown by the parameter β. The unaware individuals become aware of a rate given by θ. A number of aware individuals are infected at a rate of $(1-\kappa)$. The individuals in class C join I at a rate ϕ. The transfer rate of the individuals in class A joins class I at a rate ν. At a rate ζ, the individuals infected with HIV lead to AIDS infection. ψ defines the rate by which HIV-infected people are treated under ART. The AIDS-infected people who die of infection are given by δ.

A biological feasible region for the HIV/AIDS infection model (1) shown by Γ and is given by

$$\Gamma = \left\{ (S_u, S_a, I, C, A) \in \mathbb{R}_+^5 : 0 \leq N \leq \frac{\Pi}{\mu} \right\}.$$

The Γ is positively invariant, well-posed, and all the related solutions for the initial values remains in Γ, remains in Γ for any time $t \geq 0$.

A fractional model in Atangana–Baleanu case

We extend the model (1) by using the concept of Atangana–Baleanu operator and obtained the following extended system:

$$\begin{cases} {}_0^{AB}D_t^p S_u = \Pi - \frac{\beta S_u I}{N} - (\theta + \mu) S_u, \\ {}_0^{AB}D_t^p S_a = \theta S_u - (1-\kappa)\frac{\beta S_a I}{N} - \mu S_a, \\ {}_0^{AB}D_t^p I = \frac{\beta I}{N}(S_u + (1-\kappa)S_a) + \phi C + \nu A - (\psi + \zeta + \mu) I, \\ {}_0^{AB}D_t^p C = \psi I - (\phi + \mu) C, \\ {}_0^{AB}D_t^p A = \zeta I - (\nu + \delta + \mu) A, \end{cases} \tag{3}$$

together with the initial values of the model variables,

$$S_u(0) = S_{u0} \geq 0, S_a(0) = S_{a0} \geq 0, I(0) = I_0 \geq 0, C(0) = C_0 \geq 0, A(0) = A_0 \geq 0. \tag{4}$$

Equilibrium points and their stability

The equilibrium points of the model (3) can be determined by equating the right side of all the equations to zero. For the case of disease-free, we denote its equilibrium by E_0 and obtained is as follows:

$$E_0 = \left(\frac{\Pi}{\mu+\theta}, \frac{\theta\Pi}{\mu(\mu+\theta)}, 0, 0, 0\right)$$

Further, the basic reproduction number $\mathcal{R}_0$ and their computation can be done using the method of the next-generation matrix shown in [56]. According to [56], we have the following matrices:

$$F = \begin{pmatrix} \frac{\beta(\mu+(1-\kappa)\theta)}{\mu+\theta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } V = \begin{pmatrix} \psi+\zeta+\mu & -\phi & -\nu \\ -\psi & \phi+\mu & 0 \\ -\zeta & 0 & \mu+\delta+\nu \end{pmatrix}.$$

Utilizing the concept of the spectral radius $\mathcal{R}_0 = \varpi(FV^{-1})$, we can obtain the required basic reproduction number,

$$\mathcal{R}_0 = \underbrace{\frac{\beta(\theta(1-\kappa)+\mu)}{Q_1Q_2}}_{\mathcal{R}_1} + \underbrace{\frac{\zeta\nu}{Q_2Q_4}}_{\mathcal{R}_2} + \underbrace{\frac{\psi\phi}{Q_2Q_3}}_{\mathcal{R}_3},$$

where $Q_1 = (\theta+\mu)$, $Q_2 = (\zeta+\mu+\psi)$, $Q_3 = (\mu+\phi)$, $Q_4 = (\delta+\mu+\nu)$. We show the following theorem for the local stability of the disease-free equilibrium E_0.

Theorem 1. *The HIV/AIDS system (3) is locally asymptotically stable at E_0 if $\mathcal{R}_0 < 1$.*

Proof. We compute the Jacobian matrix of the model (3) at the disease-free case, and obtain the following,

$$J(E_0) = \begin{pmatrix} -Q_1 & 0 & -\frac{\beta\mu}{Q_1} & 0 & 0 \\ \theta & -\mu & -\frac{\beta\theta(1-\kappa)}{Q_1} & 0 & 0 \\ 0 & 0 & \frac{\beta(\theta(1-\kappa)+\mu)}{Q_1} - Q_2 & \phi & \nu \\ 0 & 0 & \psi & -Q_3 & 0 \\ 0 & 0 & \zeta & 0 & -Q_4 \end{pmatrix}.$$

The two eigenvalues of $J(E_0)$ are $\lambda_1 = -\mu$, $\lambda_2 = -(\mu+\theta)$ has negative real part and the rest of the three eigenvalues can be determined on solving the following equation,

$$\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3 = 0, \tag{5}$$

where

$$\begin{aligned} k_1 &= Q_2(1-\mathcal{R}_1) + Q_3 + Q_4, \\ k_2 &= Q_1Q_2Q_4(1-\mathcal{R}_1-\mathcal{R}_2) + Q_1Q_2Q_3(1-\mathcal{R}_1-\mathcal{R}_3) + Q_1Q_4Q_3, \\ k_3 &= Q_2Q_3Q_4(1-\mathcal{R}_0). \end{aligned}$$

We use the Routh–Hurwitz criteria to ensure that the above polynomial gives eigenvalues with negative real part is to satisfy, $k_1, k_3 > 0$ and $k_1k_2 > k_3$. Here, $k_1 > 0$ if $\mathcal{R}_1 < 1$ and obviously $\mathcal{R}_1 < \mathcal{R}_0$, and hence k_1 and k_2 are positive. Similarly, we can see that $k_3 > 0$ if $\mathcal{R}_0 < 1$. So, all the coefficients given in (5) are positive and further it is easy to holds $k_1k_2 > k_3$. Thus, the fractional order HIV/AIDS model (3) is locally asymptotically stable if $\mathcal{R}_0 < 1$. □

Analysis of the endemic equilibrium

Here, we obtain the expression for the endemic equilibria of the fractional model (3). We denote the endemic equilibrium of the model (3) by $E_1 = (S_u^*, S_a^*, I^*, C^*, A^*)$, where

$$S_u^* = \frac{\Pi}{\lambda^* + Q_1}, \tag{6}$$

$$\begin{aligned} S_a^* &= \frac{\theta\Pi}{(\lambda^*+Q_1)(\mu+\lambda^*(1-\kappa))}, \\ I^* &= \frac{\nu A^* + \phi C^* + (1-\kappa)\lambda^* S_a{}^* + \lambda^* S_u^*}{Q_2}, \\ C^* &= \frac{\psi I^*}{Q_3}, \\ A^* &= \frac{\zeta I^*}{Q_4}, \end{aligned} \tag{7}$$

where

$$\lambda^* = \frac{\beta I^*}{N^*}. \tag{8}$$

Inserting Eq. (6) into (8), we have

$$z_1\lambda^{*2} + z_2\lambda^* + z_3 = 0,$$

where

$$\begin{aligned} z_1 &= (1-\kappa)\left(Q_3\left(\zeta+Q_4\right) + Q_4\psi\right), \\ z_2 &= Q_3\left(\zeta((1-\kappa)(\theta-\nu)+\mu) + Q_4\left((1-\kappa)(\theta-\beta)+\mu+(1-\kappa)Q_2\right)\right) \\ &\quad + Q_4\psi((1-\kappa)(\theta-\phi)+\mu), \\ z_3 &= Q_1Q_2Q_3Q_4(1-\mathcal{R}_0). \end{aligned} \tag{9}$$

We can see from (9) that $z_1 > 0$ and z_2 can be positive or negative if $\mathcal{R}_0 < 1$ or greater than one. The sign of z_2 will decide the possible roots of the endemic equilibria whether positive or negative. Such brief discussion is sort out below:

Theorem 2. *The fractional HIV/AIDS model (3) has the following:*

(i) A unique endemic equilibrium for the case when $z_3 < 0$ or $\mathcal{R}_0 > 1$,
(ii) A unique endermic equilibrium exists for the case when $z_2 < 0$ and either $z_3 = 0$ or $z_2^2 - 4z_1z_3 = 0$,
(iii) A possibility of two endemic equilibria for the case when $z_3 > 0$, $z_2 < 0$ and $z_2^2 - 4z_1z_3 > 0$,
(iv) Other than the above cases there is no further possibilities of equilibria.

Piecewise HIV/AIDS model

This section studies the model (3) into piecewise fractional differential equations in Atangana–Baleanu sense. To do this, we write the model (3) in piecewise differential equations given by:

$$\begin{cases} \frac{dS_u}{dt} = \Pi - \frac{\beta S_u I}{N} - (\theta+\mu)S_u, \\ \frac{dS_a}{dt} = \theta S_u - (1-\kappa)\frac{\beta S_a I}{N} - \mu S_a, \\ \frac{dI}{dt} = \frac{\beta I}{N}(S_u + (1-\kappa)S_a) + \phi C + \nu A - (\psi+\zeta+\mu)I, \\ \frac{dC}{dt} = \psi I - (\phi+\mu)C, \\ \frac{dA}{dt} = \zeta I - (\nu+\delta+\mu)A. \end{cases} \tag{10}$$

Table 1
Parameters values used in simulation of the model (10)–(12).

Symbol	Value	Source
Π	$\frac{229{,}800{,}000}{67.39}$	[55]
β	0.3465	[55]
θ	0.2351	[55]
μ	$\frac{1}{67.39}$	[55]
κ	0.3243	[55]
ϕ	0.2059	[55]
ν	0.7661	[55]
ζ	0.1882	[55]
ψ	3.6523e–04	[55]
δ	0.7012	[55]

$$\begin{cases} {}_0^{AB}D_t^p S_u = \Pi - \frac{\beta S_u I}{N} - (\theta+\mu)S_u, \\ {}_0^{AB}D_t^p S_a = \theta S_u - (1-\kappa)\frac{\beta S_a I}{N} - \mu S_a, \\ {}_0^{AB}D_t^p I = \frac{\beta I}{N}(S_u + (1-\kappa)S_a) + \phi C + \nu A - (\psi+\zeta+\mu)I, \\ {}_0^{AB}D_t^p C = \psi I - (\phi+\mu)C, \\ {}_0^{AB}D_t^p A = \zeta I - (\nu+\delta+\mu)A, \end{cases} \tag{11}$$

$$\begin{cases} dS_u = (\Pi - \frac{\beta S_u I}{N} - (\theta+\mu)S_u)dt + \sigma_1 S_u dB_1(t), \\ dS_a = (\theta S_u - (1-\kappa)\frac{\beta S_a I}{N} - \mu S_a) + \sigma_2 S_a dB_2(t), \\ dI = (\frac{\beta I}{N}(S_u + (1-\kappa)S_a) + \phi C + \nu A - (\psi+\zeta+\mu)I) + \sigma_3 I dB_3(t), \\ dC = (\psi I - (\phi+\mu)C) + \sigma_4 C dB_4(t), \\ dA = (\zeta I - (\nu+\delta+\mu)A) + \sigma_5 A dB_5(t). \end{cases} \tag{12}$$

In above models (10)–(12), we use the time interval respectively $t \in [0, T_1]$, $t \in [T_1, T_2]$ and $t \in [T_2, T]$. The fractional order is shown by p and σ_i for $i = 1, \dots, 5$ defines the stochastic parameters (noises) which are positive, while the definition of the Atangana–Baleanu derivative and the related piecewise differential equations related results are shown below: For the function $f(t)$ and the fractional order $0 < \alpha \leq 1$, we write

$${}_0^{ABC}D_t^p f(t) = \frac{AB(p)}{1-p}\int_0^t f'(\xi)E_p\left[-\frac{p}{1-p}(t-\xi)^p\right]d\xi,$$

where $AB(p) = 1 - p + p/\Gamma(p)$ defines the normalization function.

Numerical procedure

In order to solve numerically the piecewise model (10)–(12), we use the method explained in [45] for the case of Atangana–Baleanu derivative. We start the procedure in the following:

$$\begin{cases} \frac{du_k(t)}{dt} = \Psi\left(t, u_k\right), \ u_k(0) = u_{k,0}, k = 1, 2, \dots, n \text{ if } t \in [0, T_1], \\ {}_{T_1}^{ABC}D_t^p u_k(t) = \Psi\left(t, u_k\right), \ u_k\left(T_1\right) = u_{k,1}, \ \text{if } t \in [T_1, T_2], \\ du_k(t) = \Psi\left(t, u_k\right)dt + \sigma_k u_k dB_k(t), \ u_k\left(T_2\right) = u_{k,2}, \ \text{if } t \in [T_2, T]. \end{cases} \tag{13}$$

$$u_k^{m_1} = u_k(0) + \sum_{j_1=2}^{m_1}\left[\frac{23}{12}\Psi\left(t_{j_1}, u_{j_1}\right) - \frac{4}{3}\Psi\left(t_{j_1-1}, u_{j_1-1}\right)\right.$$
$$\left. + \frac{5}{12}\Psi\left(t_{j_1-2}, u_{j_1-2}\right)\right] h, t \in [0, T_1], \tag{14}$$

$$\begin{aligned} u_k^{m_2} &= u_k\left(T_1\right) + \frac{1-p}{AB(p)}\Psi\left(t_{m_2}, u_{m_2}\right) \\ &+ \frac{p(h)^p}{AB(p)\Gamma(p+1)}\sum_{r_2=m_1+3}^{m_2}\Psi\left(t_{r_2-2}, u_{j_2-2}\right)\Theta_1 \\ &+ \frac{p(h)^p}{AB(p)\Gamma(p+2)}\sum_{j_2=m_1+3}^{m_2}\left[\Psi\left(t_{r_2-1}, u_{r_2-1}\right) - \Psi\left(t_{r_2-2}, u_{j_2-2}\right)\right]\Theta_2 \\ &+ \frac{p(h)^p}{2AB(p)\Gamma(p+3)}\sum_{r_2=m_1+3}^{m_2}\left[\Psi\left(t_{r_2}, u_{r_2}\right) - 2\Psi\left(t_{r_2-1}, u_{r_2-1}\right)\right. \\ &\left. + \Psi\left(t_{r_2-2}, u_{r_2-2}\right)\right]\Theta_3, \quad t \in [T_1, T_2] \end{aligned} \tag{15}$$

$$\begin{aligned} u_k^{m_3} &= u_k\left(T_2\right) + \sum_{j_3=m_2+3}^{m_3}\left[\frac{23}{12}\Psi\left(t_{r_3}, u_{r_3}\right) - \frac{4}{3}\Psi\left(t_{r_3-1}, u_{r_3-1}\right)\right. \\ &\left. + \frac{5}{12}\Psi\left(t_{r_3-2}, u_{r_3-2}\right)\right] h \\ &+ \sigma_k \sum_{r=m_2+3}^{m_3} u_k^{r_3}\left(B_k^{r_3} - B_k^{r_3-1}\right), t \in [T_2, T], \end{aligned} \tag{16}$$

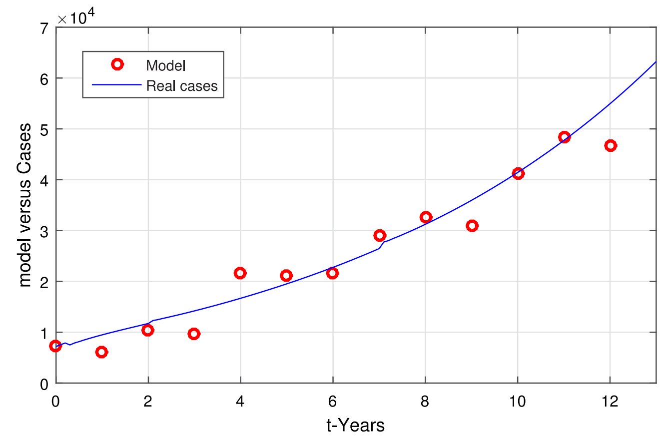

Fig. 1. Data fitting to HIV/AIDS cases using model (10)–(12) for $T_1 = 2$, $T_2 = 7$, $T_3 = 13$.

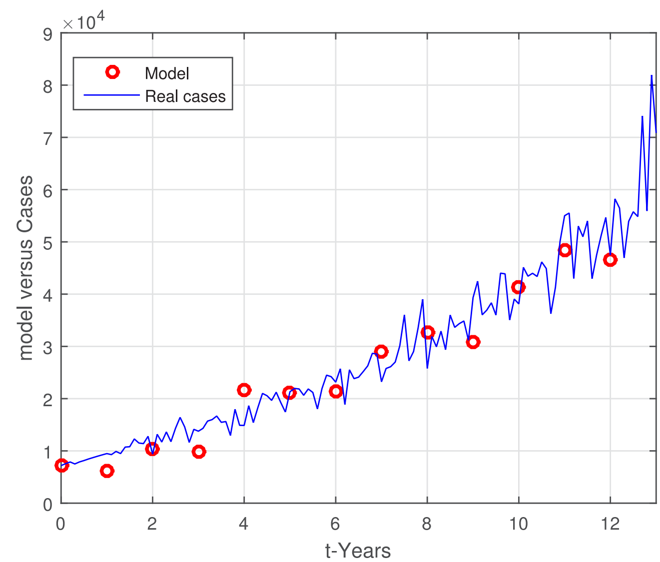

Fig. 2. Data fitting to HIV/AIDS cases using model (10)–(12) when $T_1 = 1$, $T_2 = 1$, $T_3 = 13$ and $\sigma_1 = 0.2$, $\sigma_2 = 0.40$, $\sigma_3 = 0.10$, $\sigma_4 = 0.30$, $\sigma_5 = 0.4$.

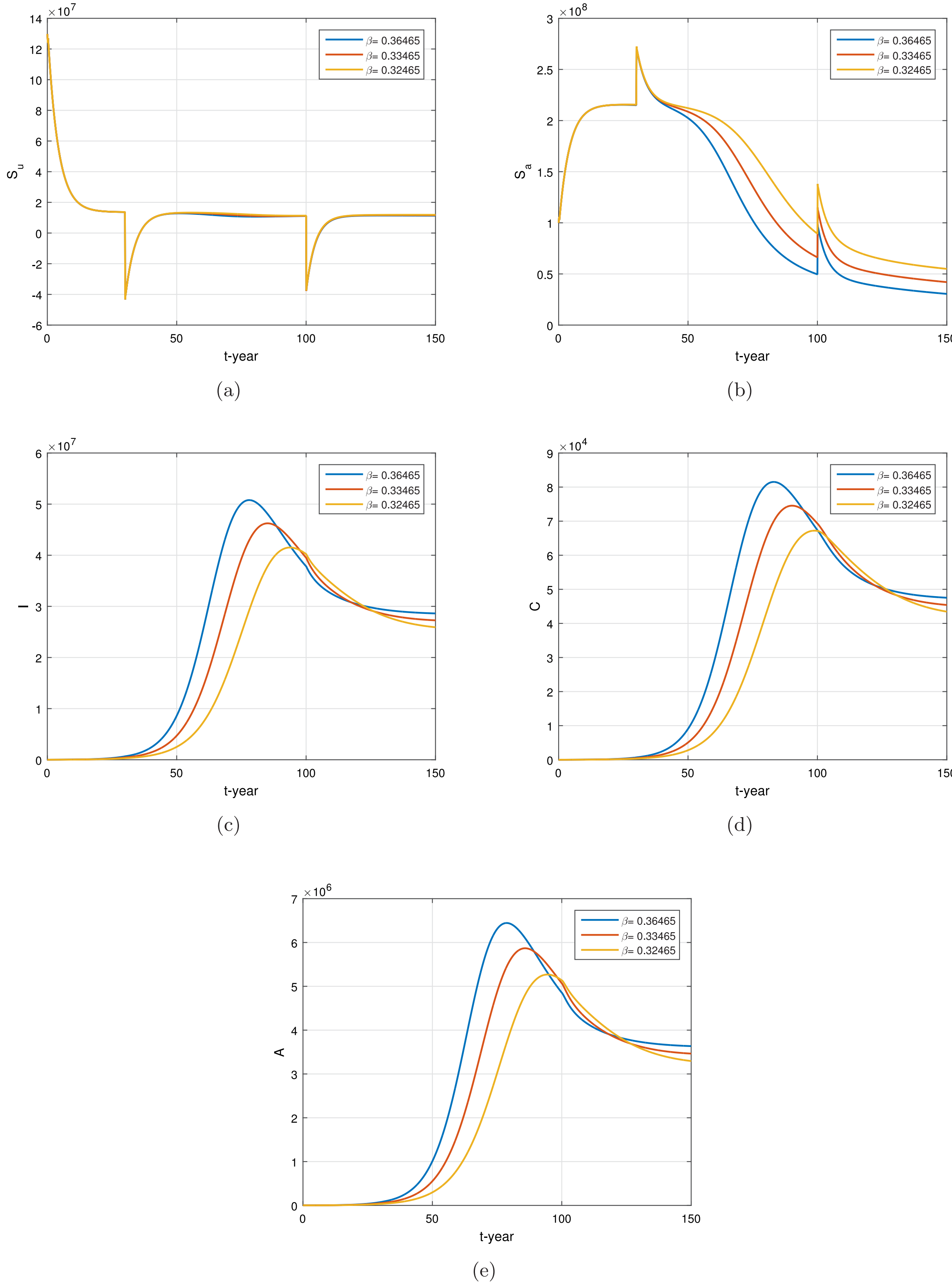

Fig. 3. Dynamical behavior of the model when $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 0$ and varying β.

where

$$\Theta_1 = \left\{ (m_2 - r_2 + 1)^p - (m_2 - r_2)^p \right\},$$

$$\Theta_2 = \left\{ \begin{array}{c} (m_2 - r_2 + 1)^p (m_2 - r_2 + 3 + 2p) \\ -(m_2 - r_2)^p (m_2 - r_2 + 3 + 3p) \end{array} \right\},$$

$$\Theta_3 = \left\{ \begin{array}{c} (m_2 - r_2 + 1)^p \left[\begin{array}{c} 2(m_2 - r_2)^2 + (3p + 10)(m_2 - r_2) \\ +2p^2 + 9p + 12 \end{array} \right] \\ -(m_2 - r_2)^p \left[\begin{array}{c} 2(m_2 - r_2)^2 + (5p + 10)(m_2 - r_2) \\ +6p^2 + 18p + 12 \end{array} \right] \end{array} \right\}.$$

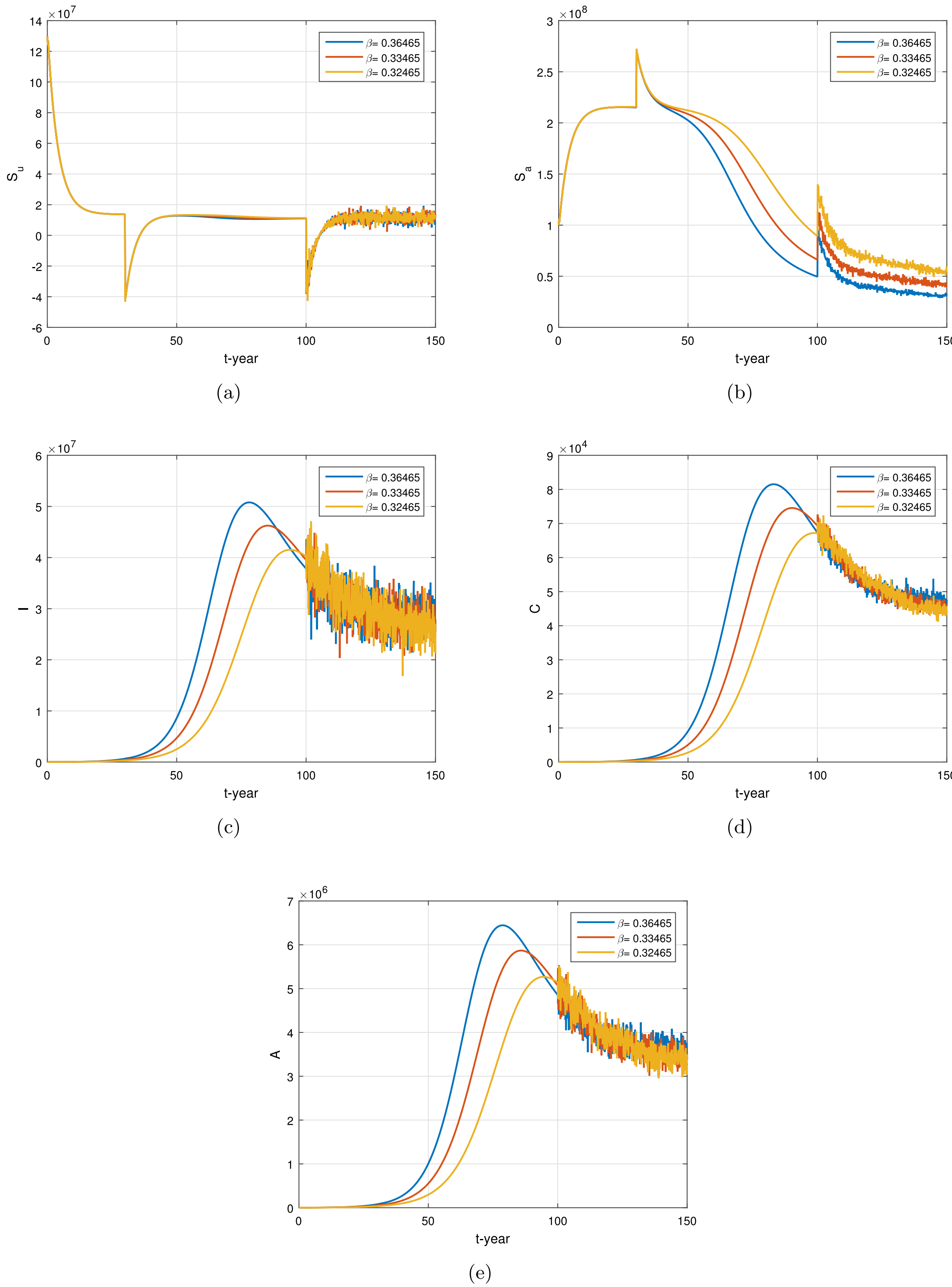

Fig. 4. Dynamical behavior of the model equations when $\sigma_1 = 0.2$, $\sigma_2 = 0.040$, $\sigma_3 = 0.10$, $\sigma_4 = 0.030$, $\sigma_5 = 0.04$ and varying the values of β.

Numerical solution

This section describes the numerical solution of the piecewise model (10)–(12) by using the numerical algorithms shown in (14)–(16). The numerical values of the model parameters are shown in Table 1 obtained from the work given in [55]. The authors provided fitting to the real data of HIV/AIDS in their work in [55], see (Figs. 2 and 3). With the new concept of the piecewise fractional differential equation model (10)–(12), we showed fitting to the data by using the piecewise model and obtained reasonable results as shown in Figs. 1 and 2. The comparison of the data fitting to HIV/AIDS model (10)–(12) is better than the fractional model results given in [55]. Further, we show simulations of the model (10)–(12) with the use of the fractional order and the stochastic noises, see Figs. 3–6. Fig. 3 presents the dynamics of

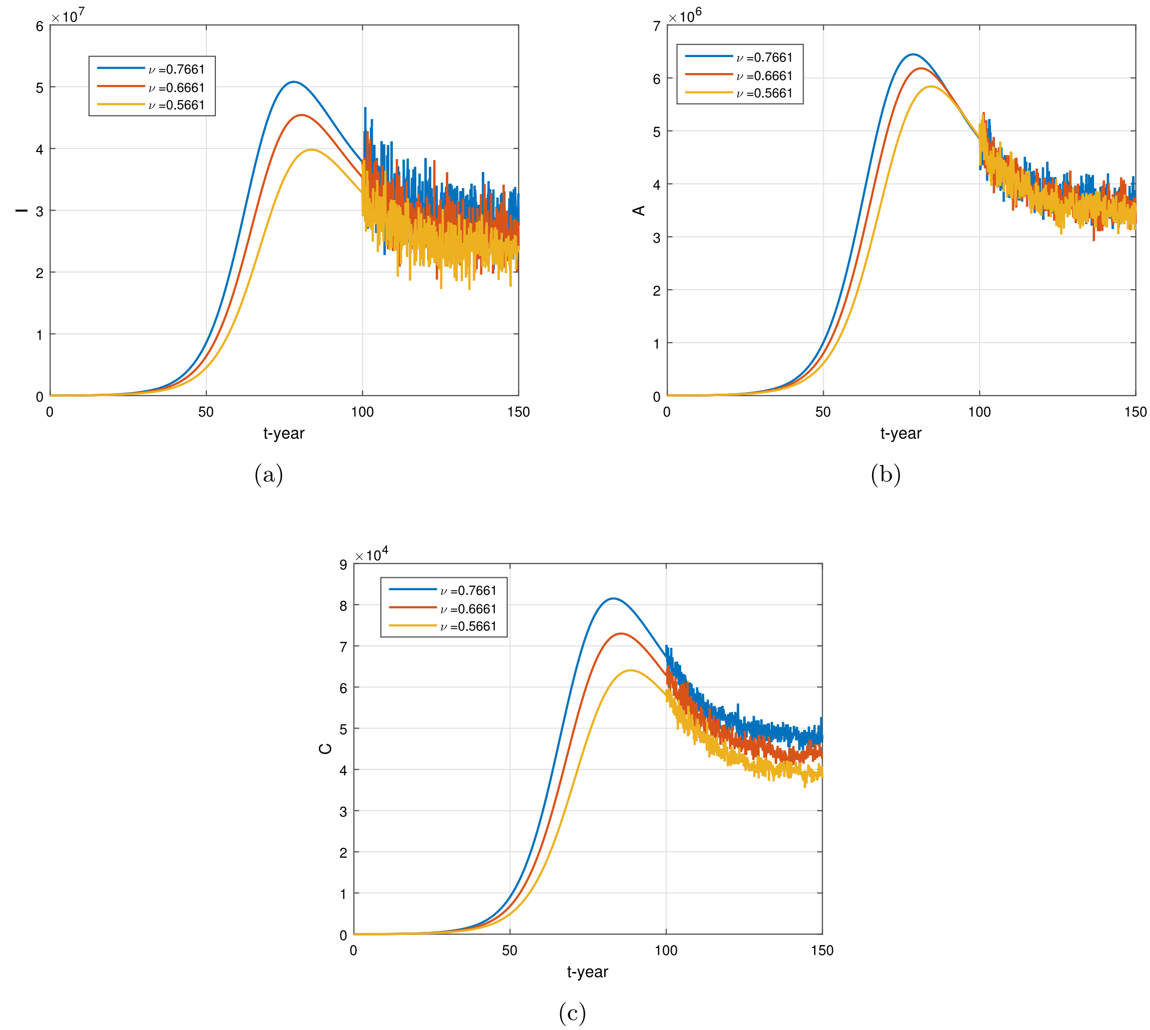

Fig. 5. Dynamical behavior of the model equations when $\sigma_1 = 0.2$, $\sigma_2 = 0.040$, $\sigma_3 = 0.10$, $\sigma_4 = 0.030$, $\sigma_5 = 0.04$ and varying the values of ν.

HIV/AIDS for different values of β in the absence of stochastic noises. It can be observed from Fig. 3 that decreasing the contact among healthy and unaware individuals decreases the number of infected cases. Fig. 4 describes the dynamics of the piecewise model for different values of the parameter β and with the suggested stochastic noises. It can be observed that by decreasing the interaction among the healthy and HIV-infected people, there is obviously a decrease in the infected people. Similarly, the behavior of the individuals for the parameter is shown in Fig. 5, and we can see the decrease in the population of the infected individuals. In Fig. 6 the behavior of the model is shown graphically for various of the fractional order p.

Conclusion

We studied the HIV/AIDS infection model using the new concept of piecewise fractional differential equation in the Atangana–Baleanu sense. The model proposed in [55] has been extended to a piecewise model and obtained its mathematical results. The model is proven to be locally asymptotically stable. The data fitting for the HIV/AIDS cases in Indonesia for the period 2006–2018 are considered and obtained fitting with the new concept of the piecewise model. The results suggest that the fitting with the piecewise model is better than in the previous study. We used the same model with the same parameters and the initial conditions reported in [55] and proved that the results of the current study are better than the previous. The computed basic reproduction using the piecewise fitting to the data is $\mathcal{R}_0 \approx 2.2763$. Further, we observed the fitting when stochastic case and found the results better. The extended piecewise model is solved numerically using the recently reported algorithm for the numerical solution of piecewise stochastic fractional differential equations and provided graphical results to illustrate the theoretical findings. The results indicate this new concept of piecewise differential provides better results for the proposed model and maybe it is better for other engineering and scientific problems.

CRediT authorship contribution statement

Yi Zhao: Methodology, Software, Data curation, Writing – review & editing. **Ehab E. Elattar:** Methodology, Software, Data curation, Writing – review & editing. **Muhammad Altaf Khan:** Conceptualization, Methodology, Software, Data curation, Writing – original draft, Visualization, Investigation, Validation, Writing – review & editing. **Fatmawati:** Conceptualization, Methodology, Software, Data curation, Writing – original draft, Visualization, Investigation, Validation, Writing – review & editing. **Mohammed Asiri:** Methodology, Software, Data curation, Writing – review & editing. **Pongsakorn Sunthrayuth:** Methodology, Software, Data curation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

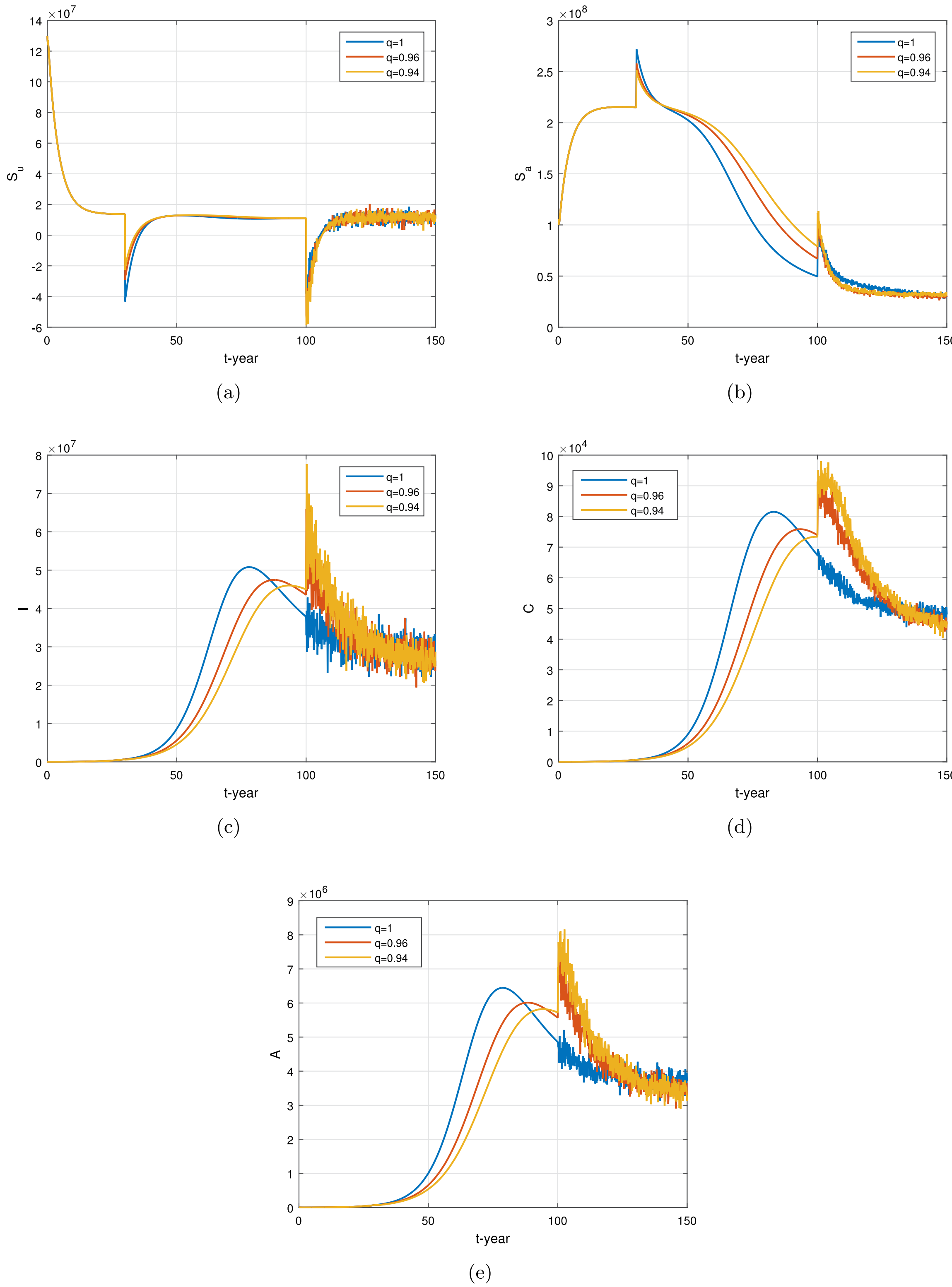

Fig. 6. Dynamical behavior of the infected compartment of the model when $\sigma_1 = 0.2$, $\sigma_2 = 0.040$, $\sigma_3 = 0.10$, $\sigma 4 = 0.030$, $\sigma_5 = 0.04$, and $T_1 = 30$, $T_2 = 100$, $T_3 = 150$.

Data availability

Data will be made available on request.

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References

[1] Hunaids. unaids data 2019. 2019, unaids.org/sites/default/files/media_asset/2019-unaids-data_en.pdf. [Accessed 16 November 2019].

[2] Departemen kesehatan republik indonesia. situasi hiv/aids di indonesia tahun 1987-2006. 2006, pusat data dan informasi departemen kesehatan ri, jakarta.

[3] Lashley Felissa R. Hiv infection and aids: Etiology, epidemiology, and transmission. In: The person with HIV/AIDS: nursing perspectives. 2010, p. 7–15.

[4] Vita-Finzi Leo, et al. Preventing chronic diseases: A vital investment. World Health Organization; 2005.

[5] Poletti Piero, Ajelli Marco, Merler Stefano. The effect of risk perception on the 2009 h1n1 pandemic influenza dynamics. PLoS One 2011;6(2).

[6] Deeks Steven G, Lewin Sharon R, Havlir Diane V. The end of aids: Hiv infection as a chronic disease. Lancet 2013;382(9903):1525–33.

[7] Sharma Anupama, Misra AK. Modeling the impact of awareness created by media campaigns on vaccination coverage in a variable population. J Biol Systems 2014;22(02):249–70.

[8] Miner Michael H, Peterson John L, Welles Seth L, Jacoby Scott M, Simon Rosser BR. How do social norms impact hiv sexual risk behavior in hiv-positive men who have sex with men? multiple mediator effects. J Health Psychol 2009;14(6):761–70.

[9] Who. hiv/aids. 2019, https://www.who.int/news-room/fact-sheets/detail/hiv-aids. [Accessed 9 November 2019].

[10] Kementerian kesehatan republik indonesia. perkembangan hiv-aids & penyakit infeksi menular (pims) triwulan i tahun 2017, direktorat jendral pencegahan dan pengendalian penyakit, jakarta. 2017.

[11] Zhao Tie-Hong, He Zai-Yin, Chu Yu-Ming. Sharp bounds for the weighted hölder mean of the zero-balanced generalized complete elliptic integrals. Comput Methods Funct Theory 2021;21(3):413–26.

[12] Zhao Tie-Hong, Wang Miao-Kun, Chu Yu-Ming. Concavity and bounds involving generalized elliptic integral of the first kind. J Math Inequal 2021;15(2):701–24.

[13] Ladra Manuel, Páez-Guillán Pilar, Recio Tomás. Dealing with negative conditions in automated proving: tools and challenges. the unexpected consequences of rabinowitsch's trick. Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM 2020;114(4):1–16.

[14] Chu Hong-Hu, Zhao Tie-Hong, Chu Yu-Ming. Sharp bounds for the toader mean of order 3 in terms of arithmetic, quadratic and contraharmonic means. Math Slovaca 2020;70(5):1097–112.

[15] Zhao Tie-Hong, He Zai-Yin, Chu Yu-Ming. On some refinements for inequalities involving zero-balanced hypergeometric function. AIMS Math 2020;5(6):6479–95.

[16] Tie-Hong Zhao, Zhou Bu-Chuan, Miao-Kun Wang, Yu-Ming Chu. On approximating the quasi-arithmetic mean. J Inequal Appl 2019;2019(1):1–12.

[17] Zha Tie-Hong, Castillo Oscar, Jahanshahi Hadi, Yusuf Abdullahi, Alassafi Madini O, Alsaadi Fawaz E, et al. A fuzzy-based strategy to suppress the novel coronavirus (2019-ncov) massive outbreak. Appl Comput Math 2021;160–76.

[18] Chu Yu-Ming, Nazir Umar, Sohail Muhammad, Selim Mahmoud M, Lee Jung-Rye. Enhancement in thermal energy and solute particles using hybrid nanoparticles by engaging activation energy and chemical reaction over a parabolic surface via finite element approach. Fractal Fract 2021;5(3):119.

[19] Khan Muhammad Altaf, Ullah Saif, Farhan Muhammad. The dynamics of zika virus with caputo fractional derivative. AIMS Math 2019;4(1):134–46.

[20] Khan MA, Ali K, Bonyah E, Okosun KO, Islam S, Khan A. Mathematical modeling and stability analysis of pine wilt disease with optimal control. Sci Rep 2017;7(1):1–19.

[21] Wang Shuihua, Du Sidan, Atangana Abdon, Liu Aijun, Lu Zeyuan. Application of stationary wavelet entropy in pathological brain detection. Multimedia Tools Appl 2018;77(3):3701–14.

[22] Karthikeyan Kulandhivel, Karthikeyan Panjaiyan, Baskonus Haci Mehmet, Venkatachalam Kuppusamy, Chu Yu-Ming. Almost sectorial operators on ψ-hilfer derivative fractional impulsive integro-differential equations. Math Methods Appl Sci 2021.

[23] Rashid Saima, Abouelmagd Elbaz I, Khalid Aasma, Farooq Fozia Bashir, Chu YM. Some recent developments on dynamical h-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels. Fractals 2022;30(2):2240110.

[24] Hajiseyedazizi Sayyedeh Narges, Samei Mohammad Esmael, Alzabut Jehad, ming Chu Yu. On multi-step methods for singular fractional q-integro-differential equations. Open Math 2021;19(1):1378–405.

[25] Jin Fang, Qian Zi-Shan, Chu Yu-Ming, Rahman Mati ur. On nonlinear evolution model for drinking behavior under caputo-fabrizio derivative. J Appl Anal Comput 2022;12(2):790–806.

[26] Rashid Saima, Sultana Sobia, Karaca Yeliz, Khalid Aasma, Chu Yu-Ming. Some further extensions considering discrete proportional fractional operators. Fractals 2022;30(01):2240026.

[27] Wang Fuzhang, Khan Muhammad Nawaz, Ahmad Imtiaz, Ahmad Hijaz, Abu-Zinadah Hanaa, Chu Yu-Ming. Numerical solution of traveling waves in chemical kinetics: time-fractional fishers equations. Fractals 2022;30(2). 2240051–34.

[28] Rashid Saima, Abouelmagd Elbaz I, Sultana Sobia, Chu Yu-Ming. New developments in weighted n-fold type inequalities via discrete generalized $\hat{\hbar}$-proportional fractional operators. Methods 2022;9:22.

[29] Iqbal Sayed Allamah, Hafez Md Golam, Chu Yu-Ming, Park Choonkil. Dynamical analysis of nonautonomous rlc circuit with the absence and presence of atangana-baleanu fractional derivative. J Appl Anal Comput 2022;12(2):770–89.

[30] Aghababa Mohammad Pourmahmood, Borjkhani Mehdi. Chaotic fractional-order model for muscular blood vessel and its control via fractional control scheme. Complexity 2014;20(2):37–46.

[31] Taheri Mostafa, Zhang Chongqi, Berardehi Zahra Rasooli, Chen Yucheng, Roohi Majid. No-chatter model-free sliding mode control for synchronization of chaotic fractional-order systems with application in image encryption. Multimedia Tools Appl 2022;1–31.

[32] Rida Saad Z, Farghaly Ahmed A, Hussien Fatma. The effect of feedback controls on stability in a fractional-order si epidemic model. Int J Appl Comput Math 2021;7(4):1–12.

[33] Chu Yu-Ming, Ali Aatif, Khan Muhammad Altaf, Islam Saeed, Ullah Saif. Dynamics of fractional order covid-19 model with a case study of saudi arabia. Results Phys 2021;21:103787.

[34] Khan Muhammad Altaf, Ullah Saif, Kumar Sunil. A robust study on 2019-ncov outbreaks through non-singular derivative. Eur Phys J Plus 2021;136(2):1–20.

[35] Silva Cristiana J, Torres Delfim FM. A sica compartmental model in epidemiology with application to hiv/aids in cape verde. Ecol Complex 2017;30:70–5.

[36] Ghosh Indrajit, Tiwari Pankaj Kumar, Samanta Sudip, Elmojtaba Ibrahim M, Al-Salti Nasser, Chattopadhyay Joydev. A simple si-type model for hiv/aids with media and self-imposed psychological fear. Math Biosci 2018;306:160–9.

[37] Shaiful Endrik Mifta, Utoyo Mohammad Imam, et al. A fractional-order model for hiv dynamics in a two-sex population. Int J Math Math Sci 2018;2018.

[38] Silva Cristiana J, Torres Delfim FM. Stability of a fractional hiv/aids model. Math Comput Simulation 2019;164:180–90.

[39] Owolabi Kolade M, Atangana Abdon. Mathematical analysis and computational experiments for an epidemic system with nonlocal and nonsingular derivative. Chaos Solitons Fractals 2019;126:41–9.

[40] Atangana Abdon, Araz Seda Igret. Atangana-seda numerical scheme for labyrinth attractor with new differential and integral operators. Fractals 2020.

[41] Atangana Abdon, Bouallegue Ghaith, Bouallegue Kais. New multi-scroll attractors obtained via julia set mapping. Chaos Solitons Fractals 2020;134:109722.

[42] Khan Muhammad Altaf, Atangana Abdon. Modeling the dynamics of novel coronavirus (2019-ncov) with fractional derivative. Alex Eng J 2020.

[43] Hashemi MS, Atangana A, Hajikhah S. Solving fractional pantograph delay equations by an effective computational method. Math Comput Simulation 2020.

[44] Atangana Abdon, Araz Seda İğret. Extension of atangana-seda numerical method to partial differential equations with integer and non-integer order. Alex Eng J 2020.

[45] Atangana Abdon, Araz Seda İğret. Modeling third waves of covid-19 spread with piecewise differential and integral operators: Turkey, spain and czechia. Results Phys 2021;29:104694.

[46] Atangana Abdon, Araz Seda Igret. New numerical scheme with newton polynomial: theory, methods, and applications. Academic Press; 2021.

[47] Chu Yu-Ming, Bashir Seemab, Ramzan Muhammad, Malik Muhammad Yousaf. Model-based comparative study of magnetohydrodynamics unsteady hybrid nanofluid flow between two infinite parallel plates with particle shape effects. Math Methods Appl Sci 2022.

[48] Nazeer Mubbashar, Hussain Farooq, Khan M Ijaz, El-Zahar Essam Roshdy, Chu Yu-Ming, Malik MY, et al. Theoretical study of mhd electro-osmotically flow of third-grade fluid in micro channel. Appl Math Comput 2022;420:126868.

[49] Chu Yu-Ming, Shankaralingappa BM, Gireesha BJ, Alzahrani Faris, Khan M Ijaz, Khan Sami Ullah. Combined impact of cattaneo-christov double diffusion and radiative heat flux on bio-convective flow of maxwell liquid configured by a stretched nano-material surface. Appl Math Comput 2022;419:126883.

[50] Zhao Tie-Hong, Khan M Ijaz, Chu Yu-Ming. Artificial neural networking (ann) analysis for heat and entropy generation in flow of non-newtonian fluid between two rotating disks. Math Methods Appl Sci 2021.

[51] Sene Ndolane. Sir epidemic model with mittag–leffler fractional derivative. Chaos Solitons Fractals 2020;137:109833.

[52] Higazy M. Novel fractional order sidarthe mathematical model of covid-19 pandemic. Chaos Solitons Fractals 2020;138:110007.

[53] Sun Tingzhe, Wang Yan. Modeling covid-19 epidemic in heilongjiang province, China. Chaos Solitons Fractals 2020;138:109949.

[54] Atangana Abdon. Modelling the spread of covid-19 with new fractal-fractional operators: can the lockdown save mankind before vaccination? Chaos Solitons Fractals 2020;136:109860.

[55] Khan Muhammad Altaf, Odinsyah Hafidz Putra, et al. Fractional model of hiv transmission with awareness effect. Chaos Solitons Fractals 2020;138:109967.

[56] den Driessche Pauline Van, Watmough James. Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission. Math Biosci 2002;180(1–2):29–48.

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A NEW GENERALIZED FORWARD-BACKWARD SPLITTING METHOD IN REFLEXIVE BANACH SPACES

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Abstract. In this article, we propose a new generalized forward-backward splitting method for finding a common zero of a finite family of the sum of maximal monotone and Bregman inverse strongly monotone operators in the framework of a reflexive Banach space. We then prove the strong convergence result of the sequence generated by our proposed method under suitable conditions. Some numerical experiments are presented to illustrate the efficiency of the proposed algorithm. The results presented in this paper improve and generalize many known results in this research field.

1. Introduction

Let E be a real Banach space. Let $A : E \to E$ and $B : E \multimap E$ single and set-valued operators, respectively. Consider the following *quasi-inclusion problem*: find $z \in E$ such that

$$0 \in (A+B)z, \tag{1.1}$$

where 0 is a zero vector in E. We denote by $(A+B)^{-1}0$ set of solutions of the Problem (1.1). When $A = 0$, the Problem (1.1) becomes the inclusion problem introduced by Rockafellar [51] and when $E = \mathbb{R}^n$, the Problem (1.1) becomes the generalized equation introduced by Robinson [49]. Many practical nonlinear problems arising in applied sciences such as in image recovery, signal processing and machine learning can be formulated as finding zeros of the operator decomposed as the sum of two operators (1.1). Furthermore, the Problem (1.1) is a generalization of variational inequalities, equilibrium problem, split feasibility problem, convex minimization problem, Nash equilibrium problem in noncooperative games and so on (see for instance [23,33,56,59]).

An efficient method for solving the Problem (1.1) in a real Hilbert space H, is known as *forward-backward splitting method* [23,33]. This method is defined as the following formula:

$$x_{n+1} = (I+\lambda B)^{-1}(I-\lambda A)x_n, \ \forall n \geq 1, \tag{1.2}$$

where $\lambda > 0$ is a step size. In fact, under appropriate conditions, the sequence generated by (1.2) converges weakly to a point in $(A+B)^{-1}0$. Note that, this

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method includes, as special cases, the proximal point algorithm (when $A = 0$) [25, 51] and the gradient method [11, 24].

In order to get strong convergence, Takahashi et al. [59] introduced the following modified forward-backward method:

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A x_n)), \ \forall n \geq 1, \tag{1.3}$$

where $u \in H$, $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$, A is an α-inverse strongly monotone mapping on H, B is a maximal monotone operator on H, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2\alpha)$. They proved that the sequence $\{x_n\}$ converges strongly to a point in $(A+B)^{-1}0$ under suitable conditions on the parameters.

For solving the Problem (1.1) in a q-uniformly smooth and uniformly convex Banach spaces E, López et al. [34] introduced the following modified forward-backward method:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(J^B_{\lambda_n}(x_n - \lambda_n(Ax_n + a_n)) + b_n), \ \forall n \geq 1, \tag{1.4}$$

where $u \in E$, A is an α-inverse strongly accretive mapping on E, B is an m-accretive operator on E, $\{\alpha_n\} \subset (0,1]$ and $\{a_n\}, \{b_n\}$ are the error sequences in E. They proved that the sequence $\{x_n\}$ converges strongly to a point in $(A+B)^{-1}0$ provided that the sequence of step size is chosen to satisfy the following inequality:

$$0 < a \leq \lambda_n \leq b < \Big(\frac{\alpha q}{\kappa_q}\Big)^{\frac{1}{q-1}}, \tag{1.5}$$

where $\alpha > 0$, $1 < q \leq 2$ and κ_q is the q-uniform smoothness coefficient of E (see [61] for more detail).

Cholamjiak [20] introduced the following generalized forward-backward splitting method for solving the Problem (1.1) in a q-uniformly smooth and uniformly convex Banach spaces E:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \delta_n J^B_{\lambda_n}(x_n - \lambda_n A x_n), \ \forall n \geq 1, \tag{1.6}$$

where $u \in E$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ with $\alpha_n + \beta_n + \delta_n = 1$. If the sequence of step size satisfies inequality (1.5), then it was shown that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a point in $(A+B)^{-1}0$.

However, the results of López et al. [34] and Cholamjiak [20] may be not applicable in L_q(or l_q) for $q > 2$ since E is a q-uniformly smooth Banach space. For this reason, it is necessary to extend those results to more general settings. Moreover, the sequence of step size is chosen dependently of q-uniform smoothness coefficient κ_q. This might be too hard to implement in practice.

Many problems in applied sciences can be formulated as the following fixed point problem: find a point $z \in E$ such that

$$z = Sz, \tag{1.7}$$

where $S : E \to E$ is a nonlinear mapping. The set of solutions of the fixed point problem is denoted by $F(S)$. Various modifications of iterative methods have been constructed and developed in order to get weak and strong convergence results for finding fixed points of several kinds of nonlinear mappings by many authors (see for instance [5, 22, 28–31, 36, 39, 53–55]).

In 2020, Ogbuisi and Izuchukwu [35] introduced a forward-backward splitting method for finding a zero of the sum of maximal monotone and Bregman inverse strongly monotone operators which is also a solution of a fixed point problem for a Bregman strongly nonexpansive mapping S. They proved a strong convergence theorem for approximating an element in $F(S) \cap (A+B)^{-1}0$ in a real reflexive Banach space.

In recent years, forward-backward splitting algorithms have been studied and developed by many authors in several setting (see for instance [19, 21, 32, 52]). In this paper, we study the following *generalized quasi-inclusion problem*: find $z \in E$ such that

$$0 \in \bigcap_{i=1}^{N} (A_i + B_i)z, \tag{1.8}$$

where $A_i : E \to E$ and $B_i : E \multimap E$, $i = 1, 2, \ldots, N$ are finite family of Bregman inverse strongly monotone and maximal monotone operators, respectively.

Inspired by the above works, we propose a generalized forward-backward splitting method for finding a common zero of a finite family of the sum of maximal monotone and Bregman inverse strongly monotone operators in the framework of reflexive Banach spaces. The main advantage of the proposed method is that the step size does not require the prior knowledge of the q-uniform smoothness coefficient. The outline of the paper is divided as follows: In Section 2, we recall some basic notations and some preliminary results which are need for in our work. In section 3, we prove the strong convergence result of the sequence generated by our proposed method. In Section 4, we give some deduced results related to the main result and finally, in Section 5, we provide some numerical examples of the proposed method.

2. Notations and preliminaries

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a reflexive Banach space with its dual space E^*. We write $\langle x^*, x\rangle$ for the value of a functional x^* in E^* at x in E, that is, $\langle x^*, x\rangle := x^*(x)$. In what follows, we shall use the following notations:

- $x_n \to x$ means that $\{x_n\}$ converges strongly to x.
- $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x.

Throughout this paper, we assume that $f : E \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function. The effective domain of f is denoted by $\mathrm{dom} f := \{x \in E : f(x) < \infty\}$. The function f on E is said to be *cofinite* if $\mathrm{dom} f^* = E^*$. The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, \infty]$ defined by

$$f^*(x^*) := \sup_{x \in E}\{\langle x^*, x\rangle - f(x)\}.$$

The *subdifferential* of f is the set-valued mapping $\partial f : E \multimap E^*$ defined by

$$\partial f(x) := \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y),\ \forall y \in E\},\ x \in E.$$

We know that the *Young-Fenchel inequality* holds, that is, $f(x) + f^*(x^*) \ge \langle x^*, x\rangle$ for all $x \in E$ and $x^* \in E^*$. It is also known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x^*, x\rangle$ (see [50]). For any $x \in \mathrm{int}(\mathrm{dom} f)$, the *directional derivative* of f at x in the direction $y \in E$ given by

$$f'_+(x,y) := \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteaux differentiable* at x if the limit (2.1) exists for each y. In this case, $f'_+(x,y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. The function f is said to be *Fréchet differentiable* at x if the limit (2.1) is attained uniformly in $\|y\| = 1$ and f is said to be *uniformly Fréhet differentiable* on a subset C of E it the limit (2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E (see [46, Proposition 1]).

Definition 2.1 ([62])**.** Let $B_r := \{x \in E : \|x\| \leq r\}$ for all $r > 0$. A function f on E is said to be:

(1) *strongly coercive* if
$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty.$$
(2) *locally bounded* if $f(B_r)$ is bounded for all $r > 0$.
(3) *uniformly convex* on bounded subsets of E if $\rho_r(t) > 0$ for all r, $t > 0$, where $\rho_r : [0,\infty) \to [0,\infty)$ is defined by
$$\rho_r(t) := \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$
for all $t \geq 0$.

Definition 2.2 ([10])**.** The function f is said to be:

(1) *essentially smooth* if f is both locally bounded and single-valued on its domain.
(2) *essentially strictly convex* if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} f$.
(3) *Legendre* if it is both essentially smooth and essentially strictly convex.

Remark 2.3. Let E be a reflexive Banach space. For a Legendre function f on E, we have the following properties:

(a) f is essentially smooth if and only if f^* is essentially strictly convex (see [10, Theorem 5.4]).
(b) $(\partial f)^{-1} = \partial f^*$ (see [12]).
(c) f is Legendre if and only if f^* is Legendre (see [12, Corollary 5.5]).
(d) If f is Legendre, then ∇f is a bijection and it satisfying
$$\nabla f = (\nabla f^*)^{-1}, \operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f^*) \text{ and } \nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f)$$
(see [12, Theorem 5.10] and [48]).

We recommend reader for various examples of a Legendre function (see [7, 10]). One important and interesting example of a Legendre function is $f = \frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping J_p $(1 < p < \infty)$ which is given by

$$J_p(x) := \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\},\ \forall x \in E.$$

In particular, $J_2 = J$ is called the *normalized duality mapping*. If E is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$, then $\nabla f = I$ is the identity mapping on E.

Definition 2.4 ([14])**.** Let $f : E \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \mathrm{dom}f \times \mathrm{int}(\mathrm{dom}f) \to [0, \infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle$$

is called *Bregman distance* respect to f .

We note that $D_f(x, y) \geq 0$ for all $x, y \in E$. In general, the Bregman distance is like to a metric but does not satisfy the triangle inequality nor symmetry.

Example 2.5. If E is smooth Banach space and $f = \frac{1}{p}\|\cdot\|^p$, where $\frac{1}{p} + \frac{1}{q} = 1$ then, we have

$$\begin{aligned}
D_{\frac{1}{p}\|\cdot\|^p}(x, y) &= \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle x - y, J_p(y)\rangle \\
&= \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle x, J_p(y)\rangle + \langle y, J_p(y)\rangle \\
&= \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle x, J_p(y)\rangle + \|y\|^p \\
&= \frac{1}{p}\|x\|^p + \frac{1}{q}\|y\|^p - \langle x, J_p(y)\rangle \\
&= \phi_p(x, y),\ \forall x, y \in E.
\end{aligned}$$

Such ϕ_p is called the p-Lyapunov function which was studied in [13]. In the particular case $p = 2$, we have $D_{\frac{1}{2}\|\cdot\|^2}(x, y) = \frac{1}{2}\phi(x, y)$, $\forall x, y \in E$ and ϕ is called Lyapunov function studied by Alber [3] and Reich [44]. Also, if E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in E$.

Example 2.6. Let $K := \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ and $f(x) = \sum_{i=1}^n x_i \log x_i$, then $D_f(x, y) = \sum_{i=1}^n (x_i \log \frac{x_i}{y_i} + y_i - x_i)$ for $x, y \in K$. This is called Kullback-Leibler divergence between probability distributions x and y.

Moreover, the Bregman distance has the following important properties [45]:

(1) *two point identity*: for each $x, y \in \mathrm{int}(\mathrm{dom}f)$,

$$D_f(x, y) + D_f(y, x) = \langle \nabla f(x) - \nabla f(y), x - y\rangle.$$

(2) *three point identity*: for each $x \in \mathrm{dom}f$, $y, z \in \mathrm{int}(\mathrm{dom}f)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle.$$

(3) *four point identity*: for each $y, w \in \mathrm{dom}f$, $x, z \in \mathrm{int}(\mathrm{dom}f)$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) - D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w\rangle.$$

We next give the definition of a totally convex function which was introduced in [16].

Definition 2.7. The function $f : E \to (-\infty, \infty]$ is said to be:

(1) *totally convex at a point* $x \in \text{int}(\text{dom}f)$ if its *modulus of total convexity* at x, $v_f(x, \cdot)$ is positive whenever $t > 0$, where $v_f(x, \cdot) : [0, \infty) \to [0, \infty]$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom}f, \|y - x\| = t\}.$$

The function f is called *totally convex* when it is totally convex at every point of $\text{int}(\text{dom}f)$.

(2) *totally convex on bounded sets* if for any nonempty and bounded set X of E, the *modulus of total convexity of* f on E, $v_f(X, t)$ is positive for any $t > 0$, where $v_f : \text{int}(\text{dom}f) \times [0, \infty) \to [0, \infty]$ defined by

$$v_f(X, t) := \inf\{v_f(x, t) : x \in X \cap \text{dom}f\}.$$

It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [17, Theorem 2.10]).

Recall that the *Bregman projection* ([14]) of $x \in \text{int}(\text{dom}f)$ onto the nonempty, closed and convex set $C \subset \text{dom}f$ is the unique point $P_C^f(x) \subset C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

In the case when E is a uniformly convex and uniformly smooth Banach space and $f = \frac{1}{2}\| \cdot \|^2$, then P_C^f coincides with the generalized projection Π_C studied by Alber [2], if E is a Hilbert space, then P_C^f coincides the metric projection P_C.

Lemma 2.8 ([48])**.** *Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then, we have*

(1) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \ \forall y \in C$.

(2) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \ \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a Legendre function. Following [3, 18], we make use of the function $V_f : E \times E^* \to [0, \infty)$ which is given by

$$V_f(x, x^*) := f(x) - \langle x, x^*\rangle + f^*(x^*), \ \forall x \in E, \ x^* \in E^*. \tag{2.2}$$

Clearly, V_f is nonnegative and V_f has the following properties:

$$D_f(x, \nabla f^*(x^*)) = V_f(x, x^*), \ \forall x \in E, \ x^* \in E^* \tag{2.3}$$

and

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \ \forall x \in E, \ x^*, y^* \in E^*. \tag{2.4}$$

In addition, if $f : E \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function, then $f^* : E^* \to (-\infty, \infty]$ is a proper weak*, lower semicontinuous and convex function (see [43]). Hence, V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\Bigg(z, \nabla f^*\bigg(\sum_{i=1}^N t_i \nabla f(x_i)\bigg)\Bigg) \leq \sum_{i=1}^N t_i D_f(z, x_i), \tag{2.5}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N$ with $\sum_{i=1}^N t_i = 1$.

For a set-valued operator $A : E \multimap E^*$, we denote by

$$\mathrm{dom}A := \{x \in E : Ax \neq \emptyset\},$$

the domain of A;

$$\mathrm{ran}A := \bigcup_{x \in \mathrm{dom}A} \{Ax\},$$

range of A; and

$$G(A) := \{(x, x^*) \in E \times E^* : x^* \in Ax\},$$

the graph of A.

Definition 2.9 ([48])**.** An operator A is said to be *monotone* if for each $(x, x^*), (y, y^*) \in G(A)$,

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

We next give some examples of monotone operator. The following classical examples played an important role for the development of monotone operator theory.

Example 2.10 ([4])**.** Let $G \subset \mathbb{R}^n$ be a bounded measurable domain. Define the operator $A : L_p(G) \to L_q(G)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, by the formula

$$Ay(x) := \phi(x, |y(x)|^{p-1})|y(x)|^{p-2}y(x), \ x \in G,$$

where the function $\phi(x, s)$ is measurable as a function on x for every $s \geq 0$ and continuous for almost all $x \in G$ as a function on s, $|\phi(x, s)| \leq M$ for all $s \geq 0$ and for almost all $x \in G$. Note that the operator A really maps $L_p(G) \to L_q(G)$ because of the inequality $|Ay| \leq M|y|^{p-1}$. It can be shown that A is a monotone mapping on $L_p(G)$.

Example 2.11 ([4])**.** This example is a one part from quantum mechanics. Define the operator

$$Au := -a^2\nabla^2 u + (f(x) + b)u(x) + u(x) \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy,$$

where $\nabla^2 = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^3$, a and b are constants, $f(x) = f_0(x) + f_1(x)$, $f_0(x) \in L_\infty(\mathbb{R}^3)$, $f_1(x) \in L_2(\mathbb{R}^3)$. Let $A := L + B$, where the operator L is the linear part of A (it is the Schrödinger operator) and B is defined by the last term. It can be shown that B is a monotone mapping. This implies that $A : L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)$ is also a monotone operator.

Example 2.12 ([60])**.** This example is an importance for the study of optimization theory. Let E be a Banach space with dual space E^* and C be a nonempty, closed and convex subset of E. Let $\rho > 0$ and $G : E^* \times C \to (-\infty, \infty]$ be a functional defined by

$$G(\varphi, x) := \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2 + 2\rho f(x),$$

where $\varphi \in E^*$, $x \in E$ and $f : C \to (-\infty, \infty]$ is proper, lower semicontinuous and convex function. The generalized f-projection operator $\Pi_C^f : E^* \to C$ defined by

$$\Pi_C^f(\varphi) := \{u \in C : G(\varphi, u) = \inf_{y \in C} G(\varphi, y)\}, \ \forall \varphi \in E^*.$$

In particular, if $f(x) = 0$ for all $x \in C$, then the generalized f-projection operator reduces to the generalized projection operator Π_C. It was shown in [60] that Π_C^f is monotone.

Definition 2.13. A monotone operator A is said to be *maximal*, if its graph is not contained in the graph of any other monotone operators on E.

Remark 2.14. It is known that if $f : E \to \mathbb{R}$ is Gâteaux differentiable, strictly convex and cofinite, then A is maximal monotone if and only if $\operatorname{ran}(\nabla f + \lambda A) = E^*$ for $\lambda > 0$ (see [8, Corollary 2.4]) and if A is maximal monotone, then the set $A^{-1}0 = \{x \in E : 0 \in Ax\}$ is closed and convex (see [58, Theorem 6.5.2]).

For a maximal monotone operator $A : E \multimap E^*$. Let $x \in E$ and $\lambda > 0$, we consider the following equation:

$$\nabla f(x) \in \nabla f(x_\lambda) + \lambda A x_\lambda.$$

This equation has a unique solution x_λ and hence we can define the resolvent of A corresponding to f by $Res_{\lambda A}^f(x) = x_\lambda$ for all $x \in E$. In other word, $Res_{\lambda A}^f = (\nabla f + \lambda A)^{-1} \circ \nabla f$. It is known that $Res_{\lambda A}^f$ is single-valued and $F(Res_{\lambda A}^f) = A^{-1}0$, where $F(Res_{\lambda A}^f)$ denotes the set of all fixed points of $Res_{\lambda A}^f$ (see [9]). The *Yosida approximation* $A_\lambda : E \to E^*$, associated with A for $\lambda > 0$ is the mapping defined by

$$A_\lambda(x) = \frac{1}{\lambda}\big(\nabla f(x) - \nabla f\big(Res_{\lambda A}^f(x)\big)\big), \ \forall x \in E.$$

From [47, Proposition 2.7], we know that $\big(Res_{\lambda A}^f(x), A_\lambda(x)\big) \in G(A)$ and $0 \in Ax$ if and only if $0 \in A_\lambda(x)$ for all $x \in E$ and $\lambda > 0$.

Definition 2.15 ([40])**.** A mapping $A : E \to E^*$ is called *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x$ implies that $Ax_n \rightharpoonup^* Ax$.

Definition 2.16 ([48])**.** Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable function. A mapping $A : E \multimap E^*$ satisfying $\operatorname{ran}(\nabla f - \lambda A) \subset \operatorname{ran}(\nabla f)$ is called *Bregman inverse strongly monotone* if $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f) \neq \emptyset$ and for any $x, y \in \operatorname{int}(\operatorname{dom} f)$ and each $u \in Ax, v \in Ay$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geq 0.$$

Remark 2.17. If E is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$, then the Bregman inverse strongly monotone mapping becomes the class of inverse strongly monotone mapping (see [48]) .

Example 2.18 ([15])**.** Let f has a minimizer in $\operatorname{int}(\operatorname{dom} f)$ (that is, the equation $\nabla f(x) = 0$ has a solution) and if $\alpha \in (0, 1)$, then the operator $Ax = \alpha \nabla f(x)$ is Bregman inverse strongly monotone on $\operatorname{dom} A = \operatorname{int}(\operatorname{dom} f)$.

For any operator $A_\lambda^f : E \multimap E^*$, the *anti-resolvent* $A_\lambda^f : E \multimap E$ of A for $\lambda > 0$ is defined by

$$A_\lambda^f := \nabla f^* \circ (\nabla f - \lambda A).$$

Observe that dom$A_\lambda^f \subset$ dom$A \ \cap$ int(dom$f$) and ran$A_\lambda^f \subset$ int(dom$f$). It is known that the operator A is Bregman inverse strongly monotone if and only if A_λ^f is a single-valued mapping (see [15, Lemma 3.5 (c) and (d), p. 2109]). Let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping and $B : E \multimap E^*$ be a maximal monotone operator. Define a mapping $T_\lambda x := Res_{\lambda B}^f \circ A_\lambda^f(x)$ for $x \in E$ and $\lambda > 0$. We see that

$$\begin{aligned}
x = T_\lambda x \quad &\Leftrightarrow \quad x = Res_{\lambda B}^f \circ A_\lambda^f(x)\\
&\Leftrightarrow \quad x = (\nabla f + \lambda B)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda A)x)\\
&\Leftrightarrow \quad x = (\nabla f + \lambda B)^{-1} \circ (\nabla f - \lambda A)x)\\
&\Leftrightarrow \quad \nabla f(x) - \lambda Ax \in \nabla f(x) + \lambda Bx\\
&\Leftrightarrow \quad 0 \in (A + B)x\\
&\Leftrightarrow \quad x \in (A + B)^{-1}0.
\end{aligned}$$

Lemma 2.19 ([35])**.** *Let $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping and $B : E \multimap E^*$ be a maximal monotone operator. Then, we have*

$$D_f(z, Res_{\lambda B}^f \circ A_\lambda^f(x)) + D_f(Res_{\lambda B}^f \circ A_\lambda^f(x), x) \le D_f(z, x)$$
$$\textit{for all } z \in (A + B)^{-1}0,\ x \in E \textit{ and } \lambda > 0.$$

Lemma 2.20 ([47])**.** *Let $f : E \to \mathbb{R}$ be a totally convex function. Suppose that $x \in E$, if $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.21 ([42])**.** *Let E be a Banach space and $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then, $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.22 ([37])**.** *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real number such that*

$$a_{n+1} \le (1 - \delta_n)a_n + b_n + c_n,\ n \ge 1,$$

where $\{\delta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^\infty c_n < \infty$, $\sum_{n=1}^\infty \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \le 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.23 ([38])**.** *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_\ell}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_\ell} < \Gamma_{n_\ell+1}$ for all $\ell \in \mathbb{N}$. Define the sequence $\{\sigma(n)\}_{n\ge n_0}$ of integers as follows:*

$$\sigma(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \ne \emptyset$. Then, the following results hold:

(i) *$\sigma(n_0) \le \sigma(n_0 + 1) \le \cdots$ and $\sigma(n) \to \infty$;*
(ii) *$\Gamma_{\sigma(n)} \le \Gamma_{\sigma(n)+1}$ and $\Gamma_n \le \Gamma_{\sigma(n)+1}$, $\forall n \ge n_0$.*

3. Algorithm and its convergence

In this section, we propose a generalized forward-backward splitting method for solving the Problem (1.8) in a reflexive Banach spaces. In order to establish the convergence result, we suppose the following assumptions are satisfied:

Assumption 1 The Banach space E is reflexive.

Assumption 2 The function $f : E \to \mathbb{R}$ is strongly coercive Legendre which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E.

Assumption 3 The mapping $A_i : E \to E^*$ $(i = 1, 2, \ldots, N)$ is Bregman inverse strongly monotone and weakly sequentially continuous.

Assumption 4 The mapping $B_i : E \multimap E^*$ $(i = 1, 2, \ldots, N)$ is maximal monotone.

Assumption 5 The solution set of (1.8) is nonempty, that is, $\Omega := \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0 \neq \emptyset$.

Remark 3.1. For Assumption 3, if E is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$, then it is not necessary to assume the weak sequentially continuity of A.

We now propose the Algorithm 3.2 for solving the Problem (1.8).

Algorithm 3.2. *Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$\begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(Res^f_{\lambda_{n,1}B_1} \circ A^f_{\lambda_{n,1}}(x_n))), \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(y_{n,1}) + (1-\beta_{n,2})\nabla f(Res^f_{\lambda_{n,2}B_2} \circ A^f_{\lambda_{n,2}}(y_{n,1}))), \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(y_{n,N-1}) + (1-\beta_{n,N})\nabla f(Res^f_{\lambda_{n,N}B_N} \circ A^f_{\lambda_{n,N}}(y_{n,N-1}))), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})), \ \forall n \geq 1, \end{cases}$$

where $y_{n,0} = x_n$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_{n,i}\}_{i=1}^N \subset [0,1)$ and $\{\lambda_{n,i}\}_{i=1}^N \subset (0,\infty)$.

Lemma 3.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Then, $\{x_n\}$ is bounded.*

Proof. Let $w \in \Omega := \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0$. Define a mapping $T^i_{\lambda_n} := Res^f_{\lambda_{n,i}B} \circ A^f_{\lambda_{n,i}}$ for $i = 1, 2, \ldots, N$ and $n \in \mathbb{N}$. From (2.5), we see that

$$\begin{aligned} D_f(w, y_{n,i}) &= D_f(w, \nabla f^*(\beta_{n,i}\nabla f(y_{n,i-1}) + (1-\beta_{n,i})\nabla f(T^i_{\lambda_n} y_{n,i-1})) \\ &\leq \beta_{n,i} D_f(w, y_{n,i-1}) + (1-\beta_{n,i}) D_f(w, T^i_{\lambda_n} y_{n,i-1}) \\ &\leq \beta_{n,i} D_f(w, y_{n,i-1}) + (1-\beta_{n,i}) D_f(w, y_{n,i-1}) \\ &= D_f(w, y_{n,i-1}). \end{aligned} \tag{3.1}$$

It follows that

$$D_f(w, y_{n,N}) \leq D_f(w, y_{n,N-1}) \leq \cdots \leq D_f(w, x_n). \tag{3.2}$$

Then, we have

$$\begin{aligned} D_f(w, x_{n+1}) &= D_f(w, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N}))) \\ &\leq \alpha_n D_f(w, u) + (1-\alpha_n) D_f(w, y_{n,N}) \\ &\leq \alpha_n D_f(w, u) + (1-\alpha_n) D_f(w, x_n) \end{aligned}$$

$$
\begin{aligned}
&\leq \max\{D_f(w,u), D_f(w,x_n)\} \\
&\vdots \\
&\leq \max\{D_f(w,u), D_f(w,x_1)\}. \qquad (3.3)
\end{aligned}
$$

Hence, $\{D_f(w,x_n)\}$ is bounded and $\{x_n\}$ is also bounded by Lemma 2.20. □

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Then, we have*

$$
\begin{aligned}
D_f(z,x_{n+1}) \leq{}& \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z,x_n) \\
&- (1-\alpha_n)\sum_{i=1}^{N}(1-\beta_{n,i}) D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}),
\end{aligned}
$$

where $z = P^f_\Omega(u)$ and $T^i_{\lambda_n} := Res^f_{\lambda_{n,i}B} \circ A^f_{\lambda_{n,i}}$ for $i = 1,2,\dots,N$ and $n \in \mathbb{N}$.

Proof. Let $z = P^f_\Omega(u)$. From Lemma 2.19, we see that

$$
\begin{aligned}
D_f(z,y_{n,i}) &= D_f(z, \nabla f^*(\beta_{n,i}\nabla f(y_{n,i-1}) + (1-\beta_{n,i})\nabla f(T^i_{\lambda_n} y_{n,i-1})) \\
&\leq \beta_{n,i} D_f(z,y_{n,i-1}) + (1-\beta_{n,i}) D_f(z, T^i_{\lambda_n} y_{n,i-1}) \\
&\leq \beta_{n,i} D_f(z,y_{n,i-1}) + (1-\beta_{n,i}) D_f(z, y_{n,i-1}) \\
&\quad -(1-\beta_{n,i}) D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}) \\
&= D_f(z,y_{n,i-1}) - (1-\beta_{n,i}) D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}). \qquad (3.4)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D_f(z,y_{n,N}) &\leq D_f(z,y_{n,N-1}) - (1-\beta_{n,N}) D_f(T^N_{\lambda_n} y_{n,N-1}, y_{n,N-1}) \\
&\vdots \\
&\leq D_f(z,y_{n,1}) - (1-\beta_{n,2}) D_f(T^2_{\lambda_n} y_{n,1}, y_{n,1}) \\
&\quad - \cdots - (1-\beta_{n,N}) D_f(T^N_{\lambda_n} y_{n,N-1}, y_{n,N-1}) \\
&\leq D_f(z,x_n) - (1-\beta_{n,1}) D_f(T^1_{\lambda_n} x_n, x_n) \\
&\quad -(1-\beta_{n,2}) D_f(T^2_{\lambda_n} y_{n,1}, y_{n,1}) \\
&\quad - \cdots - (1-\beta_{n,N}) D_f(T^N_{\lambda_n} y_{n,N-1}, y_{n,N-1}) \\
&= D_f(z,x_n) - \sum_{i=1}^{N}(1-\beta_{n,i}) D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}) \qquad (3.5)
\end{aligned}
$$

for $i = 1,2,\dots,N$. Hence, we have

$$
\begin{aligned}
D_f(z,x_{n+1}) &= D_f(z, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})) \\
&\leq \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z,y_{n,N}) \\
&\leq \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z,x_n) \\
&\quad -(1-\alpha_n)\sum_{i=1}^{N}(1-\beta_{n,i}) D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}). \qquad (3.6)
\end{aligned}
$$

□

Theorem 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $\limsup_{n\to\infty} \beta_{n,i} < 1$ *for* $i = 1, 2, \ldots, N$*;*
(C3) $\liminf_{n\to\infty} \lambda_{n,i} > 0$ *for* $i = 1, 2, \ldots, N$*.*

Then, $\{x_n\}$ *converges strongly to* $z = P^f_\Omega(u)$*.*

Proof. From (3.6), we have

$$(1-\alpha_n)\sum_{i=1}^{N}(1-\beta_{n,i})D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}) \le D_f(z, x_n) - D_f(z, x_{n+1}) + \alpha_n M, \tag{3.7}$$

where $M = \sup_{n\in\mathbb{N}}\{|D_f(z, u) - D_f(z, x_n)|\}$.

The rest of the proof will be divided into two parts:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(z, x_n)\}_{n=n_0}^{\infty}$ is non-increasing. Then, we have

$$D_f(z, x_n) - D_f(z, x_{n+1}) \to 0. \tag{3.8}$$

By our assumptions, we have from (3.7) that

$$D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}) \to 0 \tag{3.9}$$

for $i = 1, 2, \ldots, N$. It follows that

$$\begin{aligned} D_f(y_{n,i}, y_{n,i-1}) &\le \beta_{n,i} D_f(y_{n,i-1}, y_{n,i-1}) + (1-\beta_{n,i}) D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}) \\ &= (1-\beta_{n,i}) D_f(T^i_{\lambda_n} y_{n,i-1}, y_{n,i-1}) \\ &\to 0. \end{aligned} \tag{3.10}$$

By Lemma 2.21, we have

$$\|T^i_{\lambda_n} y_{n,i-1} - y_{n,i-1}\| \to 0 \tag{3.11}$$

and

$$\|y_{n,i} - y_{n,i-1}\| \to 0 \tag{3.12}$$

for $i = 1, 2, \ldots, N$. Since ∇f is uniformly continuous on bounded subset of E, we have

$$\|\nabla f(y_{n,i}) - \nabla f(y_{n,i-1})\| \to 0 \tag{3.13}$$

and

$$\|\nabla f(T^i_{\lambda_n} y_{n,i-1}) - \nabla f(y_{n,i-1})\| \to 0 \tag{3.14}$$

for $i = 1, 2, \ldots, N$. It follows from (3.13) that

$$\begin{aligned} \|\nabla f(y_{n,N}) - \nabla f(x_n)\| &\le \|\nabla f(y_{n,N}) - \nabla f(y_{n,N-1})\| \\ &\quad + \|\nabla f(y_{n,N-1}) - \nabla f(y_{n,N-2})\| \\ &\quad + \cdots + \|\nabla f(y_{n,1}) - \nabla f(x_n)\| \\ &\to 0. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we get

$$\|\nabla f(y_{n,i-1}) - \nabla f(x_n)\| \le \|\nabla f(y_{n,i-1}) - \nabla f(y_{n,i})\|$$

$$\begin{aligned} & +\cdots+\|\nabla f(y_{n,N-1})-\nabla f(y_{n,N})\| \\ & +\|\nabla f(y_{n,N})-\nabla f(x_n)\| \\ \to\ & 0. \end{aligned} \tag{3.16}$$

Hence, we have

$$\begin{aligned} \|\nabla f(T^i_{\lambda_n}y_{n,i-1})-\nabla f(x_n)\| &\leq \|\nabla f(T^i_{\lambda_n}y_{n,i-1})-\nabla f(y_{n,i-1})\| \\ &\quad +\|\nabla f(y_{n,i-1})-\nabla f(x_n)\| \\ &\to 0 \end{aligned} \tag{3.17}$$

for $i=1,2,\dots,N$. From (3.15), we also get

$$\begin{aligned} \|\nabla f(x_{n+1})-\nabla f(x_n)\| &\leq \|\nabla f(x_{n+1})-\nabla f(y_{n,N})\|+\|\nabla f(y_{n,N})-\nabla f(x_n)\| \\ &= \alpha_n\|\nabla f(u)-\nabla f(y_{n,N})\|+\|\nabla f(y_{n,N})-\nabla f(x_n)\| \\ &\to 0. \end{aligned} \tag{3.18}$$

Since f is strongly coercive and uniformly convex on bounded sets, then ∇f^* is uniformly continuous on bounded subset of E^* (see [6, Proposition 3.6.4]), then

$$\|y_{n,i-1}-x_n\|\to 0, \tag{3.19}$$

$$\|T^i_{\lambda_n}y_{n,i-1}-x_n\|\to 0 \tag{3.20}$$

and

$$\|x_{n+1}-x_n\|\to 0 \tag{3.21}$$

for $i=1,2,\dots,N$. By the reflexivity of a Banach space E and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup \hat{x}\in E$ as $k\to\infty$ and

$$\limsup_{n\to\infty}\langle\nabla f(u)-\nabla f(z),x_n-z\rangle=\lim_{k\to\infty}\langle\nabla f(u)-\nabla f(z),x_{n_k}-z\rangle. \tag{3.22}$$

Since $T^i_{\lambda_n}y_{n,i-1}=Res^f_{\lambda_{n,i}B_i}\circ A^f_{\lambda_{n,i}}(y_{n,i-1})$ for $i=1,2,\dots,N$. Then, we have

$$\begin{aligned} & \nabla f(y_{n,i-1})-\lambda_{n,i}A_iy_{n,i-1}\in\nabla f(T^i_{\lambda_n}y_{n,i-1})+\lambda_{n,i}B_iT^i_{\lambda_n}y_{n,i-1} \\ \Leftrightarrow\ & \frac{1}{\lambda_{n,i}}(\nabla f(y_{n,i-1})-\nabla f(T^i_{\lambda_n}y_{n,i-1})-A_iy_{n,i-1}\in B_iT^i_{\lambda_n}y_{n,i-1}. \end{aligned} \tag{3.23}$$

By the monotonicity of B_i for $i=1,2,\dots,N$, we have

$$\left\langle v^*-\frac{1}{\lambda_{n,i}}(\nabla f(y_{n,i-1})-\nabla f(T^i_{\lambda_n}y_{n,i-1}))+A_iy_{n,i-1},v-T^i_{\lambda_n}y_{n,i-1}\right\rangle\geq 0 \tag{3.24}$$

for all $(v,v^*)\in G(B_i)$. It follows that

$$\begin{aligned} \langle v^*+A_i\hat{x},v-\hat{x}\rangle &\geq \frac{1}{\lambda_{n_k,i}}\langle\nabla f(y_{n_k,i-1})-\nabla f(T^i_{\lambda_{n_k}}y_{n_k,i-1}),v-T^i_{\lambda_{n_k}}y_{n_k,i-1}\rangle \\ &\quad -\langle A_iy_{n_k,i-1}-A_i\hat{x},v-\hat{x}\rangle+\langle v^*+A_iy_{n_k,i-1},T^i_{\lambda_{n_k}}y_{n_k,i-1}-\hat{x}\rangle \\ &\geq -\frac{1}{\lambda_{n_k,i}}\|\nabla f(y_{n_k,i-1})-\nabla f(T^i_{\lambda_{n_k}}y_{n_k,i-1})\|\|v-T^i_{\lambda_{n_k}}y_{n_k,i-1}\| \\ &\quad -\langle A_iy_{n_k,i-1}-A_i\hat{x},v-\hat{x}\rangle+\langle v^*+A_iy_{n_k,i-1},T^i_{\lambda_{n_k}}y_{n_k,i-1}-\hat{x}\rangle. \end{aligned} \tag{3.25}$$

Since A_i is weakly sequentially continuous for $i = 1, 2, \dots, N$ and $x_{n_k} \rightharpoonup \hat{x}$, then from (3.14), (3.19) and (3.20), we obtain

$$\langle v^* + A_i\hat{x}, v - \hat{x}\rangle \geq 0 \tag{3.26}$$

for $i = 1, 2, \dots, N$ and $(v, v^*) \in G(B_i)$. Since B_i is maximal monotone, we have $-A_i\hat{x} \in B_i\hat{x}$, that is, $\hat{x} \in (A_i + B_i)^{-1}0$ for $i = 1, 2, \dots, N$ and so $\hat{x} \in \bigcap_{i=1}^N (A_i + B_i)^{-1}0$. Then, we have

$$\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_n - z\rangle = \langle \nabla f(u) - \nabla f(z), \hat{x} - z\rangle \leq 0. \tag{3.27}$$

Furthermore, from (3.21), we also have

$$\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \leq 0. \tag{3.28}$$

Finally, we show the strong convergence of $\{x_n\}$. From (2.4), we have

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f(z, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})) \\ &= V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})) \\ &\leq V_f\big(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N}) - \alpha_n(\nabla f(u) - \nabla f(z))\big) \\ &\quad + \langle \alpha_n(\nabla f(u) - \nabla f(z)), x_{n+1} - z\rangle \\ &= V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(y_{n,N}) \\ &\quad + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &\leq \alpha_n V_f(z, \nabla f(z)) + (1-\alpha_n)V_f(z, \nabla f(y_{n,N})) \\ &\quad + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &= \alpha_n D_f(z, z) + (1-\alpha_n)D_f(z, y_{n,N}) \\ &\quad + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)D_f(z, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle. \end{aligned} \tag{3.29}$$

Thus by Lemma 2.22, we can conclude that $D_f(z, x_n) \to$ as $n \to \infty$. Therefore, $x_n \to z$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_\ell}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_\ell} < \Gamma_{n_\ell + 1}$ for all $\ell \in \mathbb{N}$. Then, by Lemma 2.23, we can define an integer sequence $\{\sigma(n)\}$ for all $n \geq n_0$ by

$$\sigma(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}. \tag{3.30}$$

Moreover, $\{\sigma(n)\}$ is a non-decreasing sequence such that $\sigma(n) \to \infty$ as $n \to \infty$ with $\Gamma_{\sigma(n)} \leq \Gamma_{\sigma(n)+1}$ and $\Gamma_n \leq \Gamma_{\sigma(n)+1}$ for all $n \geq n_0$. Put $\Gamma_n = D_f(z, x_n)$ for all $n \in \mathbb{N}$. Thus by (3.7), we have

$$D_f(T^i_{\lambda_{\sigma(n)}} y_{\sigma(n),i-1}, y_{\sigma(n),i-1}) \to 0 \tag{3.31}$$

for $i = 1, 2, \dots, N$. By the similar argument as in Case 1, we can show that

$$T^i_{\lambda_{\sigma(n)}} y_{\sigma(n),i-1}, y_{\sigma(n),i-1} \to 0, \tag{3.32}$$

$$\|T^i_{\lambda_{\sigma(n)}} y_{\sigma(n),i-1} - x_{\sigma(n)}\| \to 0, \tag{3.33}$$

$$\|y_{\sigma(n),i-1} - x_{\sigma(n)}\| \to 0, \tag{3.34}$$

and

$$\|x_{\sigma(n)+1} - x_{\sigma(n)}\| \to 0 \tag{3.35}$$

for $i = 1, 2, \dots, N$. Furthermore, we can show that

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_{\sigma(n)+1} - z\rangle \leq 0. \tag{3.36}$$

Since $\Gamma_{\sigma(n)} \leq \Gamma_{\sigma(n)+1}$ and $\alpha_{\sigma(n)} > 0$, it follows from (3.29) that

$$\begin{aligned} D_f(z, x_{\sigma(n)}) &\leq \frac{D_f(z, x_{\sigma(n)}) - D_f(z, x_{\sigma(n)+1})}{\alpha_{\sigma(n)}} + \langle \nabla f(u) - \nabla f(z), x_{\sigma(n)+1} - z\rangle \\ &\leq \langle \nabla f(u) - \nabla f(z), x_{\sigma(n)+1} - z\rangle. \qquad (3.37) \end{aligned}$$

Then above inequality gives $\limsup_{n\to\infty} D_f(z, x_{\sigma(n)}) \leq 0$ and so $\lim_{n\to\infty} D_f(z, x_{\sigma(n)}) = 0$. On the other hand, from (3.29), we have

$$D_f(z, x_{\sigma(n)+1}) \leq D_f(z, x_{\sigma(n)}) + \alpha_{\sigma(n)}\langle \nabla f(u) - \nabla f(z), x_{\sigma(n)+1} - z\rangle. \tag{3.38}$$

Hence $\lim_{n\to\infty} D_f(z, x_{\sigma(n)+1}) = 0$. This together with $\Gamma_n \leq \Gamma_{\sigma(n)+1}$, we have

$$D_f(z, x_n) \leq D_f(z, x_{\sigma(n)+1}) \to 0, \tag{3.39}$$

which implies that $D_f(z, x_n) \to 0$ as $n \to \infty$. Therefore, $x_n \to z$. This completes the proof. □

We next apply our main result to the variational inequality problem (VIP). Let $f : E \to (-\infty, \infty]$ be a Legendre and totally convex function. Let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping and let C be a nonempty, closed and convex subset of dom$A$. The *variational inequality problem* (VIP) corresponding to such a mapping A is to find $z \in C$ such that

$$\langle Az, x - z\rangle \geq 0, \ \forall x \in C.$$

The set of solutions of VIP is denoted by $VI(C, A)$. Recall that the *indicator function* of C given by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases}$$

It is known that $\partial i_C(x) = N_C(x)$, where N_C is the normal cone of C (see [1]). Thus, we can define the resolvent associated with ∂i_C for $\lambda > 0$ by $Res^f_{\lambda\partial i_C}(x) = (\nabla f + \lambda\partial i_C)^{-1} \circ \nabla f(x)$, $\forall x \in E$. It is easy to show that for any $x \in E$ and $y \in C$, $z = Res^f_{\lambda\partial i_C}(x) \Leftrightarrow z = P^f_C(x)$, where P^f_C is the Bregman projection from E onto C. We also know that If C is a nonempty, closed and convex subset of dom$A \cap \mathrm{int}(\mathrm{dom} f)$, then $VI(C, A) = F(P^f_C \circ A^f_\lambda)$ [48]).

Put $B_i = \partial i_C$ for all $i = 1, 2, \dots, N$ in Theorem 3.5, we obtain the following result.

Theorem 3.6. *Let C be a nonempty, closed and convex subsets of a reflexive Banach space E and $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of*

E. Let $A_i : C \to E^*$ $(i = 1, 2, \dots, N)$ be a Bregman inverse strongly monotone mapping which is weakly sequentially continuous. Suppose that $\Omega := \bigcap_{i=1}^N VI(C, A_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and

$$(3.40)\quad \begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(P_C^f \circ A_{\lambda_{n,1}}^f(x_n))), \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(y_{n,1}) + (1-\beta_{n,2})\nabla f(P_C^f \circ A_{\lambda_{n,2}}^f(y_{n,1}))), \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(y_{n,N-1}) + (1-\beta_{n,N})\nabla f(P_C^f \circ A_{\lambda_{n,N}}^f(y_{n,N-1}))), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})),\ \forall n \geq 1, \end{cases}$$

where $y_{n,0} = x_n$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_{n,i}\}_{i=1}^N \subset [0,1)$ and $\{\lambda_{n,i}\}_{i=1}^N \subset (0,\infty)$ satisfy Conditions $(C1) - (C3)$ in Theorem 3.5. Then, $\{x_n\}$ converges strongly to $z = P_\Omega^f(u)$.

Remark 3.7. Compare our work with the works in [19, 20, 32, 34] and other related works. We remark the following main points:

(1) Our result extends those works from the problem of finding a zero of the sum of two monotone operators in q-uniformly smooth Banach spaces which is also uniformly convex (or 2-uniformly convex which is also uniformly smooth) to the problem of finding a common zero of the sum of a finite family of Bregman inverse strongly monotone and maximal monotone operators in a reflexive Banach space.

(2) The method of our result is very different from the method of proof of those works in the sense that we use the Bregman distance while the method of proof in those works used the norm distance.

(3) The convergence results of those works are proved provided that the sequences of step size are chosen to satisfy the following inequality:

$$0 < \liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < \Big(\frac{\alpha q}{\kappa_q}\Big)^{\frac{1}{q-1}},$$

where $\alpha > 0$, $1 < q \leq 2$ and κ_q is the q-uniform smoothness coefficient of E while our convergence result is established without the assumption $\limsup_{n\to\infty} \lambda_{n,i} < \big(\frac{\alpha q}{\kappa_q}\big)^{\frac{1}{q-1}}$ for $i = 1, 2, \dots, N$.

Remark 3.8. Compare our work with other works in [41, 56, 59]. The assumption $\liminf_{n\to\infty} \beta_n > 0$ is not necessary for our result.

4. Deduced results

The following results can be obtained from Theorem 3.5.

(1) If $A_i = 0$ for all $i = 1, 2, \dots, N$, then we have the following result for the generalized proximal point algorithm in a reflexive Banach space.

Corollary 4.1. *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. For $i = 1, 2, \dots, N$, let $B_i : E \multimap E^*$ be a maximal monotone mapping. Suppose that $\Omega := \bigcap_{i=1}^N B_i^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a* *sequence generated by* $u, x_1 \in E$ *and*

$$(4.1)\quad \begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(Res^f_{\lambda_{n,1}B_1}x_n)), \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(y_{n,1}) + (1-\beta_{n,2})\nabla f(Res^f_{\lambda_{n,2}B_2}y_{n,1})), \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(y_{n,N-1}) + (1-\beta_{n,N})\nabla f(Res^f_{\lambda_{n,N}B_N}y_{n,N-1})), \\ x_{n+1} = \nabla f^*(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})), \ \forall n \geq 1, \end{cases}$$

where $y_{n,0} = x_n$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_{n,i}\}_{i=1}^N \subset [0,1)$ *and* $\{\lambda_{n,i}\}_{i=1}^N \subset (0,\infty)$ *satisfy Conditions* $(C1) - (C3)$ *in Theorem 3.5. Then,* $\{x_n\}$ *converges strongly to* $z = P^f_\Omega(u)$.

(2) If $\beta_{n,i} = 0$ for all $i = 1, 2, \dots, N$ and $n \in \mathbb{N}$, then we have the following result for the Halpern-type generalized forward-backward splitting method in a reflexive Banach space.

Corollary 4.2. *Let* E *be a reflexive Banach space and* $f : E \to \mathbb{R}$ *be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* E*. For* $i = 1, 2, \dots, N$*, let* $A_i : E \to E^*$ *be a Bregman inverse strongly monotone mapping which is weakly sequentially continuous and* $B_i : E \multimap E^*$ *be maximal monotone mapping. Suppose that* $\Omega := \bigcap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by* $u, x_1 \in E$ *and*

$$(4.2)\quad \begin{cases} y_{n,1} = Res^f_{\lambda_{n,1}B_1} \circ A^f_{\lambda_{n,1}}(x_n)), \\ y_{n,2} = Res^f_{\lambda_{n,2}B_2} \circ A^f_{\lambda_{n,2}}(y_{n,1})), \\ \vdots \\ y_{n,N} = Res^f_{\lambda_{n,N}B_N} \circ A^f_{\lambda_{n,N}}(y_{n,N-1})), \\ x_{n+1} = \nabla f^*(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})), \ \forall n \geq 1, \end{cases}$$

where $y_{n,0} = x_n$, $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_{n,i}\}_{i=1}^N \subset (0,\infty)$ *satisfy Conditions* $(C1)$ *and* $(C3)$ *in Theorem 3.5. Then,* $\{x_n\}$ *converges strongly to* $z = P^f_\Omega(u)$.

(3) If $N = 1$, then we have the following result for the Halpern-type forward-backward splitting method in a reflexive Banach space.

Corollary 4.3. *Let* E *be a reflexive Banach space and* $f : E \to \mathbb{R}$ *be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* E*. Let* $A : E \to E^*$ *be a Bregman inverse strongly monotone mapping which is weakly sequentially continuous and* $B : E \multimap E^*$ *be a maximal monotone mapping. Suppose that* $\Omega := (A+B)^{-1}0 \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by* $u, x_1 \in E$ *and*

$$(4.3)\quad x_{n+1} = \nabla f^*(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(Res^f_{\lambda_n B} \circ A^f_{\lambda_n}(x_n))), \ \forall n \geq 1,$$

where $y_{n,0} = x_n$, $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\liminf_{n\to\infty}\lambda_n > 0$*. Then,* $\{x_n\}$ *converges strongly to* $z = P^f_\Omega(u)$.

(4) If E is a uniformly convex Banach space which is also uniformly smooth and $f = \frac{1}{2}\|\cdot\|^2$, then we have the following result.

Corollary 4.4. *Let E be a uniformly convex and uniformly smooth Banach space. For $i = 1, 2, \ldots, N$, let $A_i : E \to E^*$ be a Bregman inverse strongly monotone mapping respect to the function $f = \frac{1}{2}\|\cdot\|^2$ and $B_i : E \multimap E^*$ be a maximal monotone operator. Let $Q^{B_i}_{\lambda_{n,i}} = (J + \lambda_{n,i}B_i)^{-1}J$ for $i = 1, 2, \ldots, N$. Suppose that $\Omega := \bigcap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$\left\{\begin{array}{l} y_{n,1} = J^{-1}(\beta_{n,1}Jx_n + (1-\beta_{n,1})J(Q^{B_1}_{\lambda_{n,1}}J^{-1}(Jx_n - \lambda_{n,1}A_1x_n))), \\ y_{n,2} = J^{-1}(\beta_{n,2}Jy_{n,1} + (1-\beta_{n,2})J(Q^{B_2}_{\lambda_{n,2}}J^{-1}(Jy_{n,1} - \lambda_{n,2}A_2y_{n,1}))), \\ \vdots \; (4.4) \\ y_{n,N} = J^{-1}(\beta_{n,N}Jy_{n,N-1} + (1-\beta_{n,N})J(Q^{B_N}_{\lambda_{n,N}}J^{-1}(Jy_{n,N-1} - \lambda_{n,N}A_Ny_{n,N-1}))), \\ x_{n+1} = J^{-1}(\alpha_nJu + (1-\alpha_n)Jy_{n,N}), \ \forall n \geq 1, \end{array}\right.$$

where $y_{n,0} = x_n$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_{n,i}\}_{i=1}^N \subset [0,1)$ and $\{\lambda_{n,i}\}_{i=1}^N \subset (0,\infty)$ satisfy Conditions $(C1)-(C3)$ in Theorem 3.5. Then, $\{x_n\}$ converges strongly to $z = \Pi^f_\Omega(u)$.

5. Numerical example

In this section, we provide some numerical experiments to verify the effectiveness and performance of the proposed algorithm.

Example 5.1. Let $E = \ell_2 := \Big\{x = (u_1, u_2, \ldots, u_k, \ldots) : \sum_{k=1}^\infty |u_k|^2 < \infty\Big\}$ with the norm $\|x\|_{\ell_2} = \Big(\sum_{k=1}^\infty |u_k|^2\Big)^{1/2}$ and the inner product $\langle x, y\rangle = \sum_{k=1}^\infty u_kv_k$, where $x = (u_1, u_2, \ldots) \in \ell_2$ and $y = (v_1, v_2, \ldots) \in \ell_2$. Let $f : \ell_2 \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{2}\|x\|^2_{\ell_2}$, then $\nabla f(x) = \nabla f^*(x) = x$. For each $i = 1, 2, \ldots, N$, let $A_i : \ell_2 \to \ell_2$ be a mapping defined by $A_ix = \frac{x}{i}$ for all $x \in \ell_2$. It is easy to show that A_i is a Bregman inverse strongly monotone mapping. Indeed, for each $i = 1, 2, \ldots, N$ and $x, y \in \ell_2$, we have

$$\begin{aligned} &\langle A_ix - A_iy, \nabla f^*(\nabla f(x) - A_ix) - \nabla f^*(\nabla f(y) - A_iy)\rangle \\ &= \Big\langle \frac{x}{i} - \frac{y}{i}, \Big(x - \frac{x}{i}\Big) - \Big(y - \frac{y}{i}\Big)\Big\rangle \\ &= \frac{1}{i}\langle x-y, x-y\rangle - \frac{1}{i}\Big\langle x-y, \frac{x}{i} - \frac{y}{i}\Big\rangle \\ &= \Big(\frac{1}{i} - \frac{1}{i^2}\Big)\|x-y\|^2_{\ell_2} \geq 0. \end{aligned}$$

Hence, A_i is a Bregman inverse strongly monotone mapping. On the other hand, for each $i = 1, 2, \ldots, N$, let $B_i : \ell_2 \to \ell_2$ be a mapping defined by $B_ix = ix$ for all $x \in \ell_2$. It is easy to see that B_i is a monotone mapping for all $i = 1, 2, \ldots, N$ with $\text{ran}(\nabla f + \lambda B_i) = \ell_2$ for all $\lambda > 0$. Hence, B_i is a maximal monotone mapping. The explicit form of the resolvent operator of B_i associated to f can be written as

$$\begin{aligned} Res^f_{\lambda_iB_i} \circ A^f_{\lambda_i}(x) &= (\nabla f + \lambda_iB_i)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda_iA_i)x) \\ &= \frac{i - \lambda_i}{i(1 + i\lambda_i)}x, \end{aligned}$$

where $\lambda_i > 0$ for $i = 1, 2, \dots, N$. It is observe that $\bigcap_{i=1}^N (A_i+B_i)^{-1}0 = \{(0, 0, 0, \dots)\}$ and let $z = (0, 0, 0, \dots)$. The parameters of algorithm are chosen as follows: $\alpha_n = \frac{1}{n+1}$, $\beta_{n,i} = \frac{i}{i+1} - \frac{1}{n+1}$ and $\lambda_{n,i} = \frac{1}{n} + i$ for $i = 1, 2, \dots, N$. It is obvious that all parameters satisfy the Conditions $(C1) - (C3)$. In this experiment, we perform numerical tests for solving the generalized quasi-inclusion problem (1.8) with four different choices of N $(N = 5, 10, 50, 80)$. The solution of the problem are known, so we use $\|x_n - z\|_{\ell_2} < 10^{-4}$ to terminate algorithm and take $u = x_1$. The numerical results of Algorithm 3.2 are presented in Table 1.

Table 1. The numerical results for Algorithm 3.2 in each given N

The starting point x_1	$N = 5$		$N = 10$		$N = 50$		$N = 80$	
	iter.	time	iter.	time	iter.	time	iter.	time
$(1, 0, 0, 0, \dots)$	11998	0.04	10998	0.12	10198	0.29	10123	0.29
$(1, 1, 0, 0, 0, \dots)$	16968	0.05	15554	0.18	14423	0.42	14317	0.41
$(1, 1, 1, 0, \dots)$	20783	0.05	19051	0.20	17665	0.43	17535	0.51
$(1, 1, 1, 1, 0, \dots)$	23998	0.05	21998	0.18	20398	0.40	20248	0.59

Next, we provide numerical results of the proposed algorithm with $f(x) = \frac{1}{2}\|x\|^2$ and also compare them with some existing methods in solving the convex minimization problems.

Example 5.2. Consider the following minimization problem:

$$\min_{x \in \mathbb{R}^3} h(x), \tag{5.1}$$

where $h(x) = \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (4, -2, 1)^T x + 4$ for all $x = (u_1, u_2, u_3) \in \mathbb{R}^3$. Let $F(x) = \frac{1}{2}\|x\|_2^2 + (4, -2, 1)^T x + 4$ and $G(x) = \|x\|_1$. By Fermat's rule, we know that the Problem (5.1) is equivalent to the problem of finding $z \in \mathbb{R}^3$ such that

$$0 \in \nabla F(z) + \partial G(z), \tag{5.2}$$

where ∇F is a gradient of F and ∂G is a subdifferential of G. Then $\nabla F(x) = x + (4, -2, 1)$ and

$$\partial G(x) = \left\{ (\xi_1, \xi_2, \xi_3) : \ \xi_i = \begin{cases} 1 & \text{if } u_i > 0, \\ -1 & \text{if } u_i < 0, \\ [-1, 1] & \text{if } u_i = 0, \end{cases} \quad \text{for all } i = 1, 2, 3 \right\}.$$

It is easy to see that ∇F is a Bregman inverse strongly monotone mapping and ∂G is a maximal monotone operator. Then we can set $A = \nabla F$ and $B = \partial G$. From [26], we have

$$\begin{aligned} Res_\lambda^{\partial G}(x) &= (I + \lambda \partial G)^{-1}(x) \\ &= \big(\max\{|u_1| - \lambda, 0\}\mathrm{sgn}(u_1), \\ &\qquad \max\{|u_2| - \lambda, 0\}\mathrm{sgn}(u_2), \max\{|u_3| - \lambda, 0\}\mathrm{sgn}(u_3) \big) \end{aligned}$$

for $\lambda > 0$. In this experiments, we compare our Algorithm 3.2 with Algorithm (1.3) and Algorithm (1.6). The parameters of algorithms are chosen as follows:

- Our algorithm: $\lambda_n = 3.45$, $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{2}{3} - \frac{1}{n+1}$;
- Algorithm (1.3): $\lambda_n = 0.5$, $\alpha_n = \frac{1}{n+1}$ and $\beta_n = 0.8$;
- Algorithm (1.6): $\lambda_n = 0.5$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{3}{4} - \frac{1}{4n}$ and $\delta_n = \frac{1}{4} - \frac{1}{4n}$.

We perform the numerical experiments with different three cases of starting point. We use $\|x_{n+1} - x_n\| < 10^{-12}$ to terminate all algorithms and take $u = x_1$. The numerical results of all algorithms are presented in Table 2.

TABLE 2. The numerical results for Algorithm 3.2, Algorithm (1.3) and Algorithm (1.6)

The starting point x_1	Algorithm 3.2		Algorithm (1.3)		Algorithm (1.6)	
	iter.	time	iter.	time	iter.	time
$(1, 2, 3)$	801	0.03	2962068	3.56	1233413	1.25
$(-1, 5, 10)$	798	0.02	3662537	4.54	2300706	2.31
$(10, 20, 30)$	796	0.02	7412286	8.40	4382434	4.45

CONCLUSIONS

In this paper, we have proposed a modified forward-backward splitting method with Bregman distance for finding a common zero of a finite family of the sum of maximal monotone and Bregman inverse strongly monotone operators. Strong convergence theorem of the proposed algorithm has been established under suitable conditions on the parameters in reflexive Banach spaces. We also have performed some numerical experiments to show the efficiency of the proposed algorithm.

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REFERENCES

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, New York, 2009.

[2] Y. I. Alber, *Generalized projection operators in Banach spaces: Properties and Applications*, in: Proceedings of the Israel Seminar of Functional Differential Equation, vol. 1, 1993, pp. 1–21.

[3] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, Marcel Dekker, New York, 1996.

[4] Y. Alber and I. Ryazantseva, *Nonlinear ILL-Posed Problems of Monotone Type*, Springer, Dordrecht, 2006.

[5] M. Aliyari, M. Gabeleh and E. Karapınar, *Mann and Ishikawa iterative processes for cyclic relatively nonexpansive mappings in uniformly convex Banach spaces*, J. Nonlinear Convex Anal. **22** (2021), 699–713.

[6] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge University Press, Cambridge, 1993.

[7] H. H. Bauschke and J. M. Borwein, *Legendre functions and the method of random Bregman projections*, J. Convex Anal. **4** (1997), 27–67.

[8] H. H. Bauschke, X. Wang, and L. Yao, *General resolvents for monotone operators: characterization and extension*, Biomedical Mathematics: Promising Directions in Imagine, Therapy Planning and Inverse Problem, Medical Physics Publishing, Wisconsin, 2010.

[9] H. H. Bauschke, J. M. Borwein and P. L. Combettes, *Bregman monotone optimization algorithms*, SIAM J. Control Optim. **42** (2003), 596–636.

[10] H. H. Bauschke, J. M. Borwein and P. L. Combettes, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces*, Commun. Contemp. Math. **3** (2001), 615–647.

[11] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*, Athena Scientific. Belmont, Massachusetts, 1997.

[12] J.F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Verlag, New York, 2000.

[13] T. Bonesky, K. S. Kazimierski, P. Maass, F. Schöpfer F and T. Schuster, *Minimization of Tikhonov functionals in Banach spaces*, Abstr Appl. Anal. **2008** (2008): Art ID 192679.

[14] L.M. Bregman, *The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming*, USSR Comput. Math. Math. Phys. **7** (1967), 200–217.

[15] D. Butnariu and G. Kassay, *A proximal-projection method for finding zeroes of set-valued operators*, SIAM J. Control Optim. **47** (2008), 2096–2136.

[16] D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht, 2000.

[17] D. Butnariu and E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal. **2006** (2006): Art ID 84919, 39 pages.

[18] Y. Censor and A. Lent, *An iterative row-action method for interval convex programming*, J. Optim. Theory Appl. **34** (1981), 321–353.

[19] S. S. Chang, C. F. Wen and J. C. Yao, *Generalized viscosity implicit rules for solving quasi-inclusion problems of accretive operators in Banach spaces*, Optimization **66** (2017), 1105–1117.

[20] P. Cholamjiak, *A generalized forward-backward splitting method for solving quasi inclusion problems in Banach spaces*, Numer. Algorithm **71** (2016), 915–932.

[21] P. Cholamjiak, N. Pholasa, S. Suantai and P. Sunthrayuth, *The generalized viscosity explicit rules for solving variational inclusion problems in Banach spaces*, Optimization **70** (2021), 2607–2633.

[22] W. Cholamjiak, S. A. Khan, H. A. Hammad and H. Dutta, *Weak and strong convergence results for the modified Noor iteration of three quasi-nonexpansive multivalued mappings in Hilbert spaces*, Filomat **34** (2020), 2495–2510.

[23] P. L. Combettes and V. R. Wajs, *Signal recovery by proximal forward-backward splitting*, Multiscale Model. Simul. **4** (2005), 1168–1200.

[24] J. C. Dunn, *Convexity, monotonicity, and gradient processes in Hilbert space*, J. Math. Anal. Appl. **53** (1976), 145–158.

[25] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29** (1991), 403–419.

[26] E. T. Hale, W. Yin and Y. Zhang, *A fixed-point continuation method for ℓ_1-regularized minimization with applications to compressed sensing*, Tech. rep. CAAM TR07–07, 2007.

[27] S. He and C. Yang, *Solving the variational inequality problem defined on intersection of finite level sets*, Abstr. Appl. Anal. **2013** (2013): Art. ID 942315.

[28] A. Hussain, D. Ali and E. Karapınar, *Stability data dependency and errors estimation for a general iteration method*, Alex. Eng. J. **60** (2021), 703–710.

[29] N. Jun-on, R. Suparatulatorn, M. Gamal and W. Cholamjiak, *An inertial parallel algorithm for a finite family of G-nonexpansive mappings applied to signal recovery*, AIMS Math. **7** (2022), 1775–1790.

[30] S. A. Khan, W. Cholamjiak and K. R. Kazmi, *An inertial forward-backward splitting method for solving combination of equilibrium problems and inclusion problems*, Comput. Appl. Math. **37** (2018), 6283–6307.

[31] E. Karapınar and K. Taş, *Generalized (C)-conditions and related fixed point theorems*, Comput. Math. Appl., **61** (2011), 3370–3380.

[32] Y. Kimura and K. Nakajo, *Strong convergence for a modified forward-backward splitting method in Banach spaces*, J. Nonlinear Var. Anal. **3** (2019), 5–18.

[33] P. L. Lions and B. Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM J. Numer. Anal. **16** (1979), 964–979.

[34] G. López, V. Martín-Márquez, F. Wang and H.–K. Xu, *Forward-Backward splitting methods for accretive operators in Banach spaces*, Abstr. Appl. Anal. **2012** (2012): Art ID 109236.

[35] F. U. Ogbuisi and C. Izuchukwu, *Approximating a Zero of Sum of Two Monotone Operators Which Solves a Fixed Point Problem in Reflexive Banach Spaces*, Numer. Funct. Anal. Optim. **41** (2020), 322–343.

[36] F. U. Ogbuisi, L. O. Jolaoso and Y. Shehu, *Weak and strong convergence results for sum of two monotone operators in reflexive Banach spaces*, Fixed Point Theory **21** (2020), 281–30.

[37] P. E. Maingé, *Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **325** (2007), 469–479.

[38] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.

[39] G. Marino, B. Scardamaglia, E. Karapinar *Strong convergence theorem for strict pseudo-contractions in Hilbert spaces*, J. Inequalities Appl. **2016** (2016): 134.

[40] V. Martín-Márqueza, S. Reich and S. Sabach, *Bregman strongly nonexpansive operators in reflexive Banach spaces*, J. Math. Anal. Appl. **400** (2013), 597–614.

[41] E. Naraghirad, *Halpern's Iteration for Bregman Relatively Nonexpansive Mappings in Banach Spaces*, Numer. Funct. Anal. Optim. **34** (2013), 1129–1155.

[42] E. Naraghirad and J. C. Yao, *Bregman weak relatively non expansive mappings in Banach space*, Fixed Point Theory Appl. **2013** (2013): 141.

[43] R. P. Phelps, *Convex functions, monotone operators, and differentiability*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1993.

[44] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996.

[45] S. Reich and S. Sabach, *A projection method for solving nonlinear problems in reflexive Banach spaces*, J. Fixed Point Theory Appl. **9** (2011), 101–116.

[46] S. Reich and S. Sabach, *A strong convergence Theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 471–485.

[47] S. Reich and S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **31** (2010), 22–44.

[48] S. Reich and S. Sabach, *Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces*, Nonlinear Anal. **73** (2010), 122–135.

[49] S. M. Robinson, *Generalized Equations*, Mathematical Programming The State of the Art, Springer, Berlin, 1983.

[50] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, New York, 1970.

[51] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.

[52] S. Suantai, P. Cholamjiak and P. Sunthrayuth, *Iterative methods with perturbations for the sum of two accretive operators in q-uniformly smooth Banach spaces*, RACSAM. **113** (2019), 203–223.

[53] P. Sunthrayuth and P. Kumam, *A New Composite General Iterative Scheme for Nonexpansive Semigroups in Banach Spaces*, Int. J. Math. Math. Sci. **2011** (2011): Art ID 560671, 18 pages.

[54] P. Sunthrayuth and P. Kumam, *Fixed point solutions of variational inequalities for a semigroup of asymptotically nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2012** (2012): Art ID 177.

[55] S. Suantai, K. Kankam, P. Cholamjiak and W. Cholamjiak, *A parallel monotone hybrid algorithm for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with a graph applicable in signal recovery*, Comput. Appl. Math. **40** (2021), 1–17.

[56] S. Takahashi, W. Takahashi and M. Toyoda, *Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl. **147** (2010), 27–41.

[57] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, 2000.

[58] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[59] W. Takahashi, N. C. Wong and J. C. Yao, *Two generalized strong convergence theorems of Halpern's type in Hilbert spaces and applications*, Taiwanese J. Math. **16** (2012), 1151–1172.

[60] K. G. Wu and N. J. Huang, *Properties of the generalized f-projection operator and its applications in Banach spaces*, Comput. Math. Appl. **54** (2007), 399–406.

[61] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.

[62] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, New Jersey, 2002.

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ORIGINAL PAPER

Self-adaptive algorithms for solving split feasibility problem with multiple output sets

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Abstract

In this paper, we study the split feasibility problem with multiple output sets in Hilbert spaces. For solving the aforementioned problem, we propose two new self-adaptive relaxed CQ algorithms which involve computing of projections onto half-spaces instead of computing onto the closed convex sets, and it does not require calculating the operator norm. We establish a weak and a strong convergence theorems for the proposed algorithms. We apply the new results to solve some other problems. Finally, we present some numerical examples to show the efficiency and accuracy of our algorithm compared to some existing results. Our results extend and improve some existing methods in the literature.

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1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a nonzero bounded linear operator and let $A^* : H_2 \to H_1$ be its adjoint. The `split feasibility problem (SFP)` is formulated to find a point $x^* \in H_1$ satisfying

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{1}$$

The SFP was first introduced in 1994 by Censor and Elfving [1] in finite-dimensional Hilbert spaces for modeling certain inverse problems and has received a great attention since then. This is because the SFP can be used to model several inverse problems arising from, for example, phase retrievals and in medical image reconstruction [1, 2], intensity-modulated radiation therapy (IMRT) [3–5], gene regulatory network inference [6], just to mention but few, for more details one can, see, e.g., [7–14] and the references therein. In the span of the last twenty five years, focusing on real world applications, several iterative methods for solving the SFP (1) have been introduced and analyzed. Among them, Byrne [2, 9] introduced the first applicable and most celebrated method called the well-known `CQ-algorithm` as follows: for $x_0 \in H_1$;

$$x_{n+1} := P_C(x_n - \tau_n A^*(I - P_Q)Ax_n)), \tag{2}$$

where P_C and P_Q are the metric projections onto C and Q, respectively, and the stepsize $\tau_n \in \left(0, \frac{2}{\|A\|^2}\right)$ where $\|A\|^2$ is the spectral radius of the matrix A^*A.

The CQ algorithm proposed by Byrne [2, 9], requires the computation of metric projection onto the sets C and Q (in some cases, it is impossible or is too expensive to exactly compute the metric projection). In addition, the determination of the stepsize depends on the operator norm which computation (or at least estimate) is not easy task. In practical applications, the sets C and Q are usually the level sets of convex functions which are given by

$$C := \{x \in H_1 : c(x) \leq 0\} \text{ and } Q = \{y \in H_2 : q(y) \leq 0\}, \tag{3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex and subdifferentiable functions on H_1 and H_2, respectively, and that subdifferentials $\partial c(x)$ and $\partial q(y)$ of c and q, respectively, are bounded operators (i.e., bounded on bounded sets).

Later, in 2004, Yang [12] generalized the CQ method to the so-called `relaxed CQ algorithm`, needing computation of the metric projection onto (relaxed sets) `half-spaces` C_n and Q_n, where

$$C_n := \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{4}$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n := \{y \in H_2 : q(Ax_n) \leq \langle \eta_n, Ax_n - y \rangle\}, \tag{5}$$

where $\eta_n \in \partial q(Ax_n)$. It is easy to see that $C \subseteq C_n$ and $Q \subseteq Q_n$ for all $n \geq 1$. Moreover, it is known that projections onto half-spaces C_n and Q_n have closed forms. In what follows, define

$$f_n(x_n) := \frac{1}{2}\|(I - P_{Q_n})Ax_n\|^2, \tag{6}$$

where Q_n is given as in (5). f_n is a convex and differentiable function with its gradient ∇f_n defined by

$$\nabla f_n(x_n) := A^*(I - P_{Q_n})Ax_n. \tag{7}$$

More precisely, Yang [12] introduced the following relaxed CQ algorithm for solving the SFP (1) in a finite-dimensional Hilbert space: for $x_0 \in H_1$;

$$x_{n+1} := P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{8}$$

where $\tau_n \in \left(0, \frac{2}{\|A\|^2}\right)$. Since P_{C_n} and P_{Q_n} are easily calculated, this method appears to be very practical. However, to compute the norm of A turns out to be complicated and costly. To overcome this difficulty, in 2012, López et al. [15] introduced a relaxed CQ algorithm for solving the SFP (1) with a new adaptive way of determining the stepsize sequence τ_n defined as follows:

$$\tau_n := \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \tag{9}$$

where $\rho_n \in (0, 4), \forall n \geq 1$ such that $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$. It was proved that the sequence $\{x_n\}$ generated by (8) with τ_n defined by (9) converges weakly to a solution of the SFP (1). That is, their algorithm has only weak convergence in the framework of infinite-dimensional Hilbert spaces. But, in the infinite-dimensional spaces norm (strong) convergence is more desirable than the weak convergence for solving our problems. In this regard, many authors proposed algorithms that generate a sequence $\{x_n\}$, converges strongly to a point in the solution set of the SFP (1), see, e.g., [15–19]. In particular, López et al. [15] proposed a `Halpern's iterative` scheme for solving the SFP (1) in the setting of infinite-dimensional Hilbert spaces as follows: for $u, x_0 \in H_1$;

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) P_{C_n}\left(x_n - \tau_n \nabla f_n(x_n)\right), \forall n \geq 1, \tag{10}$$

where $\{\alpha_n\} \subset (0, 1)$, and $\nabla f_n(x_n)$ and τ_n are given by (7) and (9), respectively. In 2013, He et al. [16] also introduced a new relaxed CQ algorithm for solving the SFP (1) such that strong convergence is guaranteed in infinite-dimensional Hilbert space. Their algorithm generates a sequence $\{x_n\}$ by the following manner: for $u, x_0 \in H_1$;

$$x_{n+1} := P_{C_n}\left(\alpha_n u + (1 - \alpha_n)\big(x_n - \tau_n \nabla g_n(x_n)\big)\right), \tag{11}$$

where C_n and τ_n are given as in (4) and (9), respectively, and $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Under some standard conditions, it was shown that the sequence $\{x_n\}$ generated by (10) and (11) converges strongly to $p^* = P_\Omega(u) \in \Omega = \{p \in H_1 : p \in C \text{ such that } Ap \in Q\}$ of the SFP (1). Both schemes (10) and (11) do not require any prior knowledge of the operator norm and compute the projections onto the half-spaces C_n and Q_n (which have closed-form), and thus both are easily implementable.

Some generalizations of the SFP have also been studied by many authors. We mention, for instance, the multiple-sets SFP (MSSFP) [3, 20–34], the split common fixed point problem (SFPP) [35, 36], the split variational inequality problem (SVIP) [37], and the split common null point problem (SCNPP) [38–42].

Very recently, Reich et al. [43] considered and studied the following split feasibility problem with multiple output sets in real Hilbert spaces.

Let H, H_i, $i = 1, 2, \ldots, N$, be real Hilbert spaces and let $A_i : H \to H_i$, $i = 1, 2, \ldots, N$, be bounded linear operators. Let C and Q_i, $i = 1, 2, \ldots, N$, be nonempty, closed, and convex subsets of H and H_i, $i = 1, 2, \ldots, N$, respectively. Given H, H_i, and A_i as above, the `split feasibility problem with multiple output sets` (`SFPMOS`, for short) is to find an element p^* such that

$$p^* \in \Omega := C \cap \left(\cap_{i=1}^N A_i^{-1}(Q_i)\right) \neq \emptyset. \tag{12}$$

That is $p^* \in C$ and $A_i p^* \in Q_i$ for each $i = 1, 2, \ldots, N$.

In 2020, Reich et al. [43] introduced the following two methods for solving the SFPMOS (12).

For any given points, $x_0, y_0 \in H$, $\{x_n\}$, and $\{y_n\}$ are sequences generated by

$$x_{n+1} := P_C\left(x_n - \lambda_n \sum_{i=1}^N A_i^*(I - P_{Q_i})A_i x_n\right), \tag{13}$$

$$y_{n+1} := \alpha_n f(y_n) + (1 - \alpha_n)P_C\left(y_n - \lambda_n \sum_{i=1}^N A_i^*(I - P_{Q_i})A_i y_n\right), \tag{14}$$

where $f : C \to C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0, 1)$, $\lambda_n \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$. It was proved that if the sequence $\{\lambda_n\}$ satisfies the condition:

$$0 < a \leq \lambda_n \leq b < \frac{2}{N \max_{i=1,2,\ldots,N}\{\|A_i\|^2\}}$$

for all $n \geq 1$, then the sequence $\{x_n\}$ generated by (13) converges weakly to a solution point $p^* \in \Omega$ of the SFPMOS (12). Furthermore, if the sequence $\{\alpha_n\}$ satisfies the conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^\infty \alpha_n = \infty,$$

then the sequence $\{y_n\}$ generated by (14) converges strongly to a solution point $p^* \in \Omega$ of the SFPMOS (12), which is a unique solution of the variational inequality

$$\langle (I - f)p^*, x - p^* \rangle \geq 0 \ \forall x \in \Omega.$$

An important observation here is that the iterative methods (Scheme (13) and Scheme (14)) introduced by Reich et al. [43] requires to compute the metric projections on to the sets C and Q_i. Moreover, it needs to compute the operator norm. Due to this reason, the following question naturally arises. *Question:* Can we design two new iterative algorithms (a weakly convergent and strongly convergent methods, different from Scheme (13) and Scheme (14)) for solving the SFPMOS (12) which mainly involves a self-adaptive step-size and requires to compute the projections onto half-spaces so that the algorithm is easily implementable?.

We have a positive answer for the above question which is motivated by the iterative schemes (13) and (14) proposed by Reich et al. [43] for solving the SFPMOS (12), the Halpern's-type iterative schemes (10) and (11) proposed by López et al. [15] and He et al. [16], respectively, to solve the SFP (1). In this paper, we propose two new self-adaptive relaxed CQ algorithms for solving the SFPMOS (12) in infinite-dimensional Hilbert spaces.

In the next section, we recall some necessary tools which are used in establishing our main results. In Section 3, we propose self-adaptive relaxed CQ algorithms for solving the SFPMOS (12), and we establish and analyze weak and strong convergence theorems for the proposed algorithms. In the same section, we also present some newly derived results for solving the SFP (1). In Section 4, we present the application of our methods to solve the generalized split feasibility problem (another generalization of the SFP). Finally, in the last section, we provide several numerical examples to illustrate the implementation of our algorithms compared to some existing results.

2 Preliminaries

In this section, we recall some definitions and basic results which are needed in the sequel. Let H be a real Hilbert space with the inner product $\langle ., . \rangle$, and induced norm $\|.\|$. Let I stands for the identity operator on H. Let the symbols "$\rightharpoonup$" and "$\rightarrow$", denote the weak and strong convergence, respectively. For any sequence $\{x_n\} \subset H$, $\omega_w(x_n) = \{x \in H : \exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$.

Definition 1 ([44]) Let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a given operator. Then, T is called

(1) `Lipschitz continuous` with constant $\lambda > 0$ on C if

$$\|Tx - Ty\| \leq \lambda \|x - y\|, \forall x, y \in C; \tag{15}$$

(2) `nonexpansive` on C if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C; \tag{16}$$

(3) `firmly nonexpansive` on C if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - ||(I - T)x - (I - T)y\|^2, \forall x, y \in C, \tag{17}$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \forall x, y \in C; \tag{18}$$

(4) `averaged` if there exists a number $\lambda \in (0, 1)$ and a nonexpansive operator $F : C \to H$ such that

$$T = \lambda F + (1 - \lambda)I, \text{ where } I \text{ is the identity operator.} \tag{19}$$

In this case, we say that T is λ-*averaged*.

Definition 2 ([44]) Let $C \subset H$ be a nonempty, closed and convex set. For every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}. \tag{20}$$

The operator P_C (mapping $P_C : H \to C$) is called a `metric projection` of H onto C and it has the following well-known properties.

Lemma 1 *([44, 45]) Let $C \subset H$ be a nonempty, closed and convex set. Then, the following assertions hold for any $x, y \in H$ and $z \in C$:*

(1) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0$;
(2) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$;
(3) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle$;
(4) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|x - P_C(x)\|^2$.

We see from Lemma 1 that the metric projection mapping is firmly nonexpansive and nonexpansive. Moreover, it is not hard to show that $I - P_C$ is also firmly nonexpansive and nonexpansive.

Lemma 2 *For all $x, y \in H$ and for all $\alpha \in \mathbb{R}$, we have*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y,\ x + y\rangle$;
(2) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,\ y\rangle$;
(3) $\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2$;
(4) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Fejér-monotone sequences are very useful in the analysis of optimization iterative algorithms.

Definition 3 ([44]) Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H. Then, $\{x_n\}$ is Fejér monotone with respect to C if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \ \forall z \in C.$$

It is easy to see that a Fejér monotone sequence $\{x_n\}$ is bounded and the limit $\lim_{n\to\infty} \|x_n - z\|$ exists.

Lemma 3 *(Demiclosedness principle of nonexpansive mappings [44]) Let C be a closed convex subset of H, $T : C \to C$ be a nonexpansive mapping with nonempty fixed point sets. If $\{x_n\}$ is a sequence in C converging weakly to x and $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x = Tx$.*

Lemma 4 *([44, 46, 47]) Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H satisfying the properties:*

(1) $\lim_{n\to\infty} \|x_n - x^*\|$ exists for every $x^* \in C$;
(2) $\omega_w(x_n) \subset C$.

Then, there exists a point $\hat{x} \in C$ such that $\{x_n\}$ converges weakly to $\hat{x}$.

Definition 4 Let $f : H \to \mathbb{R}$ be a function and $\lambda \in [0, 1]$. Then,

(1) f is `convex` if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y), \forall x, y \in H.$$

(2) A vector $\xi \in H$ is a `subgradient` of f at a point x if

$$f(y) \geq f(x) + \langle \xi, y - x\rangle, \ \forall y \in H.$$

(3) The set of all subgradients of a convex function $f : H \to \mathbb{R}$ at $x \in H$, denoted by $\partial f(x)$, is called the `subdifferential` of f, and is defined by

$$\partial f(x) = \{\xi \in H : f(y) \geq f(x) + \langle \xi, y - x\rangle, \ \text{for each } y \in H\}.$$

(4) If $\partial f(x) \neq \emptyset$, f is said to be `subdifferentiable` at x. If the function f is continuously differentiable then $\partial f(x) = \{\nabla f(x)\}$. The convex function is subdifferentiable everywhere [44].
(5) f is called `weakly lower semicontinuous` at x_0 if for a sequence $\{x_n\}$ weakly converging to x_0 one has

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

A function which is weakly lower semicontinuous at each point of H is called `weakly lower semicontinuous` on H.

Lemma 5 *([48]) Let H_1 and H_2 be real Hilbert spaces and $f : H_1 \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ where Q is a nonempty, closed convex subset of H_2 and $A : H_1 \to H_2$ be a bounded linear operator. Then, the following assertions hold:*

(1) f is convex and differentiable;
(2) f is weakly lower semicontinuous on H_1;
(3) $\nabla f(x) = A^*(I - P_Q)Ax$, for $x \in H_1$;
(4) ∇f is $\|A\|^2$-Lipschitz, i.e., $\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2\|x - y\|, \forall x, y \in H_1$.

Lemma 6 *([49]) Let $\{\Lambda_n\}$ be a sequence of real numbers that does not decrease at infinity. Also consider the sequence of integers $\{\varphi(n)\}_{n\geq n_0}$ defined by*

$$\varphi(n) = \max\{m \in \mathbb{N} : m \leq n, \Lambda_m \leq \Lambda_{m+1}\}.$$

Then, $\{\varphi(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\varphi(n)=\infty$, and for all $n\geq n_0$, the following two estimates hold:

$$\Lambda_{\varphi(n)} \leq \Lambda_{\varphi(n)+1} \text{ and } \Lambda_n \leq \Lambda_{\varphi(n)+1}.$$

Lemma 7 *([50]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$s_{n+1} \leq (1-\varrho_n)s_n + \varrho_n\mu_n + \theta_n, n \geq 1,$$

where $\{\varrho_n\}$, $\{\mu_n\}$ and $\{\theta_n\}$ satisfying the conditions:

(1) $\{\varrho_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\varrho_n = \infty$;
(2) $\limsup_{n\to\infty}\mu_n \leq 0$;
(3) $\theta_n \geq 0$, $\sum_{n=1}^{\infty}\theta_n < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

3 The iterative algorithms for solving SFPMOS

In this section, we propose new self-adaptive relaxed iterative methods for solving the SFPMOS (12) in the infinite-dimensional Hilbert spaces, and we prove a weak and strong convergence theorems of the proposed methods.

The relaxed projection methods use metric projections onto half-spaces instead of projections onto the original closed convex sets. In what follows, we consider a general case of the SFPMOS (12), where the nonempty, closed and convex sets C and $Q_i (i = 1, 2, \dots, N)$ are given by level sets of convex functions defined as follows:

$$C := \{x \in H : c(x) \leq 0\} \text{ and } Q_i := \{y \in H_i : q_i(y) \leq 0\} \tag{21}$$

where, $c : H \to \mathbb{R}$ and $q_i : H_i \to \mathbb{R}$, $i = 1, 2, \dots, N$ are lower semicontinuous convex functions. We assume that both c and each q_i are subdifferentiable on H and H_i, respectively, with subdifferential ∂c and ∂q_i, respectively. Moreover, assume that for any $x \in H$ a subgradient $\xi \in \partial c(x)$ can be calculated, and for any $y \in H_i$ and for each $i \in \{1, 2, \dots, N\}$, a subgradient $\eta_i \in \partial q_i(y)$ can be calculated. Again, assume that both ∂c and $\partial q_i (i = 1, 2, \dots, N)$ are bounded operators (i.e., bounded on bounded sets). The subdifferentials ∂c and ∂q_i are defined by

$$\partial c(x) := \{\xi \in H : c(z) \geq c(x) + \langle \xi, z - x\rangle,\ \forall z \in H\}$$

for all $x \in C$ and

$$\partial q_i(y) := \{\eta_i \in H_i : q_i(u) \geq q_i(y) + \langle \eta_i, u - y\rangle,\ \forall u \in H_i\}$$

for all $y \in Q_i$, $i = 1, 2, \dots, N$.

In this situation, the projections onto C and Q_i are not easily implemented in general. To avoid this difficulty, we introduce a relaxed projection gradient methods, in which the projections onto the half-spaces are adopted in stead of the projections onto C and Q_i. In particular for $n \in \mathbb{N}$, we define the relaxed sets (half-spaces) C_n

and $Q_i^n (i = 1, 2, \ldots, N)$ of C and Q_i, respectively, at x_n as follows:

$$C_n := \{x \in H : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{22}$$

where $\xi_n \in \partial c(x_n)$ is subgradient of c at x_n and

$$Q_i^n := \{y \in H_i : q_i(A_i x_n) \leq \langle \eta_i^n, A_i x_n - y\rangle\}, \tag{23}$$

where $\eta_i^n \in \partial q_i(A_i x_n)$. By the definition of the subgradient, it is easy to see that $C \subseteq C_n$ and $Q_i \subseteq Q_i^n$ (see [51]), and the metric projections onto C_n and Q_i^n can be directly calculated (since the projections onto C_n and Q_i^n have closed-form expressions), for example, for $\xi_n \in \partial c(x_n)$

$$P_{C_n}(x_n) = \begin{cases} x_n - \frac{c(x_n)}{\|\xi_n\|^2}\xi_n, & \text{if} \quad \xi_n \neq 0, \\ x_n, & \text{otherwise .} \end{cases}$$

Now, we present the following easily implementable algorithms.

3.1 Weak convergence theorems

In this subsection, we propose a new self-adaptive relaxed iterative method for solving the SFPMOS (12) in the infinite-dimensional Hilbert spaces, and we prove a weak convergence theorem of the proposed method.

Algorithm 1 Weakly convergent self-adaptive CQ algorithm for solving SFPMOPS.

Choose a constant $\beta > 0$ and three sequences $\{\rho_1^n\}, \{\rho_2^n\} \subset (0, 1)$ and $\{\vartheta_i\}_{i=1}^N > 0$. Select an arbitrary starting point $x_0 \in H$, and set $n = 0$. Given the current iterate $x_n \in H$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = x_n - \rho_1^n \left(I - P_{C_n}\right) x_n - \tau_n \sum_{i=1}^N \vartheta_i A_i^* \left(I - P_{Q_i^n}\right) A_i x_n$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\| \left(I - P_{Q_i^n}\right) A_i x_n \right\|^2}{\bar{\tau}_n^2} \tag{24}$$

where

$$\bar{\tau}_n := \max \left\{ \left\| \sum_{i=1}^N \vartheta_i A_i^* \left(I - P_{Q_i^n}\right) A_i x_n \right\|, \beta \right\},$$

and C_n and Q_i^n are the half-spaces given as in (22) and (23), respectively.

Theorem 1 *Assume that the SFPMOS (12) is consistent (i.e., $\Omega \neq \emptyset$). Suppose the sequences $\{\rho_1^n\}$ and $\{\rho_2^n\}$ in Algorithm 1 are in $(0, 1)$ such that $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, respectively. Then, the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a solution $p^* \in \Omega$ of the SFPMOS (12).*

Proof For convenience, we set the following notations first (for $i = 1, 2, \ldots, N$)

$$f_{C_n}^n := \left(I - P_{C_n}\right) x_n, \quad f_{Q_i^n}^n := \left(I - P_{Q_i^n}\right) A_i x_n. \tag{25}$$

Consequently, the step-size τ_n given by (24) can be written as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \| f_{Q_i^n}^n \|^2}{\bar{\tau}_n^2} \tag{26}$$

where

$$\bar{\tau}_n := \max \left\{ \| \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|, \ \beta \right\}.$$

Then, the iterative sequence $\{x_n\}$ in Algorithm 1 can be rewritten as follows:

$$x_{n+1} = x_n - \rho_1^n f_{C_n}^n - \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n. \tag{27}$$

Let $p^* \in \Omega$ (Ω is the solution set of the SFPMOS (12)). By (27), we have

$$\begin{aligned}
\|x_{n+1} - p^*\|^2 &= \|x_n - \rho_1^n f_{C_n}^n - \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n - p^*\|^2 \\
&= \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho_1^n f_{C_n}^n + \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \right\rangle \\
&+ \ \|\rho_1^n f_{C_n}^n + \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|^2 \\
&= \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho_1^n f_{C_n}^n \right\rangle - 2\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \right\rangle \\
&+ \ \|\rho_1^n f_{C_n}^n\|^2 + \|\tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|^2 + 2\left\langle \rho_1^n f_{C_n}^n, \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \right\rangle \\
&\leq \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho_1^n f_{C_n}^n \right\rangle - 2\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \right\rangle \\
&+ \ \|\rho_1^n f_{C_n}^n\|^2 + \|\tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|^2 + 2\|\rho_1^n f_{C_n}^n\| \|\tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \| \\
&\leq \|x_n - p^*\|^2 - 2\left\langle x_n - p^*, \rho_1^n f_{C_n}^n \right\rangle - 2\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \right\rangle \\
&+ \ 2\|\rho_1^n f_{C_n}^n\|^2 + 2\tau_n^2 \| \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|^2.
\end{aligned} \tag{28}$$

Using Lemma 1 (1), we obtain the following two estimations.

$$\begin{aligned}\left\langle x_n - p^*, \rho_1^n f_{C_n}^n \right\rangle &= \rho_1^n \left\langle x_n - p^*, f_{C_n}^n \right\rangle \\ &= \rho_1^n (\langle x_n - P_{C_n}(x_n), f_{C_n}^n \rangle + \langle P_{C_n}(x_n) - p^*, f_{C_n}^n \rangle) \\ &= \rho_1^n (\langle f_{C_n}^n, f_{C_n}^n \rangle + \langle P_{C_n}(x_n) - p^*, f_{C_n}^n \rangle) \\ &\geq \rho_1^n \| f_{C_n}^n \|^2. \end{aligned} \tag{29}$$

$$\begin{aligned}\left\langle x_n - p^*, \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \right\rangle &= \tau_n \sum_{i=1}^N \vartheta_i \left\langle x_n - p^*, A_i^* f_{Q_i^n}^n \right\rangle \\ &= \tau_n \sum_{i=1}^N \vartheta_i \left\langle A_i x_n - A_i p^*, f_{Q_i^n}^n \right\rangle \\ &= \tau_n \sum_{i=1}^N \vartheta_i \left(\left\langle f_{Q_i^n}^n, f_{Q_i^n}^n \right\rangle + \left\langle P_{Q_i^n}(A_i x_n) - A_i p^*, f_{Q_i^n}^n \right\rangle \right) \\ &\geq \tau_n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2. \end{aligned} \tag{30}$$

Substituting (29) and (30) into (28) and since $\| \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \| \leq \bar{\tau}_n$, we obtain that

$$\begin{aligned}\|x_{n+1} - p^*\|^2 &\leq \|x_n - p^*\|^2 - 2\rho_1^n \| f_{C_n}^n \|^2 - 2\tau_n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 \\ &\quad + 2\|\rho_1^n f_{C_n}^n\|^2 + 2\tau_n^2 \| \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n \|^2 \\ &\leq \|x_n - p^*\|^2 - 2\rho_1^n \left(1 - \rho_1^n\right) \| f_{C_n}^n \|^2 - 2\tau_n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 + 2\tau_n^2 \bar{\tau}_n^2 \\ &= \|x_n - p^*\|^2 - 2\rho_1^n \left(1 - \rho_1^n\right) \| f_{C_n}^n \|^2 \\ &\quad - 2 \left(\frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2}{\bar{\tau}_n^2} \right) \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 + 2 \left(\frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2}{\bar{\tau}_n^2} \right)^2 \bar{\tau}_n^2 \\ &= \|x_n - p^*\|^2 - 2\rho_1^n \left(1 - \rho_1^n\right) \| f_{C_n}^n \|^2 - 2\rho_2^n \left(1 - \rho_2^n\right) \frac{\left(\sum_{i=1}^N \vartheta_i \left\| f_{Q_i^n}^n \right\|^2 \right)^2}{\bar{\tau}_n^2}. \end{aligned} \tag{31}$$

Since $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, we have from (31) that

$$\|x_{n+1} - p^*\|^2 \leq \|x_n - p^*\|^2.$$

Therefore, the sequence $\{x_n\}$ is Fejér-monotone with respect to Ω. As a consequence, $\lim_{n\to\infty} \|x_n - p^*\|$ exists. That is, $\{x_n\}$ is bounded, and hence the sequence $\{A_i x_n\}_{i=1}^N$ is also bounded.

Noticing that $\rho_1^n \in [a_1, b_1] \subset (0, 1)$, we can obtain from (31) that

$$\begin{aligned} 2a_1(1-b_1)\|f_{C_n}^n\|^2 &\le 2\rho_1^n\left(1-\rho_1^n\right)\|f_{C_n}^n\|^2 \\ &\le \|x_n - p^*\|^2 - \|x_{n+1} - p^*\|^2. \end{aligned} \tag{32}$$

Since $\{x_n\}$ is bounded and $f_{C_n}^n$ is 1-Lipschitz continuous, there exists a real number $R > 0$ such that $\|f_{C_n}^n\|^2 \le R$. Thus, we can obtain from (32) that

$$\lim_{n\to\infty} \|f_{C_n}^n\|^2 = 0. \tag{33}$$

Hence, we obtain from (33)

$$\lim_{n\to\infty} \|f_{C_n}^n\| = 0. \tag{34}$$

Noticing that $\rho_2^n \in [a_2, b_2] \subset (0, 1)$, we can obtain from (31) that

$$\begin{aligned} 2a_2(1-b_2)\frac{\left(\sum_{i=1}^N \vartheta_i \left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} &\le 2\rho_2^n\left(1-\rho_2^n\right)\frac{\left(\sum_{i=1}^N \vartheta_i \left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} \\ &\le \|x_n - p^*\|^2 - \|x_{n+1} - p^*\|^2. \end{aligned} \tag{35}$$

Letting $n \to \infty$ on both sides of (35), we have

$$\lim_{n\to\infty} \frac{\left(\sum_{i=1}^N \vartheta_i \left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} = 0. \tag{36}$$

Since the iterative sequence $\{x_n\}$ is bounded and by the Lipschitz continuity of $f_{Q_i^n}^n$, the sequence $\left\{\left\|\sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right\|\right\}_{n=1}^{\infty}$ is bounded and so the sequence $\{\bar{\tau}_n\}$ is bounded too. Therefore, we can get from (36) that

$$\lim_{n\to\infty} \|f_{Q_i^n}^n\| = 0 \ \text{ for } i = 1, 2, \ldots, N. \tag{37}$$

Next, we will prove that $\omega_w(x_n) \subset \Omega$. For each $i = 1, 2, \ldots, N$, since ∂q_i is bounded on bounded sets, there exists a constant $\gamma > 0$ such that $\|\eta_i^n\| \le \gamma$, where $\eta_i^n \in \partial q_i(A_i x_n)$. Then, for $i = 1, 2, \ldots, N$, notice that $P_{Q_i^n}(A_i x_n) \in Q_i^n$, we have

$$\begin{aligned} q_i(A_i x_n) &\le \langle \eta_i^n, A_i x_n - P_{Q_i^n}(A_i x_n)\rangle \\ &\le \|\eta_i^n\|\|A_i x_n - P_{Q_i^n}(A_i x_n)\| \\ &\le \gamma\left\|\left(I - P_{Q_i^n}\right)A_i x_n\right\|. \end{aligned} \tag{38}$$

By (37), we have for any $i = 1, 2, \ldots, N$, that

$$\limsup_{n\to\infty} q_i(A_i x_n) \le 0. \tag{39}$$

Let $\hat{p} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_m}\} \subset \{x_n\}$ such that $x_{n_m} \rightharpoonup \hat{p}$ as $m \to \infty$. By the weak lower semicontinuity of the function q_i and (39), we get

$$q_i(A_i \hat{p}) \le \liminf_{m\to\infty} q_i(A_i x_{n_m}) \le 0, \tag{40}$$

which means that $A_i \hat{p} \in Q_i$ for $i = 1, 2, \ldots, N$.

Since ∂c is bounded, there exists a constant $\delta > 0$ such that $\|\xi_n\| \leq \delta$, where $\xi_n \in \partial c(x_n)$. Then, notice that $P_{C_n}(x_n) \in C_n$, we have

$$\begin{aligned} c(x_n) &\leq \langle \xi_n, x_n - P_{C_n}(x_n) \rangle \\ &\leq \|\xi_n\| \|x_n - P_{C_n}(x_n)\| \\ &\leq \delta \| \left(I - P_{C_n}\right) x_n \|. \end{aligned} \tag{41}$$

By (34), we have that

$$\limsup_{n\to\infty} c(x_n) \leq 0. \tag{42}$$

By the weak lower semicontinuity of the convex function c and (42), we obtain

$$c(\hat{p}) \leq \liminf_{m\to\infty} c(x_{n_m}) \leq 0. \tag{43}$$

Consequently, $\hat{p} \in C$. Therefore, $\hat{p} \in \Omega$.

Notice that for any $p^* \in \Omega$, $\lim_{n\to\infty} \|x_n - p^*\|$ exists and $\omega_w(x_n) \subset \Omega$. Therefore, applying Lemma 4, we conclude that the iterative sequence $\{x_n\}$ converges weakly to a solution of the SFPMOS (12). This completes the proof. □

For $N = 1$, we note the following iterative method for solving the SFP (1).

Algorithm 2 Weakly convergent self-adaptive CQ algorithm for solving SFP.

Choose a constant $\beta > 0$ and two real sequences $\{\rho_1^n\}, \{\rho_2^n\} \subset (0, 1)$. Select an arbitrary starting point $x_0 \in H_1$, and set $n = 0$. Given the current iterate $x_n \in H_1$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = x_n - \rho_1^n \left(I - P_{C_n}\right) x_n - \tau_n A^*(I - P_{Q_n})Ax_n$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \|(I - P_{Q_n})Ax_n\|^2}{\bar{\tau}_n^2} \tag{44}$$

where

$$\bar{\tau}_n := \max\left\{\left\| A^*\left(I - P_{Q_n}\right) Ax_n\right\|,\ \beta\right\},$$

and C_n and Q_n are the half-spaces given as in (4) and (5), respectively.

As an immediate consequence of Theorem 1, we obtain the following corollary.

Corollary 1 *Assume that the SFP (1) is consistent. Suppose the sequences $\{\rho_1^n\}$ and $\{\rho_2^n\}$ in Algorithm 2 are in $(0, 1)$ such that $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, respectively. Then, the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to a solution $p^* \in \Omega = \{p \in H_1 : p \in C$ such that $Ap \in Q\}$.*

3.2 Strong convergence theorem

In this subsection, we propose a new iterative method for solving the SFPMOS (12) in the infinite-dimensional Hilbert spaces, and we prove a strong convergence theorem of the proposed method.

Algorithm 3 Strongly convergent self-adaptive CQ algorithm for solving SFP-MOPS.

Choose a constant $\beta > 0$ and sequences $\{\rho_1^n\}$, $\{\rho_2^n\}$, $\{\alpha_n\} \subset (0, 1)$ and $\{\vartheta_i\}_{i=1}^N > 0$. Let $u \in H$ be a fixed point, take an arbitrary starting point $x_0 \in H$, and set $n = 0$. Given the current iterate $x_n \in H$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left(x_n - \rho_1^n \left(I - P_{C_n}\right) x_n - \tau_n \sum_{i=1}^{N} \vartheta_i A_i^*(I - P_{Q_i^n}) A_i x_n\right),$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \left\| \left(I - P_{Q_i^n}\right) A_i x_n \right\|^2}{\bar{\tau}_n^2} \tag{45}$$

where

$$\bar{\tau}_n := \max\left\{ \left\| \sum_{i=1}^{N} \vartheta_i A_i^*(I - P_{Q_i^n}) A_i x_n \right\|, \beta \right\},$$

and C_n and Q_i^n are the half-spaces given as in (22) and (23), respectively.

Theorem 2 *Assume that the SFPMOS* (12) *is consistent (i.e.,* $\Omega \neq \emptyset$*). Suppose the sequences* $\{\rho_1^n\}$, $\{\rho_2^n\}$, *and* $\{\alpha_n\}$ *in Algorithm 3 are in* $(0, 1)$ *such that* $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ *and* $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, *and* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 3 converges strongly to the point* $p^* \in \Omega$, *where* $p^* = P_\Omega u$.

Proof For simplicity, the same as we did in the proof of Theorem 1, we introduce some notations first.

$$f_{C_n}^n := \left(I - P_{C_n}\right) x_n, \quad f_{Q_i^n}^n := \left(I - P_{Q_i^n}\right) A_i x_n \ \text{ for } i = 1, 2, \ldots, N,$$

$$y_n = x_n - \rho_1^n f_{C_n}^n - \tau_n \sum_{i=1}^{N} \vartheta_i A_i^* f_{Q_i^n}^n,$$

where τ_n is the stepsize given in the Algorithm 3 and can be defined as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \| f_{Q_i^n}^n \|^2}{\bar{\tau}_n^2} \tag{46}$$

where

$$\bar{\tau}_n := \max\left\{\left\|\sum_{i=1}^{N}\vartheta_i A_i^* f_{Q_i^n}^n\right\|,\ \beta\right\}.$$

Then, the iterative sequence $\{x_n\}$ in Algorithm 3 can be rewritten as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n. \tag{47}$$

Let $p^* \in \Omega$. Using Lemma 2 (1) and by (47), we have that

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &= \|\alpha_n u + (1-\alpha_n)y_n - p^*\|^2 \\
&= \|\alpha_n u + (1-\alpha_n)y_n - p^* + \alpha_n p^* - \alpha_n p^*\|^2 \\
&= \|\alpha_n(u-p^*) + (1-\alpha_n)(y_n-p^*)\|^2 \\
&\le (1-\alpha_n)^2\|y_n-p^*\|^2 + 2\langle\alpha_n(u-p^*),\ x_{n+1}-p^*\rangle \\
&\le (1-\alpha_n)\|y_n-p^*\|^2 + 2\alpha_n\langle u-p^*,\ x_{n+1}-p^*\rangle.
\end{aligned} \tag{48}$$

From (31), we have

$$\|y_n-p^*\|^2 \le \|x_n-p^*\|^2 - 2\rho_1^n\left(1-\rho_1^n\right)\|f_{C_n}^n\|^2 - 2\rho_2^n\left(1-\rho_2^n\right)\frac{\left(\sum_{i=1}^N\vartheta_i\left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2}. \tag{49}$$

From (48) and (49), we obtain

$$\begin{aligned}
&\|x_{n+1}-p^*\|^2 \le (1-\alpha_n)\|x_n-p^*\|^2 + 2\alpha_n\langle u-p^*, x_{n+1}-p^*\rangle - \\
&(1-\alpha_n)\left[2\rho_1^n\left(1-\rho_1^n\right)\|f_{C_n}^n\|^2 + 2\rho_2^n\left(1-\rho_2^n\right)\frac{\left(\sum_{i=1}^N\vartheta_i\|f_{Q_i^n}^n\|^2\right)^2}{\bar{\tau}_n^2}\right].
\end{aligned} \tag{50}$$

Now, we prove the sequence $\{x_n\}$ is bounded. Indeed, using the assumptions imposed on $\{\rho_1^n\}$, $\{\rho_2^n\}$ and $\{\alpha_n\}$, we have from (50) that

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &\le (1-\alpha_n)\|x_n-p^*\|^2\| + 2\alpha_n\langle u-p^*, x_{n+1}-p^*\rangle \\
&\le (1-\alpha_n)\|x_n-p^*\|^2\| + 4\alpha_n\|u-p^*\|^2 + \frac{1}{4}\alpha_n\|x_{n+1}-p^*\|^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &\le \frac{1-\alpha_n}{1-\frac{1}{4}\alpha_n}\|x_n-p^*\|^2 + \frac{\frac{3}{4}\alpha_n}{1-\frac{1}{4}\alpha_n}\frac{16}{3}\|u-p^*\|^2 \\
&\le \max\left\{\|x_n-p^*\|, \frac{16}{3}\|u-p^*\|\right\} \\
&\ \ \vdots \\
&\le \max\left\{\|x_0-p^*\|, \frac{16}{3}\|u-p^*\|\right\}.
\end{aligned} \tag{51}$$

This shows that the sequence $\{x_n\}$ is bounded, and $\{y_n\}$ and $\{A_i x_n\}_{i=1}^N$ as well.

Next, with no loss of generality, we may assume that there exist $\sigma_1, \sigma_2 > 0$ such that $2\rho_1^n\left(1-\rho_1^n\right)(1-\alpha_n) \geq \sigma_1$ and $2\rho_2^n\left(1-\rho_2^n\right)(1-\alpha_n) \geq \sigma_2$ for all n.

Setting $s_n = \|x_n - p^*\|^2$, we get from (50) that

$$\begin{aligned} s_{n+1}-(1-\alpha_n)s_n+\sigma_1\|f_{C_n}^n\|^2+\frac{\sigma_2\left(\sum_{i=1}^N \vartheta_i\|f_{Q_i^n}^n\|^2\right)^2}{\bar{\tau}_n^2} &\leq 2\alpha_n\langle u-p^*, x_{n+1}-p^*\rangle \\ &\leq 2\alpha_n\|u-p^*\|\|x_{n+1}-p^*\|. \quad (52) \end{aligned}$$

Now, we prove $s_n \to 0$ by distinguishing two cases.

Case 1: Assume that $\{s_n\}$ is eventually decreasing. That is, there exists $k \geq 0$ such that $s_{n+1} < s_n$ holds for all $n \geq k$. In this case, $\{s_n\}$ must be convergent, and from (52) it follows that

$$\left(\sigma_1\|f_{C_n}^n\|^2+\frac{\sigma_2\left(\sum_{i=1}^N \vartheta_i\|f_{Q_i^n}^n\|^2\right)^2}{\bar{\tau}_n^2}\right) \leq \alpha_n K + (s_n - s_{n+1}), \quad (53)$$

where $K > 0$ is a constant such that $2\|u-p^*\|\|x_{n+1}-p^*\| \leq K$ for all $n \in \mathbb{N}$. Since $\sigma_1, \sigma_2 > 0$, and $\alpha_n \to 0$ as $n \to \infty$, we have from (53) that

$$\lim_{n\to\infty}\|f_{C_n}^n\|^2 = 0 \Rightarrow \lim_{n\to\infty}\|f_{C_n}^n\| = 0 \Rightarrow \lim_{n\to\infty}\|\left(I-P_{C_n}\right)x_n\| = 0, \quad (54)$$

and

$$\lim_{n\to\infty}\frac{\left(\sum_{i=1}^N \vartheta_i\left\|f_{Q_i^n}^n\right\|^2\right)^2}{\bar{\tau}_n^2} = 0. \quad (55)$$

Next, we show that $\left\{f_{Q_i^n}^n\right\} \to 0$. To do so, it suffices to verify that $\left\{\|\sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\|^2\right\}$ is bounded. Since $p^* \in \Omega$, we note that $A_i^*\left(I-P_{Q_i^n}\right)A_i p^* = 0$. Hence, it follows from Lemma 5 that

$$\|A_i^*\left(I-P_{Q_i^n}\right)A_i x_n - A_i^*\left(I-P_{Q_i^n}\right)A_i p^*\| \leq \left(\max_{1\leq i\leq N}\|A_i\|^2\right)\|x_n-p^*\| \quad (56)$$

and since $\{x_n\}$ is bounded, for all $i = 1, 2, \ldots, N$, we have the sequence $\left\{\left\|A_i^*\left(I-P_{Q_i^n}\right)A_i x_n\right\|\right\}_{n=1}^\infty$ is bounded. This implies that the sequence $\left\{\left\|\sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right\|\right\}_{n=1}^\infty$ is also bounded. Consequently, $\{\bar{\tau}_n\}$ is bounded too. Therefore, we can get from (55) that

$$\lim_{n\to\infty}\|f_{Q_i^n}^n\| = 0 \Rightarrow \lim_{n\to\infty}\|\left(I-P_{Q_i^n}\right)A_i x_n\| = 0 \quad (57)$$

for i=1, 2, ..., N.

Next, we verify that $\omega_w(x_n) \subset \Omega$. For each $i = 1, 2, \ldots, N$, since ∂q_i is bounded on bounded sets, there exists a constant $\delta > 0$ such that $\|\eta_i^n\| \leq \delta$ for all $n \geq 0$, where $\eta_i^n \in \partial q_i(A_i x_n)$. Then, from the fact that $P_{Q_i^n}(A_i x_n) \in Q_i^n$ and (23), it follows (for $i = 1, 2, \ldots, N$) that

$$
\begin{aligned}
q_i(A_i x_n) &\leq \langle \eta_i^n, A_i x_n - P_{Q_i^n}(A_i x_n)\rangle \\
&\leq \|\eta_i^n\| \|A_i x_n - P_{Q_i^n}(A_i x_n)\| \\
&\leq \delta \| \left(I - P_{Q_i^n}\right) A_i x_n\|.
\end{aligned} \tag{58}
$$

Let $\hat{p} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \rightharpoonup \hat{p}$ as $m \to \infty$. Then, the weak lower semicontinuity of q_i and (58) imply that

$$
q_i(A_i \hat{p}) \leq \liminf_{m\to\infty} q_i(A_i x_{n_m}) \leq 0. \tag{59}
$$

It turns out that $A_i \hat{p} \in Q_i$ for $i = 1, 2, \ldots, N$. Next, we turn to prove that $\hat{p} \in C$. Since ∂c is bounded, there exists a constant $\gamma > 0$ such that $\|\xi_n\| \leq \gamma$ for all $n \geq 0$, where $\xi_n \in \partial c(x_n)$. Then, from that trivial fact that $P_{C_n}(x_n) \in C_n$ and (22), it follows that

$$
\begin{aligned}
c(x_n) &\leq \langle \xi_n, x_n - P_{C_n}(x_n)\rangle \\
&\leq \|\xi_n\| \|x_n - P_{C_n}(x_n)\| \\
&\leq \gamma \| \left(I - P_{C_n}\right) x_n\|.
\end{aligned} \tag{60}
$$

The weak lower semicontinuity of c then implies that

$$
c(\hat{p}) \leq \liminf_{m\to\infty} c(x_{n_m}) \leq 0. \tag{61}
$$

Consequently, $\hat{p} \in C$. Therefore, $\hat{p} \in \Omega$. Hence, $\omega_w(x_n) \subset \Omega$.

Moreover, we have the following estimation

$$
\begin{aligned}
\|x_n - x_{n+1}\| &= \left\| x_n - \left[\alpha_n u + (1-\alpha_n)\left(x_n - \rho_1^n f_{C_n}^n - \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right)\right]\right\| \\
&= \left\|\alpha_n(x_n - u) + (1-\alpha_n)\left(\rho_1^n f_{C_n}^n + \tau_n \sum_{i=1}^N \vartheta_i A_i^* f_{Q_i^n}^n\right)\right\| \\
&\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\left(\rho_1^n \|f_{C_n}^n\| + \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \|f_{Q_i^n}^n\|^2}{\bar{\tau}_n}\right),
\end{aligned}
$$

since $\{x_n\}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, we have that $\lim_{n\to\infty} \alpha_n \|x_n - u\| = 0$. Noting that $\{\|A_i^*(I - P_{Q_i^n})A_i x_n\|\}_{n=1}^\infty$ is bounded, (57) together with the conditions $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, we have that

$$
\lim_{n\to\infty}(1-\alpha_n)\left(\rho_1^n \|f_{C_n}^n\| + \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \|f_{Q_i^n}^n\|^2}{\bar{\tau}_n}\right) = 0,
$$

which implies that

$$
\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{62}
$$

Furthermore, due to Lemma 1 (1), we get

$$\limsup_{n\to\infty}\langle u-p^*, x_n-p^*\rangle = \max_{z\in\omega_w(x_n)}\langle u-P_\Omega(u), z-P_\Omega(u)\rangle \le 0. \tag{63}$$

Taking into account of (52), we have

$$s_{n+1} \le (1-\alpha_n)s_n + 2\alpha_n\langle u-p^*, x_{n+1}-p^*\rangle. \tag{64}$$

Applying Lemma 7 to (64), we obtain $s_n = \|x_n - p^*\|^2 \to 0$.

Case 2: Assume that $\{s_n\}$ is not eventually decreasing. That is, we can find an integer n_0 such that $s_{n_0} \le s_{n_0+1}$. Now we define

$$M_n := \{n_0 \le m \le n : s_m \le s_{m+1}\}, n > n_0. \tag{65}$$

It is easy to see that M_n is nonempty and satisfies $M_n \subseteq M_{n+1}$. Let

$$\phi(n) := \max M_n, n > n_0. \tag{66}$$

It is clear that $\phi(n) \to \infty$ as $n \to \infty$ (otherwise, $\{s_n\}$ is eventually decreasing). It is also clear that $s_{\phi(n)} \le s_{\phi(n)+1}$ for all $n > n_0$. Moreover,

$$s_n \le s_{\phi(n)+1},\ \ n > n_0. \tag{67}$$

In fact, if $\phi(n) = n$, then (67) is trivial: if $\phi(n) < n$, from (66), there exists some $j \in \mathbb{N}$ such that $\phi(n) + j = n$, we deduce that

$$s_n = s_{\phi(n)+j} < \cdots < s_{\phi(n)+2} < s_{\phi(n)+1}, \tag{68}$$

and (67) holds again. Since $s_{\phi(n)} < s_{\phi(n)+1}$ for all $n > n_0$, it follows from (53) that

$$\left(\sigma_1\|f^{\phi(n)}_{C_{\phi(n)}}\|^2 + \frac{\sigma_2(\sum_{i=1}^N \vartheta_i\|f^{\phi(n)}_{Q_i^{\phi(n)}}\|^2)^2}{\bar{\tau}^2_{\phi(n)}}\right) \le \alpha_{\phi(n)}K \to 0, \tag{69}$$

so that

$$\lim_{n\to\infty}\|f^{\phi(n)}_{C_{\phi(n)}}\| = 0 \Rightarrow \lim_{n\to\infty}\|(I-P_{C_{\phi(n)}})x_{\phi(n)}\| = 0, \tag{70}$$

and

$$\lim_{n\to\infty}\frac{(\sum_{i=1}^N \vartheta_i\|f^{\phi(n)}_{Q_i^{\phi(n)}}\|^2)^2}{\bar{\tau}^2_{\phi(n)}} = 0. \tag{71}$$

Noting that $\left\{\left\|A_i^*(I-P_{Q_i^{\phi(n)}})A_i x_{\phi(n)}\right\|\right\}_{n=1}^{\infty}$ is bounded, for $i = 1, 2, \ldots, N$, we also have that

$$\lim_{n\to\infty}\left\|f^{\phi(n)}_{Q_i^{\phi(n)}}\right\| = 0 \Rightarrow \lim_{n\to\infty}\left\|(I-P_{Q_i^{\phi(n)}})A_i x_{\phi(n)}\right\| = 0. \tag{72}$$

By the same argument to the proof in Case 1, we have $\omega_w(x_{\phi(n)}) \subset \Omega$.

Furthermore, by the same argument to the proof in Case 1, from (62), we have that

$$\lim_{n\to\infty}\|x_{\phi(n)} - x_{\phi(n)+1}\| = 0. \tag{73}$$

Thus, one can deduce that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-p^*, x_{\phi(n)+1}-p^*\rangle &= \limsup_{n\to\infty}\langle u-p^*, x_{\phi(n)}-p^*\rangle \\ &= \max_{z\in\omega_w(x_{\phi(n)})}\langle u-P_\Omega(u), z-P_\Omega(u)\rangle \\ &\leq 0. \end{aligned} \tag{74}$$

Since $s_{\phi(n)} \leq s_{\phi(n)+1}$, it follows from (52) that

$$s_{\phi(n)} \leq 2\langle u-p^*, x_{\phi(n)+1}-p^* >, \ n > n_0. \tag{75}$$

(74) and (75) together gives

$$\limsup_{n\to\infty} s_{\phi(n)} \leq 0. \tag{76}$$

Hence, $\lim\limits_{n\to\infty} s_{\phi(n)} = 0$, which together with (73)

$$\begin{aligned}\sqrt{s_{\phi(n)+1}} &\leq \|x_{\phi(n)+1}-p^*\| \\ &= \|(x_{\phi(n)}-p^*)+(x_{\phi(n)+1}-x_{\phi(n)})\| \\ &\leq \|x_{\phi(n)}-p^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\| \\ &= \sqrt{s_{\phi(n)}}+\|x_{\phi(n)+1}-x_{\phi(n)}\| \to 0, \end{aligned} \tag{77}$$

which, together with (67), in turn implies that $s_n \to 0$, that is, $x_n \to p^*$. Therefore, the full iterative sequence $\{x_n\}$ converges strongly to the solution $p^* = P_\Omega(u)$ of SFPMOS (12). This completes the proof. □

Corollary 2 *Assume that the SFPMOS (12) is consistent (i.e., $\Omega \neq \emptyset$). Let $x_0 \in H$ be an arbitrary initial point, and set $n = 0$. Let $\{x_n\}$ be a sequence generated via the manner*

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\left(x_n - \rho_1^n\left(I-P_{C_n}\right)x_n - \tau_n\sum_{i=1}^N \vartheta_i A_i^*\left(I-P_{Q_i^n}\right)A_i x_n\right)$$

s where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \|(I-P_{Q_i^n})A_i x_n\|^2}{\bar{\tau}_n^2} \tag{78}$$

where for a constant $\beta > 0$

$$\bar{\tau}_n := \max\left\{\|\sum_{i=1}^N \vartheta_i A_i^*\left(I-P_{Q_i^n}\right)A_i x_n\|, \ \beta\right\},$$

and C_n and Q_i^n are the half-spaces given as in (22) and (23), respectively. Suppose the parameters $\{\rho_1^n\}$, $\{\rho_2^n\}$, $\{\alpha_n\}$ are in $(0, 1)$ satisfying the conditions in Theorem 2, and $\{\vartheta_i\}_{i=1}^N > 0$. Then, the sequence $\{x_n\}$ converges strongly to the point $p^* \in \Omega$, where $p^* = P_\Omega(x_0)$.

Similarly as in Subsection 3.1, for $N = 1$, we again obtain the following strongly convergent result regarding the SFP (1).

Algorithm 4 Strongly convergent self-adaptive CQ algorithm for solving SFP.

Choose a constant $\beta > 0$ and three real sequences $\{\rho_1^n\}, \{\rho_2^n\}, \{\alpha_n\} \subset (0, 1)$. Let $u \in H_1$ be a fixed point, take an arbitrary starting point $x_0 \in H_1$, and set $n = 0$. Given the current iterate $x_n \in H_1$. Compute the next iterate x_{n+1} via the rule

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \rho_1^n \left(I - P_{C_n}\right) x_n - \tau_n A^* \left(I - P_{Q_n}) x_n\right),$$

where the step-size τ_n is updated self-adaptively as

$$\tau_n := \frac{\rho_2^n \left\|(I - P_{Q_n}) A x_n\right\|^2}{\bar{\tau}_n^2} \tag{79}$$

where

$$\bar{\tau}_n := \max \left\{\|A^* \left(I - P_{Q_n}\right) A x_n\|,\ \beta\right\},$$

and C_n and Q_n are the half-spaces given as in (4) and (5), respectively.

As an immediate consequence of Theorem 2, we obtain the following corollary.

Corollary 3 *Assume that the SFP* (1) *is consistent. Suppose the sequences* $\{\rho_1^n\}$, $\{\rho_2^n\}$ *and* $\{\alpha_n\}$ *in Algorithm 4 are in* $(0, 1)$ *such that* $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ *and* $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, *and* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 4 converges strongly to the point* $p^* = P_\Omega(u) \in \Omega = \{p \in H_1 : p \in C$ *such that* $Ap \in Q\}$.

4 Application to the generalized split feasibility problem

In this section, we present an application of Theorems 1 and 2 for solving generalized split feasibility problem (another generalization of the SFP) in Hilbert spaces. We recall the generalized split feasibility problem first.

In 2020, Reich and Tuyen [52] first introduced and studied the following generalized split feasibility problem (GSFP).

Let H_i, $i = 1, 2, \ldots, N$, be real Hilbert spaces and C_i, $i = 1, 2, \ldots, N$, be closed and convex subsets of H_i, respectively. Let $B_i : H_i \to H_{i+1}$, $i = 1, 2, \ldots, N-1$, be bounded linear operators such that

$$S := C_1 \cap B_1^{-1}(C_2) \cap \cdots \cap B_1^{-1}\left(B_2^{-1} \ldots \left(B_{N-1}^{-1}(C_N)\right)\right) \neq \emptyset. \tag{80}$$

Given H_i, C_i and A_i as above, the generalized SFP (GSFP)([52]) is to

$$\text{find an element } p^* \in S. \tag{81}$$

That is $p^* \in C_1,\ B_1 p^* \in C_2, \ldots, B_{N-1}B_{N-2}\ldots B_1 p^* \in C_N$. In [52], Reich and Tuyen proved a strong convergence theorem for a modification of the CQ method which solves the GSFP (81). For more details on the GSFP (81), one can read the paper [52].

Remark 1 ([43, Remark 1.1]) Letting $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $A_1 = B_1, A_2 = B_2B_1, \ldots,$ and $A_{N-1} = B_{N-1}B_{N-2}B_{N-3}\ldots B_2B_1$, then the SFPMOS (12) becomes the GSFP (81).

From Theorem 1 and Remark 1, we note the following theorem for solving the GSFP (81).

Theorem 3 *Let $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $A_1 = B_1, A_2 = B_2B_1, \ldots,$ and $A_{N-1} = B_{N-1}B_{N-2}B_{N-3}\ldots B_2B_1$. Assume that the GSFP (81) is consistent (i.e., $S \neq \emptyset$). Let $x_0 \in C_1$ be an arbitrary initial point and set $n = 0$. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = x_n - \rho_1^n \left(I - P_{C_1^n}\right) x_n - \tau_n \sum_{i=1}^{N-1} \vartheta_i A_i^* \left(I - P_{C_{i+1}^n}\right) A_i x_n \tag{82}$$

where C_1^n and C_{i+1}^n are half-spaces of C_1 and C_{i+1} (at the nth iterate), respectively,

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^{N-1} \vartheta_i \left\| \left(I - P_{C_{i+1}^n}\right) A_i x_n \right\|^2}{\bar{\tau}_n^2}$$

where for a constant $\beta > 0$

$$\bar{\tau}_n := \max \left\{ \left\| \sum_{i=1}^{N-1} \vartheta_i A_i^* \left(I - P_{C_{i+1}^n}\right) A_i x_n \right\|, \ \beta \right\},$$

and the sequences $\{\rho_1^n\}$, $\{\rho_2^n\} \subset (0, 1)$ such that $0 < a_1 \leq \rho_1^n \leq b_1 < 1$ and $0 < a_2 \leq \rho_2^n \leq b_2 < 1$, and the parameter $\{\vartheta_i\}_{i=1}^N > 0$. Then, the sequence $\{x_n\}$ generated by the iterative scheme (82) converges weakly to a solution $p^* \in S$.

Again, using Theorem 2 and Remark 1, we note the following result to solve the GSFP (81).

Theorem 4 *Let $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $A_1 = B_1, A_2 = B_2B_1, \ldots,$ and $A_{N-1} = B_{N-1}B_{N-2}B_{N-3}\ldots B_2B_1$. Assume that the GSFP (81) is consistent (i.e., $S \neq \emptyset$). Let $u \in C_1$ be a fixed point and $x_0 \in C_1$ is an arbitrary initial point, and set $n = 0$. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \left(x_n - \rho_1^n \left(I - P_{C_1^n}\right) x_n - \tau_n \sum_{i=1}^{N-1} \vartheta_i A_i^* \left(I - P_{C_{i+1}^n}\right) A_i x_n \right) \tag{83}$$

where C_1^n and C_{i+1}^n are half-spaces of C_1 and C_{i+1}, respectively,

$$\tau_n := \frac{\rho_2^n \sum_{i=1}^{N-1} \vartheta_i \left\| \left(I - P_{C_{i+1}^n}\right) A_i x_n \right\|^2}{\bar{\tau}_n^2}$$

where for a constant $\beta > 0$ *and*

$$\bar{\tau}_n := \max\left\{\left\|\sum_{i=1}^{N-1}\vartheta_i A_i^*\left(I - P_{C_{i+1}^n}\right)A_i x_n\right\|,\ \beta\right\},$$

the sequences $\{\rho_1^n\}$, $\{\rho_2^n\}$, $\{\alpha_n\} \subset (0, 1)$ *such that* $0 < a_1 \le \rho_1^n \le b_1 < 1,\ 0 < a_2 \le \rho_2^n \le b_2 < 1,\ \lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$, *and the parameter* $\{\vartheta_i\}_{i=1}^N > 0$. *Then, the sequence* $\{x_n\}$ *generated by the iterative scheme* (83) *converges strongly to the solution* $p^* \in S$, *where* $p^* = P_S(u)$.

5 Numerical results

In this section, we present some numerical examples to illustrate the implementation and efficiency of our proposed methods compared to some existing results by solving some problems. The numerical results are completed on a standard FUJITSUNOTEBOOK laptop with 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80GHz 2.80 GHz with memory 16GB. The code is implemented in MATLAB R2022a. In our numerical experiments, Iter. (n) stands for the number of iterations and CPU(s) for the Elapsed time-run in seconds.

Example 1 ([43]) Consider $H = \mathbb{R}^{10}$, $H_1 = \mathbb{R}^{20}$, $H_2 = \mathbb{R}^{30}$ and $H_3 = \mathbb{R}^{40}$. Find a point $p^* \in \mathbb{R}^{10}$ such that

$$p^* \in \Omega := C \cap A_1^{-1}(Q_1) \cap A_2^{-1}(Q_2) \cap A_3^{-1}(Q_3) \neq \emptyset, \tag{84}$$

where the sets C and Q_i, and the linear bounded operators A_i are defined by

$$\begin{aligned} C &= \left\{x \in \mathbb{R}^{10} : \|x - \mathbf{c}\|^2 \le \mathbf{r}^2\right\},\\ Q_1 &= \left\{A_1x \in \mathbb{R}^{20} : \|A_1x - \mathbf{c}_1\|^2 \le \mathbf{r}_1^2\right\},\\ Q_2 &= \left\{A_2x \in \mathbb{R}^{30} : \|A_2x - \mathbf{c}_2\|^2 \le \mathbf{r}_2^2\right\},\\ Q_3 &= \left\{A_3x \in \mathbb{R}^{40} : \|A_3x - \mathbf{c}_3\|^2 \le \mathbf{r}_3^2\right\}. \end{aligned} \tag{85}$$

where $\mathbf{c} \in \mathbb{R}^{10}$, $\mathbf{c}_1 \in \mathbb{R}^{20}$, $\mathbf{c}_2 \in \mathbb{R}^{30}$, $\mathbf{c}_3 \in \mathbb{R}^{40}$, $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \mathbb{R}$, and $A_1 : \mathbb{R}^{10} \to \mathbb{R}^{20}$, $A_2 : \mathbb{R}^{10} \to \mathbb{R}^{30}$, $A_3 : \mathbb{R}^{10} \to \mathbb{R}^{40}$. In this case, for any $x \in \mathbb{R}^{10}$ we have $c(x) = \|x - \mathbf{c}\|^2 - \mathbf{r}^2$ and $q_i(A_ix) = \|A_ix - \mathbf{c}_i\|^2 - \mathbf{r}_i^2$ for $i = 1, 2, 3$. According to (22) and (23), the half-spaces C_n and Q_i^n $(i = 1, 2, 3)$, respectively of the sets C and Q_i are determined at a point x_n and A_ix_n, respectively as follows:

$$\begin{aligned} C_n &= \{x \in \mathbb{R}^{10} : \|x_n - \mathbf{c}\|^2 - \mathbf{r}^2 \le 2\langle x_n - \mathbf{c}, x_n - x\rangle\},\\ Q_1^n &= \{y \in \mathbb{R}^{20} : \|A_1x_n - \mathbf{c}_1\|^2 - \mathbf{r}_1^2 \le 2\langle A_1x_n - \mathbf{c}_1, A_1x_n - y\rangle\},\\ Q_2^n &= \{y \in \mathbb{R}^{30} : \|A_2x_n - \mathbf{c}_2\|^2 - \mathbf{r}_2^2 \le 2\langle A_2x_n - \mathbf{c}_2, A_2x_n - y\rangle\},\\ Q_3^n &= \{y \in \mathbb{R}^{40} : \|A_3x_n - \mathbf{c}_3\|^2 - \mathbf{r}_3^2 \le 2\langle A_3x_n - \mathbf{c}_3, A_3x_n - y\rangle\}. \end{aligned} \tag{86}$$

Then, the metric projections onto the half-spaces C_n and Q_i^n $(i = 1, 2, 3)$, can be easily calculated. The elements of the representing matrices A_i are randomly generated in the closed interval $[-5, 5]$, the coordinates of the centers $\mathbf{c}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ are randomly generated in the closed interval $[-1, 1]$, and the radii $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are randomly generated in the closed intervals $[10, 20]$, $[20, 40]$, $[30, 60]$ and $[40, 80]$, respectively. For simplicity, denote $e_1 = (1, 1, \ldots, 1)^T \in \mathbb{R}^{10}$.

In this example, we examine the convergence of the sequence $\{x_n\}$ which is defined by Algorithms 1 and 3 by solving problem (84) compared to the recently introduced iterative methods for solving the SFPMOS (12) given by Scheme (13), Scheme (14), and with the following viscosity approximation an optimization approach method proposed by Reich et al. [53] for solving the SFPMOS (12). For any given point $x_0 \in H$, $\{x_n\}$ is a sequence generated by the iterative method

$$x_{n+1} := \alpha_n f(x_n) + (1-\alpha_n) P_C \left(x_n - \lambda_n \sum_{i \in I(x_n)} \gamma_{i,n} A_i^* \left(I - P_{Q_i}\right) A_i x_n \right), \quad (87)$$

where $f : C \to C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0, 1)$, $\{\alpha_n\} \subset (0, 1)$, $I(x_n) = \left\{ i : \left\| A_i x_n - P_{Q_i} A_i x_n \right\| = \max_{i=1,2,\ldots,N} \left\| A_i x_n - P_{Q_i} A_i x_n \right\| \right\}$, $\gamma_{i,n} \geq 0$ for all $i \in I(x_n)$ with $\sum_{i \in I(x_n)} \gamma_{i,n} = 1$, and for $\{\rho_n\} \subset [\bar{a}, \bar{a}] \subset (0, 2)$ $\{\lambda_n\} \subset [0, \infty)$ such that

$$\lambda_n = \begin{cases} \rho_n \frac{(\max_{i=1,2,\ldots,N} \|A_i x_n - P_{Q_i} A_i x_n\|)^2}{\| \sum_{i \in I(x_n)} \gamma_{i,n} A_i^* (I - P_{Q_i}) A_i x_n \|^2}, & \text{if } \| \sum_{i \in I(x_n)} \gamma_{i,n} A_i^* (I - P_{Q_i}) A_i x_n \| > 0, \\ 0, & \text{otherwise} . \end{cases} \quad (88)$$

For comparison purpose, we consider the values of the parameters appeared in the methods as follows. For Algorithms 1 and 3, we take $\beta = 0.05$, $\rho_1^n = \frac{1}{10^4 n + 1} = \rho_2^n$ and $\vartheta_i = \frac{i}{12}$, $i = 1, 2, 3$. For Algorithm 3, Scheme (14), and Scheme (87) $\alpha_n = \frac{1}{n+1}$. For Scheme (13) and Scheme (14), we take $\lambda_n = 0.00005$. For Scheme (14) and Scheme (87), we take $f(x) = 0.975x$. Moreover, for Scheme (87), we take $\gamma_{1,n} = \frac{1}{6}$, $\gamma_{2,n} = \frac{1}{3}$, $\gamma_{3,n} = \frac{1}{2}$ and $\rho_n = \frac{1}{10^4 n + 1}$. Using $E_n = \|x_{n+1} - x_n\|^2 < 10^{-8}$ as stopping criteria, for different choices of the fixed point u and the initial point x_0, the results of numerical experiments are reported in Table 1 and Fig. 1.

It can be observed from Table 1 and Fig. 1 that for each choices of (u, x_0), our proposed methods Algorithms 1 and 3 have better performance interms of the iteration numbers (Iter. (n)) and comparatively the CPU-run time in seconds (CPU(s)) than of the compared methods. More precisely, Algorithms 1 and 3 have less number of iterations and take small CPU-time to run than of the iterative methods given by Scheme (13), Scheme (14), and Scheme (87).

Example 2 Let $H_1 = H_2 = L_2([0, 2\pi])$ with the inner product $\langle . \rangle$ defined by

$$\langle x, y \rangle = \int_0^{2\pi} x(t) y(t) dt, \quad \forall x, y \in L_2([0, 2\pi])$$

and with the norm $\|.\|$ defined by

$$\|x\|_2 := \sqrt{\int_0^{2\pi} |x(t)|^2 dt}, \quad \forall x, y \in L^2([0, 2\pi]).$$

Table 1 Comparison of Algorithms 1 and 3 with Scheme (13), Scheme (14), and Scheme (87) for different choices of u and x_0

(u, x_0)		Algorithm 1	Algorithm 3	Scheme (13)	Scheme (14)	Scheme (87)
	Iter. (n)	21	397	580	709	611
$(10e_1, 15e_1)$	CPU(s)	0.000814	0.007049	0.009757	0.012382	0.007107
	E_n	9.7443e-09	9.9995e-09	9.9612e-09	9.9995e-09	9.9689e-09
	Iter. (n)	24	548	654	662	571
$(10e_1, \frac{1}{2}e_1)$	CPU(s)	0.000873	0.007506	0.009706	0.011476	0.008185
	E_n	9.5908e-09	9.9589e-09	9.9869e-09	9.9834e-09	9.9686e-09
	Iter. (n)	49	297	393	543	470
$(3e_1, \frac{1}{5}e_1)$	CPU(s)	0.010865	0.019628	0.028973	0.036331	0.019807
	E_n	9.7579e-09	9.9626e-09	9.7810e-09	9.9688e-09	9.9801e-09
	Iter. (n)	45	318	441	685	590
$(3e_1, -\frac{1}{5}e_1)$	CPU(s)	0.001141	0.006589	0.008854	0.011453	0.007842
	E_n	9.9649e-09	9.9562e-09	9.9854e-09	9.9735e-09	9.9709e-09
	Iter. (n)	53	180	349	611	528
$(e_1, -\frac{1}{50}e_1)$	CPU(s)	0.001559	0.004430	0.008021	0.011801	0.005567
	E_n	9.8328e-09	9.8907e-09	9.8595e-09	9.9901e-09	9.9765e-09
	Iter. (n)	43	178	272	470	409
$(e_1, -\frac{1}{1000}e_1)$	CPU(s)	0.002600	0.003973	0.006847	0.009095	0.006656
	E_n	9.7746e-09	9.8789e-09	9.8990e-09	9.9887e-09	9.9616e-09

Furthermore, we consider the following half-spaces

$$C := \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} x(t)dt \leq 1\right\} \text{ and } Q := \left\{y \in L_2([0, 2\pi]) : \int_0^{2\pi} |y(t) - sin(t)|^2 dt \leq 16\right\}.$$

In addition, we consider a linear continuous operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$, where $(Ax)(t) = x(t)$. Then, $(A^*x)(t) = x(t)$ and $\|A\| = 1$. That is, A is an identity operator. The metric projection onto C and Q have an explicit formula [54]. We can also write the projections onto C and the projections onto Q as follows:

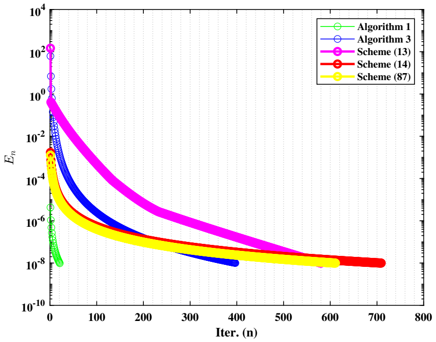
$$P_C(x(t)) = \begin{cases} x(t) + \frac{1-\int_0^{2\pi} x(t)dt}{4\pi^2}, & \text{if } \int_0^{2\pi} x(t)dt > 1, \\ x(t), & \text{if } \int_0^{2\pi} x(t)dt \leq 1. \end{cases}$$

$$P_Q(y(t)) = \begin{cases} sin(t) + \frac{4(y(t)-sin(t))}{\sqrt{\int_0^{2\pi} |y(t)-sin(t)|^2 dt}}, & \text{if } \int_0^{2\pi} |y(t) - sin(t)|^2 dt > 16, \\ y(t), & \text{if } \int_0^{2\pi} |y(t) - sin(t)|^2 dt \leq 16. \end{cases}$$

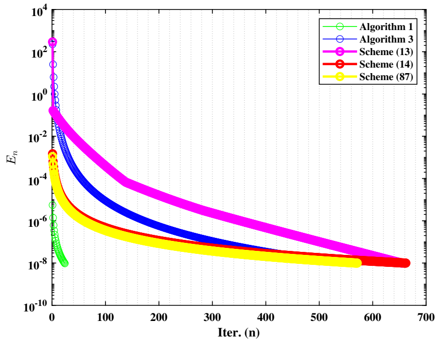
Now, we solve the following problem

$$\text{find } p^* \in C \text{ such that } Ap^* \in Q. \tag{89}$$

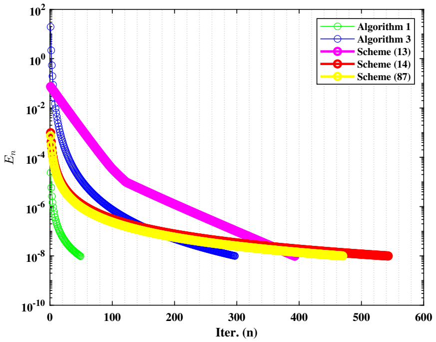
In this example, we examine the numerical behaviour of our proposed method: Algorithm 4 and compare it with the strongly convergent iterative algorithms given



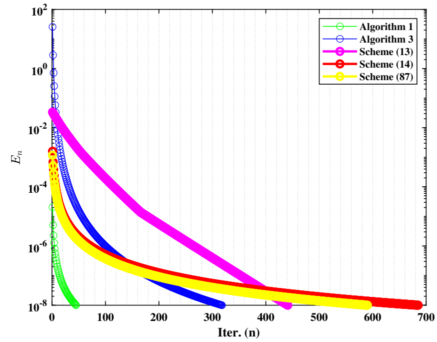
(a) $(u, x_0) = (10e_1, 15e_1)$



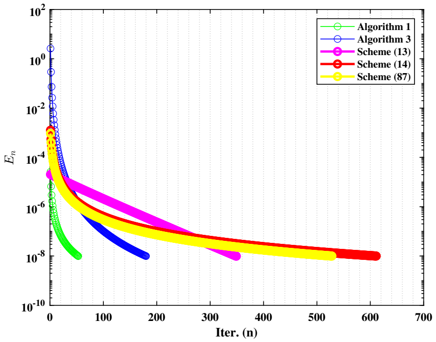
(b) $(u, x_0) = (10e_1, \frac{1}{2}e_1)$



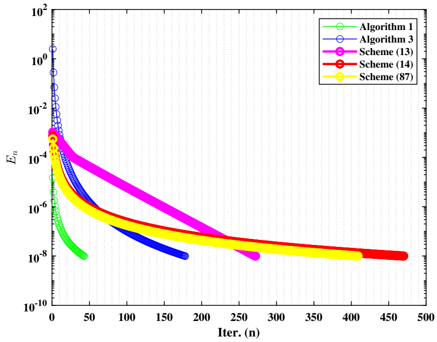
(c) $(u, x_0) = (3e_1, \frac{1}{5}e_1)$



(d) $(u, x_0) = (3e_1, -\frac{1}{5}e_1)$



(e) $(u, x_0) = (e_1, -\frac{1}{50}e_1)$



(f) $(u, x_0) = (e_1, -\frac{1}{1000}e_1)$

Fig. 1 Comparison of Algorithm 1 and Algorithm 3 with Scheme (13), Scheme (14), and Scheme (87) for different choices of u and x_0

by Scheme (10) and Scheme (11) by solving problem (89). For comparison purpose, we take the following data: For Algorithm 4, we take, $\beta = 0.05$, $\rho_1^n = \rho_2^n = \frac{n}{n+1}$ and $\alpha_n = \frac{1}{n+1}$. For Schemes (10) and (11), we take $\rho_n = \frac{n}{n+1}$ and $\alpha_n = \frac{1}{n+1}$.

Now, using $E_n = \|x_{n+1} - x_n\| < 10^{-4}$ as stopping criteria for all methods, for different choices of the fixed point u and the initial point x_0, the outcomes of the numerical experiments of the compared methods are reported in Table 2 and Fig. 2.

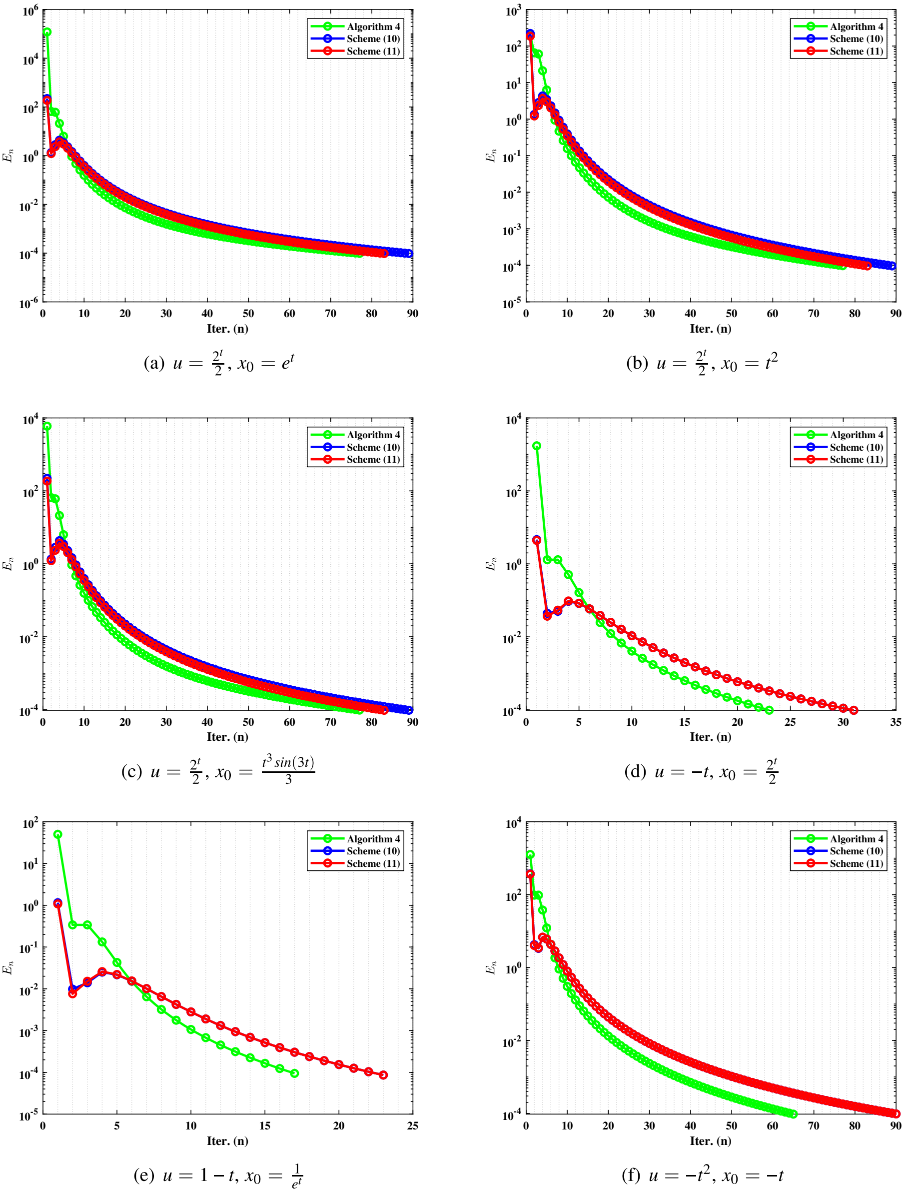
It can be observed from Table 2 and Fig. 2 that for each choices of u and x_0, Algorithm 4 is faster in terms of less number of iterations (Iter. (n)) and CPU-run time in seconds (CPU(s)) than the compared algorithms.

Example 3 The problem of computing sparse solutions (i.e., solutions where only a very small number of entries are nonzero) to linear inverse problems arises in a large number of application areas, for instance, in image restoration [55], channel equalization [56], echo cancellation [57], and stock market analysis [58]. The linear inverse problem consists of computing sparse solutions of a vector that has been digitized and has been degraded by an additive noise. Without loss of generality, for a vector $x \in H_1$ and an observed vector $y \in H_2$, a model including an additive noise can be written as

$$y = Ax + \eta,$$

Table 2 Comparison of Algorithm 4 with Scheme (10) and Scheme (11) for different choices of u and x_0

		Algorithm 4	Scheme (10)	Scheme (11)
$u = \frac{2^t}{2}, x_0 = e^t$	Iter. (n)	77	89	83
	CPU(s)	197.820341	258.683440	271.607658
	E_n	0.000097596	0.0000965296	0.000096184
$u = \frac{2^t}{2}, x_0 = t^2$	Iter. (n)	77	89	83
	CPU(s)	171.863335	195.831745	252.749699
	E_n	0.000097596	0.0000965297	0.000096184
$u = \frac{2^t}{2}, x_0 = \frac{t^3 sin(3t)}{3}$	Iter. (n)	77	89	83
	CPU(s)	154.680836	182.982568	290.567723
	E_n	0.000097596	0.0000965297	0.000096184
$u = -t, x_0 = \frac{2^t}{2}$	Iter. (n)	23	31	31
	CPU(s)	67.111117	97.401316	98.183831
	E_n	0.000097482	0.000096914	0.000096914
$u = 1 - t, x_0 = \frac{1}{e^t}$	Iter. (n)	17	23	23
	CPU(s)	33.258399	45.372658	45.323811
	E_n	0.000094479	0.000086201	0.0000862013
$u = -t^2, x_0 = -t$	Iter. (n)	65	90	90
	CPU(s)	128.514409	184.037896	297.695904
	E_n	0.000095464	0.000096829	0.000096829



(a) $u = \frac{2^t}{2}, x_0 = e^t$

(b) $u = \frac{2^t}{2}, x_0 = t^2$

(c) $u = \frac{2^t}{2}, x_0 = \frac{t^3 sin(3t)}{3}$

(d) $u = -t, x_0 = \frac{2^t}{2}$

(e) $u = 1 - t, x_0 = \frac{1}{e^t}$

(f) $u = -t^2, x_0 = -t$

Fig. 2 Comparison of Algorithm 4 with Scheme (10) and Scheme (11) for different choices of $u,\ x_0$

where A is a bounded linear operator between the two Hilbert spaces H_1 and H_2 and $\eta \in H_2$ denotes the additive noise.

Suppose that $H_1 = H_2 = L^2([0, 1])$ with norm $\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt,\ x, y \in L^2([0, 1])$. Define the Volterra integral operator $A : L^2([0,1]) \to L^2([0,1])$ by

$$Ax(t) := \int_0^t x(s)ds, \quad \forall x \in L^2([0,1]), t \in [0,1].$$

Then, A is bounded linear monotone and $\|A\| = \frac{2}{\pi}$ (see [59, Problem 188, p100; Solution 188, p300]). Using Algorithm 4, we develop an iterative algorithm to recover the solution of the linear equation $Ax = y - \eta$. Furthermore, we compare the performance of our proposed Algorithm 4 and Scheme (10).

We are interested in solutions $x^* \in \{x \in C : Ax \in Q\}$, where C is the cone of functions $x(t)$ that are negative for $t \in [0, 0.25]$ and positive for $t \in [0.25, 1]$ and $Q = [a(t), b(t)] := \{y(t) : a(t) \le y(t) \le b(t), 0 \le t \le 1\}$ is a box delimited by the functions $a(t)$ and $b(t)$. The metric projection P_Q can be computed by formula:

$$P_Q(y) := max\{a, min\{y, b\}\}.$$

For some problems, the solution is almost sparse. To ensure the existence of the solution of the consider problem, K-sparse vector $x^*(t)$ is generated randomly in C. Taking $y(t) = Ax^*(t)$ and $a(t) = y(t) - 0.01$, $b(t) = y(t) + 0.01$, we have $Q = [a(t), b(t)]$. We take $K = 30$, $K = 55$ and $K = 70$ (see Fig. 3). The problem of interest is to find $x \in C$ such that $Ax \in Q$.

We compare the behavior of Algorithm 4 and Scheme (10) for the same initial point $x_0 = e^{4t^2}$ and same fixed point $u = t^2$. Set $\rho_1^n = \frac{2n}{3n+1} = \rho_2^n$ and $\alpha_n = \frac{1}{n}$ in Algorithm 4 and $\rho_n = \frac{2n}{3n+1}$ and $\alpha_n = \frac{1}{n}$ in Scheme (10). In the implementation, we

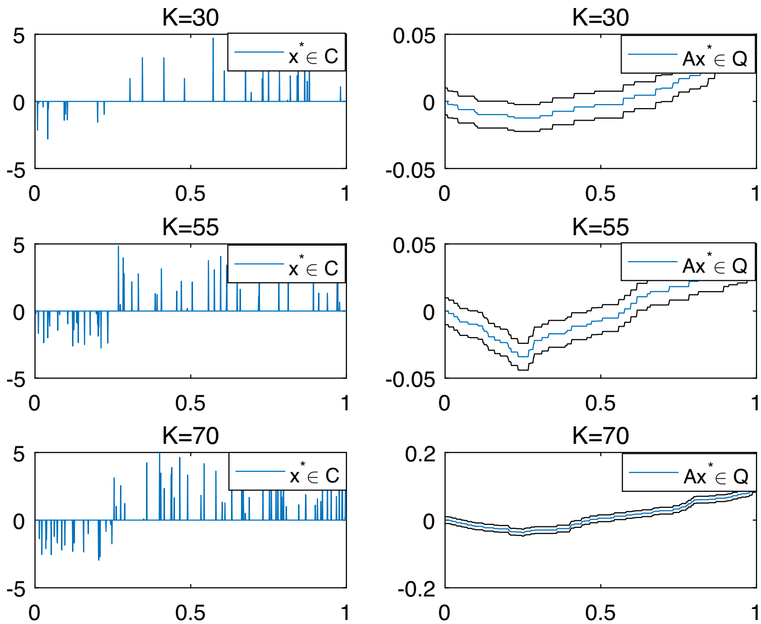


Fig. 3 $x^* \in C$ (left) and $Ax^* \in Q$ (right) for $K = 30, 55, 70$

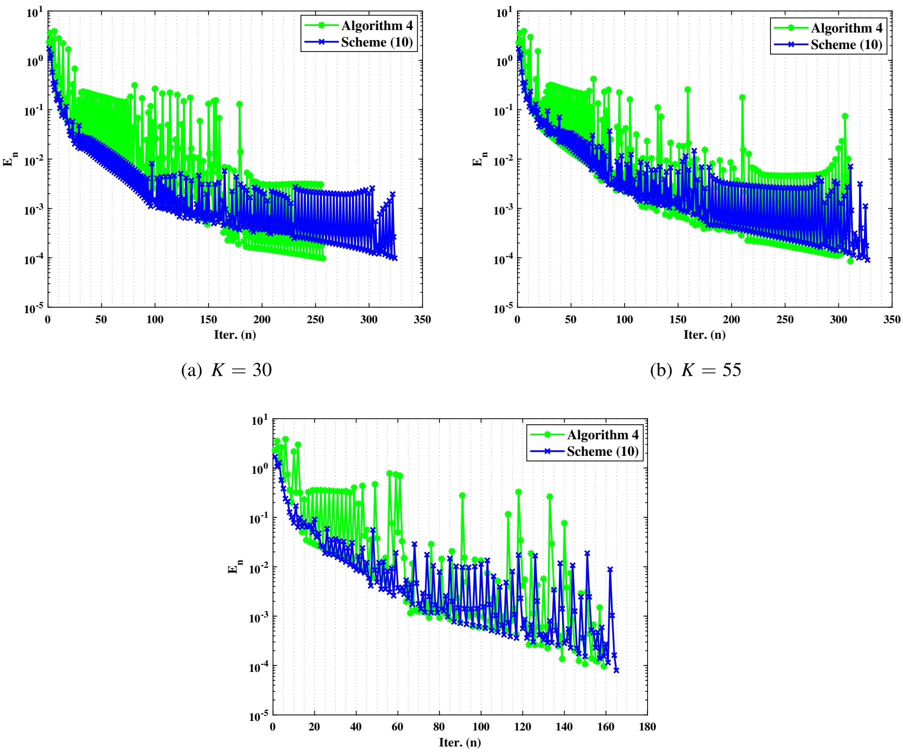
Table 3 Numerical results with different K-sparse

	Algorithm 4		Scheme (10)	
	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)
$K = 30$	259	4.0322	324	9.0005
$K = 55$	311	5.0901	327	10.2111
$K = 70$	159	2.5093	165	5.1393

take $E_n < \varepsilon = 10^{-4}$ as the stopping criterion, where

$$E_n = \|x - P_C x\|^2 + \|Ax - P_Q Ax\|^2.$$

In Table 3, we present our numerical results with different K-sparse ($K = 30$, 55, 70). Table 3 shows the number of iterations and the time of execution in seconds (CPU(s)) of Algorithm 4 and Scheme (10). In Fig. 4, we report the behavior of Algorithm 4 and Scheme (10) for $K = 30, 55, 70$. Furthermore, Fig. 4 presents error value



(a) $K = 30$

(b) $K = 55$

(c) $K = 70$

Fig. 4 Number of iterations and error estimate for Algorithm 4 and Scheme (10) versus the iteration numbers. It can be seen that Algorithm 4 is significantly faster than Scheme (10). This shows the effectiveness of our proposed algorithms.

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Code availability Not available.

Declarations

Competing interests The authors declare no competing interests.

References

1. Censor, Y., Elfving, T.: A multiprojection algorithm using bregman projections in a product space. Numer. Algorithms **8**(2), 221–239 (1994)
2. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**(2), 441 (2002)
3. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**(6), 2071 (2005)
4. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity-modulated radiation therapy. Phys. Med. Biol. **51**(10), 2353 (2006)
5. Censor, Y., Segal, A.: Iterative projection methods in biomedical inverse problems. Mathematical methods in biomedical imaging and intensity-modulated radiation therapy (IMRT) **10**, 65–96 (2008)
6. Wang, J., Hu, Y., Li, C., Yao, J.-C.: Linear convergence of CQ algorithms and applications in gene regulatory network inference. Inverse Probl. **33**(5), 055017 (2017)
7. Ansari, Q.H., Rehan, A.: Split feasibility and fixed point problems. In: Nonlinear Analysis, Springer, pp. 281–322 (2014)
8. Xu, H.-K.: Iterative methods for the split feasibility problem in infinite-dimensional hilbert spaces. Inverse Probl. **26**(10), 105018 (2010)
9. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**(1), 103 (2003)
10. Takahashi, W.: The split feasibility problem and the shrinking projection method in banach spaces. J. Nonlinear Convex Anal. **16**(7), 1449–1459 (2015)
11. Xu, H.-K.: A variable krasnosel'skii–mann algorithm and the multiple-set split feasibility problem. Inverse Probl. **22**(6), 2021 (2006)

12. Yang, Q.: The relaxed CQ algorithm solving the split feasibility problem. Inverse Probl. **20**(4), 1261 (2004)
13. Gibali, A., Mai, D.T., et al.: A new relaxed CQ algorithm for solving split feasibility problems in hilbert spaces and its applications. J. Ind. Manag. Optim. **15**(2), 963 (2019)
14. Sahu, D.R., Cho, Y.J., Dong, Q.L., Kashyap, M.R., Li, X.H.: Inertial relaxed CQ algorithms for solving a split feasibility problem in hilbert spaces. Numer. Algorithms 1–21 (2020)
15. López, G., Martín-Márquez, V., Wang, F., Xu, H.-K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Probl. **28**(8), 085004 (2012)
16. He, S., Zhao, Z.: Strong convergence of a relaxed CQ algorithm for the split feasibility problem. J. Inequal. Appl. **2013**(1), 197 (2013)
17. Yao, Y., Postolache, M., Liou, Y.-C.: Strong convergence of a self-adaptive method for the split feasibility problem. Fixed Point Theory and Applications **2013**(1), 201 (2013)
18. Dang, Y., Sun, J., Xu, H.: Inertial accelerated algorithms for solving a split feasibility problem. Journal of Industrial and Management Optimization **13**(3), 1383–1394 (2017)
19. Gibali, A., Liu, L.-W., Tang, Y.-C.: Note on the modified relaxation CQ algorithm for the split feasibility problem. Optim. Lett. **12**(4), 817–830 (2018)
20. Zhang, W., Han, D., Li, Z.: A self-adaptive projection method for solving the multiple-sets split feasibility problem. Inverse Probl. **25**(11), 115001 (2009)
21. Zhao, J., Yang, Q.: A simple projection method for solving the multiple-sets split feasibility problem. Inverse Probl. Sci. Eng. **21**(3), 537–546 (2013)
22. López, G., Martin, V., Xu, H.K., et al.: Iterative algorithms for the multiple-sets split feasibility problem. Biomed. Math.: Promising Directions in Imag, Therapy Planning Inverse Problems 243–279 (2009)
23. Dang, Y., Yao, J., Gao, Y.: Relaxed two points projection method for solving the multiple-sets split equality problem. Numer. Algorithms **78**(1), 263–275 (2018)
24. Iyiola, O.S., Shehu, Y.: A cyclic iterative method for solving multiple sets split feasibility problems in banach spaces. Quaest. Math. **39**(7), 959–975 (2016)
25. Shehu, Y.: Strong convergence theorem for multiple sets split feasibility problems in banach spaces. Numer. Funct. Anal. Optim. **37**(8), 1021–1036 (2016)
26. Suantai, S., Pholasa, N., Cholamjiak, P.: Relaxed CQ algorithms involving the inertial technique for multiple-sets split feasibility problems. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **113**(2), 1081–1099 (2019)
27. Wang, J., Hu, Y., Yu, C.K.W., Zhuang, X.: A family of projection gradient methods for solving the multiple-sets split feasibility problem. J. Optim. Theory Appl. **183**(2), 520–534 (2019)
28. Mewomo, O.T., Ogbuisi, F.U.: Convergence analysis of an iterative method for solving multiple-set split feasibility problems in certain banach spaces. Quaest. Math. **41**(1), 129–148 (2018)
29. Buong, N., Hoai, P.T.T., Binh, K.T.: Iterative regularization methods for the multiple-sets split feasibility problem in hilbert spaces. Acta Appl. Math. **165**(1), 183–197 (2020)
30. Yao, Y., Postolache, M., Zhu, Z.: Gradient methods with selection technique for the multiple-sets split feasibility problem. Optimization (2019)
31. Wang, X.: Alternating proximal penalization algorithm for the modified multiple-sets split feasibility problems. J. Inequal. Appl. **2018**(1), 1–8 (2018)
32. Taddele, G.H., Kumam, P., Gebrie, A.G., Sitthithakerngkiet, K.: Half-space relaxation projection method for solving multiple-set split feasibility problem. Math. Comput. Appl. **25**(3), 47 (2020)
33. Taddele, G.H., Kumam, P., Gebrie, A.G.: An inertial extrapolation method for multiple-set split feasibility problem. J. Inequal. Appl. **2020**(1), 1–22 (2020)
34. Sunthrayuth, P., Tuyen, T.M.: A generalized self-adaptive algorithm for the split feasibility problem in banach spaces. Bull. Iran. Math. Soc. 1–25 (2021)
35. Censor, Y., Segal, A.: The split common fixed point problem for directed operators. J. Convex Anal **16**(2), 587–600 (2009)
36. Moudafi, A.: The split common fixed-point problem for demicontractive mappings. Inverse Probl. **26**(5), 055007 (2010)
37. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms **59**(2), 301–323 (2012)
38. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal **13**(4), 759–775 (2012)

39. Tuyen, T.M., Ha, N.S., Thuy, N.T.T.: A shrinking projection method for solving the split common null point problem in banach spaces. Numer. Algorithms **81**(3), 813–832 (2019)
40. Reich, S., Tuyen, T.M.: Two projection algorithms for solving the split common fixed point problem. J. Optim. Theory Appl. **186**(1), 148–168 (2020)
41. Reich, S., Tuyen, T.M.: Two new self-adaptive algorithms for solving the split common null point problem with multiple output sets in hilbert spaces. J. Fixed Point Theory Appl. **23**(2), 1–19 (2021)
42. Reich, S., Tuyen, T.M.: Projection algorithms for solving the split feasibility problem with multiple output sets. J. Optim. Theory Appl. **190**(3), 861–878 (2021)
43. Reich, S., Truong, M.T., Mai, T.N.H.: The split feasibility problem with multiple output sets in hilbert spaces. Optim. Lett 1–19 (2020)
44. Bauschke, H.H., Combettes, P.L., et al.: Convex analysis and monotone operator theory in hilbert spaces, vol. 408. Springer, Berlin (2011)
45. Goebel, K., Simeon, R.: Uniform convexity, hyperbolic geometry, and nonexpansive mappings. Dekker (1984)
46. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**(4), 591–597 (1967)
47. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM journal on control and optimization **14**(5), 877–898 (1976)
48. Cholamjiak, P., Suantai, S., et al.: A new CQ algorithm for solving split feasibility problems in hilbert spaces. Bull. Malays. Math. Sci. Soc. **42**(5), 2517–2534 (2019)
49. Maingé, P.-E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Analysis **16**(7-8), 899–912 (2008)
50. Xu, H.-K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**(1), 240–256 (2002)
51. Fukushima, M.: A relaxed projection method for variational inequalities. Math. Program. **35**(1), 58–70 (1986)
52. Reich, S., Tuyen, T.M.: Iterative methods for solving the generalized split common null point problem in hilbert spaces. Optimization **69**(5), 1013–1038 (2020)
53. Reich, S., Tuyen, T.M., Ha, M.T.N.: An optimization approach to solving the split feasibility problem in hilbert spaces. J. Glob. Optim 1–16 (2020)
54. Cegielski, A.: Iterative methods for fixed point problems in hilbert spaces, vol. 2057. Springer, Berlin (2012)
55. Jeffs, B.D., Gunsay, M.: Restoration of blurred star field images by maximally sparse optimization. IEEE Trans. Image Process. **2**(2), 202–211 (1993)
56. Fevrier, I.J., Gelfand, S.B., Fitz, M.P.: Reduced complexity decision feedback equalization for multipath channels with large delay spreads. IEEE Trans. Commun. **47**(6), 927–937 (1999)
57. Duttweiler, D.L.: Proportionate normalized least-mean-squares adaptation in echo cancelers. IEEE Transactions on speech and audio processing **8**(5), 508–518 (2000)
58. Ramsey, J.B., Zhang, Z.: The application of wave form dictionaries to stock market index data. In: Predictability of complex dynamical systems, Springer, pp. 189–205 (1996)
59. Halmos, P.R.: A hilbert space problem book. Springer-Verlag, New York (1982)

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Weak and strong convergence results for solving monotone variational inequalities in reflexive Banach spaces

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ABSTRACT

In this paper, we introduce two modified Tseng's extragradient algorithms with a new generalized adaptive stepsize for solving monotone variational inequalities (VI) in reflexive Banach spaces. The advantage of our methods is that stepsizes do not require prior knowledge of the Lipschitz constant of the cost mapping. Based on Bregman projection-type methods, we prove weak and strong convergence of the proposed algorithms to a solution of VI. Some numerical experiments to show the efficiency of our methods including a comparison with related methods are provided.

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1. Introduction

Let E be a real Banach space with a norm $\|\cdot\|$ and E^* be a dual of E. We denote by $\langle x, f\rangle$ the value of $f \in E^*$ at $x \in E$, that is, $\langle x, f\rangle = f(x)$. Let C be a nonempty, closed and convex subset of E and $A : C \to E^*$ be a continuous mapping. The *variational inequalities* (VI) is to find a point $z \in C$ such that

$$\langle x - z, Az\rangle \geq 0 \quad \forall\, x \in C. \tag{1}$$

The solution set of VI (1) is denoted by $VI(C, A)$. Variational inequality theory is an important tool in physics, control theory, engineering, economics, management science, mathematical programming, and so on. Several iterative methods have been proposed for solving the variational inequalities. A classical method for solving the VI in a Hilbert space H is the *gradient projection method* which is given by

$$x_{n+1} = P_C(x_n - \lambda A x_n), \tag{2}$$

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where P_C is the projection operator onto the convex and closed subset C of H and $\lambda > 0$ is a suitable stepsize. However, the convergence of this method requires a strong (or inverse strong) monotonicity of A. To avoid this hypothesis, Korpelevich [1] and Antipin [2] proposed the following so-called *extragradient method* for solving VI in a finite-dimensional Euclidean space $\mathbb{R}^m$:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{3}$$

where $A : C \to \mathbb{R}^m$ is monotone and L-Lipschitz continuous and $\lambda \in (0, \frac{1}{L})$. Note that algorithm (3) involves two projections onto the set C per iteration. In order to reduce the number of evaluations of projection per iteration, Tseng [3] introduced the following method and later was known as *Tseng's extragradient method*:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n). \end{cases} \tag{4}$$

The weak convergence of this method was proved provided $\lambda \in (0, \frac{1}{L})$. It is remarkable that the Tseng's extragradient method (4) only needs to calculate one projection, which is simpler than the extragradient method. By these reasons, the Tseng's extragradient method has received great attention in various ways to obtain the weak and strong convergence of these methods. However, the convergence of (4) has been established only in Hilbert spaces (see, e.g. [4–11]). It was of great interest to extend Tseng's result to Banach spaces. Very recently, Shehu [12] first extend Tseng's result to a 2-uniformly convex Banach space E. He proposed the following algorithm:

$$\begin{cases} y_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\ x_{n+1} = J^{-1}(J y_n - \lambda_n(A y_n - A x_n)), \end{cases} \tag{5}$$

where $A : E \to E^*$ is monotone and L-Lipschitz continuous, Π_C is the generalized projection from E onto C, J is the normalized duality mapping on E. Also, he proposed the following algorithm which is a variant of (5) based on Halpern-type iteration:

$$\begin{cases} y_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\ z_n = J^{-1}(J y_n - \lambda_n(A y_n - A x_n)), \\ x_{n+1} = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J z_n). \end{cases} \tag{6}$$

He proved weak convergence and strong convergence theorems of algorithms (5) and (6), respectively, in a 2-uniformly convex Banach space under the sequences of stepsize satisfy the following inequality:

$$0 < a \le \lambda_n \le b < \frac{1}{\sqrt{2\mu\kappa L}}, \tag{7}$$

where $\mu > 0$ is the 2-uniform convexity constant of E and $\kappa > 0$ is the 2-uniform smoothness constant of E^*. However, the 2-uniform convex Banach spaces are too restrictive for applications in L_p(or ℓ_p) with $p > 2$. Moreover, the sequence of stepsize is chosen dependently of the Lipschitz constant of the mapping. From a practical point of view the Lipschitz constant is very difficult to estimate.

Question: Can we extend and modify the Shehu's methods (5) and (6) for solving the monotone VI in more general reflexive Banach spaces which stepsizes does not require prior estimates of the Lipschitz constants?

The purpose of this paper is to give an answer to the above question. We introduce two modified Tseng's extragradient algorithms with a new generalized adaptive stepsize for solving monotone VI in the framework of reflexive Banach spaces. The stepsizes of our methods are updated over each iteration by a cheap computation. This allows the algorithms to be computed more easily without the prior knowledge the Lipschitz constant. The weak and strong convergence of the proposed methods are establish under some suitable conditions.

Our paper is organized as follows: In Section 2, we present some preliminaries which will be needed in the sequel. In Section 3, we propose two algorithms and analyze their convergence. Finally, some numerical examples are provided in Section 4.

2. Preliminaries and lemmas

Throughout this paper, let E be a real reflexive Banach space with its dual E^* and $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function with its domain is denoted by $\mathrm{dom} f := \{x \in E : f(x) < \infty\}$. We use the notations $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence and weak convergence of the sequence $\{x_n\} \subset E$ to x, respectively. We also denote by $\langle x, j\rangle$ the value of functional $j \in E^*$ at $x \in E$. The *subdifferential* of f defined by

$$\partial f(x) := \{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \le f(y)\ \forall\, y \in E\}, \quad x \in E.$$

The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, \infty]$ defined by

$$f^*(x^*) := \sup_{x\in E}\{\langle x, x^*\rangle - f(x)\}.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x, x^*\rangle$ (see [13, Theorem 23.5]).

For any $x \in \mathrm{int}(\mathrm{dom} f)$ and $y \in E$, the *directional derivative* of f at x in the direction $y \in E$ is given by

$$f'(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{8}$$

The function f is said to be *Gâteaux differentiable* at x if the limit as $t \to 0$ in (8) exists for each y. In this case, the *gradient* of f at x is the linear function $\nabla f(x) : E \to E^*$ defined by $\langle y, \nabla f(x)\rangle = f'(x, y)$ for any $y \in E$. For more details about gradient, we recommend [14, Remark 3.32]. The function f is said to be

Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. It is known that if f is continuous at x and $\partial f(x)$ is single valued, then f is Gâteaux differentiable at x and $\nabla f(x) = \partial f(x)$ (see [15, Proposition 2.40]). The function f is said to be *Fréchet differentiable* at x if the limit (8) is attained uniformly in $\|y\| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (8) is attained uniformly for $x \in C$ and $\|y\| = 1$. We know that every Fréchet differentiable function is Gâteaux differentiable and if f is Fréchet differentiable, then it is continuous (see [16, p.142]). It is also known that if $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* (see [17, Proposition 1])

Definition 2.1: The function $f : E \to \mathbb{R}$ is said to be:

(1) *uniformly convex* with modulus ϕ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)\phi(\|x-y\|),$$

for all $x, y \in \operatorname{dom} f$ and $t \in (0,1)$, where ϕ is an increasing function vanishing only at 0;

(2) *strongly convex* with a constant $\sigma > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\sigma}{2}t(1-t)\|x-y\|^2 \tag{9}$$

for all $x, y \in \operatorname{dom} f$ and $t \in (0,1)$.

We know that f is uniformly convex if and only if f^* is Fréchet differentiable and ∇f^* is uniformly continuous (see [18, Theorem 3.5.10]). Obviously, f is strongly convex with a constant σ if and only if it is uniformly convex with modulus $\phi(t) = \frac{\sigma}{2}t^2$ and it is also equivalent to the following inequality (see [14, Theorem 5.24]):

$$f(x) \geq f(y) + \langle x-y, \nabla f(y)\rangle + \frac{\sigma}{2}\|x-y\|^2 \tag{10}$$

for all $x \in \operatorname{dom} f$ and $y \in \operatorname{int}(\operatorname{dom} f)$.

Definition 2.2 ([19, p.25]): The function $f : E \to (-\infty, \infty]$ is said to be *Legendre* if it satisfies the following two conditions:

(L1) $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f)$;

(L2) $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$.

Remark 2.3: In a reflexive Banach space, we always have $(\partial f)^{-1} = \partial f^*$ (see [20, p.83]). This fact, when combined with conditions (L1) and (L2), implies the following two facts:

(i) ∇f is a bijection from int(dom$f$) into int(domf^*) satisfying $\nabla f = (\nabla f^*)^{-1}$ (see [21, Theorem 5.10]);
(ii) ran∇f = dom∇f^* = int(domf^*) and ran∇f^* = dom∇f = int(dom$f$) (see [22, p.123]),

where ran∇f denotes the range of ∇f. Also, conditions (L1) and (L2), in conjunction with [21, Theorem 5.4], imply that the functions f and f^* are essentially strictly convex on the interior of their respective domains.

One important and interesting Legendre function is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. For more examples of Legendre functions, we recommend [21,23,24].

Definition 2.4 ([25]): Let $f : E \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function. The bifunction $D_f : \text{dom } f \times \text{int}(\text{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$$

is called the *Bregman distance* with respect to *f*.

The geometric of Bregman distance is shown in Figure 1. In particular, if E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$ for all

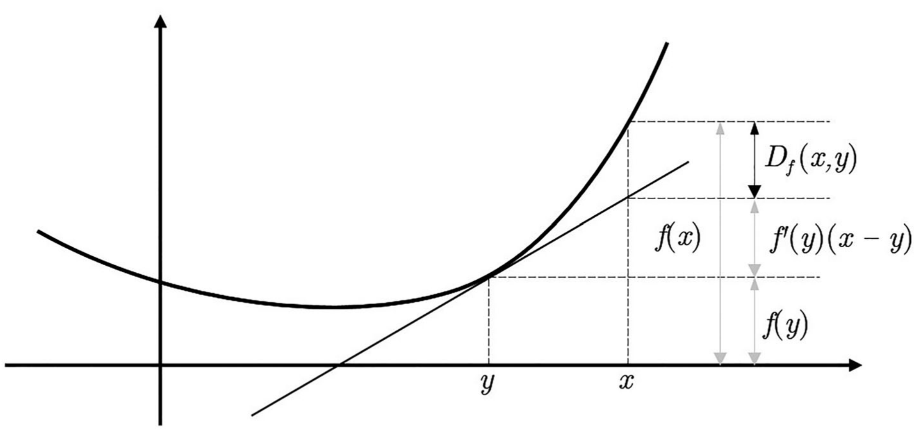

Figure 1. Bregman distance with respect to *f*.

$x \in E$, then $\nabla f(x) = Jx$, where J is the normalized duality mapping defined by

$$Jx := \{j \in E^* : \langle x, j\rangle = \|x\|^2 = \|j\|^2\}.$$

Then we have $D_f(x,y) = \frac{1}{2}\phi(x,y)$, where ϕ is called the *Lyapunov functional* which is defined by $\phi(x,y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ (see [26,27]). For a 2-uniformly convex and smooth Banach space E, the Lyapunov functional satisfies the following inequality:

$$\phi(x,y) \geq c\|x-y\|^2, \tag{11}$$

where $c > 0$ is the 2-uniformly convexity constant of E (see [28, Lemma 2.3]). It is well known that $\phi(x,y) = \|x-y\|^2$ and $c = 1$ whenever E is a Hilbert space.

From the definition of Bregman distance and (10), it is clear that

$$D_f(x,y) \geq \frac{\sigma}{2}\|x-y\|^2. \tag{12}$$

The following important properties follow from the definition of Bregman distance:

(i) (two-point indentity) for any $x, y \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(x,y) + D_f(y,x) = \langle x-y, \nabla f(x) - \nabla f(y)\rangle; \tag{13}$$

(ii) (three-point indentity) for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(x,y) = D_f(x,z) - D_f(y,z) + \langle x-y, \nabla f(z) - \nabla f(y)\rangle. \tag{14}$$

The *modulus of total convexity* of f at $x \in \operatorname{int}(\operatorname{dom} f)$ is the function $v_f : \operatorname{int}(\operatorname{dom} f) \times [0,\infty) \to [0,\infty]$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f, \|y-x\| = t\}.$$

The function f is called *totally convex* at x if $v_f(x,t) > 0$ whenever $t > 0$. It is called *totally convex* if it is totally convex at any point $x \in \operatorname{int}(\operatorname{dom} f)$. The *modulus of total convexity* of the function f on the set X is the function $v_f : \operatorname{int}(\operatorname{dom} f) \times [0,\infty) \to [0,\infty]$ defined by

$$v_f(X,t) := \inf\{v_f(x,t) : x \in X \cap \operatorname{dom} f\}.$$

The function f is said to be *totally convex on bounded sets* of E if $v_f(X,t) > 0$ for any nonempty bounded subset X of E and $t > 0$. It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [29, Theorem 2.10]).

The *Bregman projection* with respect to f of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed and convex set $C \subset \operatorname{dom} f$ is the minimizer over C defined by

$$\Pi_C^f(x) := \operatorname{argmin}\{D_f(y,x) : y \in C\}.$$

The geometric of Bregman projection is shown in Figure 2. If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then

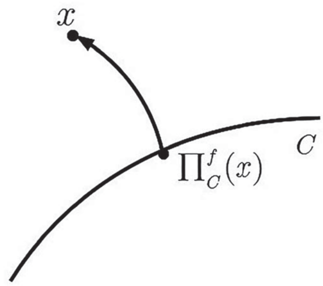

Figure 2. Bregman projection with respect to *f*.

Π_C^f coincides with the generalized projection Π_C (see [30, Definition 7.2]) and if E is a Hilbert space, then Π_C^f coincides the metric projection P_C.

Lemma 2.5 ([29, Corollary 4.4]): *Suppose that f is Gâteaux differentiable and totally convex on* int(dom$f$). *Let $x \in$* int(dom$f$) *and let C be a nonempty, closed and convex subset of* int(dom$f$). *If $z \in C$, then the following statements are equivalent:*

(i) $z = \Pi_C^f(x)$ *is the Bregman projection of x onto C with respect to f;*
(ii) *z is the unique solution of the following variational inequality:*

$$\langle y - z, \nabla f(x) - \nabla f(z) \rangle \leq 0 \quad \forall\, y \in C;$$

(iii) *z is the unique solution of the following inequality:*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \quad \forall\, y \in C.$$

Let $f : E \to \mathbb{R}$ be a Legendre function. We define the function $V_f : E \times E^* \to [0, \infty)$ associated with f by

$$V_f(x, x^*) := f(x) - \langle x, x^* \rangle + f^*(x^*) \quad \forall\, x \in E, \quad x^* \in E^*.$$

From [31, Proposition 1], we know the following properties:

(i) V_f is nonnegative and convex in the second variable;
(ii) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))\ \ \forall x \in E,\ x^* \in E^*$;
(iii) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)\ \ \forall x \in E,\ x^*, y^* \in E^*$;
(iv) $D_f(x, \nabla f^*(t\nabla f(y) + (1-t)\nabla f(z))) \leq tD_f(x, y) + (1-t)D_f(x, z)$ for all $t \in [0, 1]$ and for all $x, y, z \in E$.

Recall that the function f is called *sequentially consistent* [29, p.9], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in dom f and int(dom$f$), respectively, such that the first one is bounded and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6 ([32, Lemma 2.1.2]): *The function $f : E \to (-\infty, \infty]$ is sequentially consistent if and only if it is totally convex on bounded subsets of E.*

From Lemma 2.6, if f is additionally assumed to be Fréchet differentiable which is bounded on bounded subsets of E, then for any two sequences $\{x_n\}$ and $\{y_n\}$ in E,

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Longrightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0 \Longrightarrow \lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0.$$

Definition 2.7: Let C be a nonempty subset of E. Recall that a mapping $A : C \to E^*$ is said to be:

(1) *monotone* if $\langle x - y, Ax - Ay\rangle \geq 0$ for all $x, y \in C$;
(2) *Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$;
(3) *weakly sequentially continuous* if for any sequence $\{x_n\} \subset C$, $x_n \rightharpoonup x$ implies that $Ax_n \rightharpoonup^* Ax$;
(4) *hemicontinuous* if for each $x, y \in C$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) := A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Let $A : E \to 2^{E^*}$ be a multi-valued mapping. The domain of A is denoted by $D(A) := \{x \in E : Ax \neq \emptyset\}$ and the set of zeros of A is denoted by $A^{-1}0 := \{x \in D(A) : 0 \in Ax\}$. The graph of A is denoted by $G(A) := \{(x, y) \in E \times E^* : x \in D(A), y \in Ax\}$. A multi-valued mapping A is called *monotone* if

$$\langle x - y, u - v\rangle \geq 0 \quad \forall\, (x, u), (y, v) \in G(A).$$

A monotone mapping A on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone mapping on E. If A is maximal monotone, then $A^{-1}0$ is closed and convex.

The following lemma can be found in [33] (see also [34, Theorem 2.9]).

Lemma 2.8: *Let C be a nonempty, closed convex subset of a Banach space E. Let $A : C \to E^*$ be a monotone and hemicontinuous operator and $T : E \to 2^{E^*}$ be an operator defined as follows:*

$$Tx := \begin{cases} Ax + N_C(x) & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

where $N_C(x)$ is the normal cone for C at a point $x \in C$, that is,

$$N_C(x) := \{u \in E^* : \langle y - x, u\rangle \leq 0\ \forall\, y \in C\}.$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.9 ([35, Theorem 7.1.8]): *Let C be a nonempty, closed and convex subset of a Banach space E. Let A be a monotone and hemicontinuous mapping of C into E^*. Then $VI(C, A)$ is nonempty, closed and convex.*

Lemma 2.10 ([31, Proposition 9]): *Let $f : E \to \mathbb{R}$ be a Legendre function such that ∇f is weakly sequentially continuous. Suppose that the sequence $\{x_n\}$ is bounded and that $\lim_{n\to\infty} D_f(u, x_n)$ exists for any weak subsequential limit u of $\{x_n\}$. Then $\{x_n\}$ converges weakly to u.*

Lemma 2.11 ([36, Lemma 2.5]): *Assume that $\{a_n\}$ is a nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 ([37, Lemma 3.1]): *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k := \max\{j \leq k : a_j \leq a_{j+1}\}$.

Lemma 2.13 ([38, Lemma 7]): *Assume that $\{\Gamma_n\}$ is a nonnegative real sequence such that*

$$\Gamma_{n+1} \leq (1 - \delta_n)\Gamma_n + \delta_n\tau_n$$

and

$$\Gamma_{n+1} \leq \Gamma_n - \eta_n + \rho_n,$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$, $\{\eta_n\}$ is a nonnegative real sequence and $\{\tau_n\}$, and $\{\rho_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(ii) $\lim_{n\to\infty} \rho_n = 0$;
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$, *implies* $\lim_{k\to\infty} \sup \tau_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then $\lim_{n\to\infty} \Gamma_n = 0$.

3. Main results

In this section, we propose two Bregman projection-type methods for solving monotone VI in reflexive Banach spaces. In order to prove convergence results of our methods, we make the following assumption:

Assumption 3.1: (i) *The feasible set C is a closed and convex subset of a real reflexive Banach space E.*

(ii) *The function $f : E \to \mathbb{R}$ is σ-strongly convex, Legendre which is bounded and uniformly Fréchet differentiable on bounded subsets of E.*

(iii) *The mapping $A : E \to E^*$ is monotone and Lipschitz continuous with a constant $L > 0$.*

(iv) *The solution set of VI is nonempty, that is, $\Omega := VI(C, A) \neq \emptyset$.*

3.1. Weak convergence

In this subsection, we propose a Bregman projection-type method for solving VI in reflexive Banach spaces which is constructed based on Tseng's extragradient method. The algorithm is shown as below.

Remark 3.2: In view of Lemma 2.5, if Algorithm 1 stops in the n-th step of iterations, then y_n is a solution of VI. In what follows, we assume that the Algorithm 1 does not stop in any finite iterations and generates an infinite sequence $\{x_n\}$.

Lemma 3.3: *Let $\{\lambda_n\}$ be a sequence generated by* (15). *Then there exists $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \theta]$ such that $\lambda = \lim_{n\to\infty} \lambda_n$, where $\theta = \sum_{n=1}^{\infty} \theta_n$.*

Proof: Using the fact that A is Lipschitz-continuous with $L > 0$, in the case of $\langle x_{n+1} - y_n, Ax_n - Ay_n\rangle > 0$, we obtain

$$\frac{\epsilon_n \mu(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\langle x_{n+1} - y_n, Ax_n - Ay_n\rangle} \geq \frac{2\epsilon_n \mu \|x_n - y_n\| \|x_{n+1} - y_n\|}{2\|x_{n+1} - y_n\| \|Ax_n - Ay_n\|}$$
$$\geq \frac{\mu \|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

The rest of the proof is same as in [7, Lemma 3.1]. ∎

Remark 3.4: It is observe that the stepsize λ_n generated in Algorithm 1 is large which allowed to increase when the iteration increases. Therefore, the use of this new stepsize reduces the dependence on the initial stepsize λ_1. In particular, if $\epsilon_n = 1$ for all $n \geq 1$, then the stepsize λ_n generated in Algorithm 1 is similar to the methods in [7,39] and if $\epsilon_n = 1$ and $\theta_n = 0$ for all $n \geq 1$, then the stepsize λ_n generated in Algorithm 1 is similar to the methods in [10,40–42].

Algorithm 1:

Initialization: Choose $\lambda_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (12). Choose sequences $\{\theta_n\}$ and $\{\epsilon_n\}$ satisfy the following conditions:

(1) $\{\theta_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$;

(2) $\{\epsilon_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 1$.

Let $x_1 \in E$ be arbitrary.

Iterative steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = \Pi_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A x_n).$$

If $x_n = y_n$, then stop and y_n is a solution of VI. Otherwise,

Step 2. Compute

$$x_{n+1} = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\epsilon_n\mu \dfrac{\|x_{n+1} - y_n\|^2 + \|x_n - y_n\|^2}{2\langle x_{n+1} - y_n, Ax_n - Ay_n\rangle}, \lambda_n + \theta_n\right\} & \text{if } \langle x_{n+1} - y_n, Ax_n - Ay_n\rangle > 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{15}$$

Set $n := n + 1$ go to *Step 1*.

Lemma 3.5: *Assume that Assumption 3.1 is satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then for all $p \in \Omega$ and $n \geq 1$, we have*

$$D_f(p, x_{n+1}) \leq D_f(p, x_n) - \left(1 - \frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) - \left(1 - \frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(x_{n+1}, y_n).$$

Proof: Let $p \in \Omega$. Then we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n))) \\ &= f(p) - f(x_{n+1}) - \langle p - x_{n+1}, \nabla f(y_n) - \lambda_n(Ay_n - Ax_n)\rangle \end{aligned}$$

$$
\begin{aligned}
&= f(p) - f(x_{n+1}) - \langle p - x_{n+1}, \nabla f(y_n)\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle + \langle p - y_n, \nabla f(y_n)\rangle \\
&\quad + f(y_n) - f(x_{n+1}) - \langle p - x_{n+1}, \nabla f(y_n)\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle - f(x_{n+1}) + f(y_n) \\
&\quad + \langle x_{n+1} - y_n, \nabla f(y_n)\rangle + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= D_f(p, y_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle. \qquad (16)
\end{aligned}
$$

From the three-point identity of D_f, we have

$$
D_f(p, y_n) = D_f(p, x_n) - D_f(y_n, x_n) + \langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle. \qquad (17)
$$

Substituting (17) into (16), we have

$$
\begin{aligned}
D_f(p, x_{n+1}) = D_f(p, x_n) &- D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\
&+ \langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \\
&+ \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle. \qquad (18)
\end{aligned}
$$

Note that $y_n = \Pi_C^f \nabla f^*(\nabla f(x_n) - \lambda_n Ax_n)$, by Lemma 2.5, we have

$$
\langle p - y_n, \nabla f(x_n) - \lambda_n Ax_n - \nabla f(y_n)\rangle \leq 0.
$$

This implies that

$$
\langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \leq \lambda_n \langle p - y_n, Ax_n\rangle. \qquad (19)
$$

Substituting (19) into (18), we get

$$
\begin{aligned}
D_f(p, x_{n+1}) &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle p - y_n, Ax_n\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle p - y_n, Ax_n\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n\rangle - \lambda_n \langle p - x_{n+1}, Ax_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle x_{n+1} - y_n, Ax_n\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle x_{n+1} - y_n, Ax_n\rangle \\
&\quad - \lambda_n \langle y_n - p, Ay_n\rangle + \lambda_n \langle y_n - x_{n+1}, Ay_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\
&\quad + \lambda_n \langle x_{n+1} - y_n, Ax_n - Ay_n\rangle - \lambda_n \langle y_n - p, Ay_n\rangle. \qquad (20)
\end{aligned}
$$

Since $y_n \in C$ and $p \in \Omega$, we have $\langle y_n - p, Ap\rangle \geq 0$ and the monotonicity of A implies that $\langle y_n - p, Ay_n\rangle \geq 0$. From (15), we see that

$$\begin{aligned}\lambda_{n+1} &= \min\left\{\epsilon_n\mu\frac{\|x_{n+1}-y_n\|^2+\|x_n-y_n\|^2}{2\langle x_{n+1}-y_n, Ax_n-Ay_n\rangle}, \lambda_n+\theta_n\right\}\\ &\leq \epsilon_n\mu\frac{\|x_{n+1}-y_n\|^2+\|x_n-y_n\|^2}{2\langle x_{n+1}-y_n, Ax_n-Ay_n\rangle}.\end{aligned}$$

This implies that

$$\langle x_{n+1}-y_n, Ax_n-Ay_n\rangle \leq \frac{\epsilon_n\mu}{2\lambda_{n+1}}\|x_{n+1}-y_n\|^2+\frac{\epsilon_n\mu}{2\lambda_{n+1}}\|x_n-y_n\|^2. \quad (21)$$

Then from (20), (21) and (12), we have

$$\begin{aligned}D_f(p,x_{n+1}) &\leq D_f(p,x_n)-D_f(y_n,x_n)-D_f(x_{n+1},y_n)\\ &\quad+\lambda_n\langle x_{n+1}-y_n, Ax_n-Ay_n\rangle\\ &\leq D_f(p,x_n)-D_f(y_n,x_n)-D_f(x_{n+1},y_n)\\ &\quad+\frac{\epsilon_n\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}\|x_{n+1}-y_n\|^2+\frac{\epsilon_n\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}\|x_n-y_n\|^2\\ &\leq D_f(p,x_n)-D_f(y_n,x_n)-D_f(x_{n+1},y_n)\\ &\quad+\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}D_f(x_{n+1},y_n)+\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}D_f(y_n,x_n)\\ &= D_f(p,x_n)-\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)\\ &\quad-\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(x_{n+1},y_n).\end{aligned}$$

Thus this lemma is proved. ■

Theorem 3.6: *Assume that Assumption 3.1 is satisfied. Suppose, in addition, that ∇f is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in Ω.*

Proof: We first prove that $\{x_n\}$ is bounded. Since $\lim_{n\to\infty}\lambda_n = \lambda > 0$, $\lim_{n\to\infty}\epsilon_n = 1$ and $\mu \in (0,\sigma)$, we have

$$\lim_{n\to\infty}\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) = 1-\frac{\mu}{\sigma} = \frac{\sigma-\mu}{\sigma} > 0.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}} > 0 \quad \forall\, n \geq n_0.$$

This implies that

$$\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)+\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(x_{n+1},y_n)\geq 0 \quad \forall\, n\geq n_0.$$

Then from Lemma 3.5, we have

$$D_f(p,x_{n+1})\leq D_f(p,x_n) \quad \forall\, n\geq n_0.$$

This show that $\lim_{n\to\infty} D_f(p,x_n)$ exists and hence $\{D_f(p,x_n)\}$ is bounded. Applying Lemma 12, we have $\{x_n\}$ is bounded and, in consequence $\{y_n\}$ is bounded. On the other hand, from Lemma 3.5, we see that

$$\begin{aligned}&\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)+\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(x_{n+1},y_n)\\ &\leq D_f(p,x_n)-D_f(p,x_{n+1}).\end{aligned}$$

Since $\lim_{n\to\infty} D_f(p,x_n)$ exists, there exists a nonnegative constant γ such that $\gamma=\lim_{n\to\infty} D_f(p,x_n)=\lim_{n\to\infty} D_f(p,x_{n+m})$ for all $m\in\mathbb{N}$. This implies that $\lim_{n\to\infty} D_f(y_n,x_n)=\lim_{n\to\infty} D_f(x_{n+1},y_n)=0$. Thus we have

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{22}$$

By the reflexivity of E and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\rightharpoonup z\in E$. Since $\|x_n-y_n\|\to 0$ and $y_n\in C$, we have $y_{n_k}\rightharpoonup z\in C$. We next prove $z\in\Omega$. Let T be a mapping defined as in Lemma 2.8. Then T is maximal monotone and $T^{-1}0=\Omega$. Let $(v,w)\in G(T)$. Since $w\in Tv=Av+N_C(v)$, we have $w-Av\in N_C(v)$ and hence $\langle v-x,w-Av\rangle\geq 0$ for all $x\in C$. Again, since $y_n\in C$, we have

$$\langle v-y_n,w-Av\rangle\geq 0. \tag{23}$$

From the definition of y_n and Lemma 2.5, we have $\langle v-y_n,\nabla f(x_n)-\lambda_nAx_n-\nabla f(y_n)\rangle\leq 0$ or, equivalently,

$$\left\langle v-y_n,\frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}-Ax_n\right\rangle\leq 0. \tag{24}$$

Then from (23) and (24), we have

$$\begin{aligned}\langle v-y_n,w\rangle &\geq \langle v-y_n,Av\rangle\\ &\geq \langle v-y_n,Av\rangle+\left\langle v-y_n,\frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}-Ax_n\right\rangle\\ &=\langle v-y_n,Av-Ax_n\rangle+\left\langle v-y_n,\frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}\right\rangle\\ &=\langle v-y_n,Av-Ay_n\rangle+\langle v-y_n,Ay_n-Ax_n\rangle\end{aligned}$$

$$+\left\langle v-y_n, \frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}\right\rangle$$
$$\geq \langle v-y_n, Ay_n-Ax_n\rangle+\left\langle v-y_n, \frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}\right\rangle.$$

Since A is Lipschitz continuous and from (22), we have $\langle v-z, w\rangle \geq 0$. By the maximality of T, we obtain $z \in T^{-1}0$ and hence $z \in \Omega$. In summary, we have shown that $\lim_{n\to\infty} D_f(z, x_n)$ exists for any weak subsequential limit z of $\{x_n\}$. Thus by Lemma 2.10, we conclude $\{x_n\}$ converges weakly to a point in Ω. The proof is completed. ■

If E is a 2-uniformly convex and uniformly smooth Banach space, and $f(x)=\frac{1}{2}\|x\|^2$, then we have the following weak convergence result.

Corollary 3.7: *Let C be a nonempty, closed and convex subset of E, $A: E\to E^*$ be a monotone and Lipschitz continuous with a constant $L>0$. Assume that $\Omega\neq\emptyset$. Choose $\lambda_1>0$ and $\mu\in(0,c)$, where c is a constant given by (11). Choose a real sequence $\{\theta_n\}\subset[0,\infty)$ such that $\sum_{n=1}^{\infty}\theta_n<\infty$ and a real sequence $\{\epsilon_n\}\subset[1,\infty)$ such that $\lim_{n\to\infty}\epsilon_n=1$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n=\Pi_C J^{-1}(Jx_n-\lambda_n Ax_n), \\ x_{n+1}=J^{-1}(Jy_n-\lambda_n(Ay_n-Ax_n)) \quad \forall\, n\geq 1, \end{cases}$$

where

$$\lambda_{n+1}=\begin{cases} \min\left\{\epsilon_n\mu\dfrac{\|x_{n+1}-y_n\|^2+\|x_n-y_n\|^2}{2\langle x_{n+1}-y_n, Ax_n-Ay_n\rangle},\right. \\ \qquad \left.\lambda_n+\theta_n\right\} & \text{if } \langle x_{n+1}-y_n, Ax_n \\ & \quad -Ay_n\rangle>0, \\ \lambda_n+\theta_n & \text{otherwise.} \end{cases}$$

Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ converges weakly to a point in Ω.

3.2. Strong convergence

In this subsection, we propose another Bregman projection-type method for solving VI in reflexive Banach spaces, which is constructed based on Tseng's extragradient method and Halpern-type iteration.

Theorem 3.8: *Suppose that Assumption 3.1 is satisfied. If $\{\alpha_n\}\subset(0,1)$ with $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$, then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z=\Pi_\Omega^f(u)$.*

Algorithm 2:

Initialization: Choose $\lambda_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (12). Choose sequences $\{\theta_n\}$ and $\{\epsilon_n\}$ satisfy the following conditions:

(1) $\{\theta_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$;

(2) $\{\epsilon_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 1$.

Let $u, x_1 \in E$ be arbitrary.

Iterative steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = \Pi_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A x_n).$$

If $x_n = y_n$, then stop and y_n is a solution of VI. Otherwise,

Step 2. Compute

$$z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)).$$

Step 3. Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(z_n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\epsilon_n \mu \dfrac{\|z_n - y_n\|^2 + \|x_n - y_n\|^2}{2\langle z_n - y_n, Ax_n - Ay_n\rangle}, \right. \\ \left. \quad \lambda_n + \theta_n \right\} & \text{if } \langle z_n - y_n, Ax_n \\ & \quad -Ay_n\rangle > 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ go to *Step 1*.

Proof: We first prove that $\{x_n\}$ is bounded. Using the same arguments as in the proof of Lemma 3.5, we can show that

$$D_f(p, z_n) \le D_f(p, x_n) - \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) - \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n). \tag{25}$$

Moreover, as proved in Theorem 3.6, we can deduce that

$$\left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) + \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n) \ge 0 \quad \forall\, n \ge n_0.$$

Then from (25), we have

$$D_f(p, z_n) \leq D_f(p, x_n) \quad \forall\, n \geq n_0. \tag{26}$$

Hence

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, z_n) \\
&\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, x_n) \\
&\leq \max\{D_f(z, u), D_f(z, x_n)\} \\
&\quad \vdots \\
&\leq \max\{D_f(z, u), D_f(z, x_{n_0})\}.
\end{aligned}$$

This implies that $\{D_f(p, x_n)\}$ is bounded. Applying (12), we have $\{x_n\}$ is bounded and, in consequence $\{y_n\}$ and $\{z_n\}$ are bounded. Let $z = \Pi^f_\Omega(u)$. From (25), we have

$$\begin{aligned}
D_f(z, x_{n+1}) &\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, z_n) \\
&\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, x_n) \\
&\quad - (1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) \\
&\quad - (1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n).
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) + (1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n) \\
&\quad \leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, z_n) - D_f(z, x_{n+1}) \\
&\quad = D_f(z, x_n) - D_f(z, x_{n+1}) + \alpha_n (D_f(z, u) - D_f(z, x_n)) \\
&\quad \leq D_f(z, x_n) - D_f(z, x_{n+1}) + \alpha_n M,
\end{aligned} \tag{27}$$

where $M = \sup_{n\geq 1}\{|D_f(z, u) - D_f(z, x_n)|\}$.

We now consider the following two possible cases to prove $x_n \to z$.

Case 1. Suppose that there exists $N \in \mathbb{N}$ such that $\{D_f(z, x_n)\}$ is nonincreasing for all $n \geq N$. From this we have $\lim_{n\to\infty} D_f(z, x_n)$ exists and hence $\lim_{n\to\infty}(D_f(z, x_n) - D_f(z, x_{n+1})) = 0$. Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}(1 - \frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}) > 0$ and from (27), we have $\lim_{n\to\infty} D_f(y_n, x_n) = \lim_{n\to\infty} D_f(z_n, y_n) = 0$. Hence

$$\lim_{n\to\infty}\|\nabla f(y_n) - \nabla f(x_n)\| = \lim_{n\to\infty}\|\nabla f(z_n) - \nabla f(y_n)\| = 0. \tag{28}$$

We also have

$$\|\nabla f(z_n) - \nabla f(x_n)\| \leq \|\nabla f(z_n) - \nabla f(y_n)\| + \|\nabla f(y_n) - \nabla f(x_n)\|$$

$$\to 0, \quad n \to \infty. \tag{29}$$

On the other hand, we observe that

$$\begin{aligned}\|\nabla f(x_{n+1}) - \nabla f(x_n)\| &\leq \|\nabla f(x_{n+1}) - \nabla f(z_n)\| + \|\nabla f(z_n) - \nabla f(x_n)\| \\ &= \alpha_n \|\nabla f(u) - \nabla f(z_n)\| + \|\nabla f(z_n) - \nabla f(x_n)\|.\end{aligned}$$

Again, since $\lim_{n\to\infty} \alpha_n = 0$ and from (29), we have

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0.$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{30}$$

By the reflexivity of E and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$ as $k \to \infty$ and

$$\limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle = \lim_{k\to\infty} \langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle.$$

Since $\|\nabla f(y_n) - \nabla f(x_n)\| \to 0$, we have $\|y_n - x_n\| \to 0$. Further, by the same arguments as in the proof of Theorem 3.6, we can show that $\hat{x} \in \Omega$. Then from Lemma 2.5, we have

$$\limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle = \langle \hat{x} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0. \tag{31}$$

From (30) and (31), we have

$$\begin{aligned}&\limsup_{n\to\infty} \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\quad \leq \limsup_{n\to\infty} \langle x_{n+1} - x_n, \nabla f(u) - \nabla f(z)\rangle \\ &\qquad + \limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.\end{aligned} \tag{32}$$

From the properties of V_f and (26), we have

$$\begin{aligned}D_f(z, x_{n+1}) &= V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)) \\ &\leq V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n) - \alpha_n(\nabla f(u) - \nabla f(z))) \\ &\quad + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &= V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(z_n)) \\ &\quad + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &= D_f(z, \nabla f^*(\alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(z_n)) \\ &\quad + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\leq \alpha_n D_f(z, z) + (1-\alpha_n) D_f(z, z_n) + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle\end{aligned}$$

$$\leq (1-\alpha_n)D_f(z,x_n) + \alpha_n\langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle. \tag{33}$$

From (32) and Lemma 2.11, we conclude $\lim_{n\to\infty} D_f(z,x_n) = 0$. Therefore, $x_n \to z$.

Case 2. Suppose that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $D_f(z,x_{n_j}) < D_f(z,x_{n_j+1})$ for all $j \in \mathbb{N}$. Then by Lemma 2.12, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$D_f(z,x_{m_k}) \leq D_f(z,x_{m_k+1}) \quad \text{and} \quad D_f(z,x_k) \leq D_f(z,x_{m_k+1}).$$

From (27), we note that

$$\begin{aligned}
&(1-\alpha_{m_k})\left(1 - \frac{\epsilon_{m_k}\mu}{\sigma}\frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right) D_f(y_{m_k}, x_{m_k}) \\
&\quad + (1-\alpha_{m_k})\left(1 - \frac{\epsilon_{m_k}\mu}{\sigma}\frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right) D_f(z_{m_k}, y_{m_k}) \\
&\leq D_f(z,x_{m_k}) - D_f(z,x_{m_k+1}) + \alpha_{m_k}M \\
&\leq \alpha_{m_k}M,
\end{aligned} \tag{34}$$

where $M > 0$. Thus we have

$$\lim_{k\to\infty} D_f(y_{m_k}, x_{m_k}) = \lim_{k\to\infty} D_f(z_{m_k}, y_{m_k}) = 0.$$

Following the line in the proof of Case 1, we can show that

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty}\langle x_{m_k+1} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Moreover, we can show that

$$\begin{aligned}
D_f(z,x_{m_k+1}) &\leq (1-\alpha_{m_k})D_f(z,x_{m_k}) + \alpha_{m_k}\langle x_{m_k+1} - z, \nabla f(u) - \nabla f(z)\rangle \\
&\leq (1-\alpha_{m_k})D_f(z,x_{m_k+1}) + \alpha_{m_k}\langle x_{m_k+1} - z, \nabla f(u) - \nabla f(z)\rangle.
\end{aligned}$$

This implies that

$$D_f(z,x_k) \leq D_f(z,x_{m_k+1}) \leq \langle x_{m_k+1} - z, \nabla f(u) - \nabla f(z)\rangle.$$

From this, we have $\limsup_{k\to\infty} D_f(z,x_k) = 0$. Therefore, $x_k \to z$. The proof is completed. ■

Remark 3.9: Another proof of Theorem 3.8 can be used Lemma 2.13. As proved in Theorem 3.8 and from (25), we have

$$\begin{aligned} D_f(z,x_{n+1}) \le (1-\alpha_n)D_f(z,x_n) - (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n) \\ - (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(z_n,y_n) \\ + \alpha_n\langle x_{n+1}-z, \nabla f(u)-\nabla f(z)\rangle. \end{aligned} \tag{35}$$

For each $n \ge 1$, let

$$\begin{aligned} \Gamma_n &:= D_f(z,x_n), \\ \tau_n &:= \langle x_{n+1}-z, \nabla f(u)-\nabla f(z)\rangle, \\ \eta_n &:= (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n) \\ &\quad + (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(z_n,y_n) \end{aligned}$$

and

$$\rho_n := \alpha_n\langle x_{n+1}-z, \nabla f(u)-\nabla f(z)\rangle.$$

Then (35) reduces to the following inequalities:

$$\Gamma_{n+1} \le (1-\alpha_n)\Gamma_n + \alpha_n\tau_n, \quad n \ge 1$$

and

$$\Gamma_{n+1} \le \Gamma_n - \eta_n + \rho_n, \quad n \ge 1.$$

Since $\lim_{n\to\infty}\alpha_n = 0$, we have $\lim_{n\to\infty}\rho_n = 0$. In order to complete the proof, it is sufficient to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \le 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. Hence $\lim_{k\to\infty} D_f(y_{n_k},x_{n_k}) = \lim_{k\to\infty} D_f(z_{n_k},y_{n_k}) = 0$ and so

$$\lim_{k\to\infty}\|\nabla f(y_{n_k})-\nabla f(x_{n_k})\| = \lim_{k\to\infty}\|\nabla f(z_{n_k})-\nabla f(y_{n_k})\| = 0.$$

Then we can show that

$$\lim_{k\to\infty}\|\nabla f(z_{n_k})-\nabla f(x_{n_k})\| = \lim_{k\to\infty}\|\nabla f(x_{n_k+1})-\nabla f(x_{n_k})\| = 0$$

and

$$\limsup_{k\to\infty}\langle x_{n_k+1}-z, \nabla f(u)-\nabla f(z)\rangle \le 0.$$

Hence $\limsup_{k\to\infty}\tau_{n_k} \le 0$. We conclude by Lemma 2.13 that $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, $x_n \to z$ as $n \to \infty$.

If E is a 2-uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have the following strong convergence result.

Corollary 3.10: *Let C be a nonempty, closed and convex subset of E, $A : E \to E^*$ be a monotone and Lipschitz continuous with a constant $L > 0$. Assume that $\Omega \neq \emptyset$. Choose $\lambda_1 > 0$ and $\mu \in (0, c)$, where c is a constant given by (11). Choose a real sequence $\{\theta_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$ and a real sequence $\{\epsilon_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 1$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u, x_1 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n) \quad \forall\, n \geq 1, \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\epsilon_n \mu \dfrac{\|z_n - y_n\|^2 + \|x_n - y_n\|^2}{2\langle z_n - y_n, Ax_n - Ay_n\rangle}, \; \lambda_n + \theta_n\right\} & \text{if } \langle z_n - y_n, Ax_n - Ay_n\rangle > 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases}$$

If $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ converges strongly to a point in Ω.

Remark 3.11: Our results generalize and improve the main results of Shehu [12] in the following senses:

- We generalize the Shehu's results [12] from 2-uniformly convex Banach spaces to reflexive Banach spaces.
- For the structure of Banach spaces, we generalize duality mapping in Shehu's results [12] to a differentiable of Legendre-type function.
- The sequence of stepsizes of our methods is chosen without the prior knowledge of the Lipschitz constant of the mapping and without the uniform smoothness constant of E^* while the sequence of stepsizes of Sehu's results [12] requires the prior knowledge of them.

4. Numerical examples

In this section, we perform some numerical experiments to illustrate the numerical behaviour of Algorithms 1 and 2 and compare the proposed algorithms with Algorithms 5 and 6. For our experiments, we take $\theta_n = \frac{1000}{(n+1)^{1.1}}$ and $\epsilon_n =$

Table 1. Numerical results for Example 4.1.

		Algorithm 1	Algorithm 5	Algorithm 2	Algorithm 6
$m = 100$	Iter.	25	795	25	23599
	Time (s)	0.03	0.02	0.01	0.50
$m = 1000$	Iter.	22	383	22	23541
	Time (s)	1.20	2.09	0.93	73.6

$1 + \frac{1000}{(n+1)^{1.1}}$ in Algorithms 1 and 2. To terminate the algorithms, we use the condition $\|x_n - y_n\| \leq \varepsilon$ for all the algorithms. In what follows, we use 'Iter.' and 'Time (s)' to denote the number of iterations and the CPU time in seconds, respectively.

Example 4.1: We consider the HpHard problem which is taken from [43]. Let A be an operator defined by $Ax = Mx + q$ with $q \in \mathbb{R}^m$ and

$$M := NN^T + S + D,$$

where every entry of the $m \times m$ matrix N and of the $m \times m$ skew-symmetric matrix S is uniformly generated from $(-5, 5)$, and every diagonal entry of the $m \times m$ diagonal D is uniformly generated from $(0, 0.3)$ (we get M is positive definite), with every entry of q uniformly generated from $(-500, 0)$. Then A is monotone and Lipschitz continuous with $L = \|M\|$. The feasible set C is the nonnegative orthant $\mathbb{R}^m_+ := \{x = (x^{(1)}, x^{(2)}, \dots, x^{(m)})^T : x^{(i)} \geq 0\}$. We take $\lambda_n = \lambda = \frac{0.7}{\|L\|}$ in Algorithms 5 and 6, we take $\lambda_1 = \frac{0.7}{\|L\|}$ in our methods. For Algorithms 6 and 2, we take $\alpha_n = \frac{1}{n+2}$. For all tests, we take $u = x_1 = (1, 1, \dots, 1)^T$ and $\varepsilon = 10^{-3}$. For every m, as shown in Table 1, we have generated two random samples with different choice of M and q.

Example 4.2: We consider a classical problem considered in [44,45]. The feasible set C is the nonnegative orthant $\mathbb{R}^m_+$. Let $Ax: = Qx$, where Q is a square matrix $m \times m$ given by the following condition:

$$q_{i,j} = \begin{cases} -1, & \text{if } j = m + 1 - i \text{ and } j > i, \\ 1, & \text{if } j = m + 1 - i \text{ and } j < i, \\ 0, & \text{otherwise.} \end{cases}$$

This is a classical example of a problem where usual gradient method does not converge. For even m, the zero vector is the solution of this problem. We take $\lambda_n = \lambda = 0.7$ in Algorithms 5 and 6, we take $\lambda_1 = 0.7$ in our methods. For Algorithms 6 and 2, we take $\alpha_n = \frac{1}{n+2}$. For all tests, we take $u = x_1 = (1, 1, \dots, 1)^T$ and $\varepsilon = 10^{-3}$. The numerical results are showed in Table 2.

Finally, we give a concrete example in a non-Euclidean distance to support our main results.

Table 2. Numerical results for Example 4.2.

		Algorithm 1	Algorithm 5	Algorithm 2	Algorithm 6
$m = 300$	Iter.	46	1424	46	1436
	Time (s)	0.006	0.172	0.006	0.125
$m = 2000$	Iter.	46	1424	46	1436
	Time (s)	0.561	16.45	0.536	17.47
$m = 5000$	Iter.	46	1424	46	1436
	Time (s)	3.592	111.7	3.537	110.4

Example 4.3: In this example, we consider the negative entropy function $f : \mathbb{R}^m \to (-\infty, \infty]$ over the nonnegative orthant $C = \mathbb{R}^m_+$ defined by

$$f(x) := \begin{cases} \sum_{i=1}^{m} x^{(i)} \ln x^{(i)} & \text{if } x \in \mathbb{R}^m_+, \\ \infty & \text{otherwise,} \end{cases}$$

In this case $0 \ln 0 = 0$. It is known that f satisfy (ii) in Assumption 3.1 (see [24, Example 2.1]) and f is 1-strongly convex [14, Remark 5.18]. The Bregman distance with respect to f defined for any $x \in \mathbb{R}^m_+$ and $y \in \mathbb{R}^m_{++}$ by

$$D_f(x, y) = \sum_{i=1}^{m} \left(x^{(i)} \ln \frac{x^{(i)}}{y^{(i)}} + y^{(i)} - x^{(i)} \right),$$

which is the so-called Kullback-Leibler divergence distance measure. Then we have the Bregman projection with respect to f,

$$\Pi^f_C(y) = \operatorname{argmin}\{D_f(x, y) : x \in \mathbb{R}^m_+\}.$$

Moreover, we have

$$\nabla f(x) = (1 + \ln x^{(1)}, 1 + \ln x^{(2)}, \ldots, 1 + \ln x^{(m)})^T \quad \text{and}$$
$$\nabla f^*(x) = (e^{x^{(1)}-1}, e^{x^{(2)}-1}, \ldots, e^{x^{(m)}-1})^T.$$

Let A be the same as in Example 4.1. We can choose $\lambda_1 > 0$ and $\mu \in (0, 1)$. Then Algorithms 1 and 2 converge weakly and strongly, respectively, to a point in $VI(C, A)$.

5. Conclusions

In this work, we have proposed two modified Bregman projection-type methods with a generalized adaptive stepsize for solving monotone variational inequalities in reflexive Banach spaces. Our algorithms are motivated by the results of Shehu [12] and Bregman [25]. The weak and strong convergence of the proposed methods have been established without the prior knowledge of the Lipschitz constant of the cost mapping. Some numerical experiments have been performed to illustrate the effectiveness of the proposed algorithms.

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References

[1] Korpelevich GM. The extragradient method for finding saddle points and other problem. Ekonomika i Matematicheskie Metody. 1976;12:747–756.
[2] Antipin AS. On a method for convex programs using a symmetrical modification of the Lagrange function. Ekonomika i Matematicheskie Metody. 1976;12(6):1164–1173.
[3] Tseng P. A modified forward-backward splitting method for maximal monotone mapping. SIAM J Control Optim. 2000;38:431–446.
[4] Anh PK, Thong DV, Vinh NT. Improved inertial extragradient methods for solving pseudo-monotone variational inequalities. Optimization. 2020;71:505–528. https://doi.org/10.1080/02331934.2020.1808644.
[5] Chen J, Liu S, Chang X. Modified Tseng's extragradient methods for variational inequality on Hadamard manifolds. Appl Anal. 2019;100:2627–2640. https://doi.org/10.1080/00036811.2019.1695783.
[6] Hammad HA, Rehman H, la Sen MD. Shrinking projection methods for accelerating relaxed inertial tseng-type algorithm with applications. Math Probl Eng. 2020;2020:7487–383.
[7] Liu HW, Yang J. Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput Optim Appl. 2020;77:491–508.
[8] Thong DV, Hieu DV. Weak and strong convergence throrems for variational inequality problems. Numer Algor. 2018;78(4):1045–1060.
[9] Thong DV, Vinh NT, Hieu DV. Accelerated hybrid and shrinking projection methods for variational inequality problems. Optimization. 2019;68(5):981–998.
[10] Thong DV, Yang J, Cho YJ, et al. Explicit extragradient-likemethod with adaptive stepsizes for pseudomonotone variational inequalities. Optim Lett. 2021;15:2181–2199.
[11] Yang J, Liu H. Strong convergence result for solving monotone variational inequalities in Hilbert space. Numer Algorithms. 2019;80(3):741–752.
[12] Shehu Y. Single projection algorithm for variational inequalities in Banach spaces with application to contact problem. Acta Math Sci. 2020;40:1045–1063.
[13] Rockafellar RT. Convex analysis. Princeton: Princeton University Press; 1970.

[14] Beck A. First-order methods in optimization. Philadelphia: Society for Industrial and Applied Mathematics; 2017.
[15] Barbu V, Precupanu T. Convexity and optimization in Banach spaces. London New York: Springer Dordrecht Heidelberg; 2010.
[16] Pathak HK. An introduction to nonlinear analysis and fixed point theory. Springer Nature Singapore Pte Ltd. 2018.
[17] Reich S, Sabach S. A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J Nonlinear Convex Anal. 2009;1:471–485.
[18] Zălinescu C. Convex analysis in general vector spaces. River Edge: World Scientific, Publishing Co. Inc.; 2002.
[19] Reich S, Sabach S. Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numerical Funct Anal Optim. 2010;31(1):2–44.
[20] Bonnans JF, Shapiro A. Perturbation analysis of optimization problems. New York: Springer Verlag; 2000.
[21] Bauschke HH, Borwein JM, Combettes PL. Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun Contemp Math. 2001;3:615–647.
[22] Reich S, Sabach S. Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. TMA. 2010;73:122–135.
[23] Bauschke HH, Borwein JM. Legendre functions and the method of random Bregman projections. J Convex Anal. 1997;4:27–67.
[24] Bauschke HH, Wang X, Ye J, et al. Bregman distances and Chebyshev sets. J Approximation Theory. 2009;159:3–25.
[25] Bregman LM. The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput Math Math Phys. 1967;7:200–217.
[26] Alber YI. Metric and generalized projection operators in Banach spaces: properties and applications. In: A.G. Kartsatos editiors. Theory and applications of nonlinear operator of accretive and monotone type. New York: Marcel Dekker; 1996. p. 15–50.
[27] Reich S. A weak convergence theorem for the alternating method with Bregman distances. In: Theory and applications of nonlinear operators of accretive and monotone type. New York: Marcel Dekker; 1996. p. 313–318.
[28] Nakajo K. Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces. Appl Math Comput. 2015;271:251–258.
[29] Butnariu D, Resmerita E. Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr Appl Anal. 2006;2006:84919.
[30] Alber YI. Generalized projection operators in Banach spaces: properties and applications. In: Proceedings of the Israel Seminar, Vol. 1 of Functional Differential Equation, Ariel. 1993. p. 1–21.
[31] Martín-Márquez V, Reich S, Sabach S. Iterative methods for approximating fixed points of Bregman nonexpansive operators. Discrete Continuous Dyn Syst Ser S. 2013;6(4):1043–1063.
[32] Butnariu D, Iusem AN. Totally convex functions for fixed points computation and infinite dimensional optimization. Dordrecht: Kluwer Academic Publishers; 2000.
[33] Rockafellar RT. On the maximality of sums of nonlinear monotone operators. Trans Amer Math Soc. 1970;149:75–88.
[34] Iiduka H, Takahashi W. Weak convergence of a projection algorithm for variational inequalities in a Banach space. J Math Anal Appl. 2008;339:668–679.
[35] Takahashi W. Nonlinear functional analysis. Yokohama: Yokohama Publishers; 2000.

[36] Xu HK. Iterative algorithms for nonlinear operators. J London Math Soc. 2002;66(1): 240–256.
[37] Maingé PE. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. 2008;16:899–912.
[38] He S, Yang C. Solving the variational inequality problem defined on intersection of finite level sets. Abstr Appl Anal. 2013;2013:8.
[39] Yang J, Cholamjiak P, Sunthrayuth P. Modified Tseng's splitting algorithms for the sum of two monotone operators in Banach spaces. AIMS Math. 2021;6(5):4873–4900.
[40] Ma F. A subgradient extragradient algorithm for solving monotone variational inequalities in Banach spaces. J Inequalities Appl.. 2020;2020:26.
[41] Yang J. Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities. Appl Anal. 2021;100(5):1067–1078.
[42] Yang J, Liu HW, Liu ZX. Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization. 2018;67(12):2247–2258.
[43] Harker PT, Pang JS. A damped-Newton method for the linear complementarity problem. In: Allgower G, Georg K, editors. Computational solution of nonlinear systems of equations. Providence: AMS; 1990. p. 265–284. (Lectures in Applied Mathematics; 26).
[44] Malitsky YV. Projected reflected gradient methods for variational inequalities. SIAM J Optim. 2015;25(1):50–520.
[45] Mainge PE, Gobinddass ML. Convergence of one-step projected gradient methods for variational inequalities. J Optim Theory Appl. 2016;171:146–168.

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Strong Convergence for the Modified Split Monotone Variational Inclusion and Fixed Point Problem

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Abstract The purpose of this research is to modify the split monotone variational inclusion problem and prove a strong convergence theorem for finding a common element of the set of solutions of this problem and the set of fixed points of a nonexpansive mapping in Hilbert space. We also apply our main result involving a κ-strictly pseudo-contractive mapping. Moreover, we give the numerical example to support some of our results.

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1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in C : Tx = x\}$.
A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

It is well known that if $T : H \to H$ is a nonexpansive mapping, we have

$$\langle Ty - Tx, (I - T)x - (I - T)y \rangle \leq \frac{1}{2}\|(I - T)x - (I - T)y\|^2, \forall x, y \in H.$$

Moreover, we also know that

$$\langle y - Tx, (I - T)x \rangle \leq \frac{1}{2}\|(I - T)x\|^2,$$

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for all $x \in H$ and $y \in F(T)$.
A mapping $h : C \to C$ is said to be a *contraction* if there exists a constant $\alpha \in (0,1)$ such that

$$\|h(x) - h(y)\| \leq \alpha\|x - y\|, \forall x, y \in C.$$

A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \left\|Ax - Ay\right\|^2, \forall x, y \in C.$$

In 2006, Marino and Xu [1] introduced the general iterative method based on the viscosity approximation method proposed by Moudafi [2] in 2000 as folows:

$$\begin{cases} x_0 \in H_1 \text{ arbitrary chosen,} \\ x_{n+1} = (I - \alpha_n D)\, Tx_n + \alpha_n \xi h\left(x_n\right), \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where T is a nonexpansive mapping, h is a contractive mapping on H, D is a strongly positive bounded linear self-adjoint operator and $\{\alpha_n\}$ is a sequence in (0,1). They also proved a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.1).

Let $f : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The *variational inclusion problem* is to find $u \in H$ such that

$$\theta \in f(u) + Mu, \tag{1.2}$$

where θ is zero vector in H. The set of the solution of (1.2) is denoted by $VI(H, f, M)$. A multi-valued mapping $M : H \to 2^H$ is called *monotone*, if for all $x, y \in H$, $u \in Mx$ and $v \in My$ implies that $\langle u - v, x - y\rangle \geq 0$. A multi-valued mapping $M : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in \text{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_\lambda^M : H \to H$ defined by

$$J_\lambda^{M_1}(u) = (I + \lambda M_1)^{-1}(u), \forall u \in H,$$

is called the *resolvent operator* associated with M where λ is a positive number and I is an identity mapping, see [3].

In 2008, Zhang et al. [3] proved a strong convergence theorem for finding a common element of the set of solutions of the variational inclusion problem and the set of fixed points of nonexpansive mappings in Hilbert space. They introduced the iterative scheme as follows:

$$\begin{aligned} &y_n = J_\lambda^{M_1}(x_n - \lambda Ax_n), \\ &x_{n+1} = \alpha_n x + (1 - \alpha_n)Sy_n, \forall n \in \mathbb{N}, \end{aligned}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and λ. In 2014, Khuangsatung and Kangtunyakarn [4] modified a variational inclusion problem as follows: Finding $u \in H$ such that

$$\theta \in \sum_{i=1}^{N} a_i f_i(u) + Mu, \tag{1.3}$$

where $f_i : H \to H$ is a single valued mapping, $M : H \to 2^H$ is a multi-valued mapping, $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$, and θ is a zero vector for all $i = 1, 2, ..., N$. Such a problem is called *the modified variational inclusion*. The set of solutions (1.3) is denoted by $VI(H, \sum_{i=1}^{N} a_i A_i, M)$. If $A_i \equiv A$ for all $i = 1, 2, ..., N$, then (1.3) reduces to (1.2).

Let H_1 and H_2 be real Hilbert spaces. Let $M_1 : H_1 \to 2^{H_1}$ be a multi-valued mapping on a Hilbert space H_1, $M_2 : H_2 \to 2^{H_2}$ be a multi-valued mapping on a Hilbert space H_2, $A : H_1 \to H_2$ be a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be two given single-valued operators. In 2011, Moudafi [5] introduced the *split monotone variational inclusion problem (SMVIP)* as follows: Find $x^* \in H_1$ such that

$$\theta \in f(x^*) + M_1 x^*, \tag{1.4}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } \theta \in g(y^*) + M_2 y^*. \tag{1.5}$$

The set of all solutions of (1.4) and (1.5) is denoted by $\Theta = \{x^* \in H_1 : x^* \in VI(H_1, f, M_1)$ and $Ax^* \in VI(H_2, g, M_2)\}$. In order to solve the SMVIP, he introduced the following iterative algorithm:

$$x_{n+1} = J_\lambda^{M_1}(I - \lambda f)(x_n + \gamma A^*(J_\lambda^{M_2}(I - \lambda g) - I)Ax_n), \forall n \in \mathbb{N},$$

where $J_\lambda^{M_1}$ and $J_\lambda^{M_2}$ are the resolvents of M_1 and M_2, respectively, $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A, and f, g are α_1 and α_2 inverse strongly monotone operators, respectively. He also proved that the sequence generated by the proposed algorithm weakly converges to a solution of SMVIP under suitable conditions. Many research papers have increasingly investigated SMVIP, see, for instance, [6, 7], and the references therein. We know that spacial cases of SMVIP include the split feasibility problem, the proximal split feasibility problem, the split common fixed point problem, the split variational inclusion problem, the split variational inequality problem, and so on, see for instance, [8–12], and the references therein. The split feasibility problem can be applied to solving important real world problems in medical fields such as intensity-modulated radiation therapy (IMRT) (see, [13])

In this paper, motivated by [1], [4], and [5], we introduce the modified split monotone variational inclusion problem (MSMVIP) which is to find $x^* \in H_1$ such that

$$\theta \in \sum_{i=1}^{N} a_i f_i(x^*) + M_1 x^*, \tag{1.6}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } \theta \in \sum_{i=1}^{N} b_i g_i(y^*) + M_2 y^*, \tag{1.7}$$

where $f_i : H_1 \to H_1$ is a single valued mapping, $g_i : H_2 \to H_2$ is a single valued mapping, for all $i = 1, 2, ..., N$, $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, $b_i \in (0, 1)$ with $\sum_{i=1}^{N} b_i = 1$, $M_j : H_j \to 2^{H_j}$ be a multi-valued mapping on a Hilbert space H_j, for all $j = 1, 2$, $A : H_1 \to H_2$ is a bounded linear operator, and θ is a zero vector. The set of all solutions of (1.6) and (1.7) is denoted by $\Omega = \{x^* \in H_1 : x^* \in VI(H_1, \sum_{i=1}^{N} a_i f_i, M_1)$ and $Ax^* \in VI(H_2, \sum_{i=1}^{N} b_i g_i, M_2)\}$, where $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, $b_i \in (0, 1)$ with $\sum_{i=1}^{N} b_i = 1$.

The purpose of this article is to prove a strong convergence theorem for finding a common element of the set of solutions of the modified split monotone variational inclusion problem (MSMVIP) and the set of fixed points of a nonexpansive mapping in Hilbert space. Moreover, we also apply our main result involving a κ-strictly pseudo-contractive mapping. In the last section, we give the numerical example to support some of our results.

2. Preliminaries

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subset of H_1 and H_2, respectively. Recall that H_1 satisfies *Opial's condition* [14], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H_1$ with $y \neq x$.
For a proof of the our main results, we will use the following lemmas.

Lemma 2.1 ([15]). *Given $x \in H_1$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.2 ([16]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1): $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2): $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([3]). *$u \in H_1$ is a solution of variational inclusion (1.2) if and only if $u = J_\lambda^{M_1}(u - \lambda f(u)), \forall \lambda > 0$, i.e.,*

$$VI(H_1, f, M_1) = F(J_\lambda^{M_1}(I - \lambda f)), \forall \lambda > 0.$$

Further, if $\lambda \in (0, 2\alpha]$, then $VI(H_1, f, M_1)$ is closed convex subset in H_1.

Lemma 2.4 ([4]). *Let H_1 be a real Hilbert space and let $M_1 : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, ..., N$, let $f_i : H_1 \to H_1$ be α_i-inverse strongly monotone mapping with $\eta = \min_{i=1,2,...,N}\{\alpha_i\}$ and $\bigcap_{i=1}^N VI(H_1, f_i, M_1) \neq \emptyset$. Then*

$$VI(H_1, \sum_{i=1}^N a_i f_i, M_1) = \bigcap_{i=1}^N VI(H_1, f_i, M_1),$$

where $\sum_{i=1}^N a_i = 1$, and $0 < a_i < 1$ for every $i = 1, 2, \dots, N$. Moreover, we have $J_\lambda^{M_1}(I - \lambda\sum_{i=1}^N a_i f_i)$ is a nonexpansive mapping, for all $0 < \lambda < 2\eta$.

Lemma 2.5 ([1]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.6. *For every $j = 1, 2$, $M_j : H_j \to 2^{H_j}$ be a multi-valued maximal monotone mapping on a Hilbert space H_j and let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, ..., N$, let $f_i : H_1 \to H_1$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,...,N}\{\mu_i\}$ and $g_i : H_2 \to H_2$ be ν_i-inverse strongly monotone mapping with $\nu = min_{i=1,2,...,N}\{\nu_i\}$. Assume that Ω is a nonempty. Then the following are equivalent:*

(1) $x^* \in \Omega$,

(2) $x^* = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^N a_i f_i)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^N b_i g_i))Ax^*)$,

where $0 < \lambda_1 < 2\mu$, $0 < \lambda_2 < 2\nu$, $\gamma \in (0, \frac{1}{L})$ *with L is the spectral radius of the operator* A^*A, $a_i \in (0, 1)$ *with* $\sum_{i=1}^N a_i = 1$, *and* $b_i \in (0, 1)$ *with* $\sum_{i=1}^N b_i = 1$, *for all* $i = 1, 2, ..., N$.

Proof. Let the condition holds. Put $G_1 = \sum_{i=1}^N a_i f_i$, and $G_2 = \sum_{i=1}^N b_i g_i$.
(1) $\Longrightarrow$ (2) Let $x^* \in \Omega$, we have $x^* \in VI(H_1, \sum_{i=1}^N a_i f_i, M_1) = VI(H_1, G_1, M_1)$ and $Ax^* \in VI(H_2, \sum_{i=1}^N b_i g_i, M_2) = VI(H_2, G_2, M_2)$. From Lemma 2.3, we have $x^* \in F(J_{\lambda_1}^{M_1}(I - \lambda_1 G_1))$ and $Ax^* \in F(J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))$. This implies that

$$x^* = J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*).$$

(2) $\Longrightarrow$ (1) Let $x^* = J_{\lambda_1}^{M_1}(I-\lambda_1 G_1)(x^*-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2 G_2))Ax^*)$ and let $z \in \Omega$. From Lemma 2.4, we have the mapping $J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)$ and $J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)$ are nonexpansive mappings. Since $z \in \Omega$ and (1) $\Longrightarrow$ (2), we have

$$z = J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(z + \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az).$$

Since $J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)$ and $J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)$ are nonexpansive mappings, we have

$$\begin{aligned}
\|x^* - z\|^2 &= \|J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*) \\
&\quad - J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(z - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az)\|^2 \\
&\leq \|(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*) \\
&\quad - \ (z - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az)\|^2 \\
&= \|(x^* - z) - \gamma\big(A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^* \\
&\quad - A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az\big)\|^2 \\
&= \|x^* - z\|^2 - 2\gamma\langle x^* - z, A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle \\
&\quad + \ \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 - 2\gamma\langle Ax^* - Az, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle \\
&\quad + \ \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 + 2\gamma\langle Az - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax^* + J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax^* \\
&\quad - Ax^*, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle + \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 + 2\gamma\big(\langle Az - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax^*, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle \\
&\quad - \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2\big) + \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \|x^* - z\|^2 + 2\gamma\Big(\frac{1}{2}\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&\quad - \ \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2\Big) + \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 - \gamma\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2. \qquad (2.1)
\end{aligned}
$$

Applying (2.1), we have

$$Ax^* \in F(J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)). \qquad (2.2)$$

From Lemma (2.3), we have

$$Ax^* \in VI(H_2, G_2, M_2). \qquad (2.3)$$

From the definition of x^* and (2.2), we have

$$
\begin{aligned}
x^* &= J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*) \\
&= J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)x^*.
\end{aligned}
$$

Then $x^* \in F(J_{\lambda_1}^{M_1}(I - \lambda_1 G_1))$. From Lemma (2.3), we have

$$x^* \in VI(H_1, G_1, M_1). \qquad (2.4)$$

From (2.3) and (2.4), we have $x^* \in \Omega$. ∎

3. Main Theorem

In this section, we prove a strong convergence theorem for the modified split monotone variational inclusion and the set of fixed point of a nonexpansive mapping in Hilbert space.

Theorem 3.1. *For every $j = 1, 2$, $M_j : H_j \to 2^{H_j}$ be a multi-valued maximal monotone mapping on a Hilbert space H_j and let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, ..., N$, let $f_i : H_1 \to H_1$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,...,N}\{\mu_i\}$ and $g_i : H_2 \to H_2$ be ν_i-inverse strongly monotone mapping with $\nu = min_{i=1,2,...,N}\{\nu_i\}$. Let Ω be a solution of (1.6) and (1.7) and $\Omega \neq \emptyset$. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $\Psi = F(T) \cap \Omega$ is nonempty. Let $h : H_1 \to H_1$ be a contractive mapping with $\alpha \in (0, 1)$ and let D be a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0, 1)$ with $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$
\begin{cases}
u_n = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^N a_i f_i)(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^N b_i g_i))Ax_n), \\
x_{n+1} = \alpha_n \xi h(x_n) + (I - \alpha_n D)\left(\beta_n T x_n + (1 - \beta_n) u_n\right), \forall n \in \mathbb{N},
\end{cases} \qquad (3.1)
$$

where $0 < \lambda_1 < 2\mu$, $0 < \lambda_2 < 2\nu$, and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii): $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii): $a_i \in (0,1)$ *with* $\sum_{i=1}^{N} a_i = 1$, *and* $b_i \in (0,1)$ *with* $\sum_{i=1}^{N} b_i = 1$, *for all* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ *converges strongly to* $z = P_\Psi(I - D + \xi h)(z)$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n \leq \|D\|^{-1}$, for all $n \in \mathbb{N}$.
We divide the proof into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded. Let $z \in \Psi$. From Lemma 2.6, we have

$$z = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^{N} a_i f_i)(z - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^{N} b_i g_i))Az).$$

Put $y_n = \beta_n T x_n + (1 - \beta_n) u_n$ and applying (2.1) in Lemma 2.6, we have

$$\begin{aligned} \|y_n - z\| &= \|\beta_n T x_n + (1 - \beta_n) u_n - z\| \\ &\leq \beta_n \|T x_n - z\| + (1 - \beta_n)\|u_n - z\| \\ &\leq \beta_n \|x_n - z\| + (1 - \beta_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \tag{3.2}$$

From the definition of x_n and (3.2), we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n \xi h(x_n) + (I - \alpha_n D) y_n - z\| \\ &= \|\alpha_n(\xi h(x_n) - Dz) + (I - \alpha_n D)(y_n - z)\| \\ &\leq \alpha_n \|\xi h(x_n) - Dz\| + \|I - \alpha_n D\| \, \|y_n - z\| \\ &\leq \alpha_n (\xi \|h(x_n) - h(z)\| + \|\xi h(z) - Dz\|) + (1 - \alpha_n \bar{\xi})\|x_n - z\| \\ &\leq (1 - \alpha_n(\bar{\xi} - \xi\alpha)) \|x_n - z\| + \alpha_n \|\xi h(z) - Dz\| \\ &\leq \max\left\{\|x_1 - z\|, \frac{\|\xi h(z) - Dz\|}{\bar{\xi} - \xi\alpha}\right\}. \end{aligned}$$

By mathematical induction, we have $\|x_n - z\| \leq K, \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Put $G_1 = \sum_{i=1}^{N} a_i f_i$, and $G_2 = \sum_{i=1}^{N} b_i g_i$. From the definition of u_n, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &= \|J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n) \\ &\quad - J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x_{n-1} - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1})\|^2 \\ &\leq \|(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n) \\ &\quad - (x_{n-1} - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1})\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|(x_n - x_{n-1}) - \gamma A^*\big((I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\big)\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma\langle x_n - x_{n-1}, A^*\big((I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\big)\rangle \\
&\quad + \ \gamma^2\|A^*\big((I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\big)\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma\Big\langle Ax_n - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n + J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n \\
&\quad - \ Ax_{n-1} + J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1} - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1}, \\
&\quad (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n) - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1})\Big\rangle \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 \\
&\quad - \ 2\gamma\langle(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1} \\
&\quad + \ J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1}, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\rangle \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma\big(\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\quad + \ \langle J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1}, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\rangle\big) \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 \\
&\quad + \ 2\gamma\big(- \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\quad + \ \langle J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1} - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\rangle\big) \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\le \|x_n - x_{n-1}\|^2 \\
&\quad + \ 2\gamma\big(- \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\quad + \ \frac{1}{2}\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2\big) \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 \\
&\quad - \ \gamma(1 - \gamma L)\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\le \|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.3}
$$

From the definition of y_n and (3.3), we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|\beta_n T x_n + (1-\beta_n)u_n - \beta_{n-1}Tx_{n-1} - (1-\beta_{n-1})u_{n-1}\| \\
&= \|\beta_n(Tx_n - Tx_{n-1}) + (\beta_n - \beta_{n-1})Tx_{n-1} + (1-\beta_n)(u_n - u_{n-1}) \\
&\quad + \ (\beta_{n-1} - \beta_n)u_{n-1}\| \\
&\le \beta_n\|Tx_n - Tx_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| + (1-\beta_n)\|u_n - u_{n-1}\| \\
&\quad + \ |\beta_n - \beta_{n-1}|\|u_{n-1}\| \\
&\le \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + \ |\beta_n - \beta_{n-1}|\|u_{n-1}\| \\
&= \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| + |\beta_n - \beta_{n-1}|\|u_{n-1}\|. \qquad (3.4)
\end{aligned}$$

From the definition of x_n and (3.4), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| =& \ \|\alpha_n\xi h(x_n) + (I - \alpha_n D)y_n - \alpha_{n-1}\xi h(x_{n-1}) - (I - \alpha_{n-1}D)y_{n-1}\| \\
\le& \ \alpha_n\xi\|h(x_n) - h(x_{n-1})\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| \\
&+ \ \|(I - \alpha_n D)\|\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| \\
\le& \ \alpha_n\xi\alpha\|x_n - x_{n-1}\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| \\
&+ \ (1 - \alpha_n\bar{\xi})\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| \\
\le& \ \alpha_n\xi\alpha\|x_n - x_{n-1}\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| \\
&+ \ (1 - \alpha_n\bar{\xi})\big(\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| \\
&+ \ |\beta_n - \beta_{n-1}|\|u_{n-1}\|\big) \\
\le& \ (1 - \alpha_n(\bar{\xi} - \xi\alpha))\|x_n - x_{n-1}\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| \\
&+ \ |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| \\
&+ \ |\beta_n - \beta_{n-1}|\|u_{n-1}\|. \qquad (3.5)
\end{aligned}$$

Applying Lemma 2.2, (3.5) and the conditions (i), (ii), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (3.6)$$

Step 3. We show that $\lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|u_n - x_n\| = 0$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - y_n\| =& \ \|\alpha_n\xi h(x_n) + (I - \alpha_n D)y_n - y_n\| \\
\le& \ \alpha_n\|\xi h(x_n) - \alpha_n Dy_n\|.
\end{aligned}$$

Based on the above equation and the condition (i), we have

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \qquad (3.7)$$

Observe that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|. \qquad (3.8)$$

From (3.6), (3.7), and (3.8), we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \qquad (3.9)$$

From the definition of y_n, we have

$$\begin{aligned}\|y_n - z\|^2 &= \|\beta_n Tx_n + (1-\beta_n)u_n - z\|^2 \\ &= \beta_n\|Tx_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 - \beta_n(1-\beta_n)\|Tx_n - u_n\|^2 \\ &\leq \|x_n - z\|^2 - \beta_n(1-\beta_n)\|Tx_n - u_n\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\beta_n(1-\beta_n)\|Tx_n - u_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\ &= (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.\end{aligned}$$

From the condition (ii) and (3.9), we have

$$\lim_{n\to\infty} \|Tx_n - y_n\| = 0. \tag{3.10}$$

Observe that

$$\|Tx_n - x_n\| \leq \|Tx_n - y_n\| + \|y_n - x_n\|. \tag{3.11}$$

From (3.9), (3.10), and (3.11), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.12}$$

Since

$$y_n - Tx_n = \beta_n Tx_n + (1-\beta_n)u_n - Tx_n,$$

then

$$y_n - Tx_n = (1-\beta_n)(u_n - Tx_n).$$

From the equation above, (3.10), and the condition (ii), we have

$$\lim_{n\to\infty} \|Tx_n - u_n\| = 0. \tag{3.13}$$

Observe that

$$\|u_n - x_n\| \leq \|u_n - Tx_n\| + \|Tx_n - x_n\|. \tag{3.14}$$

From (3.12), (3.13), and (3.14), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.15}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle(\xi h - D)z, x_n - z\rangle \leq 0$, where $z = P_\Psi(I - D + \xi h)z$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle(\xi h - D)z, x_n - z\rangle = \lim_{k\to\infty} \langle(\xi h - D)z, x_{n_k} - z\rangle. \tag{3.16}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From (3.15), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.

Next, we will show that $\omega \in \Omega$. Assume that $\omega \notin \Omega$. By Lemma 2.6, we have $\omega \neq J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(\omega - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))A\omega)$. By the Opial's condition and

(3.15), we obtain

$$\begin{aligned}
&\liminf_{k\to\infty}\|x_{n_k}-\omega\| \\
&\quad < \liminf_{k\to\infty}\|x_{n_k}-J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(\omega-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))A\omega)\| \\
&\quad \leq \liminf_{k\to\infty}(\|x_{n_k}-J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(x_{n_k}-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))Ax_{n_k})\| \\
&\qquad + \|J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(x_{n_k}-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))Ax_{n_k}) \\
&\qquad - J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(\omega-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))A\omega)\|) \\
&\quad \leq \liminf_{k\to\infty}\|x_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega\in\Omega.$$

Next, we will show that $\omega\in F(T)$. Assume that $\omega\notin F(T)$. Then $\omega\neq T\omega$. By the nonexpansiveness of T, the Opial's condition, and (3.12), we obtain

$$\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\|x_{n_k}-T\omega\| \\
&\leq \liminf_{k\to\infty}(\|x_{n_k}-Tx_{n_k}\|+\|Tx_{n_K}-T\omega\|) \\
&\leq \liminf_{k\to\infty}\|x_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega\in F(T).$$

Therefore $\omega\in\Psi=\Omega\cap F(T)$.
Since $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$ and $\omega\in\Psi$. By (3.16) and Lemma 2.1, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\xi h-D)z,x_n-z\rangle &= \lim_{k\to\infty}\langle(\xi h-D)z,x_{n_k}-z\rangle \\
&= \langle(\xi h-D)z,\omega-z\rangle \\
&\leq 0. \qquad (3.17)
\end{aligned}$$

Step 5. Finally, we show that $\lim_{n\to\infty}x_n=z$, where $z=P_\Psi(I-D+\xi h)z$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n(\xi h(x_n)-Dz)+(I-\alpha_nD)(y_n-z)\|^2 \\
&\leq \|(I-\alpha_nD)(y_n-z)\|^2+2\alpha_n\langle\xi h(x_n)-Dz,x_{n+1}-z\rangle \\
&\leq (1-\alpha_n\bar{\xi})^2\|y_n-z\|^2+2\alpha_n\langle\xi h(x_n)-\xi h(z),x_{n+1}-x_0\rangle \\
&\quad + 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle \\
&\leq (1-\alpha_n\bar{\xi})^2\|x_n-z\|^2+2\alpha_n\xi\|h(x_n)-h(z)\|\|x_{n+1}-z\| \\
&\quad + 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle \\
&\leq (1-\alpha_n\bar{\xi})^2\|x_n-z\|^2+2\alpha_n\xi\alpha\|x_n-z\|\|x_{n+1}-z\| \\
&\quad + 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle \\
&= (1-\alpha_n\bar{\xi})^2\|x_n-z\|^2+\alpha_n\xi\alpha\|x_n-z\|^2+\alpha_n\xi\alpha\|x_{n+1}-z\|^2 \\
&\quad + 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-2\alpha_n\bar{\xi}+(\alpha_n\bar{\xi})^2+\alpha_n\xi\alpha}{1-\alpha_n\xi\alpha}\|x_n-z\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\xi\alpha}\langle \xi h(z)-Dz, x_{n+1}-z\rangle \\
&= \left(1-\frac{2\alpha_n(\bar{\xi}-\xi\alpha)}{1-\alpha_n\xi\alpha}\right)\|x_n-z\|^2 \\
&\quad + \frac{2\alpha_n(\bar{\xi}-\xi\alpha)}{1-\alpha_n\xi\alpha}\Bigg(\frac{\alpha_n\bar{\xi}^2}{2(\bar{\xi}-\xi\alpha)}\|x_n-z\|^2 \\
&\quad + \frac{1}{\bar{\xi}-\xi\alpha}\langle \xi h(z)-Dz, x_{n+1}-z\rangle\Bigg).
\end{aligned}$$

From the condition (i), (3.17) and Lemma 2.2, we can conclude that the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I-D+\xi h)z$. This completes the proof. ■

As direct proof of Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces, $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings and let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $g : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let Θ be a solution of (1.4) and (1.5) and $\Theta \neq \emptyset$. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $\Psi = F(T) \cap \Theta$ is nonempty. Let $h : H_1 \to H_1$ be a contractive mapping with $\alpha \in (0,1)$ and let D be a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0,1)$ with $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$\begin{cases} u_n = J_{\lambda_1}^{M_1}(I-\lambda_1 f)(x_n - \gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2 g))Ax_n), \\ x_{n+1} = \alpha_n\xi h(x_n) + (I-\alpha_n D)\left(\beta_n Tx_n + (1-\beta_n)u_n\right), \forall n \in \mathbb{N}, \end{cases} \tag{3.18}$$

where $0 < \lambda_1 < 2\mu$, $0 < \lambda_2 < 2\nu$, and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying the following conditions:

(i): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$;

(ii): $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I-D+\xi h)(z)$.

Proof. Put $f_i \equiv f$ and $g_i \equiv g$ for all $i = 1, 2, ..., N$ in Theorem 3.1. So, from Theorem 3.1, we obtain the desired result. ■

4. Application

In this section, we utilize our main theorem to prove a strong convergence theorem for finding a common element of the set of solutions of the modified split monotone variational inclusion problem (MSMVIP) and the set of fixed points of a κ-strictly pseudo-contractive mapping.

A mapping $T : C \to C$ is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2,$$

for all $x, y \in C$. Note that the class of strictly pseudo-contractions strictly includes the class of nonexpansive mapping.

Lemma 4.1 (See [17]). *Let $T : C \to H_1$ be a κ-strict pseudo-contraction. Define $S : C \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

Theorem 4.2. *Let H_1 and H_2 be two real Hilbert spaces, $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings and let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, ..., N$, let $f_i : H_1 \to H_1$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,...,N}\{\mu_i\}$ and $g_i : H_2 \to H_2$ be ν_i-inverse strongly monotone mapping with $\nu = min_{i=1,2,...,N}\{\nu_i\}$. Let Ω be a solution of (1.6) and (1.7) and $\Omega \neq \emptyset$. Let $T : H_1 \to H_1$ be a κ-strict pseudo-contraction with $\Psi = F(T) \cap \Omega$ is nonempty. Define the mapping $S = H_1 \to H_1$ by $Sx = \sigma x + (1 - \sigma)Tx$ for every $x \in H_1$ and $\sigma \in (k, 1)$. Let $h : H_1 \to H_1$ be a contractive mapping with $\alpha \in (0, 1)$ and let D be a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0, 1)$ and $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$\begin{cases} u_n = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^N a_i f_i)(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^N b_i g_i))Ax_n) \\ x_{n+1} = \alpha_n \xi h(x_n) + (I - \alpha_n D)\left(\beta_n S x_n + (1 - \beta_n)u_n\right), \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $0 < \lambda_1 < 2\mu$, $0 < \lambda_2 < 2\nu$, and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii): $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii): $a_i \in (0, 1)$ *with* $\sum_{i=1}^N a_i = 1$, *and* $b_i \in (0, 1)$ *with* $\sum_{i=1}^N b_i = 1$, *for all* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I - D + \xi h)(z)$.

Proof. From Lemma 4.1 and Theorem 3.1, we obtain the desired result. ∎

5. Numerical Result

The purpose of this section, we give a numerical example to support our main result. The following example is given to support Theorem 3.1.

Example 5.1. Let $\mathbb{R}$ be the set of real numbers and let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2$, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ and $H_1 = H_2 = \mathbb{R}^2$. For every $i = 1, 2, \ldots, N$, let $D, f_i, g_i, h : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$D\mathbf{x} = \left(\frac{x_1}{2}, \frac{x_2}{2}\right), \quad f_i\mathbf{x} = \left(\frac{ix_1}{12}, \frac{ix_2}{12}\right), \quad g_i\mathbf{x} = \left(\frac{ix_1}{9}, \frac{ix_2}{9}\right), \quad h(\mathbf{x}) = \left(\frac{x_1}{3}, \frac{x_2}{3}\right),$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\mathbf{x} = \left(\frac{x_1}{5}, \frac{x_2}{5}\right),$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Observe that, for every $i = 1, 2, ..., N$, f_i and g_i are inverse strongly monotone mappings, D is a strongly positive bounded linear operator, h is a contractive mapping, and also we have T is a nonexpansive mapping. We also define $A : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$A\mathbf{x} = (5x_1 - 3x_2, 3x_1 + 5x_2),$$

and $A^* : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$A^*\mathbf{x} = (5x_1 + 3x_2, 5x_2 - 3x_1).$$

Then A is a bounded linear operator. Moreover, the spectral radius of the operator A^*A is 34 and also we have $\gamma \in (0, \frac{1}{34})$.

For every $i = 1, 2, \ldots, N$. Suppose that $J_{\lambda_1}^{M_1} = J_{\lambda_2}^{M_2} = I$, $a_i = \frac{14}{15^i} + \frac{1}{N15^N}$, $b_i = \frac{15}{16^i} + \frac{1}{N16^N}$. Let $\mathbf{x}_n = \left(x_n^1, x_n^2\right)$ and $\mathbf{u}_n = \left(u_n^1, u_n^2\right)$ be generated by (3.1), where $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{n}{5n+6}$ for every $n \in \mathbb{N}$. Put $\lambda_1 = \lambda_2 = \frac{1}{N}$ and $\xi = \frac{1}{2}$. It is easy to see that all sequences satisfy conditions of Theorem 3.1. For every $n \in \mathbb{N}$, we rewrite (3.1) as follows:

$$\begin{aligned}
\mathbf{u}_n &= \left(I - \frac{1}{N}\left(\sum_{i=1}^{N} \frac{14}{15^i} + \frac{1}{N15^N}\right) f_i\right) \\
&\quad \times \left(\mathbf{x}_n - \gamma A^* \left(I - \left(I - \frac{1}{N}\left(\sum_{i=1}^{N} \frac{15}{16^i} + \frac{1}{N15^N}\right) g_i\right)\right) A\mathbf{x}_n\right), \\
\mathbf{x}_{n+1} &= \frac{1}{2n}\left(\frac{1}{2}\right) h(\mathbf{x}_n) + (I - \frac{1}{2n}D)\left(\frac{n}{5n+6}T\mathbf{x}_n + (1 - \frac{n}{5n+6})\mathbf{u}_n\right),
\end{aligned} \tag{5.1}$$

where $\mathbf{x}_n = \left(x_n^1, x_n^2\right)$ and $\mathbf{u}_n = \left(u_n^1, u_n^2\right)$. Then the sequences $\mathbf{x}_n = \left(x_n^1, x_n^2\right)$ and $\mathbf{u}_n = \left(u_n^1, u_n^2\right)$ generated by (5.1) converge strongly to $\mathbf{0}$, where $\mathbf{0} = (0, 0)$.
Using the algorithm (5.1) and choosing $\mathbf{x}_1 = (5, -5)$ and $n = N = 100$, the numerical results for the sequences x_n and u_n are shown the following table and figure.

n	u_n^1	u_n^2	x_n^1	x_n^2
1	4.9915097	-4.9915097	5.0000000	-5.0000000
2	3.8815482	-3.8815482	3.8881505	-3.8881505
3	3.2134041	-3.2134041	3.2188699	-3.2188699
4	2.6939520	-2.6939520	2.6985343	-2.6985343
5	2.2672346	-2.2672346	2.2710910	-2.2710910
⋮	⋮	⋮	⋮	⋮
50	0.0009374	-0.0009374	0.0009390	-0.0009390
⋮	⋮	⋮	⋮	⋮
96	0.0000003	-0.0000003	0.0000003	-0.0000003
97	0.0000003	-0.0000003	0.0000003	-0.0000003
98	0.0000002	-0.0000002	0.0000002	-0.0000002
99	0.0000002	-0.0000002	0.0000002	-0.0000002
100	0.0000001	-0.0000001	0.0000001	-0.0000001

Table 1. The values of the sequences $\{\mathbf{u}_n\}$ and $\{\mathbf{x}_n\}$ with initial values $\mathbf{x}_1 = (5, -5)$ and $n = N = 100$.

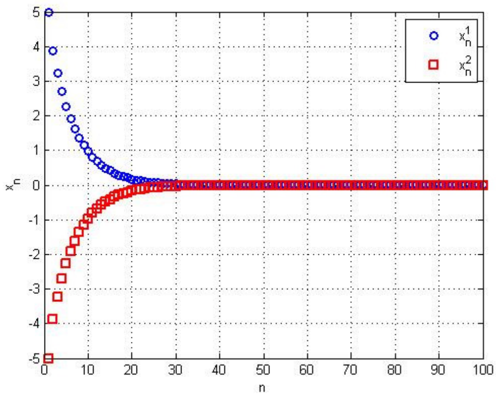

FIGURE 1. The behavior of $\{\mathbf{x}_n\}$ with initial values $\mathbf{x}_1 = (5, -5)$ and $n = N = 100$.

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References

[1] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318 (2006) 43–52.

[2] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46–55.

[3] S.S. Zhang, J.H.W. Lee, C.K. Chan, Algorithms of common solutions for quasi variational inclusion and fixed point problems, Appl. Math. Mech. 29 (2008) 571–581.

[4] W. Khuangsatung, A. Kangtunyakarn, Algorithm of a new variational inclusion problem and strictly pseudononspreading mapping with application, Fixed Point Theory Appl. 2014 (2014) Article no. 209.

[5] A. Moudafi, Split monotone variational inclusions, J. Optim. Theory Appl. 150 (2011) 275–283.

[6] Y. Shehu, F.U. Ogbuisi, An iterative method for solving split monotone variational inclusion and fixed point problems, RACSAM. 110 (2) (2016) 503–518.

[7] Q.H. Ansari, A. Rehan, An iterative method for split hierarchical monotone variational inclusions, Fixed Point Theory and Appl. 2015 (2015) Article no. 121.

[8] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, Inverse probl. 18 (2002) 441–453.

[9] W. Khuangsatung, P. Jailoka, S. Suantai, An iterative method for solving proximal split feasibility problems and fixed point problems, Comput. Appl. Math. 38 (2019) Article no. 177.

[10] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009) 587–600.

[11] K.R. Kazmi, S.H. Rizvi, An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping, Optim. Lett. 8 (3) (2014) 1113–1124.

[12] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, Numer. Algorithms. 59 (2012) 301–323.

[13] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51 (2006) 2353–2365.

[14] Z. Opial, Weak convergence of the sequence of successive approximation of nonexpansive mappings, Bull. Amer. Math. Soc 73 (1967) 591–597.

[15] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[16] H.K. Xu, An iterative approach to quadric optimization, J. Optim. Theory Appl. 116 (2003) 659–678.

[17] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197–228.

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On an Open Problem in Complex Valued Rectangular b-Metric Spaces with an Application

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ABSTRACT

The purpose of the paper is to solve problem 1. Moreover, we prove fixed point theorems for contraction mappings in complete rectangular b-metrics and give examples as a satisfying the theorems in such spaces and give examples as a satisfying the theorems in rectangular b-metric spaces. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

Keywords: Fixed point; Contraction mapping; Rectangular b-metric spaces; Integral equation; Fredholm type

1. Introduction

In 2015, George et al. [1] established the concept of rectangular b-metric space as a generalization of metric space (MS) [2], rectangular metric space (RMS) [3] and b-metric space (bMS) [4].

In the same year, Ege [6] established the complex valued rectangular b-metric space (CRbMS) as a generalization of a complex valued metric space (CMS) [5] and rectangular b-metric space (RbMS) [1] and proved an analogue of Banach contraction principle. Author also proved a different contraction principle with a new condition and a fixed point theorem in this space. Finally, author gave an application of Banach contraction principle to linear equations.

The complex metric space was initiated by Azam et al. [5]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$.

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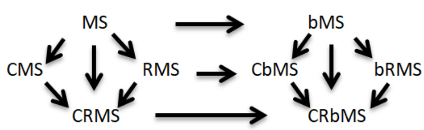
Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
$(C_1) Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$;
$(C_2) Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$;
$(C_3) Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$;
$(C_4) Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$.

Particularly, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C_2), (C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:
(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \precsim bz$ for all $z \in \mathbb{C}$.
(2) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.
(3) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.1 ([6]). Let X be a nonempty set and the mapping $d : X \times X \to \mathbb{C}$ satisfies:
(i) $0 \precsim d(x, y)$ for all $x = y$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iv) there exists a real number $s \geq 1$ such that $d(x, y) \precsim s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$, and all distinct points $u, v \in X \backslash \{x, y\}$. Then d is called a complex valued rectangular b-metric on X and (X, d) is called a *complex valued rectangular b-metric space (in short CRbMS)* with coefficient .



Note that every complex valued space is a complex valued rectangular b-metric space with coefficient $s = 1$.

Example 1.2 ([6]). Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \mathbb{Z}^+$ and $d : X \times X \to \mathbb{C}$ be defined as follows: $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x, y) = \begin{cases} 0, & \text{if } \ x = y; \\ 2t, & \text{if } \ x, y \in A; \\ \frac{t}{2n}, & \text{if } \ x \in A \text{ and } y \notin \{2, 3\}, \\ t, & \text{otherwise,} \end{cases}$$

where $t > 0$ is a constant. Then (X, d) is a complex valued rectangular b-metric space with coefficient $s = 2$, but (X, d) is not a complex valued rectangular metric space.

Now, we review definition in complex valued rectangular b-metric spaces as follows:

Definition 1.3 ([6]). Let (X, d) be a complex valued rectangular b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) The sequence $\{x_n\}$ is said to be *complex valued convergent* in (X, d) and converges to x, if for every $\epsilon \succ 0$ there exists $n_0 \in N$ such that $d(x_n, x) \prec \epsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

(ii) The sequence $\{x_n\}$ is said to be *complex valued Cauchy sequence* in X if for every $\epsilon \succ 0$ there exists $n_0 \in N$ such that $d(x_n, x_{n+p}) \prec \epsilon$ for all $n > n_0,\ p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(iii) X is said to be a *complete complex valued rectangular b-metric space* if every Cauchy sequence in X converges to some $x \in X$.

Lemma 1.4 ([6]). *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.5 ([6]). *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then*

$\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

The main result in paper [6] is the following theorem (The Banach contraction principle theorem in complex valued rectangular b-metric spaces).

Theorem 1.6. *Let (X, d) be a complex valued complete rectangular b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \precsim \alpha d(x, y) \qquad (1.1)$$

for all $x, y \in X$, where $\alpha \in [0, \frac{1}{s})$. Then T has a unique fixed point.

In 2017, Mitrović [7], improved $[0, \frac{1}{s})$ to $[0, 1)$ for the Banach contraction principle theorem of George et al.[1] in rectangular b-metric spaces.

The above results naturally bring us to the following open problem.

Open Problem 1. In Theorem 1.6, we can extend the range of α to the case $\alpha \in [\frac{1}{s}, 1)$?

The purpose of this paper is to give some affirmative answers to the questions raised. We prove fixed point theorems for contraction mappings in complete rectangular b-metrics and give examples extend the theorems in such spaces. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

2. Main Results

In this section, we prove a fixed point theorem for contraction mappings in complete rectangular b-metric space and give an example that satisfies main theorem in such spaces.

Theorem 2.1. *Let (X, d) be a complex valued complete rectangular b-metric space. Suppose that $T : X \to X$ is a mapping satisfying: There exists constant α with $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \precsim \alpha d(x, y) \qquad (2.1)$$

for all $x, y \in X$. Then T has a unique fixed point. Moreover, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, converges to fixed point for any $x_0 \in X$, where $n = 0, 1, 2, \dots$.

Proof. Let $\alpha \in [0, 1)$. Since $\lim_{n\to\infty} \alpha^n = 0$, there exists a natural number k_0 such that

$$0 < \alpha^k s < 1, \qquad (2.2)$$

for all $k > k_0$. Let x_0 be a arbitrary in X such that $Tx_0 = x_1 \in X$. Define a sequence $\{x_n\}$ by $x_n = Tx_{n-1} \in X$, where $n = 1, 2, \dots$. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Then $x_n \neq x_{n+k}$ for all $n \geq 0$, $k \geq 1$. Namely, if $x_n = x_{n+k}$ for some $n \geq 0$ and $k \geq 1$ we have that $Tx_n = Tx_{n+k}$ and $x_{n+1} = x_{n+k+1}$. By (2.1), we have

$$\begin{aligned} d(x_{n+q}, x_{m+q}) &= d(Tx_{n+q-1}, Tx_{m+q-1}) \\ &\precsim \alpha d(x_{n+q-1}, x_{m+q-1}) \\ &= \alpha d(Tx_{n+q-2}, Tx_{m+q-2}) \\ &\precsim \alpha^2 d(x_{n+q-2}, x_{m+q-2}) \\ &\precsim \vdots \\ &\precsim \alpha^q d(x_n, x_m), \quad n, m, q \in \mathbb{N}. \end{aligned} \qquad (2.3)$$

Similarly, we get for any $n, r \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+r}) &= d(Tx_{n-1}, Tx_{n+r-1}) \\ &\precsim \alpha d(x_{n-1}, x_{n+r-1}) \\ &\precsim \alpha^2 d(x_{n-2}, x_{n+r-2}) \\ &\precsim \vdots \\ &\precsim \alpha^n d(x_0, x_r). \end{aligned} \qquad (2.4)$$

Since $0 \leq \alpha < 1$, we get $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. We next prove that $\{x_n\}$ is a Cauchy sequence by letting $m, n \in \mathbb{N}$ with $m, n > k_0$,

$$\begin{aligned} d(x_n, x_m) &\precsim s[d(x_n, x_{n+k_0}) \\ &\quad + d(x_{n+k_0}, x_{m+k_0}) + d(x_{m+k_0}, x_m)] \\ &\precsim s[\alpha^n d(x_0, x_{k_0}) \\ &\quad + \alpha^{k_0} d(x_n, x_m) + \alpha^m d(x_{k_0}, x_0)]. \end{aligned} \tag{2.5}$$

It follows that

$$|d(x_n, x_m)| \leq \frac{s\alpha^n + s\alpha^m}{1 - \alpha^{k_0}} |d(x_0, x_{k_0})|. \tag{2.6}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of a complete rectangular b-metric space (X, d), there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$.
Next, we will show that p is a fixed point of T. Let $n \in \mathbb{N} \bigcup \{0\}$. We have

$$\begin{aligned} d(p, Tp) &\preceq s[d(p, x_n) + d(x_n, x_{n+1}) \\ &\quad + d(x_{n+1}, Tp)] \\ &\preceq s[d(p, x_n) + d(x_n, x_{n+1}) \\ &\quad + d(Tx_n, Tp)] \\ &\preceq s[d(p, x_n) + d(x_n, x_{n+1}) \\ &\quad + \alpha d(x_n, p)]. \end{aligned} \tag{2.7}$$

Taking limit as $n \to \infty$ in (2.7), we get $p = Tp$. Thus p is a fixed point of T. To prove uniqueness, suppose that there exists $p^* \in X$ such that $p^* = Tp^*$. We consider

$$\begin{aligned} d(p, p^*) &= d(Tp, Tp^*) \\ &\preceq \alpha d(p, p^*), \end{aligned} \tag{2.8}$$

which implies

$$|d(p, p^*)| \leq \alpha |d(p, p^*)|, \tag{2.9}$$

and then $p = p^*$. So, the Picard iteration $\{x_n\}_{n=0}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ converges to p for any $x_0 \in X$. $\square$

We give examples in order to validate the proved result.

Example 2.2. Let $X = \mathbb{R}$ with $d(x, y) = |x - y|^2 + i|x - y|^2$. Let $T : X \to X$ be given by

$$Tx = \begin{cases} \frac{x^2}{3} & \text{if } x, y \in [-1, 1]; \\ \frac{3x}{4}, & \text{if } x, y \in X \setminus [-1, 1]. \end{cases}$$

Then (X, d) is a complete CRbMS with coefficient $s = 2$. Let $x, y \in X$. Now, we consider, for any $x, y \in X \setminus [-1, 1]$,

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty|^2 + i|Tx - Ty|^2 \\ &= \frac{9}{16}(|x - y|^2 + i|x - y|^2) \\ &\precsim \alpha d(x, y), \end{aligned} \tag{2.10}$$

where $\alpha = \frac{9}{16}$ and $\alpha \in [\frac{1}{s}, 1)$. If $x, y \in [-1, 1]$, $|x| \neq 1$ and $|y| \neq 1$ then

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty|^2 + i|Tx - Ty|^2 \\ &= (|x^2 - y^2|^2 + i|x^2 - y^2|^2) \\ &= \frac{|x + y|^2}{9}(|x - y|^2 + i|x - y|^2) \\ &\precsim \frac{|x + y|^2}{|x + y|^2 + 5}(|x - y|^2 + i|x - y|^2) \\ &= \alpha d(x, y), \end{aligned} \tag{2.11}$$

where $\alpha = \frac{|x+y|^2}{|x+y|^2+5}$ and $\alpha \in [\frac{1}{s}, 1)$, which implies that T has a unique fixed point $0 \in X$.

3. Application

In this section, we endeavor to apply Theorem 2.1 to prove the existence and uniqueness of solution of the following integral equation of Fredholm type:

$$x(t) = \int_a^b G(t, s, x(s)) ds + h(t) \tag{3.1}$$

for $t, s \in [a, b]$ where, $G, h \in C([a, b], \mathbb{R})$ (say $X = C([a, b], \mathbb{R})$ Define $d : X \times X \to$

$\mathbb{C}$ by $d(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2 + i \sup_{t\in[a,b]} |x(t) - y(t)|^2$ for all $x, y \in X$, Then, (X, d) is a complete extended rectangular b-metric space, see example 2.2. Now, we are equipped to state and prove our result as follows:

Theorem 3.1. *For all* $x, y \in X := C([a, b], \mathbb{R})$,

$$G(t, s, x(t), G(t, s, x(y))) \leq \frac{1}{2(b-a)} \tag{3.2}$$

for all $t, s \in [a, b]$. *Then the integral equation (3.1) has a unique solution.*

Proof. Define $T : X \to X$ by $Tx(t) = G(t, s, x(s))ds + h(t)$ for all $t, s \in [a, b]$ It is clear that, x is a fixed point of the operator T if and only if it is a solution of the integral equation For all $x, y \in X$, we get

$$\begin{aligned} &|fx(t) - fy(t)|^2 \\ &\leq \int_a^b |G(t, s, x(s)) - G(t, s, y(s))|ds \\ &\leq \int_a^b \frac{1}{2(b-a)}|x(s) - y(s)|ds \\ &\leq \frac{1}{4(b-a)^2} \sup_{t\in[a,b]} (\int_a^b ds)^2, \end{aligned} \tag{3.3}$$

then $d(fx(t) - fy(t)) \to 0$ as $n \to \infty$ with $\frac{1}{2(b-a)} \in [0, 1]$. Hence, the operator T has a unique fixed point, that is, the Fredholm integral Equation (3.1) has a unique solution

□

4. Conclusion

The purpose of this paper is to give some affirmative answers to the questions raised. We proved fixed point theorems for contraction mappings in complete rectangular b-metrics and gave examples that satisfy the theorems in such spaces. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation:

Let (X, d) be a complex valued complete rectangular b-metric space. Suppose that $T : X \to X$ is a mapping satisfying: There exists constants α with $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \precsim \alpha d(x, y) \tag{4.1}$$

for all $x, y \in X$. Then T has a unique fixed point and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, converges to fixed point for any $x_0 \in X$, where $n = 0, 1, 2, \ldots$.

$$x(t) = \int_a^b G(t, s, x(s))ds + h(t) \tag{4.2}$$

for $t, s \in [a, b]$ where, $G, h \in C([a, b], \mathbb{R})$ (say $X = C([a, b], \mathbb{R})$ For all $x, y \in X := C([a, b], \mathbb{R})$,

$$G(t, s, x(t), G(t, s, x(y))) \leq \frac{1}{2(b-a)} \tag{4.3}$$

for all $t, s \in [a, b]$. Then the integral equation 3.1 has a unique solution.

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References

[1] George, R., Radenovic, S. Reshma, K.P. and Shukla. S., Rectangular b-metric space and contraction principles, Journal of Nonlinear Sciences and Applications, 2015, 8:1005-13.

[2] Frechet, M., Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo, 1906, 22:1-72.

[3] Branciari, A. A fixed point theorem of Banach-Caccippoli type on a class of generalised metric spaces, Publ. Math. Debrecen, 2000, 57:31-7.

[4] Bakhtin, IA., The contraction principle in quasimetric spaces, In: Functional Analysis, 1989, 30:26-37.

[5] Azam, A., Fisher, B. and Khan, M., Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim, 2011, 32(3):243-53.

[6] Ege, O., Complex valued rectangular b-metric spaces and an application to linear equations, J. Nonlinear Sci. Appl., 2015, 8:1014-21.

[7] Mitrović,Z.D., On an open problem in rectangular b-metric space, J Anal., 2017, 25:135-7. DOI 10.1007/s41478-017-0036-7

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Research Article

An Iterative Method for Solving Split Monotone Variational Inclusion Problems and Finite Family of Variational Inequality Problems in Hilbert Spaces

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The purpose of this paper is to study the convergence analysis of an intermixed algorithm for finding the common element of the set of solutions of split monotone variational inclusion problem (SMIV) and the set of a finite family of variational inequality problems. Under the suitable assumption, a strong convergence theorem has been proved in the framework of a real Hilbert space. In addition, by using our result, we obtain some additional results involving split convex minimization problems (SCMPs) and split feasibility problems (SFPs). Also, we give some numerical examples for supporting our main theorem.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For a mapping $S: C \longrightarrow C$, we denoted by $F(S)$ the set of fixed points of S (i.e., $F(S) = \{x \in C: Sx = x\}$). Let $A: C \longrightarrow H$ be a nonlinear mapping. The variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1}$$

and the solution set of problem (1) is denoted by $VI(C, A)$. It is known that the variational inequality, as a strong and great tool, has already been investigated for an extensive class of optimization problems in economics and equilibrium problems arising in physics and many other branches of pure and applied sciences. Recall that a mapping $A: C \longrightarrow C$ is said to be α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2}$$

A multivalued mapping $M: H_1 \longrightarrow 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $\langle x - y, u - v \rangle \geq 0$, for any $u \in Mx$ and $v \in My$. A monotone mapping $M: H_1 \longrightarrow 2^{H_1}$ is maximal if the graph $G(M)$ for M is not properly contained in the graph of any other monotone mapping. It is generally known that M is maximal if and only if for $(x, u) \in H_1 \times H_1, \langle x - y, u - v \rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in Mx$. Let $M: H_1 \longrightarrow 2^{H_1}$ be a multivalued maximal monotone mapping. The resolvent mapping $J_\lambda^M: H_1 \longrightarrow H_1$ associated with M is defined by

$$J_\lambda^M(x) \coloneqq (I + \lambda M)^{-1}(x), \quad \forall x \in H_1, \lambda > 0, \tag{3}$$

where I stands for the identity operator on H_1. We note that for all $\lambda > 0$, the resolvent J_λ^M is single-valued, nonexpansive, and firmly nonexpansive.

In 2011, Moudafi [1] introduced the following split monotone variational inclusion problem (SMVI):

$$\text{find } x^* \in H_1 \text{ such that } \theta \in A_1(x^*) + M_1(x^*) \tag{4}$$

$$\text{and such that } y^* = Tx^* \in H_2 \text{ solves } \theta \in A_2(y^*) + M_2(y^*), \tag{5}$$

where θ is the zero vector in H_1 and H_2, $M_1: H_1 \longrightarrow 2^{H_1}$ and $M_2: H_2 \longrightarrow 2^{H_2}$ are multivalued mappings on H_1 and H_2, $A_1: H_1 \longrightarrow H_1$ and $A_2: H_2 \longrightarrow H_2$ are two given single-valued mappings, and $T: H_1 \longrightarrow H_2$ is a bounded linear operator with adjoint T^* of T. We note that if (4) and (5) are considered separately, we have that (4) is a variational inclusion problem with its solution set $VI(H_1, A_1, M_1)$ and (5) is a variational inclusion problem with its solution set $VI(H_1, A_2, M_2)$. We denoted the set of all solutions of (SMVI) by $\Omega = \{x^* \in VI(H_1, A_1, M_1): Tx^* \in VI(H_2, A_2, M_2)\}$.

It is worth noticing that by taking $M_1 = N_C$ and $M_2 = N_Q$ normal cones to closed convex sets C and Q, then (SMVI) (4) and (5) reduce to the split variational inequality problem (SVIP) that was introduced by Censor et at. [2]. In [1], they mentioned that (SMVI) (4) and (5) contain many special cases, such as split minimization problem (SMP), split minimax problem (SMMP), and split equilibrium problem (SEP). Some related works can be found in [1, 3–10].

For solving (SMVI) (4) and (5), Modafi [1] proposed the following algorithm.

Algorithm 1. Let $\lambda > 0$, $x_0 \in H_1$, and the sequence $\{x_n\}$ be generated by

$$x_{n+1} = J_\lambda^{M_1}(1 - \lambda f)\left(x_n + \gamma T^*\left(J_\lambda^{M_2}(I - \lambda g) - I\right)Tx_n\right), \quad n \in \mathbb{N}, \tag{6}$$

where $\gamma \in (1, 1/L)$ with L being the spectral radius of the operator T^*T.

He obtained the following weak convergence theorem for algorithm (6).

Theorem 1 (see [1]). *Let H_1, H_2 be real Hilbert spaces. Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^*. For $i = 1, 2$, let $A_i: H_i \longrightarrow H_i$ be α_i-inverse strongly monotone with $\alpha = \min\{\alpha_1, \alpha_2\}$ and let $M_i: H_i \longrightarrow 2^{M_i}$ be two maximal monotone operators. Then, the sequence generated by (6) converges weakly to an element $x^* \in \Omega$ provided that $\Omega \neq \emptyset$, $\lambda \in (0, 2\alpha)$, and $\gamma \in (1, 1/L)$ with L being the spectral radius of the operator T^*T.*

Since then, because of a lot of applications of (SMVI), it receives much attention from many authors. They presented many approximation methods for solving (SMVI) (4) and (5). Also the iterative methods for solving (SMVIP) (4) and (5) and fixed-point problems of some nonlinear mappings have been investigated (see [11–19]).

On the other hand, Yao et al. [20] presented an intermixed Algorithm 1.3 for two strict pseudo-contractions in real Hilbert spaces. They also showed that the suggested algorithms converge strongly to the fixed points of two strict pseudo-contractions, independently. As a special case, they can find the common fixed points of two strict pseudo-contractions in Hilbert spaces (i.e., a mapping $S: C \longrightarrow C$ is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0, 1)$ such that $\|Sx - Sy\|^2 \leq \|x - y\|^2 + k \|(I - S)x - (I - S)\|$, 0.3 cm $\forall x, y \in C$).

Algorithm 2. For arbitrarily given $x_0, y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n], & n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C[\alpha_n f(x_n) + (1 - k - \alpha_n)y_n + kSy_n], & n \geq 0, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of the real number in $(0, 1)$, $T, S: C \longrightarrow C$ are λ-strictly pseudo-contractions, $f: C \longrightarrow H$ is a ρ_1-contraction, $g: C \longrightarrow H$ is a ρ_2-contraction, and $k \in (0, 1 - \lambda)$ is a constant.

Under some control conditions, they proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}f(y^*)$ and $\{y_n\}$ converges strongly to $P_{F(S)}f(x^*)$, respectively, where $x^* \in F(T)$, $y^* \in F(S)$, and $P_{F(T)}$ and $P_{F(S)}$ are the metric projection of H onto $F(T)$ and $F(S)$, respectively. After that, many authors have developed and used this algorithm to solve the fixed-point problems of many nonlinear operators in real Hilbert spaces (see for example [21–27]). Question: can we prove the strong convergence theorem of two sequences of split monotone variational inclusion problems and fixed-point problems of nonlinear mappings in real Hilbert spaces?

The purpose of this paper is to modify an intermixed algorithm to answer the question above and prove a strong convergence theorem of two sequences for finding a common element of the set of solutions of (SMVI) (4) and (5) and the set of solutions of a finite family of variational inequality problems in real Hilbert spaces. Furthermore, by applying our main result, we obtain some additional results involving split convex minimization problems (SCMPs) and split feasibility problems (SFPs). Finally, we give some numerical examples for supporting our main theorem.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote the strong convergence of $\{x_n\}$ to x and the weak convergence of $\{x_n\}$ to x by notations "$x_n \longrightarrow x$ as $n \longrightarrow \infty$" and "$x_n \rightharpoonup x$ as $n \longrightarrow \infty$," respectively. For each $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have

$$\begin{aligned} \|x+y\|^2 &\leq \|x\|^2 + 2\langle y, x+y\rangle, \\ \|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|x\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2. \end{aligned} \tag{8}$$

Definition 1. Let H be a real Hilbert space and C be a closed convex subset of H. Let $S\colon C \longrightarrow C$ be a mapping. Then, S is said to be

(1) Monotone, if $\langle Sx - Sy, x-y\rangle \geq 0, \forall x, y \in H$
(2) Firmly nonexpansive, if $\langle Sx - Sy, x-y\rangle \geq \|Sx - Sy\|^2, \forall x, y \in H$
(3) Lipschitz continuous, if there exists a constant $L > 0$ such that $\|Sx - Sy\| \leq L\|x-y\|, \forall x, y \in H$
(4) Nonexpansive, if $= \|Sx - Sy\| \leq \|x-y\|, \forall x, y \in H$

It is well known that if S is α-inverse strongly monotone, then it is $1/\alpha$-Lipschitz continuous and every nonexpansive mapping S is 1-Lipschitz continuous. We note that if $S\colon H \longrightarrow H$ is a nonexpansive mapping, then it satisfies the following inequality (see Theorem 3 in [28] and Theorem 1 in [29]):

$$\langle Sy - Sx, (I-S)x - (I-S)y\rangle \leq \frac{1}{2}\|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in H. \tag{9}$$

Particularly, for every $x \in H$ and $y \in F(S)$, we have

$$\langle y - Sx, (I-S)x\rangle \leq \frac{1}{2}\|(I-S)x\|^2. \tag{10}$$

For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\left\|x - P_C x\right\| \leq \|x-y\|, \quad \forall y \in C. \tag{11}$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 1 (see [30]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \quad \forall v \in C. \tag{12}$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\left\|P_C x - P_C y\right\|^2 \leq \langle P_C x - P_C y, x-y\rangle, \quad \forall x, y \in H. \tag{13}$$

Moreover, we also have the following lemma.

Lemma 2 (see [31]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u, \tag{14}$$

where P_C is the metric projection of H onto C.

Lemma 3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let $A_i\colon C \longrightarrow H$ be the α_i-inverse strongly monotone with $\overline{\alpha} = \min_{i=1,2,\ldots,N}\{\alpha_i\}$. If $\cap_{i=1}^N VI(C, A_i) \neq \emptyset$, then*

$$VI\left(C, \sum_{i=1}^N a_i A_i\right) = \bigcap_{i=1}^N VI(C, A_i), \tag{15}$$

where $0 < a_i < 1$ for all $i = 1, 2, \ldots, N$ and $\sum_{i=1}^N a_i = 1$. Moreover, $I - \lambda\sum_{i=1}^N a_i A_i$ is a nonexpansive mapping for all $\lambda \in (0, 2\overline{\alpha})$.

Proof. By Lemma 4.3 of [32], we have that $VI(C, \sum_{i=1}^N a_i A_i) = \cap_{i=1}^N VI(C, A_i)$. Let $\lambda \in (0, 2\overline{\alpha})$ and let $x, y \in C$. As the same argument as in the proof of Lemma 8 in [16], we have $I - \lambda\sum_{i=1}^N a_i A_i$ as nonexpansive. □

Lemma 4 (see [33]). *Let H be a real Hilbert space, $A\colon H \longrightarrow H$ be a single-valued nonlinear mapping, and $M\colon H \longrightarrow 2^H$ be a set-valued mapping. Then, a point $u \in H$ is a solution of variational inclusion problem if and only if $u = J_\lambda^M(I - \lambda A)u, \forall \lambda > 0$, i.e.,*

$$VI(H, A, M) = F\left(J_\lambda^M(I - \lambda A)\right), \quad \forall \lambda > 0. \tag{16}$$

Furthermore, if A is α-inverse strongly monotone and $\lambda \in (0, 2\alpha]$, then $VI(H, A, M)$ is a closed convex subset of H.

Lemma 5 (see [33]). *ie resolvent operator J_λ^M associated with M is single-valued, nonexpansive, and 1-inverse strongly monotone for all $\lambda > 0$.*

The following two lemmas are the particular case of Lemmas 7 and 8 in [16].

Lemma 6 (see [16]). *For every $i = 1, 2$, let H_i be real Hilbert spaces, let $M_i\colon H_i \longrightarrow 2^{H_i}$ be a multivalued maximal monotone mapping, and let $A_i\colon H_i \longrightarrow H_i$ be an α_i-inverse strongly monotone mapping. Let $T\colon H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^* of T, and let $\widetilde{G}\colon H_1 \longrightarrow H_1$ be a mapping defined by $\widetilde{G}(x) = J_{\lambda_1}^{M_1}(I - \lambda_1 A_1)(x - \gamma T^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2))Tx)$, for all $x \in H_1$. Then, $\|\widetilde{G}x - \widetilde{G}y\|^2 \leq \|x-y\|^2 - \gamma\ (1 - \gamma L)\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2))Tx - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2))Ty\|^2$, for all $x, y \in H_1$, where L is the spectral radius of the operator T^*T, $\lambda_1 \in (0, 2\alpha_1)$, $\lambda_2 \in (0, 2\alpha_2)$, and $\gamma > 0$. Furthermore, if $0 < \gamma < 1/L$, then $\widetilde{G}$ is a nonexpansive mapping.*

Lemma 7 (see [16]). *Let H_1 and H_2 be Hilbert spaces. For $i = 1, 2$, let $M_i: H_i \longrightarrow 2^{H_i}$ be a multivalued maximal monotone mapping and let $A_i: H_i \longrightarrow H_i$ be an α_i-inverse strongly monotone mapping. Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^*. Assume that $\Omega \neq \phi$. Then, $x^* \in \Omega$ if and only if $x^* = \widetilde{G}(x^*)$, where $\widetilde{G}: H_1 \longrightarrow H_1$ is a mapping defined by*

$$\widetilde{G}(x) = J_{\lambda_1}^{M_1}(I - \lambda_1 A_1)\Big(x - \gamma T^*\Big(I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2)\Big)Tx\Big), \tag{17}$$

for all $x \in H_1$, $\lambda_1 \in (0, 2\alpha_1)$, $\lambda_2 \in (0, 2\alpha_2)$, and $0 < \gamma < 1/L$, where L is the spectral radius of the operator T^*T.

Next, we give an example to support Lemma 7.

Example 1. Let $\mathbb{R}$ be a set of real number and $H_1 = H_2 = \mathbb{R}^2$, and let $\langle \cdot, \cdot \rangle: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be inner product defined by $\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and the usual norm $\|\cdot\|: \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\|x\| = \sqrt{x_1^2 + x_2^2}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let $T: H_1 \longrightarrow H_2$ be defined by $Tx = (2x_1, 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $T^*: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T^*z = (2z_1, 2z_2)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let $M_1, M_2: \mathbb{R}^2 \longrightarrow 2^{\mathbb{R}^2}$ be defined by $M_1 x = \{(3x_1 - 5, 3x_2 - 5)\}$ and $M_2 x = \{(x_1/3 - 2, x_2/3 - 2)\}$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A_1, A_2: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $A_1 x = ((x_1 - 4)/2, (x_2 - 4)/2)$ and $A_2 x = ((x_1 - 2)/3, (x_2 - 2)/3)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then, $(2, 2)$ is a fixed point of $\widetilde{G}$. That is, $(2, 2) \in F(\widetilde{G})$.

Proof. It is obvious to see that $\Omega = \{(2, 2)\}$, A_1 is 2-inverse strongly monotone, and A_2 is 3-inverse strongly monotone. Choose $\lambda_1 = 1/3$. Since $M_1 x = \{(3x_1 - 5, 3x_2 - 5)\}$ and the resolvent of $M_1, J_{\lambda_1}^{M_1} x = (I + \lambda_1 M_1)^{-1} x$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, we obtain that

$$J_{\lambda_1}^{M_1} x = \frac{x}{2} + \frac{5}{6}, \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{18}$$

Choose $\lambda_2 = 1$. Since $M_2 x = \{(x_1/3 - 2, x_2/3 - 2)\}$ and the resolvent of $M_2, J_{\lambda_2}^{M_2} x = (I + \lambda_2 M_2)^{-1} x$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, we obtain that

$$J_{\lambda_2}^{M_2} x = \frac{3x}{4} + \frac{3}{2}, \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{19}$$

Since the spectral radius of the operator T^*T is 4, we choose $\gamma = 0.1$. Then, from (18) and (19), we get that

$$\begin{aligned} \widetilde{G}(x) &= J_{1/3}^{M_1}\Big(I - \frac{1}{3}A_1\Big)\Big(x - 0.1T^*\Big(I - J_1^{M_2}(I - A_2)\Big)Tx\Big), \\ &= \frac{x}{3} + \frac{4}{3}, \end{aligned} \tag{20}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then, by Lemma 7, we have that $(2, 2) \in F(\widetilde{G})$. □

Lemma 8 (see [34]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying $s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0$ where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(1) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(2) $\limsup_{n \longrightarrow \infty} \alpha_n/\delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$. Then $\lim_{n \longrightarrow \infty} s_n = 0$.

3. Main Results

In this section, we introduce an iterative algorithm of two sequences which depend on each other by using the intermixed method. Then, we prove a strong convergence theorem for solving two split monotone variational inclusion problems and a finite family of variational inequality problems.

Theorem 2. *Let H_1 and H_2 be Hilbert spaces, and let C be a nonempty closed convex subset of H_1. Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator, and let $f, g: H_1 \longrightarrow H_1$ be ρ_f, ρ_g-contraction mappings with $\rho = \max\{\rho_f, \rho_g\}$. For $i = 1, 2$, let $M_i^x, M_i^y: H_i \longrightarrow 2^{H_i}$ be multivalued maximal monotone mappings and let $A_i^x, A_i^y: H_i \longrightarrow H_i$ be α_i^x, α_i^y–inverse strongly monotone mappings, respectively. For $i = 1, 2, \ldots, N$, let $B_i^x, B_i^y: H_1 \longrightarrow H_1$ be β_i^x, β_i^y–inverse strongly monotone mappings, respectively, $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$, and $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. Let $\widetilde{G}^x, \widetilde{G}^y: H_1 \longrightarrow H_1$ be defined by $\widetilde{G}^x x = J_{\lambda_1^x}^{M_1^x}(I - \lambda_1^x A_1^x)(x - \gamma^x T^*(I - J_{\lambda_2^x}^{M_2^x}(I - \lambda_2^x A_2^x))Tx), \forall x \in H_1$, and $\widetilde{G}^y y = J_{\lambda_1^y}^{M_1^y}(I - \lambda_1^y A_1^y)(y - \gamma^y T^*(I - J_{\lambda_2^y}^{M_2^y}(I - \lambda_2^y A_2^y))Ty), \forall\ y \in H_1$, respectively, where $\lambda_i^x \in (0, 2\alpha_i^x), \lambda_i^y \in (0, 2\alpha_i^y)$, and $0 < \gamma^x, \gamma^y < 1/L$ with L being a spectral radius of T^*T. Assume that $\mathscr{F}^x = \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x)) \neq \varnothing$ and $\mathscr{F}^y = \Omega^y \cap (\cap_{i=1}^N VI(C, B_i^y)) \neq \varnothing$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1, y_1 \in H_1$ and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right)x_n + \eta_n\big(\alpha_n f(y_n) + (1 - \alpha_n)\widetilde{G}^x x_n\big), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^y \sum_{i=1}^N a_i^y B_i^y\right)y_n + \eta_n\big(\alpha_n g(x_n) + (1 - \alpha_n)\widetilde{G}^y y_n\big), \end{cases} \tag{21}$$

for all $n \geq 1$ where $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots, a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, and $\{\mu_n^x\}, \{\mu_n^y\} \subset (0,\infty)$. Assume the following condition holds:

(1) $\sum_{n=1}^{\infty} \mu_n^x < \infty, \sum_{n=1}^{\infty} \mu_n^y < \infty$, and $0 < a < \mu_n^x \leq 2\beta_x, 0 < b < \mu_n^y \leq 2\overline{\beta_y}$, for some $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty, lim_{n \longrightarrow \infty} \alpha_n = 0$.

(3) $\sum_{i=1}^{N} a_i^x = \sum_{n=1}^{N} a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, for all $n \in \mathbb{N}$, for some $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n|, \sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ and $\{y_n\}$ converges strongly to $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$.

Proof. We divided the proof into five steps. □

Step 1. We will show that $\{x_n\}$ and $\{y_n\}$ are bounded. Let $x^* \in \mathcal{F}^x$ and $y^* \in \mathcal{F}^y$. Then, from Lemma 7 and Lemma 6, we get

$$\begin{aligned}\left\|\widetilde{G}^x x_n - x^*\right\| &= \left\|J_{\lambda_1^x}^{M_1^x}(I - \lambda_1^x A_1^x)\left(x_n - \gamma^x T^*\left(I - J_{\lambda_2^x}^{M_2^x}(I - \lambda_2^x A_2^x)\right)Tx_n\right) - x^*\right\|,\\ &\leq \left\|x_n - x^*\right\|.\end{aligned} \tag{22}$$

From (21), Lemma 3, and (22), we have

$$\begin{aligned}\left\|x_{n+1} - x^*\right\| &= \left\|\delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x\right)x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right) - x^*\right\|,\\ &\leq \delta_n\left\|x_n - x^*\right\| + \sigma_n\left\|P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x\right)x_n - x^*\right\| + \eta_n\left\|\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right) - x^*\right\|\\ &\leq \delta_n\left\|x_n - x^*\right\| + \sigma_n\left\|x_n - x^*\right\| + \eta_n\left\|\alpha_n\left(f(y_n) - x^*\right) + (1-\alpha_n)\left(\widetilde{G}^x x_n - x^*\right)\right\|\\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\left[\alpha_n\left\|f(y_n) - x^*\right\| + (1-\alpha_n)\left\|\widetilde{G}^x x_n - x^*\right\|\right]\\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\left[\alpha_n\left\|f(y_n) - f(y^*)\right\| + \alpha_n\left\|f(y^*) - x^*\right\| + (1-\alpha_n)\left\|x_n - x^*\right\|\right]\\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\left[\alpha_n\rho_f\left\|y_n - y^*\right\| + \alpha_n\left\|f(y^*) - x^*\right\| + (1-\alpha_n)\left\|x_n - x^*\right\|\right]\\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\alpha_n\rho\left\|y_n - y^*\right\| + \eta_n\alpha_n\left\|f(y^*) - x^*\right\| + \eta_n(1-\alpha_n)\left\|x_n - x^*\right\|\\ &= (1-\eta_n\alpha_n)\left\|x_n - x^*\right\| + \eta_n\alpha_n\rho\left\|y_n - y^*\right\| + \eta_n\alpha_n\left\|f(y^*) - x^*\right\|.\end{aligned} \tag{23}$$

Similarly, from definition of y_n, we have

$$\left\|y_{n+1} - y^*\right\| \leq (1-\eta_n\alpha_n)\left\|y_n - y^*\right\| + \eta_n\alpha_n\rho\left\|x_n - x^*\right\| + \eta_n\alpha_n\left\|g(x^*) - y^*\right\|. \tag{24}$$

Hence, from (23) and (24), we obtain

$$\begin{aligned}\left\|x_{n+1} - x^*\right\| + \left\|y_{n+1} - y^*\right\| &\leq (1-\eta_n\alpha_n)\left(\left\|x_n - x^*\right\| + \left\|y_n - y^*\right\|\right) + \eta_n\alpha_n\rho\left(\left\|x_n - x^*\right\| + \left\|y_n - y^*\right\|\right)\\ &\quad + \eta_n\alpha_n\left(\left\|f(y^*) - x^*\right\| + \left\|g(x^*) - y^*\right\|\right)\\ &= (1-(1-\rho)\eta_n\alpha_n)\left(\left\|x_n - x^*\right\| + \left\|y_n - y^*\right\|\right) + \eta_n\alpha_n\left(\left\|f(y^*) - x^*\right\| + \left\|g(x^*) - y^*\right\|\right).\end{aligned} \tag{25}$$

By induction, we have

$$\|x_n - x^*\| + \|y_n - y^*\| \le \max\left\{\|x_1 - x^*\| + \|y_1 - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1-\rho}\right\}, \tag{26}$$

for every $n \in \mathbb{N}$. Thus, $\{x_n\}$ and $\{y_n\}$ are bounded.

Step 2. We will show that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}\|y_{n+1} - y_n\| = 0$. Put $u_n = P_C(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x)x_n$, $v_n = P_C(I - \mu_n^y \sum_{i=1}^N a_i^y B_i^y)y_n$, $z_n = \alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n$, and $w_n = \alpha_n f(x_n) + (1-\alpha_n)\widetilde{G}^y y_n$, for all $n \ge 1$. From Lemma 6, we have

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \left\|\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right) - \left(\alpha_{n-1} f(y_{n-1}) + (1-\alpha_{n-1})\widetilde{G}^x x_{n-1}\right)\right\|,\\
&\le \alpha_n\|f(y_n) - f(y_{n-1})\| + (1-\alpha_n)\|\widetilde{G}^x x_n - \widetilde{G}^x x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|\widetilde{G}^x x_{n-1}\|\\
&\le \alpha_n\rho_f\|y_n - y_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|\widetilde{G}^x x_{n-1}\|\\
&\le \alpha_n\rho\|y_n - y_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|\widetilde{G}^x x_{n-1}\|\right).
\end{aligned} \tag{27}$$

By applying Lemma 3, we get that

$$\begin{aligned}
\|u_n - u_{n-1}\| &= \left\|P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n - P_C\left(I - \mu_{n-1}^x\sum_{i=1}^N a_i^x B_i^x\right)x_{n-1}\right\|,\\
&\le \|x_n - x_{n-1}\| + \left\|P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_{n-1} - P_C\left(I - \mu_{n-1}^x\sum_{i=1}^N a_i^x B_i^x\right)x_{n-1}\right\| a_i^x\|B_i^x x_{n-1}\|.\\
&\le \|x_n - x_{n-1}\| + |\mu_n^x - \mu_{n-1}^x|\sum_{i=1}^N
\end{aligned} \tag{28}$$

From the definition of $\{x_n\}$, (27), and (28), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(\delta_n x_n + \sigma_n u_n + \eta_n z_n) - (\delta_{n-1}x_{n-1} + \sigma_{n-1}u_{n-1} + \eta_{n-1}z_{n-1})\|,\\
&\le \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + \sigma_n\|u_n - u_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|u_{n-1}\| + \eta_n\|z_n - z_{n-1}\| + |\eta_n - \eta_{n-1}|\|z_{n-1}\|\\
&\le \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + \sigma_n\left[\|x_n - x_{n-1}\| + |\mu_n^x - \mu_{n-1}^x|\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\|\right] + |\sigma_n - \sigma_{n-1}|\|u_{n-1}\|\\
&\quad + \eta_n\left[\alpha_n\rho\|y_n - y_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|\widetilde{G}^x x_{n-1}\|\right)\right] + |\eta_n - \eta_{n-1}|\|z_{n-1}\|\\
&= (1 - \eta_n\alpha_n)\|x_n - x_{n-1}\| + \eta_n\alpha_n\rho\|y_n - y_{n-1}\| + \sigma_n|\mu_n^x - \mu_{n-1}^x|\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\|\\
&\quad + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|u_{n-1}\| + \eta_n|\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|\widetilde{G}^x x_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\|z_{n-1}\|.
\end{aligned} \tag{29}$$

By the same argument as in (27) and (29), we also have

$$\|w_n - w_{n-1}\| \leq \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\|g(x_{n-1})\| + \|\widetilde{G}^y y_{n-1}\|\right). \tag{30}$$

$$\begin{aligned}\|y_{n+1} - y_n\| \leq{}& (1-\eta_n\alpha_n)\|y_n - y_{n-1}\| + \eta_n\alpha_n\rho\|x_n - x_{n-1}\| + \sigma_n|\mu_n^y - \mu_{n-1}^y|\sum_{i=1}^N a_i^y\|B_i^y y_{n-1}\| \\ &+ |\delta_n - \delta_{n-1}|\|y_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|v_{n-1}\| + \eta_n|\alpha_n - \alpha_{n-1}|\left(\|g(x_{n-1})\| + \|\widetilde{G}^y y_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\|w_{n-1}\|.\end{aligned} \tag{31}$$

From (29) and (31), we obtain that

$$\begin{aligned}\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq{}& (1-(1-\rho)\eta_n\alpha_n)\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right) \\ &+ \sigma_n|\mu_n^x - \mu_{n-1}^x|\left(\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\| + \sum_{i=1}^N a_i^y\|B_i^y y_{n-1}\|\right) + |\delta_n - \delta_{n-1}\|\left(\|x_{n-1}\| + \|y_{n-1}\|\right) + |\sigma_n - \sigma_{n-1}|\left(\|u_{n-1}\| + \|v_{n-1}\|\right) \\ &+ \eta_n|\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|g(x_{n-1})\| + \|\widetilde{G}^x x_{n-1}\| + \|\widetilde{G}^y y_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\left(\|z_{n-1}\| + \|w_{n-1}\|\right) \\ \leq{}& (1-(1-\rho)\overline{a}\alpha_n)\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right) + \overline{b}|\mu_n^x - \mu_{n-1}^x|\left(\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\| + \sum_{i=1}^N a_i^y\|B_i^y y_{n-1}\|\right) \\ &+ |\delta_n - \delta_{n-1}\|\left(\|x_{n-1}\| + \|y_{n-1}\|\right) + |\sigma_n - \sigma_{n-1}|\left(\|u_{n-1}\| + \|v_{n-1}\|\right) \\ &+ \overline{b}|\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|g(x_{n-1})\| + \|\widetilde{G}^x x_{n-1}\| + \|\widetilde{G}^y y_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\left(\|z_{n-1}\| + \|w_{n-1}\|\right).\end{aligned} \tag{32}$$

From (32), conditions (1), (2), and (5), and Lemma 8, we obtain that

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0, \tag{33}$$

$$\lim_{n\longrightarrow\infty}\|y_{n+1} - y_n\| = 0. \tag{34}$$

Step 3. We show that $\lim_{n\longrightarrow\infty}\|z_n - P_C(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x)z_n\| = \lim_{n\longrightarrow\infty}\|z_n - \widetilde{G}^x z_n\| = 0$. From (21), we have that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 ={}& \left\|\delta_n(x_n - x^*) + \sigma_n\left(P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n - x^*\right) + \eta_n(z_n - x^*)\right\|^2, \\ \leq{}& \delta_n\|x_n - x^*\|^2 + \sigma_n\left\|P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n - x^*\right\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + \eta_n\|z_n - x^*\|^2 \\ \leq{}& (1-\eta_n)\|x_n - x^*\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + \eta_n\left\|\alpha_n\left(f\left(y_n - \widetilde{G}^x x_n\right) + \left(\widetilde{G}^x x_n - x^*\right)\right)\right\|^2 \\ \leq{}& (1-\eta_n)\|x_n - x^*\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + \eta_n\left[\|\widetilde{G}^x x_n - x^*\|^2 + 2\alpha_n\langle f(y_n) - \widetilde{G}^x x_n, z_n - x^*\rangle\right] \\ \leq{}& \|x_n - x^*\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + 2\eta_n\alpha_n\|f(y_n) - \widetilde{G}^x x_n\|\|z_n - x^*\|,\end{aligned} \tag{35}$$

$$\begin{aligned}\delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 \leq{}& \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\eta_n\alpha_n\|f(y_n) - \widetilde{G}^x x_n\|\|z_n - x^*\|, \\ \leq{}& \|x_n - x_{n+1}\|\left[\|x_n - x^*\| + \|x_{n+1} - x^*\|\right] + 2\eta_n\alpha_n\|f(y_n) - \widetilde{G}^x x_n\|\|z_n - x^*\|.\end{aligned} \tag{36}$$

Then, we have

$$\left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{37}$$

Observe that

$$x_{n+1} - x_n = \sigma_n\left(P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - x_n \right) + \eta_n (z_n - x_n). \tag{38}$$

From (33) and (37), we have

$$\| z_n - x_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{39}$$

By the same argument as above, we also have that

$$\| w_n - x_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{40}$$

Note that

$$\begin{aligned}
\left\| z_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\| &\leq \| z_n - x_n \| + \left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| \\
&\quad + \left\| P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\|, \\
&\leq \| z_n - x_n \| + \left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| + \| x_n - z_n \| \\
&= 2\| z_n - x_n \| + \left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\|,
\end{aligned} \tag{41}$$

By (37) and (39), we get that

$$\left\| z_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{42}$$

By the same argument as (41), we also obtain

$$\left\| w_n - P_C\left(I - \mu_n^y \sum_{i=1}^{N} a_i^y B_i^y \right) w_n \right\| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{43}$$

Consider

$$\| x_{n+1} - z_n \| \leq \| x_{n+1} - x_n \| + \| x_n - z_n \|. \tag{44}$$

By (33) and (39), we get that

$$\| x_{n+1} - z_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{45}$$

However,

$$\begin{aligned}
\| x_n - \widetilde{G}^x x_n \| &\leq \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| + \| z_n - \widetilde{G}^x x_n \|, \\
&= \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| + \| \alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n - \widetilde{G}^x x_n \| \\
&= \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| + \alpha_n \| f(y_n) - \widetilde{G}^x x_n \|,
\end{aligned} \tag{46}$$

It follows from (33) and (45) that

$$\| x_n - \widetilde{G}^x x_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{47}$$

Consider

$$\begin{aligned}
\| z_n - \widetilde{G}^x z_n \| &\leq \| z_n - x_n \| + \| x_n - \widetilde{G}^x x_n \| + \| \widetilde{G}^x x_n - \widetilde{G}^x z_n \|, \\
&\leq 2\| z_n - x_n \| + \| x_n - \widetilde{G}^x x_n \|.
\end{aligned} \tag{48}$$

From (39) and (47), we obtain

$$\| z_n - \widetilde{G}^x z_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{49}$$

Applying the same method as (48), we also have

$$\| w_n - \widetilde{G}^y w_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{50}$$

Step 4. We will show that $\limsup_{n\longrightarrow\infty}\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle \leq 0$ and $\limsup_{n\longrightarrow\infty}\langle g(\widetilde{x}) - \widetilde{y}, z_n - \widetilde{y}\rangle \leq 0$, where $\widetilde{x} = P_{\mathscr{F}^x} f(\widetilde{y})$ and $\widetilde{y} = P_{\mathscr{F}^y} f(\widetilde{x})$. First, we take a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle = \lim_{k\longrightarrow\infty}\langle f(\widetilde{y}) - \widetilde{x}, z_{n_k} - \widetilde{x}\rangle. \tag{51}$$

Since $\{x_n\}$ is bounded, there exists a subsequence x_{n_k} of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q_1$ as $k \longrightarrow \infty$. From (39), we get that $z_{n_k} \rightharpoonup q_1$. Next, we need to show that $q_1 \in \mathscr{F}^x = \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x))$. Assume that $q_1 \notin \Omega^x$. By Lemma 7, we get that $q_1 \neq \widetilde{G}^x q_1$. Applying Opial's condition and (49), we get that

$$\begin{aligned}
\liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\| &< \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - \widetilde{G}^x q_1\right\|,\\
&\leq \liminf_{n\longrightarrow\infty}\left\|z_{n_k} - \widetilde{G}^x z_{n_k}\right\| + \liminf_{k\longrightarrow\infty}\left\|\widetilde{G}^x z_{n_k} - \widetilde{G}^x q_1\right\|\\
&\leq \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\|.
\end{aligned} \tag{52}$$

This is a contradiction. Thus, $q_1 \in \Omega^x$.

Assume that $q_1 \notin \cap_{i=1}^N VI(C, B_i^x)$. Then, from Lemma 3 and Lemma 2, we have $q_1 \notin F(P_C(I - \mu_n^x \sum_{i=1}^N a_i B_i^x))$. From Opial's condition and (42), we obtain

$$\begin{aligned}
\liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\| &< \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - P_C\left(I - \mu_{n_k}^x \sum_{i=1}^N a_i^x B_i^x\right)q_1\right\|,\\
&\leq \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - P_C\left(I - \mu_{n_k}^x \sum_{i=1}^N a_i^x B_i^x\right)z_{n_k}\right\| + \liminf_{k\longrightarrow\infty}\left\|P_C\left(I - \mu_{n_k}^x \sum_{i=1}^N a_i^x B_i^x\right)z_{n_k} - P_C\left(I - \mu_{n_k} \sum_{i=1}^N a_i^x B_i^x\right)q_1\right\|\\
&\leq \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\|.
\end{aligned} \tag{53}$$

This is a contradiction. Thus, $q_1 \in \cap_{i=1}^N VI(C, B_i^x)$, and so,

$$q_1 \in \mathscr{F}^x = \Omega^x \cap \left(\bigcap_{i=1}^N VI\left(C, B_i^x\right)\right). \tag{54}$$

However, $z_{n_k} \rightharpoonup q_1$. From (54) and Lemma 1, we can derive that

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle &= \lim_{k\longrightarrow\infty}\langle f(\widetilde{y}) - \widetilde{x}, z_{n_k} - \widetilde{x}\rangle,\\
&= \langle f(\widetilde{y}) - \widetilde{x}, q_1 - \widetilde{x}\rangle\\
&\leq 0.
\end{aligned} \tag{55}$$

By the same method as (55), we also obtain that

$$\limsup_{n\longrightarrow\infty}\langle g(\widetilde{x}) - \widetilde{y}, z_n - \widetilde{y}\rangle \leq 0. \tag{56}$$

Step 5. Finally, we show that the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to $\widetilde{x} = P_{\mathscr{F}^x} f(\widetilde{y})$ and $\widetilde{y} = P_{\mathscr{F}^y} f(\widetilde{x})$, respectively. From the definition of z_n, we have

$$\begin{aligned}
\left\|z_n - \widetilde{x}\right\|^2 &= \langle \alpha_n (f(y_n) - \widetilde{x}) + (1 - \alpha_n)\left(\widetilde{G}^x x_n - \widetilde{x}\right), z_n - \widetilde{x}\rangle,\\
&= \alpha_n\langle f(y_n) - \widetilde{x}, z_n - \widetilde{x}\rangle + (1 - \alpha_n)\langle \widetilde{G}^x x_n - \widetilde{x}, z_n - \widetilde{x}\rangle\\
&\leq \alpha_n\langle f(y_n) - f(\widetilde{y}), z_n - \widetilde{x}\rangle + \alpha_n\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle + (1 - \alpha_n)\left\|\widetilde{G}^x x_n - \widetilde{x}\right\|\left\|z_n - \widetilde{x}\right\|\\
&\leq \alpha_n\rho\left\|y_n - \widetilde{y}\right\|\left\|z_n - \widetilde{x}\right\| + \alpha_n\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle + (1 - \alpha_n)\left\|x_n - \widetilde{x}\right\|\left\|z_n - \widetilde{x}\right\|\\
&\leq \frac{\alpha_n\rho}{2}\left[\left\|y_n - \widetilde{y}\right\|^2 + \left\|z_n - \widetilde{x}\right\|^2\right] + \alpha_n\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle + \frac{(1 - \alpha_n)}{2}\left[\left\|x_n - \widetilde{x}\right\|^2 + \left\|z_n - \widetilde{x}\right\|^2\right],
\end{aligned} \tag{57}$$

which implies that

$$\|z_n - \tilde{x}\|^2 \leq \frac{\alpha_n \rho}{1 + \alpha_n (1 - \rho)} \|y_n - \tilde{y}\|^2 + \frac{(1 - \alpha_n)}{1 + \alpha_n (1 - \rho)} \|x_n - \tilde{x}\|^2 + \frac{2\alpha_n}{1 + \alpha_n (1 - \rho)} \langle f(\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle. \tag{58}$$

From the definition of $\{x_n\}$ and (58), we get

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \delta_n \|x_n - \tilde{x}\|^2 + \sigma_n \left\| P_C \left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - \tilde{x} \right\|^2 + \eta_n \|z_n - \tilde{x}\|^2, \\
&\leq (1 - \eta_n) \|x_n - \tilde{x}\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1 - \rho)} \|y_n - \tilde{y}\|^2 + \frac{(1 - \alpha_n)\eta_n}{1 + \alpha_n (1 - \rho)} \|x_n - \tilde{x}\|^2 + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \langle f(\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle \\
&= \left(1 - \frac{\alpha_n \eta_n (2 - \rho)}{1 + \alpha_n (1 - \rho)} \right) \|x_n - \tilde{x}\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1 - \rho)} \|y_n - \tilde{y}\|^2 + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \langle f(\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle.
\end{aligned} \tag{59}$$

Applying the same argument as in (58) and (59), we get

$$\|y_{n+1} - \tilde{y}\|^2 \leq \left(1 - \frac{\alpha_n \eta_n (2 - \rho)}{1 + \alpha_n (1 - \rho)} \right) \|y_n - \tilde{y}\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1 - \rho)} \|x_n - \tilde{x}\|^2 + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \langle g(\tilde{x}) - \tilde{y}, z_n - \tilde{y} \rangle. \tag{60}$$

From (58) and (59), we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 + \|y_{n+1} - \tilde{y}\|^2 &\leq \left(1 - \frac{\alpha_n \eta_n (2 - \rho)}{1 + \alpha_n (1 - \rho)} \right) \left[\|x_n - \tilde{x}\|^2 + \|y_n - \tilde{y}\|^2 \right] \\
&\quad + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1 - \rho)} \left[\|x_n - \tilde{x}\|^2 + \|y_n - \tilde{y}\|^2 \right] + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \left[\langle f(\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle + \langle g(\tilde{x}) - \tilde{y}, z_n - \tilde{y} \rangle \right], \\
&\leq \left(1 - \frac{2\alpha_n \eta_n (1 - \rho)}{1 + \alpha_n (1 - \rho)} \right) \left[\|x_n - \tilde{x}\|^2 + \|y_n - \tilde{y}\|^2 \right] + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \left[\langle f(\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle + \langle g(\tilde{x}) - \tilde{y}, z_n - \tilde{y} \rangle \right].
\end{aligned} \tag{61}$$

According to condition (2) and (4), (61), and Lemma 8, we can conclude that $\{x_n\}$ and $\{y_n\}$ converge strongly to $\tilde{x} = P_{\mathcal{F}^x} f(\tilde{y})$ and $\tilde{y} = P_{\mathcal{F}^y} g(\tilde{x})$, respectively. Furthermore, from (39) and (40), we get that $\{z_n\}$ and $\{w_n\}$ converge strongly to $\tilde{x} = P_{\mathcal{F}^x} f(\tilde{y})$ and $\tilde{y} = P_{\mathcal{F}^y} g(\tilde{x})$, respectively. This completes the proof. □

One of the great special cases of the SMVIP is the split variational inclusion problem that has a wide variety of application backgrounds, such as split minimization problems and split feasibility problems.

If we set $A_i^x = 0$ and $A_i^y = 0$ in Theorem 2, for all $i = 1, 2$, then we get the strong convergence theorem for the split variational inclusion problem and the finite families of the variational inequality problems as follows:

Corollary 1. *Let H_1 and H_2 be Hilbert spaces, and let C be a nonempty closed convex subset of H_1. Let $T\colon H_1 \longrightarrow H_2$ be a bounded linear operator, and let $f, g\colon H_1 \longrightarrow H_1$ be ρ_f, ρ_g-contraction mappings with $\rho = \max\{\rho_f, \rho_g\}$. For every $i = 1, 2$, let $M_i^x, M_i^y\colon H_i \longrightarrow 2^{H_i}$ be multivalued maximal monotone mappings. For $i = 1, 2, \ldots, N$, let B_i^x, B_i^y:*

$H_1 \longrightarrow H_1$ *be* β_i^x, β_i^y*–inverse strongly monotone with* $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$ *and* $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. *Let* $\mathcal{S}^x = \{x^* \in H_1\colon 0 \in M_1^x x^*, \tilde{x} = Tx^* \in H_2\colon 0 \in M_2^x \tilde{x}\}$ *and* $\mathcal{S}^y = \{y^* \in H_1\colon 0 \in M_1^y y^*, \tilde{y} = Ty^* \in H_2\colon 0 \in M_2^y \tilde{y}\}$. *Assume that* $\mathcal{F}^x = \mathcal{S}^x \cap (\cap_{i=1}^N VI(C, B_i^x)) \neq \varnothing$ *and* $\mathcal{F}^y = \mathcal{S}^y \cap (\cap_{i=1}^N VI(C, B_i^y)) \neq \varnothing$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_1, y_1 \in H_1$ *and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right)x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n) J_{\lambda_1^x}^{M_1^x}\left(x - \gamma^x T^*\left(I - J_{\lambda_2^x}^{M_2^x}\right)Tx_n\right)\right), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^y B_i^y\right)y_n + \eta_n\left(\alpha_n g(x_n) + (1-\alpha_n) J_{\lambda_1^y}^{M_1^y}\left(y - \gamma^y T^*\left(I - J_{\lambda_2^y}^{M_2^y}\right)Ty_n\right)\right), \end{cases} \tag{62}$$

for all $n \geq 1$, *where* $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, $\lambda_i^x, \lambda_i^y \in (0,\infty)$ *for all* $i = 1,2$, *and* $0 < \gamma^x, \gamma^y < 1/L$ *with* L *being a spectral radius of* T^*T. *Assume the following conditions hold:*

(1) $\sum_{n=1}^\infty \mu_n^x < \infty, \sum_{n=1}^\infty \mu_n^y < \infty$, *and* $0 < a < \mu_n^x \leq 2\overline{\beta_x}$, $0 < b < \mu_n^y \leq 2\overline{\beta}_y$, *for some* $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^\infty \alpha_n = \infty, \lim_{n\longrightarrow\infty}\alpha_n = 0$.

(3) $\sum_{n=1}^N a_i^x = \sum_{n=1}^N a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, *for all* $n \in \mathbb{N}$, *for some* $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n|, \sum_{n=1}^\infty |\sigma_{n+1} - \sigma_n|$, *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $\tilde{x} = P_{\mathcal{F}^x} f(\tilde{y})$ *and* $\{y_n\}$ *converges strongly to* $\tilde{y} = P_{\mathcal{F}^y} g(\tilde{x})$.

4. Applications

In this section, by applying our main result in Theorem 2, we can prove strong convergence theorems for approximating the solution of the split convex minimization problems and split feasibility problems.

4.1. Split Convex Minimization Problems. Let $\varphi\colon H \longrightarrow \mathbb{R}$ be a convex and differentiable function and $\psi\colon H \longrightarrow (-\infty, \infty]$ be a proper convex and lower semicontinuous function. It is well known that if $\nabla\varphi$ is $1/\alpha$-Lipschitz continuous, then it is α-inverse strongly monotone, where $\nabla\varphi$ is the gradient of φ (see [10]). It is also known that the subdifferential $\partial\psi$ of ψ is maximal monotone (see [35]). Moreover,

$$\varphi(x^*) + \psi(x^*) = \min_{x\in H}[\varphi(x) + \psi(x)] \Leftrightarrow 0 \in \nabla\varphi(x^*) + \partial\varphi(x^*). \tag{63}$$

Next, we consider the following the split convex minimization problem (SCMP): find

$$x^* \in H_1, \quad \text{such that } \varphi_1(x^*) + \psi_1(x^*) = \min_{x\in H_1}[\varphi_1(x) + \psi_1(x)] \tag{64}$$

and such that $y^* = Tx^* \in H_2$ solves

$$\varphi_2(y^*) + \psi_2(y^*) = \min_{y\in H_2}[\varphi_2(y) + \psi_2(y)], \tag{65}$$

where $T\colon H_1 \longrightarrow H_2$ is a bounded linear operator with adjoint T^*, φ_i, and ψ_i defined as above, for $i = 1,2$. We denoted the set of all solutions of (64) and (65) by Θ. That is, $\Theta = \{x^* \text{ which solves (64)}\colon Tx^* \text{ solves (65)}\}$.

If we set $A_i^x = \nabla\varphi_i^x, A_i^y = \nabla\varphi_i^y$, and $M_i^x = \partial\psi_i^x, M_i^y = \partial\psi_i^y$, for $i = 1,2$, in Theorem 2, then we get the strong convergence theorem for finding the common solution of the split convex minimization problems and the finite families of the variational inequality problems as follows.

Theorem 3. *Let* H_1 *and* H_2 *be Hilbert spaces, and let* C *be a nonempty closed convex subset of* H_1. *Let* $T\colon H_1 \longrightarrow H_2$ *be a bounded linear operator, and let* $f, g\colon H_1 \longrightarrow H_1$ *be* ρ_f, ρ_g*-contraction mappings with* $\rho = \max\{\rho_f, \rho_g\}$. *For* $i = 1,2$, *let* $\psi_i^x, \psi_i^y\colon H_i \longrightarrow (-\infty, \infty]$ *be proper convex and lower semicontinuous functions, and let* $\varphi_i^x, \varphi_i^y\colon H_i \longrightarrow \mathbb{R}$ *be convex and differentiable function such that* $\nabla\varphi_i^x$ *and* $\nabla\varphi_i^y$ *be* $1/\alpha_i^x$*-Lipschitz continuous and* $1/\alpha_i^y$*-Lipschitz continuous, respectively. For* $i = 1,2,\ldots,N$, *let* $B_i^x, B_i^y\colon H_1 \longrightarrow H_1$ *be* β_i^x, β_i^y*–inverse strongly monotone with* $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$ *and* $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. *Let* $\tilde{G}^x, \tilde{G}^y\colon H_1 \longrightarrow H_1$ *be defined by* $\tilde{G}^x x = J_{\lambda_1^x}^{\partial\psi_1^x}(I - \lambda_1^x \nabla\varphi_1^x)(x - \gamma^x T^*(I - J_{\lambda_2^x}^{\partial}\ \psi_2^x(I - \lambda_2^x \nabla\varphi_2^x))Tx), \forall x \in H_1$, *and* $\tilde{G}^y y = J_{\lambda_1^y}^{\partial\psi_1^y}(I - \lambda_1^y \nabla\ \varphi_1^y)(y - \gamma^y T^*(I - J_{\lambda_2^y}^{\partial\psi_2^y}(I - \lambda_2^y \nabla\varphi_2^y))Ty), \forall y \in H_1$, *respectively, where* $\lambda_i^x \in (0, 2\alpha_i^x), \lambda_i^y \in (0, 2\alpha_i^y)$ *and* $0 < \gamma^x, \gamma^y < 1/L$ *with* L *is a spectral radius of* T^*T. *Assume that* $\mathcal{F}^x = \Theta^x \cap (\cap_{i=1}^N$

$VI(C,B_i^x)) \neq \emptyset$ and $\mathcal{F}^y = \Theta^y \cap (\cap_{i=1}^N VI(C,B_i^y)) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1, y_1 \in H_1$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right)x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^y \sum_{i=1}^N a_i^y B_i^y\right)y_n + \eta_n\left(\alpha_n g(x_n) + (1-\alpha_n)\widetilde{G}^y y_n\right), \end{cases} \tag{66}$$

for all $n \geq 1$ where $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, and $\{\mu_n^x\}, \{\mu_n^y\} \subset (0,\infty)$. Assume the following condition holds:

(1) $\sum_{n=1}^\infty \mu_n^x < \infty, \sum_{n=1}^\infty \mu_n^y < \infty$, and $0 < a < \mu_n^x \leq 2\overline{\beta}_x$, $0 < b < \mu_n^y \leq 2\overline{\beta}_y$, for some $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^\infty \alpha_n = \infty$, $lim_{n\longrightarrow\infty}\alpha_n = 0$.

(3) $\sum_{n=1}^N a_i^x = \sum_{n=1}^N a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, for all $n \in \mathbb{N}$, for some $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n|$, $\sum_{n=1}^\infty |\sigma_{n+1} - \sigma_n|$, and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ and $\{y_n\}$ converges strongly to $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$.

4.2. The Split Feasibility Problem. Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be the nonempty closed convex subset of H_1 and H_2, respectively. The split feasibility problem (SFP) is to find

$$\text{a point } x \in C, \quad \text{such that } Ax \in Q. \tag{67}$$

The set of all solutions (SFP) is denoted by $\Psi = \{x \in C\colon Ax \in Q\}$. This problem was introduced by Censor and Elfving [8] in 1994. The split feasibility problem was investigated extensively as a widely important tool in many fields such as signal processing, intensity-modulated radiation therapy problems, and computer tomography (see [36–38] and the references therein).

Let H be a real Hilbert space, and let h be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. The subdifferential ∂h of h is defined by $\partial h(x) = \{z \in H\colon h(x) + \langle z, u - x\rangle \leq h(u), \forall u \in H\}$ for all $x \in H$. Then, ∂h is a maximal monotone operator [39]. Let C be a nonempty closed convex subset of H, and let i_C be the indicator function of C, i.e., $i_C(x) = 0$ if $x \in C$ and $i_C(x) = \infty$ if $x \notin C$. Then, i_C is a proper, lower semicontinuous and convex function on H, and so the subdifferential ∂i_C of i_C is a maximal monotone operator. Then, we can define the resolvent operator $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$, by $J_\lambda^{\partial i_C}(x) = (I + \lambda\, \partial i_C)^{-1}(x)$, for all $x \in H$.

Recall that the normal cone $N_C(u)$ of C at a point u in H is defined by $N_C(u) = \{z \in H\colon \langle z, u - v \leq 0\rangle, \forall v \in C\}$ if $u \in C$ and $N_C(u) = \emptyset$ if $u \notin C$. We note that $\partial i_C = N_C$, and for $\lambda > 0$, we have that $u = J_\lambda^{\partial i_C} x$ if and only if $u = P_C x$ (see [31]).

Setting $M_1 = \partial i_C$, $M_2 = \partial i_Q$, and in (SMVI) (4) and (5), then (SMVI) (4) and (5) are reduced to the split feasibility problem (SFP) (67)

Now, by applying Theorem 2, we get the following strong convergence theorem to approximate a common solution of SFP (67) and a finite family of variational inequality problems.

Theorem 4. *Let H_1 and H_2 be Hilbert spaces, and let C and Q be the nonempty closed convex subset of H_1 and H_2, respectively. Let $T\colon H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^*, and let $f, g\colon H_1 \longrightarrow H_1$ be ρ_f, ρ_g-contraction mappings with $\rho = \max\{\rho_f, \rho_g\}$. For $i = 1, 2, \ldots, N$, let $B_i^x, B_i^y\colon H_1 \longrightarrow H_1$ be β_i^x, β_i^y−inverse strongly monotone with $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$ and $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. Assume that $\mathcal{F}^x = \Psi^x \cap (\cap_{i=1}^N VI(C,B_i^x)) \neq \emptyset$ and $\mathcal{F}^y = \Psi^y \cap (\cap_{i=1}^N VI(C,B_i^y)) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1, y_1 \in H_1$ and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right)x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n)P_C\left(x - \gamma^x T^*\left(I - P_Q\right)T x_n\right)\right), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^y B_i^y\right)y_n + \eta_n\left(\alpha_n g(x_n) + (1-\alpha_n)P_C\left(y - \gamma^y T^*\left(I - P_Q\right)T y_n\right), \end{cases} \tag{68}$$

for all $n \geq 1$, *where* $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots, a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, $\{\mu_n^x\}, \{\mu_n^y\} \subset (0,\infty)$, $\lambda_i^x, \lambda_i^y \in (0,\infty)$ *for all* $i = 1,2$, *and* $0 < \gamma^x, \gamma^y < 1/L$ *with L being a spectral radius of* T^*T. *Assume the following condition holds:*

(1) $\sum_{n=1}^{\infty} \mu_n^x < \infty, \sum_{n=1}^{\infty} \mu_n^y < \infty$, *and* $0 < a < \mu_n^x \leq 2\overline{\beta_x}, 0 < b < \mu_n^y \leq 2\overline{\beta_y}$, *for some* $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty, lim_{n\longrightarrow\infty}\alpha_n = 0$.

(3) $\sum_{n=1}^{N} a_i^x = \sum_{n=1}^{N} a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, *for all* $n \in \mathbb{N}$, *for some* $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n|$, $\sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ *and* $\{y_n\}$ *converges strongly to* $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$.

Proof. Set $M_1^x = \partial i_C, M_1^y = \partial i_C$, $M_2^x = \partial i_Q, M_2^y = \partial i_Q$, and $A_i^x = 0$ and $A_i^y = 0$ in Theorem 2. Then, we get the result. □

The split feasibility problem is a significant part of the split monotone variational inclusion problem. It is extensively used to solve practical problems in numerous situations. Many excellent results have been obtained. In what follows, an example of a signal recovery problem is introduced.

Example 2. In signal recovery, compressed sensing can be modeled as the following under-determined linear equation system:

$$y = Ax + \delta, \tag{69}$$

where $x \in \mathbb{R}^N$ is a vector with m non-zero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy δ, and $A: \mathbb{R}^N \longrightarrow \mathbb{R}^M$ $(M < N)$ is a bounded linear observation operator. An essential point of this problem is that the signal x is sparse; that is, the number of nonzero elements in the signal x is much smaller than the dimension of the signal x. To solve this situation, a classical model, convex constraint minimization problem, is used to describe the above problem. It is known that problem (69) can be seen as solving the following LASSO problem [40]:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y - Ax\|_2^2, \quad \text{subject to } \|x\|_1 \leq t, \tag{70}$$

where $t > 0$ is a given constant and $\|\cdot\|_1$ is ℓ_1 norm. In particular, LASSO problem (70) is equivalent to the split feasibility problem (SFP) (67) when $C = \{x \in \mathbb{R}^N: \|x\|_1 \leq t\}$ and $Q = \{y\}$.

5. Numerical Examples

In this section, we give some examples for supporting Theorem 2. In example 3, we give the computer programming to support our main result.

Example 3. Let $\mathbb{R}$ be a set of real number and $H_1 = H_2 = \mathbb{R}^2$. Let $C = [-20, 20] \times [-20, 20]$, and let $\langle \cdot, \cdot \rangle: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be inner product defined by $\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and the usual norm $\|\cdot\|: \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\|x\| = \sqrt{x_1^2 + x_2^2}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $Tx = (2x_1, 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $T^*: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T^*z = (2z_1, 2z_2)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let $M_1^x, M_1^y, M_2^x, M_2^y: \mathbb{R}^2 \longrightarrow 2^{\mathbb{R}^2}$ be defined by $M_1^x x = \{(3x_1 - 2, 3x_2 - 2)\}, M_1^y x = \{(2x_1, 2x_2)\}, M_2^x = \{((x_1/3) + 2, (x_2/3) + 2)\}$, and $M_2^y x = \{((x_1/3) + 3, (x_2/3) + 3)\}$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A_1^x, A_1^y, A_2^x, A_2^y: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $A_1^x x = ((x_1 - 3)/2, (x_2 - 3)/2), A_1^y x = (x_1 + 6, x_2 + 6)$, $A_2^x x = ((x_1 - 2)/3, (x_2 - 2)/3)$, and $A_2^y = ((x_1 - 1)/3, (x_2 - 1)/3)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. For every $i = 1, 2, \ldots, N$, let the mappings $B_i^x, B_i^y: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $B_i^x x = ((x_1 - 1)/3i, (x_2 - 1)/3i)$ and $B_i^y x = ((x_1 + 2)/5i, (x_2 + 2)/5i)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$, and let $a_i^x = (2/3^i + 1/N3^N)$ and $a_i^y = (4/5^i + 1/N5^N)$. Let the mappings $f, g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $f(x) = (x_1/7, x_2/7)$ and $g(x) = (x_1/9, x_2/9)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Choose γ^x and $\gamma^y = 0.1$, $\lambda_1^x = 2, \lambda_1^y = 1.2, \lambda_2^x = 0.1$, and $\lambda_2^y = 1.9$. Setting $\{\delta_n\} = \{n/(9n + 3)\}$, $\{\sigma_n\} = \{(4n + (2/3))/(9n + 3)\}$, $\{\eta_n\} = \{(4n + (7/3))/(9n + 3)\}$, $\{\alpha_n\} = \{1/20n\}$, $\{\mu_n^x\} = \{1/7n^2\}$, and $\{\mu_n^y\} = \{1/5n^2\}$. Let $x_1 = (x_1^1, x_1^2)$ and $y_1 = (y_1^1, y_1^2) \in \mathbb{R}^2$, and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (21) as follows:

Table 1: Values of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-10, 10)$, $y_1 = (-10, 10)$, and $n = N = 50$.

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−10.000000, 10.000000)	(−10.000000, 10.000000)
2	(−4.093342, 5.126154)	(−5.481385, 3.338182)
3	(−1.544910, 3.037093)	(−3.639877, 0.575101)
4	(−0.307347, 2.029037)	(−2.789106, −0.719370)
⋮	⋮	⋮
30	(0.998745, 0.996821)	(−1.996611, −1.998243)
⋮	⋮	⋮
47	(0.998973, 0.998235)	(−1.998394, −1.999022)
48	(0.998986, 0.998280)	(−1.998454, −1.999055)
49	(0.999000, 0.998324)	(−1.998511, −1.999087)
50	(0.999013, 0.998365)	(−1.998566, −1.999118)

$$\begin{cases} x_{n+1} = \dfrac{n}{9n+3}x_n + \dfrac{4n+(2/3)}{9n+3}P_C\left(I - \dfrac{1}{7n^2}\sum_{i=1}^{N}\left(\dfrac{2}{3^i} + \dfrac{1}{N3^N}\right)B_i^x\right)x_n + \dfrac{4n+7/3}{9n+3}\left(\dfrac{1}{20n}f(y_n) + \dfrac{20n-1}{20n}\widetilde{G}^x x_n\right), \\ y_{n+1} = \dfrac{n}{9n+3}y_n + \dfrac{4n+(2/3)}{9n+3}P_C\left(I - \dfrac{1}{5n^2}\sum_{i=1}^{N}\left(\dfrac{4}{5^i} + \dfrac{1}{N5^N}\right)B_i^y\right)y_n + \dfrac{4n+7/3}{9n+3}\left(\dfrac{1}{20n}g(x_n) + \dfrac{20n-1}{20n}\widetilde{G}^y y_n\right), \\ \widetilde{G}^x x_n = J_2^{M_1^x}(I - 2A_1^x)\left(x - 0.1T^*\left(I - J_{0.1}^{M_2^x}(I - 0.1A_2^x)\right)Tx_n\right), \\ \widetilde{G}^y y_n = J_{1.2}^{M_1^y}(I - 1.2A_1^y)\left(y - 0.1T^*\left(I - J_{1.9}^{M_2^y}(I - 1.9A_2^y)\right)Ty_n\right), \end{cases} \tag{71}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2)$ and $y_n = (y_n^1, y_n^2)$. By the definition of M_i^x, M_i^y, A_i^x, and A_i^y, for all $i = 1, 2$, B_i^x and B_i^y, for all $i = 1, 2, \ldots, N$, and f and g, we have that $(1,1) \in \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x))$ and $(-2,-2) \in \Omega^y \cap (\cap_{i=1}^N VI(C, B_i^y))$. Also, it is easy to see that all parameters satisfy all conditions in Theorem 2. Then, by Theorem 2, we can conclude that the sequence $\{x_n\}$ converges strongly to $(1,1)$ and $\{y_n\}$ converges strongly to $(-2,-2)$.

Table 1 and Figure 1 show the numerical results of $\{x_n\}$ and $\{y_n\}$ where $x_1 = (-10, 10)$, $y_1 = (-10, 10)$, and $n = N = 50$.

Next, in Example 4, we only show an example in infinite-dimensional Hilbert space for supporting Theorem 2. We omit the computer programming.

Example 4. Let $H_1 = H_2 = C = \ell_2$ be the linear space whose elements consist of all 2-summable sequence $(x_1, x_2, \ldots, x_j, \ldots)$ of scalars, i.e.,

$$\ell_2 = \left\{x\colon x = \left(x_1, x_2, \ldots, x_j, \ldots\right) \text{ and } \sum_{j=1}^{\infty}\left|x_j\right|^2 < \infty\right\}, \tag{72}$$

with an inner product $\langle\cdot,\cdot\rangle\colon \ell_2 \times \ell_2 \longrightarrow \mathbb{R}$ defined by $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j$, where $x = \{x_j\}_{j=1}^{\infty}$ and $y = \{y_j\}_{j=1}^{\infty} \in \ell_2$, and a norm $\|\cdot\|\colon \ell_2 \longrightarrow \mathbb{R}$ defined by $\|x\|_2 = (\sum_{j=1}^{\infty}|x_j|^2)^{1/2}$ where $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let $T\colon \ell_2 \longrightarrow \ell_2$ be defined by $Tx = (x_1/2, x_2/2, \ldots, x_j/2, \ldots)$ for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. And, $T^*\colon \ell_2 \longrightarrow \ell_2$ be defined by $Tx = (z_1/2, z_2/2, \ldots, z_j/2, \ldots)$ for all $x = \{z_j\}_{j=1}^{\infty} \in \ell_2$. Let $M_1^x, M_1^y, M_2^x, M_2^y\colon \ell_2 \longrightarrow \ell_2$ be defined by $M_1^x x = \{(2x_1, 2x_2, \ldots, 2x_j, \ldots)\}$, $M_1^y x = \{(x_1 - 1, x_2 - 1, \ldots, x_j - 1, \ldots)\}$, $M_2^x = \{3x_1, 3x_2, \ldots, 3x_j, \ldots\}$, and $M_2^y x = \{(2x_1 - 1, 2x_2 - 1, \ldots, 2x_j - 1, \ldots)\}$, respectively, for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let the mapping A_1^x, A_1^y, A_2^x, A_2^y: $\ell_2 \longrightarrow \ell_2$ be defined by $A_1^x x = (x_1/3, x_2/3, \ldots, x_j/3, \ldots)$, $A_1^y x = ((x_1 - 1)/2, (x_2 - 1)/2, \ldots, (x_j - 1)/2, \ldots)$, $A_2^x x = (x_1/4, x_2/4, \ldots, x_j/4, \ldots)$, and $A_2^y = ((2x_1 - 1)/3, (2x_2 - 1)/3, \ldots, (2x_j - 1)/3)$ respectively, for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. For every $i = 1, 2, \ldots, N$, let the mappings $B_i^x, B_i^y\colon \ell_2 \longrightarrow \ell_2$ be defined by $B_i^x x = (2x_1/3i, 2x_2/3i, \ldots, 2x_j/3i, \ldots)$ and $B_i^y x = ((2x_1 - 1)/4i, (2x_2 - 1)/4i, \ldots, (2x_j - 1)/4i)$, respectively,

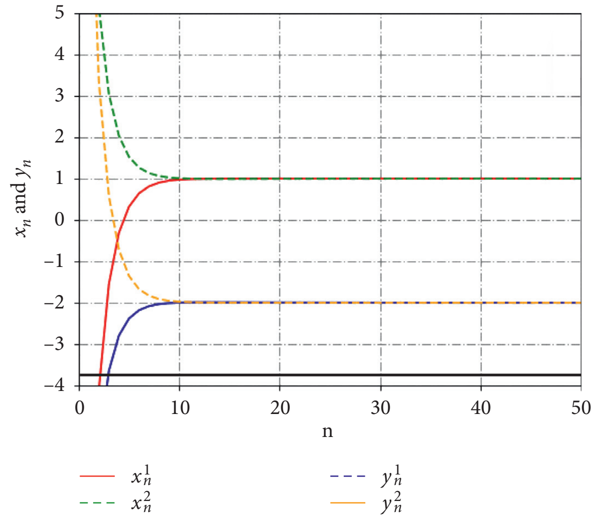

FIGURE 1: Convergence of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1=(-10,10)$, $y_1=(-10,10)$, and $n=N=50$.

for all $x=\{x_j\}_{j=1}^{\infty}\in \ell_2$, and let $a_i^x=(5/6^i+1/N6^N)$ and $a_i^y=(7/8^i+1/N8^N)$. Let the mappings $f,g\colon \ell_2\longrightarrow \ell_2$ be defined by $f(x)=(x_1/5,x_2/5,\dots,x_j/5,\dots)$, $g(x)=(x_1/4,x_2/4,\dots,x_j/4,\dots)$, respectively, for all $x=\{x_j\}_{j=1}^{\infty}\in \ell_2$.

Let $\lambda_1^x=1,\lambda_1^y=1,\lambda_2^x=0.5$, and $\lambda_2^y=2$. Since $L=1/4$, we choose γ^x and $\gamma^y=0.5$. Let $x_1=(x_1^1,x_1^2,\dots,x_1^j,\dots)$ and $y_1=(y_1^1,y_1^2,\dots,y_1^j,\dots)\in \ell_2$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by (2) as follows:

$$\begin{cases} x_{n+1}=\dfrac{n}{5n+2}x_n+\dfrac{2n+2/3}{5n+2}P_C\left(I-\dfrac{1}{3n^2}\displaystyle\sum_{i=1}^{N}\left(\dfrac{5}{6^i}+\dfrac{1}{N6^N}\right)B_i^x\right)x_n+\dfrac{2n+4/3}{5n+3}\left(\dfrac{1}{10n}f(y_n)+\dfrac{10n-1}{20n}\widetilde{G}^x x_n\right),\\[2ex] y_{n+1}=\dfrac{n}{5n+2}y_n+\dfrac{2n+2/3}{5n+2}P_C\left(I-\dfrac{1}{4n^2}\displaystyle\sum_{i=1}^{N}\left(\dfrac{7}{8^i}+\dfrac{1}{N8^N}\right)B_i^y\right)y_n+\dfrac{2n+4/3}{5n+2}\left(\dfrac{1}{10n}g(x_n)+\dfrac{10n-1}{20n}\widetilde{G}^y y_n\right),\\[2ex] \widetilde{G}^x x_n=J_1^{M_1^x}(I-A_1^x)\left(x-0.5T^*\left(I-J_{0.5}^{M_2^x}(I-0.5A_2^x)\right)Tx_n\right),\\[2ex] \widetilde{G}^y y_n=J_1^{M_1^y}(I-A_1^y)\left(y-0.5T^*\left(I-J_2^{M_2^y}(I-2A_2^y)\right)Ty_n\right), \end{cases} \quad (73)$$

for all $n\geq 1$, where $x_n=(x_n^1,x_n^2,\dots,x_n^j,\dots)$ and $y_n=(y_n^1,y_n^2,\dots,y_n^j,\dots)$. It is easy to see that M_i^x, M_i^y, A_i^x, and A_i^y, $\forall i=1,2$, B_i^x and B_i^y, $\forall i=1,2,\dots,N$, T,f,g, and all parameters satisfy Theorem 2. Furthermore, we have that $0\in\Omega^x\cap(\cap_{i=1}^N VI(C,B_i^x))$ and $1\in\Omega^y\cap(\cap_{i=1}^N VI(C,B_i^y))$. Then, by Theorem 2, we can conclude that the sequence $\{x_n\}$ converges strongly to 0 and $\{y_n\}$ converges strongly to 1.

6. Conclusion

(1). Table 1 and Figure 1 in Example 3 show that the sequence $\{x_n\}$ converges to $(1,1)\in\Omega^x\cap(\cap_{i=1}^N VI(C,B_i^x))$ and $\{y_n\}$ converges to $(-2,-2)\in\Omega^y\cap(\cap_{i=1}^N VI(C,B_i^y))$

(2) Example 4 is an example in infinite-dimensional Hilbert space for supporting Theorem 2

(3) Theorem 2 guarantees the convergence of $\{x_n\}$ and $\{y_n\}$ in Example 3 and Example 4

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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References

[1] A. Moudafi, "Split monotone variational inclusions," *Journal of Optimization Theory and Applications*, vol. 150, no. 2, pp. 275–283, 2011.

[2] Y. Censor, A. Gibali, and S. Reich, "Algorithms for the split variational inequality problem," *Numerical Algorithms*, vol. 59, pp. 301–323, 2012.

[3] A. S. Alofi, S. M. Alsulami, and W. Takahashi, "The split common null point problem and halpern-type strong convergence theorem in hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 16, no. 5, pp. 775–789, 2015.

[4] C. Byrne, Y. Censor, A. Gibali, and S. Reich, "The split common null point problem," *Journal Nonlinear Convex Anal*, vol. 13, no. 4, pp. 759–775, 2012.

[5] H. Che and M. Li, "The conjugate gradient method for split variational inclusion and constrained convex minimization problems," *Applied Mathematics and Computation*, vol. 290, pp. 426–438, 2016.

[6] L.-J. Lin, Y.-D. Chen, and C.-S. Chuang, "Solutions for a variational inclusion problem with applications to multiple sets split feasibility problems," *Fixed Point Theory and Applications*, vol. 2013, no. 1, pp. 1–21, 2013.

[7] S. Takahashi and W. Takahashi, "The split common null point problem and the shrinking projection method in banach spaces," *Optimization*, vol. 65, no. 2, pp. 281–287, 2016.

[8] Y. Censor and T. Elfving, "A multiprojection algorithm using bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.

[9] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.

[10] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2003.

[11] Z.-T. Yu, L.-J. Lin, and C.-S. Chuang, "Mathematical programming with multiple sets split monotone variational inclusion constraints," *Fixed Point Theory and Applications*, vol. 2014, no. 1, pp. 1–27, 2014.

[12] Q. H. Ansari and A. Rehan, "An iterative method for split hierarchical monotone variational inclusions," *Fixed Point Theory and Applications*, vol. 2015, no. 1, pp. 1–10, 2015.

[13] K. Kazmi, S. Rizvi, and R. Ali, "A hybrid-extragradient iterative method for split monotone variational inclusion, mixed equilibrium problem and fixed point problem for a nonexpansive mapping," *Journal of the Nigerian Mathematical Society*, vol. 35, no. 2, pp. 312–338, 2016.

[14] Y. Shehu and F. U. Ogbuisi, "An iterative method for solving split monotone variational inclusion and fixed point problems," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas*, vol. 110, no. 2, pp. 503–518, 2016.

[15] K. R. Kazmi, R. Ali, and M. Furkan, "Hybrid iterative method for split monotone variational inclusion problem and hierarchical fixed point problem for a finite family of nonexpansive mappings," *Numerical Algorithms*, vol. 79, no. 2, pp. 499–527, 2018.

[16] B. Chaloemyotphong and A. Kangtunyakarn, "Modified halpern iterative method for solving hierarchical problem and split combination of variational inclusion problem in hilbert space," *Mathematics*, vol. 7, no. 11, Article ID 1037, 2019.

[17] X. Zhao, J.-C. Yao, and Y. Yao, "A proximal algorithm for solving split monotone variational inclusions," *University Politehnica of Bucharest Scientific, Bulletin-Series A-Applied Mathematics and Physics*, vol. 82, no. 3, pp. 43–52, 2020.

[18] F. U. Ogbuisi and O. T. Mewomo, "Solving split monotone variational inclusion problem and fixed point problem for certain multivalued maps in hilbert spaces," *Thai Journal of Mathematics*, vol. 19, no. 2, pp. 503–520, 2021.

[19] Y. Yao, Y. Shehu, X.-H. Li, and Q.-L. Dong, "A method with inertial extrapolation step for split monotone inclusion problems," *Optimization*, vol. 70, no. 4, pp. 741–761, 2021.

[20] Z. Yao, S. M. Kang, and H.-J. Li, "An intermixed algorithm for strict pseudo-contractions in hilbert spaces," *Fixed Point Theory and Applications*, vol. 2015, no. 1, Article ID 206, 2015.

[21] S. Suwannaut, "The s-intermixed iterative method for equilibrium problems," *Thai Journal of Mathematics*, pp. 60–74, 2018.

[22] S. Suwannaut, "The general intermixed iteration for equilibrium problems and variational inequality problems in hilbert spaces," *Thai Journal of Mathematics*, vol. 18, no. 3, pp. 1497–1518, 2020.

[23] W. Nazeer, M. Munir, and S. M. Kang, "An intermixed algorithm for three strict pseudo-contractions in hilbert spaces," *Journal of Computational Analysis and Applications*, vol. 24, pp. 1322–1333, 2018.

[24] L. O. Jolaoso, F. Ogbuisi, and O. Mewomo, "An intermixed algorithm for two strict pseudocontractions in q-uniformly smooth banach space," *Nonlinear Studies*, vol. 26, no. 1, pp. 27–41, 2019.

[25] A. Sripattanet and A. Kangtunyakarn, "Convergence theorem for solving a new concept of the split variational inequality problems and application," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas*, vol. 114, no. 4, pp. 1–33, 2020.

[26] A. Sripattanet and A. Kangtunyakarn, "Convergence theorem of two sequences for solving the modified generalized system of variational inequalities and numerical analysis," *Mathematics*, vol. 7, no. 10, Article ID 916, 2019.

[27] K. Saechou and A. Kangtunyakarn, "An intermixed iteration for constrained convex minimization problem and split feasibility problem," *Journal of Inequalities and Applications*, vol. 2019, no. 1, Article ID 269, 2019.

[28] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general banach spaces," *Fixed Point Theory and Applications*, vol. 2005, no. 1, pp. 1–21, 2005.

[29] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and

applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.
[30] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishere, Yokohama, Japan, 2000.
[31] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, Japan, 2009.
[32] W. Khuangsatung and A. Kangtunyakarn, "The applications of modified generalized mixed equilibrium problems to nonlinear problems," *Thai Journal of Mathematics*, vol. 14, no. 3, pp. 771–796, 2016.
[33] S.-s. Zhang, J. H. W. Lee, and C. K. Chan, "Algorithms of common solutions to quasi variational inclusion and fixed point problems," *Applied Mathematics and Mechanics*, vol. 29, no. 5, pp. 571–581, 2008.
[34] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.
[35] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, no. 1, pp. 75–88, 1970.
[36] B. Qu and N. Xiu, "A note on the cq algorithm for the split feasibility problem," *Inverse Problems*, vol. 21, no. 5, pp. 1655–1665, 2005.
[37] Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, "A unified approach for inversion problems in intensity-modulated radiation therapy," *Physics in Medicine and Biology*, vol. 51, no. 10, pp. 2353–2365, 2006.
[38] Y. Censor, A. Motova, and A. Segal, "Perturbed projections and subgradient projections for the multiple-sets split feasibility problem," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1244–1256, 2007.
[39] R. Rockafellar, "On the maximal monotonicity of subdifferential mappings," *Pacific Journal of Mathematics*, vol. 33, no. 1, pp. 209–216, 1970.
[40] R. Tibshirani, "Regression shrinkage and selection via the lasso," *Journal of the Royal Statistical Society: Series B*, vol. 58, no. 1, pp. 267–288, 1996.

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CONVERGENCE RESULTS OF ITERATIVE ALGORITHMS FOR THE SUM OF TWO MONOTONE OPERATORS IN REFLEXIVE BANACH SPACES

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Abstract. The aim of this paper is to propose two modified forward-backward splitting algorithms for zeros of the sum of a maximal monotone operator and a Bregman inverse strongly monotone operator in reflexive Banach spaces. We prove weak and strong convergence theorems of the generated sequences by the proposed methods under some suitable conditions. We apply our results to study the variational inequality problem and the equilibrium problem. Finally, a numerical example is given to illustrate the proposed methods. The results presented in this paper improve and generalize many known results in recent literature.

Keywords: maximal operator; Bregman distance; reflexive Banach space; weak convergence; strong convergence

MSC 2020: 47H09, 47H10, 47J25, 47J05

1. Introduction

Let E be a real Banach space with its dual space E^*. We study the so-called *quasi-inclusion problem*: find $z \in E$ such that

$$0 \in (A + B)z, \tag{1.1}$$

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where $A\colon E \to E^*$, $B\colon E \multimap E^*$ are single- and set-valued mappings, respectively. In particular, if $E = H$, where H is a real Hilbert space, then $E^* = H$. This problem (1.1) has received considerable attention in recent decades due to its wide theoretical value in nonlinear analysis or optimization theory and wide spectrum of applications such as in signal processing, image reconstruction and machine learning.

One of the simplest methods for solving (1.1) is the *forward-backward splitting method* (see [19], [25], [48]), which is of the following form: for any $x_1 \in H$ and $\lambda > 0$,

$$x_{n+1} = \mathrm{Res}_{\lambda}^{B} \circ A_{\lambda}(x_n) \quad \forall\, n \geqslant 1, \tag{1.2}$$

where $\mathrm{Res}_{\lambda}^{B} := (I + \lambda B)^{-1}$ is the resolvent of B, $A_{\lambda} := I - \lambda A$ and I denotes the identity mapping on H. In the context of this method, the operators $\mathrm{Res}_{\lambda}^{B}$ and A_{λ} are also often referred to as the backward and forward operators, respectively. Note that this method includes, as special cases, the proximal point algorithm (when $A = 0$) (see [17], [21], [41]) and the gradient method (see [8], [20]). However, from the numerical point of view, the weak convergence of this method is not enough to make it efficient.

In order to obtain the strong convergence result, Takahashi et al. [47] introduced the following *Halpern-type forward-backward splitting method* for solving (1.1) in a Hilbert space H: for any x_1, $u \in H$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\mathrm{Res}_{\lambda_n}^{B} \circ A_{\lambda_n}(x_n) \quad \forall\, n \geqslant 1, \tag{1.3}$$

where $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, A is an α-inverse strongly monotone mapping and B is a maximal monotone operator. It was shown that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a solution of (1.1).

In 2012, López et al. [26] extended the above result to a q-uniformly smooth and uniformly convex Banach space E. They proposed the following iterative process with errors a_n and b_n: for any x_1, $u \in E$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\mathrm{Res}_{\lambda_n}^{B}(x_n - \lambda_n(Ax_n + a_n)) + b_n) \quad \forall\, n \geqslant 1, \tag{1.4}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$, B is a maximal monotone operator and A is an α-inverse strongly monotone mapping. They also proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a solution of (1.1) under appropriate assumptions.

Very recently in 2019, Kimura and Nakajo [24] proposed the following iterative process for approximating a solution of (1.1) in a 2-uniformly convex and uniformly smooth Banach space E: for any x_1, $u \in E$,

$$x_{n+1} = \mathrm{Res}_{\lambda_n}^{B} \circ A_{\lambda_n} J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jx_n - \lambda_n Bx_n) \quad \forall\, n \geqslant 1, \tag{1.5}$$

where $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1]$, A is an α-inverse strongly monotone mapping, $\mathrm{Res}^B_{\lambda_n} := (J+\lambda_n B)^{-1}J$ is the resolvent of a maximal monotone operator B, $A_\lambda := J^{-1}(J-\lambda_n A)$ and J denotes the normalized duality mapping from E into 2^{E^*}. Under appropriate assumptions, they proved that the sequence generated by (1.5) converges strongly to a solution of (1.1).

In recent years, various modified forward-backward splitting algorithms have been studied and developed by many researchers in Hilbert spaces and extended to Banach spaces (see e.g. [14], [13], [15], [16], [18], [24], [26], [43], [44], [45], [46], [49]). The following important question arises here:

Question: Can we extend the above-mentioned results to a more general class of forward-backward operators in more general Banach spaces which are not necessarily uniformly convex and uniformly smooth?

To answer the above question in this paper, we introduce two modified forward-backward splitting algorithms for solving (1.1), where A is a Bregman inverse strongly monotone mapping and B is a maximal monotone mapping in the framework of reflexive Banach spaces. The paper is organized as follows. In Section 2, we collect definitions and results which are needed for our further analysis. The weak and strong convergence theorems of the proposed algorithms are established in Section 3. Some applications of our results to the variational inequality problem and the equilibrium problem are considered in Section 4, and a numerical example is given in Section 5.

2. Preliminaries

Throughout this paper, let E be a real reflexive Banach space with its dual E^* and $f\colon E \to (-\infty,\infty]$ be a proper, lower semicontinuous and convex function. We denote by $\langle x, j\rangle$ the value of the functional $j \in E^*$ at $x \in E$. The *subdifferential* of f is defined by

$$\partial f(x) = \{x^* \in E^*\colon\ f(x) + \langle y-x, x^*\rangle \leqslant f(y)\ \forall\, y \in E\}, \quad x \in E.$$

The *Fenchel conjugate* of f is the function $f^*\colon E^* \to (-\infty,\infty]$ defined by

$$f^*(x^*) = \sup_{x\in E}\{\langle x, x^*\rangle - f(x)\}.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x, x^*\rangle$ (see [39], Theorem 23.5 (d)). We denote by $\mathrm{dom} f = \{x \in E\colon\ f(x) < \infty\}$ the domain of f. The function f on E is said to be *cofinite* if $\mathrm{dom} f^* = E^*$ and f is said to be *strongly coercive* if $\lim\limits_{\|x\|\to\infty} f(x)/\|x\| = \infty$.

For any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, the *directional derivative* of f at x in the direction $y \in E$ is given by

$$f'(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteux differentiable* at x if the limit as $t \to 0$ in (2.1) exists for each y. In this case, the *gradient* of f at x is the linear function $\nabla f(x)\colon E \to E^*$ defined by $\langle y, \nabla f(x)\rangle = f'(x, y)$ for any $y \in E$. For more details about gradient, we recommend [7], Remark 3.32. The function f is said to be *Gâteux differentiable* if it is Gâteux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. It is known that if f is Gâteux differentiable at x, then $\partial f(x)$ is single-valued. Conversely, if f is continuous at x and $\partial f(x)$ is single-valued, then f is Gâteux differentiable at x and $\nabla f(x) = \partial f(x)$ (see [4], Proposition 2.40). The function f is said to be *Fréchet differentiable* at x if the limit (2.1) is attained uniformly in $\|y\| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that every Fréchet differentiable function is Gâteux differentiable and if f is Fréchet differentiable, then it is continuous, but if f is Gâteux differentiable, then it is not necessary that f is continuous (see [34], p. 142).

The function $f\colon E \to (-\infty, \infty]$ is said to be *Legendre* ([36], p. 25) if and only if it satisfies the following two conditions:

(L1) $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ and f is Gâteux differentiable with $\operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f)$,
(L2) $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ and f^* is Gâteux differentiable with $\operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$.

In a reflexive Banach space, we always obtain $(\partial f)^{-1} = \partial f^*$ (see [9], p. 83). This fact, when combined with conditions (L1) and (L2), implies the following facts:

(i) ∇f is a bijection with $\nabla f = (\nabla f^*)^{-1}$ (see [5], Theorem 5.10),
(ii) $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f)$ (see [37], p. 123),

where $\operatorname{ran}\nabla f$ denotes the range of ∇f.

Also, conditions (L1) and (L2), in conjunction with [5], Theorem 5.4, imply that the functions f and f^* are strictly convex on the interior of their respective domains.

If $f\colon E \to (-\infty, \infty]$ is additionally assumed to be Gâteux differentiable, then the function $D_f\colon \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle \tag{2.2}$$

is called the *Bregman distance* with respect to f [11]. In general, D_f is not symmetric and it does not satisfy the triangle inequality. Clearly, $D_f(x, x) = 0$, but $D_f(x, y) = 0$

may not imply $x = y$. It is known that the Bregman distance is a certain useful substitute for a distance. Next, we clarify several examples of the Bregman distance which are shown in the following:

Example 2.1. Let E is a uniformly convex and uniformly smooth Banach space. Define $f(x) = \|x\|^2$ for all $x \in E$. Then $\nabla f(x) = 2Jx$, where J is the normalized duality mapping defined by $Jx = \{j \in E^*\colon \langle x, j\rangle = \|x\|^2 = \|j\|^2\}$. So, we obtain

$$D_f(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 := \varphi(x, y),$$

where φ is called the Lyapunov function, which was studied in [3], [35]. Also, if E is a Hilbert space, then $\varphi(x, y) = \|x - y\|^2$, which is the Euclidean norm distance.

Example 2.2. Define $f(x) = -\sum\limits_{i=1}^{m} \ln x_i$ for all $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m_+ := \{x \in \mathbb{R}^m\colon x_i > 0\}$. Then $\nabla f(x) = -(1/x_1, 1/x_2, \ldots, 1/x_m)^\top$. So, we obtain the Itakura-Saito distance given by

$$D_f(x, y) = \sum_{i=1}^{m} \Big(\frac{x_i}{y_i} - \ln\Big(\frac{x_i}{y_i}\Big) - 1\Big).$$

Example 2.3. Define $f(x) = x_i \ln x_i$ for all $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m_+ := \{x \in \mathbb{R}^m\colon x_i > 0\}$. Then $\nabla f(x) = (1 + \ln(x_1), 1 + \ln(x_2), \ldots, 1 + \ln(x_m))^\top$. So, we obtain the Kullback-Leibler distance given by

$$D_f(x, y) = \sum_{i=1}^{m} \Big(x_i \ln\Big(\frac{x_i}{y_i}\Big) + y_i - x_i\Big).$$

Example 2.4. Define $f(x) = \frac{1}{2} x^\top Q x$ for all $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m$. Then $\nabla f(x) = Qx$, where $Q = \operatorname{diag}(1, 2, \ldots, m)$. So, we obtain the squared Mahalanobis distance given by

$$D_f(x, y) = \frac{1}{2}(x - y)^\top Q (x - y).$$

For more examples of Bregman distances, we recommend [23], [31], [32].

The *modulus of total convexity* of f at $x \in \operatorname{dom} f$ is the function $v_f(x, \cdot)\colon [0, \infty) \to [0, \infty]$ defined by

$$v_f(x, t) = \inf\{D_f(y, x)\colon y \in \operatorname{dom} f,\ \|y - x\| = t\}.$$

The function f is called *totally convex at* x if $v_f(x, t) > 0$, whenever $t > 0$, and is called *totally convex* if it is totally convex at any point $x \in \operatorname{int}(\operatorname{dom} f)$. It is well known that if f is totally convex and Fréchet differentiable, then f is cofinite (see [36], Proposition 2.3, p. 39). The function f is said to be *totally convex on bounded sets* if $v_f(X,t) > 0$ for any nonempty bounded subset X of E and $t > 0$, where the *modulus of total convexity* of the function f on the set X is the function $v_f\colon \operatorname{int}(\operatorname{dom} f) \times [0,\infty) \to [0,\infty]$ defined by

$$v_f(X,t) = \inf\{v_f(x,t)\colon\ x \in X \cap \operatorname{dom} f\}.$$

Let $B_r := \{x \in E\colon\ \|x\| \leqslant r\}$ for all $r > 0$. Then a function $f\colon E \to \mathbb{R}$ is said to be *uniformly convex* on bounded subsets of E, if $\varrho_r(t) > 0$ for all r, $t > 0$, where $\varrho_r\colon [0,\infty) \to [0,\infty)$ is defined by

$$\varrho_r(t) = \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geqslant 0$. It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [12], Theorem 2.10).

Let C be a nonempty subset of E. Let $S\colon C \to C$ be a mapping. A point $p \in C$ is a fixed point of S if $p = Sp$ and we denote by $F(S)$ the set of fixed points of S, i.e., $F(S) = \{x \in C\colon\ x = Sx\}$. A point p in C is said to be an *asymptotic fixed point* [35] of S, if C contains a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to p and $\lim\limits_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widehat{F}(S)$.

Let C be a nonempty subset of $\operatorname{int}(\operatorname{dom} f)$. The mapping $S\colon C \to \operatorname{int}(\operatorname{dom} f)$ is said to be:

(i) *Bregman firmly nonexpansive* (BFNE) if

$$\langle Sx - Sy, \nabla f(Sx) - \nabla f(Sy)\rangle \leqslant \langle Sx - Sy, \nabla f(x) - \nabla f(y)\rangle \quad \forall\, x,y \in C.$$

(ii) *Bregman strongly nonexpansive* (BSNE) with $\widehat{F}(S) \neq \emptyset$ if

$$D_f(p, Sx) \leqslant D_f(p,x) \quad \forall\, x \in C,\ p \in \widehat{F}(S)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(S)$ and

$$\lim_{n\to\infty} (D_f(p,x_n) - D_f(p,Sx_n)) = 0,$$

it follows that $\lim\limits_{n\to\infty} D_f(x_n, Sx_n) = 0$.

As shown in [38], Lemma 15.6, p. 308, if S is BFNE and f is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $F(S) = \widehat{F}(S)$ and $F(S)$ is closed and convex. It also follows that every BFNE mapping is BSNE with respect to $F(S) = \widehat{F}(S)$.

Recall that the *Bregman projection with respect to* f of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed and convex set $C \subset \operatorname{dom} f$ is the unique $P_C^f(x) \subset C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x)\colon\ y \in C\}. \tag{2.3}$$

It can be characterized by the following variational inequality [12], Corollary 4.4:

$$\langle y - P_C^f(x), \nabla f(x) - \nabla f(P_C^f(x))\rangle \leqslant 0 \quad \forall\, y \in C. \tag{2.4}$$

Moreover,

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leqslant D_f(y, x) \quad \forall\, y \in C. \tag{2.5}$$

If E is a uniformly convex and uniformly smooth Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then P_C^f coincides with the generalized projection Π_C (see [2], Definition 7.2), and if E is a Hilbert space, then P_C^f coincides with the metric projection P_C.

Let $f\colon E \to \mathbb{R}$ be a Legendre function. Let $V_f\colon E \times E^* \to [0, \infty)$ associated with f be defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*) \quad \forall\, x \in E,\ x^* \in E^*. \tag{2.6}$$

From [28], Proposition 1, we know the following properties:

(i) V_f is nonnegative and convex in the second variable.

(ii)

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \quad \forall\, x \in E,\ x^* \in E^*. \tag{2.7}$$

(iii)

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \leqslant V_f(x, x^* + y^*) \quad \forall\, x \in E,\ x^*, y^* \in E^*. \tag{2.8}$$

Since V_f is convex in the second variable, we have for all $z \in E$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leqslant \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.9}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N$ with $\sum\limits_{i=1}^{N} t_i = 1$.

For a set-valued operator $A\colon E \multimap E^*$, we define its domain, range and graph as follows: $\operatorname{dom} A = \{x \in E\colon Ax \neq \emptyset\}$, $\operatorname{ran} A = \bigcup\{Ax\colon x \in \operatorname{dom} A\}$ and $G(A) = \{(x, x^*) \in E \times E^*\colon x^* \in Ax\}$. An operator A is said to be *monotone*, if for each $(x, x^*), (y, y^*) \in G(A)$, we have $\langle x - y, x^* - y^*\rangle \geqslant 0$. A monotone operator A is said to be *maximal*, if its graph is not contained in the graph of any other monotone operator on E. It is known that if $f\colon E \to \mathbb{R}$ is Gâteux differentiable, strictly convex and cofinite, then A is maximal monotone if and only if $\operatorname{ran}(\nabla f + \lambda A) = E^*$ for $\lambda > 0$ (see [6], Corollary 2.4).

Let $f\colon E \to (-\infty, \infty]$ be a Fréchet differentiable function which is bounded on bounded subsets of E and let A be a maximal monotone operator. Then the *resolvent* $\operatorname{Res}^f_{\lambda A}\colon E \multimap E$ of A for $\lambda > 0$, is defined by

$$\operatorname{Res}^f_{\lambda A} = (\nabla f + \lambda A)^{-1} \circ \nabla f.$$

The mapping A satisfying $\operatorname{ran}(\nabla f - \lambda A) \subset \operatorname{ran}(\nabla f)$ is called *Bregman inverse strongly monotone* if $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f) \neq \emptyset$ and for any $x, y \in \operatorname{int}(\operatorname{dom} f)$ and each $u \in Ax$, $v \in Ay$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geqslant 0.$$

For any operator $A\colon E \multimap E^*$, the *anti-resolvent* $A^f_\lambda\colon E \multimap E$ of A, for $\lambda > 0$, is defined by

$$A^f_\lambda = \nabla f^* \circ (\nabla f - \lambda A).$$

Lemma 2.5. *Let $f\colon E \to \mathbb{R}$ be a Fréchet differentiable Legendre function which is bounded on bounded subsets of E. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and $B\colon E \multimap E^*$ be a maximal monotone operator. Define a mapping $T_\lambda x := \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x)$ for $x \in E$ and $\lambda > 0$. Then $F(T_\lambda) = (A + B)^{-1}0$.*

Proof. Let $x \in E$ and $\lambda > 0$. We see that

$$\begin{aligned}
x = T_\lambda x &\Leftrightarrow x = \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x)\\
&\Leftrightarrow x = (\nabla f + \lambda B)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda A)x)\\
&\Leftrightarrow x = (\nabla f + \lambda B)^{-1} \circ (\nabla f - \lambda A)x)\\
&\Leftrightarrow \nabla f(x) - \lambda Ax \in \nabla f(x) + \lambda Bx\\
&\Leftrightarrow 0 \in \lambda(A + B)x\\
&\Leftrightarrow x \in (A + B)^{-1}0.
\end{aligned}$$

The proof is completed. $\square$

Lemma 2.6 ([33], Theorem 3.1)**.** *Let $f\colon E \to \mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and $B\colon E \multimap E^*$ be a maximal monotone operator. Then the following statements hold:*

(i) $D_f(z, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x)) + D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x), x) \leqslant D_f(z, x)$ *for all* $z \in (A+B)^{-1}0$, $x \in E$ *and* $\lambda > 0$.

(ii) $\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda$ *is a BSNE operator such that* $F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x)) = \widehat{F}(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x))$.

We shall use the following notation in this paper:

- $x_n \to x$ means that $\{x_n\}$ converges strongly to x,
- $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x.

The mapping $A\colon E \to E^*$ is called *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ implies that $Ax_n \rightharpoonup^* Ax$.

Lemma 2.7 ([28], Proposition 9)**.** *Let $f\colon E \to \mathbb{R}$ be a Legendre function such that ∇f is weakly sequentially continuous. Suppose that the sequence $\{x_n\}$ is bounded and that $\lim\limits_{n\to\infty} D_f(u, x_n)$ exists for any weak subsequential limit u of $\{x_n\}$. Then $\{x_n\}$ converges weakly to u.*

Lemma 2.8 ([36], Lemma 3.1)**.** *Let $f\colon E \to \mathbb{R}$ be a Gâteux differentiable and totally convex function. Suppose that $x \in E$. If $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.9 ([30], Lemma 2.4)**.** *Let E be a Banach space and $f\colon E \to \mathbb{R}$ be a Gâteux differentiable function which is uniformly convex on bounded subsets of E. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then $\lim\limits_{n\to\infty} D_f(x_n, y_n) = 0$ if and only if $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.10 ([27], Lemma 3.1)**.** *Let $\{a_n\}$ and $\{c_n\}$ be nonnegative real sequences such that*

$$a_{n+1} \leqslant (1-\delta_n)a_n + b_n + c_n,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume that $\sum\limits_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $b_n/\delta_n \leqslant M$ for some $M \geqslant 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum\limits_{n=1}^{\infty} \delta_n = \infty$ and $\limsup\limits_{n\to\infty} b_n/\delta_n \leqslant 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

Lemma 2.11 ([22], Lemma 7). *Assume that* $\{s_n\}$ *is a nonnegative real sequence such that*

$$s_{n+1} \leqslant (1-\delta_n)s_n + \delta_n \tau_n$$

and

$$s_{n+1} \leqslant s_n - \eta_n + \varrho_n,$$

where $\{\delta_n\}$ *is a sequence in* $(0,1)$, $\{\eta_n\}$ *is a nonnegative real sequence and* $\{\tau_n\}$, $\{\varrho_n\}$ *are real sequences such that*

(i) $\sum\limits_{n=1}^{\infty} \delta_n = \infty$,

(ii) $\lim\limits_{n\to\infty} \varrho_n = 0$,

(iii) $\lim\limits_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup\limits_{k\to\infty} \tau_{n_k} \leqslant 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then $\lim\limits_{n\to\infty} s_n = 0$.

3. Main results

In this section, we propose two modifications of a forward-backward splitting method for solving (1.1) in reflexive Banach spaces. In order to prove the convergence results, we assume the following:

Assumption 3.1.

(i) *Let* E *be a real reflexive Banach space.*

(ii) *Let* $f\colon E \to \mathbb{R}$ *be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of* E.

(iii) *Let* $A\colon E \to E^*$ *be a Bregman inverse strongly monotone mapping and* $B\colon E \multimap E^*$ *be a maximal monotone mapping.*

(iv) *The solution set* $(A+B)^{-1}0$ *of* (1.1) *is nonempty.*

3.1. Weak convergence theorem.

In this subsection, we propose a modification of the Mann-type forward-backward splitting method and prove its weak convergence.

Algorithm 3.2. *For a given* $x_1 \in E$, *let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \quad \forall\, n \geqslant 1, \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\lambda > 0$.

We now prove a weak convergence theorem for Algorithm 3.2.

Theorem 3.3. *Assume that Assumption* 3.1 *is satisfied. Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.2. *Assume that* $\limsup_{n\to\infty} \alpha_n < 1$. *Suppose, in addition, that* ∇f *is weakly sequentially continuous on* E. *Then* $\{x_n\}$ *converges weakly to a point in* $(A+B)^{-1}0$.

P r o o f. Let $w \in (A+B)^{-1}0$. Then from (2.9) and Lemma 2.6 (i), we have

$$\begin{aligned} D_f(w, x_{n+1}) &= D_f(w, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\ &\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n) D_f(w, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\ &\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n) D_f(w, x_n) \\ &= D_f(w, x_n). \end{aligned} \tag{3.2}$$

This shows that $\lim_{n\to\infty} D_f(w, x_n)$ exists and, consequently, $\{D_f(w, x_n)\}$ is bounded. Hence, by Lemma 2.8, we have that $\{x_n\}$ is bounded. This implies that $\{\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)\}$ is bounded. From Lemma 2.6 (i), we have

$$\begin{aligned} D_f(w, x_{n+1}) &\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n) D_f(w, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\ &\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n)(D_f(w, x_n) - D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n)), \end{aligned} \tag{3.3}$$

which implies that

$$(1-\alpha_n) D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n) \leqslant D_f(w, x_n) - D_f(w, x_{n+1}). \tag{3.4}$$

Since $\lim_{n\to\infty} D_f(w, x_n)$ exists, we have

$$\lim_{n\to\infty} D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n) = 0. \tag{3.5}$$

Moreover, from Lemma 2.9, we have

$$\lim_{n\to\infty} \|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n) - x_n\| = 0. \tag{3.6}$$

By the reflexivity of the Banach space E and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$ as $k \to \infty$. From (3.6), we note that $\|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}) - x_{n_k}\| \to 0$ as $k \to \infty$, then we have $\hat{x} \in \widehat{F}(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda) = F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda)$, and hence $\hat{x} \in (A+B)^{-1}0$. By Lemma 2.7, we conclude that $\{x_n\}$ converges weakly to a point in $(A+B)^{-1}0$. This completes the proof. □

If $\alpha_n = 0$ for all $n \geqslant 1$, then we have the following result for the forward-backward splitting method in a reflexive Banach space.

Corollary 3.4. *Let $\{x_n\}$ be a sequence generated by $x_1 \in E$ and*

$$x_{n+1} = \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n) \quad \forall\, n \geqslant 1. \tag{3.7}$$

Then $\{x_n\}$ converges weakly to a point in $(A+B)^{-1}0$.

If E is a uniformly convex Banach space which is also uniformly smooth and $f(x) = \|x\|^2$ for all $x \in E$, then we have the following result for the Mann-type forward-backward splitting method.

Corollary 3.5. *Let J be a duality mapping from E into E^* such that J is weakly sequentially continuous. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping with respect to the function $f(x) = \|x\|^2$ and let $B\colon E \multimap E^*$ be a maximal monotone mapping. Let $\{x_n\}$ be a sequence generated by $x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J(\operatorname{Res}^B_\lambda \circ A_\lambda(x_n))) \quad \forall\, n \geqslant 1, \tag{3.8}$$

where $\operatorname{Res}^B_\lambda := (J+\lambda B)^{-1}J$, $A_\lambda := J^{-1}(J-\lambda A)$ for $\lambda > 0$ and $\{\alpha_n\} \subset [0,1)$. Suppose that $\limsup\limits_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges weakly to a point in $(A+B)^{-1}0$.

3.2. Strong convergence theorem. In this subsection, we propose a strong convergence theorem for another modification of the forward-backward splitting method based on the Halpern-type iteration.

Algorithm 3.6. *For given $u, x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(y_n)) \quad \forall\, n \geqslant 1, \end{cases} \tag{3.9}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\lambda > 0$.

Theorem 3.7. *Assume that Assumption 3.1 is satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 3.6. Suppose that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges strongly to $z = P^f_{(A+B)^{-1}0}(u)$, where $P^f_{(A+B)^{-1}0}$ is the Bregman projection of E onto $(A+B)^{-1}0$.*

Proof. Let $w \in (A+B)^{-1}0$. Then we have

$$\begin{aligned} D_f(w, y_n) &= D_f(w, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\ &\leqslant \alpha_n D_f(w,u) + (1-\alpha_n) D_f(w, \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\ &\leqslant \alpha_n D_f(w,u) + (1-\alpha_n) D_f(w, x_n). \end{aligned} \tag{3.10}$$

It follows that

$$
\begin{aligned}
(3.11)\quad D_f(w, x_{n+1}) &= D_f(w, \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(y_n))) \\
&\leqslant \beta_n D_f(w, x_n) + (1-\beta_n) D_f(w, y_n) \\
&\leqslant \beta_n D_f(w, x_n) + (1-\beta_n)(\alpha_n D_f(w, u) + (1-\alpha_n) D_f(w, x_n)) \\
&= (1-(1-\beta_n)\alpha_n) D_f(w, x_n) + (1-\beta_n)\alpha_n D_f(w, u).
\end{aligned}
$$

By Lemma 2.10 (i), we have that $\{D_f(w, x_n)\}$ is bounded. Hence, $\{x_n\}$ is bounded by Lemma 2.8. Let $z = P^f_{(A+B)^{-1}0}(u)$. From (2.7), (2.8), (2.9), and Lemma 2.6 (i), we have

$$
\begin{aligned}
(3.12)\qquad D_f(z, y_n) &= D_f(z, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\
&= V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \\
&\leqslant V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\
&\qquad - \alpha_n(\nabla f(u) - \nabla f(z))) + \langle \alpha_n(y_n - z), \nabla f(u) - \nabla f(z)\rangle \\
&= V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \\
&\qquad + \langle \alpha_n(y_n - z), \nabla f(u) - \nabla f(z)\rangle \\
&= D_f(z, \nabla f^*(\alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\
&\qquad + \langle \alpha_n(y_n - z), \nabla f(u) - \nabla f(z)\rangle \\
&\leqslant \alpha_n D_f(z, z) + (1-\alpha_n) D_f(z, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\
&\qquad + \alpha_n \langle y_n - z, \nabla f(u) - \nabla f(z)\rangle \\
&\leqslant (1-\alpha_n)(D_f(z, x_n) - D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n)) \\
&\qquad + \alpha_n \langle y_n - z, \nabla f(u) - \nabla f(z)\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(3.13)\qquad D_f(z, x_{n+1}) &\leqslant \beta_n D_f(z, x_n) + (1-\beta_n) D_f(z, y_n) \\
&\leqslant (1-(1-\beta_n)\alpha_n) D_f(z, x_n) \\
&\qquad - (1-\beta_n)(1-\alpha_n) D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n) \\
&\qquad + \alpha_n(1-\beta_n)\langle y_n - z, \nabla f(u) - \nabla f(z)\rangle.
\end{aligned}
$$

For each $n \geqslant 1$, we put $s_n = D_f(z, x_n)$, $\delta_n = (1-\beta_n)\alpha_n$, $\tau_n = \langle y_n - z, \nabla f(u) - \nabla f(z)\rangle$, $\eta_n = (1-\beta_n)(1-\alpha_n) D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n)$ and $\varrho_n = \alpha_n(1-\beta_n)\langle y_n - z, \nabla f(u) - \nabla f(z)\rangle$. Then (3.13) reduces to the following formulae:

$$
(3.14)\qquad s_{n+1} \leqslant (1-\delta_n)s_n + \delta_n \tau_n, \quad n \geqslant 1,
$$

and

$$
(3.15)\qquad s_{n+1} \leqslant s_n - \eta_n + \varrho_n, \quad n \geqslant 1.
$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, it follows that $\sum_{n=1}^{\infty} \delta_n = \infty$. Since $\lim_{n\to\infty} \alpha_n = 0$, we have $\lim_{n\to\infty} \varrho_n = 0$. In order to complete the proof, using Lemma 2.11, it is sufficient to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leqslant 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$. Then, we have

$$\lim_{k\to\infty} D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}), x_{n_k}) = 0. \tag{3.16}$$

By Lemma 2.9, we also have

$$\lim_{k\to\infty} \|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}) - x_{n_k}\| = 0. \tag{3.17}$$

Since f is uniformly Fréchet differentiable, it follows that ∇f is uniformly continuous on bounded subsets of E (see [51], Proposition 3.6.3), then

$$\|\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k})) - \nabla f(x_{n_k})\| \to 0. \tag{3.18}$$

From (3.18), we have

$$\begin{aligned} \|\nabla f(y_{n_k}) - \nabla f(x_{n_k})\| &\leqslant \|\nabla f(y_{n_k}) - \nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}))\| \\ &\quad + \|\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k})) - \nabla f(x_{n_k})\| \\ &= \alpha_{n_k}\|\nabla f(u) - \nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}))\| \\ &\quad + \|\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k})) - \nabla f(x_{n_k})\| \to 0. \end{aligned} \tag{3.19}$$

Since f is strongly coercive and uniformly convex on bounded sets, it follows that ∇f^* is uniformly continuous on bounded subsets of E^* (see [51], Theorem 3.5.10), then

$$\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|\nabla f^*(\nabla f(y_{n_k})) - \nabla f^*(\nabla f(x_{n_k}))\| = 0. \tag{3.20}$$

By the reflexivity of the Banach space E and the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup \hat{x} \in E$ as $i \to \infty$ and

$$\limsup_{k\to\infty} \langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle = \lim_{i\to\infty} \langle x_{n_{k_i}} - z, \nabla f(u) - \nabla f(z)\rangle. \tag{3.21}$$

Since $\|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_{k_i}}) - x_{n_{k_i}}\| \to 0$ as $i \to \infty$, it follows that $\hat{x} \in \widehat{F}(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda) = F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda)$. This implies that $\hat{x} \in (A+B)^{-1}0$. So we obtain

$$\limsup_{k\to\infty} \langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle = \langle \hat{x} - z, \nabla f(u) - \nabla f(z)\rangle \leqslant 0. \tag{3.22}$$

Furthermore, from (3.20), we also have

$$\limsup_{k\to\infty} \langle y_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle \leqslant 0. \tag{3.23}$$

This means that $\limsup_{k\to\infty} \tau_{n_k} \leqslant 0$. We conclude by Lemma 2.11 that $\lim_{n\to\infty} s_n = 0$. Therefore, $x_n \to z$ as $n \to \infty$. This completes the proof. $\square$

R e m a r k 3.8. (1) Compare the results in [14], [13]. Our results are proved without the strict assumption $\lambda < (\alpha q/\kappa_q)^{1/(q-1)}$, where $\alpha > 0$, $1 < q \leqslant 2$ and κ_q is the q-uniform smoothness coefficient of E (see [50] for more detail).

(2) Compare Theorem 3.7 with other works in [29], [47], [52], [53]. The assumption $\liminf\limits_{n\to\infty} \beta_n > 0$ can be dropped.

If $\beta_n = 0$ for all $n \geqslant 1$, then we have the following result for the Halpern-type forward-backward splitting method in a reflexive Banach space.

Corollary 3.9. *Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \quad \forall\, n \geqslant 1, \tag{3.24}$$

where $\{\alpha_n\} \subset (0,1)$ and $\lambda > 0$. Suppose that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z = P^f_{(A+B)^{-1}0}(u)$, where $P^f_{(A+B)^{-1}0}$ is the Bregman projection of E onto $(A+B)^{-1}0$.

If E is a uniformly convex Banach space which is also uniformly smooth and $f(x) = \|x\|^2$ for all $x \in E$, then we have the following result for the Halpern-type forward-backward splitting method.

Corollary 3.10. *Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping with respect to the function $f(x) = \|x\|^2$ for all $x \in E$ and let $B\colon E \multimap E^*$ be a maximal monotone mapping. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$\begin{cases} y_n = J^{-1}(\alpha_n Ju + (1-\alpha_n)J(\mathrm{Res}^B_\lambda \circ A_\lambda(x_n))), \\ x_{n+1} = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jy_n) \quad \forall\, n \geqslant 1, \end{cases} \tag{3.25}$$

where $\mathrm{Res}^B_\lambda := (J+\lambda B)^{-1}J$, $A_\lambda := J^{-1}(J-\lambda A)$ for $\lambda > 0$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$. Suppose that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges strongly to $z = \Pi_{(A+B)^{-1}0}(u)$, where $\Pi_{(A+B)^{-1}0}$ is the generalized projection of E onto $(A+B)^{-1}0$.

4. Some applications

In this section, we apply our results to the variational inequality problem and the equilibrium problem.

4.1. Variational inequality problem. Let E be a real reflexive Banach space. Let $f\colon E \to (-\infty, \infty]$ be a Legendre and totally convex function. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and C be a nonempty, closed and convex subset of $\operatorname{dom} A$. The *variational inequality problem* (VIP) is to find $z \in C$ such that

$$\langle x - z, Az\rangle \geqslant 0 \quad \forall\, x \in C.$$

The set of solutions of (VIP) is denoted by $\mathrm{VI}(C, A)$. Recall that the *indicator function* of C is given by

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

It is known that i_C is a proper, lower semicontinuous and convex function and its subdifferential ∂i_C is maximal monotone (see [40], Theorem A). Moreover, from [1], Proposition 2.5.13, we know that

$$\partial i_C(x) = \begin{cases} N_C(x) & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

where N_C is the normal cone of C given by

$$N_C(x) = \{x^* \in E^*\colon \langle y - x, x^*\rangle \leqslant 0 \quad \forall\, y \in C\}.$$

Thus, we can define the resolvent associated with ∂i_C for $\lambda > 0$ by

$$\mathrm{Res}^f_{\lambda \partial i_C}(x) = (\nabla f + \lambda \partial i_C)^{-1} \circ \nabla f(x) \quad \forall\, x \in E.$$

So we have for any $x \in E$ and $y \in C$,

$$\begin{aligned} z = \mathrm{Res}^f_{\lambda \partial i_C}(x) &\Leftrightarrow \nabla f(x) \in \nabla f(z) + \lambda \partial i_C(z) \\ &\Leftrightarrow \nabla f(x) \in \nabla f(z) + \lambda N_C(z) \\ &\Leftrightarrow \nabla f(x) - \nabla f(z) \in \lambda N_C(z) \\ &\Leftrightarrow \frac{1}{\lambda}\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leqslant 0 \quad \forall\, y \in C \\ &\Leftrightarrow \langle y - z, \nabla f(x) - \nabla f(z)\rangle \leqslant 0 \quad \forall\, y \in C \\ &\Leftrightarrow z = P^f_C(x), \end{aligned}$$

where P^f_C is the Bregman projection from E onto C.

Proposition 4.1 ([37], Proposition 8)**.** *Let* $f\colon E \to (-\infty, \infty]$ *be a Legendre and totally convex function. Let* $A\colon E \to E^*$ *be a Bregman inverse strongly monotone mapping. If* C *is a nonempty, closed and convex subset of* $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f)$*, then* $\mathrm{VI}(C, A) = F(P_C^f \circ A_\lambda^f)$ *for* $\lambda > 0$.

Setting $B = \partial i_C$ in Theorems 3.3 and 3.7, we obtain by Proposition 4.1 the following results.

Theorem 4.2. *Let* $f\colon E \to \mathbb{R}$ *be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of* E *such that* ∇f *is weakly sequentially continuous. Let* $A\colon E \to E^*$ *be a Bregman inverse strongly monotone mapping and* C *be a nonempty, closed and convex subset of* $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f)$*. Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in C$ *and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(P_C^f \circ A_\lambda^f(x_n))) \quad \forall\, n \geqslant 1, \tag{4.1}$$

where $\{\alpha_n\} \subset [0, 1)$ *and* $\lambda > 0$*. Suppose that* $\limsup_{n\to\infty} \alpha_n < 1$*. If* $\mathrm{VI}(C, A) \neq \emptyset$*, then* $\{x_n\}$ *converges weakly to a point in* $\mathrm{VI}(C, A)$.

Theorem 4.3. *Let* $f\colon E \to \mathbb{R}$ *be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of* E*. Let* $A\colon E \to E^*$ *be a Bregman inverse strongly monotone mapping and* C *be a nonempty, closed and convex subset of* $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f)$*. Let* $\{x_n\}$ *be a sequence generated by* $u, x_1 \in C$ *and*

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(P_C^f \circ A_\lambda^f(x_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(y_n)) \quad \forall\, n \geqslant 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\} \subset (0, 1)$*,* $\{\beta_n\} \subset [0, 1)$ *and* $\lambda > 0$*. Suppose that* $\lim_{n\to\infty} \alpha_n = 0$*,* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\limsup_{n\to\infty} \beta_n < 1$*. If* $\mathrm{VI}(C, A) \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to* $z = P_{\mathrm{VI}(C,A)}^f(u)$.

4.2. Equilibrium problem. Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $G\colon C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* (EP) is to find $z \in C$ such that

$$G(z, x) \geqslant 0 \quad \forall\, x \in C.$$

The set of solutions of (EP) is denoted by $\mathrm{EP}(G)$. For solving this problem, let us assume that the bifunction satisfies the following conditions:

(A1) $G(x,x) = 0$ for all $x \in C$,

(A2) G is monotone, i.e., $G(x,y) + G(y,x) \leqslant 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$, $\limsup_{t \to 0} G(tz + (1-t)x, y) \leqslant G(x,y)$,

(A4) for each $x \in C$, the function $y \mapsto G(x,y)$ is convex and lower semicontinuous.

For $\lambda > 0$ and $x \in E$, we define the resolvent operator $\mathrm{Res}^f_{\lambda G}\colon E \multimap C$ by

$$\mathrm{Res}^f_{\lambda G}(x) = \Big\{z \in C\colon\ G(z,x) + \frac{1}{\lambda}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geqslant 0\ \forall\, y \in C\Big\}.$$

Proposition 4.4 ([37], Lemma 2)**.** *Let $f\colon E \to (-\infty,\infty]$ be a Legendre function. Let C be a closed and convex subset of E. Let $G\colon C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)–(A4)*. Then the following hold:*

(i) $\mathrm{Res}^f_{\lambda G}$ *is single-valued,*

(ii) $\mathrm{Res}^f_{\lambda G}$ *is BFNE,*

(iii) $F(\mathrm{Res}^f_{\lambda G}) = \mathrm{EP}(G)$,

(iv) $\mathrm{EP}(G)$ *is closed and convex,*

(v) *for all $x \in E$ and $z \in F(\mathrm{Res}^f_{\lambda G})$, we have*

$$D_f(z, \mathrm{Res}^f_{\lambda G}(x)) + D_f(\mathrm{Res}^f_{\lambda G}(x), x) \leqslant D_f(z,x).$$

Remark 4.5. If f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then from Proposition 4.4, we have $F(\mathrm{Res}^f_{\lambda G}) = \widehat{F}(\mathrm{Res}^f_{\lambda G})$.

Proposition 4.6 ([42], Proposition 4.2)**.** *Let $f\colon E \to (-\infty,\infty]$ be a strongly coercive and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E. Let C be a closed and convex subset of E. Let $G\colon C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)–(A4)*. Let $A_G\colon E \multimap E^*$ be defined by*

$$A_G(x) = \begin{cases} \{z \in E^*\colon\ G(x,y) \geqslant \langle y - x, z\rangle\ \forall\, y \in C\} & \text{if } x \in C,\\ \emptyset & \text{if } x \notin C.\end{cases}$$

Then the following hold:

(i) *A_G is maximal monotone with* $\mathrm{EP}(G) = A_G^{-1}0$,

(ii) $\mathrm{Res}^f_{\lambda G}$ *is the resolvent of A_G, i.e.,* $\mathrm{Res}^f_{\lambda G} = (\nabla f + \lambda A_G)^{-1}$ *for* $\lambda > 0$.

If we set $B = A_G$ and $A = 0$ in Theorems 3.3 and 3.7, then by Proposition 4.6, we obtain the following results.

Theorem 4.7. *Let $f\colon E \to \mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E such that ∇f is weakly sequentially continuous. Let C be a closed and convex subset of E.*

Let $G\colon C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)–(A4). *Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda G}(x_n))) \quad \forall\, n \geqslant 1, \tag{4.3}$$

where $\{\alpha_n\} \subset [0,1)$ and $\lambda > 0$. Suppose that $\limsup\limits_{n\to\infty} \alpha_n < 1$. If $\mathrm{EP}(G) \neq \emptyset$, then $\{x_n\}$ converges weakly to a point in $\mathrm{EP}(G)$.

Theorem 4.8. *Let $f\colon E \to \mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $G\colon C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)–(A4). *Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and*

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda G}(x_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(y_n)) \quad \forall\, n \geqslant 1, \end{cases} \tag{4.4}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\lambda > 0$. Suppose that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \beta_n < 1$. If $\mathrm{EP}(G) \neq \emptyset$, then $\{x_n\}$ converges strongly to $z = P^f_{\mathrm{EP}(G)}(u)$.

5. Numerical example

In this section, we provide a numerical example to illustrate the behaviour of our Algorithms 3.2 and 3.6.

Example 5.1. Let $E = \mathbb{R}^3$ with Euclidean norm. Let $f(x) = \|x\|^2$ for all $x = (y_1, y_2, y_3)^\top \in \mathbb{R}^3$, then f satisfies Assumption 3.1 (see [10]). Thus we have $\nabla f(x) = 2x$ and $\nabla f^*(x^*) = \frac{1}{2}x^*$. Let $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Ax = \frac{1}{2}x + (-1,2,0)^\top$ and $B\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Bx = 4x$ for all $x \in \mathbb{R}^3$. We see that A is a Bregman inverse strongly monotone mapping and B is a maximal monotone mapping. Indeed, let $x, y \in \mathbb{R}^3$, then

$$\begin{aligned} &\langle Ax - Ay, \nabla f^*(\nabla f(x) - Ax) - \nabla f^*(\nabla f(y) - Ay)\rangle \\ &\qquad = \Big\langle \frac{1}{2}x - \frac{1}{2}y, \nabla f^*\Big(2x - \frac{1}{2}x - (-1,2,0)\Big) - \nabla f^*\Big(2y - \frac{1}{2}y - (-1,2,0)\Big)\Big\rangle \\ &\qquad = \Big\langle \frac{1}{2}x - \frac{1}{2}y, \frac{3}{4}x - \frac{3}{4}y\Big\rangle = \frac{3}{8}\|x-y\|^2 \geqslant 0. \end{aligned}$$

We also have

$$\langle Bx - By, x - y\rangle = 4\|x-y\|^2 \geqslant 0$$

and $\operatorname{ran}(\nabla f + \lambda B) = \mathbb{R}^3$ for all $\lambda > 0$. Then the explicit form of the resolvent operator of B corresponding to f can be written by

$$\begin{aligned}\operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x) &= (\nabla f + \lambda B)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda A)x) \\ &= \frac{4-\lambda}{2(2+4\lambda)}x - \frac{\lambda}{2+4\lambda}(-1,2,0),\end{aligned}$$

where $\lambda > 0$. It is not hard to check that $(A+B)^{-1}0 = \{(\frac{2}{9}, -\frac{4}{9}, 0)^\top\}$. Let us denote $z = (\frac{2}{9}, -\frac{4}{9}, 0)^\top$. We choose $\alpha_n = 1/(100n+1)$, $\beta_n = n/(3n+1)$ and $u = x_1 = (5,-4,9)^\top$ and use the stopping rule $\|x_n - z\| < \varepsilon$ to terminate the iterative processes. We consider three different cases of λ ($\lambda = 0.01$, $\lambda = 0.05$ and $\lambda = 0.25$). The numerical results are reported in Table 1.

		Algorithm 3.2		Algorithm 3.6	
λ	ε	No. of Iter.	Time (sec.)	No. of Iter.	Time (sec.)
	10^{-6}	558	2.805×10^{-3}	810	5.390×10^{-3}
0.01	10^{-9}	867	4.358×10^{-3}	1276	8.510×10^{-3}
	10^{-15}	1487	7.569×10^{-3}	2209	1.518×10^{-2}
	10^{-6}	132	6.630×10^{-4}	193	1.276×10^{-3}
0.05	10^{-9}	196	9.737×10^{-4}	291	1.923×10^{-3}
	10^{-15}	324	1.863×10^{-3}	486	3.246×10^{-3}
	10^{-6}	35	1.788×10^{-4}	53	3.526×10^{-4}
0.25	10^{-9}	50	2.575×10^{-4}	77	4.884×10^{-4}
	10^{-15}	79	4.481×10^{-4}	125	8.284×10^{-4}

Table 1. The numerical experiments for Example 5.1 in each given λ and ε.

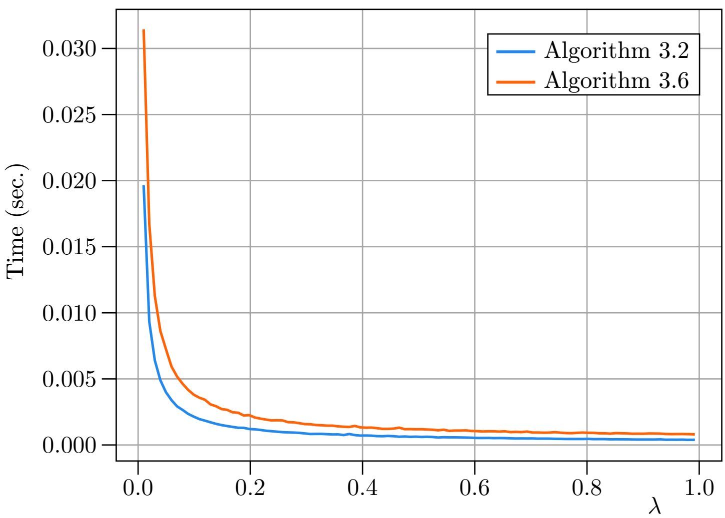

Figure 1. The relation between $\lambda \in (0,1)$ and the average running time with $\varepsilon = 10^{-15}$.

According to the experiments, we see that the average running times are significantly dependent on the values of λ. One observes that when λ increases, less iterations are required. We illustrate the relation between $\lambda \in (0,1)$ and the average running time in Figure 1 with $\varepsilon = 10^{-15}$.

6. Conclusions

In this paper, we introduced two modified forward-backward splitting algorithms for the problem of finding zeros of the sum of a maximal monotone operator and a Bregman inverse strongly monotone operator in reflexive Banach spaces. Our algorithms are based on two well-known methods, which are the Mann-type iteration and the Halpern-type iteration. We study weak and strong convergence results of the proposed algorithms for solving such a problem. Some applications related to the obtained results are presented. A numerical example is performed to illustrate the convergence results.

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References

[1] *R. P. Agarwal, D. O'Regan, D. R. Sahu*: Fixed Point Theory for Lipschitzian-type Mappings with Applications. Topological Fixed Point Theory and Its Applications 6. Springer, New York, 2009. zbl MR doi
[2] *Y. I. Alber*: Generalized projection operators in Banach spaces: Properties and applications. Funct. Differ. Equ. *1* (1993), 1–21. zbl MR
[3] *Y. I. Alber*: Metric and generalized projection operators in Banach spaces: Properties and applications. Theory and Applications of Nonlinear Operator of Accretive and Monotone Type. Lecture Notes in Pure and Applied Mathematics 178. Marcel Dekker, New York, 1996, pp. 15–50. zbl MR
[4] *V. Barbu, T. Precupanu*: Convexity and Optimization in Banach Spaces. Springer Monographs in Mathematics. Springer, Dordrecht, 2012. zbl MR doi
[5] *H. H. Bauschke, J. M. Borwein, P. L. Combettes*: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun. Contemp. Math. *3* (2001), 615–647. zbl MR doi
[6] *H. H. Bauschke, X. Wang, L. Yao*: General resolvents for monotone operators: Characterization and extension. Biomedical Mathematics: Promising Directions in Imagine, Therapy Planning and Inverse Problem (Huangguoshu 2008) (Y. Censor, M. Jiang, G. Wang, eds.). Medical Physics Publishing, Madison, 2010, pp. 57–74.
[7] *A. Beck*: First-Order Methods in Optimization. MOS-SIAM Series on Optimization 25. SIAM, Philadelphia, 2017. zbl MR doi
[8] *D. P. Bertsekas, J. N. Tsitsiklis*: Parallel and Distributed Computation: Numerical Methods. Athena Scientific, Belmont, 1997. zbl MR

[9] *J. F. Bonnans, A. Shapiro*: Perturbation Analysis of Optimization Problems. Springer Series in Operations Research. Springer, New York, 2000. zbl MR doi
[10] *J. M. Borwein, S. Reich, S. Sabach*: A characterization of Bregman firmly nonexpansive operators using a new monotonicity concept. J. Nonlinear Convex Anal. *12* (2011), 161–184. zbl MR
[11] *L. M. Bregman*: The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. U.S.S.R. Comput. Math. Math. Phys. *7* (1967), 200–217; translation from Zh. Vychisl. Mat. Mat. Fiz. *7* (1967), 620–631. zbl MR doi
[12] *D. Butnariu, E. Resmerita*: Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. *2006* (2006), Article ID 84919, 39 pages. zbl MR doi
[13] *S.-S. Chang, C.-F. Wen, J.-C. Yao*: Generalized viscosity implicit rules for solving quasi-inclusion problems of accretive operators in Banach spaces. Optimization *66* (2017), 1105–1117. zbl MR doi
[14] *S.-S. Chang, C.-F. Wen, J.-C. Yao*: Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces. Optimization *67* (2018), 183–1196. zbl MR doi
[15] *S.-S. Chang, C.-F. Wen, J.-C. Yao*: A generalized forward-backward splitting method for solving a system of quasi variational inclusions in Banach spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM *113* (2019), 729–747. zbl MR doi
[16] *S.-S. Chang, C.-F. Wen, J.-C. Yao*: Zero point problem of accretive operators in Banach spaces. Bull. Malays. Math. Sci. Soc. (2) *42* (2019), 105–118. zbl MR doi
[17] *G. H.-G. Chen, R. T. Rockafellar*: Convergence rates in forward-backward splitting. SIAM J. Optim. *7* (1997), 421–444. zbl MR doi
[18] *P. Cholamjiak, N. Pholasa, S. Suantai, P. Sunthrayuth*: The generalized viscosity explicit rules for solving variational inclusion problems in Banach spaces. To appear in Optimization. doi
[19] *P. L. Combettes, V. R. Wajs*: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. *4* (2005), 1168–1200. zbl MR doi
[20] *J. C. Dunn*: Convexity, monotonicity, and gradient processes in Hilbert space. J. Math. Anal. Appl. *53* (1976), 145–158. zbl MR doi
[21] *O. Güler*: On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. *29* (1991), 403–419. zbl MR doi
[22] *S. He, C. Yang*: Solving the variational inequality problem defined on intersection of finite level sets. Abstr. Appl. Anal. *2013* (2013), Article ID 942315, 8 pages. zbl MR doi
[23] *R. Iyer, J. A. Bilmes*: Submodular-Bregman and the Lovász-Bregman divergences with applications. Advances in Neural Information Processing Systems 25 (NIPS 2012). MIT Press, Cambridge, 2012, 9 pages.
[24] *Y. Kimura, K. Nakajo*: Strong convergence for a modified forward-backward splitting method in Banach spaces. J. Nonlinear Var. Anal. *3* (2019), 5–18. zbl doi
[25] *P.-L. Lions, B. Mercier*: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. *16* (1979), 964–979. zbl MR doi
[26] *G. López, V. Martín-Márquez, F. Wang, H.-K. Xu*: Forward-backward splitting methods for accretive operators in Banach spaces. Abstr. Appl. Anal. *2012* (2012), Article ID 109236, pages 25 pages. zbl MR doi
[27] *P. E. Maingé*: Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. *325* (2007), 469–479. zbl MR doi
[28] *V. Martín-Márquez, S. Reich, S. Sabach*: Iterative Methods for approximating fixed points of Bregman nonexpansive operators. Discrete Contin. Dyn. Syst., Ser. S *6* (2013), 1043–1063. zbl MR doi

[29] *E. Naraghirad*: Halpern's iteration for Bregman relatively nonexpansive mappings in Banach spaces. Numer. Funct. Anal. Optim. *34* (2013), 1129–1155. zbl MR doi
[30] *E. Naraghirad, J.-C. Yao*: Bregman weak relatively nonexpansive mappings in Banach space. Fixed Point Theory Appl. *2013* (2013), Article ID 141, 43 pages. zbl MR doi
[31] *F. Nielsen, S. Boltz*: The Burbea-Rao and Bhattacharyya centroids. IEEE Trans. Inf. Theory *57* (2011), 5455–5466. zbl MR doi
[32] *F. Nielsen, R. Nock*: Generalizing skew Jensen divergences and Bregman divergences with comparative convexity. IEEE Signal Process. Lett. *24* (2017), 1123–1127. doi
[33] *F. U. Ogbuisi, C. Izuchukwu*: Approximating a zero of sum of two monotone operators which solves a fixed point problem in reflexive Banach spaces. Numer. Funct. Anal. Optim. *41* (2020), 322–343. zbl MR doi
[34] *H. K. Pathak*: An Introduction to Nonlinear Analysis and Fixed Point Theory. Springer, Singapore, 2018. zbl MR doi
[35] *S. Reich*: A weak convergence theorem for the alternating method with Bregman distances. Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lecture Notes in Pure and Applied Mathematics 178. Marcel Dekker, New York, 1996, pp. 313–318. zbl MR
[36] *S. Reich, S. Sabach*: Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim. *31* (2010), 22–44. zbl MR doi
[37] *S. Reich, S. Sabach*: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal., Theory Methods Appl., Ser. A *73* (2010), 122–135. zbl MR doi
[38] *S. Reich, S. Sabach*: Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces. Fixed-Point Algorithms for Inverse Problems in Science and Engineering. Springer Optimization and Its Applications 49. Springer, New York, 2011, pp. 301–316. zbl MR doi
[39] *R. T. Rockafellar*: Convex Analysis. Princeton University Press, Princeton, 1970. zbl MR doi
[40] *R. T. Rockafellar*: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. *33* (1970), 209–216. zbl MR doi
[41] *R. T. Rockafellar*: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. *14* (1976), 877–898. zbl MR doi
[42] *S. Sabach*: Products of finitely many resolvents of maximal monotone mappings in reflexive Banach spaces. SIAM J. Optim. *21* (2011), 1289–1308. zbl MR doi
[43] *Y. Shehu*: Convergence results of forward-backward algorithms for sum of monotone operators in Banach spaces. Result. Math. *74* (2019), Article ID 138, 24 pages. zbl MR doi
[44] *Y. Shehu, G. Cai*: Strong convergence result of forward-backward splitting methods for accretive operators in Banach spaces with applications. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM *112* (2018), 71–87. zbl MR doi
[45] *S. Suantai, P. Cholamjiak, P. Sunthrayuth*: Iterative methods with perturbations for the sum of two accretive operators in q-uniformly smooth Banach spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM *113* (2019), 203–223. zbl MR doi
[46] *P. Sunthrayuth, P. Cholamjiak*: Iterative methods for solving quasi-variational inclusion and fixed point problem in q-uniformly smooth Banach spaces. Numer. Algorithms *78* (2018), 1019–1044. zbl MR doi
[47] *W. Takahashi, N.-C. Wong, J.-C. Yao*: Two generalized strong convergence theorems of Halpern's type in Hilbert spaces and applications. Taiwanese J. Math. *16* (2012), 1151–1172. zbl MR doi
[48] *P. Tseng*: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. *38* (2000), 431–446. zbl MR doi

[49] *Y. Wang, F. Wang*: Strong convergence of the forward-backward splitting method with multiple parameters in Hilbert spaces. Optimization *67* (2018), 493–505. zbl MR doi

[50] *H.-K. Xu*: Inequalities in Banach spaces with applications. Nonlinear Anal., Theory Methods Appl. *16* (1991), 1127–1138. zbl MR doi

[51] *C. Zălinescu*: Convex Analysis in General Vector Spaces. World Scientific, Singapore, 2002. zbl MR doi

[52] *H. Zegeye, N. Shahzad*: An algorithm for finding a common point of the solution set of a variational inequality and the fixed point set of a Bregman relatively nonexpansive mapping. Appl. Math. Comput. *248* (2014), 225–234. zbl MR doi

[53] *J.-H. Zhu, S.-S. Chang*: Halpern-Mann's iterations for Bregman strongly nonexpansive mappings in reflexive Banach spaces with applications. J. Inequal. Appl. *2013* (2013), Article ID 146, 14 pages. zbl MR doi

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Research article

Numerical solution of stochastic and fractional competition model in Caputo derivative using Newton method

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Abstract: Many useful numerical algorithms of the numerical solution are proposed due to the increasing interest of the researchers in fractional calculus. A new discretization of the competition model for the real statistical data of banking finance for the years 2004–2014 is presented. We use a novel numerical method that is more reliable and accurate which is introduced recently for the solution of ordinary differential equations numerically. We apply this approach to solve our model for the case of Caputo derivative. We apply the Caputo derivative on the competition system and obtain its numerical results. For the numerical solution of the competition model, we use the Newton polynomial approach and present in detail a novel numerical procedure. We utilize the numerical procedure and present various numerical results in the form of graphics. A comparison of the present method versus the predictor corrector method is presented, which shows the same solution behavior to the Newton Polynomial approach. We also suggest that the real data versus model provide good fitting for both the data for the fractional-order parameter value $\rho = 0.7$. Some more values of ρ are used to obtain graphical results. We also check the model in the stochastic version and show the model behaves well when fitting to the data.

Keywords: Caputo derivative; Newton polynomial approach; real data 2004–2014; numerical algorithm; stochastic version
Mathematics Subject Classification: 34A08, 37N30

1. Introduction

The problems arising in science and engineering have been successfully modeled through the mathematical models tools [1–5]. Besides this, the application of mathematical models to the social sciences and other financial areas are getting attention these days from researchers around the world. In the country's economic growth, the role of the banks is considered to be a backbone. Strong banking policy and its benefits to the citizens play an important role in the overall development of the country, the bank's related terms and explanations can be found in [6]. The banking sector may be Islamic or conventional or maybe both of these. The banks that follow the rule of syriah law or conventionally are defined as per the rule Act No. 10 of 1998. The activities of rural and commercial banks in Indonesia are observed and documented that commercial banks have more business activates than the former [7, 8]. Further, in Indonesia, it is reported that commercial banks are more than the rural, but the products of both the banks are consider to be the same [9]. Although, the commercial banks are greater but the rural are still improving their business activities to attract customers. Due to the improvements in their business products, the rural banks may have competition with commercial banks.

If the rural banks continue their efforts to improve their products, then definitely there may have close competition in Indonesia. This competition can be effectively studied through evolutionary differential equations known as the Lotka Volterra system [10]. The researchers utilized the Lotka Volterra equations in order to study the competition in many real-word problems, see [11–18]. For an instant, one of the applications of the Korean mobile company data through the competition system is explored in [11]. The competition system as a technological substitution, see [12], modeling the policy and their implications to the Korean stock market [13], dynamics estimations of the market exhibition, see [14], competition dynamics in the operating system market, see [15], and the analysis of the banking dynamics system, see [16]. Moreover, some recent works regarding this, can be seen in [17–20].

In the above-mentioned studies, it is worthy to mention that the Lotka Volterra system was used to obtain the dynamics of different problems with the integer-order study, except [17–20]. From the last years, it is observed that fractional calculus plays a vital role in the dynamic modeling of such practical problems. One of the reasons is considered to be the memory and the heredity properties. The model of real-life problems is often nonlinear and the crossover behavior makes it difficult to solve it exactly. The model formulated in fractional derivatives has many advantages, such as the data fitting, the memory index, the heredity properties, and the cross-over behavior. The dynamics of the model with fractional order derivatives and its applications to the real-life problems can be seen in [21, 22, 26–31]. For example, the competition among rural and commercial banks through fractional derivatives is discussed in [21]. The dynamics of TB using the fractional-order derivative are explored in [22]. An introduction to fractional derivative, fractional differential equations, and its numerical solution is discussed in [26]. A fractional lotka Volterra mathematical model and its analysis are studied in [27]. The Hepatitis E model in fractional Caputo Fabrizio derivative is explored in [28]. A fractional-order TB model with relapse is considered in [29]. The authors in [30] considered the HIV infection model using two-sex populations. The RLC circuit model using the fractional derivative and its numerical investigations is considered in [31]. The authors used the concept of fractional derivatives to study different problems, see for example, the training model for football movement trajectory [32], the educational reforms through fractional differential equations [33], a fractional model for the sugar [34], a fractional model for the impact of financial repression [35], and the university education model using fractional differential equations and its numerical solution [36]. Some more mathematical model formulated in fractional derivatives and their applications to COVID-19 infection is explored in [37,38,40], where the authors studied a mathematical model for COVID-19 with isolation class [37], COVID-19 with stochastic perturbation [38], COVID-19 model with control in fractional environment [39] and the COVID-19 model with crowding effect in [40]. Application of fractional derivative to Hepatitis B [41,42], and COVID-19 [43]. The fractional-order model to study the leakage delay [44], the fractional-order with neural networks using multi-delays [45], bifurcation analysis in a fractional-order [46], a delayed BAM fractional-order system [47], fractional-order model with control [48], and fractional-order neural networks with mixed delays [49]. It is important that real-life problems with realistic data, give more accurate information about that particular phenomena than the assumed data, such useful results can be seen in [19–22], where a particular focus is given to show the data fitting to the system of equations. The works mentioned above, have been solved numerically by using the novel numerical technique.

The goal of this work is to study the competition of the banking data between two banks in Indonesia with real statistical data through a fractional model. The fractional derivative considered in this work is the Caputo derivative. For the numerical solution of the fractional model, we use the recently developed new numerical method using the approach of Newton polynomial. This novel technique was introduced recently in literature to get the numerical solution of fractional ordinary differential equations, that has been used for many scientific problems, see [23–25]. For example, the authors introduced this technique for the solution of the COVID-19 model in fractional derivative in Atangana-Baleanu derivative [23]. The HIV dynamics and its mathematical analysis through a fractional model with real cases using this new approach is considered by the authors in [24]. The application of this new method to groundwater flow is considered by the authors in [25]. This novel technique is more accurate and reasonable than the other technique available in the literature for the factional models. The proposed technique will show how the data fits well with the consideration of the specific fractional orders. The rest of the results in this paper are as follows: The model and their descriptions are shown in Section 2. The related definition and the integral are shown in Section 3. The solution of the model numerically by giving the algorithm is shown in Section 4. Section 5 explain the numerical investigations of the model while Sections 6 and 7 respectively show the formulation of the stochastic problem and the summary of the results.

2. Concepts related to fractional operator

We provide here the related concepts for Caputo derivative [59].

Definition 1. *[59] Suppose $g:\mathbb{R}^+\to\mathbb{R}$ and $\rho\in(m-1,m)$, $j\in\mathbb{K}$. Then, the representation of Caputo derivative with order ρ for a function g can be defined by the way below,*

$$ {}_0^C D_t^{\rho} g(t) = \frac{1}{\Gamma(m-\rho)}\int_0^t (t-\psi)^{m-\rho-1} g^{(m)}(\psi)d\psi. \tag{2.1}$$

Definition 2. *[59] The integral for the above is given by the following,*

$$ {}_0^C I_t^{\rho}(g(t)) = \frac{1}{\Gamma(\rho)}\int_0^t g(\psi)(t-\psi)^{\rho-1}d\psi. \tag{2.2}$$

3. Mathematical model

The competition among the commercial and rural banks can be described here through the useful mathematical model called the Lotka-Volterra system. The Lotka-Volterra system has two equations, which were designed to have competition among two species for food etc. This competition model has been used effectively in many studies by the authors for real statistical data and found it interesting. So, on the basis of the competition model, it aims to describe the two equations by commercial and rural banks dynamics. At any time t, the dynamics of commercial banks is shown by $x(t)$ while the rural banks dynamics is given by $y(t)$, with the assumptions of maximum profit having both the banks. Further, the banking limited funds behave logistically their growths. So, with this hypothesis, the dynamical analysis of the competition model among the two banks system is shown by,

$$\begin{aligned} \frac{dx}{dt} &= \alpha_1 x\Big(1-\frac{x}{K_1}\Big)-\psi_1 xy, \\ \frac{dy}{dt} &= \alpha_2 y\Big(1-\frac{y}{K_2}\Big)-\psi_2 xy, \end{aligned} \tag{3.1}$$

the initial conditions to the (3.1) are $x(0)=x_0\geq 0$, and $y(0)=y_0\geq 0$. The parameters α_i for $i=1,2$ describe the growth factor respectively for commercial and rural banks and so is positive. The others parameters K_i for $i=1,2$ that define the maximum profit gained respectively, by the commercial and the rural banks. The coefficients ψ_i for $i=1,2$ are used as a competition parameter respectively for commercial and rural banks. So, it is obvious from this discussion that these parameters are positive.

3.1. *A fractional Caputo model*

We use the model presented above (3.2) and apply the Caputo derivative to generalize it. With this generalization, the model (3.2) takes the following fractional form below:

$$\begin{aligned} {}_0^C D_0^\rho\big(x(t)\big) &= \alpha_1 x\Big(1-\frac{x}{K_1}\Big)-\psi_1 xy, \\ {}_0^C D_0^\rho\big(y(t)\big) &= \alpha_2 y\Big(1-\frac{y}{K_2}\Big)-\psi_2 xy, \end{aligned} \tag{3.2}$$

where $\rho\in(0,1)$ that defines the fractional order is considered in Caputo sense.

3.2. *Equilibria and their stability*

The equilibrium points for the considered model (3.2) can be obtained by the following way,

$${}_0^C D_0^\rho\big(x(t)\big)=0,\quad {}_0^C D_0^\rho\big(y(t)\big)=0. \tag{3.3}$$

The condition (3.3), gives the following:

$$\begin{aligned} &\alpha_1 x\Big(1-\frac{x}{K_1}\Big)-\psi_1 xy=0, \\ &\alpha_2 y\Big(1-\frac{y}{K_2}\Big)-\psi_2 xy=0. \end{aligned} \tag{3.4}$$

Solving the equations in (3.4), we have:

$$\begin{aligned} E_0 &= (0,0), \quad E_1 = (0, K_2), \quad E_2 = (K_1, 0), \\ E_3 &= \Big(\frac{\alpha_2 K_1 (K_2\psi_1 - \alpha_1)}{K_1K_2\psi_1\psi_2 - \alpha_1\alpha_2}, \frac{\alpha_1 K_2 (K_1\psi_2 - \alpha_2)}{K_1K_2\psi_1\psi_2 - \alpha_1\alpha_2}\Big). \end{aligned}$$

Consider the aforementioned equilibrium points, we discuss its stability results for the system (3.2) below: Initially, we find the Jacobian matrix of the system (3.2) given by the following:

$$J = \begin{pmatrix} \left(1 - \frac{2x^*}{K_1}\right)\alpha_1 - \psi_1 y^* & -\psi_1 x^* \\ -\psi_2 y^* & \left(1 - \frac{2y^*}{K_2}\right)\alpha_2 - \psi_2 x^* \end{pmatrix}.$$

We consider now the stability at the above fixed points. First, we consider $E_0 = (0,0)$. At this equilibrium point, we get the eigenvalues α_1, α_2 which are positive, and so at this point the system is not stable. At E_1, we have the eigenvalues, $-\alpha_2, \alpha_1 - K_2\psi_1$, where the second one can be negative if $\alpha_1 < K_2\psi_1$, and the equilibrium point of the system will be stable. The equilibrium point E_2 gives the eigenvalues, $-\alpha_1, \alpha_2 - K_1\psi_2$. If the second one becomes negative, then the proposed model will be locally asymptotically stable. For the last equilibrium point, which is feasible, and their stability is shown below by having the characteristics polynomial:

$$\lambda^2 + \varpi_1\lambda + \varpi_2 = 0,$$

where

$$\begin{aligned} \varpi_1 &= \frac{\alpha_1\alpha_2 (\alpha_1 - K_2\psi_1 + \alpha_2 - K_1\psi_2)}{\alpha_1\alpha_2 - K_1K_2\psi_1\psi_2}, \\ \varpi_2 &= \frac{\alpha_1\alpha_2 (\alpha_1 - K_2\psi_1)(\alpha_2 - K_1\psi_2)}{\alpha_1\alpha_2 - K_1K_2\psi_1\psi_2}. \end{aligned}$$

The coefficients ϖ_1 and ϖ_2 can be positive if $(\alpha_1 - K_2\psi_1) > 0$, $(\alpha_2 - K_1\psi_2) > 0$ and $\alpha_1\alpha_2 - K_1K_2\psi_1\psi_2 > 0$. If these conditions hold then the system will be locally asymptotically stable.

3.3. Existence and uniqueness

Consider the system (3.2), we write for simplicity the model (3.2),

$$\begin{aligned} {}_0^C D_t^\rho x(t) &= f_1(x, y, t), \\ {}_0^C D_t^\rho y(t) &= f_2(x, y, t). \end{aligned}$$

We define the following norm,

$$\|x\|_\infty = \sup_{t \in D} |x(t)|,$$

where $D = [0, T]$. We assume to show that $x(t)$ and $y(t)$ are bounded in $[0, T]$, therefore, $\forall t \in [0, T]$ there exists λ_1 and λ_2 such that $||x||_\infty < \lambda_1$ and $||y||_\infty < \lambda_2$. We first show that f_1 and f_2 are bounded

$$\begin{aligned}
|f_1(x, y, t)| &= |\alpha_1 x(1 - \frac{x}{K_1}) - \psi_1 xy|, \\
&\leq \alpha_1 |x\Big(1 - \frac{x}{K_1}\Big)| + |\psi xy|, \\
&\leq \alpha_1 |x||\Big(1 - \frac{x}{K_1}\Big)| + |\psi_1 xy|, \\
&\leq \alpha_1 \sup_{t\in[0,T]} |x(t)|| \sup_{t\in[0,T]} \Big(1 - \frac{x(t)}{K_1}\Big)| + \psi_1 \sup_{t\in[0,T]} |x(t)| \sup_{t\in[0,T]} |y(t)|, \\
&\leq \alpha_1 ||x||_\infty \Big(1 + \frac{||x||_\infty}{K_1}\Big) + \psi_1 ||x||_\infty ||y||_\infty, \\
&< \alpha_1 \lambda_1 \Big(1 + \frac{\lambda_1}{K_1}\Big) + \psi_1 \lambda_1 \lambda_2, \\
&< \infty.
\end{aligned}$$

Similarly, we can show

$$|f_2(x, y, t)| < \alpha_1 \lambda_2 \Big(1 + \frac{\lambda_2}{K_2}\Big) + \psi_2 xy < \infty.$$

Therefore, if x and y are bounded then there exists M_1 and M_2 such that

$$\sup_{t\in[0,T]} |f_1(x, y, t)| < M_1, \quad \sup_{t\in[0,T]} |f_2(x, y, t)| < M_2.$$

On the other hand if $\forall t \in [0, T]$, $|f_1(x, y, t)| < M_1$ and $|f_2(x, y, t)| < M_2$, then $||x||_\infty < \infty$ and $||y||_\infty < \infty$.

Proof. Assuming that f_1 and f_2 are bounded, then

$$\begin{aligned}
x(t) &= x(0) + \frac{1}{\Gamma(\rho)} \int_0^t f_1(x, y, \tau)(t - \tau)^{\rho-1} d\tau, \\
y(t) &= y(0) + \frac{1}{\Gamma(\rho)} \int_0^t f_2(x, y, \tau)(t - \tau)^{\rho-1} d\tau.
\end{aligned}$$

$$\begin{aligned}
|x(t)| &\leq |x(0)| + \frac{1}{\Gamma(\rho)} | \int_0^t f_1(x, y, \tau)(t - \tau)^{\rho-1} d\tau|, \\
|y(t)| &\leq |y(0)| + \frac{1}{\Gamma(\rho)} | \int_0^t f_2(x, y, \tau)(t - \tau)^{\rho-1} d\tau|.
\end{aligned}$$

Then without loss of generality, we present for $x(t)$

$$|x(t)| \leq |x(0)| + \frac{1}{\Gamma(\rho)} \int_0^t |f_1(x, y, \tau)|(t - \tau)^{\rho-1} d\tau,$$

$$
\begin{aligned}
&< |x(0)| + \frac{1}{\Gamma(\rho)}\int_0^t \sup_{t\in[0,T]} |f_1(x,y,\tau)|(t-\tau)^{\rho-1}d\tau, \\
&< |x(0)| + \frac{M_1 t^\rho}{\Gamma(\rho+1)} < |x(0)| + \frac{M_1 t^\rho}{\Gamma(\rho+1)}.
\end{aligned}
$$

Therefore,

$$|x(t)| < \|x\|_\infty < |x(0)| + \frac{M_1 T^\rho}{\Gamma(\rho+1)}.$$

Similarly, we can show

$$|y(t)| \;<\; \|y\|_\infty < |y(0)| + \frac{M_2 t^\rho}{\Gamma(\rho+1)}.$$

□

To prove that our system admits a unique solution, we need to show that

- $\forall t \in [0,T]$, f_1 and f_2 satisfy the following condition, $|f_1(x,y,t)|^2 < \beta_1(1+|x|^2)$, $|f_2(x,y,t)|^2 < \beta_2(1+|y|^2)$,
- $\forall t \in [0,T]$, f_1 and f_2 satisfy the Lipschitz condition $|f_1(x_1,y,t)-f_1(x_2,y,t)|^2 < \overline{\beta_1}|x_1-x_2|^2$, $|f_2(x,y_1,t)-f_2(x,y_2,t)|^2 < \overline{\beta_2}|y_1-y_2|^2$.

Without loss of generality, we show he proof for f_1,

$$
\begin{aligned}
|f_1(x_1,y,t)-f_1(x_2,y,t)|^2 &= |\alpha_1 x_1\Big(1-\frac{x_1}{K_1}\Big) - \psi_1 x_1 y - \alpha_2 x_2\Big(1-\frac{x_2}{K_1}\Big) + \psi_1 x_2 y|^2, \\
&= |\alpha_1(x_1-x_2) - \alpha_1\Big(\frac{x_1^2}{K_1} - \frac{x_2^2}{K_1}\Big) - \psi_1(x_1-x_2)y| \\
&< 3\alpha_1^2|x_1-x_2|^2 + 3\alpha_1^2|\frac{x_1^2}{K_1} - \frac{x_2^2}{K_1}|^2 + 3\psi_1^2|x_1-x_2|^2|y| \\
&< 3\{\alpha_1^2 + 2\alpha_1^2\Big(|\frac{x_1}{K_1}|^2 + |\frac{x_2}{K_1}|^2\Big) + \psi_1^2|y|^2\}|x_1-x_2|^2, \\
&< 3\{\alpha_1^2 + 2\alpha_1^2\Big(\sup_{t\in[0,T]}|\frac{x_1}{K_1}|^2 + \sup_{t\in[0,T]}|\frac{x_2}{K_1}|^2\Big) + \psi_1^2\sup_{t\in[0,T]}|y|^2\}|x_1-x_2|^2, \\
&< 3\Big(\alpha_1^2 + 2\alpha_1^2\Big(\frac{2\lambda_1^2}{K_1^2}\Big) + \psi_1^2\lambda_2^2\Big)|x_1-x_2|^2, \\
&< \overline{\beta_1}|x_1-x_2|^2
\end{aligned}
$$

where $\overline{\beta_1} = \Big(\alpha_1^2 + 2\alpha_1^2\Big(\frac{2\lambda_1^2}{K_1^2}\Big) + \psi_1^2\lambda_2^2\Big)$. On the other hand

$$
\begin{aligned}
|f_1(x,y,t)|^2 &= |\alpha_1 x\Big(1-\frac{x}{K_1}\Big) - \psi_1 xy|^2, \\
&\le 3\alpha_1^2|x|^2 + 3\alpha_1^2|\frac{x}{K_1}|^2 + 3\psi_1^2|x|^2|y|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big(3\alpha_1^2 + 3\frac{\alpha_1^2}{K_1^2}|x|^2 + 3\psi_1^2|y|^2\Big)|x|^2, \\
&\leq \Big(3\alpha_1^2 + 3\frac{\alpha_1^2}{K_1^2}\sup_{t\in[0,T]}|x(t)^2| + 3\psi_1^2\sup_{t\in[0,T]}|y(t)^2|\Big)|x|^2, \\
&< \Big(3\alpha_1^2 + \frac{3\alpha_1^2}{K_1^2}\lambda_1^2 + 3\psi_1^2\lambda_2^2\Big)|x|^2, \\
&< \beta_1\Big(1 + |x|^2\Big),
\end{aligned}
$$

where $\beta_1 = \Big(3\alpha_1^2 + \frac{3\alpha_1^2}{K_1^2}\lambda_1^2 + 3\psi_1^2\lambda_2^2\Big)$. Therefore, the system (3.2) has a unique system of solution.

4. Solution procedure with Caputo fractional derivative

This section presents a new discretization for the Caputo derivative through Newton polynomial, which was established recently in [60]. We explain below briefly the procedure by considering a general fractional differential equation in the form given by:

$$
{}_0^C D_t^\rho z(t) = f(t, z(t)), \tag{4.1}
$$

where D_t^ρ shows the Caputo derivative and the f is the nonlinear function. To present a numerical solution procedure to get the solution of fractional differential equation using Newton descritization, we have to rewrite the problems is as follows:

$$
z(t) - z(0) = \frac{1}{\Gamma(\rho)}\int_0^t f(\tau, z(\tau))(t-\tau)^{\rho-1}d\tau. \tag{4.2}
$$

At $t_{n+1} = (n+1)\Delta t$, the following can be written,

$$
z(t_{n+1}) - z(0) = \frac{1}{\Gamma(\rho)}\int_0^{t_{n+1}} f(\tau, z(\tau))(t_{n+1}-\tau)^{\rho-1}d\tau. \tag{4.3}
$$

Also, we can write

$$
z(t_{n+1}) = z(0) + \frac{1}{\Gamma(\rho)}\sum_{j=2}^{n}\int_{t_j}^{t_{j+1}} f(\tau, z(\tau))(t_{n+1}-\tau)^{\rho-1}d\tau. \tag{4.4}
$$

We now using the Newton approach to approximate the $f(\tau, z(\tau))$ which is inside the integral, and has the following

$$
\begin{aligned}
P_n(\tau) = {} & f(t_{n-2}, z(t_{n-2})) + \frac{f(t_{n-1}, z(t_{n-1})) - f(t_{n-2}, z(t_{n-2}))}{\Delta t}(\tau - t_{n-2}) \\
& + \frac{f(t_n, z(t_n)) - 2f(t_{n-1}, z(t_{n-1})) + f(t_{n-2}, z(t_{n-2}))}{2(\Delta t)^2}(\tau - t_{n-2})(\tau - t_{n-1}).
\end{aligned} \tag{4.5}
$$

Substitution of the result (4.5) into (4.4), the following is obtained,

$$
\begin{aligned}
z^{n+1} &= z^0 + \frac{1}{\Gamma(\rho)} \sum_{j=2}^{n} \int_{t_j}^{t_{j+1}} \Big[f(t_{j-2}, z^{j-2}) + \frac{f(t_{j-1}, z^{j-1}) - f(t_{j-2}, z^{j-2})}{\Delta t} (\tau - t_{j-2}) \\
&\quad + \frac{f(t_j, z^j) - 2f(t_{j-1}, z^{j-1}) + f(t_{j-2}, z^{j-2})}{2(\Delta t)^2} (\tau - t_{j-2})(\tau - t_{j-1}) \Big] \\
&\quad \times (t_{n+1} - \tau)^{\rho-1} d\tau. \qquad (4.6)
\end{aligned}
$$

Re-arranging Eq (4.6), leads to the following,

$$
\begin{aligned}
z^{n+1} &= z^0 + \frac{1}{\Gamma(\rho)} \sum_{j=2}^{n} \Big[\int_{t_j}^{t_{j+1}} f(t_{j-2}, z^{j-2}) (t_{n+1} - \tau)^{\rho-1} d\tau \\
&\quad + \int_{t_j}^{t_{j+1}} \frac{f(t_{j-1}, z^{j-1}) - f(t_{j-2}, z^{j-2})}{\Delta t} (\tau - t_{j-2})(t_{n+1} - \tau)^{\rho-1} d\tau \\
&\quad + \int_{t_j}^{t_{j+1}} \frac{f(t_j, z^j) - 2f(t_{j-1}, z^{j-1}) + f(t_{j-2}, z^{j-2})}{2(\Delta t)^2} (\tau - t_{j-2})(\tau - t_{j-1}) \\
&\quad \times (t_{n+1} - \tau)^{\rho-1} d\tau \Big]. \qquad (4.7)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
z^{n+1} &= z^0 + \frac{1}{\Gamma(\rho)} \sum_{j=2}^{n} f(t_{j-2}, z^{j-2}) \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\rho-1} d\tau \\
&\quad + \frac{1}{\Gamma(\rho)} \sum_{j=2}^{n} \frac{f(t_{j-1}, z^{j-1}) - f(t_{j-2}, z^{j-2})}{\Delta t} \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1} - \tau)^{\rho-1} d\tau \\
&\quad + \frac{1}{\Gamma(\rho)} \sum_{j=2}^{n} \frac{f(t_j, z^j) - 2f(t_{j-1}, z^{j-1}) + f(t_{j-2}, z^{j-2})}{2(\Delta t)^2} \times \\
&\quad \int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1} - \tau)^{\rho-1} d\tau. \qquad (4.8)
\end{aligned}
$$

We simplify the integrals in Eq (4.8) below,

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\rho-1} d\tau &= \frac{h^\rho}{\rho} [(n - j + 1)^\rho - (n - j)^\rho], \\
\int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(t_{n+1} - \tau)^{\rho-1} d\tau &= \frac{h^{\rho+1}}{\rho(\rho + 1)} [(n - j + 1)^\rho (n - j + 3 + 2\rho) \\
&\quad - (n - j)^\rho (n - j + 3 + 3\rho)], \\
\int_{t_j}^{t_{j+1}} (\tau - t_{j-2})(\tau - t_{j-1})(t_{n+1} - \tau)^{\rho-1} d\tau &= \frac{h^{\rho+2}}{\rho(\rho + 1)(\rho + 2)}
\end{aligned}
$$

$$\times \left[\begin{array}{c} (n-j+1)^{\rho} \left[\begin{array}{c} 2(n-j)^2 + (3\rho+10)(n-j) \\ +2\rho^2 + 9\rho + 12 \end{array} \right] \\ -(n-j)^{\rho} \left[\begin{array}{c} 2(n-j)^2 + (5\rho+10)(n-j) \\ +6\rho^2 + 18\rho + 12 \end{array} \right] \end{array} \right], \tag{4.9}$$

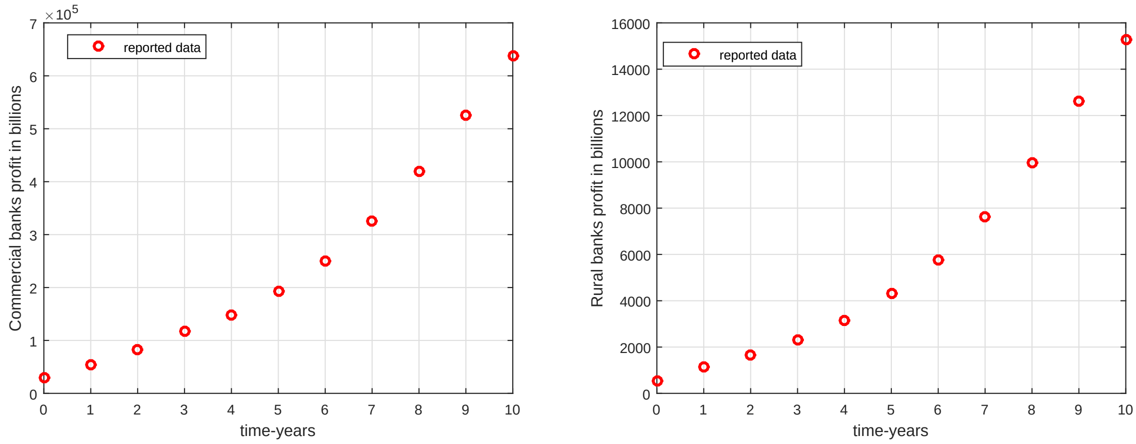
the following formula is obtained after inserting the above solution into Eq (4.9), the following is presented,

$$\begin{aligned} z^{n+1} &= z^0 + \frac{h^{\rho}}{\Gamma(\rho+1)} \sum_{j=2}^{n} f(t_{j-2}, z^{j-2})[(n-j+1)^{\rho} - (n-j)^{\rho}] \\ &\quad + \frac{h^{\rho}}{\Gamma(\rho+2)} \sum_{j=2}^{n} [f(t_{j-1}, z^{j-1}) - f(t_{j-2}, z^{j-2})] \\ &\quad \times [(n-j+1)^{\rho}(n-j+3+2\rho) - (n-j)^{\rho}(n-j+3+3\rho)] \\ &\quad + \frac{h^{\rho}}{2\Gamma(\rho+3)} \sum_{j=2}^{n} [f(t_j, z^j) - 2f(t_{j-1}, z^{j-1}) + f(t_{j-2}, z^{j-2})] \\ &\quad \times \left[\begin{array}{c} (n-j+1)^{\rho} \left[\begin{array}{c} 2(n-j)^2 + (3\rho+10)(n-j) \\ +2\rho^2 + 9\rho + 12 \end{array} \right] \\ -(n-j)^{\rho} \left[\begin{array}{c} 2(n-j)^2 + (5\rho+10)(n-j) \\ +6\rho^2 + 18\rho + 12 \end{array} \right] \end{array} \right]. \end{aligned} \tag{4.10}$$

The procedure explained above to obtained the graphical results for the model of competition system will be used in next section.

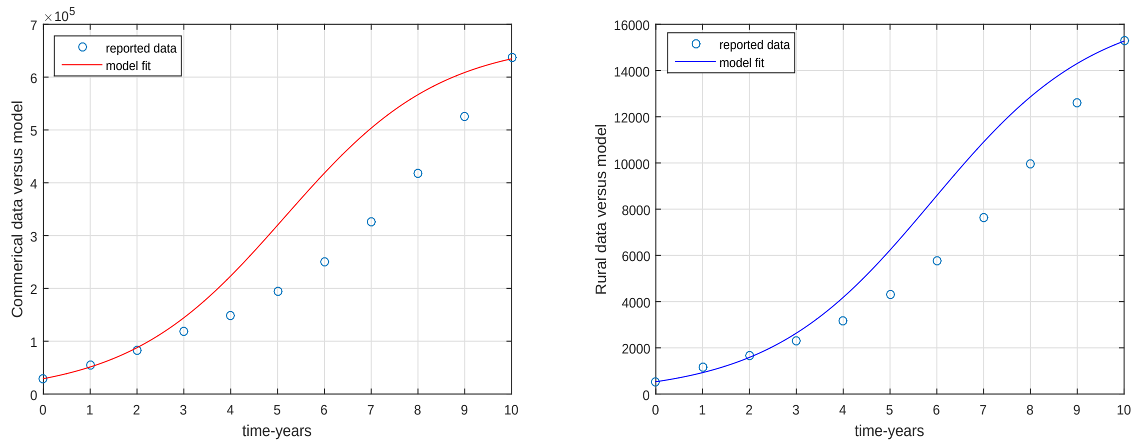
5. Numerical results

The efficient numerical algorithm described above is considered to obtain the numerical solution of the fractional model in Caputo derivative. We used the real statistical data to investigate the model parameters for the competition between the two banking systems in Indonesia for the given years 2004–2014. Using the statistical data and the parameters obtained that is considered are below: $\psi_1 = 2.90 \times 10^{-10}$, $\psi_2 = 3.9 \times 10^{-8}$, $K_1 = 669318.198$, $K_2 = 17540.6219$, $\alpha_1 = 0.6$, and $\alpha_2 = 0.58$. These parameters are also the same estimations presented in the work published in [51]. Using these parameters values, we presented the graphical results. The real data of commercial and rural banks in cumulative form is shown in Figure 1. The comparison of real data with the model is shown in Figure 2 when $\rho = 1$. Figures 3 and 4 are plotted in order to show model fitting with real data for the arbitrary order $\rho = 0.9, 0.8, 0.7$. It is observed from the results depicted in Figures 3 and 4, that decreasing the fractional-order ρ, we see the good agreement of the data versus model fitting. The future predictions of the real statistical data are depicted in Figure 5. It is observed a good fitting for a long time of the model versus real data. Further, we provided by choosing many values of the arbitrary order and show the results in Figure 6. We also utilized the approach of the FDE12 (predictor-corrector PECE method for fractional differential equations) to solve the present model and compare the result with the Newton polynomials method used in this work. We used the method given in [57, 58] to obtain the graphical results for the fractional-order model 3.2 using the FDE12 and the Newton polynomial method with the same parameter and initial conditions, with the step size $h = 0.01$. The results are shown in Figure 7. We show them in graphics legends, the Newton method by "present method" while the PECE method is by FDE12. One can observe that both the methods give the same results for the integer and fractional order value $\rho = 0.96$.



(a) (b)

Figure 1. The profit of commercial and rural banks, (a) commercial; (b) rural.



(a) (b)

Figure 2. Fitting of data versus model, $\rho = 1$. (a) commercial; (b) rural.

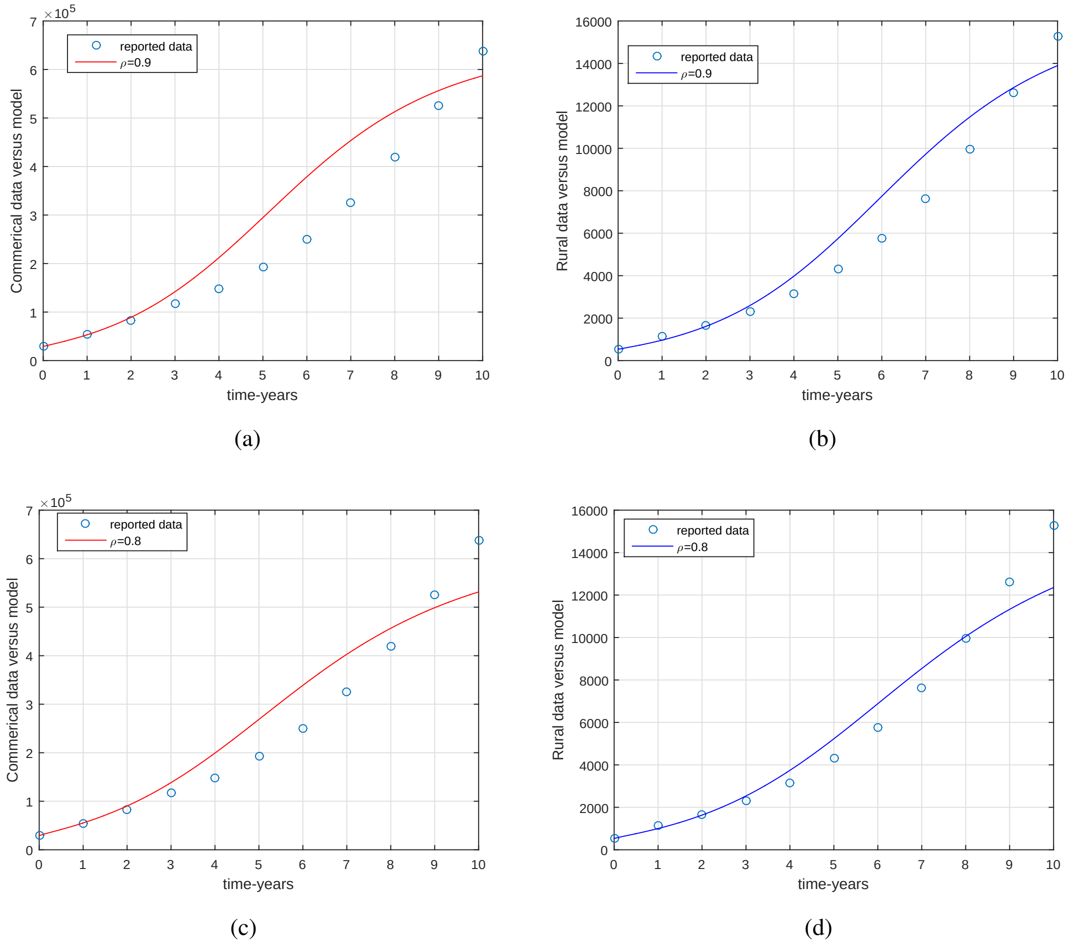


Figure 3. Comparison of data versus model fitting considering different fractional order values (a) and (c) commercial banks; (b) and (d) rural banks.

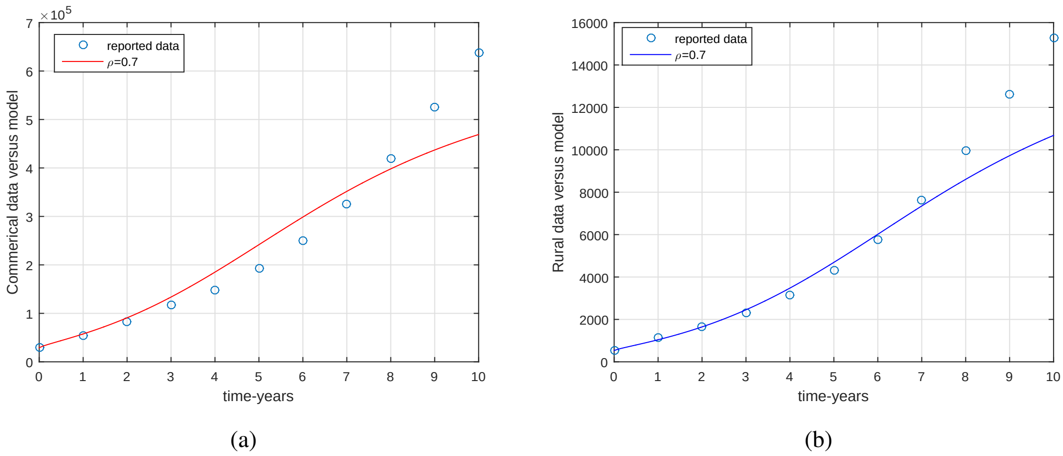


Figure 4. Comparison of data versus model fitting when $\rho = 0.7$. (a) commercial banks; (b) rural banks.

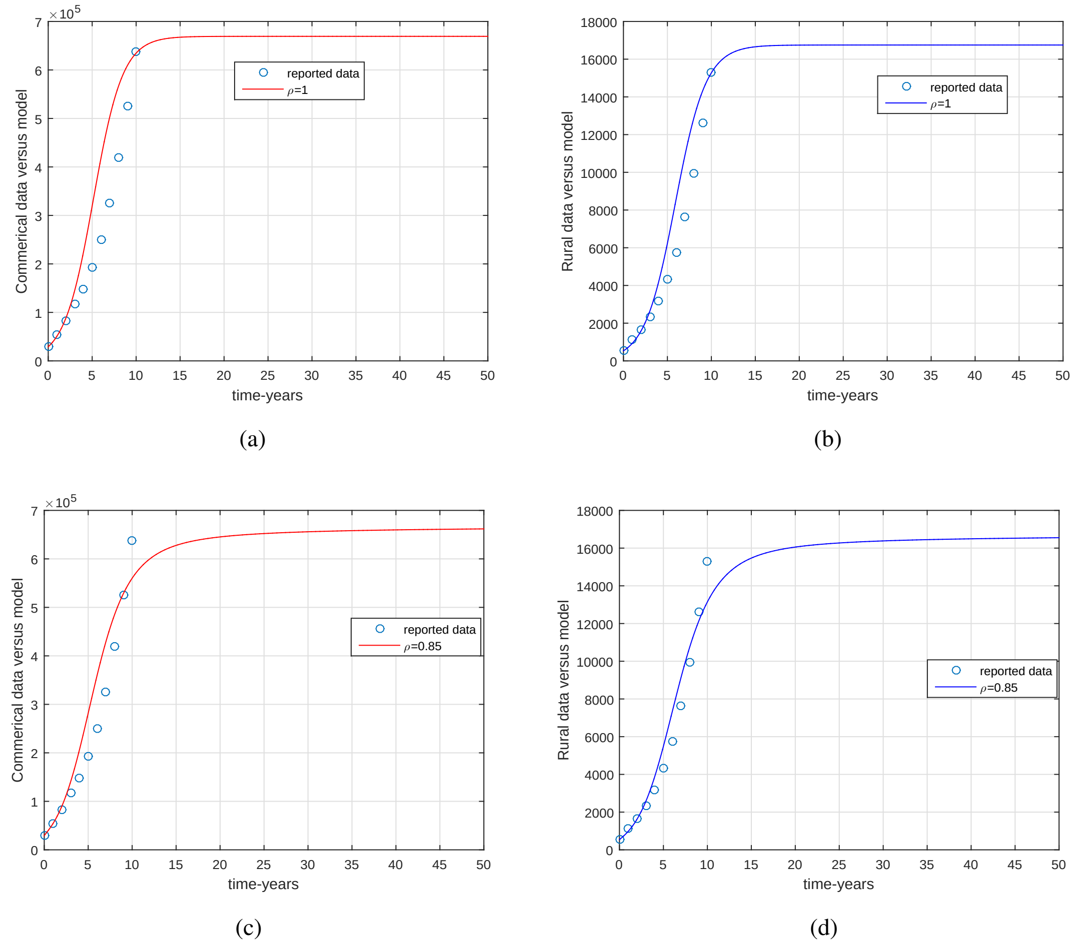


Figure 5. Comparison of data versus model fitting for long term for different value of ρ.

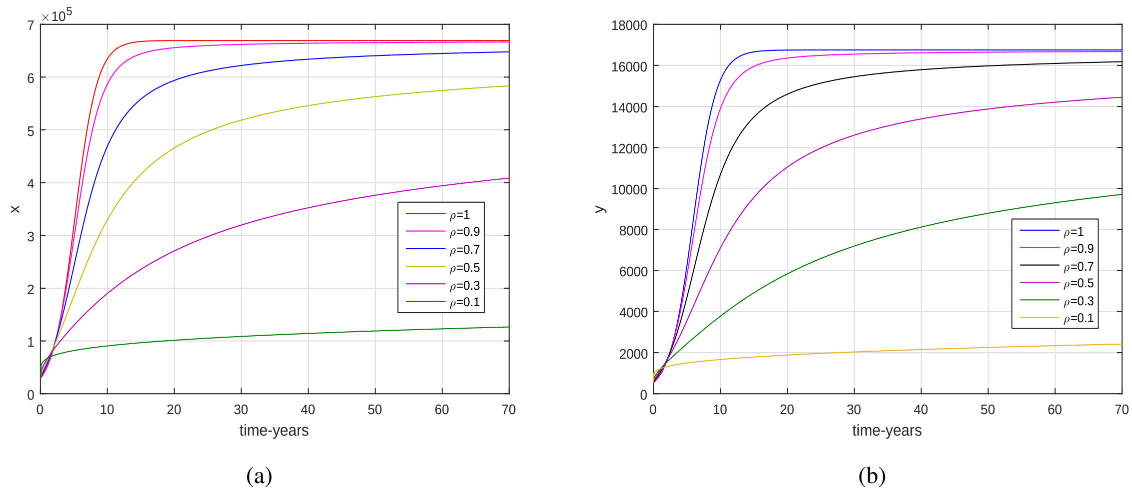


Figure 6. Dynamics of the model for different value of ρ.

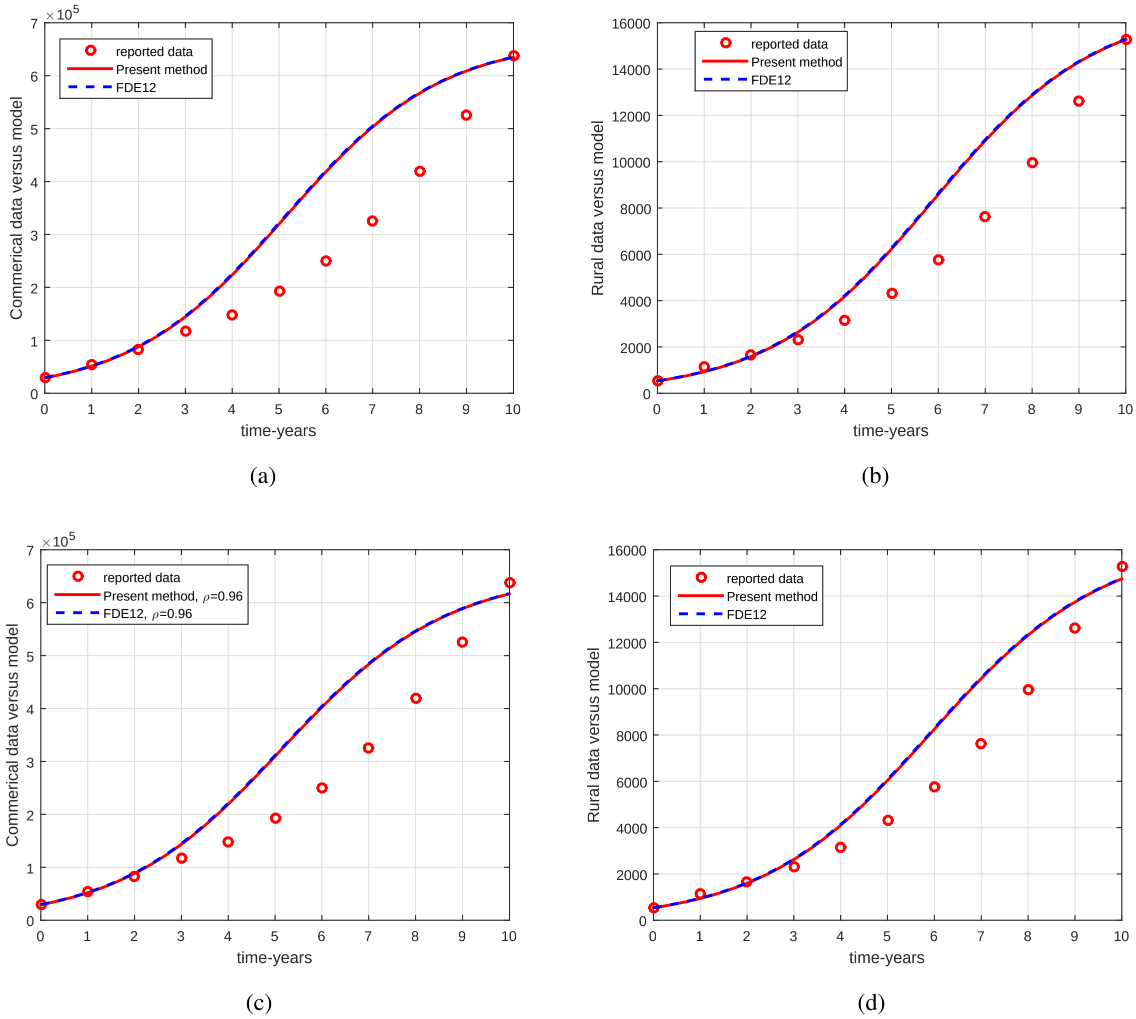


Figure 7. The comparison of the present method with FDE12. (a) Commercial data versus model; (b) rural data versus model; (c) commercial data versus model for $\rho = 0.96$; (d) rural data versus model for $\rho = 0.96$.

6. A stochastic competition model

This section studies the model in stochastic version. There are many stochastic models in literature which study different physical problems, some are listed here [52–56]. The model (3.1) is extended to the stochastic version given by

$$\begin{aligned} dx &= \Big[\alpha_1 x\Big(1 - \frac{x}{K_1}\Big) - \psi_1 xy\Big]dt + \sigma_1 x dW(t), \\ dy &= \Big[\alpha_2 y\Big(1 - \frac{y}{K_2}\Big) - \psi_2 xy\Big]dt + \sigma_2 y dW(t), \end{aligned} \tag{6.1}$$

where σ_1 and σ_2 are the real constants that represent the intensity of the stochastic differential equations, where $W(t)$ refereed to be the stochastic Brownian motion. Keeping in view the model (3.1), we use it for simulation purposes and check whether the data fit well to the stochastic model. In this regard, we keep the same numerical values as we used for the fractional case model (3.2). The simulation of the model (6.1) with data is given in Figure 8. We can see that the model (6.1) also behaves well with the real data. Figure 9 shows the model behavior for a large time level.

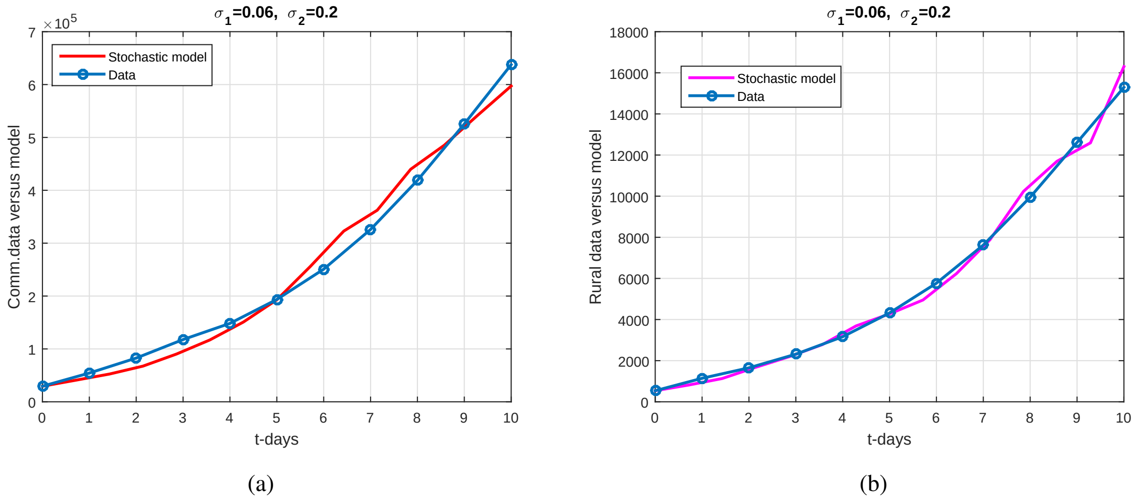


Figure 8. Simulation of the model (6.1), (a) fitting for commercial data, (b) fitting with rural data.

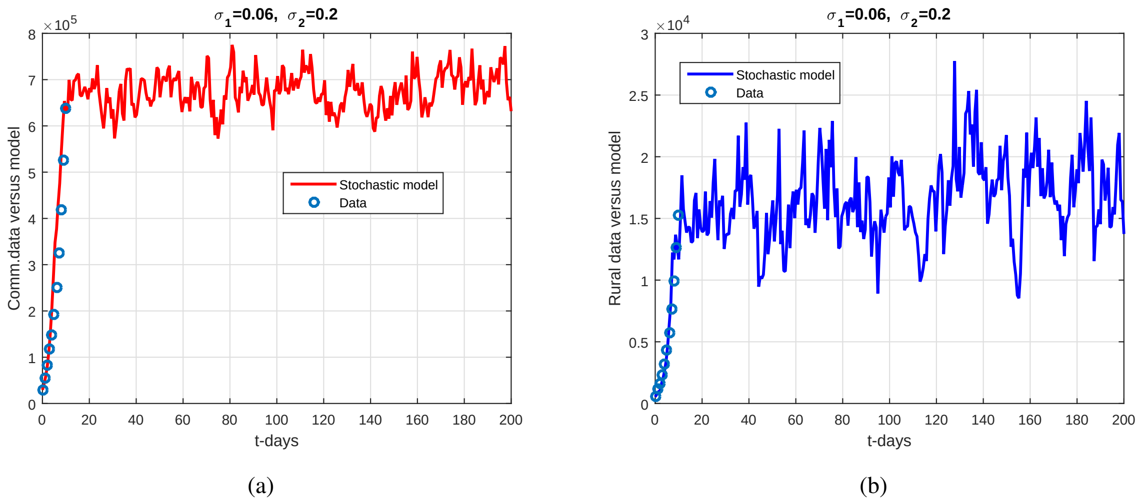


Figure 9. Prediction of the model (6.1). (a) Fitting for commercial data; (b) fitting with rural data.

7. Conclusions

We obtained some new findings regarding the competition system for the real data of the two banks in Indonesia through a fractional model with Caputo derivative through new numerical solution. Initially, we presented the model in ordinary derivative and then applied the Caputo derivative to the model for its generalization. The generalized model is then used to present a novel numerical procedure for their numerical solution. The real statistical data obtained from [50] for the year 2004–2014 are utilized and obtained various graphical results. The realistic parameters are fitted for the model of two different data sets which provide a reasonable fitting to the model. For comparison purposes, we considered also the FDE12 method and presented the results. We found that both the methods provide the same results and it can be used confidently for other physical or social problems. The suggested graphical results were tested in order to find the best fractional-order values for which the data provide good fitting is $\rho = 0.7$. While using the stochastic version of the model, we observed that the data provide reasonable fitting to the model. The stochastic version for some different values of the intensities parameters has been shown. It is obvious that the real data fitting provides useful information for the phenomenon's future prediction, where some policy and rule can be designed to obtain the future goals. Thus, we hope that these new results for the competition model for the bank data with the real statistical data will bring new information for the banking and finance sector.

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Conflict of interest

No conflict of interests exists regarding the publication of this work.

References

1. X. P. Li, Y. Wang, M. A. Khan, M. Y. Alshahrani, T. Muhammad, A dynamical study of SARS-COV-2: A study of third wave, *Res. Phys.,* **29** (2021), 1–6. https://doi.org/10.1016/j.rinp.2021.104705
2. Z. H. Shen, Y. M. Chu, M. A. Khan, S. Muhammad, O. A. AlHartomy, M. Higazy, Mathematical modeling and optimal control of the COVID-19 dynamics, *Res. Phy.,* **31** (2021), 1–9. https://doi.org/10.1016/j.rinp.2021.105028
3. X. P. Li, N. Gul, M. A. Khan, R. Bilal, A. Ali, M. Y. Alshahrani, et al., A new Hepatitis B model in light of asymptomatic carriers and vaccination study through Atangana-Baleanu derivative, *Res. Phys.,* **29** (2021), 104603. https://doi.org/10.1016/j.rinp.2021.104603
4. P. Y. Xiong, M. I. Khan, R. J. P. Gowda, R. N. Kumar, B. C. Prasannakumara, Y. M. Chu, Comparative analysis of (Zinc ferrite, Nickel Zinc ferrite) hybrid nanofluids slip flow with entropy generation, *Mod. Phys. Lett. B,* **35** (2021), 1–22.
5. P. Y. Xiong, A. Hamid, Y. M. Chu, M. I. Khan, R. J. P. Gowda, R. N. Kumar, et al., Dynamics of multiple solutions of Darcy-Forchheimer saturated flow of Cross nanofluid by a vertical thin needle point, *Eur. Phys. J. Plus,* **136** (2021), 1–22. https://doi.org/10.1140/epjp/s13360-021-01294-2
6. Laws of the republic indonesia number 10 year 1998 about amendment to law number 7 of 1992 concerning banking. Available from: `https://www.global-regulation.com/translation/indonesia/7224941/act-no.-10-of-1998.html`.

7. S. Arbi, *Lembaga perbankan keuangan dan pembiayaan*, Yogyakarta: BPFE, 2013.
8. S. Iskandar, *Bank dan lembaga keuangan lainnya,* Jakarta: Penerbit, 2013.
9. OJK, Statistik Perbankan Indonesia, 2004–2014. Available from: `http://www.ojk.go.id/datastatistikperbankan-indonesia`.
10. A. Hastings, *Population biology: Concepts and models*, Springer Science & Business Media, 2013.
11. J. Kim, D. J. Lee, J. Ahn, A dynamic competition analysis on the korean mobile phone market using competitive diffusion model, *Comput. Ind. Eng.,* **51** (2006), 174–182. https://doi.org/10.1016/j.cie.2006.07.009
12. S. A. Morris, D. Pratt, Analysis of the lotka-volterra competition equations as a technological substitution model, *Technol. Forecast. Soc.,* **70** (2003), 103–133. https://doi.org/10.1016/S0040-1625(01)00185-8
13. S. J. Lee, D. J. Lee, H. S. Oh, Technological forecasting at the korean stock market: A dynamic competition analysis using lotka-volterra model, *Technol. Forecast. Soc.,* **72** (2005), 1044–1057. https://doi.org/10.1016/j.techfore.2002.11.001
14. C. Michalakelis, C. Christodoulos, D. Varoutas, T. Sphicopoulos, Dynamic estimation of markets exhibiting a prey-predator behavior, *Expert Syst. Appl.,* **39** (2012), 7690–7700. https://doi.org/10.1016/j.eswa.2012.01.049
15. S. Lakka, C. Michalakelis, D. Varoutas, D. Martakos, Competitive dynamics in the operating systems market: Modeling and policy implications, *Technol. Forecast. Soc.,* **80** (2013), 88–105. https://doi.org/10.1016/j.techfore.2012.06.011
16. C. A. Comes, Banking system: Three level lotka-volterra model, *Proc. Econ. Financ.,* **3** (2012), 251–255. https://doi.org/10.1016/S2212-5671(12)00148-7
17. Fatmawati, M. A. Khan, M. Azizah, Windarto, S. Ullah, A fractional model for the dynamics of competition between commercial and rural banks in Indonesia, *Chaos Soliton. Fract.,* **122** (2019), 32–46. https://doi.org/10.1016/j.chaos.2019.02.009
18. W. Wang, M. A. Khan, Fatmawati, P. Kumam, P. Thounthong, A comparison study of bank data in fractional calculus, *Chaos Soliton. Fract.,* **126** (2019), 369–384. https://doi.org/10.1016/j.chaos.2019.07.025
19. Z. F. Li, Z. Liu, M. A. Khan, Fractional investigation of bank data with fractal-fractional Caputo derivative, *Chaos Soliton. Fract.,* **131** (2020), 109528. https://doi.org/10.1016/j.chaos.2019.109528
20. W. Wang, M. A. Khan, Analysis and numerical simulation of fractional model of bank data with fractal-fractional Atangana-Baleanu derivative, *J. Comput. Appl. Math.,* **369** (2020), 112646. https://doi.org/10.1016/j.cam.2019.112646
21. M. A. Khan, M. Azizah, S. Ullah, A fractional model for the dynamics of competition between commercial and rural banks in indonesia, *Chaos Soliton. Fract.,* **122** (2019), 32–46. https://doi.org/10.1016/j.chaos.2019.02.009
22. S. Ullah, M. A. Khan, M. Farooq, A fractional model for the dynamics of tb virus, *Chaos Soliton. Fract.,* **116** (2018), 63–71. https://doi.org/10.1016/j.chaos.2018.09.001
23. M. A. Khan, Fatmawati, A. Atangana, E. Alzahrani, The dynamics of COVID-19 with quarantined and isolation, *Adv. Differ. Equ.,* **2020** (2020), 425. https://doi.org/10.1186/s13662-020-02882-9

24. Fatmawati, M. A. Khan, H. P. Odinsyah, Fractional model of HIV transmission with awareness effect, *Chaos Soliton. Fract.,* **138** (2020), 109967. https://doi.org/10.1016/j.chaos.2020.109967

25. A. Atangana, S. I. Araz, Atangana-Seda numerical scheme for Labyrinth attractor with new differential and integral operators, *Fractals,* 2020. https://doi.org/10.1142/S0218348X20400447

26. P. I, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications,* Academic Press, USA, 1998.

27. S. Das, P. Gupta, A mathematical model on fractional lotka-volterra equations, *J. Theor. Biol.,* **277** (2011), 1–6. https://doi.org/10.1016/j.jtbi.2011.01.034

28. M. A. Khan, Z. Hammouch, D. Baleanu, Modeling the dynamics of hepatitis E via the Caputo-Fabrizio derivative, *Math. Model. Nat. Pheno.,* **14** (2019), 311. https://doi.org/10.1051/mmnp/2018074

29. M. A. Khan, S. Ullah, M. Farooq, A new fractional model for tuberculosis with relapse via atangana-baleanu derivative, *Chaos Soliton. Fract.,* **116** (2018), 227–238. https://doi.org/10.1016/j.chaos.2018.09.039

30. F. Fatmawati, E. Shaiful, M. Utoyo, A fractional-order model for hiv dynamics in a two-sex population, *Int. J. Math. Math. Sci.,* **2018** (2018). https://doi.org/10.1155/2018/6801475

31. A. Atangana, J. J. Nieto, Numerical solution for the model of rlc circuit via the fractional derivative without singular kernel, *Adv. Mech. Eng.,* **7** (2015). https://doi.org/10.1177/1687814015613758

32. Y. Che, M. Y. A. Keir, Study on the training model of football movement trajectory drop point based on fractional differential equation, *Appl. Math. Nonlinear Sci.,* 2021, 1–6. https://doi.org/10.2478/amns.2021.2.00095

33. S. Man, R. Yang, Educational reform informatisation based on fractional differential equation, *Appl. Math. Nonlinear Sci.,* 2021, 1–10. https://doi.org/10.2478/amns.2021.2.00079

34. J. Gao, F. S. Alotaibi, R. T. Ismail, The model of sugar metabolism and exercise energy expenditure based on fractional linear regression equation, *Appl. Math. Nonlinear Sci.,* 2021, 1–9. https://doi.org/10.2478/amns.2021.2.00026

35. N. Zhao, F. Yao, A. O. Khadidos, B. M. Muwafak, The impact of financial repression on manufacturing upgrade based on fractional Fourier transform and probability, *Appl. Math. Nonlinear Sci.,* 2021, 1–11. https://doi.org/10.2478/amns.2021.2.00060

36. C. Li, N. Alhebaishi, M. A. Alhamami, Calculating university education model based on finite element fractional differential equations and macro-control analysis, *Appl. Math. Nonlinear Sci.,* 2021, 1–10. https://doi.org/10.2478/amns.2021.2.00069

37. A. Zeb, E. Alzahrani, V. S. Erturk, G. Zaman, Mathematical model for coronavirus disease 2019 (COVID-19) containing isolation class, *Biomed Res. Int.,* **2020** (2020). https://doi.org/10.1155/2020/3452402

38. Z. Zhang, A. Zeb, S. Hussain, E. Alzahrani, Dynamics of COVID-19 mathematical model with stochastic perturbation, *Adv. Differ. Equ.,* **2020** (2020), 1–2. https://doi.org/10.1186/s13662-020-02909-1

39. S. Bushnaq, T. Saeed, D. F. Torres, A. Zeb, Control of COVID-19 dynamics through a fractional-order model, *Alex. Eng. J.,* **60** (2021), 3587–3592. https://doi.org/10.1016/j.aej.2021.02.022
40. Z. Zhang, A. Zeb, E. Alzahrani, S. Iqbal, Crowding effects on the dynamics of COVID-19 mathematical model, *Adv. Differ. Equ.,* **2020** (2020), 1–3. https://doi.org/10.1186/s13662-020-03137-3
41. A. Din, Y. Li, A. Yousaf, A. I. Ali, Caputo type fractional operator applied to Hepatitis B system, *Fractals*, 2021. https://doi.org/10.1142/S0218348X22400230
42. A. Din, Y. Li, F. M. Khan, Z. U. Khan, P. Liu, On Analysis of fractional order mathematical model of Hepatitis B using Atangana-Baleanu Caputo (ABC) derivative, *Fractals*, 2021. https://doi.org/10.1142/S0218348X22400175
43. A. Din, A. Khan, A. Zeb, M. R. Sidi Ammi, M. Tilioua, D. F. Torres, Hybrid method for simulation of a fractional COVID-19 model with real case application, *Axioms,* **10** (2021), 290. https://doi.org/10.3390/axioms10040290
44. C. Xu, M. Liao, P. Li, S. Yuan, Impact of leakage delay on bifurcation in fractional-order complex-valued neural networks, *Chaos Soliton. Fract.,* **142** (2021), 110535. https://doi.org/10.1016/j.chaos.2020.110535
45. C. Xu, Z. Liu, L. Yao, C. Aouiti, Further exploration on bifurcation of fractional-order six-neuron bi-directional associative memory neural networks with multi-delays, *Appl. Math. Comput.,* **410** (2021), 126458. https://doi.org/10.1016/j.amc.2021.126458
46. C. Xu, M. Liao, P. Li, Y. Guo, Q. Xiao, S. Yuan, Influence of multiple time delays on bifurcation of fractional-order neural networks, *Appl. Math. Comput.,* **361** (2019), 565–582. https://doi.org/10.1016/j.amc.2019.05.057
47. C. Xu, M. Liao, P. Li, Y. Guo, Z. Liu, Bifurcation properties for fractional order delayed BAM neural networks, *Cogn. Comput.,* **13** (2021), 322–356. https://doi.org/10.1007/s12559-020-09782-w
48. C. J. Xu, W. Zhang, C. Aouiti, Z. X. Liu, M. X. Liao, P. L. Li, Further investigation on bifurcation and their control of fractional-order BAM neural networks involving four neurons and multiple delays, *Math. Meth. Appl. Sci.,* 2021, 75.
49. C. Xu, W. Zhang, Z. Liu, L. Yao, Delay-induced periodic oscillation for fractional-order neural networks with mixed delays, *Neurocomputing*, 2021. https://doi.org/10.1016/j.neucom.2021.11.079
50. OJK, Statistik Perbankan Indonesia 2004–2014.
51. A. Atangana, M. A. Khan, Modeling and analysis of competition model of bank data with fractal-fractional Caputo-Fabrizio operator, *Alex. Eng. J.,* **159** (2020), 1985–1998. https://doi.org/10.1016/j.aej.2019.12.032
52. A. Din, Y. Li, T. Khan, G. Zaman, Mathematical analysis of spread and control of the novel corona virus (COVID-19) in China, *Chaos Soliton Fract.,* **141** (2020). https://doi.org/10.1016/j.chaos.2020.110286
53. A. Din, Y. Li, M. A. Shah, The complex dynamics of hepatitis B infected individuals with optimal control, *J. Syst. Sci. Comput.,* **34** (2021), 1301–1323. https://doi.org/10.1007/s11424-021-0053-0

54. A. Din, T. Khan, Y. Li, H. Tahir, A. Khan, W. A. Khan, Mathematical analysis of dengue stochastic epidemic model, *Result Phys.,* **20** (2021), 103719. https://doi.org/10.1016/j.rinp.2020.103719

55. A. Din, Y. Li, The extinction and persistence of a stochastic model of drinking alcohol, *Result Phys.,* **28** (2021), 104649. https://doi.org/10.1016/j.rinp.2021.104649

56. A. Din, Y. Li, Stochastic optimal control for norovirus transmission dynamics by contaminated food and water, *Chinese Phys. B,* 2021. https://doi.org/10.1088/1674-1056/ac2f32

57. A. Atangana, Mathematical model of survival of fractional calculus, critics and their impact: How singular is our world? *Adv. Differ. Equ.,* **2021** (2021), 1–59.

58. A. Atangana, S. I. Araz, Modeling third waves of Covid-19 spread with piecewise differential and integral operators: Turkey, Spain and Czechia, *Result Phys.*, **29** (2021), 104694. https://doi.org/10.1016/j.rinp.2021.104694

59. M. Caputo, Linear models of dissipation whose Q is almost frequency independent, part II, *Geophys. J. Int.,* **13** (1967), 529–539.

60. A. Atangana, S. I. Araz, *New numerical scheme with newton polynomial: Theory, methods, and applications*, Academic Press, 2021.

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ORIGINAL PAPER

A Generalized Self-Adaptive Algorithm for the Split Feasibility Problem in Banach Spaces

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Abstract

In this paper, we propose a generalized self-adaptive method for solving the multiple-set split feasibility problem in the framework of certain Banach spaces. Under some suitable conditions, we prove the strong convergence of the sequence generated by our method with a new way to select the step-sizes without prior knowledge of the operator norm. Several numerical experiments to illustrate the convergence behavior are presented. The results presented in this paper improve and extend the corresponding results in the literature.

Keywords Metric projection · Banach space · Strong convergence · Self-adaptive method · Multiple-set split feasibility problem

Mathematics Subject Classification 47H09 · 47H10 · 47J25 · 47J05

1 Introduction

Let E and F be two real p-uniformly convex Banach spaces which are also uniformly smooth. Let C_i, $i = 1, 2, \dots, M$ and Q_j, $j = 1, 2, \dots, N$ be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator with its adjoint $A^* : F^* \to E^*$. We consider the following so-called *multiple-set split*

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feasibility problem (MSFP):

$$\text{find } x^* \in \bigcap_{i=1}^{M} C_i \text{ such that } Ax^* \in \bigcap_{j=1}^{N} Q_j. \tag{1.1}$$

We denote by $\Omega := \left(\bigcap_{i=1}^{M} C_i\right) \cap A^{-1}\left(\bigcap_{j=1}^{N} Q_j\right)$ the solution set of Problem (1.1). This problem was first introduced in finite-dimensional Hilbert spaces by Censor et al. [10]. The MSFP has broad applicability in many areas of mathematics and the physical and engineering sciences, for example, it can be applied in fields of image reconstruction and signal processing (see [33]) and in the inverse problem of intensity-modulated radiation therapy (IMRT) in the field of medical care (see [10,13,14]). Moreover, this problem is a generalization of convex feasibility problem (CFP) and as a generalization of the split feasibility problem. In particular, if $M = N = 1$, then the MSFP becomes the following well-known *split feasibility problem* (SFP) [12]:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.2}$$

There are many modification methods have been proposed for solving the MSFP and the SFP in different styles (see for instance [6,9,16,19,29–32,34–46,50]).

A one efficient method for solving the SFP in Hilbert spaces is known as *Byrne's CQ algorithm* [9] which is defined in the following manner: for given $x_1 \in C$, compute the sequences $\{x_n\}$ generated iteratively by

$$x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n), \quad \forall n \geq 1, \tag{1.3}$$

where P_C and P_Q are the metric projections onto C and Q, respectively. It was proved that the sequence $\{x_n\}$ defined by (1.3) converges weakly to a solution of the SFP provided the step-size $\tau_n \in (0, \frac{2}{\|A\|^2})$.

Note that the choice of the step-size τ_n of above work and other corresponding results depend on the operator norm $\|A\|$. In general, the implementation of such algorithms is not an easy work in practice. As a result the implementation of the iteration process inefficient when the computation of the operator norm is not explicit. To overcome this difficulty, López et al. [21] constructed a new choice to select the following step-size so that without prior knowledge of the operator norm:

$$\tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \tag{1.4}$$

where $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ with its gradient $\nabla f(x) = A^*(I - P_Q)Ax$ and $\{\rho_n\} \subset (0, 4)$ satisfies $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$. They established the weak convergence of the Byrne's CQ algorithm (1.3) to a solution of SFP with the step-size τ_n defined by (1.4).

Let C and Q be nonempty, closed and convex subsets of E and F, respectively. Schöpfer et al. [34] first introduced the following algorithm for solving SFP in Banach spaces: for given $x_1 \in E$ and

$$x_{n+1} = \Pi_C J_q^{E^*}(J_p^E(x_n) - \tau_n A^* J_p^F(I - P_Q)Ax_n), \quad \forall n \geq 1, \tag{1.5}$$

where Π_C is the generalized projection onto C, P_Q is the metric projection onto Q. They considered more general Bregman distance functions for its solution and proved that the sequence $\{x_n\}$ generated by (1.5) converges weakly to a solution of the SFP provided the duality mappings are weak-to-weak continuous and the step-size τ_n satisfies $0 < \tau_n < \left(\frac{q}{c_q\|A\|^q}\right)^{\frac{1}{q-1}}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and c_q is the uniform smoothness coefficient of E (see [48]). Clearly, the algorithm (1.5) covers the Byrne's CQ algorithm as a special case.

To obtain the strong convergence result, Shehu [35] proposed the following algorithm for solving the SFP in p-uniformly convex Banach spaces which are also uniformly smooth: for given $u, x_1 \in E$ and

$$x_{n+1} = \Pi_C J_q^{E^*}(\alpha_n J_p^E(u) + (1-\alpha_n)(J_p^E(x_n) - \tau_n A^* J_p^F(I - P_Q)Ax_n)), \quad \forall n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and the step-size τ_n satisfies $0 < a \leq \tau_n \leq b < \left(\frac{q}{\kappa_q\|A\|^q}\right)^{\frac{1}{q-1}}$ for some $a, b > 0$. He proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a solution of the SFP under some mild conditions.

Very recently, Alsulami and Takahashi [6] introduced an algorithm for solving the SFP between Hilbert space and strictly convex, reflexive and smooth Banach space. To be more precise, they obtained the following result.

Theorem 1.1 *Let H be a Hilbert space and E be a strictly convex, reflexive and smooth Banach space. Let J_E be the duality mapping on E. Let C and Q be nonempty, closed and convex subsets of H and E, respectively. Let P_C and P_Q be the metric projections of H onto C and E onto Q, respectively. Let $A : H \to E$ be a bounded linear operator with its adjoint A^* such that $A \neq 0$. Suppose that the solution set Ω of the SFP (1.2) is nonempty. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For given $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)P_C(x_n - \tau A^* J_E(I - P_Q)Ax_n)), \quad \forall n \geq 1, \tag{1.7}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(ii) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0, 1)$*;*
(iii) $0 < \tau\|A\|^2 < 2$*, where* $\tau > 0$.

Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega u$.

There are some open questions which are posed as follows:

(1) Can we extend Theorem 1.1 for solving the MSFP in two Banach spaces?
(2) It is possible to remove the conditions $0 < \tau\|A\|^2 < 2$ and $0 < a \leq \beta_n$?

In this paper, we propose a new iterative method to answer two above open questions. We prove the strong convergence of the sequence generated by our method under some suitable conditions. Finally, we give some numerical examples to illustrate for the main result and showing its performance in finite and infinite dimensional spaces.

2 Preliminaries

Let E and E^* be real Banach spaces and the dual space of E, respectively. We write $\langle x, j\rangle$ for the value of a functional j in E^* at x in E. We shall use the notations $x_n \to x$ means that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. Let $S_E = \{x \in E : \|x\| = 1\}$ and $B_E = \{x \in E : \|x\| \leq 1\}$. The *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in B_E, \|x - y\| \geq \epsilon\right\}.$$

Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The space E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and *p-uniformly convex* if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$. The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\left\{\tfrac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in S_E\right\}.$$

The space E is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$ and called *q-uniformly smooth* if there exists a $c_q > 0$ such that $\rho_E(\tau) \leq c_q\tau^q$ for all $\tau > 0$. It is known that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space and E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [1]). Furthermore, L_p (or ℓ_p) and the Sobolev spaces are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ while Hilbert space is uniformly smooth (see [48]).

Definition 2.1 A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge function* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.2 The mapping $J_\varphi : E \to 2^{E^*}$ associated with a gauge function φ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\varphi(\|x\|),\ \|f\| = \varphi(\|x\|)\},\quad x \in E$$

is called the *duality mapping with gauge* φ, where $\langle\cdot, \cdot\rangle$ denotes the duality pairing between E and E^*.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the *normalized duality mapping*. In the case $\varphi(t) = t^{p-1}$, where $p > 1$, the duality mapping $J_\varphi = J_p$ is called the *generalized duality mapping* which is defined by

$$J_p(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}.$$

It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ and $J_p(x) = \|x\|^{p-2} J(x)$, $p > 1$. It is well-known that if E is uniformly smooth, the generalized duality mapping J_p is norm-to-norm uniformly continuous on bounded subsets of E (see [27]). Furthermore, J_p is one-to-one, single-valued and satisfies $J_p = J_q^{-1}$, where J_q is the generalized duality mapping of E^* (see [15,26] for more details).

Lemma 2.3 [48] *Let E be a q-uniformly smooth Banach space. Then there exists a constant $c_q > 0$ which is called the q-uniform smoothness coefficient of E such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_q(x)\rangle + c_q\|y\|^q,$$

for all $x, y \in E$.

Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$\|x - z\| \leq \inf_{y \in C} \|x - y\|.$$

The mapping $P_C : E \to C$ defined by $z = P_C x$ is called the *metric projection* of E onto C. It is well-known that $P_C x$ is the unique minimizer of the norm distance, which can be characterized by the variational inequality:

$$\langle y - P_C x, J_\varphi(x - P_C x)\rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

For a gauge function φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds$$

is a continuous, convex and strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi'(t) = \varphi(t)$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. Therefore, Φ has a continuous inverse function Φ^{-1}.

Let E be a real smooth Banach space. The *Bregman distance* $D_\varphi : E \times E \to \mathbb{R}^+$ [7] is defined by

$$D_\varphi(x, y) = \Phi(\|y\|) - \Phi(\|x\|) - \langle y - x, J_\varphi(x)\rangle$$

for all $x, y \in E$. We note that $D_\varphi(x, y) \geq 0$ and $D_\varphi(x, y) = 0$ if and only of $x = y$. In general, the Bregman distance is not a metric due to the fact that it is not symmetric. The Bregman distance has the following important properties:

$$D_\varphi(x, y) + D_\varphi(y, x) = \langle x - y, J_\varphi(x) - J_\varphi(y)\rangle$$

and

$$D_\varphi(x, y) + D_\varphi(y, z) - D_\varphi(x, z) = \langle x - y, J_\varphi(z) - J_\varphi(y)\rangle$$

for all $x, y, z \in E$.

In the case $\varphi(t) = t^{p-1}$, $p > 1$, we have $\Phi(t) = \int_0^t \varphi(s)ds = \frac{t^p}{p}$. So we have the distance $D_\varphi = D_p$ is called the *p-Lyapunov function* which was studied in [8] and it is given by

$$D_p(x, y) = \frac{\|x\|^p}{p} - \langle x, J_p(y)\rangle + \frac{\|y\|^p}{q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 2$, then the Bregman distance becomes the Lyapunov function $\phi : E \times E \to \mathbb{R}^+$ [2,3] defined as

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2.$$

Let E be a strictly convex, smooth and reflexive Banach space. Following [2,11], we make use of the function $V_p : E \times E^* \to \mathbb{R}^+$ which is given by

$$V_p(x, \bar{x}) = \frac{\|x\|^p}{p} - \langle x, \bar{x}\rangle + \frac{\|\bar{x}\|^q}{q}$$

for all $x \in E$ and $\bar{x} \in E^*$. Then V_p is nonnegative and V_p satisfies the following properties:

$$V_p(x, \bar{x}) = D_p(x, J_q(\bar{x})), \;\; \forall x \in E, \; \bar{x} \in E^* \tag{2.2}$$

and

$$V_p(x, \bar{x}) + \langle J_q(\bar{x}) - x, \bar{y}\rangle \le V_p(x, \bar{x} + \bar{y}), \;\; \forall x \in E, \; \bar{x}, \bar{y} \in E^*. \tag{2.3}$$

Moreover, V_p is convex in the second variable. Then for all $z \in E$,

$$D_p\Big(z, J_q\Big(\sum_{i=1}^{M} t_i J_p(x_i)\Big)\Big) \le \sum_{i=1}^{M} t_i D_p(z, x_i),$$

where $\{x_i\}_{i=1}^M \subset E$ and $\{t_i\}_{i=1}^M \subset (0, 1)$ with $\sum_{i=1}^M t_i = 1$.

The *Bregman projection*, denoted by Π_C^φ, is defined as the unique solution of the following minimization problem:

$$\Pi_C^\varphi x = \mathrm{argmin}_{y\in C} D_\varphi(x, y), \;\; x \in E.$$

It can be characterized by the variational inequality [20]:

$$\langle z - \Pi_C^\varphi x, J_\varphi(x) - J_\varphi(\Pi_C^\varphi x)\rangle \le 0, \;\; \forall z \in C.$$

Moreover, we have

$$D_\varphi(y, \Pi_C^\varphi x) + D_\varphi(\Pi_C^\varphi x, x) \le D_\varphi(y, x), \;\; \forall y \in C. \tag{2.4}$$

When $\varphi(t) = t$, we have Π_C^{φ} coincides with the generalized projection which studied in [2]. When $\varphi(t) = t^{p-1}$, where $p > 1$, we have Π_C^{φ} becomes the Bregman projection with respect to p and denoted by Π_C.

Lemma 2.4 [28] *Let E be a smooth and uniformly convex real Banach space. Suppose that $x \in E$, if $\{D_p(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.5 [25] *Let E be a smooth and uniformly convex Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then $\lim_{n\to\infty} D_p(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6 [22] *Let $\{a_n\}$ and $\{c_n\}$ be nonnegative real sequences such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad \forall n \geq 1,$$

where $\{\delta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $\frac{b_n}{\delta_n} \leq M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.7 [23] *Let $\{\Gamma_n\}$ be a nonnegative real sequence that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_k} < \Gamma_{n_k+1}$ for all $k \in \mathbb{N}$. For each $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as follows:*

$$\tau(n) = \max\{n_0 \leq k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then the following results hold:

(i) *$\tau(n) \to \infty$ as $n \to \infty$;*
(ii) *$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$.*

3 Main Result

In this section, we propose a new self-adaptive algorithm to solve the multiple-set split feasibility problem in Banach spaces E and prove a convergence theorem of the generated sequences by the proposed method. Throughout this paper, we denote by J_p^E and $J_q^{E^*}$ the duality mappings of E and its dual space, respectively, where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.1 *Let E be a p-uniformly convex and uniformly smooth Banach space and F be a reflexive, strictly convex and smooth Banach space. Let C_i, $i = 1, 2, \ldots, M$ and Q_j, $j = 1, 2, \ldots, N$ be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be an adjoint of A.*

Suppose that the solution set Ω *of the MSFP* (1.1) *is nonempty. Let* $\{u_n\}$ *be a sequence in* E *such that* $u_n \to u$*. For given* $x_1 \in E$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} v_{n,1} = J_q^{E^*}(J_p^E(x_n) - \tau_{n,1}\nabla f(x_n)), \\ v_{n,2} = J_q^{E^*}(J_p^E(v_{n,1}) - \tau_{n,2}\nabla f(v_{n,1})), \\ \quad \vdots \\ v_{n,N} = J_q^{E^*}(J_p^E(v_{n,N-1}) - \tau_{n,N}\nabla f(v_{n,N-1})), \\ y_n = J_q^{E^*}(a_{n,0}J_p^E(v_{n,N}) + \sum_{i=1}^M a_{n,i}J_p^E(\Pi_{C_i}v_{n,N})), \\ x_{n+1} = J_q^{E^*}(\beta_n J_p^E(x_n) + (1-\beta_n)(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$*,* $\{a_{n,i}\}_{i=1}^M \subset (0,1)$*,* $\{\beta_n\} \subset [0,1)$*,* $f(v_{n,j}) = \frac{1}{p}\|(I - P_{Q_{j+1}})Av_{n,j}\|^p$ *for* $j = 1, 2, \ldots, N-1$ *and* $f(x_n) = \frac{1}{p}\|(I - P_{Q_1})Ax_n\|^p$ *with the step-sizes* $\tau_{n,1}$ *and* $\tau_{n,j}$*,* $j = 1, 2, \ldots, N-1$ *are chosen self-adaptively as*

$$\tau_{n,1} = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tau_{n,j+1} = \begin{cases} \frac{\rho_n f^{p-1}(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}, & \text{if } f(v_{n,j}) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

respectively, where $\{\rho_n\} \subset \left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$*. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*
(C2) $\liminf_{n\to\infty} \rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right) > 0$*;*
(C3) $\sum_{i=0}^M a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$*;*
(C4) $\limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x^* = \Pi_\Omega u$*, where* Π_Ω *is the Bregman projection from* E *onto* Ω*.*

Proof For each $j = 1, 2, \ldots, N-1$, we note that $\nabla f(v_{n,j}) = A^* J_p^F(I - P_{Q_{j+1}})Av_{n,j}$ (see [17, Proposition 5.7]). Let $z \in \Omega$, that is, $z \in \bigcap_{i=1}^M C_i$ and $Az \in \bigcap_{j=1}^N Q_j$. Then for each $j = 1, 2, \ldots, N-1$, we have from (2.1) that

$$\begin{aligned} \|v_{n,j} - z\|\|\nabla f(v_{n,j})\| &\geq \langle v_{n,j} - z, \nabla f(v_{n,j})\rangle \\ &= \langle v_{n,j} - z, A^* J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \\ &= \langle Av_{n,j} - Az, J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \\ &\geq \langle Av_{n,j} - Az, J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \end{aligned}$$

$$
\begin{aligned}
&+\langle Az - P_{Q_{j+1}}Av_{n,j}, J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \\
&= \langle Av_{n,j} - P_{Q_{j+1}}Av_{n,j}, J_p^E(I - P_{Q_{j+1}})Av_{n,j}\rangle \\
&= \|(I - P_{Q_{j+1}})Av_{n,j}\|^p = pf(v_{n,j}).
\end{aligned} \tag{3.1}
$$

We see that $\|\nabla f(v_{n,j})\| > 0$, when $f(v_{n,j}) \neq 0$. This implies that $\|\nabla f(v_{n,j})\| \neq 0$ for each $j = 1, 2, \dots, N-1$. Hence, $\tau_{n,j+1}$ is well defined. In the same manner, we also have $\tau_{n,1}$ is well defined. For each $j = 1, 2, \dots, N-1$, it follows from Lemma 2.3 and (3.1) that

$$
\begin{aligned}
D_p(z, v_{n,j+1}) &= D_p(z, J_q^{E^*}(J_p^E(v_{n,j}) - \tau_{n,j+1}\nabla f(v_{n,j}))) \\
&= V_p(z, J_p^E(v_{n,j}) - \tau_{n,j+1}\nabla f(v_{n,j})) \\
&= \frac{\|z\|^p}{p} - \langle z, J_p^E(v_{n,j})\rangle + \tau_{n,j+1}\langle z, \nabla f(v_{n,j})\rangle \\
&\quad + \frac{1}{q}\|J_p^E(v_{n,j}) - \tau_{n,j+1}\nabla f(v_{n,j})\|^q \\
&\leq \frac{\|z\|^p}{p} - \langle z, J_p^E(v_{n,j})\rangle + \tau_{n,j+1}\langle z, \nabla f(v_{n,j})\rangle \\
&\quad + \frac{1}{q}\|J_p^E(v_{n,j})\|^q - \tau_{n,j+1}\langle v_{n,j}, \nabla f(v_{n,j})\rangle \\
&\quad + \frac{c_q\tau_{n,j+1}^q}{q}\|\nabla f(v_{n,j})\|^q \\
&= \frac{\|z\|^p}{p} - \langle z, J_p^E(v_{n,j})\rangle + \frac{1}{q}\|v_{n,j}\|^p - \tau_{n,j+1}\langle v_{n,j} - z, \nabla f(v_{n,j})\rangle \\
&\quad + \frac{c_q\tau_{n,j+1}^q}{q}\|\nabla f(v_{n,j})\|^q \\
&= D_p(z, v_{n,j}) - \tau_{n,j+1}pf(v_{n,j}) + \frac{c_q\tau_{n,j+1}^q}{q}\|\nabla f(v_{n,j})\|^q \\
&= D_p(z, v_{n,j}) - \frac{\rho_n pf^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p} + \frac{\rho_n^q c_q}{q}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p} \\
&= D_p(z, v_{n,j}) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}.
\end{aligned} \tag{3.2}
$$

In the same manner, we can see that

$$
D_p(z, v_{n,1}) \leq D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}. \tag{3.3}
$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}
&D_p(z, v_{n,N})\\
&\leq D_p(z, v_{n,N-1}) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,N-1})}{\|\nabla f(v_{n,N-1})\|^p}\\
&\vdots\\
&\leq D_p(z, v_{n,1}) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,1})}{\|\nabla f(v_{n,1})\|^p} - \dots\\
&\quad -\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,N-1})}{\|\nabla f(v_{n,N-1})\|^p}\\
&\leq D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,1})}{\|\nabla f(v_{n,1})\|^p}\\
&\quad - \dots - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(v_{n,N-1})}{\|\nabla f(v_{n,N-1})\|^p}\\
&= D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big].
\end{aligned}\tag{3.4}$$

From (2.4) and (3.4), we see that

$$\begin{aligned}
D_p(z, y_n) &= D_p(z, J_q^{E^*}(a_{n,0}J_p^E(v_{n,N}) + \sum_{i=1}^{M} a_{n,i}J_p^E(\Pi_{C_i}v_{n,N})))\\
&\leq a_{n,0}D_p(z, v_{n,N}) + \sum_{i=1}^{M} a_{n,i}D_p(z, \Pi_{C_i}v_{n,N})\\
&\leq a_{n,0}D_p(z, v_{n,N}) + \sum_{i=1}^{M} a_{n,i}D_p(z, v_{n,N}) - \sum_{i=1}^{M} a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N})\\
&= D_p(z, v_{n,N}) - \sum_{i=1}^{M} a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N})\\
&\leq D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big]\\
&\quad -\sum_{i=1}^{M} a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N}),
\end{aligned}\tag{3.5}$$

which implies by the assumption of $\{\rho_n\}$ that

$$D_p(z, y_n) \leq D_p(z, x_n).$$

Put $w_n = J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))$ for all $n \geq 1$, we have

$$\begin{aligned} D_p(z, w_n) &= D_p(z, J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))) \\ &\leq \alpha_n D_p(z, u_n) + (1-\alpha_n)D_p(z, y_n) \\ &\leq \alpha_n D_p(z, u_n) + (1-\alpha_n)D_p(z, x_n). \end{aligned}$$

It follows that

$$\begin{aligned} D_p(z, x_{n+1}) &= D_p(z, J_q^{E^*}(\beta_n J_p^E(x_n) + (1-\beta_n)J_p^E(w_n))) \\ &\leq \beta_n D_p(z, x_n) + (1-\beta_n)D_p(z, w_n) \\ &\leq \beta_n D_p(z, x_n) + (1-\beta_n)(\alpha_n D_p(z, u_n) + (1-\alpha_n)D_p(z, x_n)) \\ &= (1-(1-\beta_n)\alpha_n)D_p(z, x_n) + (1-\beta_n)\alpha_n D_p(z, u_n). \end{aligned}$$

Since $\{u_n\}$ is bounded, we also have $\{D_p(z, u_n)\}$ is bounded. By induction, we have $\{D_p(z, x_n)\}$ is bounded. Hence, by Lemma 2.6, we have $\{x_n\}$ is bounded, so are $\{v_{n,j}\}$ and $\{y_n\}$ for each $j = 1, 2, \ldots, N-1$. Let $x^* = \Pi_\Omega u$. From (2.3) and (3.5), we have

$$\begin{aligned} D_p(x^*, w_n) &= D_p(x^*, J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n))) \\ &= V_p(x^*, \alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n)) \\ &\leq V_p(x^*, \alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n) - \alpha_n(J_p^E(u_n) - J_p^E(x^*)) \\ &\quad +\alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle \\ &= V_p(x^*, \alpha_n J_p^E(x^*) + (1-\alpha_n)J_p^E(y_n)) \\ &\quad +\alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle \\ &= \alpha_n D_p(x^*, x^*) + (1-\alpha_n)D_p(x^*, y_n) \\ &\quad\quad +\alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle \\ &\leq (1-\alpha_n)\left\{D_p(x^*, x_n) - \rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right)\left[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}\right.\right. \\ &\quad\quad \left. + \sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\right] \\ &\quad\quad \left. - \sum_{i=1}^{M} a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N})\right\} + \alpha_n\langle w_n - x^*, J_p^E(u_n) - J_p^E(x^*)\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} &D_p(x^*, x_{n+1}) \\ &\quad \leq \beta_n D_p(x^*, x_n) + (1-\beta_n)D_p(x^*, w_n) \end{aligned}$$

$$\begin{aligned}
&\leq (1-(1-\beta_n)\alpha_n)D_p(x^*, x_n)\\
&\quad -(1-\alpha_n)(1-\beta_n)\rho_n\Big(p-\frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}+\sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big]\\
&\quad -(1-\alpha_n)(1-\beta_n)\sum_{i=1}^{M}a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N})\\
&\quad +\alpha_n(1-\beta_n)\langle w_n-x^*, J_p^E(u_n)-J_p^E(u)\rangle\\
&\quad +\alpha_n(1-\beta_n)\langle w_n-x^*, J_p^E(u)-J_p^E(x^*)\rangle. \qquad (3.6)
\end{aligned}$$

Put $\Gamma_n = D_p(x^*, x_n)$ for all $n \geq 1$. From (3.6), we have

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\rho_n\Big(p-\frac{\rho_n^{q-1}c_q}{q}\Big)\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}+\sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big]\\
&\quad +(1-\alpha_n)(1-\beta_n)\sum_{i=1}^{M}a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N})\\
&\leq \Gamma_n-\Gamma_{n+1}+\alpha_n(1-\beta_n)\langle w_n-x^*, J_p^E(u_n)\\
&\quad -J_p^E(u)\rangle+\alpha_n(1-\beta_n)\langle w_n-x^*, J_p^E(u)-J_p^E(x^*)\rangle. \qquad (3.7)
\end{aligned}$$

We now show that $\Gamma_n \to 0$ as $n \to \infty$ by the following two possible cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq n_0$. Then we have

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

By our assumptions, we have

$$\lim_{n\to\infty}\Big[\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}+\sum_{j=1}^{N-1}\frac{f^p(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}\Big]=0$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{M}a_{n,i}D_p(\Pi_{C_i}v_{n,N}, v_{n,N})=0.$$

Since $\{\|\nabla f(x_n)\|^p\}$ and $\{\|\nabla f(v_{n,j})\|^p\}$ for all $j = 1, 2, \ldots, N-1$ are bounded, we have

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}\|(I-P_{Q_1})Ax_n\|=0$$

and

$$\lim_{n\to\infty} f(v_{n,j}) = \lim_{n\to\infty} \|(I - P_{Q_{j+1}})Av_{n,j}\| = 0 \text{ for each } j = 1, 2, \ldots, N-1. \tag{3.8}$$

Moreover, we also have

$$\lim_{n\to\infty} D_p(\Pi_{C_i} v_{n,N}, v_{n,N}) = 0 \text{ for each } i = 1, 2, \ldots, M$$

and hence

$$\begin{aligned} D_p(y_n, v_{n,N}) &\leq a_{n,0} D_p(v_{n,N}, v_{n,N}) + \sum_{i=1}^{M} a_{n,i} D_p(\Pi_{C_i} v_{n,N}, v_{n,N}) \\ &\to 0. \end{aligned}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} \|v_{n,N} - \Pi_{C_i} v_{n,N}\| = 0 \text{ for each } i = 1, 2, \ldots, M \tag{3.9}$$

and

$$\lim_{n\to\infty} \|y_n - v_{n,N}\| = 0.$$

From (3.8), we see that

$$\begin{aligned} \|J_p^E(v_{n,j+1}) - J_p^E(v_{n,j})\| &= \tau_{n,j+1} \|\nabla f(v_{n,j})\| \\ &\leq \tau_{n,j+1} \|A^*\| \|(I - P_{Q_{j+1}})Av_{n,j}\|^{p-1} \\ &\to 0 \end{aligned}$$

for each $j = 1, 2, \ldots, N-1$. In a similar way, we can see that

$$\begin{aligned} \|J_p^E(v_{n,1}) - J_p^E(x_n)\| &= \tau_{n,1} \|\nabla f(x_n)\| \\ &\leq \tau_{n,1} \|A^*\| \|(I - P_{Q_1})Ax_n\|^{p-1} \\ &\to 0. \end{aligned}$$

Since $J_q^{E^*}$ is norm-to-norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|v_{n,j+1} - v_{n,j}\| = 0 \text{ for each } j = 1, 2, \ldots, N-1 \tag{3.10}$$

and

$$\lim_{n\to\infty} \|v_{n,1} - x_n\| = 0. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\begin{aligned}\|y_n - x_n\| &\le \|y_n - v_{n,N}\| + \|v_{n,N} - v_{n,N-1}\| + \ldots + \|v_{n,1} - x_n\| \\ &\to 0. \end{aligned} \tag{3.12}$$

It follows that

$$\begin{aligned}\|x_n - v_{n,N}\| &\le \|x_n - y_n\| + \|y_n - v_{n,N}\| \\ &\to 0. \end{aligned} \tag{3.13}$$

From (3.12), we see that

$$\begin{aligned}D_p(w_n, x_n) &\le \alpha_n D_p(u_n, x_n) + (1-\alpha_n) D_p(y_n, x_n) \\ &\to 0\end{aligned}$$

and hence

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.14}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in E$ as $k \to \infty$. Also, we have a subsequence $\{v_{n_k,N}\}$ of $\{v_{n,N}\}$ such that $v_{n_k,N} \rightharpoonup v \in E$ as $k \to \infty$.

We next show that $v \in \Omega$. From (2.1) and (3.9), we have

$$\begin{aligned}D_p(v, \Pi_{C_i} v) &\le \langle v - \Pi_{C_i} v, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &= \langle v - v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\quad + \langle v_{n_k,N} - \Pi_{C_i} v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\quad + \langle \Pi_{C_i} v_{n_k,N} - \Pi_{C_i} v, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\le \langle v - v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\quad + \langle v_{n_k,N} - \Pi_{C_i} v_{n_k,N}, J_p^E(v) - J_p^E(\Pi_{C_i} v)\rangle \\ &\to 0.\end{aligned}$$

This gives $v \in C_i$ for $i = 1, 2, \ldots, M$ and so $v \in \bigcap_{i=1}^M C_i$. Form (3.10) and (3.13), for each $j = 1, 2, \ldots, N-1$, we have

$$\begin{aligned}\|x_n - v_{n,j}\| &\le \|x_n - v_{n,N}\| + \|v_{n,N} - v_{n,N-1}\| + \ldots + \|v_{n,j+1} - v_{n,j}\| \\ &\to 0.\end{aligned}$$

Since $x_{n_k} \rightharpoonup v$, we also have $v_{n_k,j} \rightharpoonup v$ as $k \to \infty$. For each $j = 1, 2, \dots, N-1$, we note that

$$\begin{aligned}
&\|Av - P_{Q_{j+1}} Av\|^p \\
&\quad = \langle Av - P_{Q_{j+1}} Av, J_p^F (Av - P_{Q_{j+1}} Av)\rangle \\
&\quad = \langle Av - Av_{n_k,j}, J_p^F (Av - P_{Q_{j+1}} Av)\rangle \\
&\qquad + \langle Av_{n_k,j} - P_{Q_{j+1}} Av_{n_k,j}, J_p^F (Av - P_{Q_{j+1}} Av)\rangle \\
&\qquad + \langle P_{Q_{j+1}} Av_{n_k,j} - P_{Q_{j+1}} Av, J_p^F (Av - P_{Q_{j+1}} Av)\rangle \\
&\quad \le \langle Av - Av_{n_k,j}, J_p^F (Av - P_{Q_{j+1}} Av)\rangle \\
&\qquad + \langle Av_{n_k,j} - P_{Q_{j+1}} Av_{n_k,j}, J_p^F (Av - P_{Q_{j+1}} Av)\rangle. \qquad (3.15)
\end{aligned}$$

By the continuity of A, we have $Av_{n_k,j} \rightharpoonup Av$ and $Av_{n_k,j} - P_{Q_{j+1}} v_{n_k,j} \to 0$. Letting $k \to \infty$ in (3.15), we have $\|Av - P_{Q_{j+1}} Av\| = 0$ for each $j = 1, 2, \dots, N-1$. In a similar way, we can see that $\|Av - P_{Q_1} Av\| = 0$. Hence, we have $Av \in Q_j$ for $j = 1, 2, \dots, N$ and so $Av \in \bigcap_{j=1}^N Q_j$. Therefore, $v \in \Omega$.

We next show that

$$\limsup_{n\to\infty} \langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle \le 0.$$

To get this inequality, we can choose a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty} \langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle = \lim_{k\to\infty} \langle w_{n_k} - x^*, J_p^E(u) - J_p^E(x^*)\rangle.$$

Since $x_{n_k} \rightharpoonup v$ and by (3.14), we also have $w_{n_k} \rightharpoonup v$. Then we have

$$\limsup_{n\to\infty} \langle w_n - x^*, J_p^E(u) - J_p^E(x^*)\rangle = \langle v - x^*, J_p^E(u) - J_p^E(x^*)\rangle \le 0. \tag{3.16}$$

Since $u_n \to u$, it follows that $\lim_{n\to\infty} \langle w_n - x^*, J_p^E(u_n) - J_p^E(u)\rangle = 0$. This together with (3.6) and (3.16), we conclude by Lemma 2.6 that $\Gamma_n \to 0$ as $n \to \infty$. Therefore, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Then by Lemma 2.7, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Moreover, $\{\tau(n)\}$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ for all $n \ge n_0$. From (3.7), we can show that

$$\lim_{n\to\infty} \|(I - P_{Q_1})Ax_{\tau(n)}\| = 0,$$
$$\lim_{n\to\infty} \|(I - P_{Q_{j+1}})Av_{\tau(n),j}\| = 0 \ \text{ for each } \ j = 1, 2, \dots, N-1$$

and

$$\lim_{n\to\infty} \|v_{\tau(n),N} - \Pi_{C_i} v_{\tau(n),N}\| = 0 \ \text{ for each } \ i = 1, 2, \dots, M.$$

By the similar argument as in Case 1, we can show that

$$\limsup_{n\to\infty} \langle w_{\tau(n)} - x^*, J_p^E(u) - J_p^E(x^*)\rangle \le 0.$$

Also, from (3.6) and the assumptions of $\{\alpha_{\tau(n)}\}$ and $\{\beta_{\tau(n)}\}$, we have

$$\Gamma_{\tau(n)} \le \langle w_{\tau(n)} - x^*, J_p^E(u_{\tau(n)}) - J_p^E(u)\rangle + \langle w_{\tau(n)} - x^*, J_p^E(u) - J_p^E(x^*)\rangle. \tag{3.17}$$

Hence, $\limsup_{n\to\infty} \Gamma_{\tau(n)} \le 0$ and so $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$. Again from (3.6), we see that

$$\begin{aligned}\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \ &\le \ \alpha_{\tau(n)}(1 - \beta_{\tau(n)})\langle w_{\tau(n)} - x^*, J_p^E(u_{\tau(n)}) - J_p^E(u)\rangle \\ &\quad + \alpha_{\tau(n)}(1 - \beta_{\tau(n)})\langle w_{\tau(n)} - x^*, J_p^E(u) - J_p^E(x^*)\rangle \\ &\to 0.\end{aligned}$$

This together with (3.17) implies that $\lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$. Thus, we have

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_n\} \le \Gamma_{\tau(n)+1} \to 0,$$

which implies that $D_p(x^*, x_n) \to 0$. Therefore, $x_n \to x^* \in \Omega$. We thus complete the proof. $\square$

Remark 3.2 We note that Theorem 3.1 improves and extends the main results of López et al. [21] and Alsulami and Takahashi [6] in the following ways:

(i) Our result extends the result of López et al. [21] (from SFP in Hilbert spaces to MSFP in Banach spaces) and Alsulami and Takahashi [6] (from SFP between Hilbert and Banach spaces to MSFP in two Banach spaces).

(ii) The step-sizes of our method are very different from Alsulami and Takahashi [6] because they do not depend on the operator norm of the bounded linear operators, while the step-size of those work depends on the operator norm.

(iii) Our result is proved with a new assumption on the control condition $\{\beta_n\}$. However, the assumption that $\liminf_{n\to\infty} \beta_n > 0$ of our result can be removed.

Taking $\beta_n = 0$ for all $n \geq 1$, we obtain the following Halpern-type iteration process in Banach spaces immediately.

Corollary 3.3 *Let E be a p-uniformly convex and uniformly smooth Banach space and F be a reflexive, strictly convex and smooth Banach space. Let C_i, $i = 1, 2, \ldots, M$ and Q_j, $j = 1, 2, \ldots, N$ be nonempty, closed and convex subsets of E and F, respectively. Let $A : E \to F$ be a bounded linear operator and $A^* : F^* \to E^*$ be the adjoint of A. Suppose that $\Omega \neq \emptyset$. Let $\{u_n\}$ be a sequence in E such that $u_n \to u$. For given $x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} v_{n,1} = J_q^{E^*}(J_p^E(x_n) - \tau_{n,1}\nabla f(x_n)), \\ v_{n,2} = J_q^{E^*}(J_p^E(v_{n,1}) - \tau_{n,2}\nabla f(v_{n,1})), \\ \quad \vdots \\ v_{n,N} = J_q^{E^*}(J_p^E(v_{n,N-1}) - \tau_{n,N}\nabla f(v_{n,N-1})), \\ y_n = J_q^{E^*}(a_{n,0}J_p^E(v_{n,N}) + \sum_{i=1}^M a_{n,i}J_p^E(\Pi_{C_i}v_{n,N})), \\ x_{n+1} = J_q^{E^*}(\alpha_n J_p^E(u_n) + (1-\alpha_n)J_p^E(y_n)), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{a_{n,i}\}_{i=1}^M \subset (0, 1)$, $f(v_{n,j}) = \frac{1}{p}\|(I - P_{Q_{j+1}})Av_{n,j}\|^p$ for $j = 1, 2, \ldots, N-1$ and $f(x_n) = \frac{1}{p}\|(I - P_{Q_1})Ax_n\|^p$ with the step-sizes $\tau_{n,1}$ and $\tau_{n,j}$, $j = 1, 2, \ldots, N-1$ are chosen self-adaptively as

$$\tau_{n,1} = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tau_{n,j+1} = \begin{cases} \frac{\rho_n f^{p-1}(v_{n,j})}{\|\nabla f(v_{n,j})\|^p}, & \text{if } f(v_{n,j}) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

respectively, where $\{\rho_n\} \subset \left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty}\rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right) > 0$*;*

(C3) $\sum_{i=0}^M a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$.

Then $\{x_n\}$ converges strongly to $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from E onto Ω.

We consequently obtain the following result in Hilbert spaces.

Corollary 3.4 *Let H_1 and H_2 be two real Hilbert spaces. Let C_i, $i = 1, 2, \ldots, M$ and Q_j, $j = 1, 2, \ldots, N$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be* *the adjoint of A. Suppose that* $\Omega \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence in* H_1 *such that* $u_n \to u$. *For given* $x_1 \in H_1$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} v_{n,1} = x_n - \tau_{n,1} \nabla f(x_n), \\ v_{n,2} = v_{n,1} - \tau_{n,2} \nabla f(v_{n,1}), \\ \ \vdots \\ v_{n,N} = v_{n,N-1} - \tau_{n,N} \nabla f(v_{n,N-1}), \\ y_n = a_{n,0} v_{n,N} + \sum_{i=1}^{M} a_{n,i} P_{C_i} v_{n,N}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n) y_n), \quad \forall n \geq 1, \end{cases} \tag{3.18}$$

where $\{\alpha_n\} \subset (0,1)$, $\{a_{n,i}\}_{i=1}^{M} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, $f(v_{n,j}) = \frac{1}{2}\|(I - P_{Q_{j+1}})Av_{n,j}\|^2$ *for* $j = 1, 2, \ldots, N-1$ *and* $f(x_n) = \frac{1}{2}\|(I - P_{Q_1})Ax_n\|^2$ *with the step-sizes* $\tau_{n,1}$ *and* $\tau_{n,j}$, $j = 1, 2, \ldots, N-1$ *are chosen self-adaptively as*

$$\tau_{n,1} = \begin{cases} \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tau_{n,j+1} = \begin{cases} \frac{\rho_n f(v_{n,j})}{\|\nabla f(v_{n,j})\|^2}, & \text{if } f(v_{n,j}) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

respectively, where $\{\rho_n\} \subset (0,4)$. *Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$;
(C3) $\sum_{i=0}^{M} a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$;
(C4) $\limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x^* = P_\Omega u$, *where* P_Ω *is the metric projection from* H_1 *onto* Ω.

We obtain the following result for the SFP in Banach spaces.

Corollary 3.5 *Let E be a p-uniformly convex and uniformly smooth Banach space and F be a reflexive, strictly convex and smooth Banach space. Let C and Q be nonempty, closed and convex subsets of E and F, respectively. Let* $A : E \to F$ *be a bounded linear operator and* $A^* : F^* \to E^*$ *be the adjoint of A. Suppose that* $\Omega \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence in E such that* $u_n \to u$. *For given* $x_1 \in E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = \Pi_C J_q^{E^*}(J_p^E(x_n) - \tau_n \nabla f(x_n)), \\ x_{n+1} = J_q^{E^*}(\beta_n J_p^E(x_n) + (1-\beta_n)(\alpha_n J_p^E(u_n) + (1-\alpha_n) J_p^E(y_n))), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ *and* $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$ *with the step-size* τ_n *is chosen self-adaptively as*

$$\tau_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases}$$

where $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$. *Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0$;
(C3) $\limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $x^* = \Pi_\Omega u$, *where* Π_Ω *is the Bregman projection from* E *onto* Ω.

4 Numerical Examples

In this section, we give some numerical examples to support our main result.

4.1 Numerical Example in Finite Dimensional Spaces

Example 4.1 We consider MSFP (1.1) with $C_i \subset \mathbb{R}^{\mathcal{N}}$ and $Q_j \subset \mathbb{R}^{\mathcal{M}}$, which are defined by

$$C_i = \{x \in \mathbb{R}^{\mathcal{N}} : \langle a_i^C, x\rangle \leq b_i^C\},$$
$$Q_j = \{x \in \mathbb{R}^{\mathcal{M}} : \langle a_j^Q, x\rangle \leq b_j^Q\},$$

where $a_i^C \in \mathbb{R}^{\mathcal{N}}, a_j^Q \in \mathbb{R}^{\mathcal{M}}$, $b_i^C, b_j^Q \in \mathbb{R}$ for all $i = 1, 2, \ldots, M$ and all $j = 1, 2, \ldots, N$, and A is a bounded linear operator from $\mathbb{R}^{\mathcal{N}}$ into $\mathbb{R}^{\mathcal{M}}$ the elements of the representing matrix of which are randomly generated in the closed interval [5, 10]. Next, we use randomly generated values of the coordinates of a_i^C, a_j^Q in the closed interval [3, 5] and of b_i^C, b_j^Q in the closed interval [1, 10], respectively. It is clear that $\Omega := \Big(\bigcap_{i=1}^M C_i\Big) \cap A^{-1}\Big(\bigcap_{j=1}^N Q_j\Big) \neq \emptyset$ because $0 \in \Omega$.

Remark 4.2 In this example, we define the function TOL_n by

$$\mathrm{TOL}_n = \frac{1}{M}\sum_{i=1}^{M} \|x_n - P_{C_i}x_n\|^2 + \frac{1}{N}\sum_{j=1}^{N} \|Ax_n - P_{Q_j}Ax_n\|^2, \quad \forall n \geq 1.$$

We use the stopping rule $\mathrm{TOL}_n < \mathrm{err}$ to stop the iterative process. Note that if at the nth step $\mathrm{TOL}_n = 0$, then $x_n \in \Omega$, that is, x_n is a solution to this problem.

Table 1 Table of numerical results for Example 4.1

Stop condition: $\mathrm{TOL}_n < \mathrm{err}$			
err	TOL_n	n	Time (s)
10^{-5}	9.996495e − 06	251	0.907
10^{-6}	9.931760e − 07	411	1.383
10^{-7}	9.923342e − 08	541	1.803
10^{-8}	9.528326e − 09	603	1.947
10^{-9}	9.759029e − 010	625	2.099

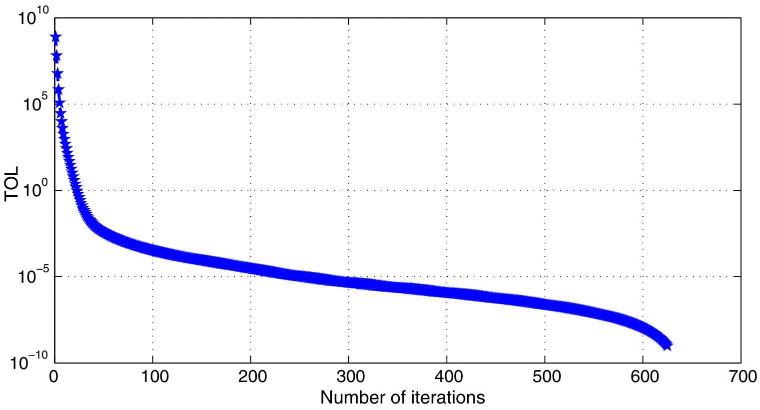

Fig. 1 The behavior of TOL_n with the stop condition $\mathrm{TOL}_n < 10^{-9}$

Applying iterative method (3.18) in Corollary 3.4 with $\mathcal{N} = 40$, $\mathcal{M} = 50$, $M = 30$, $N = 40$, $\beta_n = \frac{3}{4}$, $\alpha_n = \frac{1}{n+1}$, $\rho_n = 0.25$ and $u_n = u$ for all $n \geq 1$. Take the initial values $u, x_1 \in \mathbb{R}^{\mathcal{N}}$ where its coordinates are also randomly generated in the closed interval [10, 50], we arrive at the following table of numerical results (Table 1).

The behavior of TOL_n in the case $\mathrm{TOL}_n < 10^{-9}$ is described in Fig. 1.

4.2 Numerical Examples in Infinite Dimensional Spaces

Example 4.3 In this example, we take $E = F = L_2([0, \pi])$ with the inner product

$$\langle f, g\rangle = \int_0^\pi f(t)g(t)\mathrm{d}t$$

and the norm

$$\|f\| = \left(\int_0^\pi f^2(t)\mathrm{d}t\right)^{1/2},$$

for all $f, g \in L_2([0, \pi])$.

Now, let

$$C_i = \{x \in L_2([0, \pi]) : \langle a_i, x\rangle = b_i\},$$

where $a_i(t) = \sin(2it)$, $b_i = \dfrac{4i}{4i^2 - 1}$ for all $i = 1, 2, \ldots, M$ and $t \in [0, \pi]$,

$$Q_j = \{x \in L_2([0, \pi]) : \langle c_j, x\rangle \leq d_j\},$$

in which $c_j(t) = \exp(jt)$, $d_j = \dfrac{\exp(j\pi) - 1}{j}$ for all $j = 1, 2, \ldots, N$ and $t \in [0, \pi]$.

Let us assume that

$$A : L_2([0, \pi]) \to L_2([0, \pi]),\ (Ax)(t) = \frac{x(t)}{2}.$$

We consider the Problem (1.1) with C_i, Q_j and A are defined as the above. It is easy to check that $x(t) = \cos t + c \in \bigcap_{i=1}^{M} C_i$, with c is an arbitrary real number. Moreover, if the constant $c \in [0, 1]$, then we have

$$\int_0^\pi \exp(jt)\frac{\cos t + c}{2}\mathrm{d}t \leq \int_0^\pi \exp(jt)\mathrm{d}t = \frac{\exp(j\pi) - 1}{j},$$

for all $j = 1, 2, \ldots, N$. So, we obtain that $A(\cos t + c) \in \bigcap_{j=1}^{N} Q_j$. Thus, we arrive that

$$x(t) = \cos t + c \in \left(\bigcap_{i=1}^{M} C_i\right) \cap A^{-1}\left(\bigcap_{j=1}^{N} Q_j\right),\ \forall c \in [0, 1].$$

So, the set of the solutions of the Problem (1.1) is a nonempty set.

When $M = 50$, $N = 100$, with the same initial guess elements $x_1(t) = t^2 + 1$ and $u_n(t) = u(t) = t$ for all $n \geq 1$ and $t \in [0, \pi]$, we now consider the convergence of iterative method (3.18) with $\rho_n = 0.05$, $\beta_n = 0.25$, $\alpha_n = 1/n$, $a_{n,i} = 1/(M + 1)$ for all $n \geq 1$, $i = 0, 1, \ldots, M$, and iterative method (27) in [38, Theorem 4.1] with $\rho_n = 0.05$, $\beta_{i,n} = 1.5$, $\lambda_{j,n} = 0.5$, $\alpha_n = 1/n$ for all $n \geq 1$, $i = 1, 2, \ldots, M$, and $j = 1, 2, \ldots, N$. Note that, we define the function TOL_n as in Example 4.1 and use the stopping rule $\mathrm{TOL}_n <$ err to stop the iterative process.

The behaviors of the approximation solution $x_n(t)$ in Table 2 (with $\mathrm{TOL}_n < 10^{-3}$ and $\mathrm{TOL}_n < 10^{-4}$) are presented in Figs. 2 and 3.

Finally, we provide some connection between the MSFP and the Fredholm integral equations.

Example 4.4 Let us consider the Fredholm integral equation of the first kind as considered in [4],

$$\int_a^b K(s, t)x(t)\mathrm{d}t = g(s),\quad a \leq s \leq b, \tag{4.1}$$

Table 2 Table of numerical results for Example 4.3

Stop condition: $\mathrm{TOL}_n < \mathrm{err}$							
Iterative method (3.18)				Iterative method (27) in [38]			
err	TOL_n	n	Time(s)	err	TOL_n	n	Time (s)
10^{-3}	$9.983193e-04$	904	2.112	10^{-3}	$9.973428e-04$	1183	2.629
10^{-4}	$9.995264e-05$	2856	6.482	10^{-4}	$9.956379e-05$	3737	8.328
10^{-5}	$9.999619e-06$	9028	20.195	10^{-5}	$9.976772e-06$	11744	25.908

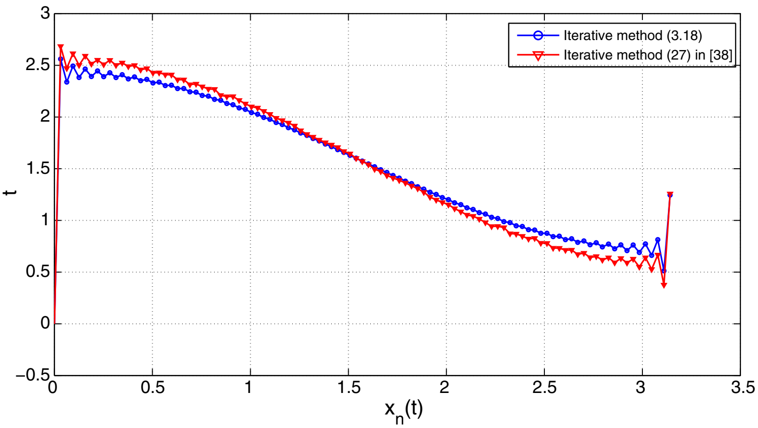

Fig. 2 The behavior of $x_n(t)$ with the stop condition $\mathrm{TOL}_n < 10^{-3}$

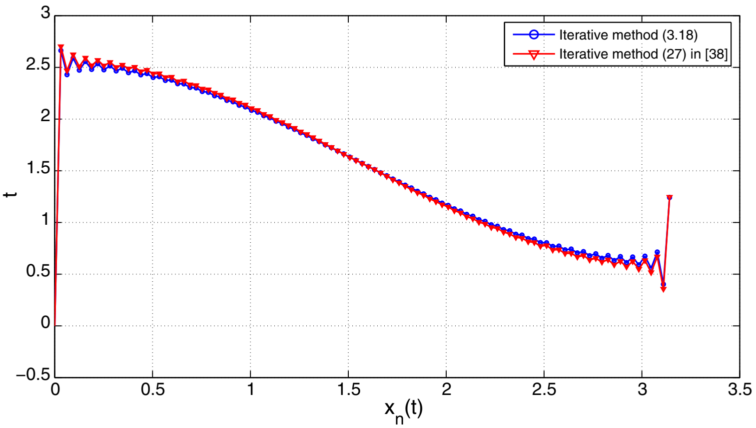

Fig. 3 The behavior of $x_n(t)$ with the stop condition $\mathrm{TOL}_n < 10^{-4}$

where $K : [a,b]^2 \to \mathbb{R}$ is the continuous kernel and $g : [a,b] \to \mathbb{R}$ is the continuous free term. Consider the computing L_p-solutions of the Problem (4.1): find $x^* \in \bigcap_{i=1}^{M} C_i$, where

Table 3 Table of numerical results for Example 4.4

Stop condition: $\text{TOL}_n = \|x_n - x^*(t)\| < \text{err}$			
err	TOL_n	n	Time (s)
10^{-3}	9.996388e − 04	2065	2.036
10^{-4}	9.999633e − 05	20722	20.108
10^{-5}	9.999961e − 06	208106	200.647

$$C_i = \{x \in L_p([a, b]) : \langle a_i, x\rangle = b_i\},$$

with $a_i(t) = K(s_i, t) \in L_q([a, b])$ and $b_i = g(s_i) \in \mathbb{R}$ for $i = 1, 2, .., M$, while $a = s_1 < s_2 < \cdots < s_M = b$ (see [18,49]). Under some hypothesis, (4.1) has solutions [24], then approximating an L_p-solution of (4.1) equivalent to solving the MSFP with $E = F = L_p([a, b])$, $A = I$ and $Q_j = L_p([a, b])$ for all $j = 1, 2, \ldots, N$.

We consider the following the Fredholm integral equations of the first kind [47, Example 2]:

$$\frac{\pi}{2}\cos s = \int_0^{\pi} \cos(t - s)x(t)\mathrm{d}t,\ 0 \le s \le \pi. \tag{4.2}$$

It follows from [47, Example 2] that the set of solutions of the Problem (4.2) is a nonempty set. Moreover, $x(t) = \cos t$ or $x(t) = \cos t + \sin(2n + 1)t$, $n = 1, 2, \ldots$ are solutions of this problem.

We now approximate the solution of the Problem (4.2) in $L_2([0, \pi])$ by solving the MSFP, that is, find $x^* \in \bigcap_{i=1}^M C_i$, where

$$C_i = \{x \in L_2([0, \pi]) : \langle a_i, x\rangle = b_i\},$$

with $a_i(t) = \cos(t - s_i)$ and $b_i = \dfrac{\pi}{2}\cos s_i$ for $i = 1, 2, .., M$, while $0 = s_1 < s_2 < \cdots < s_M = \pi$.

In this case, the sequence $\{x_n\}$ is defined by (see, iterative method (3.18) in Corollary 3.4) $x_1, u \in L_2([0, \pi])$, and

$$\begin{cases} y_n = a_{n,0}x_n + \sum_{i=1}^M a_{n,i}P_{C_i}x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)y_n),\ \forall n \ge 1. \end{cases} \tag{4.3}$$

Applying iterative method (4.3) with $a_{n,i} = 1/(M + 1)$, $\beta_n = 0.05$, $\alpha_n = 1/n$ for all $n \ge 1$ and for all $i = 0, 1, \ldots, M$. Take the initial values $x_1(t) = 1$, $u_n(t) = u(t) = \sin 3t$ for all $n \ge 1$ and $t \in [0, \pi]$, we obtain the following table of numerical results (Table 3).

Remark 4.5 Note that, in this example when $u_n(t) = u(t) = \sin 3t$ for all $n \ge 1$, then $x^*(t) = \cos t + \sin 3t$ is the projection of u onto the set of solutions of Problem (4.2).

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References

1. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Fixed Point Theory for Lipschitzian-type Mappings with Applications. Springer, New York (2009)
2. Alber, Y.I.: Generalized projection operators in Banach spaces: properties and applications. Funct. Differ Equ. Proc. Israel Seminar Ariel **1**, 1–21 (1993)
3. Alber, Y.I.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15–50. Marcel Dekker, New York (1996)
4. Alber, Y.I., Butnariu, D.: Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces. J. Optim. Theory Appl. **92**, 33–61 (1997)
5. Alber, Y.I., Ryazantseva, I.: Nonlinear Ill-posed Problems of Monotone Type. Springer, Amsterdam (2006)
6. Alsulami, S.M., Takahashi, W.: Iterative methods for the split feasibility problem in Banach spaces. J. Nonlinear Convex Anal. **16**, 585–596 (2015)
7. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Bregman monotone optimization algorithms. SIAM J. Control. Optim. **42**, 596–636 (2003)
8. Bonesky, T., Kazimierski, K.S., Maass, P., Schöpfer, F., Schuster, T.: Minimization of Tikhonov functionals in Banach spaces. Abstr. Appl. Anal. 2008 Art ID 192679, pp. 19
9. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Prob. **18**, 441–453 (2002)
10. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Prob. **21**, 2071–2084 (2005)
11. Censor, Y., Lent, A.: An iterative row-action method for interval convex programming. J. Optim. Theory Appl. **34**, 321–353 (1981)
12. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms **8**, 221–239 (1994)
13. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: The Split Feasibility Model Leading to a Unified Approach for Inversion Problems in Intensity-Modulated Radiation Therapy Technical Report, 20 April Department of Mathematics. University of Haifa, Haifa (2005)
14. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
15. Cioranescu, I.: Geometry of Banach Spaces. Duality Mappings and Nonlinear Problems. Kluwer Academic, Dordrecht (1990)
16. Combettes, P.L., Wajsm, V.R.: Signal recovery by proximal forward–backward splitting. Multiscale Model Simul. **4**, 1168–1200 (2005)
17. Ekeland, I., Temam, R.: Convex Analysis and Variational Problems. North- Holland, Amsterdam (1976)
18. Kammerer, W.J., Nashed, M.Z.: A generalization of a matrix iterative method of G. Cimmino to best approximate solutions of linear integral equations for the first kind. Rendiconti della Accademia Nazionale dei Lincei, Serie **8**(51), 20–25 (1971)
19. Kim, J.K., Tuyen, T.M.: Pareller iterative methods for solving the split common null point problem in Banach spaces. J. Nonlinear Convex Anal. **20**(10), 2075–2093 (2019)
20. Kuo, L.W., Sahu, D.R.: Bregman distance and strong convergence of proximal-type algorithms. Abstract App. Anal. Article ID 590519:12 pages (2013)
21. López, G., Martin-Márquez, V., Wang, F., Xu, H.K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Probl. **28**, 085004 (2012)
22. Maingé, P.E.: Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **325**, 469–479 (2007)
23. Maingé, P.E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899–912 (2008)

24. Mikhlin, S.G., Smolitskiy, K.L.: Approximate Methods for Solution of Differential and Integral Equations. Elsevier, New York (1967)
25. Naraghirad, E., Yao, J.C.: Bregman weak relatively non expansive mappings in Banach space. Fixed Point Theory Appl. **2013**, 141 (2013)
26. Reich, S.: Review of Geometry of Banach spaces, Duality Mappings and Nonlinear Problems by Ioana Cioranescu, Kluwer Academic Publishers, Dordrecht. Bull. Am. Math. Soc. **26**, 367–370 (1990)
27. Reich, S.: On the asymptotic behavior of nonlinear semigroups and the range of accretive operators. J. Math Anal. Appl. **79**, 113–126 (1981)
28. Reich, S., Sabach, S.: Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim. **31**(1), 22–44 (2010)
29. Reich, S., Tuyen, T.M.: A new algorithm for solving the split common null point problem in Hilbert spaces. Numer. Algorithms **83**, 789–805 (2020)
30. Reich, S., Tuyen, T.M.: Iterative methods for solving the generalized split common null point problem in Hilbert spaces. Optimization **69**(5), 1013–1038 (2020)
31. Reich, S., Tuyen, T.M., Trang, N.M.: Parallel iterative methods for solving the split common fixed point problem in Hilbert spaces. Numer. Funct. Anal. Optim. **41**(7), 778–805 (2020)
32. Reich, S., Tuyen, T.M.: Two projection methods for solving the multiple-set split common null point problem in Hilbert spaces. Optimization **69**(9), 1913–1934 (2020)
33. Stark, H.: Image Recovery: Theory and Applications, San Diego. Academic, CA (1987)
34. Schöpfer, F., Schuster, T., Louis, A.K.: An iterative regularization method for the solution of the split feasibility problem in Banach spaces. Inverse Probl. **24**, 055008 (2008)
35. Shehu, Y.: Iterative methods for split feasibility problems in certain Banach spaces. J. Nonlinear Convex Anal. **16**, 2315–2364 (2015)
36. Shehu, Y.: Strong convergence theorem for multiple sets split feasibility problems in Banach spaces. Numer. Funct. Anal. Optim. **37**(8), 1021–1036 (2016)
37. Shehu, Y., Ogbuisi, F.U., Iyiola, O.S.: Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces. Optimization **65**, 299–323 (2016)
38. Tuyen, T.M., Thuy, N.T.T., Trang, N.M.: A strong convergence theorem for a parallel iterative method for solving the split common null point problem in Hilbert spaces. J. Optim. Theory Appl. **183**(1), 271–291 (2019)
39. Tuyen, T.M.: A strong convergence theorem for the split common null point problem in Banach spaces. Appl. Math. Optim. **79**(1), 207–227 (2019)
40. Tuyen, T.M., Ha, N.S.: A strong convergence theorem for solving the split feasibility and fixed point problems in Banach spaces. J. Fixed Point Theory Appl. **20**, 140 (2018)
41. Tuyen, T.M., Ha, N.S., Thuy, N.T.T.: A shrinking projection method for solving the split common null point problem in Banach spaces. Numer. Algorithm **81**(3), 813–832 (2019)
42. Wang, F.: Polyak's gradient method for split feasibility problem constrained by level sets. Numer. Algorithms **77**, 925–938 (2018)
43. Wang, F., Xu, H.K.: Choices of variable steps of the CQ algorithm for the split feasibility problem. Fixed Point Theory **11**, 489–496 (2011)
44. Wang, F., Xu, H.K.: Cyclic algorithms for split feasibility problems in Hilbert spaces. Nonlinear Anal. **74**, 4105–4111 (2011)
45. Wang, F.: Strong convergence of two algorithms for the split feasibility problem in Banach spaces. Optimization **67**(10), 1649–1660 (2018)
46. Wang, F.: A new algorithm for solving the multiple-sets split feasibility problem in Banach spaces. Numer. Funct. Anal. Optim. **35**(1), 99–110 (2014)
47. Wazwaz, A.-M.: The regularization method for Fredholm integral equations of the first kind. Comput. Math. Appl. **61**, 2981–2986 (2011)
48. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal.: Theory Methods Appl. **16**, 1127–1138 (1991)
49. Yoshida, K.: Lectures on Differential and Integral Equations. Interscience, London (1960)
50. Zhang, W., Han, D., Li, Z.: A self-adaptive projection method for solving the multiple-sets split feasibility problem. Inverse Prob. **25**, 115001 (16) (2009)

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Analysis of non-singular fractional bioconvection and thermal memory with generalized Mittag-Leffler kernel

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ABSTRACT

This paper deals with the application of non-singular fractional operator in the bioconvection flow of a MHD viscous fluid for vertical surface. The Laplace transform method is used for dimensionless governing equations of momentum, energy and diffusion respectively. Classical governing model is extended to non-integer order approach with non-singular kernel which can be used to describe the memory for natural phenomena. The main advantage is to use this fractional operator can it measure the rate of change at all points of the considered interval, therefore, the present fractional operator incorporate the previous history/memory effects of any system. For the prediction of physical behavior of embedded parameters, some graphs are presented in the graphical section. At the end some remarkable results are found. It is found that non-singular fractional operator measures the memory better in comparison with singular fractional operator. Further, on comparison between different kinds of viscous fluid (Water, Air, Kerosene), it is found that temperature and velocity of air is higher than water and kerosene respectively. The results are validated with the recent published work.

1. Introduction

Bioconvection is a charming fact of fluid mechanics which is motivated by the spinning motion of microorganisms. The term bioconvection is characterized by hydrodynamics instability and patterns in suspensions of biased swimming microorganisms. The velocity and spatial range of fluid motions are typically considerable higher than those associated with the velocity and size of individual cells, resulting in rapid cell transfer and the formation of specialized cell concentration visualization patterns [1,2]. Biotechnology has evolved to integrate new and diverse sciences in the early 21st century introduced by Platt [3]. Bioconvection can be defined as the phenomenon of macroscopic convection motion of the fluid generated by the density gradient which was developed by directional collective swimming of microorganisms. Bioconvection can be found in wide range of applications such as biological applications and biomicrosystems, the pharmaceutical industry, biological polymer synthesis, environmentally-friendly applications, sustainable fuel cell technologies, microbial enhanced oil recovery, bio sensors, biotechnology and continuous refinements in mathematical modeling.

Recently, many researchers have presented precious works on bioconvection. Rao et al. [4] analyzed Darcy free convection with an isothermal upright cone with static apex half angle, indicating downward in a nanofluid-soaked porous medium. Khan et al. [5] examined couple stress nanofluid flow with magnetic effect and flow properties like radiation, activation energy and chemical reaction. Abdelmalek et al. [6] investigated the bioconvection movement of cross nanofluid under the magnetic dipole exposed to a cylinder numerically. Alshomrani et al. [7] studied the movement of non-Newtonian nanofluid with heat and mass transfer rates with bioconvection, activation energy and motile microorganisms numerically. Kuznetsov et al. [8] inspected bioconvection of gyrotactic motile microorganisms in a fluid saturated porous medium. Muhammad et al. [9] calculated a time-dependent movement of magnetized rheological Carreau nanofluid carrying microorganisms on a moving wedge. Mondal et al. [10] studied bioconvection movement of nanofluid holding gyrotactic microorganisms over an expanding wedge fixed in a porous medium with binary chemical reaction numerically. Furthermore readings on bioconvection can be seen in the references [11–20].

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Latterly, fractional calculus has attained much attention of scientists due to its many applications in heat transfer, robotics, biology, chemistry and genetic algorithms etc. [21]. Atangana et al. [22] established a new fractional operator with Mittag-Leffler function as a kernel. Syam et al. [23] presented analysis and applications of fractional differential equations by using (ABC) fractional operator. Ikram et al. [24] analyzed heat transfer flow of clay water based nanofluid on an exponentially moving upright plate using (ABC) fractional operator. Sheikh et al. [25] presented a comparison between (ABC) and (CF) fractional operators in casson fluid model. (ABC) fractional derivative has been used to obtain the solution of the model of groundwater within an unconfined aquifer by Alqahtani [26]. Ali et al. [27] explored the impacts of thermal radiation and MHD on the movement of Jeffrey nanoliquid through a porous medium in a rotating structure by applying (ABC) fractional derivative. Kumar et al. [28] applied (ABC) fractional operator to solve gas dynamics equation numerically. Abro et al. [29] studied heat transfer analysis of MHD Maxwell fluid over a perpendicular plate fixed in a porous medium via (ABC) fractional approach. Yavuz et al. [30] compared (ABC) and (CF) fractional operators by solving differential equations. Tlili et al. [31] presented solution of one dimensional fractional advection diffusion equation via (ABC) operator. Kumar et al. [32] examined fractional vibration equation through (ABC) fractional operator. Imran et al. [33] offered a comparison between hybrid fractional derivative and classical derivative by finding analytical solutions of heat transfer flow application. Chu et al. [34] deliberated a fractional model of viscous nanofluid containing hybrid nanoparticles with constant proportional Caputo operator.

Popolizio et al. [35] discussed the theoretical properties of Mittag-Leffler function. Gorenflo et al. [36] explained classical, two-three parametric Mittag-Leffler functions with applications and briefed historical overview. Atangana et al. [37] presented some numerical techniques for the approximation of fractional differential equations with (ABC) operator. Atangana et al. [38] formulated and investigated a new financial chaotic model with classical and (ABC) fractional operator. Atangana et al. [39] evaluated the Drude model by applying (ABC) fractional operator with different fractional orders. Atangana et al. [40] offered a fractional nonlinear mathematical model of cancer treatment and examined with system of eighteen fractional differential equations via (ABC) fractional approach. Imran et al. [41] evaluated the solution of Stoke's first problem analytically with constant proportional Caputo fractional operator. Ikram et al. [42] discussed the fractional model of Brinkman type fluid (BTF) holding hybrid nanoparticles in a bounded microchannel via constant proportional Caputo fractional operator. Aliabadi et al. [43] studied laminar flow and heat transfer features in zigzag channels for nanofluid cooled microelectronic heat sink. They calculated the impacts of different configurations of nooks and concluded that non-flat configuration improve the thermal performance of microelectronic heat sink. Aliabadi et al. [44] studied heat transfer with nanofluid flow in the helical microtube and concluded that thermal performance of the helical microtube is better than the straight microtube.

Naik et al. [45] projected and investigated non-integer order epidemic model with Caputo (C) and (ABC) fractional operators for the spread of COVID-19. Hammouch et al. [46] applied a numerical technique to study the dynamics of a fractional circulant Halvorsen system. Akgül et al. [47] evaluated different economic models based on market equilibrium and option pricing via (C), (CF) and (ABC) fractional operators. Yavuz et al. [48] studied the non-integer order Schrodinger-KdV equation through (ABC) fractional derivative. Yavuz et al. [49] calculated the existence of European-type option pricing models by classical and (ABC) fractional derivative. Veeresha et al. [50] evaluated mathematical El Nino-Southern oscillation (ENSO) fractional model through Adams-Bashforth numerical scheme. Yavuz et al. [51] presented the analysis of behaviours of normalized sinc function (NSF) and Mittag-Leffler function (MLF) with perturbation technique. Yavuz et al. [52] used Laplace homotopy analysis method (LHAM) with (CF) fractional operator to solve the fractional Black-Scholes equations (FBSEs) with the initial conditions.

Baishya et al. [53] studied the dynamics of a fractional epidemiological model with disease infection in both the populations. They noted that the non-integral order derivative had a balancing effect and it contributed in managing the co-existence between susceptible prey, infected prey, susceptible predator, and infected predator populations. Yao et al. [54] found the iterative solution for generalized quintic complex Ginzburg-Landau (GCGL) equation by applying fractional natural decomposition method (FNDM) with (C) fractional operator. Prakasha et al. [55] examined the fractional Swift-Hohenberg (S-H) equation through residual power series method (RPSM) and (C) fractional technique. Baishya et al. [56] defined a new numerical method to solve fractional differential equations via (ABC) derivative and Laguerre polynomial. Veeresha et al. [57] examined the complex behavior of the special case of Schrödinger equation known as Gross-Pitaevskii (GP) equations using q-homotopy analysis transform method (q-HATM) with non-integer order. Veeresha et al. [58] evaluated the solution for joined equations relating the projectile motion with wind-influence using q-homotopy analysis transform method (q-HATM). They used (CF) fractional derivative and Laplace transform technique. Ilhan et al. [59] obtained the series solution for the system of fractional differential equations relating to the atmospheric dynamics of carbon dioxide (CO_2) gas using the q-homotopy analysis transform method (q-HATM). They analyzed the dynamics of human population and forest biomass in the atmosphere to the concentration of CO_2 gas by (CF) fractional approach. Ishak et al. [60] investigated the steady boundary layer flow and heat transfer of a micropolar fluid on an isothermal continuously moving plane surface numerically by Keller-box technique. The following references can be seen for more study on fractional calculus [61–72].

In the above literature, researchers have addressed the effect on bioconvection for different motions and fluids. They chose the classical models of bioconvection. In order to see the memory of those models, we need fractional derivative of non-singular order. Scientists ignored the bioconvection effect in the fractional sense. Most recently, Imran et al. [73] discussed the first time fractional bioconvection effects for viscous fluid over an infinite vertical surface with power law fractional derivative. Since the power law kernel exhibits the weak memory effect of natural phenomena so for strong memory non-singular fractional operator contains Mittag-Leffler kernel used to explain the memory of flow properties of a viscous fluid. Therefore, our main target to combine these two branches in order to cover the gape which is still not reported in the existing literature. For this purpose, we generalized the ordinary model with fractional derivative and found analytical solutions of fractional energy equation, fractional bioconvection and momentum equation in series form and M-function which clearly satisfy the boundary conditions from the mathematical point of view and through graphical illustration.

2. Preliminaries

In this section, some basic definitions and their Laplace transform are discussed.

2.1. Bioconvection

Pattern forming convective motions made by suspensions of swimming micro-organisms is called bioconvection [74].

2.2. Caputo time fractional derivative

The Caputo time fractional derivative of order $\alpha \in [0,1)$ is defined as [75]

$$^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(x)(t-x)^{-\alpha}dx.$$

The Laplace transform of Caputo time fractional derivative is

$$L\left[{}^{C}D_t^{\alpha}f(t)\right] = s^{\alpha}L[f(t)] - s^{\alpha-1}f(0).$$

2.3. Caputo-Fabrizio time fractional derivative

The Caputo-Fabrizio time fractional derivative of order $\alpha \in [0,1)$ is defined as [75]

$${}^{CF}D_t^{\alpha}f(t) = \frac{1}{1-\alpha}\int_0^t f'(x)\, exp\left(-\frac{\alpha(t-x)}{1-\alpha}\right)dx.$$

The Laplace transform of Caputo-Fabrizio time derivative is

$$L\left[{}^{CF}D_t^{\alpha}f(t)\right] = \frac{sL[f(t)] - f(0)}{(1-\alpha)s+\alpha}.$$

2.4. Atangana-Baleanu time fractional derivative

The Atangana-Baleanu time fractional derivative of order $\alpha \in [0,1)$ is defined as [22]

$${}^{ABC}D_t^{\alpha}f(t) = \frac{1}{1-\alpha}\int_0^t f'(x)E_{\alpha}\left[\frac{-\alpha(t-x)^{\alpha}}{1-\alpha}\right]dx.$$

The Laplace transform of Atangana-Baleanu time derivative is

$$L\left[{}^{ABC}D_t^{\alpha}f(t)\right] = \frac{s^{\alpha}L[f(t)] - f(0)}{(1-\alpha)s^{\alpha}+\alpha}.$$

2.5. Mittag-Leffler function

2.5.1. One parameter Mittag-Leffler function

Mittag-Leffler function with one parameter is defined as [76]

$$E_{\alpha}(x) = \sum_{i=0}^{\infty}\frac{x^i}{\Gamma(\alpha i+1)}, \qquad \alpha>0.$$

2.5.2. Two parameter Mittag-Leffler function

Mittag-Leffler function with two parameters is defined as [76]

$$E_{\alpha,\beta}(x) = \sum_{i=0}^{\infty}\frac{x^i}{\Gamma(\alpha i+\beta)}, \qquad \alpha>0, \beta>0.$$

2.5.3. Three parameter Mittag-Leffler function

Mittag-Leffler function with three parameters is defined as [76]

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{i=0}^{\infty}\frac{(\gamma)_i x^i}{\Gamma(\alpha i+\beta)i!}. \qquad \alpha>0, \beta>0, \gamma>0.$$

2.5.4. M-function

Generalized Mittag-Leffler function in special form called M-function is defined as [76]

$$t^{\epsilon_q - 1}\sum_{\epsilon=0}^{\infty}\frac{(G)^{\epsilon}\Pi_{i=1}^{j}\Gamma(a_i + A_i\epsilon)}{\epsilon!\Pi_{i=1}^{k}\Gamma(b_i + B_i\epsilon)} = M_q^p\left[G\Big|_{(b_1,B_1),(b_2,B_2),\ldots\ldots,(b_k,B_k)}^{(a_1,A_1),(a_2,A_2),\ldots\ldots,(a_j,A_j)}\right].$$

3. Mathematical formulation

Let us consider an unsteady heat transfer flow of a viscous fluid over a flat surface in xy-coordinate system situated at $y = 0$. In the beginning, at $t = 0$, the plate and the fluid are at rest and reference surface temperature T_{∞} and reference concentration of microorganisms N_{∞}. After passing some times, the plate begins to move at a constant velocity and the surface temperature T_w and the concentration of microorganisms of the plate N_w raised. Since the plate is infinite length, every physical quantity is the function of y and t only as shown in Fig. 1.

The momentum equation can be seen in [77–80]

$$\rho u_t(y,t) = \Psi_y(y,t) + g[\rho\beta_T(T-T_{\infty}) - \gamma(\rho_m-\rho)(N-N_{\infty})] - \sigma B_0^2 u(y,t), \tag{1}$$

Ψ is shears tress and its relation for viscous fluid with generalized form [81–83]

$$\Psi(y,t) = \mu^{ABC}D_t^{\alpha}u_y(y,t). \tag{2}$$

The heat flux equation

$$\rho C_p T_t(y,t) = -q_y(y,t). \tag{3}$$

The generalized Fourier's law for heat flux by using the idea of [81–83]

$$q(y,t) = -k^{ABC}D_t^{\beta}T_y(y,t). \tag{4}$$

The diffusion equation for bioconvection [77–80]

$$N_t(y,t) = -J_y(y,t). \tag{5}$$

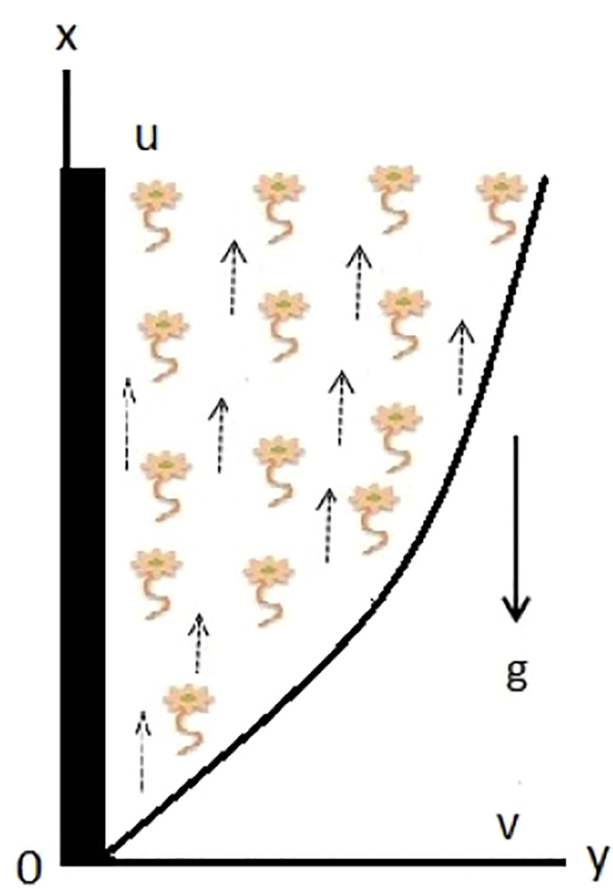

Fig. 1. Geometry of the problem.

The generalized Fick's law for diffusion is [81–83]

$$J(y,t) = -D_n{}^{ABC}D_t^{\gamma}N_y(y,t). \tag{6}$$

subject to the initial and boundary conditions

$$u(y,0)=0, \quad T(y,0)=T_\infty, \quad N(y,0)=N_\infty, \quad \forall y \ge 0, \tag{7}$$

$$u(0,t)=u_0H(t), \quad T(0,t)=T_w, \quad N(0,t)=N_w, \quad t>0, \tag{8}$$

$$u(y,t)\to 0, \quad T(y,t)\to T_\infty, \quad N(y,t)\to N_\infty, \quad y\to\infty, \quad t>0. \tag{9}$$

Introducing the non-dimensional variables

$$y^* = \frac{u_0 y}{\nu}, u^* = \frac{u}{u_0}, t^* = \frac{tu_0^2}{\nu}, \theta = \frac{T-T_\infty}{T_w-T_\infty}, N^* = \frac{N-N_\infty}{N_w-N_\infty}, q^* = \frac{q}{q_0}, J^* = \frac{J}{J_0}. \tag{10}$$

into Eqs. (1)–(9) and ignoring the (∗) notation, the dimensionless model is obtained as follows

$$u_t(y,t) = \eta_0\Psi_y(y,t) + Gr[\theta(yt) - RaN(\mathrm{yt})] - Mu(\mathrm{y,t}), \tag{11}$$

$$\Psi(y,t) = \eta_1{}^{ABC}D_t^{\alpha}u_y(y,t). \tag{12}$$

The heat equation is

$$\theta_t(y,t) = -\eta_2 q_y(y,t), \tag{13}$$

The thermal flux equation of heat conduction by Fourier's law

$$q(y,t) = -\eta_3{}^{ABC}D_t^{\beta}\theta_y(y,t), \quad 0<\beta\le 1, \tag{14}$$

The bioconvection equation is

$$N_t(y,t) = -\eta_4 J_y(y,t). \tag{15}$$

The diffusion equation is

$$J(y,t) = -\eta_5{}^{ABC}D_t^{\gamma}N_y(y,t), 0<\gamma\le 1, \tag{16}$$

subject to the conditions

$$u(y,0)=0, \quad \theta(y,0)=0, \quad N(y,0)=0, \quad y\ge 0, \tag{17}$$

$$u(0,t)=H(t), \quad \theta(0,t)=1, \quad N(0,t)=1, \quad t>0, \tag{18}$$

$$u(\infty,t)\to 0, \quad \theta(\infty,t)\to 0, \quad N(\infty,t)\to 0, \quad t>0. \tag{19}$$

where

$$\eta_0 = \frac{\nu}{\rho v_0^2}, \ \eta_1 = \frac{\rho v_0^2}{\nu}, \ \eta_2 = \frac{q_0}{\rho C_p u_0 (T_w - T_\infty)}, \ \eta_3 = \frac{k u_0 (T_w - T_\infty)}{\nu q_0},$$

$$\eta_4 = \frac{J_0}{v_0(N_w - N_\infty)}, \ \eta_5 = \frac{D_n v_0 (N_w - N_\infty)}{\nu J_0}, \ Gr = \frac{g\nu\beta_T(T_w - T_\infty)}{v_0^3},$$

$$M = \frac{\sigma B_o^2 \nu}{\rho v_0^2}, \ Pr = \frac{\mu C_p}{k}, \ Lb = \frac{\nu}{D_n}, \ Ra = \frac{\gamma(\rho_m - \rho)(N_w - N_\infty)}{\beta_T (T_w - T_\infty)\rho}.$$

Using Eqs. (12), (14) and (16) in Eqs. (11), (13) and (15) respectively, we have

$$u_t(y,t) = {}^{ABC}D_t^{\alpha}u_{yy}(y,t) + Gr\theta(y,t) - \boldsymbol{GrRaN}(y,t) - Mu(y,t), \tag{20}$$

$$Pr\theta_t(y,t) = {}^{ABC}D_t^{\beta}\theta_{yy}(y,t), \tag{21}$$

$$LbN_t(y,t) = {}^{ABC}D_t^{\gamma}N_{yy}(y,t). \tag{22}$$

4. Solution of the problem

In this section, we will find the solution of the initial value with the help of Laplace transform method.

4.1. Calculation of bioconvection field

Applying Laplace transform to Eq. (22) with conditions $(18)_3$, $(19)_3$, we have

$$\left[D^2 - \frac{\mathrm{Lb}s(s^\gamma + \gamma\Omega)}{s^\gamma\Omega}\right]\overline{N}(y,s) = 0, \tag{23}$$

satisfying

$$\overline{N}(0,s) = \frac{1}{s}, \quad \overline{N}(\infty,s)\to 0, \tag{24}$$

where

$$\Omega = \frac{1}{1-\gamma}.$$

The general solution of Eq. (23) subject to the conditions (24) given by

$$\overline{N}(y,s) = \frac{1}{s}\, exp\left(-y\sqrt{\frac{\mathrm{Lb}s(s^\gamma+\gamma\Omega)}{s^\gamma\Omega}}\right). \tag{25}$$

Eq. (25) can be written in more suitable form as

$$\overline{N}(y,s) = \frac{1}{s} + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\frac{\left(-y\sqrt{\frac{\mathrm{Lb}}{\Omega}}\right)^p(\Omega\gamma)^q\Gamma\left(\frac{p}{2}+1\right)}{p!q!s^{1-\frac{p}{2}+\gamma q}\Gamma\left(\frac{p}{2}+1-q\right)}. \tag{26}$$

By applying Laplace inverse transform on Eq. (26), by using the formula solution of bioconvection field is

$$N(y,t) = 1 + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\frac{\left(-y\sqrt{\frac{\mathrm{Lb}}{\Omega}}\right)^p(\Omega\gamma)^q t^{-\frac{p}{2}+\gamma q}\Gamma\left(\frac{p}{2}+1\right)}{p!q!\Gamma\left(1-\frac{p}{2}+\gamma q\right)\Gamma\left(\frac{p}{2}+1-q\right)}, \tag{27}$$

Further, solution given in Eq. (27) can be written in special M-function [76].

$$N(y,t) = 1 + \sum_{p=1}^{\infty}\frac{\left(-y\sqrt{\frac{\mathrm{Lb}}{\Omega}}\right)^p}{p!}M_2^1\left[(\Omega\gamma t^\gamma)|_{\left(1-\frac{p}{2},\gamma\right),\left(\frac{p}{2}+1,-1\right)}^{\left(\frac{p}{2}+1,0\right)}\right]. \tag{28}$$

4.2. Bioconvection density

Bioconvection density can be computed by the following expressions and presented in Table 1.

$$\boldsymbol{Density} = -N_y(y,t)|_{y=0}. \tag{29}$$

4.3. Calculation of temperature field

Applying Laplace transform to Eq. (21) with the conditions $(18)_2$, $(19)_2$, we have

$$\left[D^2 - \frac{Prs(s^\beta + \beta\mathfrak{R})}{s^\beta\mathfrak{R}}\right]\overline{\theta}(y,s) = 0, \tag{30}$$

Table 1
Bioconvection density for the effect of γ statistically.

γ	Density $t=1$	Density $t=2$	Density $t=3$
0.1	0.446	0.444	0.443
0.2	0.443	0.436	0.432
0.3	0.438	0.422	0.412
0.4	0.430	0.402	0.384
0.5	0.420	0.375	0.348
0.6	0.404	0.342	0.307
0.7	0.380	0.302	0.263
0.8	0.345	0.259	0.219
0.9	0.299	0.216	0.179

Table 2
Nusselt number for the effect of β statistically.

β	Nu $t=1$	Nu $t=2$	Nu $t=3$
0.1	2.483	2.474	2.469
0.2	2.466	2.427	2.403
0.3	2.438	2.348	2.291
0.4	2.397	2.236	2.136
0.5	2.337	2.088	1.940
0.6	2.249	1.903	1.711
0.7	2.116	1.683	1.465
0.8	1.920	1.442	1.220
0.9	1.664	1.204	0.999

satisfying

$$\bar{\theta}(0,s)=\frac{1}{s},\bar{\theta}(\infty,s)\to 0, \tag{31}$$

where

$$\Re=\frac{1}{1-\beta}.$$

The general solution of Eq. (30) subject to the conditions (31) can be written as

$$\bar{\theta}(y,s)=\frac{1}{s}\,exp\left(-y\sqrt{\frac{Prs(s^{\beta}+\beta\Re)}{s^{\beta}\Re}}\right). \tag{32}$$

In order to find inverse Laplace transform of Eq. (32) it can be written in more suitable form as

$$\bar{\theta}(y,s)=\frac{1}{s}+\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\frac{\left(-y\sqrt{\frac{Pr}{\Re}}\right)^{l}(\Re\beta)^{m}\Gamma\left(\frac{l}{2}+1\right)}{l!m!s^{1-\frac{l}{2}+\beta m}\Gamma\left(\frac{l}{2}+1-m\right)}. \tag{33}$$

By applying Laplace inverse transform on Eq. (33), solution of temperature field is,

$$\theta(y,t)=1+\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\frac{\left(-y\sqrt{\frac{Pr}{\Re}}\right)^{l}(\Re\beta)^{m}t^{-\frac{l}{2}+\beta m}\Gamma\left(\frac{l}{2}+1\right)}{l!m!\Gamma\left(1-\frac{l}{2}+\beta m\right)\Gamma\left(\frac{l}{2}+1-m\right)}. \tag{34}$$

Further, solution given in Eq. (34) can be written in special M-function [76].

$$\theta(y,t)=1+\sum_{l=1}^{\infty}\frac{\left(-y\sqrt{\frac{Pr}{\Re}}\right)^{l}}{l!}M_{2}^{1}\left[(\Re\beta t^{\beta})\Big|^{\left(\frac{l}{2}+1,0\right)}_{\left(1-\frac{l}{2},\beta\right),\left(\frac{l}{2}+1,-1\right)}\right]. \tag{35}$$

4.4. Nusselt number

Heat transfer rate calculated in the form of Nusselt number by using the following expression and presented in Table 2.

$$Nu=-\theta_{y}(y,t)|_{y=0}. \tag{36}$$

4.5. Calculation of velocity field

In order to obtain the solution of momentum equation given in Eq. (20) with the conditions $(18)_1$, $(19)_1$, we have the following expression after applying the Laplace transform method

$$\left[D^{2}-(s+M)\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)\right]\bar{u}(y,s)=-\mathrm{Gr}\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)\bar{\theta}(y,s)+\boldsymbol{GrRa}\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)\bar{N}(y,s), \tag{37}$$

satisfying

$$\bar{u}(0,s)=\frac{1}{s},\bar{u}(\infty,s)\to 0, \tag{38}$$

where

$$\chi=\frac{1}{1-\alpha}.$$

The general solution of Eq. (37) with boundary conditions (38), we have

$$\bar{u}(y,s)=\left[\frac{1}{s}+\frac{\mathrm{Gr}\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)}{s\left[\frac{Prs(s^{\beta}+\beta\Re)}{s^{\beta}\Re}-(s+M)\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)\right]}-\frac{\boldsymbol{GrRa}\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)}{s\left[\frac{\mathrm{Lbs}(s^{\gamma}+\gamma\Omega)}{s^{\gamma}\Omega}-(s+M)\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)\right]}\right]e^{-y\sqrt{(s+M)\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)}}$$
$$-\frac{\mathrm{Gr}\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)e^{-y\sqrt{\frac{Prs(s^{\beta}+\beta\Omega)}{s^{\beta}\Omega}}}}{s\left[\frac{Prs(s^{\beta}+\beta\Re)}{s^{\beta}\Re}-(s+M)\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)\right]}+\frac{\boldsymbol{GrRa}\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)e^{-y\sqrt{\frac{\mathbf{Lbs}(s^{\gamma}+\gamma\Omega)}{s^{\gamma}\Omega}}}}{s\left[\frac{\mathrm{Lbs}(s^{\gamma}+\gamma\Omega)}{s^{\gamma}\Omega}-(s+M)\left(\frac{s^{\alpha}+\chi\alpha}{s^{\alpha}\chi}\right)\right]}. \tag{39}$$

Eq. (39) can be written term by term

$$\bar{u}(y,s)=\bar{u}_1(y,s)+\bar{u}_2(y,s)+\bar{u}_3(y,s)+\bar{u}_4(y,s)+\bar{u}_5(y,s). \tag{40}$$

where all the components of the velocity can be written in more compact form

$$\bar{u}_1(y,s)=\frac{1}{s}+\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-y)^{n_1}(\chi\alpha)^{n_2}(\mathrm{M})^{n_3}\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_1}{2}+1\right)}{n_1!n_2!n_3!(\chi)^{\frac{n_1}{2}}s^{1-\frac{n_1}{2}+\alpha n_2+n_3}\Gamma\left(\frac{n_1}{2}+1-n_2\right)\Gamma\left(\frac{n_1}{2}+1-n_3\right)}, \tag{41}$$

$$\begin{aligned}\bar{u}_2(y,s)=&\ Gr\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\sum_{n_7=0}^{\infty}\frac{-(-y)^{n_1}(Pr)^{n_4}(\beta\mathfrak{R})^{n_5}\alpha^{n_2+n_6}\chi^{n_2+n_6-\frac{n_1}{2}}}{n_1!n_2!n_3!n_5!n_6!n_7!s^{1-\frac{n_1}{2}+\alpha n_2+n_3+n_5\beta+n_6\alpha+n_7}}\\&\times\frac{(-1)^{n_7}(M)^{n_3+n_7}\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_1}{2}+1\right)\Gamma(n_4+1)\Gamma(n_4+1)\Gamma(n_4+n_7)}{\Gamma\left(\frac{n_1}{2}+1-n_2\right)\Gamma\left(\frac{n_1}{2}+1-n_3\right)\Gamma(n_4+1-n_5)\Gamma(n_4+1-n_6)\Gamma(n_7)},\end{aligned} \tag{42}$$

$$\begin{aligned}\bar{u}_3(y,s)=&\ \mathrm{RaGr}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_8=0}^{\infty}\sum_{n_9=0}^{\infty}\sum_{n_{10}=0}^{\infty}\sum_{n_{11}=0}^{\infty}\frac{(-y)^{n_1}(\mathrm{Lb})^{n_8}(\gamma\Omega)^{n_9}\alpha^{n_2+n_{10}}\chi^{n_2+n_{10}-\frac{n_1}{2}}}{n_1!n_2!n_3!n_9!n_{10}!n_{11}!s^{1-\frac{n_1}{2}+\alpha n_2+n_3+n_9\gamma+n_{10}\alpha+n_{11}}}\\&\times\frac{(-1)^{n_{11}}(\mathrm{M})^{n_3+n_{11}}\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_1}{2}+1\right)\Gamma(n_8+1)\Gamma(n_8+1)\Gamma(n_8+n_{11})}{\Gamma\left(\frac{n_1}{2}+1-n_2\right)\Gamma\left(\frac{n_1}{2}+1-n_3\right)\Gamma(n_8+1-n_9)\Gamma(n_8+1-n_{10})\Gamma(n_{11})},\end{aligned} \tag{43}$$

$$\bar{u}_4(y,s)=\mathrm{Gr}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\sum_{n_7=0}^{\infty}\frac{\left(\frac{-y}{\mathfrak{R}}\right)^{p_1}(Pr)^{\frac{p_1}{2}+n_4}(\beta\mathfrak{R})^{p_2+n_5}(\alpha\chi)^{n_6}(-M)^{n_7}}{p_1!p_2!n_5!n_6!n_7!s^{1-\frac{p_1}{2}+p_2\beta+n_5\beta+n_6\alpha+n_7}}\times\frac{\Gamma\left(\frac{p_1}{2}+1\right)\Gamma(n_4+1)\Gamma(n_4+1)\Gamma(n_4+n_7)}{\Gamma\left(\frac{p_1}{2}+1-p_2\right)\Gamma(n_4+1-n_5)\Gamma(n_4+1-n_6)\Gamma(n_7)}, \tag{44}$$

$$\bar{u}_5(y,s)=\mathrm{GrRa}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\sum_{n_7=0}^{\infty}\frac{-\left(\frac{-y}{\Omega}\right)^{q_1}(\mathrm{Lb})^{\frac{q_1}{2}+n_4}(\gamma\Omega)^{q_2+n_5}(\alpha\chi)^{n_6}(-M)^{n_7}}{q_1!q_2!n_5!n_6!n_7!s^{1-\frac{q_1}{2}+q_2\gamma+n_5\gamma+n_6\alpha+n_7}}\times\frac{\Gamma\left(\frac{q_1}{2}+1\right)\Gamma(n_4+1)\Gamma(n_4+1)\Gamma(n_4+n_7)}{\Gamma\left(\frac{q_1}{2}+1-q_2\right)\Gamma(n_4+1-n_5)\Gamma(n_4+1-n_6)\Gamma(n_7)}. \tag{45}$$

By applying Laplace inverse transform to Eqs. (40)–(45), we have the final form of the velocity field

$$u(y,t)=u_1(y,t)+u_2(y,t)+u_3(y,t)+u_4(y,t)+u_5(y,t). \tag{46}$$

where

$$u_1(y,t)=1+\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{(-y)^{n_1}(\chi\alpha)^{n_2}(\mathrm{M})^{n_3}t^{-\frac{n_1}{2}+\alpha n_2+n_3}\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_1}{2}+1\right)}{n_1!n_2!n_3!(\chi)^{\frac{n_1}{2}}\Gamma\left(1-\frac{n_1}{2}+\alpha n_2+n_3\right)\Gamma\left(\frac{n_1}{2}+1-n_2\right)\Gamma\left(\frac{n_1}{2}+1-n_3\right)}, \tag{47}$$

$$\begin{aligned}u_2(y,t)=&\ \mathrm{Gr}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\sum_{n_7=0}^{\infty}\frac{-(-y)^{n_1}(Pr)^{n_4}(\beta\mathfrak{R})^{n_5}\alpha^{n_2+n_6}\chi^{n_2+n_6-\frac{n_1}{2}}(-1)^{n_7}}{n_1!n_2!n_3!n_5!n_6!n_7!\Gamma(n_4+1-n_6)\Gamma(n_7)}\\&\times\frac{(\mathrm{M})^{n_3+n_7}t^{-\frac{n_1}{2}+\alpha n_2+n_3+n_5\beta+n_6\alpha+n_7}\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_1}{2}+1\right)\Gamma(n_4+1)\Gamma(n_4+1)\Gamma(n_4+n_7)}{\Gamma\left(1-\frac{n_1}{2}+\alpha n_2+n_3+n_5\beta+n_6\alpha+n_7\right)\Gamma\left(\frac{n_1}{2}+1-n_2\right)\Gamma\left(\frac{n_1}{2}+1-n_3\right)\Gamma(n_4+1-n_5)},\end{aligned} \tag{48}$$

$$\begin{aligned}u_3(y,t)=&\ \mathrm{RaGr}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_8=0}^{\infty}\sum_{n_9=0}^{\infty}\sum_{n_{10}=0}^{\infty}\sum_{n_{11}=0}^{\infty}\frac{(-y)^{n_1}(\mathrm{Lb})^{n_8}(\gamma\Omega)^{n_9}\alpha^{n_2+n_{10}}\chi^{n_2+n_{10}-\frac{n_1}{2}}(-1)^{n_{11}}}{n_1!n_2!n_3!n_9!n_{10}!n_{11}!\Gamma(n_8+1-n_{10})\Gamma(n_{11})}\\&\times\frac{(\mathrm{M})^{n_3+n_{11}}t^{-\frac{n_1}{2}+\alpha n_2+n_3+n_9\gamma+n_{10}\alpha+n_{11}}\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_1}{2}+1\right)\Gamma(n_8+1)\Gamma(n_8+1)\Gamma(n_8+n_{11})}{\Gamma\left(1-\frac{n_1}{2}+\alpha n_2+n_3+n_9\gamma+n_{10}\alpha+n_{11}\right)\Gamma\left(\frac{n_1}{2}+1-n_2\right)\Gamma\left(\frac{n_1}{2}+1-n_3\right)\Gamma(n_8+1-n_9)},\end{aligned} \tag{49}$$

$$\begin{aligned}u_4(y,t)=&\ \mathrm{Gr}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\sum_{n_7=0}^{\infty}\frac{\left(\frac{-y}{\mathfrak{R}}\right)^{p_1}(Pr)^{\frac{p_1}{2}+n_4}(\beta\mathfrak{R})^{p_2+n_5}(\alpha\chi)^{n_6}(-M)^{n_7}}{p_1!p_2!n_5!n_6!n_7!\Gamma\left(1-\frac{p_1}{2}+p_2\beta+n_5\beta+n_6\alpha+n_7\right)}\\&\times\frac{t^{-\frac{p_1}{2}+p_2\beta+n_5\beta+n_6\alpha+n_7}\Gamma\left(\frac{p_1}{2}+1\right)\Gamma(n_4+1)\Gamma(n_4+1)\Gamma(n_4+n_7)}{\Gamma\left(\frac{p_1}{2}+1-p_2\right)\Gamma(n_4+1-n_5)\Gamma(n_4+1-n_6)\Gamma(n_7)},\end{aligned} \tag{50}$$

$$u_5(y,t) = \mathrm{GrRa}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\sum_{n_7=0}^{\infty}\frac{-\left(\frac{-y}{\Omega}\right)^{q_1}(\mathrm{Lb})^{\frac{q_1}{2}+n_4}(\gamma\Omega)^{q_2+n_5}(\alpha\chi)^{n_6}(-M)^{n_7}}{q_1!q_2!n_5!n_6!n_7!\Gamma\left(1-\frac{q_1}{2}+q_2\gamma+n_5\gamma+n_6\alpha+n_7\right)} \times \frac{t^{-\frac{q_1}{2}+q_2\gamma+n_5\gamma+n_6\alpha+n_7}\Gamma\left(\frac{q_1}{2}+1\right)\Gamma(n_4+1)\Gamma(n_4+1)\Gamma(n_4+n_7)}{\Gamma\left(\frac{q_1}{2}+1-q_2\right)\Gamma(n_4+1-n_5)\Gamma(n_4+1-n_6)\Gamma(n_7)}. \quad (51)$$

Eqs. (47)–(51) are expressed in the form of more general function

$$u_1(y,t) = 1+\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-y)^{n_1}(\chi\alpha)^{n_2}}{n_1!n_2!(\chi)^{\frac{n_1}{2}}}M_3^2\left[(\mathrm{M}t)\Big|^{\left(\frac{n_1}{2}+1,0\right),\left(\frac{n_1}{2}+1,0\right)}_{\left(1-\frac{n_1}{2}+\alpha n_2,1\right),\left(\frac{n_1}{2}+1-n_2,0\right),\left(\frac{n_1}{2}+1,-1\right)}\right], \quad (52)$$

$$u_2(y,t) = \mathrm{Gr}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\frac{-(-y)^{n_1}(Pr)^{n_4}(\beta\mathfrak{R})^{n_5}\alpha^{n_2+n_6}\chi^{n_2+n_6-\frac{n_1}{2}}(\mathrm{M})^{n_3}}{n_1!n_2!n_3!n_5!n_6!} \times M_6^5\left[(-\mathrm{M}t)\Big|^{\left(\frac{n_1}{2}+1,0\right),\left(\frac{n_1}{2}+1,0\right),(n_4+1,0),(n_4+1,0),(n_4,1)}_{\left(1-\frac{n_1}{2}+\alpha n_2+n_3+n_5\beta+n_6\alpha,1\right),\left(\frac{n_1}{2}+1-n_2,0\right),\left(\frac{n_1}{2}+1-n_3,0\right),(n_4+1-n_5,0),(n_4+1-n_6,0),(0,1)}\right], \quad (53)$$

$$u_3(y,t) = \mathrm{RaGr}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_8=0}^{\infty}\sum_{n_9=0}^{\infty}\sum_{n_{10}=0}^{\infty}\frac{(-y)^{n_1}(\mathrm{Lb})^{n_8}(\gamma\Omega)^{n_9}\alpha^{n_2+n_{10}}\chi^{n_2+n_{10}-\frac{n_1}{2}}(\mathrm{M})^{n_3}}{n_1!n_2!n_3!n_9!n_{10}!} \times M_6^5\left[(-\mathrm{M}t)\Big|^{\left(\frac{n_1}{2}+1,0\right),\left(\frac{n_1}{2}+1,0\right),(n_8+1,0),(n_8+1,0),(n_8,1)}_{\left(1-\frac{n_1}{2}+\alpha n_2+n_3+n_9\gamma+n_{10}\alpha,1\right),\left(\frac{n_1}{2}+1-n_2,0\right),\left(\frac{n_1}{2}+1-n_3,0\right),(n_8+1-n_9,0),(n_8+1-n_{10},0),(0,1)}\right], \quad (54)$$

$$u_4(y,t) = \mathrm{Gr}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\frac{\left(\frac{-y}{\mathfrak{R}}\right)^{p_1}(Pr)^{\frac{p_1}{2}+n_4}(\beta\mathfrak{R})^{p_2+n_5}(\alpha\chi)^{n_6}}{p_1!p_2!n_5!n_6!}\times M_5^4\left[(-\mathrm{M}t)\Big|^{\left(\frac{p_1}{2}+1,0\right),(n_4+1,0),(n_4+1,0),(n_4,1)}_{\left(1-\frac{p_1}{2}+p_2\beta+n_5\beta+n_6\alpha,1\right),\left(\frac{p_1}{2}+1-p_2,0\right),(n_4+1-n_5,0),(n_4+1-n_6,0),(0,1)}\right], \quad (55)$$

$$u_5(y,t) = \mathrm{GrRa}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}\frac{-\left(\frac{-y}{\Omega}\right)^{q_1}(\mathrm{Lb})^{\frac{q_1}{2}+n_4}(\gamma\Omega)^{q_2+n_5}(\alpha\chi)^{n_6}}{q_1!q_2!n_5!n_6!}\times M_5^4\left[(-\mathrm{M}t)\Big|^{\left(\frac{q_1}{2}+1,0\right),(n_4+1,0),(n_4+1,0),(n_4,1)}_{\left(1-\frac{q_1}{2}+q_2\gamma+n_5\gamma+n_6\alpha,1\right),\left(\frac{q_1}{2}+1-q_2,0\right),(n_4+1-n_5,0),(n_4+1-n_6,0),(0,1)}\right]. \quad (56)$$

5. Graphical results and discussion

In order to see the physical behavior of flow parameters, some graphs are plotted. Figs. 2–4 are designed for the comparison between two fractional operators. One is the Caputo (C) fractional operator of singular kernel type and second is the Atangana-Baleanu (ABC) fractional operator of non-singular kernel type. Caputo (C) fractional operator measures the memory of natural phenomena and has weak memory while (ABC) measures the strong memory due to Mittag-Leffler kernel which is a more general function. It is observed from the figures that for larger time and for different values of fractional parameters, the operator containing Mittag-Leffler kernel exhibits more memory and as a whole measure more decline the fluid properties bioconvection, temperature and velocity field respectively.

Figs. 5–7 are especially designed for the effect of fractional parameter α. As a result, it is found that for bioconvection and temperature field for larger values of fractional parameter both properties can be enhanced. Other observed aspect the boundary layer become wider and

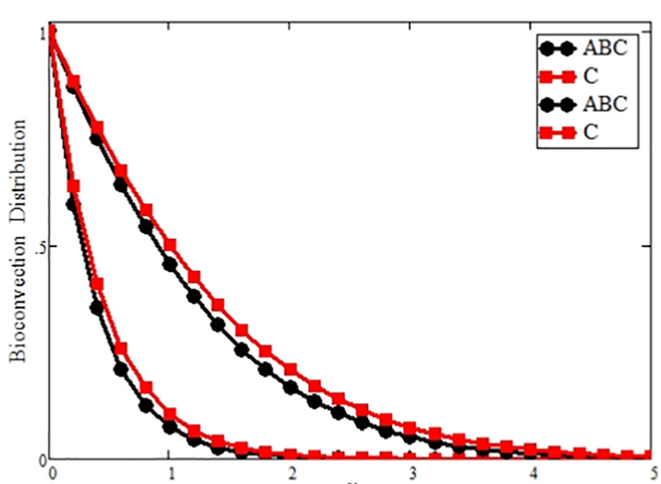

Fig. 2. Comparison between bioconvection fields with (ABC) and (C) [73] for $\alpha = 0.1, 0.8$ when $t = 15$ and Lb = 7.

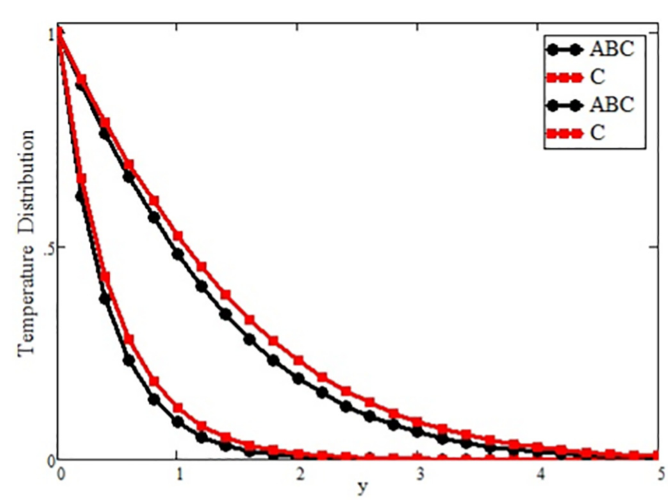

Fig. 3. Comparison between temperatures with (ABC) and (C) [73] for $\alpha = 0.1, 0.8$ when $t = 15$ and $Pr = 6.2$.

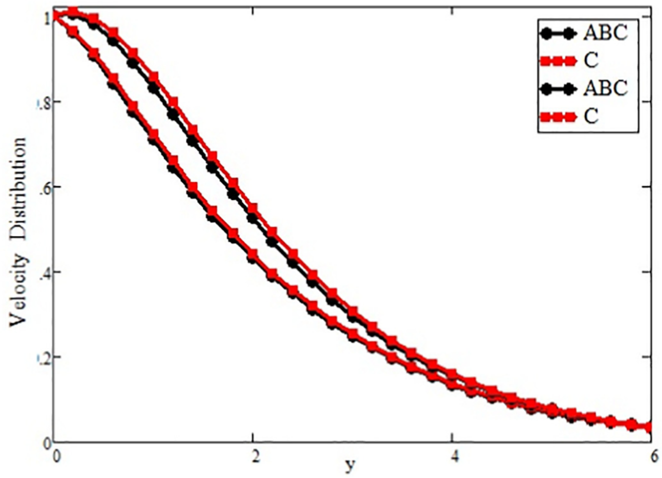

Fig. 4. Comparison between velocities with (ABC) and (C) [73] for $\alpha = 0.1, 0.8$ when $t = 5$, $\mathrm{Lb} = 8$, $\mathrm{Gr} = 1$, $\mathrm{M} = 0.2$, $\mathrm{Ra} = 0.001$ and $Pr = 6.2$.

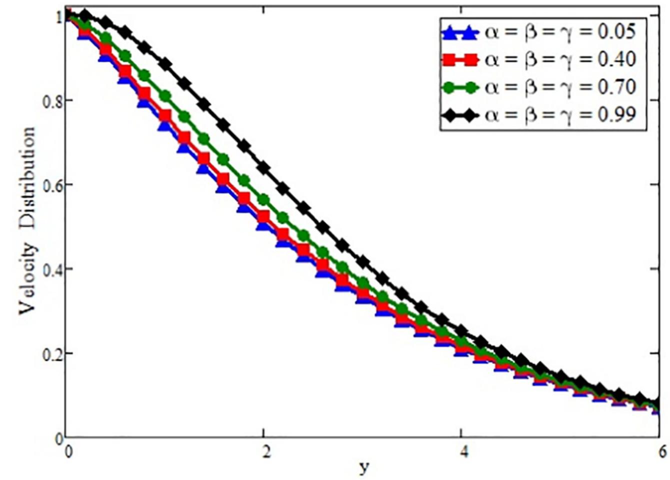

Fig. 7. Effect of fractional parameters α, β, γ on velocity when $t = 5$, $\mathrm{Lb} = 5$, $\mathrm{Gr} = 0.5$, $\mathrm{M} = 0.2$, $\mathrm{Ra} = 3$ and $Pr = 6.2$.

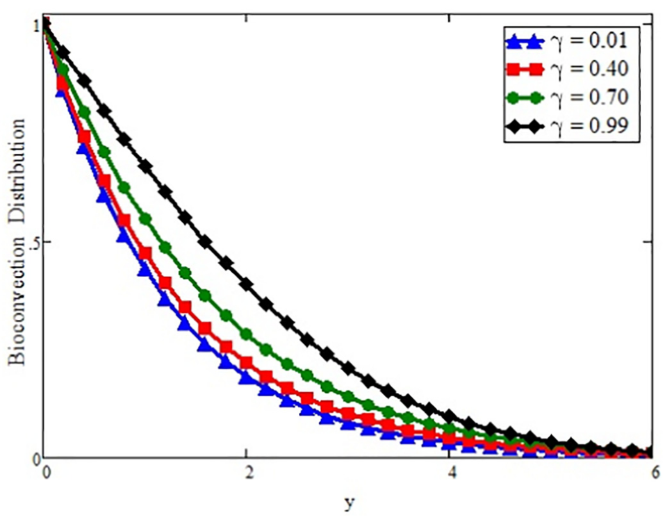

Fig. 5. Effect of fractional parameter γ on bioconvection when $t = 2$ and $\mathrm{Lb} = 0.7$.

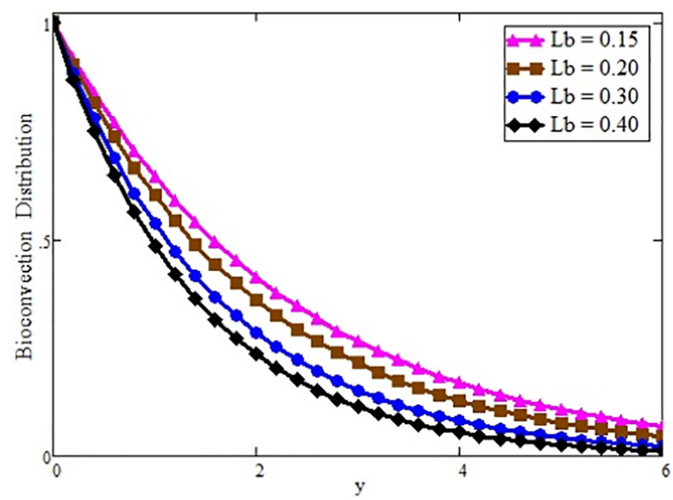

Fig. 8. Impact of bioconvection Lewis number Lb on bioconvection when $t = 0.2$ and $\gamma = 0.5$.

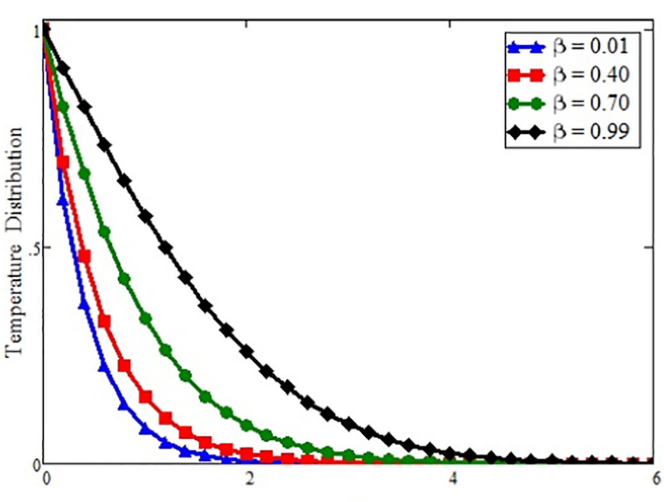

Fig. 6. Effect of fractional parameter β on temperature when $t = 10$ and $Pr = 6.2$.

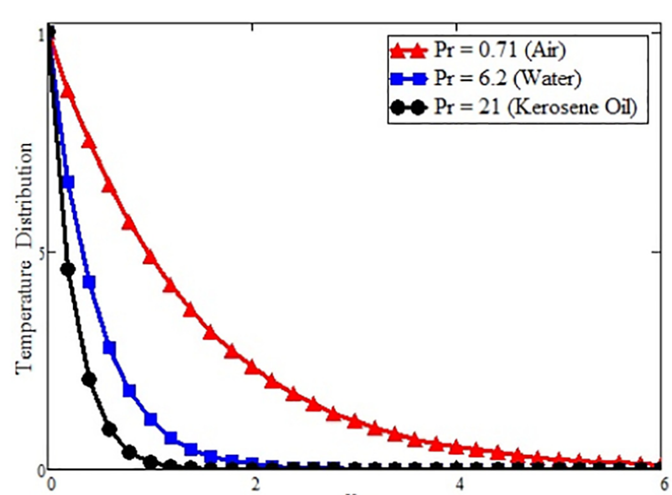

Fig. 9. Impact of Pr on temperature when $t = 2$ and $\beta = 0.5$.

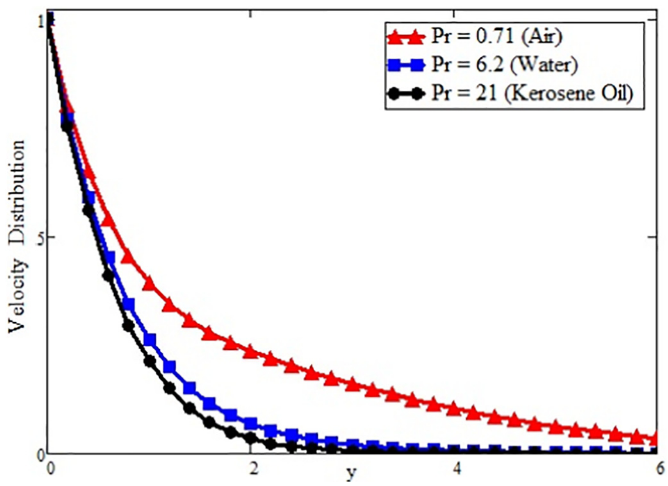

Fig. 10. Effect of Pr on velocity when $t = 5$, $\mathrm{Gr} = 0.6$, $\alpha = 0.8$, $\mathrm{Lb} = 5$, $\mathrm{M} = 1$ and $\mathrm{Ra} = 2$.

hence increases in the main stream region. While momentum boundary layer decreases for velocity field. Overall, we can say that fractional approach in fluid dynamics is very useful to control the boundary layer thickness of the fluid properties. Fig. 8 is plotted to see the effect of Lb on bioconvection field while $t = 0.2$ and $\alpha = 0.5$. It is resulted that bioconvection field is decreased when Lb is increased. It is because of that Lb is returned to mass diffusivity, so bioconvection field decreased.

Figs. 9–10 depicted for comparison of different types of Newtonian fluids like air, water and kerosene. It is clearly observed that temperature and velocity of air is higher than water and kerosene. Physically, it is due to the fact that viscosity and Prandtl number is very small for air that's why air flows faster than the other two. Fig. 11 is demonstrated the impact of Gr on velocity field while $t = 15$, $\mathrm{Lb} = 30$, $\alpha = 0.8$, $\mathrm{M} = 0.5$, $\mathrm{Ra} = 1$ and $Pr = 6.2$. It is noted that fluid velocity is upgraded by rising values of Gr. This is due to the rising thermal buoyancy influence. Fig. 12 is planned to see the influence of magnetic effect M on velocity field while $t = 15$, $\mathrm{Gr} = 0.6$, $\alpha = 0.8$, $\mathrm{Lb} = 30$, $\mathrm{M} = 0.5$ and $Pr = 6.2$. It is derived that by increasing the values of M, velocity decreases. Physically, it happened that magnetic field plays a role as a drag force which opposes the flow and ultimately responsible for this decline in the velocity field. The effect of bioconvection Rayleigh number Ra on velocity for $t = 15$, $\mathrm{Gr} = 0.6$, $\alpha = 0.8$, $\mathrm{Lb} = 30$, $\mathrm{M} = 0.5$ and $Pr = 6.2$ can be seen in Fig. 13. It is observed that velocity near the plate decreases for greater values of Ra. It is because of that the fluid velocity falls due to the decrease in the buoyancy effect from the transportation of microorganisms.

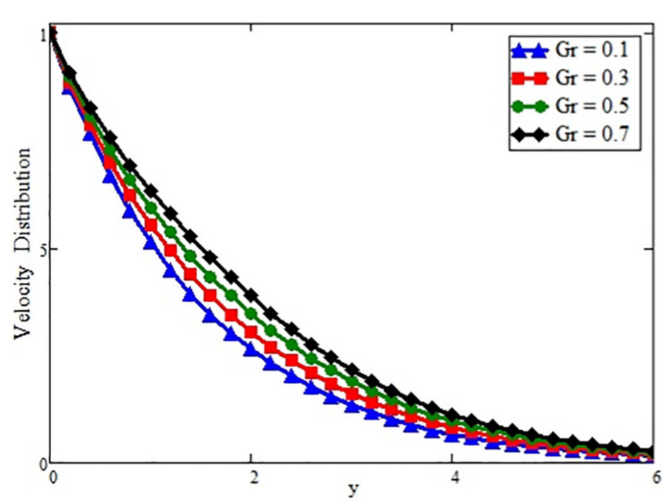

Fig. 11. Effect of Gr on velocity when $t = 15$, $\mathrm{Lb} = 30$, $\alpha = 0.8$, $\mathrm{M} = 0.5$, $\mathrm{Ra} = 1$ and $Pr = 6.2$.

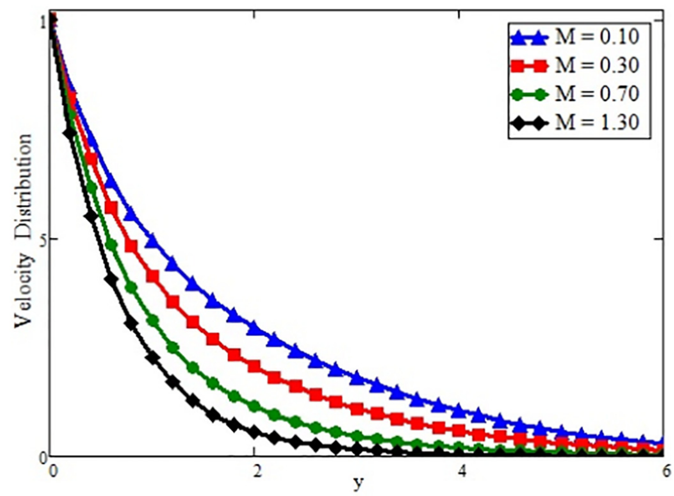

Fig. 12. Effect of magnetic parameter M on velocity when $t = 5$, $\mathrm{Gr} = 0.6$, $\alpha = 0.5$, $\mathrm{Lb} = 5$, $\mathrm{Ra} = 3$ and $Pr = 6.2$.

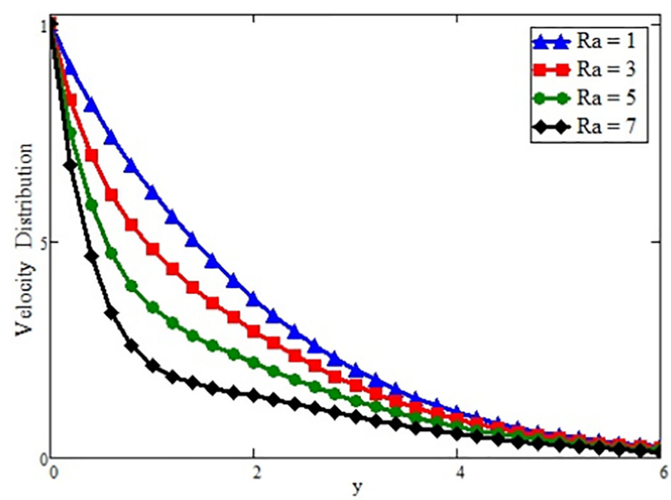

Fig. 13. Effect of Rayleigh number Ra on velocity when $t = 15$, $\mathrm{Gr} = 0.6$, $\alpha = 0.8$, $\mathrm{Lb} = 30$, $\mathrm{M} = 0.5$ and $Pr = 6.2$.

6. Conclusions

This work deals the new solutions of recently published papers with power law kernel. Usually, the kernels are responsible for the history/memory of the function. In this work, Mittag-Leffler kernel is applied for the modeling of viscous fluid and obtained series solutions with the help of Laplace transform method. Followings are the key findings of this work.

- Bioconvection Rayleigh number responsible for decline in the velocity as well as momentum boundary layer thickness.
- Fractional parameters β and γ control the diffusion, thermal and momentum boundary layers respectively.
- It is also found that fractional operator containing Mittag-Leffler kernel exhibits stronger memory than power law kernel.
- Rate of heat transfer and microorganism density can be minimized with the help of fractional parameters β and γ.
- Velocity is increasing function of Gr while it decreases with rising β and γ, M and Ra.

Nomenclature

(C)	Caputo
ρ	Fluid density
g	Gravitational acceleration
T	Fluid Temperature (K)
θ	Dimensionless Temperature
T_w	Temperature at the wall
T_∞	Ambient temperature of the fluid
f	Velocity (ms^{-1})
q	Laplace Transform
Pr	Prandtl number (Dimensionless)
Gr	Grashof number (Dimensionless)
D_n	Diffusivity of microorganisms
(ABC)	Atangana-Baleanu
(kgm^{-3})	μ Viscosity ($\mathrm{kgm^{-1}s^{-1}}$)
(ms^{-2})	β_T Volumetric coefficient of thermal expansion
N	Concentration of microorganisms
N^*	Dimensionless Concentration of microorganisms
N_w	Concentration of microorganisms at the wall
N_∞	The density of motile microorganisms
F	Dimensionless velocity
C_p	Specific heat at constant pressure ($\mathrm{jkg^{-1}K^{-1}}$)
k	Thermal conductivity ($\mathrm{Wm^{-2}K^{-1}}$)
Ra	Bioconvection Rayleigh number (Dimensionless)
Lb	Bioconvection Lewis number (Dimensionless)

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Declaration of competing interest

All the authors hereby declare that they have no conflict of interest in submission the paper titled.

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References

[1] Bearon RN, Grünbaum D. Bioconvection in a stratified environment: experiments and theory. Phys Fluids. 2006;18(12):1–15.
[2] Bees MA. Advances in bioconvection. Annu Rev Fluid Mech. 2020;52(1):449–76.
[3] Platt JR. Bioconvection patterns in cultures of free-swimming organisms. Sci. 1961; 133(3466):1766–7.
[4] Rao MVS, Gangadhar K, Chamkha AJ, Surekha P. Bioconvection in a convectional nanofluid flow containing gyrotactic microorganisms over an isothermal vertical cone embedded in a porous surface with chemical reactive species. Arab J Sci Eng. 2021;46:2493–503.
[5] Khan SU, Khaled KA, Aldabesh A, Awais M, Talili I. Bioconvection flow in accelerated couple stress nanoparticles with activation energy: bio-fuel applications. Sci Rep. 2021;11:3331.
[6] Abdelmalek Z, Khaled KA, Waqas H, Aldabesh A, Khan SU, Musmar SA, et al. Bioconvection in cross nano-materials with magnetic dipole impacted by activation energy, thermal radiation and second order slip. Symm. 2020;12(6):1–23.
[7] Alshomrani AS, Ullah MZ, Baleanu D. Importance of multiple slips on bioconvection flow of cross nanofluid past a wedge with gyrotactic motile microorganisms. Case Stud Therm Eng. 2020;22:100798.
[8] Kuznetsov AV, Avramenko AA. Stability analysis of bioconvection of gyrotactic motile microorganisms in a fluid saturated porous medium. Transp. 2003;53:95–104.
[9] Muhammad T, Alamri SZ, Waqas H, Habib D, Ellahi R. Bioconvection flow of magnetized carreau nanofluid under the influence of slip over a wedge with motile microorganisms. J Therm Anal Calorim. 2021;143:945–57.
[10] Mondal SK, Pal D. Gyrotactic mixed bioconvection flow of a nanofluid over a stretching wedge embedded in a porous media in the presence of binary chemical reaction and activation energy. Int J Ambient Energy. 2020:1–17.
[11] Kuznetsov AV. The onset of nanofluid bioconvection in a suspension containing both nanoparticles and gyrotactic microorganisms. Int Commun Heat Mass Transf. 2010; 37(10):1421–5.
[12] Kuznetsov AV. Nanofluid bioconvection in water-based suspensions containing nanoparticles and oxytactic microorganisms: oscillatory instability. Nanoscale Res Lett. 2011;6(100):1–13.
[13] Akbar NS, Khan ZH. Magnetic field analysis in a suspension of gyrotactic microorganisms and nanoparticles over a stretching surface. J Magn Magn Mater. 2016;410: 72–80.
[14] Raju CSK, Hoque MM, Sivasankar T. Radiative flow of casson fluid over a moving wedge filled with gyrotactic microorganisms. Adv Powder Technol. 2017;28: 575–83.
[15] Zhao Q, Xu H, Tao L. Unsteady bioconvection squeezing flow in a horizontal channel with chemical reaction and magnetic field effects. Math Probl Eng. 2017;2017:1–9.
[16] Uddin MJ, Khan WA, Qureshi SR, Anwar OB. Bioconvection nanofluid slip flow past a wavy surface with applications in nano-biofuel cells. Chin J Phys. 2017;55(5): 2048–63.
[17] Khaled KA, Khan SU, Khan I. Chemically reactive bioconvection flow of tangent hyperbolic nanoliquid with gyrotactic microorganisms and nonlinear thermal radiation. Heliyon. 2020;6:1.
[18] Ullah MZ, Jang TS. An efficient numerical scheme for analyzing bioconvection in von-karman flow of third-grade nanofluid with motile microorganisms. Alex Eng J. 2020;59:2739–52.
[19] Wang Y, Waqas H, Tahir M, Imran M, Jung CY. Effective prandtl aspects on bio convective thermally developed magnetized tangent hyperbolic nanoliquid with gyrotactic microorganisms and second order velocity slip. IEEE Access XX. 2017; 2017:1–22.
[20] Muhammad T, Ullah MZ, Waqas H, Alghamdi M, Riaz A. Thermo-bioconvection in stagnation point flow of third-grade nanofluid towards a stretching cylinder involving motile microorganisms. Phys Scr. 2021;96:035208.
[21] Machado JAT, Silva MF, Barbosa RS, Jesus IS, Reis CM, Marcos MG, et al. Some applications of fractional calculus in engineering. Math Probl Eng. 2010;2010:1–34.
[22] Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and applications to heat transfer model. Therm Sci. 2016;20:00.
[23] Syam MI, Refai MA. Fractional differential equations with Atangana-Baleanu fractional derivative: analysis and applications. Chaos Solitons Fractals X. 2019; 2:100013.
[24] Ikram MD, Imran MA, Ahmadian A, Ferrara M. A new fractional mathematical model of extraction nanofluids using clay nanoparticles for different based fluids. Math Methods Appl Sci. 2020;2020:1–14.
[25] Sheikh NA, Ali F, Saqib M, Khan I, Jan SAA, Alshomrani AS, et al. Comparison and analysis of the Atangana-Baleanu and Caputo-Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction. Results Phys. 2017;7:789–800.
[26] Alqahtani RT. Atangana-Baleanu derivative with fractional order applied to the model of groundwater within an unconfined aquifer. JNSA. 2016;9:3647–54.
[27] Ali F, Murtaza S, Khan I, Sheikh NA, Nisar KS. Atangana-Baleanu fractional model for the flow of Jeffrey nanofluid with diffusion-thermo effects: applications in engine oil. Adv Differ Equ. 2019;2019:346.
[28] Kumar S, Kumar A, Nieto JJ, Sharma B. Atangana-Baleanu derivative with fractional order applied to the gas dynamics equations. Fractional derivatives with Mittage-Leffler kernel; 2019. p. 235–51.
[29] Abro KA, Khan I, Tassaddiq A. Application of Atangana-Baleanu fractional derivative to convection flow of mhd Maxwell fluid in a porous medium over a vertical plate. Math Model Nat Phenom. 2018;13:1.
[30] Yavuz M, Ozdemir N. Comparing the new fractional derivative operators involving exponential and Mittage-Leffler kernel. AIMS. 2020;13(3):995–1006.
[31] Tlili I, Shah NA, Ullah S, Manzoor H. New idea of Atangana-Baleanu time-fractional derivative to advection-diffusion equation. Math Methods Appl Sci. 2019;2019: 1–11.
[32] Kumar D, Singh J, Baleanu D. On the analysis of vibration equation involving a fractionalderivative with Mittage-Leffler law. Math Methods Appl Sci. 2020;43: 443–57.
[33] Imran MA, Ikram MD, Ali R, Baleanu D, Alshomrani AS. New anayltical soltuions of heat transfer flow of clay-water base nanoparticles with the application of novel hybrid fractional derivative. Therm Sci. 2020;24(Suppl. 1):S343–50.
[34] Chu YM, Ikram MD, Imran MA, Ahmadian A, Ghaemi F. Influence of hybrid nanofluids and heat generation on coupled heat and mass transfer flow of a viscous

fluid with novel fractional derivative. J Therm Anal Calorim. 2021. https://doi.org/10.1007/s10973-021-10692-8.
[35] Popolizio M. On the matrix Mittag-Leffler function: theoretical properties and numerical computation. Math. 2019;7:1140. https://doi.org/10.3390/math7121140.
[36] Gorenflo R, Kilbas AA, Mainardi F, Rogosin S. Mittag-Leffler functions, related topics and applications. Springer; 2020; 1–540.
[37] Owolabi K, Atangana A. Numerical approximation of Atangana-Baleanu differentiation. Numerical methods for fractional differentiation. Springer; 2019https://doi.org/10.1007/987-981-15-0098-56.
[38] Atangana A, Liping C, Khan MA, Kumar S. A new financial chaotic model in Atangana-Baleanu stochastic fractional differential equations. Alex Eng J. 2021;60(6):5193–204.
[39] Atangana A, Abro KA, Gomez-Aguilar JF. Role of bi-order Atangana-Aguilar fractional differentiation on drude model: an analytic study for distinct sources. Opt Quantum Electron. 2021;53:77. https://doi.org/10.1007/s11082-021-02804-3.
[40] Atangana A, Sweilam N, Mekhlafi SMA, Assiri T. Optimal control for cancer treatment mathematical model using Atangana-Baleanu-Caputo fractional derivative. Adv Differ Equ. 2020;2020:1. https://doi.org/10.1186/s13662-020-02793-9.
[41] Imran MA, Ikram MD, Akgul A. Analysis of mhd viscous fluid flow through porous medium with novel power law fractional differential operator. Phy Scr. 2020;95:11.
[42] Ikram MD, Imran MA, Akgul A, Baleanu D. Effects of hybrid nanofluid on novel fractional model of heat transfer flow between two parallel plates. Alex Eng J. 2021;60:3593–604.
[43] Aliabadi MK, Hassani SM, Mazloumi SH, Nekoei M. Effects of nooks configuration on hydrothermal performance of zigzag channels for nanofluid-cooled microelectronic heat sink. Microelectron Reliab. 2017;79:153–65.
[44] Aliabadi MK, Pazdar S, Sartipzadeh O. Experimental investigation of water based nanofluid containing copper nanoparticles across helical microtubes. Int Commun Heat Mass Transf. 2016;70:84–92.
[45] Naik PA, Yavuz M, Qureshi S, Zu J, Townley S. Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan. Eur Phys J Plus. 2020;135:795.
[46] Hammouch Z, Yavuz M, Özdimer N. Numerical solutions and synchronization of a variable-order fractional chaotic system. Math Model Numer Sim App. 2021;1(1):11–23.
[47] Akgül EK, Akgül A, Yavuz M. New illustrative applications of integral transforms to financial models with different fractional derivatives. Chaos Solitons Fractals. 2021. https://doi.org/10.1016/j.chaos.2021.110877.
[48] Yavuz M, Sulaiman TA, Yusuf A, Abdeljawad T. The Schrödinger-KdV equation of fractional order with Mittag-Leffler nonsingular kernel. Alex Eng J. 2021;60(2):2715–24.
[49] Yavuz M. European option pricing models described by fractional operators with classical and generalized Mittag-Leffler kernels. Numer Methods Partial Differ Equ. 2020. https://doi.org/10.1002/num.22645.
[50] Veeresha P. A numerical approach to the coupled atmospheric ocean model using a fractional operator. MMNSA. 2021;1(1):1–10.
[51] Yavuz M. Characterizations of two different fractional operators without singular kernel. Math Model Nat Phenom. 2019;14(3):302. https://doi.org/10.1051/mmnp/2018070.
[52] Yavuz M, Özdemir N. European Vanilla option pricing model of fractional order without singular kernel. Fractal Fractional. 2018;2(1):3. https://doi.org/10.3390/fractalfract2010003.
[53] Baishya C, Achar SJ, Veeresha P, Prakasha DG. Dynamics of a fractional epidemiological model with disease infection in both the populations. Chaos Interdiscip J Nonlin Sci. 2021;31(4):043130. https://doi.org/10.1063/5.0028905.
[54] Yao SW, Ilhan E, Veeresha P, Baskonus HM. A powerful iterative approach for quintic complex Ginzburg-Landau equation within the frame of fractional operator. Fractals. 2021;29(5). https://doi.org/10.1142/S0218348X21400235.
[55] Prakasha DG, Veeresha P, Baskonus HM. Residual power series method for fractional Swift–Hohenberg equation. Fractal Fractional. 2019;3(1):9. https://doi.org/10.3390/fractalfract3010009.
[56] Baishya C, Veeresha P. Laguerre polynomial-based operational matrix of integration for solving fractional differential equations with non-singular kernel. Proc R Soc A. 2021;477:20210438. https://doi.org/10.1098/rspa.2021.0438.
[57] Veeresha P, Baleanu D. A unifying computational framework for fractional Gross–Pitaevskii equations. Phys Scr. 2021.;96(12).
[58] Veereasha P, Ilhan E, Baskonus HM. Fractional approach for analysis of the model describing wind-influenced projectile motion. Phys Scr. 2021.;96(7).
[59] Ilhan E, Veeresha P, Baskonus HM. Fractional approach for a mathematical model of atmospheric dynamics of CO_2 gas with an efficient method. Chaos Solitons Fractals. 2021;152:111347. https://doi.org/10.1016/j.chaos.2021.111347.
[60] Ishak A, Nazar R, Pop I. Flow of a micropolar fluid on a continuous moving surface. Arch Mech. 2006;58(6):529–41.
[61] Imran MA, Ullah MA, Raza A, Ikram MD, Naz R, Inc M. Mathematical fractional modeling of transpot phenomena of viscous fluid-flow between two plates. Therm Sci. 2021;25(2):1–5.
[62] Aleem M, Imran MA, Ahmadian A, Salimi M, Ferrara M. Heat transfer analysis of channel flow of mhd jeffrey fluid subject to generalized boundary conditions. Eur. 2020;135:1.
[63] Chu YM, Ali R, Imran MA, Ahmadian A, Senu N, Heat transfer flow of maxwell hybrid nanofluids due to pressure gradient into rectangular region. Sci Rep 10: 1.
[64] Singh J, Kumar D, Hammouch Z, Atangana A. A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. Appl Math Comput. 2018;316:504–15.
[65] Singh J, Veeresha P, Prakasha DG, Khan I, Kumar D. Analytical approach for fractional extended fisher-kolmogorov equation with Mittag-Leffler kernel. Adv Differ Equ. 2020.;1.
[66] Dubey VP, Kumar R, Kumar D, Khan I, Singh J. An efficient computational scheme for nonlinear time fractional systems of partial differential equations arising in physical sciences. Adv Differ Equ. 2020;1:46.
[67] Jarad F, Abdeljawad T, Rashid S, Hammouch Z. More properties of the proportional fractional integrals and derivatives of a function with respect to another function. Adv Differ Equ. 2020.;1.
[68] Imran MA, Aleem M, Riaz MB, Ali R, Khan I. A comprehensive report on convective flow of fractional (ABC) and (CF) MHD viscous fluid subject to generalized boundary conditions. Chaos Solitons Fractals. 2019;118:274–89.
[69] Jarad F, Abdeljawad T, Hammouch Z. On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative. Chaos Solitons Fractals. 2018;117:16–20.
[70] Imran MA, Ikram MD, Akgul A, Baleanu D. Effects of hybrid nanofluid on novel fractional model of heat transfer flow between two parallel plates. Alex Eng J. 2021;60:4.
[71] Ali R, Imran MA, Akgul A. An analysis of a mathematical fractional model of hybrid viscous nanofluids and its application in heat and mass transfer. Comput Appl Math. 2020.;383.
[72] Ahmad M, Imran MA, Baleanu D, Alshomrani AS. Thermal analysis of magnetohydrodynamic viscous fluid with innovative fractional derivative. Therm Sci. 2020;24(Suppl.1):S351–9.
[73] Imran MA, Rehman SU, Ahmadian A, Salahshour S, Salimi M. First solution of fractional bioconvection with power law kernel for vertical surface. Math. 2021;9(12):1366.
[74] Pedley TJ, Hill NA, Kessler JO. The growth of bioconvection patterns in a uniform suspension of gyrotactic micro-organisms. J Fluid Mech. 1988;195:223–37.
[75] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Prog Fract Differ Appl. 2015;1(2):73–85.
[76] Kiryakova V. The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. Comput Math Appl. 2010;59:1128–41.
[77] Raees A, Xu H, Liao SJ. Unsteady mixed nano-bioconvection flow in a horizontal channel with its upper plate expanding or contracting. Int J Heat Mass Transfer. 2015;86:174–82.
[78] Zhao Q, Xu H, Tao L. Unsteady bioconvection squeezing flow in a horizontal channel with chemical reaction and magnetic field effects. Math Probl Eng. 2017.;2017.
[79] Latiff NAA, Uddin MJ, Beg OA, Ismail AI. Unsteady forced bioconvection slip flow of a micropolar nanofluid from a stretching/shrinking sheet. Proc Inst Mech Eng N: J Nanomater Nanoeng Nanosyst. 2016;230(4):177–87.
[80] Ali L, Liu X, Ali B, Mujeed S, Abdal S. Finite element simulation of multi-slip effects on unsteady mhd bioconvective micropolar nanofluid flow over a sheet with solutal and thermal convective boundary conditions. Coat. 2019;9(12):842.
[81] Ahmed N, Vieru D, Fetecau C, Shah NA. Convective flows of generalized time-nonlocal nanofluids through a vertical rectangular channel. Phys Fluids. 2018;30(5):052002.
[82] Povstenko Y. Fractional thermoelasticity, solid mechanics and its applications. Switzerland: Springer International Publishing; 2015; 219.37.
[83] Hristov J. Derivatives with non-singular kernels. From the Caputo-Fabrizio definition and beyond: appraising analysis with emphasis on diffusion models. In: Bhalekar S, editor. Frontiers in fractional calculus. 1st ed. Bentham Science Publishers; 2017. p. 269–340. Chap. 10.

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Research Article

The Analysis of Fractional-Order System Delay Differential Equations Using a Numerical Method

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To solve fractional delay differential equation systems, the Laguerre Wavelets Method (LWM) is presented and coupled with the steps method in this article. Caputo fractional derivative is used in the proposed technique. The results show that the current procedure is accurate and reliable. Different nonlinear systems have been solved, and the results have been compared to the exact solution and different methods. Furthermore, it is clear from the figures that the LWM error converges quickly when compared to other approaches. When compared with the exact solution to other approaches, it is clear that LWM is more accurate and gets closer to the exact solution faster. Moreover, on the basis of the novelty and scientific importance, the present method can be extended to solve other nonlinear fractional-order delay differential equations.

1. Introduction

In 1965, a mathematician named L'Hopital asked Leibniz what would be the solution to the problem if the derivatives and integrals were fractional order. This L'Hopital question has resulted in the creation of new mathematical knowledge, but no one has been able to deal with it for a long time [1]. Mathematicians began to conduct study in the field of fractional derivatives, integration, and the development of a new field of fractional calculus after a period of time. In mathematics, this domain is known as fractional calculus, and it is a significant branch of mathematics that deals with the study of fractional derivatives and integration. Mathematicians have recently started working on fractional calculus because of its wide applications in all fields of research such as economics [2], viscoelastic materials [3], dynamics of interfaces between soft nanoparticles and rough substrates [4], continuum and statistical mechanics [5], solid mechanics [6], and many other topics.

Many natural problems can be solved using mathematical formulations by transforming physical facts into equation form. Differential equations (DEs) are a type of equation that is used to model a variety of phenomena. However, certain cases are too complicated to be solved using a differential equation. In this case, the researchers used fractional differential equations (FDEs), which are more accurate than differential equations with order integers in modelling the phenomenon. FDEs have realised the importance of real-world modelling challenges in recent years. Such as electrochemistry of corrosion [7], electrode-electrolyte polarization [8], heat conduction [9], optics and signal processing [10], diffusion wave [11], circuit systems [12], control theory of dynamical systems [6], probability and statistics [14, 15], fluid flow [16], and so on.

Equations with delayed arguments are known as fractional delay differential equations (FDDEs). Time delay, spatial delay, step size delay, constant delay, and so on are examples of delayed arguments. Due to various delay arguments found in nature, FDDEs are classified into distinct types. FDDEs are time delay DDEs, which are equations in which the current time derivatives are dependent on the solution and possibly its derivatives at a previous time. In the last few decades, mathematicians have paid more attention to FDDEs for modelling than simple ODEs, because a small delay has a big impact. FDDEs are employed in a variety of domains of mathematics, including infection diseases, navigation control, population dynamics, circulating blood, and the body's reaction to carbon dioxide [17–19], as well as some additional applications in advanced research studies.

It is necessary to develop accurate, time-efficient, and computationally efficient numerical algorithms for solving FDDEs. Xu and Ma [20] investigated the SEIRS epidemic model with a saturation incidence rate and a time delay that defined the latent period. Rihan et al. [21] investigated a delay differential model, numerically analysed it, and established an effective method of combining chemotherapy with therapeutic immunotherapy in 2014. The global stability of the Lotka–Volterra autonomous model with diffusion and time delay was studied by Beretta and Takeuchi [22]. Lv and Gao [23] used the well-known reproducing kernel Hilbert space approach to solve neutral functional proportional delay differential equations (RKHSM). Galach [24] investigated the time delay in the model presented by Kuznetsov and Taylor, where the time delay was included to gain better compatibility with reality. Furthermore, some researchers discussed the behaviour of delay fractional differential equations or a system of delay fractional differential equations, as well as their stability and analysis. Some works, such as in [25, 26], demonstrate this style of research.

In a number of situations, exact FDDEs solutions are difficult to get. As a result, the researchers' key goal is to develop a numerical or analytical solution to FDDEs. As a result, many strategies have been employed such as the New Predictor Corrector Method (NPCM) [27], New Iterative Method (NIM) [28], Adomian Decomposition Method (ADM) [29], Backward Differentiation Formula (BDF) [30], Chebyshev Pseudospectral Method (CPM) [31], Legendre–Gauss Collocation Method (LGCM) [32], Adams–Bashforth–Moulton Algorithm (ABMA) [33], operational matrix based on poly-Bernoulli polynomials (OMM) [34], and Runge Kutta-type Method (RKM) [35]. Overall, some of the approaches used to obtain numerical or analytical solutions to FDDEs have low accuracy of convergence, while others have great accuracy. Among all of these approaches, the wavelet approximation family is one of the more recent methods for locating FDDE solutions. For the approximate solution of FDDEs systems in the current study, we implement Laguerre Wavelets Method (LWM) in combination with the steps method. The proposed solution is shown to be entirely compatible with the complexity of such problems and to be extremely user-friendly. The error comparison shows that the suggested technique has a very high level of accuracy.

The structure of remaining paper is summarized as follows. Section 2 defines some basic definitions related to our present work. The general methodology for solving FDDEs is provided in Section 3. Section 4 presents the main results, numerical simulations, and graphical representations. The conclusion along with future research directions is drawn in Section 5.

2. Preliminaries Concept

This section introduces the basic concept and several important definitions from fractional calculus, which we will apply in our current research.

2.1. Definition. The following mathematical statement demonstrates Caputo's definition for fractional derivatives of order δ [36, 37].

$$D^{\delta}\xi(\psi) = \frac{1}{\Gamma(m-\delta)}\int_0^{\psi}(\psi-\tau)^{m-\delta-1}\xi^{(m)}(\tau)\mathrm{d}\tau, \tag{1}$$

for $n-1<\delta\leq m$, $m\in\mathbb{N}$, $\psi>0$, $\xi\in\mathbb{C}_{-1}^{n}$.

2.2. Definition. The Riemann–Liouville integral operator for order δ is given as [36, 37].

$$I^{\delta}\xi(\psi) = \frac{1}{\Gamma(\delta)}\int_0^{\psi}(\psi-\tau)^{\delta-1}\xi(\tau)\mathrm{d}\tau. \tag{2}$$

The following are the properties of the Caputo derivative and Riemann–Liouville integral operators.

$$D^{\delta}I^{\delta}\xi(\psi) = \xi(\psi),$$

$$I^{\delta}D^{\delta}\xi(\psi) = \xi(\psi) - \sum_{k=0}^{n-1}\frac{\xi^{(k)}(0^{+})}{k!}\psi^{k}, \quad \psi\geq 0\, n-1<\delta<n. \tag{3}$$

3. Laguerre Wavelets

Wavelets [38–40] are a family of functions made up of dilation and translation of a single function called the mother wavelet, $\varphi(\psi)$. The family of continuous wavelets [41] is formed when the dilation parameter a and the translation parameter b vary continuously.

$$\varphi_{a,b}(\psi) = |a|^{-1/2}\varphi\left(\frac{\psi-b}{a}\right), \quad a,b\in\mathbf{R}, a\neq 0. \tag{4}$$

The following family of discrete wavelets results from restricting the parameters a and b to discrete values as $a=a_0^{-p}$, $a=nb_0a_0^{-p}$, $a_0>1$, $b_0>0$,

$$\varphi_{p,n}(\psi) = |a|^{-p/2}\varphi\left(a_0^{p}(\psi)-nb_0\right), \quad p,n\in\mathbb{Z}, \tag{5}$$

Table 1: Comparison of the exact and MLWM solution for example 1 at $m = 9$.

ψ	Exact $\xi(\psi)$	Exact $\zeta(\psi)$	MLWM solution $\xi(\psi)$	MLWM solution $\zeta(\psi)$
0	1.000000000000000	0.000000000000000	1.000000000000000	0.000000000000000
0.1	0.900316999845194	0.998334166468281	0.900316999845194	0.998334166468281
0.2	0.802410647342520	0.198669330795061	0.802410647342527	0.198669330795061
0.3	0.707730678026351	0.295520206661339	0.707730678025662	0.295520206661339
0.4	0.617405647901646	0.389418342308650	0.617405647901653	0.389418342308637
0.5	0.532280730215671	0.479425538604203	0.532280730215273	0.479425538604203
0.6	0.452953789145250	.5646424733950353	0.452953789145497	.5646424733947035
0.7	0.379809389925154	0.644217687237691	0.37980938992536	0.644217687236876
0.8	0.313050504004480	0.717356090899522	0.313050504004346	0.717356090898501
0.9	0.252727753291169	0.783326909627483	0.252727753291868	0.783326909628249
1.0	0.198766110346413	0.841470984807896	0.198766110346480	0.841470984813237

Table 2: Error estimation of proposed method with FBPs for example 1 at $m = 9$.

ψ	Error (ξ_{MLWM})	Error (ζ_{MLWM})	Error (ξ_{FBPs})	Error (ζ_{FBPs})
0.2	7.1240361137E-15	2.1190342156276E-17	1.22E-11	3.55E-12
0.4	3.1090504913E-13	1.2920241091118E-14	9.91E-12	1.04E-11
0.6	4.0077803050E-12	3.3177676221433E-13	7.20E-12	1.59E-11
0.8	2.5602591191E-12	1.0209754593019E-12	6.56E-12	2.06E-11
1.0	4.9538867166E-11	5.3406078485367E-12	7.58E-10	1.47E-11

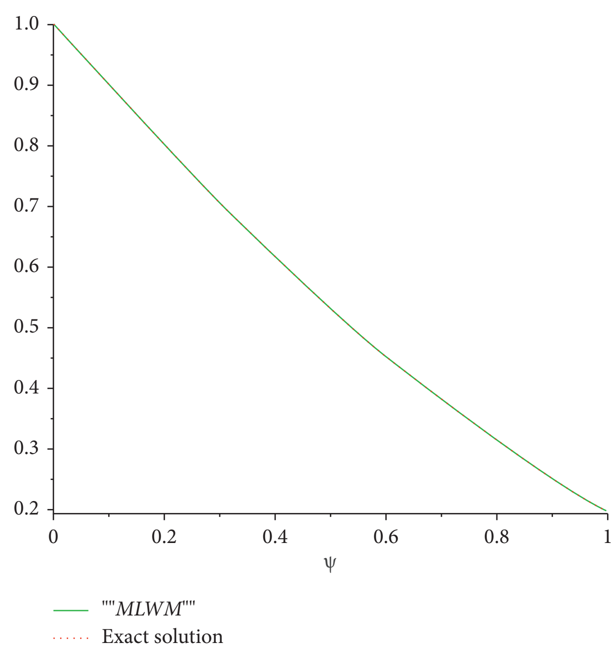

Figure 1: Behaviour of the exact solution and proposed method solution for $\xi(\psi)$ of problem 1.

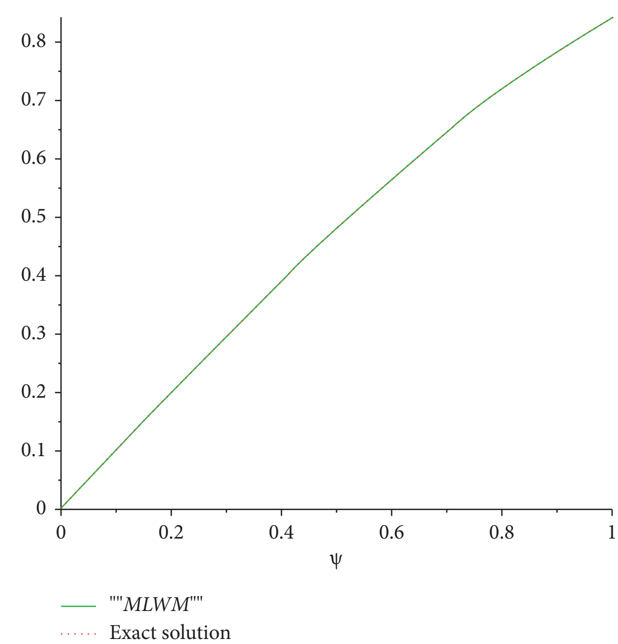

Figure 2: Behaviour of the exact solution and proposed method solution for $\zeta(\psi)$ of problem 1.

where the wavelet basis for $L^2(\mathbf{R})$ is $\varphi_{p,n}$. When $a_0 = 2$ and $b_0 = 1$, for instance, $\varphi_{p,n}(\psi)$ forms an orthonormal basis. There are four arguments in the Laguerre wavelets $\Phi_{n,m}(\psi) = \varphi(k, n, m, \psi)$, $n = 1, 2, \ldots, 2^{k-1}$, where k is non-negative integer, m represents the Laguerre polynomials degree, and represents normalized time. Over the interval $[0, 1)$, they are defined as

$$\varphi_{n,m} = \begin{cases} 2^{p/2}\widetilde{\mathscr{L}}_m\left(2^p\psi - 2n + 1\right), & \dfrac{n-1}{2^{p-1}} \le \psi < \dfrac{n}{2^{p-1}}, \\ 0, & \text{Otherwise}, \end{cases} \tag{6}$$

where

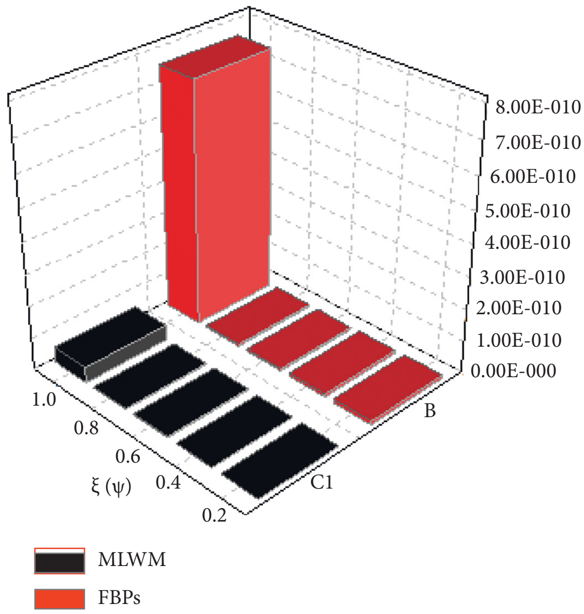

FIGURE 3: FBPs and proposed method error analysis for $\xi(\psi)$ of example 1.

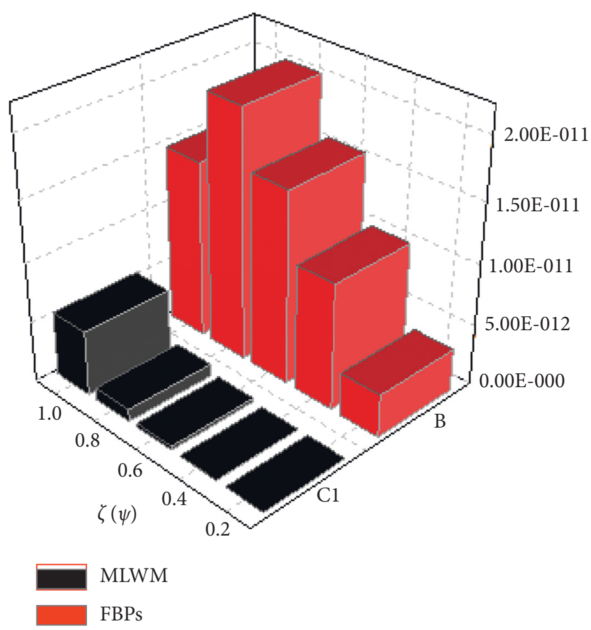

FIGURE 4: FBPs and proposed method error analysis for $\zeta(\psi)$ of example 1.

$$\widetilde{\mathscr{L}}_m = \frac{1}{m!}\mathscr{L}_m(\psi) \quad m = 0, 1, 2, \ldots A - 1. \tag{7}$$

$m = 0, 1, 2, \ldots, M - 1$. The coefficients are utilised in (10) to determine orthonormality. The Laguerre polynomials having degree m with regard to $w(\psi) = 1$ weight function on the interval $[0, \infty]$ are $L_m(\psi)$ and satisfy the recursive formula:

$$\mathscr{L}_0(\psi) = 1, \ \mathscr{L}_1(\psi) = 1 - \psi,$$
$$\mathscr{L}_{m+2} = \frac{((2m + 3 - x)\mathscr{L}_{m+1}(\psi) - (m + 1)\mathscr{L}_m)}{m + 2} \quad m = 0, 1, 2, 3, 4, \ldots. \tag{8}$$

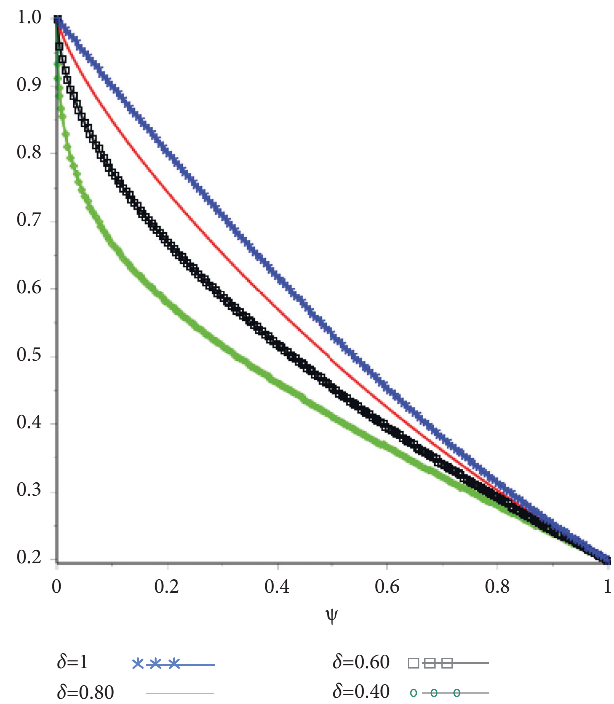

FIGURE 5: The error comparison at various fractional-orders of for $\xi(\psi)$ example 1.

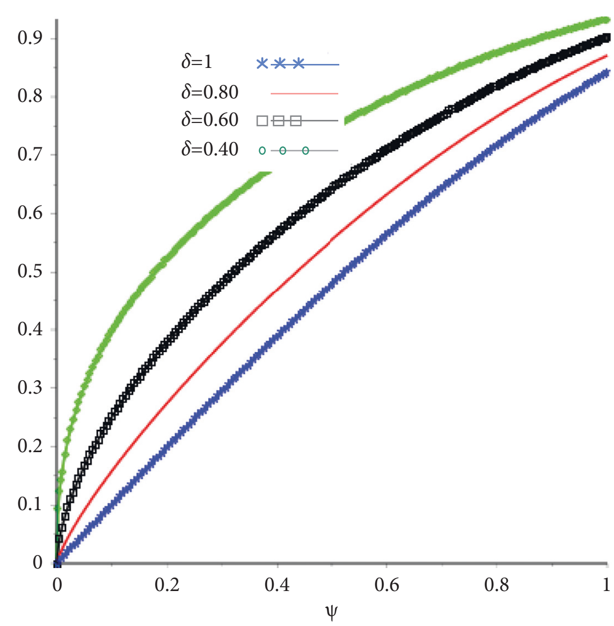

FIGURE 6: The error comparison at various fractional-orders of for $\zeta(\psi)$ example 1.

where

Modified Laguerre wavelets method (MLWM): Here, we consider the delay differential equation of the form:

$$y^{\alpha}(\psi) = f(\psi) + g(\psi)y\left(\frac{\psi}{a} - c\right), \quad 0 < \psi < b, 0 < \alpha \le 1,$$
$$y(\psi) = p(\psi), \quad -b \le \psi \le 0, \tag{9}$$

Table 3: Comparison at different fractional-order of δ on the basis of error for example 2.

δ	ψ	Exact	$MLWM$ solution	$MLWM$ error	Spline functions
	0.01	0.0001	0.00009999986375	1.3625E-10	8.2E-4
	0.02	0.0004	0.0003999998553	1.447E-10	2.5E-3
0.1	0.03	0.0009	0.0008999998624	1.376E-10	4.7E-3
	0.04	0.0016	0.001599999883	1.17E-10	7.3E-3
	0.05	0.0025	0.002499999813	1.87E-10	1.0E-2
	0.01	0.0001	0.0001000000111	1.11E-11	4.4E-4
	0.02	0.0004	0.0004000001035	1.035E-10	1.4E-3
0.2	0.03	0.0009	0.0008999999845	1.55E-11	2.7E-3
	0.04	0.0016	0.001600000057	5.7E-11	4.4E-3
	0.05	0.0025	0.002500000022	2.2E-11	6.1E-3
	0.01	0.0001	0.0001000001798	1.798E-10	2.1E-4
	0.02	0.0004	0.0004000002303	2.303E-10	7.1E-4
0.3	0.03	0.0009	0.0009000001717	1.717E-10	1.4E-3
	0.04	0.0016	0.001600000106	1.06E-10	2.4E-3
	0.05	0.0025	0.002500000136	1.36E-10	3.5E-3
	0.01	0.0001	0.0001000000189	1.89E-11	8.1E-5
	0.02	0.0004	0.0003999999502	4.98E-11	2.9E-4
0.4	0.03	0.0009	0.0008999998916	1.084E-10	6.1E-4
	0.04	0.0016	0.001599999941	5.9E-11	1.0E-3
	0.05	0.0025	0.002499999997	3.000E-12	1.0E-3
	0.01	0.0001	0.00009999988074	1.1926E-10	4.5E-6
	0.02	0.0004	0.0003999997780	2.220E-10	2.6E-5
0.5	0.03	0.0009	0.0008999996906	3.094E-10	7.0E-5
	0.04	0.0016	0.001599999717	3.83E-10	1.4E-4
	0.05	0.0025	0.002499999657	3.43E-10	2.5E-4

Table 4: Comparison at different fractional-orders of δ on the basis of error for example 2.

δ	ψ	Exact	$MLWM$ solution	$MLWM$ error	Spline functions
	0.01	0.0001	0.00009999986375	1.3625E-10	8.2E-4
	0.02	0.0004	0.0003999998553	1.447E-10	2.5E-3
	0.03	0.0009	0.0008999998624	1.376E-10	4.7E-3
	0.04	0.0016	0.001599999883	1.17E-10	7.3E-3
0.1	0.05	0.0025	0.002499999813	1.87E-10	1.0E-2
	0.01	0.0001	0.0001000000111	1.11E-11	4.4E-4
	0.02	0.0004	0.0004000001035	1.035E-10	1.4E-3
	0.03	0.0009	0.0008999999845	1.55E-11	2.7E-3
	0.04	0.0016	0.001600000057	5.7E-11	4.4E-3
0.2	0.05	0.0025	0.002500000022	2.2E-11	6.1E-3
	0.01	0.0001	0.0001000001798	1.798E-10	2.1E-4
	0.02	0.0004	0.0004000002303	2.303E-10	7.1E-4
	0.03	0.0009	0.0009000001717	1.717E-10	1.4E-3
	0.04	0.0016	0.001600000106	1.06E-10	2.4E-3
0.3	0.05	0.0025	0.002500000136	1.36E-10	3.5E-3
	0.01	0.0001	0.0001000000189	1.89E-11	8.1E-5
	0.02	0.0004	0.0003999999502	4.98E-11	2.9E-4
	0.03	0.0009	0.0008999998916	1.084E-10	6.1E-4
	0.04	0.0016	0.001599999941	5.9E-11	1.0E-3
0.4	0.05	0.0025	0.002499999997	3.000E-12	1.0E-3
	0.01	0.0001	0.00009999988074	1.1926E-10	4.5E-6
	0.02	0.0004	0.0003999997780	2.220E-10	2.6E-5
	0.03	0.0009	0.0008999996906	3.094E-10	7.0E-5
0.5	0.04	0.0016	0.001599999717	2.83E-10	1.4E-4
	0.05	0.0025	0.002499999657	3.43E-10	2.5E-4

where $f(\psi)$ is a provided continuous linear or nonlinear function and $g(\psi)$ is a source term function. Using the proposed method, transform the delay differential equation (12) to an inhomogeneous ordinary differential equation by using the initial source, $p(\psi)$, as shown in (12):

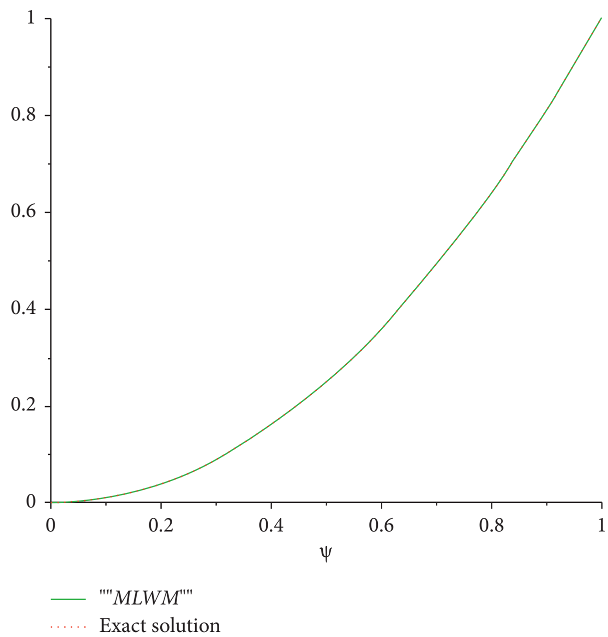

FIGURE 7: Analysis of the exact and proposed method solution for $\xi(\psi)$ of problem 2.

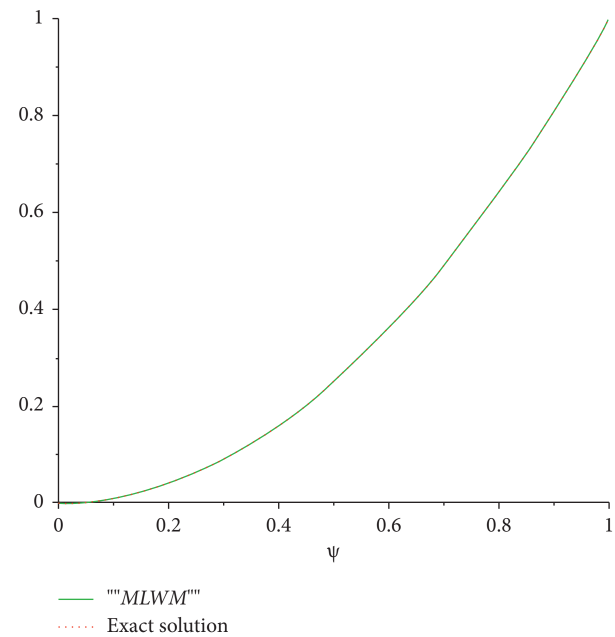

FIGURE 8: Analysis of the exact and proposed method solution for $\zeta(\psi)$ of problem 2.

$$y^{\alpha}(\psi) = f(\psi) + g(\psi)p\left(\frac{\psi}{a} - c\right), \quad 0 < \psi < b, 1 < \alpha \leq 2. \tag{10}$$

Equation (14) can be expanded as a Laguerre wavelets series as follows:

$$y(\psi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{n,m}\varphi_{n,m}(\psi), \tag{11}$$

where $\varphi_{n,m}(\psi)$ is determined by (9). The truncated series is used to approximate $y(\psi)$.

$$y_p, A = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{n,m}\varphi_{n,m}(\psi), \tag{12}$$

Then, there should be a total of $2^{p-1}A$ conditions for determining the $2^{p-1}A$ coefficient:

$$c_{10}, c_{11} \ldots c_{A-1} \ldots c_{20}, c_{2A-1} \ldots c_{2^{p-1}1} \ldots c_{2^{p-1}A-1}. \tag{13}$$

Since the initial and boundary conditions, respectively, provide the conditions.

$$y_p, A(0) = \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-1} d_{n,m}\varphi_{n,m}(0) = q(0). \tag{14}$$

$$\frac{d}{d\psi} y_p, A(1) = \frac{d}{d\psi} \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-1} d_{n,m}\varphi_{n,m}(1) = \acute{q}(1). \tag{15}$$

We see that there should be $2^{p-1}A - 2$ extra condition to recover the unknown coefficient $d_{n,m}$. These conditions can be obtained by substituting (14) in (12):

$$\frac{d^{\alpha}}{d\psi^{\alpha}} \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi) = f\left(\sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi)\right) + g(\psi)p\left(\frac{\psi}{a} - c\right). \tag{16}$$

We, now assume equation (18) is exact at $2^{p-1}A - 3$ points ψ_i as follows:

$$\frac{d^{\alpha}}{d\psi^{\alpha}} \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi_i) = f\left(\sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi_i)\right) + g(\psi_i)p\left(\frac{\psi_i}{a} - c\right). \tag{17}$$

The best choice of the ψ_i points are the zeros of the shifted Laguerre polynomials of degree $2^{p-1}A - 2$ in the interval [0, 1] that is $\psi_i = s_i - 1/2$, where $s_i = \cos((2_i - 1)\pi/2^{p-1}A - 1), i = 1, 2, 3, \ldots 2^{p-1}A - 2$. Since the initial and boundary conditions, respectively, provide the conditions. Combining equations (9) and (12) yields $2^{p-1}A$ linear equations from which the unknown coefficients, $d_{n,m}$, can be computed. The same technique is followed for first- and second-order delay differential equations.

4. Numerical Representation

4.1. Example. Consider the system of fractional ordinary delay differential equations [42],

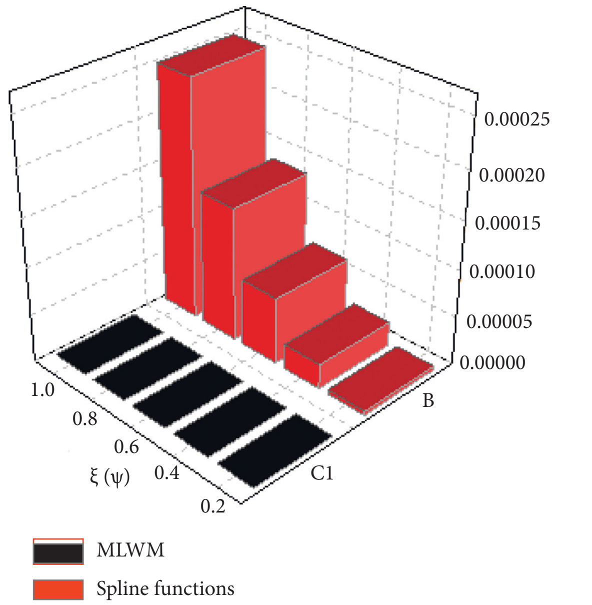

FIGURE 9: Spline functions and proposed method error analysis for $\xi(\psi)$ of example 2.

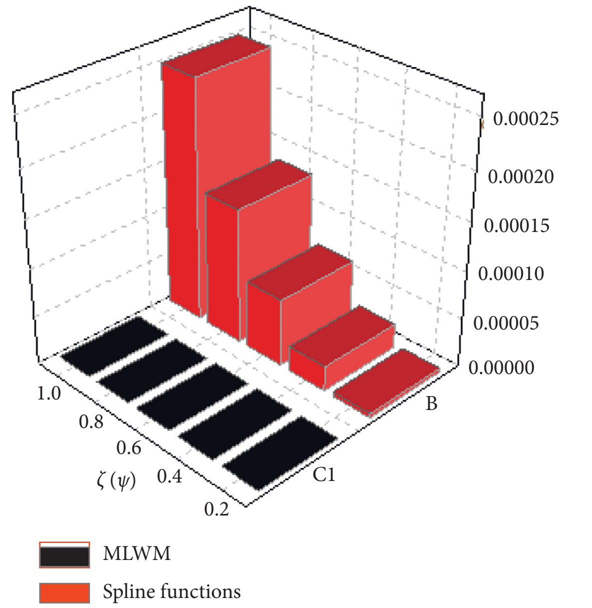

FIGURE 10: Spline functions and proposed method error analysis for $\zeta(\psi)$ of example 2.

$$
\begin{aligned}
D^{\delta}\xi(\psi) &= -\zeta(\psi) - 2e^{-\frac{3}{4}\psi}\cos\left(\frac{1}{2}\psi\right)\sin\left(\frac{1}{4}\psi\right)\xi(0.25\psi) \\
&\quad - e^{-\psi}\cos\left(\frac{1}{2}\psi\right)\zeta(0.5\psi), \\
D^{\delta}\zeta(\psi) &= e^{\psi}\xi^{2}(0.5\psi) - \zeta^{2}(0.5\psi),
\end{aligned}
\tag{18}
$$

with the initial sources $\xi(0) = 1$, $\zeta(0) = 0$, and having exact solution at $\delta = 1$ as $\xi(\psi) = e^{-\psi}\cos(\psi)$, $\zeta(\psi) = \sin(\psi)$.

Table 1 shows the exact solution and numerical results achieved using the proposed method. Table 2 shows the comparison on the basis of absolute error between our technique and those derived from FBPs. When $\delta = 1$, the behaviour of the exact solution and proposed method solution of this problem is shown in Figures 1 and 2, respectively, whereas the error comparison of CPM and FBPs is shown in Figures 3 and 4. Figures 5 and 6 show graphical representations for different fractional orders of δ, confirming that the proposed method solution converges to the exact solution as the value of δ approaches from fractional-order towards integer-order.

4.2. Example. Consider the system of fractional ordinary delay differential equations [43].

$$
\begin{aligned}
D^{\delta}\xi(\psi) &= -\xi(\psi) + \zeta\left(\frac{\psi}{2}\right) + \frac{3}{4}\psi^{2} + \frac{2}{\Gamma(3-\delta)}\psi^{2-\delta}, \\
D^{\delta}\zeta(\psi) &= \zeta(\psi) - \xi\left(\frac{\psi}{2}\right) - \frac{3}{4}\psi^{2} + \frac{2}{\Gamma(3-\delta)}\psi^{2-\delta}.
\end{aligned}
\tag{19}
$$

The exact solution is given by $\xi(\psi) = \psi^{2}$ and $\zeta(\psi) = \psi^{2}$.

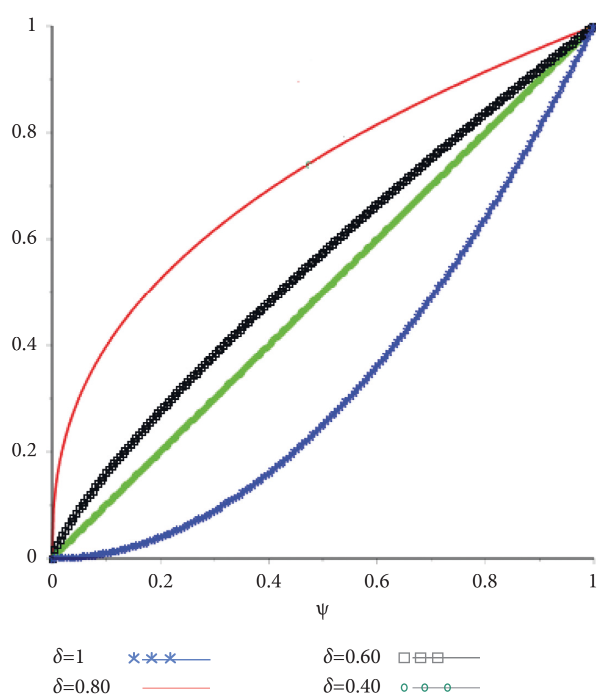

FIGURE 11: The error comparison at various fractional-orders for $\xi(\psi)$ of example 2.

The comparison among the exact solution and the Spline function polynomial technique solution are shown in Table 3. In Table 4, the errors acquired by the current technique are compared to those obtained by the Spline function polynomial method. In Figures 7 and 8, we compare the exact and approximated solutions, which shows that they are very close to each other. In addition, Figures 9 and 10 show the MLWM and Spline function error comparisons,

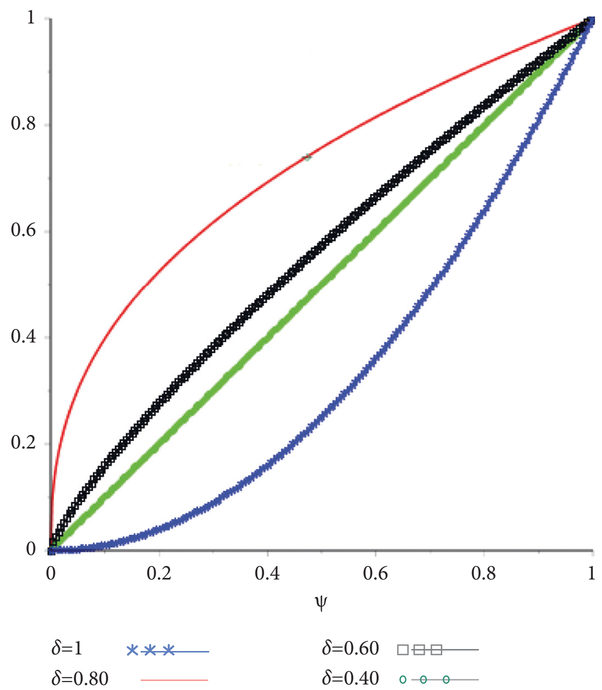

Figure 12: The error comparison at various fractional-orders for $\zeta(\psi)$ of example 2.

demonstrating that suggested approach is in best agreement with the exact solution.

5. Conclusion

We used the MLWM to solve fractional delay differential equations systems in this research. The proposed method's convergence is given special consideration. As demonstrated in Figures 1–12, the fractional-order delay differential equation solution approaches towards the solution of the integer-order delay differential equation. The results obtained by implementing the proposed method are in great agreement with the exact solution and are more accurate than those obtained by implementing other techniques. The proposed method (MLWM) is extremely user-friendly but extremely accurate, according to computational effort and numerical results. The computations work in this article are done using Maple.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors jointly worked on the results, and they read and approved the final manuscript.

References

[1] L. Adam, "Fractional Calculus: History, Definitions and Applications for the Engineer," *Rapport technique*, pp. 1–28, Univeristy of Notre Dame: Department of Aerospace and Mechanical Engineering, Notre Dame, In, USA, 2004.

[2] R. T. Baillie, "Long memory processes and fractional integration in econometrics," *Journal of Econometrics*, vol. 73, no. 1, pp. 5–59, 1996.

[3] R. L. Bagley and P. J. Torvik, "Fractional calculus in the transient analysis of viscoelastically damped structures," *AIAA Journal*, vol. 23, no. 6, pp. 918–925, 1985.

[4] T. S. Chow, "Fractional dynamics of interfaces between soft-nanoparticles and rough substrates," *Physics Letters A*, vol. 342, no. 1-2, pp. 148–155, 2005.

[5] F. Mainardi, "Fractional calculus," in *Fractals and Fractional Calculus in Contin- Uum Mechanics*, A. Carpinteri and F. Mainardi, Eds., Springer-Verlag, New York, NY, USA, pp. 291–348, 1997.

[6] Y. A. Rossikhin and M. V. Shitikova, "Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids," *Applied Mechanics Reviews*, vol. 50, no. 1, pp. 15–67, 1997.

[7] K. B. Oldham, "The reformulation of an infinite sum via semiintegration," *SIAM Journal on Mathematical Analysis*, vol. 14, no. 5, pp. 974–981, 1983.

[8] W. H. Deng and C. P. Li, "Chaos synchronization of the fractional Lü system," *Physica A: Statistical Mechanics and Its Applications*, vol. 353, pp. 61–72, 2005.

[9] Y. Z. Povstenko, "Thermoelasticity that uses fractional heat conduction equation," *Journal of Mathematical Sciences*, vol. 162, no. 2, pp. 296–305, 2009.

[10] E. Baskin and A. Iomin, "Electro-chemical manifestation of nanoplasmonics in fractal media," *Open Physics*, vol. 11, no. 6, pp. 676–684, 2013.

[11] H. M. Srivastava, R. Shah, H. Khan, and M. Arif, "Some analytical and numerical investigation of a family of fractional–order Helmholtz equations in two space dimensions," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 1, pp. 199–212, 2020.

[12] T. T. Hartley, C. F. Lorenzo, and H. Killory Qammer, "Chaos in a fractional order Chua's system," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 485–490, 1995.

[13] H. Khan, U. Farooq, R. Shah, D. Baleanu, P. Kumam, and M. Arif, "Analytical solutions of (2+ time fractional order) dimensional physical models, using modified decomposition method," *Applied Sciences*, vol. 10, no. 1, p. 122, 2019.

[14] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, . *Theory and applications of frac- tional differential equations*, Vol. 204, Elsevier Science Limited, Netherlands, 2006.

[15] I. B. Bapna and N. Mathur, "Application of fractional calculus in statistics," *Int. J. Contemp. Math. Sciences*, vol. 7, no. 18, pp. 849–856, 2012.

[16] V. V. Kulish and J. L. Lage, "Application of fractional calculus to fluid mechanics," *Journal of Fluids Engineering*, vol. 124, no. 3, pp. 803–806, 2002.

[17] F. A. Rihan, C. Tunc, S. H. Saker, S. Lakshmanan, and R. Rakkiyappan, "Applications of Delay Differential Equations in Biological Systems," *Complexity*, vol. 2018, Article ID 4584389, 3 pages, 2018.

[18] Y. Ding and H. Ye, "A fractional-order differential equation model of hiv infection of cd4+ t-cells," *Mathematical and Computer Modelling*, vol. 50, no. 3-4, pp. 386–392, 2009.

[19] V. Daftardar-Gejji, S. Bhalekar, and P. Gade, "The comparative study for solving fractional-order fornberg-whitham equation via ρ-laplace transform," *Symmtery*, vol. 79, no. 1, pp. 61–69.
[20] R. Xu and Z. Ma, "Numerical investigation of fractional-order Swift-Hohenberg equations via a Novel transform," *Symmtery*, vol. 61, pp. 229–239.
[21] F. A. Rihan, D. H. Abdelrahman, F. Al-Maskari, F. Ibrahim, and M. A. Abdeen, "Delay differential model for tumour-immune response with chemoimmunotherapy and optimal control," *Computational and Mathematical Methods in Medicine*, vol. 2014, Article ID 982978, 15 pages, 2014.
[22] E. Beretta and Y. Takeuchi, "Global stability of single-species diffusion Volterra models with continuous time delays," *Bulletin of Mathematical Biology*, vol. 49, no. 4, pp. 431–448, 1987.
[23] X. Lv and Y. Gao, "The RKHSM for solving neutral functional-differential equations with proportional delays," *Mathematical Methods in the Applied Sciences*, vol. 36, no. 6, pp. 642–649, 2013.
[24] M. Galach, "An analytical technique, based on natural transform to solve fractional-order parabolic equations," *Entropy*, vol. 13, pp. 395–406.
[25] D. Baleanu, R. L. Magin, S. Bhalekar, and V. Daftardar-Gejji, "Chaos in the fractional order nonlinear Bloch equation with delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 25, no. 1-3, pp. 41–49, 2015.
[26] B. Parsa Moghaddam, S. Yaghoobi, and J. A. Tenreiro Machado, "An extended predictor-corrector algorithm for variable-order fractional delay differential equations," *Journal of Computational and Nonlinear Dynamics*, vol. 11, no. 6, 2016.
[27] V. Daftardar-Gejji, Y. Sukale, and S. Bhalekar, "A new predictor-corrector method for fractional differential equations," *Applied Mathematics and Computation*, vol. 244, pp. 158–182, 2014.
[28] F. Awawdeh, "On new iterative method for solving systems of nonlinear equations," *Numerical Algorithms*, vol. 54, no. 3, pp. 395–409, 2010.
[29] D. J. Evans and K. R. Raslan, "The Adomian decomposition method for solving delay differential equation," *International Journal of Computer Mathematics*, vol. 82, no. 1, pp. 49–54, 2005.
[30] Z. B. Ibrahim, K. I. Othman, and M. Suleiman, "Implicit r-point block backward differentiation formula for solving first-order stiff ODEs," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 558–565, 2007.
[31] P. Sunthrayuth, R. Ullah, A. Khan et al., "Numerical analysis of the fractional-order nonlinear system of Volterra integro-differential equations," *Journal of Function Spaces*, vol. 2021, Article ID 1537958, 10 pages, 2021.
[32] S. A. Rakhshan and S. Effati, "A generalized Legendre-Gauss collocation method for solving nonlinear fractional differential equations with time varying delays," *Applied Numerical Mathematics*, vol. 146, pp. 342–360, 2019.
[33] D. Aksim and D. Pavlov, "On the extension of adams-bashforth-moulton methods for numerical integration of delay differential equations and application to the moon's orbit," *Mathematics in Computer Science*, vol. 14, no. 1, pp. 103–109, 2020.
[34] C. Phang, Y. T. Toh, and F. S. Md Nasrudin, "An operational matrix method based on poly-Bernoulli polynomials for solving fractional delay differential equations," *Computation*, vol. 8, no. 3, p. 82, 2020.
[35] N. Senu, K. C. Lee, A. Ahmadian, and S. N. I. Ibrahim, "Numerical solution of delay differential equation using two-derivative Runge-Kutta type method with Newton interpolation," *Alexandria Engineering Journal*, vol. 61, no. 8, pp. 5819–5835, 2022.
[36] M. Yi, L. Wang, and J. Huang, "Legendre wavelets method for the numerical solution of fractional integro-differential equations with weakly singular kernel," *Applied Mathematical Modelling*, vol. 40, no. 4, pp. 3422–3437, 2016.
[37] J.-H. He, Z.-B. Li, and Q.-l. Wang, "A new fractional derivative and its application to explanation of polar bear hairs," *Journal of King Saud University Science*, vol. 28, no. 2, pp. 190–192, 2016.
[38] J. H. He, "Some applications of nonlinear fractional differentialequations and their approximations," *Bulletin of Science and Technology*, vol. 15, no. 2, p. 86e90, 1999.
[39] R. L. Bagley, P. J. Torvik, N. H. Aljahdaly, R. P. Agarwal, R. Shah, and T. Botmart, "Analysis of the time fractional-order coupled burgers equations with non-singular kernel operators," *Mathematics*, vol. 9, no. 18, p. 2326, 2021.
[40] R. Panda and M. Dash, "On solutions of fractional-order gas dynamics equation by effective techniques," *Journal of function space*, vol. 8.
[41] T. Insperger and G. Stepan, "Remote control of periodic robotic motion," in *Proceedings of the Thirt Symp Theory and Practice of Robots and Manipulators*, Zakopane, Poland, Article ID 197e203, 2000.
[42] S. Davaeifar and J. Rashidinia, "Solution of a system of delay differential equations of multi pantograph type," *Journal of Taibah University for Science*, vol. 11, no. 6, pp. 1141–1157, 2017.
[43] M. N. Sherif, "Numerical solution of system of fractional delay differential equations using polynomial spline functions," *Applied Mathematics*, vol. 7, no. 6, pp. 518–526, 2016.

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Impact of nanoparticle aggregation on heat transfer phenomena of second grade nanofluid flow over melting surface subject to homogeneous-heterogeneous reactions

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ABSTRACT

It is well known that the inclusion of a certain quantity of nanoparticles boosts the thermal conductivity of the nanofluid. The reason for this tremendous improvement is yet unknown. Consequently, finding the proper thermal effect of nanoscale particles requires an understanding of nanoparticle aggregation kinematics. The utilization of nanomaterials may be seen in a variety of technological and industrial applications. The influence of homogeneous and heterogeneous chemical reactions on an incompressible flow of second-grade nanofluid through a stretched cylinder with NP aggregation is investigated in this work. Similarity transformations are used to change partial differential equations (PDEs) into a system of ordinary differential equations (ODEs). The Runge Kutta Fehlberg fourth fifth-order (RKF 45) technique and shooting approach are used to numerically solve these ODEs. The influence of major elements on flow fields and heat transfer rates is investigated and addressed using graphical representations. The results suggest that the fluid flow without NPs aggregation has better heat transmission than when the melting parameter increases. Furthermore, the higher mass transfer for fluid flow with aggregation condition is detected for increased values of strength of heterogeneous and homogeneous reaction parameters.

Nomenclature:

α_1	Normal stress moduli
α	Thermal diffusivity
T_∞	Ambient temperature
k	Thermal conductivity

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k_r *and* k_s	Rate constant
T_0	Solid temperature
R	Radius
Sc	Schmidt number
l	Reference length
$f'(\eta)$	Dimensionless velocity profile
M	Melting parameter
ψ	Stream function
u	& v Constituents of Velocity
Nu	Nusselt number
T	Temperature
T_m	Temperature of the melting surface
β	Second grade fluid parameter
Pr	Prandtl number
ρ	Density
U_w	Stretching velocity
x, r	Directions
ρC_p	Heat capacitance
A^*	and B^* Chemical species
D_A	and D_B Diffusion coefficients
I	The identity tensor
ν	Dynamic viscosity
λ	Latent heat of the fluid
A_1	and A_2 The first two Rivlin- Ericksen tensors
η	Dimensionless variable
α_2	and $\alpha_2{}^*$ The cross-viscosity and viscoelasticity of the fluid
C_s	Heat capacity of the solid surface
$\theta(\eta)$	Dimensionless thermal profile
φ	Volume fraction
$-pI$	Spherical stress due to the incompressibility constraint
Re	Local Reynolds number
γ	Curvature parameter
μ	Dynamic viscosity
H	Strength of homogeneous reaction parameters
K	Strength of heterogeneous reaction parameters

Subscripts

f	Fluid
agg	Aggregate
nf	Nanofluid
s	solid nanoparticle

1. Introduction

Nanofluids are heat transfer fluids with nanoparticles (NPs) suspended in the liquid. Water, organic fluids, motor oil, bio-liquids, and other basic liquids are usually found in base liquids, which are usually made up of these and other base liquids. Nanoparticles with a significantly higher surface area and smaller size have the potential to substantially enhance heat-transfer capabilities and fluid stability. Choi was the first person to look into improving the thermal conductivity of nanofluids. Nadeem et al. [1] swotted the forced bio-convection stream of a viscous nanofluid. Here, they are mainly concentrated on Brownian motion and thermophoresis diffusion characteristics of NPs and conclude that their study is significant for the enlightening achievement of microbial fuel cells. Ahmad and Nadeem [2,3] swotted the mass and heat transport behavior of single and multiwalled carbon nanotubes suspended in different base fluids using the non-Fourier heat flux model. They declared that hybrid nanofluid offers a higher thermal conductivity and heat transport capability than simple nanofluid. Ahmad et al. [4,5] swotted the heat and mass transport behavior of different nanoparticles suspended in different base fluids by using different nanoliquid models. These papers majorly concentrated on the behaviour of improved volume fraction on the mass and heat transport rate. Alzahrani et al. [6] quizzed the wall jet stream of a Casson nanoliquid with thermal radiation across a stretchable surface. They have done a relative study on two dissimilar kinds of nanoliquids. Khan et al. [7] considered the three different NPs, such as copper, titania and alumina with water as the base liquid to discuss the viscous nanofluid flow. Here, they swotted the impact of volume fraction on flow and heat transport. Talbi et al. [8] used a moving permeable longitudinal fin to convey the thermal radiative streams of hybrid nanofluid. Boundary circumstances in the case of convective and insulated tips. They found that the thermal profile decreases for all investigated shape parameters. Ouni et al. [9] done a case study on gold-copper/engine oil based nanofluid passes through parabolic trough solar collector. They determined that hybrid nanofluid offers a advanced thermal conductivity and heat transport capability than nanofluid. Jamshed et al. [10] discussed the single-phase optimized entropy analysis on hybrid nanoliquid flow using solar energy. They concluded that ethylene glycol-based hybrid nanofluid offers a higher thermal conductivity and heat transfer capability than ethylene glycol based nanofluid.

In addition, experiments show that aggregation of NPs play a big role in the rheological and thermal properties. Because of the surface charge and van der Waals force, NPs form groups. These NPs also form percolating networks and linear chains to make it easier for heat to move through them. Heat moves faster in NPs aggregation than in nanoparticles, so this is why it moves more quickly. Motlagh and Kalteh [11] analyzed the heat transport in a nanochannel by considering the shape and aggregation of NPs. Here, they declared that nanoliquid velocity is exaggerated by the aggregation and shape of the NPs. Swain and Mahanthesh [12] educed the Joule heating and NPs aggregation impact on the nanoliquid flow. It has been shown that aggregating NPs increases the thermal field dramatically. Sabu et al. [13] conferred the kinematics of nanoparticle aggregation in a nanomaterial stream through an angled flat plate. The velocity profile was drastically reduced as the plate's tilt increased, but the temperature profile strengthened. Mahanthesh [14] quizzed the heat conveyance of NPs with aggregation of NPs. Here, they declared that nanoliquid velocity is exaggerated by NPs aggregation. Mackolil and Mahanthesh [15] swotted the magnetic field impact on the Marangoni convective stream of nanoliquid by considering NPs aggregation. They declared that nanofluid velocity is affected by NPs aggregation and magnetic field.

When it comes to modern scientific research, non-Newtonian fluids are quickly becoming one of the most important topics. This is because they have many practical applications, such as in manufacturing processes and in industries like plastics processing and lubrication and the mining industry. There are three types of liquids; differential, integral, and rate type. Differential type liquids are called viscoelastic fluids, and they have a lot of different properties than viscous liquids. The second-grade fluid (SGF) is known as a viscoelastic non-Newtonian fluid subclass because it is very elastic and doesn't flow very quickly. Today, many researchers are interested in studying how SGF flows. Nadeem et al. [16] educed the influence of Newtonian heating on viscoelastic liquid on a sheet using non-Fourier's law. It was discovered that increasing the viscoelastic parameter causes axial velocity to rise. The influence of a magnetic field on a SGF flow traveling through a surface was investigated by Hayat et al. [17]. It was discovered that increasing the SGF parameter causes heat transfer enhancement. Anwar et al. [18] quizzed the upshot of radiation on a SGF passing through a plate. It was discovered that increasing the second-grade liquid parameter causes velocity to decline. Gowda et al. [19,20] analyzed the upshot of Stefan blowing on the SGF stream past a diverse surface. It was discovered that increasing the second-grade liquid parameter causes heat transfer enhancement.

There are going to be a lot of fluid dynamics problems with how fluid moves around a cylinder in a lot of different situations. These include things like electronic cooling systems, oil drilling and heat exchanges on the rotating tube. There have been a lot of studies about how the cylinder shows how heat and mass transfer and where it moves. Sowmya et al. [21] quizzed the dusty fluid flow on a stretching cylinder with thermal radiation. They found that when the curvature parameter rises, the heat transport rate improves. Kumar et al. [22] employed a non-Fourier's law to delineate the nanoliquid flow past an extending cylinder. The PDE equations that represent the hybrid nanofluid flow on a stretching cylinder rooted in a porously natured medium was quizzed by Gowda et al. [23]. Song et al. [24] examined the flow of Williamson nanoliquid generated by a stretching cylinder. They concluded that increasing the curvature parameter's cumulative values increases the velocity and heat transport inside the boundary. Redouane et al. [25] utilized Brinkman–Forchheimer model to study the nanofluid flow induced a cylinder while accounting for porous medium. They concluded that a positive rotating speed has a beneficial impact on heat transfer dispersion over the cavity.

Burning, biochemical frameworks, and catalysis are examples of chemically reacting systems that include homogeneous-heterogeneous (H–H) processes. The relationship between homogeneous and heterogeneous processes is particularly puzzling. Except in the presence of a catalyst, a percentage of the reactions have the capacity to ensue gradually. Chemical reactions are also used in food processing, the hydrometallurgical sector, dispersion and fog generation, polymer manufacturing, and ceramics, among other uses. Nadeem et al. [26] quizzed the H–H reactions in the Maxwell liquid stream across surface. They came to the conclusion that the strength of H–H reactions had less impact on the concentration field. Rashid et al. [27] swotted how H–H reactions affected the Oldroyd-B liquid flow on a permeable material. Hayat et al. [28] conferred the upshot of H–H reactions on a fluid stream embedded with CNTs flowing across an exponentially increasing surface. Abbas et al. [29] educed the upshot of H–H reactions on a fluid stream initiated by a rotating disk with a heat sink or generator. They came to the conclusion that the strength of H–H reactions had less impact on the concentration field. Bejawada et al. [30] quizzed the chemically reacting flow of a viscous magneto nanofluid. Here, they mainly concentrate on the upshot of pertinent parameters on heat and mass transport behaviour.

Melting is a phase shift that is accompanied by thermal energy absorption. Oil extraction, geothermal energy recovery, magma solidification, permafrost melting, and semi-conductor material preparation are just a few of the thermal engineering uses of melting heat transmission in various liquids. Epstein paved the way for melting transfer research by devising a straightforward and systematic method for determining melting rates in complicated flow situations. Khan et al. [31] simulated the Falkner-Skan nanofluid flow while accounting for the melting process. The cross nanoliquid velocity increases as the melting and velocity ratio parameters increase, according to their statistical data. The thermophysical properties of MHD nanofluid are quizzed by Amirsom et al. [32]. The melting heat transport capacities of an unstable compressed nanoliquid moving through a Darcy porous material were studied by Farooq et al. [33]. They also considered the stretched sheet's melting potential. The melting heat transport in $Cu–Fe_2SO_4– H_20$ hybrid nanofluid was investigated by Radhika et al. [34]. Mabood [35] evaluated melting heat transfer using a nanofluid model. Hybrid nanofluids have been shown to have reduced entropy production rates.

Ongoing through the above-cited articles, we inspect the effect of H–H chemical reactions effect on an incompressible stream of second grade nanoliquid through a stretching cylinder with NPs aggregation has not yet been investigated. However, no numerical

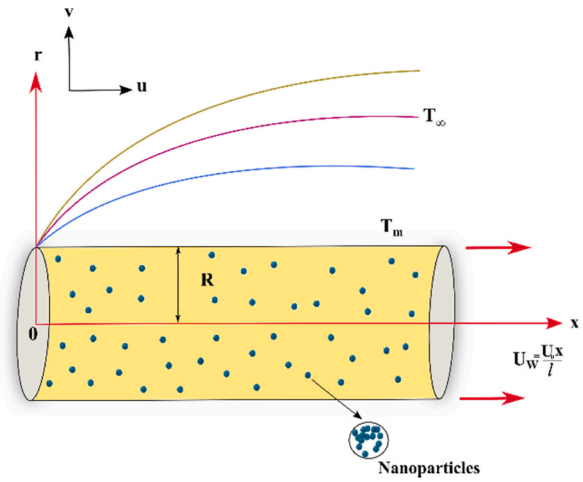

Fig. 1. Flow geometry.

solution has previously been conferred for the impact of the H–H chemical reactions effect on the flow of SGF past a stretching cylinder. The present papers main focus is on numerically examining the earlier stated flow. The most important goal of this research is to look at the previously described flow numerically. Finally, the persistence of this work was to address the below-mentioned research questions:

- ➢ What influence do varied dimensionless parameters have on nanofluid flow, heat and mass transport behaviour?
- ➢ To study the SGF flow behaviour in the presence and absence of NPs aggregation.
- ➢ What impact does an upsurge in different parameters have on the skin friction and Nusselt number?

2. Description of the problem

In the absenteeism of any body forces and with all physical fluid parameters held constant, the fundamental equations for mass, linear momentum, and temperature conservation for an incompressible liquid are provided by.

Continuity equation

$$\nabla . V = 0. \tag{1}$$

Momentum equation

$$\rho_f \frac{dV}{dt} = \nabla . \tau + \rho b. \tag{2}$$

Energy equation

$$(\rho C_p)_f \frac{dT}{dt} = \frac{1}{r}\frac{\partial}{\partial r}\left(K(T) r \frac{\partial T}{\partial r}\right). \tag{3}$$

The Cauchy stress tensor τ of SGF is (see Refs.[36–38]):

$$\tau = -pI + \mu_f A_1 + {\alpha_2}^* A_2 + \alpha_2 A_1^2, \tag{3a}$$

The first two Rivlin- Ericksen tensors defined through

$$A_1 = (\nabla V)^T + \nabla V, A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + A_1(\nabla V)^T, \tag{4}$$

here, V is the liquid velocity, $\frac{d}{dt}$ the material derivative and ∇ denote the gradient operator.

In cylindrical coordinates, we have

$$\nabla V = \begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{1}{r}\frac{\partial V_r}{\partial \varphi} - \frac{V_\varphi}{r} & \frac{\partial V_r}{\partial x} \\ \frac{\partial V_\varphi}{\partial r} & \frac{1}{r}\frac{\partial V_\varphi}{\partial \varphi} + \frac{V_r}{r} & \frac{\partial V_\varphi}{\partial x} \\ \frac{\partial V_x}{\partial r} & \frac{1}{r}\frac{\partial V_x}{\partial \varphi} & \frac{\partial V_x}{\partial x} \end{bmatrix} \text{ and} \nabla . \tau = \begin{bmatrix} \frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr}}{r} + \frac{1}{r}\frac{\partial \tau_{\varphi r}}{\partial \varphi} - \frac{\tau_{\varphi\varphi}}{r} \\ \frac{\partial \tau_{r\varphi}}{\partial r} + \frac{\tau_{r\varphi}}{r} + \frac{1}{r}\frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} - \frac{\tau_{\varphi r}}{r} \\ \frac{\partial \tau_{rx}}{\partial r} + \frac{\tau_{rx}}{r} + \frac{1}{r}\frac{\partial \tau_{\varphi x}}{\partial \varphi} - \frac{\partial \tau_{xx}}{\partial x} \end{bmatrix} \tag{5}$$

In order to satisfy the thermodynamic analysis, the below circumstances must hold.

$$\mu_f \geq 0, \alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0. \tag{6}$$

Further, the laminar boundary layer flow generated by a non-linear stretched surface and heat transformation generalized second grade nanofluid (TiO_2–ethylene glycol) is considered on a stretching cylinder along the x-axis with the velocity $U_w = \frac{U_0 x}{l}$ as outlined in Fig. 1. The flow is analyzed using the cylindrical coordinates (x, r) where r is normal to the axis of the cylinder. To simulate nanoliquids with NP aggregation, the modified Krieger and Dougherty viscosity model and the Bruggeman thermal conductivity model are used. We assume that the temperature of the melting surface is less than the ambient temperature ($T_m < T_\infty$). Using the velocity field $V = (v(r,x), 0, u(r,x))$ the governing equations after using boundary layer assumptions reduce to the below mentioned forms (Refs. [39–43]):

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0, \tag{7}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \nu_{nf}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] + \frac{\alpha_1}{\rho_{nf}}\left(\begin{array}{l}\frac{1}{r}\left(v\frac{\partial^2 u}{\partial r^2} + u\frac{\partial^2 u}{\partial x \partial r} - \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial r}\right)\\ +v\frac{\partial^3 u}{\partial r^3} + u\frac{\partial^3 u}{\partial x \partial r^2} - \frac{\partial^2 v}{\partial r^2}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\frac{\partial u}{\partial x}\end{array}\right), \tag{8}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \alpha_{nf}\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right]. \tag{9}$$

Moreover, the H–H reactions and melting effect are also considered. Here, we use Chaudhary and Merkin's [44,45] simple boundary layer flow model for the interaction among H–H processes involving dual chemical species A^* and B^*, as follows:

$$A^* + 2B^* \rightarrow 3B^*, rate = k_r a b^2, \tag{10}$$

$$A^* \rightarrow B^*, rate = k_s a. \tag{11}$$

Due to this irreversible reaction, the reaction rate is lowest at the external flow and in the boundary layer's outer edge. The expressions are listed as follows

$$u\frac{\partial a}{\partial x} + v\frac{\partial a}{\partial r} = D_A\left[\frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{\partial a}{\partial r}\right] - k_r a b^2 \tag{12}$$

$$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial r} = D_B\left[\frac{\partial^2 b}{\partial r^2} + \frac{1}{r}\frac{\partial b}{\partial r}\right] + k_r a b^2 \tag{13}$$

The associate boundary conditions for the current study are as follow:

$$\left.\begin{array}{l} r = R : u = U_w = \frac{U_0 x}{l}, k_{nf}\left(\frac{\partial T}{\partial r}\right) = \rho_{nf}(c_s(T_m - T_0) + \lambda)v, T = T_m,\\ D_A\frac{\partial a}{\partial r} = k_s a, D_B\frac{\partial b}{\partial r} = -k_s a,\\ r \rightarrow \infty : u \rightarrow 0, T \rightarrow T_\infty, a \rightarrow a_0, b \rightarrow 0. \end{array}\right\} \tag{14}$$

There are many approaches for reducing the PDE to an ODE in PDE system (or at least a PDE in a smaller number of independent variables). Various integral transformations and eigenfunction expansions are among them. When dealing with nonlinear PDE's, such strategies are much less common. There is, however, a method for identifying equations in which the solution is reliant on particular groupings of the independent variables rather than each independent variable individually. We used the similarity transformation from Refs. [46–48] in this specific investigation. The similarity solutions for the governing equations can be attained by employing a following apt transformation

$$\begin{array}{l}\psi = R\sqrt{(U_w \nu_f x)}f(\eta), \eta = \left(\frac{r^2 - R^2}{2R}\right)\sqrt{\left(\frac{U_0}{\nu_f l}\right)}, u = \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{U_0 x}{l}f'(\eta) = U_w f'(\eta), v = -\frac{1}{r}\frac{\partial \psi}{\partial x} = -\frac{R}{r}\sqrt{\left(\frac{U_0 \nu_f}{l}\right)}f(\eta),\\ \theta(\eta) = \frac{T - T_m}{T_\infty - T_m}, g(\eta) = \frac{a}{a_0}, h(\eta) = \frac{b}{a_0}.\end{array} \tag{15}$$

3. Thermophysical properties for aggregation approach

It is commonly known that nanoFluids have a high thermal conductivity based on experimental evidence. Furthermore, haphazard motion of NPs or aggregation of NPs generating percolation behaviour may be used to improve the thermal property. When compared to aggregation, which increases the mass of aggregates, Brownian randomness deteriorates, but aggregate percolation behaviour may increase thermal conductivity. As a result, the effective viscosity, density, heat capacitance, and thermal conductivity of nanofluid for nanoparticle aggregation are stated as follows (see Refs. [49–51]):

$$\mu_{nf} = \mu_f \left(1 - \frac{\varphi_{agg}}{\varphi_{\max}}\right)^{-2.5^*\varphi_{\max}} \tag{16}$$

$$\rho_{nf} = \left(1 - \varphi_{agg}\right)\rho_f + (\varphi\rho)_{agg} \tag{17}$$

$$\left(\rho C_p\right)_{nf} = \left(1 - \varphi_{agg}\right)\left(\rho C_p\right)_f + \varphi_{agg}\left(\rho C_p\right)_{agg} \tag{18}$$

$$k_{nf} = k_f \left(\frac{k_{agg} + 2k_f + 2\varphi_{agg}\left(k_{agg} - k_f\right)}{k_{agg} + 2k_f - \varphi_{agg}\left(k_{agg} - k_f\right)}\right) \tag{19}$$

3.1. Thermal characteristics of particles aggregation

The correlations for the effective viscosity and thermal conductivity are taken from the modified Krieger and Dougherty model and the modified Maxwell model, respectively (see Refs. [49–51]):

$$\varphi_{agg} = \frac{\varphi}{\varphi_{\text{int}}}, \varphi_{\text{int}} = \left(\frac{R_{agg}}{R_p}\right)^{D-3} \tag{20}$$

$$\rho_{agg} = (1 - \varphi_{\text{int}})\rho_f + \varphi_{\text{int}}\rho_s \tag{21}$$

$$\left(\rho C_p\right)_{agg} = (1 - \varphi_{\text{int}})\left(\rho C_p\right)_f + \varphi_{\text{int}}\left(\rho C_p\right)_s \tag{22}$$

$$k_{agg} = \frac{k_f}{4}\left(\begin{array}{c} [3\varphi_{\text{int}} - 1]\dfrac{k_s}{k_f} + [3(1 - \varphi_{\text{int}}) - 1]+ \\ \left[\left[(3\varphi_{\text{int}} - 1)\dfrac{k_s}{k_f} + (3(1 - \varphi_{\text{int}}) - 1)\right]^2 + \dfrac{8k_s}{k_f}\right]^{\frac{1}{2}} \end{array}\right) \tag{23}$$

The maximum particle packing fraction ($\varphi_{\max} = 0.605$ for particles of spherical shape). From the theory of fractal, R_p and R_{agg} correspond to primary NPs and radii of aggregates (the value of $\frac{R_{agg}}{R_p}$ is taken as 3.34). D is the fractal index, which takes the value 1.8 in general for spherical particles. The more realistic Maxwell and Bruggeman model can be used to precisely estimate the effective thermal conductivity of TiO_2-ethylene glycol nanoliquid.

Now, by using similarity transformations and nanoliquid thermophysical properties for aggregation approach, the continuity equation Eq. (7) satisfies trivially, and Eqs. (8), (9), (12) and (13) are reduced into the following forms:

$$\left.\begin{array}{l} \varepsilon_1\left[(2\eta\gamma + 1)f''' + 2\gamma f''\right] - \left[f'^2 - ff''\right] + \varepsilon_2 4\gamma\beta\left[f'f'' - ff'''\right] + \\ \varepsilon_2\beta(2\eta\gamma + 1)\left[2f'f''' + f''^2 - ff^{iv}\right] = 0 \end{array}\right\}, \tag{24}$$

$$\varepsilon_3 \frac{1}{\Pr}\frac{k_{nf}}{k_f}\left[(2\eta\gamma + 1)\theta'' + 2\gamma\theta'\right] + f\theta' = 0, \tag{25}$$

$$\frac{1}{Sc}\left[(2\eta\gamma + 1)g'' + 2\gamma g'\right] + fg' - Hgh^2 = 0, \tag{26}$$

$$\frac{\delta}{Sc}\left[(2\eta\gamma + 1)h'' + 2\gamma h'\right] + fh' + Hgh^2 = 0. \tag{27}$$

Where

$$\varepsilon_1 = \frac{\left(1 - \frac{\varphi_{agg}}{\varphi_{\max}}\right)^{-2.5^*\varphi_{\max}}}{\left(1 - \varphi_{agg}\right) + \varphi_{agg}\frac{\rho_{agg}}{\rho_f}}, \varepsilon_2 = \frac{1}{\left(1 - \varphi_{agg}\right) + \varphi_{agg}\frac{\rho_{agg}}{\rho_f}}, \varepsilon_3 = \frac{1}{\left(1 - \varphi_{agg}\right) + \varphi_{agg}\frac{\left(\rho C_p\right)_{agg}}{\left(\rho C_p\right)_f}}.$$

The relevant boundary constraints then take the form

$$\left.\begin{array}{l} f'(0) = 1, \varepsilon_2\dfrac{k_{nf}}{k_f}M\theta'(0) + \Pr f(0) = 0, \theta(0) = 0, g'(0) = Kg(0), \\ \delta h'(0) = -Kg(0), f'(\infty) \to 0, \theta(\infty) \to 1, g(\infty) \to 1, h(\infty) \to 0. \end{array}\right\} \tag{28}$$

The dimensionless parameters in the above system of reduced equations are as follow:

Table 1
Computational values for $-\theta'(0)$ with respect to varied Pr values for some reduced cases.

Pr / $-\theta'(0)$	0.2	0.7	2.0	7.0	20
Wang [52]	0.1691	0.4539	0.9114	1.8954	3.3539
Gorla and Sidawi [53]	0.1691	0.5349	0.9114	1.8905	3.3539
Mahanthesh et al. [54]	0.1691	0.4539	0.9113	1.8954	3.3539
Present results	0.169102	0.453901	0.911303	1.895402	3.353903

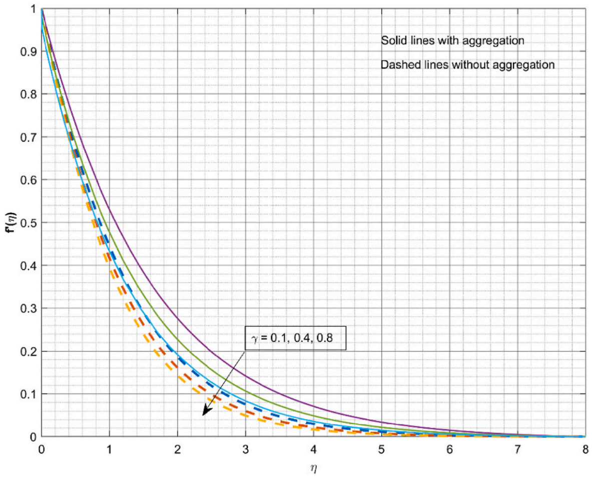

Fig. 2. The upshot of γ on $f'(\eta)$.

$$\gamma=\sqrt{\frac{\nu_f l}{U_0R^2}},\beta=\frac{\alpha_1U_0}{\rho_f\nu_f l},H=\frac{k_r{a_o}^2l}{U_0},\delta=\frac{D_B}{D_A},M=\frac{C_{pf}(T_\infty-T_m)}{\lambda+C_s(T_m-T_0)},K=\frac{k_s}{D_A}\sqrt{\frac{\nu_f l}{U_0}},$$

$$Sc=\frac{\nu_f}{D_A},\Pr=\frac{\nu_f(\rho C_p)_f}{k_f}.$$

The A^* and B^* are assumed to be of identical magnitude in this case. As a result of this logic, we must assume that the D_A and D_B are equivalent, i.e. $\delta=1$ and thus

$$g(\eta)+h(\eta)=1. \tag{29}$$

Now by using Eq. (29), in Eqs. (26) and (27) we obtain

$$\frac{1}{Sc}[(2\eta\gamma+1)g''+2\gamma g']+fg'-Hg(1-g)^2=0. \tag{30}$$

along with the boundary cons

$$g'(0)=Kg(0),g(\infty)\to 1. \tag{31}$$

3.2. Physical quantities of engineering interests

The physical quantity of interest, i.e., the local Nusselt number in dimensionless form is given below

$$\frac{Nu}{\mathrm{Re}^{0.5}}=-\frac{k_{nf}}{k_f}\theta'(0). \tag{32}$$

Where Re $=\frac{xU_w}{\nu_f}$.

4. Results and discussions

The purpose of this section is to go through the physical characteristics of the graphs and elucidate the mechanism behind the flow, heat, and mass transport changes that are generated by the dominant factors. The RKF-45 procedure and the shooting method are utilized to solve the ODEs.Two different cases namely, presence and absence of NPs aggregation are considered in this present investigation. The dashed curves show flow without NPs aggregation and solid lines indicate flow in presence NPs aggregation in the

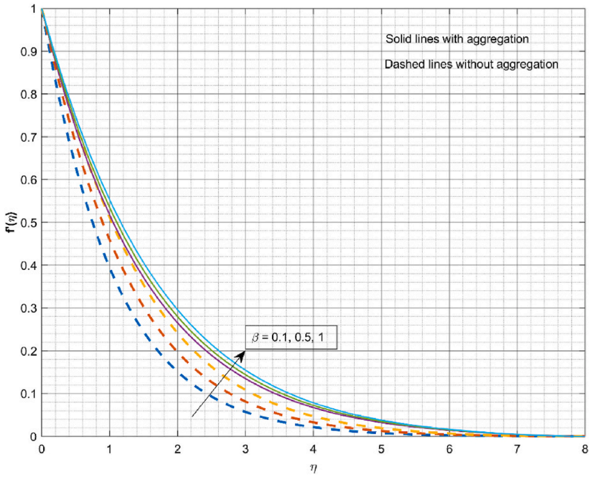

Fig. 3. The upshot of β on $f'(\eta)$.

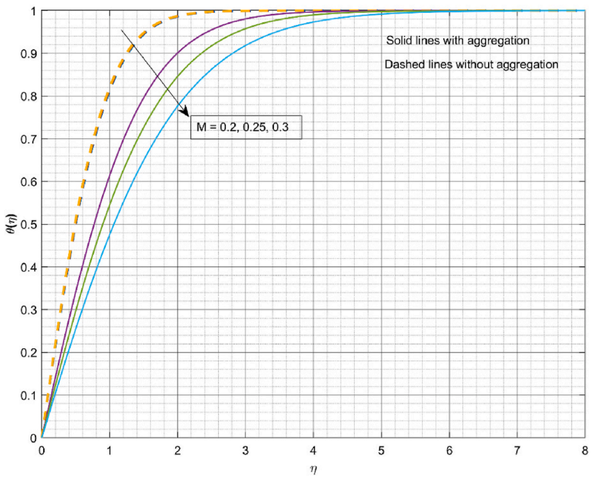

Fig. 4. The upshot of M on $\theta(\eta)$.

graphs. With proper plots, change in velocity, heat and mass transport with Nusselt number are also elucidated. Furthermore, the numerical solutions are compared to current studies, and the findings are found to be quite near to each other (see Table 1).

The influence of γ on $f'(\eta)$ is displayed in Fig. 2 for two different cases namely with and without NPs aggregation. The increase in value of γ declines the $f'(\eta)$ for both the cases. The cylinder's radius R decreases as the curvature parameter increases. The fluid-filled cylinder close to it contracts. As a result of the increased resistance to liquid stream, the $f'(\eta)$ decreases. Furthermore, the velocity of the liquid declines faster in the absence of nanoparticles aggregation. The encouragement of β on $f'(\eta)$ is showed in Fig. 3. The improvement in value of β improves the $f'(\eta)$ for both the cases. The β is inversely proportional to the liquid's viscosity. As a result of the increase in β, the viscosity reduces, which improves the velocity profile. Furthermore, $f'(\eta)$ for fluid flow in presence of NPs aggregation inclines faster for growing values of β than the other case.

Fig. 4 displays the consequence of M on $\theta(\eta)$ for both cases. The upsurge in value of M declines the $\theta(\eta)$ for both cases. This is because bigger values of the M correlate to more convective flow from the heated fluid to the cold surface, resulting in a decrease in heat transfer. Moreover, the fluid flow in absence of NPs aggregation case shows improved heat transfer than the remaining case. Here, we observe least heat transfer for fluid flow with NPs aggregation. Fig. 5 displays the consequence of γ on $\theta(\eta)$ for both the cases. The upsurge in value of γ declines the $\theta(\eta)$ for both cases. Advanced γ values tend to reduce the cylinder radius from a physical standpoint. As a result, the surface provides less resistance and improved viscosity, allowing for less heat transfer. Moreover, the fluid flow in absence of NPs aggregation case shows improved heat transfer than the remaining case. Here, we observe least heat transfer for fluid flow with NPs aggregation.

Fig. 6 displays the effect of Sc on $g(\eta)$ for both cases. The upsurge in value of Sc inclines the $g(\eta)$ for both cases. The smallest Sc indicates the highest concentration of nanoparticles. Momentum diffusivity increases as Sc escalations initiating mass transfer to decline. Moreover, the fluid flow in presence of NPs aggregation shows improved mass transfer than other. Here, we observe least mass

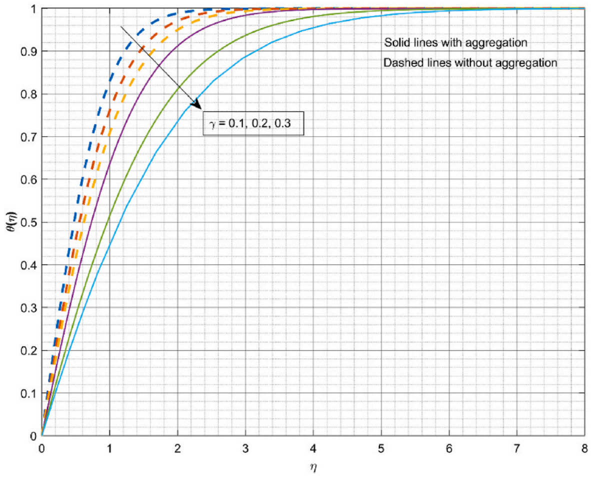

Fig. 5. The upshot of γ on $\theta(\eta)$.

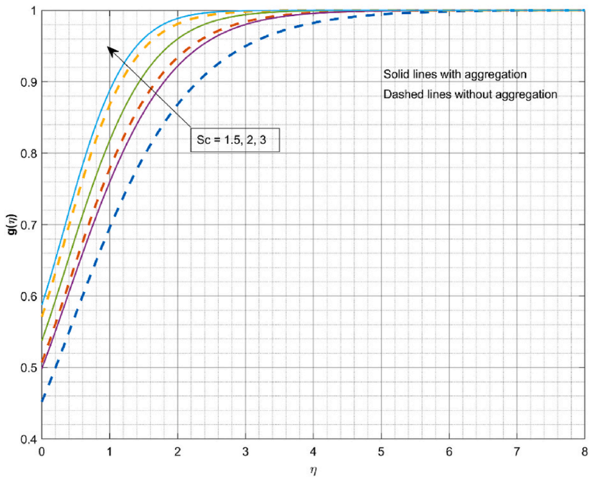

Fig. 6. The upshot of Sc on $g(\eta)$.

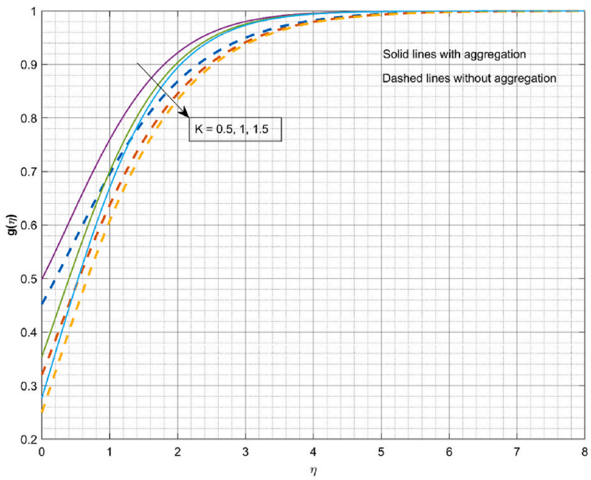

Fig. 7. The upshot of K on $g(\eta)$.

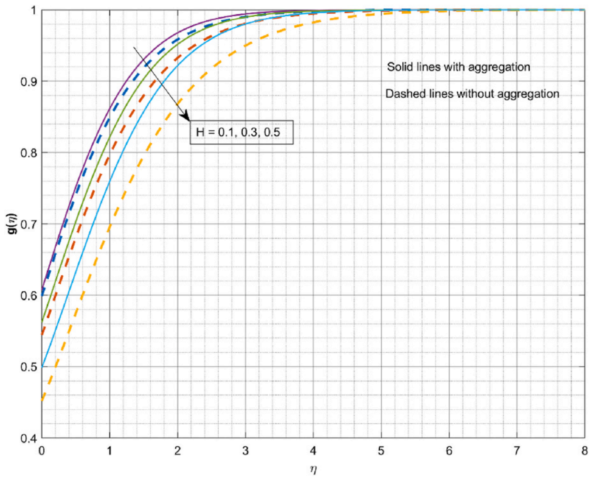

Fig. 8. The upshot of H on $g(\eta)$.

Table 2
Numerical values of the Nusselt number for varied γ, β and M.

γ	β	M	Without aggregation		With aggregation	
			$\theta'(0)$□	Nu	$\theta'(0)$	Nu
0.1	0.4	0.1	1.6758	−1.85896	1.126	−2.95631
0.3			1.5906	−1.76445	1.1107	−2.91614
0.5			1.5237	−1.69024	1.0962	−2.87807
	0.1		1.649	−1.82924	1.1228	−2.94791
	0.2		1.6598	−1.84122	1.1239	−2.9508
	0.3		1.6686	−1.85098	1.125	−2.95369
		0.2	1.5942	−1.76845	1.0703	−2.81007
		0.3	1.5214	−1.68769	1.0206	−2.67959
		0.4	1.4559	−1.61503	0.9759	−2.56223

transfer for fluid flow without NPs aggregation. The encouragement of K on $g(\eta)$ is showed in Fig. 7 for both flow cases. The increase in value of K declines the $g(\eta)$ for both flow cases. An increase in the K is helpful in increasing the chemical species concentration in this case. When the rate of change in velocity of a heterogeneous reaction increases, the concentration of chemical species increases, indicating that more chemical species are taking part in the chemical reaction. Furthermore, $g(\eta)$ for fluid flow in presence of aggregation condition declines slower for growing values of K than the other. Here, we observe higher mass transfer for fluid flow in presence of aggregation condition. The impact of H on $g(\eta)$ is presented in Fig. 8 for both flow cases. The increase in value of H declines the $g(\eta)$ for both flow cases. Because reactants are consumed throughout the homogeneous reaction, the mass transport depreciates, as seen in this figure. Furthermore, $g(\eta)$ for fluid flow in presence of aggregation condition declines slower for growing values of H than the other. Here, we observe higher mass transfer for fluid flow in presence of aggregation condition. Table 2 shows the change in Nusselt number for varied γ, β and M. Moreover, it is shown from the table that, fluid flow without NPs aggregation shows improved heat transport rate for upward values of γ, β and M.

5. Conclusions

The present research investigates the effect of NPs aggregation on SGF flow through a stretching cylinder. The H–H chemical reactions and melting effects are also considered in the modelling. The current work is intended to serve as a springboard for future modelling of stretching flows, particularly in polymeric and paper manufacturing processes. Graphically, the behaviour of concentration, temperature, and velocity fields is examined. The following are the study's main conclusions:

- ❖ The velocity of the fluid declines faster for upward values of γ which result in the increased resistance to liquid stream without NPs aggregation.
- ❖ The $f'(\eta)$ for fluid flow with NPs aggregation inclines faster due to the reduction in liquid viscosity caused by growing values of β.
- ❖ The fluid flow without NPs aggregation case displays enhanced heat transfer for augmented values of M.
- ❖ The $g(\eta)$ with aggregation condition declines slower for upward values of K due to more chemical species are taking part in the chemical reaction.
- ❖ The $g(\eta)$ in the presence of aggregation condition declines slower for increasing values of H because reactants are consumed throughout the homogeneous reaction.

❖ The fluid flow without NPs aggregation shows better-quality heat transport rate for upward values of γ, β and M.

Author statement

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] S. Nadeem, M.N. Khan, N. Muhammad, S. Ahmad, Mathematical analysis of bio-convective micropolar nanofluid, J. Comput. Des. Eng. 6 (3) (Jul. 2019) 233–242, https://doi.org/10.1016/j.jcde.2019.04.001.

[2] S. Ahmad, S. Nadeem, Cattaneo-Christov-based study of SWCNT-MWCNT/EG Casson hybrid nanofluid flow past a lubricated surface with entropy generation, Appl. Nanosci. 10 (12) (Dec. 2020) 5449–5458, https://doi.org/10.1007/s13204-020-01367-1.

[3] S. Ahmad, S. Nadeem, Flow analysis by Cattaneo-Christov heat flux in the presence of Thomson and Troian slip condition, Appl. Nanosci. 10 (12) (2020) 4673–4674.

[4] S. Ahmad, S. Nadeem, N. Muhammad, M.N. Khan, Cattaneo-Christov heat flux model for stagnation point flow of micropolar nanofluid toward a nonlinear stretching surface with slip effects, J. Therm. Anal. Calorim. 143 (2) (Jan. 2021) 1187–1199, https://doi.org/10.1007/s10973-020-09504-2, 10.1007/s13204-020-01267-4.

[5] S. Ahmad, S. Nadeem, N. Ullah, Entropy generation and temperature-dependent viscosity in the study of SWCNT-MWCNT hybrid nanofluid, Appl. Nanosci. 10 (12) (Dec. 2020) 5107–5119, https://doi.org/10.1007/s13204-020-01306-0.

[6] H.A.H. Alzahrani, A. Alsaiari, J.K. Madhukesh, R. Naveen Kumar, B.M. Prasanna, Effect of thermal radiation on heat transfer in plane wall jet flow of Casson nanofluid with suction subject to a slip boundary condition, Waves Random Complex Media (Feb. 2022) 1–18, https://doi.org/10.1080/17455030.2022.2030502, 0, no. 0.

[7] U. Khan, S. Ahmad, A. Hayyat, I. Khan, K.S. Nisar, D. Baleanu, On the Cattaneo-Christov heat flux model and OHAM analysis for three different types of nanofluids, Appl. Sci. 10 (3) (Jan. 2020), https://doi.org/10.3390/app10030886. Art. no. 3.

[8] N. Talbi, et al., Increment of heat transfer by graphene-oxide and molybdenum-disulfide nanoparticles in ethylene glycol solution as working nanofluid in penetrable moveable longitudinal fin, Waves Random Complex Media (Jan. 2022) 1–23, https://doi.org/10.1080/17455030.2022.2026527, 0, no. 0.

[9] M. Ouni, L.M. Ladhar, M. Omri, W. Jamshed, M.R. Eid, Solar water-pump thermal analysis utilizing copper-gold/engine oil hybrid nanofluid flowing in parabolic trough solar collector: thermal case study, Case Stud. Therm. Eng. 30 (Feb. 2022) 101756, https://doi.org/10.1016/j.csite.2022.101756.

[10] W. Jamshed, et al., Thermal efficiency enhancement of solar aircraft by utilizing unsteady hybrid nanofluid: a single-phase optimized entropy analysis, Sustain. Energy Technol. Assess. 52 (Aug. 2022) 101898, https://doi.org/10.1016/j.seta.2021.101898.

[11] M. Bagheri Motlagh, M. Kalteh, Molecular dynamics simulation of nanofluid convective heat transfer in a nanochannel: effect of nanoparticles shape, aggregation and wall roughness, J. Mol. Liq. 318 (Nov. 2020) 114028, https://doi.org/10.1016/j.molliq.2020.114028.

[12] K. Swain, B. Mahanthesh, Thermal enhancement of radiating magneto-nanoliquid with nanoparticles aggregation and Joule heating: a three-dimensional flow, Arabian J. Sci. Eng. 46 (6) (Jun. 2021) 5865–5873, https://doi.org/10.1007/s13369-020-04979-5.

[13] A.S. Sabu, J. Mackolil, B. Mahanthesh, A. Mathew, Nanoparticle aggregation kinematics on the quadratic convective magnetohydrodynamic flow of nanomaterial past an inclined flat plate with sensitivity analysis, Proc. Inst. Mech. Eng. Part E J. Process Mech. Eng., Dec. (2021), https://doi.org/10.1177/09544089211056235.

[14] B. Mahanthesh, Flow and heat transport of nanomaterial with quadratic radiative heat flux and aggregation kinematics of nanoparticles, Int. Commun. Heat Mass Tran. 127 (Oct. 2021) 105521, https://doi.org/10.1016/j.icheatmasstransfer.2021.105521.

[15] J. Mackolil, B. Mahanthesh, Inclined magnetic field and nanoparticle aggregation effects on thermal Marangoni convection in nanoliquid: a sensitivity analysis, Chin. J. Phys. 69 (Feb. 2021) 24–37, https://doi.org/10.1016/j.cjph.2020.11.006.

[16] S. Nadeem, S. Ahmad, N. Muhammad, Cattaneo-Christov flux in the flow of a viscoelastic fluid in the presence of Newtonian heating, J. Mol. Liq. 237 (Jul. 2017) 180–184, https://doi.org/10.1016/j.molliq.2017.04.080.

[17] T. Hayat, W.A. Khan, S.Z. Abbas, S. Nadeem, S. Ahmad, Impact of induced magnetic field on second-grade nanofluid flow past a convectively heated stretching sheet, Appl. Nanosci. 10 (8) (Aug. 2020) 3001–3009, https://doi.org/10.1007/s13204-019-01215-x.

[18] T. Anwar, P. Kumam, Asifa, I. Khan, P. Thounthong, An exact analysis of radiative heat transfer and unsteady MHD convective flow of a second-grade fluid with ramped wall motion and temperature, Heat Transf 50 (1) (Jan. 2021) 196–219, https://doi.org/10.1002/htj.21871.

[19] R.J. Punith Gowda, R. Naveen Kumar, A.M. Jyothi, B.C. Prasannakumara, I.E. Sarris, Impact of binary chemical reaction and activation energy on heat and mass transfer of Marangoni Driven boundary layer flow of a non-Newtonian nanofluid, Processes 9 (4) (Apr. 2021), https://doi.org/10.3390/pr9040702. Art. no. 4.

[20] R.J. Punith Gowda, H.M. Baskonus, R. Naveen Kumar, B.C. Prasannakumara, D.G. Prakasha, Computational investigation of Stefan blowing effect on flow of second-grade fluid over a curved stretching sheet, Int. J. Appl. Comput. Math. 7 (3) (May 2021) 109, https://doi.org/10.1007/s40819-021-01041-2.

[21] G. Sowmya, B. Saleh, R.J. Punith Gowda, R. Naveen Kumar, R.S. Varun Kumar, M. Radhika, Analysis of radiative nonlinear heat transfer in a convective flow of dusty fluid by capitalizing a non-Fourier heat flux model, Proc. Inst. Mech. Eng. Part E J. Process Mech. Eng. (Sep. 2021), https://doi.org/10.1177/09544089211041192, 09544089211041192.

[22] R.S. Varun Kumar, R.J. Punith Gowda, R. Naveen Kumar, M. Radhika, B.C. Prasannakumara, Two-phase flow of dusty fluid with suspended hybrid nanoparticles over a stretching cylinder with modified Fourier heat flux, SN Appl. Sci. 3 (3) (Feb. 2021) 384, https://doi.org/10.1007/s42452-021-04364-3.

[23] R.J. Punith Gowda, R. Naveen Kumar, B.C. Prasannakumara, Two-phase Darcy-forchheimer flow of dusty hybrid nanofluid with viscous Dissipation over a cylinder, Int. J. Appl. Comput. Math. 7 (3) (May 2021) 95, https://doi.org/10.1007/s40819-021-01033-2.

[24] Y.-Q. Song, et al., Unsteady mixed convection flow of magneto-Williamson nanofluid due to stretched cylinder with significant non-uniform heat source/sink features, Alex. Eng. J., Jun. (2021), https://doi.org/10.1016/j.aej.2021.04.089.

[25] F. Redouane, et al., Influence of entropy on Brinkman-Forchheimer model of MHD hybrid nanofluid flowing in enclosure containing rotating cylinder and undulating porous stratum, Sci. Rep. 11 (1) (Dec. 2021), https://doi.org/10.1038/s41598-021-03477-4. Art. no. 1.
[26] S. Nadeem, S. Ahmad, N. Muhammad, M.T. Mustafa, Chemically reactive species in the flow of a Maxwell fluid, Results Phys. 7 (Jan. 2017) 2607–2613, https://doi.org/10.1016/j.rinp.2017.06.017.
[27] S. Rashid, M.I. Khan, T. Hayat, M. Ayub, A. Alsaedi, Theoretical and analytical analysis of shear rheology of Oldroyd-B fluid with homogeneous-heterogeneous reactions, Appl. Nanosci. 10 (8) (Aug. 2020) 3035–3043, https://doi.org/10.1007/s13204-019-01037-x.
[28] T. Hayat, A. Aziz, T. Muhammad, A. Alsaedi, Significance of homogeneous-heterogeneous reactions in Darcy-Forchheimer three-dimensional rotating flow of carbon nanotubes, J. Therm. Anal. Calorim. 139 (1) (Jan. 2020) 183–195, https://doi.org/10.1007/s10973-019-08316-3.
[29] S.Z. Abbas, W.A. Khan, M. Waqas, M. Irfan, Z. Asghar, Exploring the features for flow of Oldroyd-B liquid film subjected to rotating disk with homogeneous/heterogeneous processes, Comput. Methods Progr. Biomed. 189 (Jun. 2020) 105323, https://doi.org/10.1016/j.cmpb.2020.105323.
[30] S.G. Bejawada, et al., Chemical reactive and viscous Dissipative flow of magneto nanofluid via natural convection by employing galerkin finite element technique, Coatings 12 (2) (Feb. 2022), https://doi.org/10.3390/coatings12020151. Art. no. 2.
[31] W.A. Khan, M. Ali, M. Irfan, M. Khan, M. Shahzad, F. Sultan, A rheological analysis of nanofluid subjected to melting heat transport characteristics, Appl. Nanosci. 10 (8) (Aug. 2020) 3161–3170, https://doi.org/10.1007/s13204-019-01067-5.
[32] N.A. Amirsom, M.J. Uddin, M.F. Md Basir, A. Kadir, O.A. Bég, A. I. Md Ismail, Computation of melting Dissipative magnetohydrodynamic nanofluid bioconvection with second-order slip and variable thermophysical properties, Appl. Sci. 9 (12) (Jan. 2019), https://doi.org/10.3390/app9122493. Art. no. 12.
[33] M. Farooq, S. Ahmad, M. Javed, A. Anjum, Melting heat transfer in Squeezed nanofluid flow through Darcy forchheimer medium, J. Heat Tran. 141 (1) (Oct. 2018), https://doi.org/10.1115/1.4041497.
[34] M. Radhika, R.J.P. Gowda, R. Naveenkumar, Siddabasappa, B.C. Prasannakumara, Heat transfer in dusty fluid with suspended hybrid nanoparticles over a melting surface, Heat Transf 50 (3) (2021) 2150–2167, https://doi.org/10.1002/htj.21972.
[35] F. Mabood, T.A. Yusuf, W.A. Khan, Cu-Al2O3-H2O hybrid nanofluid flow with melting heat transfer, irreversibility analysis and nonlinear thermal radiation, J. Therm. Anal. Calorim. 143 (2) (Jan. 2021) 973–984, https://doi.org/10.1007/s10973-020-09720-w.
[36] W. Tan, T. Masuoka, Stokes' first problem for a second-grade fluid in a porous half-space with heated boundary, Int. J. Non Lin. Mech. 40 (4) (May 2005) 515–522, https://doi.org/10.1016/j.ijnonlinmec.2004.07.016.
[37] C. Fetecău, C. Fetecău, J. Zierep, Decay of a potential vortex and propagation of a heat wave in a second-grade fluid, Int. J. Non Lin. Mech. 37 (6) (Sep. 2002) 1051–1056, https://doi.org/10.1016/S0020-7462(01)00028-2.
[38] C. Fetecau, C. Fetecau, Starting solutions for some unsteady unidirectional flows of a second-grade fluid, Int. J. Eng. Sci. 43 (10) (Jun. 2005) 781–789, https://doi.org/10.1016/j.ijengsci.2004.12.009.
[39] B. Mahanthesh, K. Thriveni, Nanoparticle aggregation effects on radiative heat transport of nanoliquid over a vertical cylinder with sensitivity analysis, Appl. Math. Mech. 42 (3) (Mar. 2021) 331–346, https://doi.org/10.1007/s10483-021-2687-7.
[40] T. Hayat, Z. Hussain, A. Alsaedi, B. Ahmad, Heterogeneous-homogeneous reactions and melting heat transfer effects in flow with carbon nanotubes, J. Mol. Liq. 220 (Aug. 2016) 200–207, https://doi.org/10.1016/j.molliq.2016.04.012.
[41] T. Hayat, M.S. Anwar, M. Farooq, A. Alsaedi, MHD stagnation point flow of second grade fluid over a stretching cylinder with heat and mass transfer, Int. J. Nonlinear Sci. Numer. Stimul. 15 (6) (Oct. 2014) 365–376, https://doi.org/10.1515/ijnsns-2013-0104.
[42] R. Malik, M. Khan, Numerical study of homogeneous-heterogeneous reactions in Sisko fluid flow past a stretching cylinder, Results Phys. 8 (Mar. 2018) 64–70, https://doi.org/10.1016/j.rinp.2017.10.047.
[43] A. Shojaei, A.J. Amiri, S.S. Ardahaie, Kh Hosseinzadeh, D.D. Ganji, Hydrothermal analysis of Non-Newtonian second grade fluid flow on radiative stretching cylinder with Soret and Dufour effects, Case Stud. Therm. Eng. 13 (Mar. 2019) 100384, https://doi.org/10.1016/j.csite.2018.100384.
[44] M.A. Chaudhary, J.H. Merkin, A simple isothermal model for homogeneous-heterogeneous reactions in boundary-layer flow. I Equal diffusivities, Fluid Dynam. Res. 16 (6) (Nov. 1995) 311, https://doi.org/10.1016/0169-5983(95)00015-6.
[45] M.A. Chaudhary, J.H. Merkin, Homogeneous-heterogeneous reactions in boundary-layer flow: effects of loss of reactant, Math. Comput. Model. 24 (3) (Aug. 1996) 21–28, https://doi.org/10.1016/0895-7177(96)00097-0.
[46] A. Pathan and M. G. Timol, "Similarity Solution of MHD Boundary Layer Flow of Prandtl-Eyring Fluids".
[47] Y. Cho, A. Aessopos, Similarity transformation methods in the analysis of the two-dimensional steady compressible laminar boundary layer, Term Pap 2 (2004).
[48] M. Hashim, Khan, A. Saleh Alshomrani, Characteristics of melting heat transfer during flow of Carreau fluid induced by a stretching cylinder, Eur. Phys. J. E 40 (1) (Jan. 2017) 8, https://doi.org/10.1140/epje/i2017-11495-6.
[49] R. Ellahi, M. Hassan, A. Zeeshan, Aggregation effects on water base Al2O3-nanofluid over permeable wedge in mixed convection, Asia Pac. J. Chem. Eng. 11 (2) (2016) 179–186, https://doi.org/10.1002/apj.1954.
[50] N. Acharya, K. Das, P.K. Kundu, Effects of aggregation kinetics on nanoscale colloidal solution inside a rotating channel, J. Therm. Anal. Calorim. 138 (1) (Oct. 2019) 461–477, https://doi.org/10.1007/s10973-019-08126-7.
[51] L. Th Benos, E.G. Karvelas, I.E. Sarris, Crucial effect of aggregations in CNT-water nanofluid magnetohydrodynamic natural convection, Therm. Sci. Eng. Prog. 11 (Jun. 2019) 263–271, https://doi.org/10.1016/j.tsep.2019.04.007.
[52] C.Y. Wang, Free convection on a vertical stretching surface, ZAMM - J. Appl. Math. Mech. Z. Für Angew. Math. Mech. 69 (11) (1989) 418–420, https://doi.org/10.1002/zamm.19890691115.
[53] R.S. Reddy Gorla, I. Sidawi, Free convection on a vertical stretching surface with suction and blowing, Appl. Sci. Res. 52 (3) (Apr. 1994) 247–257, https://doi.org/10.1007/BF00853952.
[54] B. Mahanthesh, K. Thriveni, P. Rana, T. Muhammad, Radiative heat transfer of nanomaterial on a convectively heated circular tube with activation energy and nanoparticle aggregation kinematic effects, Int. Commun. Heat Mass Tran. 127 (Oct. 2021) 105568, https://doi.org/10.1016/j.icheatmasstransfer.2021.105568.

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ORIGINAL RESEARCH

New Bregman projection methods for solving pseudo-monotone variational inequality problem

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Abstract
In this work, we introduce two Bregman projection algorithms with self-adaptive stepsize for solving pseudo-monotone variational inequality problem in a Hilbert space. The weak and strong convergence theorems are established without the prior knowledge of Lipschitz constant of the cost operator. The convergence behavior of the proposed algorithms with various functions of the Bregman distance are presented. More so, the performance and efficiency of our methods are compared to other related methods in the literature.

Keywords Bregman projection · Hilbert space · Strong convergence · Variational inequality · Pseudo-monotone mapping

Mathematics Subject Classification 47H09 · 47H10 · 47J25 · 47J05

1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H and $A : C \to H$ be a given mapping. The *variational inequality problem* (VIP) is to find a point $z \in C$ such that

$$\langle Az, x - z \rangle \geq 0, \ \forall x \in C. \tag{1.1}$$

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The solution set of VIP is denoted by $VI(C, A)$. The VIP was introduced and studied by Stampacchia [36] in 1964. It is a classical tool for the study of many physical science problems such as equilibrium problems in economics, transportation science, contact problems in elasticity and partial differential equations with unilateral boundary conditions and free boundary-value problems of elliptic type. For related works, we recommend [3,4,12,22,23,29]. Several methods have been developed for solving VIP and related optimization problems (see, for instance, [2,5,10,24,27,28,30–32,43]).
In 1976, Korpelevich [24] introduced the following *extragradient method* for solving VIP in a finite dimensional Euclidean space $\mathbb{R}^m$:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{1.2}$$

where A is a monotone and L-Lipschitz continuous mapping of C into $\mathbb{R}^m$ and $\lambda \in \left(0, \frac{1}{L}\right)$. It was proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.2) converge to a point in $VI(C, A)$. In recent years, the extragradient method was widely studied in infinite dimensional Hilbert spaces by many authors in various ways (see, for instance, [10,17,33,43]).

In order to improve the extragradient method, Tseng [43] proposed the following *Tseng's extragradient method* for solving VIP in a real Hilbert space H:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n), \end{cases} \tag{1.3}$$

where A is monotone and L-Lipschitz continuous from C into H and $\lambda \in \left(0, \frac{1}{L}\right)$. It was proved that this method converges weakly to a point in $VI(C, A)$. Note that this method requires the computation on a projection at each iterative step which is simpler than the extragradient method does.
In [10], Censor et al. proposed the *subgradient extragradient method* for solving VIP in a real Hilbert space H. They replaced the second projection in extragradient method by a projection onto a half space which is easy to calculate in practice. This algorithm is of the following form:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \\ T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \le 0\}, \end{cases} \tag{1.4}$$

where A is monotone and L-Lipschitz continuous from C into H and $\lambda \in \left(0, \frac{1}{L}\right)$. The weak convergence of the sequence $\{x_n\}$ generated by (1.4) has been proved under some suitable conditions. However, the choice of the stepsize λ of aforementioned methods depends on the Lipschitz constant of the cost operator. In practice, it is not easy to approximate the Lipschitz constant of the operator. In recent years, there have been many authors who have improved and modified iterative methods with a way of selecting the stepsizes such that the algorithms do not depend on the Lipschitz constant of the operator (see, for instance, [37,38,44,45]).

Recently, Yang et al. [46] introduced the following modification of the subgradient extragradient method with adaptive stepsize for solving monotone VIP:

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda_n Ay_n), \\ T_n = \{x \in H : \langle x_n - \lambda_n Ax_n - y_n, x - y_n\rangle \le 0\}, \end{cases} \tag{1.5}$$

where $\lambda_1 > 0$, $\mu \in (0, 1)$ and λ_n is adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{2\langle Ax_n - Ay_n, x_{n+1} - y_n\rangle}, \lambda_n\right\} & \text{if } \langle Ax_n - Ay_n, x_{n+1} - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{1.6}$$

The weak convergence of the algorithm was established without the knowledge of the Lipschitz constant of the mapping.

It would be interesting to extend the algorithms to solve VIP in a more general class of monotone mappings. Thong and Vuong [39] proposed a variant of (1.3) with a linesearch procedure for solving pseudo-monotone VIP. The algorithm is of following form:

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n), \end{cases} \tag{1.7}$$

where $\gamma > 0, l \in (0, 1), \mu \in (0, 1)$ and $\lambda_n = \gamma l^{m_n}$ with m_n is the smallest nonnegative integer satisfying

$$\gamma l^m \|Ax_n - Ay\| \le \mu\|x_n - y_n\|. \tag{1.8}$$

The weak convergence of the sequence $\{x_n\}$ generated by (1.7) was established.

Very recently, Khanh et al. [21] proposed the following modification of the subgradient extragradient method with a linesearch procedure for solving pseudo-monotone VIP:

$$\begin{cases} y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda_n Ay_n), \\ T_n = \{x \in H : \langle x_n - \lambda_n Ax_n - y_n, x - y_n\rangle \le 0\}, \end{cases} \tag{1.9}$$

where λ_n is defined as in (1.8). They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to a point in $VI(C, A)$.

It is noted that most of all methods obtained only weak convergence. However, from the numerical point of view, the weak convergence of these methods are not enough to make it efficient. In order to get the strong convergence, these methods are often based on Halpern's iteration, viscosity approximation method, shrinking projection method and hybrid projection method (see, for instance, [18,25,37,38,40,42,44,47]).

However, most of the above methods are based on the usual norm distance. In this case, it does not allow in applications to the structure of metric projection. In this approach, the Bregman distance has initiated a new field in optimization [8]. It is known that the Bregman distance is a useful substitute for a distance, obtained from the various choices of functions. The applications of Bregman distance instead of the norm gives us alternative ways for more flexibility in the selection of projections. Several methods for solving VIP with Bregman projections can be found, for instance, in [13–15,19,45]).

Motivated and inspired by the previous works, in this paper, we introduce two new efficient methods for solving pseudo-monotone VIP in a Hilbert space. In our first algorithm, we modify the Tseng's algorithm (1.7) by employing the Bregman projection and in the second one, we modify the Tseng's algorithm (1.7) by employing the hybrid projection introduced by Nakajo and Takahashi [26]. We prove the weak and strong convergence theorems of the proposed methods under mild conditions. The major advantage of our methods is that the sequence of stepsizes is chosen without the prior knowledge of Lipschitz constant of the mapping and without any linesearch procedure which can be time-consuming. Finally, we provide several numerical experiments to illustrate the efficiency and advantage of our methods. The results in this paper improve and generalize many previous results in literature.

2 Mathematical preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. For a sequence $\{x_n\} \subset H$, the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The weak ω-limit set of the sequence $\{x_n\}$ is denoted by

$$\omega_w(x_n) = \{x \in H : x_{n_k} \rightharpoonup x \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}.$$

Definition 2.1 Let C be a nonempty subset of H. A mapping $A : C \to H$ is said to be:

(i) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in C$;
(ii) *pseudo-monotone* if $\langle Ax, y - x \rangle \geq 0$ implies $\langle Ay, y - x \rangle \geq 0$ for all $x, y \in C$;
(iii) *sequential weak-to-weak continuous* on C if for each sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$;
(iv) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$.

Remark 2.2 Every monotone mapping is a pseudo-monotone mapping but the converse is not true. The example of a pseudo-monotone mapping that is not monotone can be found in [20].

Let $f : H \to \mathbb{R}$ be convex and differentiable function with its domain denoted by

$$\mathrm{dom} f = \{x \in E : f(x) < \infty\}.$$

The *subdifferential* of f at x is defined by

$$\partial f(x) = \{u \in H : f(y) - f(x) \geq \langle u, y - x \rangle, \ \forall y \in H\}.$$

The element in $\partial f(x)$ are called the *subgradient* of f at x. It is known that if f is differentiable, then $\partial f(x) = \{\nabla f(x)\}$ which is the gradient of f at x.

Definition 2.3 The *Bregman bifunction (distance)* D_f : $\operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, \infty)$ corresponding to the strictly convex and differentiable function f with its gradient ∇f is defined by

$$D_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle.$$

Various examples of Bregman distance corresponding to f can be seen in Sect. 4. The Bregman distance has the following two important properties called the *two-point identity* and the *three-point identity*, respectively: for any $x, y \in \operatorname{int}(\operatorname{dom} f)$

$$D_f(x, y) + D_f(y, x) = \langle \nabla f(x) - \nabla f(y), x - y \rangle, \tag{2.1}$$

for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$

$$D_f(x, y) = D_f(x, z) - D_f(y, z) + \langle \nabla f(z) - \nabla f(y), x - y \rangle. \tag{2.2}$$

The Bregman projection with respect to f of $x \in \operatorname{int}(\operatorname{dom} f)$ is denoted by Π_C^f and

$$\Pi_C^f(x) = \arg\min\{D_f(y, x) : y \in C\}.$$

Moreover, $\Pi_C^f(x)$ has the following properties (see [9]): for each $x \in H$,

$$z = \Pi_C^f(x) \text{ if and only if } \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \ \ \forall y \in C \tag{2.3}$$

and

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \ \ \forall y \in C. \tag{2.4}$$

A convex and differentiable function f is said to be σ-*strongly convex* if there exists a constant $\sigma > 0$ such that

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\sigma}{2}\|x - y\|^2 \tag{2.5}$$

for any $x \in \operatorname{dom} f$ and $y \in \operatorname{int}(\operatorname{dom} f)$. From the definition of the Bregman distance, we get the following inequality:

$$D_f(x, y) \geq \frac{\sigma}{2}\|x - y\|^2. \tag{2.6}$$

For more examples of σ-strongly convex function, we recommend [6]. We also know that if $f : H \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of H, then ∇f is uniformly continuous on bounded subsets of H (see [34]). Moreover, if f is assumed to be σ-strongly convex, Fréchet differentiable and bounded on bounded subsets of H, then for any two sequences $\{x_n\}$ and $\{y_n\}$ in H, we have

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Longrightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0 \Longrightarrow \lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{2.7}$$

Lemma 2.4 ([13]) *Let $f : H \to \mathbb{R}$ be a strongly convex and differentiable function with its gradient ∇f is sequentially weak-to-weak continuous. Suppose that $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, then*

$$\liminf_{n\to\infty} D_f(x, x_n) < \liminf_{n\to\infty} D_f(y, x_n)$$

for all $y \in H$ with $y \neq x$.

Lemma 2.5 ([11]) *Let C be a nonempty, closed and convex subset of H and A be a pseudo-monotone and continuous of C into H. Then z is a solution of VIP if and only if*

$$\langle Ax, x - z\rangle \geq 0, \quad \forall x \in C.$$

3 Main results

In this section, we discuss weak and strong convergence of two Bregman projection algorithms for solving pseudo-monotone variational inequalities. To establish the convergence results, we need the following assumptions:

(A1) The feasible set C is a closed and convex subset of a real Hilbert space H.
(A2) The function $f : H \to \mathbb{R}$ is σ-strongly convex which is bounded and uniformly Fréchet differentiable on bounded subsets of H.
(A3) The mapping $A : H \to H$ is pseudo-monotone and Lipschitz continuous with a constant $L > 0$.
(A4) The mapping $A : H \to H$ satisfies the following property: for each $\{q_n\} \subset C$,

$$q_n \rightharpoonup x \text{ implies } \|Ax\| \leq \liminf_{n\to\infty} \|Aq_n\|.$$

(A5) The solution set of VIP is nonempty, that is, $VI(C, A) \neq \emptyset$.

Remark 3.1 (1) In Assumption **(A3)**, if H is a finite-dimensional space, then it suffices to assume that the mapping A is continuous pseudo-monotone.
(2) It is clear that Assumption **(A4)** is weaker than the sequential weak-to-weak continuity of the mapping A (see [1,35]). The example of the mapping A which satisfy Assumption **(A4)** but is not sequential weak-to-weak continuous can be found in [41]. However, if A is monotone, then Assumption **(A4)** can be dropped.

3.1 Weak convergence

Now, we propose a new Bregman projection algorithm for solving VIP of pseudo-monotone mappings.

Algorithm 1:

Given $\lambda_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.6). Let $x_1 \in H$ be arbitrary. Set $n = 1$.
Step 1. Compute

$$y_n = \Pi_C^f (\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A x_n).$$

If $x_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of VIP. Otherwise,
Step 2. Compute

$$x_{n+1} = (\nabla f)^{-1}(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{2\langle Ax_n - Ay_n, x_{n+1} - y_n\rangle}, \lambda_n\right\} & \text{if } \langle Ax_n - Ay_n, x_{n+1} - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.1}$$

Set $n := n + 1$ go to **Step 1**.

Lemma 3.2 ([46]) *Let $\{\lambda_n\}$ be a sequence generated by (3.1), then $\lambda \geq \min\{\lambda_1, \frac{\mu}{L}\}$ with $\lambda = \lim_{n\to\infty} \lambda_n$.*

Lemma 3.3 *Suppose that Assumptions **(A1)–(A5)** are satisfied. Let $\{x_n\}$ be generated by Algorithm 1. If there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some point in H and $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, then*

(i) $\liminf_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0, \forall x \in C$;
(ii) $\omega_w(x_n) \subset VI(C, A)$.

Proof (i) Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $w \in H$ and $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$. By the definition of y_{n_k} and (2.3), we have

$$\langle \nabla f(y_{n_k}) - \nabla f(x_{n_k}) + \lambda_{n_k} Ax_{n_k}, x - y_{n_k}\rangle \geq 0, \ \forall x \in C,$$

which implies that

$$\lambda_{n_k}\langle Ax_{n_k}, x - y_{n_k}\rangle \geq \langle \nabla f(x_{n_k}) - \nabla f(y_{n_k}), x - y_{n_k}\rangle, \ \forall x \in C.$$

Moreover, we have

$$\langle Ax_{n_k}, x - x_{n_k}\rangle \geq \left\langle \frac{\nabla f(x_{n_k}) - \nabla f(y_{n_k})}{\lambda_{n_k}}, x - y_{n_k}\right\rangle + \langle Ax_{n_k}, y_{n_k} - x_{n_k}\rangle, \ \forall x \in C. \tag{3.2}$$

Note that since f is uniformly Fréchet differentiable, ∇f is uniformly continuous on bounded subsets of H and so $\|\nabla f(x_{n_k}) - \nabla f(y_{n_k})\| \to 0$ as $k \to \infty$. Hence from (3.2) with the fact that $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$ and $\{Ax_{n_k}\}$ is bounded, we have

$$\liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0, \ \forall x \in C. \tag{3.3}$$

On the other hand, we observe that

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Ax_{n_k}, x - x_{n_k}\rangle + \langle Ax_{n_k}, x - x_{n_k}\rangle + \langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle.$$

By the uniform continuity of A and (3.3), we have

$$\liminf_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0. \tag{3.4}$$

(ii) Let $\{\epsilon_k\}$ be a decreasing sequence of positive real numbers such that $\epsilon_k \to 0$ as $k \to \infty$. For each ϵ_k, we denote by N_k the smallest positive integer such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \epsilon_k \geq 0, \ \ \forall j \geq N_k,$$

where the existence of N_k follows from (i). In particular, letting $j = N_k$, we have

$$\langle Ay_{n_{N_k}}, x - y_{n_{N_k}}\rangle + \epsilon_k \geq 0. \tag{3.5}$$

Since $\{\epsilon_k\}$ is decreasing, it follows that the sequence $\{N_k\}$ is increasing. Moreover, for each k, since $y_{N_k} \in C$, we can suppose $Ay_{N_k} \neq 0$ (otherwise, y_{N_k} is a solution of VIP). Setting

$$u_{N_k} = \frac{Ay_{N_k}}{\|Ay_{N_k}\|^2}.$$

It is clear that $\langle Ay_{N_k}, u_{N_k}\rangle = 1$ for each k. Thus we can write (3.5) as

$$\langle Ay_{N_k}, x + \epsilon_k u_{N_k} - y_{N_k}\rangle \geq 0.$$

The pseudo-monotonicity of A implies that

$$\langle A(x + \epsilon_k u_{N_k}), x + \epsilon_k u_{N_k} - y_{N_k}\rangle \geq 0.$$

It follows that

$$\langle Ax, x - y_{N_k}\rangle \geq \langle Ax - A(x + \epsilon_k u_{N_k}), x + \epsilon_k u_{N_k} - y_{N_k}\rangle - \epsilon_k\langle Ax, u_{N_k}\rangle. \quad (3.6)$$

Now, we show that $\lim_{k\to\infty} \epsilon_k u_{N_k} = 0$. Since $\{y_{n_k}\} \subset C$, we have $w \in C$. We assume that $Aw \neq 0$ (otherwise, w is a solution of VIP). Then the Assumption **(A4)** implies that

$$0 < \|Aw\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|.$$

Thus we have

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k u_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ay_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} \leq \frac{0}{\|Aw\|} = 0.$$

Hence $\lim_{k\to\infty} \epsilon_k u_{N_k} = 0$. Since $\{y_{N_k}\} \subset \{y_{n_k}\}$, we have $y_{N_k} \rightharpoonup w$. Hence from (3.6), we obtain

$$\langle Ax, x - w\rangle \geq 0, \ \forall x \in C.$$

By Lemma 2.5, we get $w \in VI(C, A)$, that is, $\omega_w(x_n) \subset VI(C, A)$. □

Theorem 3.4 *Suppose that Assumptions **(A1)–(A5)** are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $VI(C, A)$.*

Proof First, we prove that $\{x_n\}$ is bounded. Let $p \in VI(C, A)$. By the definition of Bregman distance, we have

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, (\nabla f)^{-1}(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n))) \\
&= f(p) - f(x_{n+1}) - \langle\nabla f(y_n) - \lambda_n(Ay_n - Ax_n), p - x_{n+1}\rangle \\
&= f(p) - f(x_{n+1}) - \langle\nabla f(y_n), p - x_{n+1}\rangle + \lambda_n\langle Ay_n - Ax_n, p - x_{n+1}\rangle \\
&= f(p) - f(y_n) - \langle\nabla f(y_n), p - y_n\rangle + \langle\nabla f(y_n), p - y_n\rangle \\
&\quad + f(y_n) - f(x_{n+1}) - \langle\nabla f(y_n), p - x_{n+1}\rangle \\
&\quad + \lambda_n\langle Ay_n - Ax_n, p - x_{n+1}\rangle \\
&= f(p) - f(y_n) - \langle\nabla f(y_n), p - y_n\rangle - f(x_{n+1}) + f(y_n) + \langle\nabla f(y_n), x_{n+1} - y_n\rangle \\
&\quad + \lambda_n\langle Ay_n - Ax_n, p - x_{n+1}\rangle \\
&= D_f(p, y_n) - D_f(x_{n+1}, y_n) + \lambda_n\langle Ay_n - Ax_n, p - x_{n+1}\rangle. \quad (3.7)
\end{aligned}$$

From (2.2), we see that

$$D_f(p, y_n) = D_f(p, x_n) - D_f(y_n, x_n) + \langle\nabla f(x_n) - \nabla f(y_n), p - y_n\rangle. \quad (3.8)$$

Substituting (3.8) into (3.7), we get

$$D_f(p, x_{n+1}) = D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \langle\nabla f(x_n) - \nabla f(y_n), p - y_n\rangle$$

$$+\lambda_n\langle Ay_n - Ax_n, p - x_{n+1}\rangle. \tag{3.9}$$

By the definition of y_n, we have

$$\langle \nabla f(x_n) - \lambda_n Ax_n - \nabla f(y_n), p - y_n\rangle \le 0,$$

which implies that

$$\langle \nabla f(x_n) - \nabla f(y_n), p - y_n\rangle \le \lambda_n\langle Ax_n, p - y_n\rangle. \tag{3.10}$$

Substituting (3.10) into (3.9), we get

$$\begin{aligned}
D_f(p, x_{n+1}) &\le D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\
&\quad + \lambda_n\langle Ax_n, p - y_n\rangle + \lambda_n\langle Ay_n - Ax_n, p - x_{n+1}\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n\langle Ax_n, p - y_n\rangle + \lambda_n\langle Ay_n, p - x_{n+1}\rangle \\
&\quad - \lambda_n\langle Ax_n, p - x_{n+1}\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n\langle Ax_n, x_{n+1} - y_n\rangle + \lambda_n\langle Ay_n, p - x_{n+1}\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n\langle Ax_n, x_{n+1} - y_n\rangle - \lambda_n\langle Ay_n, y_n - p\rangle \\
&\quad + \lambda_n\langle Ay_n, y_n - x_{n+1}\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n\langle Ax_n - Ay_n, x_{n+1} - y_n\rangle \\
&\quad - \lambda_n\langle Ay_n, y_n - p\rangle.
\end{aligned}$$

Using the fact that $\langle Ap, y_n - p\rangle \ge 0$ and the pseudo-monotonicity of A, we have $\langle Ay_n, y_n - p\rangle \ge 0$. By the definition of λ_{n+1}, we have

$$\begin{aligned}
D_f(p, x_{n+1}) &\le D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n\langle Ax_n - Ay_n, x_{n+1} - y_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \frac{\lambda_n}{\lambda_{n+1}}\lambda_{n+1}\langle Ax_n - Ay_n, x_{n+1} - y_n\rangle \\
&\le D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \frac{\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2) \\
&= D_f(p, x_n) - D_f(y_n, x_n) + \frac{\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}\|x_n - y_n\|^2 \\
&\quad - D_f(x_{n+1}, y_n) + \frac{\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}\|x_{n+1} - y_n\|^2.
\end{aligned}$$

Using (2.6), we have

$$D_f(p, x_{n+1}) \le D_f(p, x_n) - \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n, x_n) - \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(x_{n+1}, y_n). \tag{3.11}$$

Since $\lim_{n\to\infty}\lambda_n = \lambda > 0$ and $\mu \in (0, \sigma)$, we have

$$\lim_{n\to\infty}\left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \frac{\mu}{\sigma} = \frac{\sigma - \mu}{\sigma} > 0.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}} > 0, \quad \forall n \geq n_0.$$

Consequently,

$$\left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) + \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(x_{n+1}, y_n) \geq 0, \quad \forall n \geq n_0.$$

It follows from (3.11) that

$$D_f(p, x_{n+1}) \leq D_f(p, x_n).$$

This shows that $\lim_{n\to\infty} D_f(p, x_n)$ exists and hence $\{D_f(p, x_n)\}$ is bounded. Applying (2.6), we have $\{x_n\}$ is bounded. On the other hand, from (3.11), we have

$$\begin{aligned}\left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) &+ \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(x_{n+1}, y_n) \\ &\leq D_f(p, x_n) - D_f(p, x_{n+1}).\end{aligned}$$

Since $\lim_{n\to\infty} D_f(p, x_n)$ exists, there exists a nonnegative constant γ such that $\gamma = \lim_{n\to\infty} D_f(p, x_n) = \lim_{n\to\infty} D_f(p, x_{n+i})$ for all $i \in \mathbb{N}$. Thus we have $\lim_{n\to\infty} D_f(y_n, x_n) = \lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$ and hence

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0.$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z \in C$. Similar to Lemma 3.3, we can show that $z \in VI(C, A)$. Finally, we prove that the sequence $\{x_n\}$ converges weakly to z. In order to do this, it is sufficient to show that $\{x_n\}$ has a unique weak cluster point in $VI(C, A)$. Let $\{x_{m_k}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_k} \rightharpoonup z'$. As it has been proven above, we have $z' \in VI(C, A)$. Since $\lim_{n\to\infty} D_f(u, x_n)$ exists for any $u \in VI(C, A)$, it follows from Lemma 2.4 that

$$\begin{aligned}\lim_{n\to\infty} D_f(z, x_n) = \lim_{k\to\infty} D_f(z, x_{n_k}) &= \liminf_{k\to\infty} D_f(z, x_{n_k}) \\ &< \liminf_{k\to\infty} D_f(z', x_{n_k}) = \lim_{k\to\infty} D_f(z', x_{m_k}) \\ &= \lim_{n\to\infty} D_f(z', x_n).\end{aligned}$$

In a similar way as above, we have

$$\lim_{n\to\infty} D_f(z', x_n) < \lim_{n\to\infty} D_f(z, x_n).$$

This is a contradiction. Hence $z = z'$ and we conclude that the sequence $\{x_n\}$ converges weakly to a point in $VI(C, A)$. This completes the proof. □ ‘

3.2 Strong convergence

Next, we propose another Bregman projection algorithm for solving VIP of pseudo-monotone mappings which is based on the hybrid projection method.

Algorithm 2:

Given $\lambda_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.6). Let $x_1 \in H$ be arbitrary. Set $n = 1$.
Step 1. Compute

$$y_n = \Pi_C^f (\nabla f)^{-1}(\nabla f(x_n) - \lambda_n A x_n).$$

If $x_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of VIP. Otherwise,
Step 2. Compute

$$z_n = (\nabla f)^{-1}(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\|x_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle Ax_n - Ay_n, z_n - y_n\rangle}, \lambda_n\right\} & \text{if } \langle Ax_n - Ay_n, z_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.12}$$

Step 3. Compute

$$x_{n+1} = \Pi^f_{C_n \cap Q_n}(x_1),$$

where

$$C_n = \{u \in H : D_f(u, z_n) \le D_f(u, x_n) - \theta_n\},$$
$$Q_n = \{u \in H : \langle \nabla f(x_1) - \nabla f(x_n), u - x_n\rangle \le 0\}$$

and

$$\theta_n = \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) + \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n).$$

Set $n := n + 1$ go to **Step 1**.

Theorem 3.5 *Suppose that Assumptions **(A1)–(A5)** are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z \in VI(C, A)$ with $z = \Pi^f_{VI(C,A)}(x_1)$.*

Proof

Claim 1 We show that $\{x_n\}$ is well-defined.
First, we show that C_n and Q_n are closed and convex for all $n \in \mathbb{N}$.

(i) From the definition of C_n, we see that

$$D_f(u, z_n) \leq D_f(u, x_n) - \theta_n$$

is equivalent to

$$\langle \nabla f(z_n) - \nabla f(x_n), u\rangle \leq f(z_n) - f(x_n) - \langle \nabla f(z_n), z_n\rangle + \langle \nabla f(x_n), x_n\rangle - \theta_n. \tag{3.13}$$

Let $u_1, u_2 \in C_n$ and $u = tu_1 + (1-t)u_2$ for $t \in (0, 1)$. Then we have $u \in H$ and

$$\begin{aligned}\langle \nabla f(z_n) - \nabla f(x_n), u\rangle &= \langle \nabla f(z_n) - \nabla f(x_n), tu_1 + (1-t)u_2\rangle \\ &= t\langle \nabla f(z_n) - \nabla f(x_n), u_1\rangle + (1-t)\langle \nabla f(z_n) - \nabla f(x_n), u_2\rangle \\ &\leq f(z_n) - f(x_n) - \langle \nabla f(z_n), z_n\rangle + \langle \nabla f(x_n), x_n\rangle - \theta_n.\end{aligned}$$

This implies that $u \in C_n$, that is, C_n is convex. Let $\{u_m\}$ be a sequence in C_n such that $u_m \to u$. Then we have

$$\langle \nabla f(z_n) - \nabla f(x_n), u_m\rangle \leq f(z_n) - f(x_n) - \langle \nabla f(z_n), z_n\rangle + \langle \nabla f(x_n), x_n\rangle - \theta_n$$

for all $m \in \mathbb{N}$. Taking limit as $m \to \infty$ in above inequality, we have (3.13) holds. This implies that $u \in C_n$, that is, C_n is closed.

(ii) Let $u_1, u_2 \in Q_n$ and $u = tu_1 + (1-t)u_2$ for $t \in (0, 1)$. Then we have $u \in H$ and

$$\begin{aligned}&\langle \nabla f(x_1) - \nabla f(x_n), u - x_n\rangle = t\langle \nabla f(x_1) - \nabla f(x_n), u_1 - x_n\rangle + (1-t)\langle \nabla f(x_1) \\ &\quad - \nabla f(x_n), u_2 - x_n\rangle \leq 0.\end{aligned}$$

This implies that $u \in Q_n$, that is, Q_n is convex. Let $\{u_m\}$ be a sequence in Q_n such that $u_m \to u$. We see that $\langle \nabla f(x_1) - \nabla f(x_n), u_m - x_n\rangle \leq 0$ for all $m \in \mathbb{N}$ and hence $\langle \nabla f(x_1) - \nabla f(x_n), u - x_n\rangle \leq 0$. This implies that $u \in Q_n$, that is, Q_n is closed. Therefore, $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. Let $p \in VI(C, A)$. Using the same arguments as in Theorem 3.4, we can show that

$$D_f(p, z_n) \leq D_f(p, x_n) - \theta_n. \tag{3.14}$$

This means that $VI(C, A) \subset C_n$ for all $n \in \mathbb{N}$. For $n = 1$, we have $Q_1 = H$ and hence $VI(C, A) \subset C_1 \cap Q_1$. Suppose that $VI(C, A) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. Then there exists a unique element x_{k+1} such that $x_{k+1} = \Pi^f_{C_k \cap Q_k}(x_1)$. Also by (2.3), we have

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), w - x_{k+1}\rangle \leq 0, \quad \forall w \in C_k \cap Q_k.$$

Since $VI(C, A) \subset C_k \cap Q_k$, we have

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), v - x_{k+1}\rangle \leq 0, \quad \forall v \in VI(C, A).$$

This implies that $VI(C, A) \subset Q_{k+1}$ and hence $VI(C, A) \subset C_{k+1} \cap Q_{k+1}$. By induction, we conclude $VI(C, A) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$ and hence $\{x_n\}$ is well-defined.

Claim 2 We show that $\{x_n\}$ is bounded.
By the definition of Q_n, we have

$$\langle \nabla f(x_1) - \nabla f(x_n), v - x_n \rangle \leq 0, \quad \forall v \in Q_n. \tag{3.15}$$

From (2.3), we have $x_n = \Pi^f_{Q_n}(x_1)$ and so

$$D_f(x_n, x_1) \leq D_f(v, x_1) - D_f(v, x_n) \leq D_f(v, x_1) \tag{3.16}$$

for all $v \in Q_n$. Since $VI(C, A) \subset Q_n$, we have $D_f(x_n, x_1) \leq D_f(p, x_1)$ for all $p \in VI(C, A)$. This implies that $\{D_f(x_n, x_1)\}$ is bounded. Now applying (2.6), we also have $\{x_n\}$ is bounded.

Claim 3 We show that $\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0$.
Since $x_{n+1} \in Q_n$ and from (3.16), we have $D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1)$. This shows that $\{D_f(x_n, x_1)\}$ is nondecreasing. So $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Again since $x_{n+1} \in Q_n$, it follows from (2.2) and (3.15) that

$$\begin{aligned} D_f(x_{n+1}, x_n) &= D_f(x_{n+1}, x_1) - D_f(x_n, x_1) + \langle \nabla f(x_1) - \nabla f(x_n), x_{n+1} - x_n \rangle \\ &\leq D_f(x_{n+1}, x_1) - D_f(x_n, x_1) \\ &\to 0, \quad n \to \infty \end{aligned}$$

and hence

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0.$$

From (2.1), we see that

$$\begin{aligned} D_f(x_{n+1}, x_n) &\leq \langle \nabla f(x_{n+1}) - \nabla f(x_n), x_{n+1} - x_n \rangle \\ &\leq \|\nabla f(x_{n+1}) - \nabla f(x_n)\| M \\ &\to 0, \quad n \to \infty, \end{aligned}$$

where $M > 0$. Again since $x_{n+1} \in C_n$, it follows that

$$D_f(x_{n+1}, z_n) \leq D_f(x_{n+1}, x_n) - \theta_n \leq D_f(x_{n+1}, x_n) \to 0, \quad n \to \infty$$

and hence

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(z_n)\| = 0.$$

Consequently,

$$\begin{aligned}\|\nabla f(z_n)-\nabla f(x_n)\| &\le \|\nabla f(z_n)-\nabla f(x_{n+1})\|+\|\nabla f(x_{n+1})-\nabla f(x_n)\|\\ &\to 0,\quad n\to\infty. \end{aligned}\tag{3.17}$$

From (3.14), we have

$$\Big(1-\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(y_n,x_n)+\Big(1-\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(z_n,y_n)\le D_f(p,x_n)-D_f(p,z_n). \tag{3.18}$$

As proved in Theorem 3.4, we can deduce that

$$\Big(1-\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(y_n,x_n)+\Big(1-\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(z_n,y_n)\ge 0,\quad \forall n\ge n_0.$$

From (2.2), we see that

$$D_f(p,x_n)=D_f(p,z_n)-D_f(x_n,z_n)+\langle\nabla f(z_n)-\nabla f(x_n),p-x_n\rangle.$$

This implies that

$$\begin{aligned}D_f(p,x_n)-D_f(p,z_n)&=-D_f(x_n,z_n)+\langle\nabla f(z_n)-\nabla f(x_n),p-x_n\rangle\\ &\le\langle\nabla f(z_n)-\nabla f(x_n),p-x_n\rangle\\ &\le\|\nabla f(z_n)-\nabla f(x_n)\|K\end{aligned}$$

for some $K>0$. Thus we have

$$\lim_{n\to\infty}(D_f(p,x_n)-D_f(p,z_n))=0. \tag{3.19}$$

Combining (3.18) and (3.19), we get $\lim_{n\to\infty}D_f(y_n,x_n)=\lim_{n\to\infty}D_f(z_n,y_n)=0$ and hence

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\|z_n-y_n\|=0.$$

Claim 4 We show that $\omega_w(x_n)\subset VI(C,A)$.
By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup v\in C$. Note that $\lim_{n\to\infty}\|y_n-x_n\|=0$. Using the same arguments as in Lemma 3.3, we can show that $v\in VI(C,A)$, that is, $\omega_w(x_n)\subset VI(C,A)$.

Claim 5 We show that $x_n\to z$ with $z=\Pi^f_{VI(C,A)}(x_1)$.

Let $z=\Pi^f_{VI(C,A)}(x_1)$. Since $x_n=\Pi^f_{C_n}(x_1)$ and $VI(C,A)\subset C_n\cap Q_n$, we have $D_f(x_{n+1},x_1)\le D_f(z,x_1)$. Also, from Claim 3, we know that $\{D_f(x_n,x_1)\}$ is nondecreasing. This implies that

$$D_f(x_n,x_1)\le D_f(x_{n+1},x_1)\le D_f(z,x_1).$$

Hence from (2.2), we have

$$D_f(x_n, z) = D_f(x_n, x_1) - D_f(z, x_1) + \langle \nabla f(x_1) - \nabla f(z), x_n - z\rangle \\ \leq \langle \nabla f(x_1) - \nabla f(z), x_n - z\rangle.$$

In particular, we have

$$D_f(x_{n_k}, z) \leq \langle \nabla f(x_1) - \nabla f(z), x_{n_k} - z\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in C$. From Claim 4, we have $v \in VI(C, A)$. Thus by (2.3), we have

$$\limsup_{k\to\infty} D_f(x_{n_k}, z) \leq \limsup_{k\to\infty} \langle \nabla f(x_1) - \nabla f(z), x_{n_k} - z\rangle \leq 0.$$

This implies that $\lim_{k\to\infty} D_f(x_{n_k}, z) = 0$ and so $x_{n_k} \to z$. Suppose that $\{x_{m_k}\}$ is another subsequence of $\{x_n\}$ such that $x_{m_k} \rightharpoonup z'$. In a similar way as above, we get $x_{m_k} \to z'$. Therefore, $\{x_n\}$ converges strongly to z. The proof is completed. □

4 Numerical examples

4.1 Numerical behavior of Algorithms 1 and 2

In this subsection, we provide some numerical experiments to illustrate the numerical convergence of Algorithms 1 and 2 for various examples of the Bregman distance. The following list are various functions with its Bregman distances:

(i) Define the function $f^{KL}(x) = \sum_{i=1}^m x_i \ln x_i$ with domain $\operatorname{dom} f^{KL} = \{x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m : x_i > 0, i = 1, 2, \ldots, m\}$ and range $\operatorname{ran} f^{KL} = (-\infty, \infty)$. Then

$$\nabla f^{KL}(x) = (1 + \ln(x_1), 1 + \ln(x_2), \ldots, 1 + \ln(x_m))^T$$

and

$$\left(\nabla f^{KL}\right)^{-1}(x) = (\exp(x_1 - 1), \exp(x_2 - 1), \ldots, \exp(x_m - 1))^T.$$

So we have the Kullback–Leibler distance given by

$$D_f^{KL}(x, y) = \sum_{i=1}^m \left(x_i \ln\left(\frac{x_i}{y_i}\right) + y_i - x_i\right).$$

(ii) Define the function $f^{IS}(x) = -\sum_{i=1}^m \ln x_i$ with domain $\operatorname{dom} f^{IS} = \{x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m : x_i > 0, i = 1, 2, \ldots, m\}$ and range $\operatorname{ran} f^{IS} =$

$(-\infty, \infty)$. Then

$$\nabla f^{IS}(x) = -\Big(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_m}\Big)^T$$

and

$$\left(\nabla f^{IS}\right)^{-1}(x) = -\Big(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_m}\Big)^T.$$

So we have the Itakura–Saito distance given by

$$D_f^{IS}(x, y) = \sum_{i=1}^{m} \Big(\frac{x_i}{y_i} - \ln\Big(\frac{x_i}{y_i}\Big) - 1\Big).$$

(iii) Define the function $f^{SE}(x) = \frac{1}{2}\|x\|^2$ with domain $\operatorname{dom} f^{SE} = H$ and range $\operatorname{ran} f^{SE} = [0, \infty)$. Then $\nabla f^{SE}(x) = x$ and $\left(\nabla f^{SE}\right)^{-1}(x) = x$. So we have the squared Euclidean distance given by

$$D_f^{SE}(x, y) = \frac{1}{2}\|x - y\|^2.$$

It is clear that f^{KL}, f^{IS} and f^{SE} satisfy the Assumption **(A2)** with $\sigma = 1$ (see [14]).

Example 4.1 Consider a problem which is taken from [7,41]. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be an operator defined by

$$Ax = \left(e^{-x^T Qx} + \alpha\right)(Mx + p),$$

where Q is a positive definite matrix, M is a positive semi-definite matrix, $p \in \mathbb{R}^m$ and $\alpha > 0$. It can be shown that A is pseudo-monotone but not monotone. Moreover, A is Lipschitz continuous and sequentially weak-to-weak continuous (hence A satisfies the Assumption **(A4)**). The feasible set C is given by

$$C = \{x = (x_1, x_2, \dots, x_m)^T \in \mathbb{R}^m : \|x\| \le 1 \text{ and } x_i > 0, i = 1, 2, \dots, m\}.$$

In this experiment, we take $\alpha = 0.36$, $M = N^T N$, $Q = P^T P$ with $N, P \in \mathbb{R}^{m\times m}$, $p \in \mathbb{R}^m$ are generated randomly from a normal distribution with mean zero and unit variance. In all methods, we take $\lambda_1 = 0.24$ and the starting point is generated randomly in $\mathbb{R}^m$. We perform the numerical tests of Algorithms 1 and 2 with $m = 10, 30, 50, 100$ and use $E_n = \|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion. The numerical results for the performance of Algorithm 1 are shown in Table 1 and Fig. 1, while the numerical results for the performance of Algorithm 2 are shown in Table 2 and Fig. 2.

Table 1 Numerical results of Algorithm 1 for Example 4.1

	KL	IS	SE
$m = 10$			
Iter.	5	17	30
Time (s)	0.0134	0.0279	0.0279
$m = 30$			
Iter.	5	17	30
Time (s)	0.0359	0.0371	0.0572
$m = 50$			
Iter.	5	18	32
Time (s)	0.0529	0.0609	0.0991
$m = 100$			
Iter.	5	18	31
Time (s)	0.0383	0.0827	0.1425

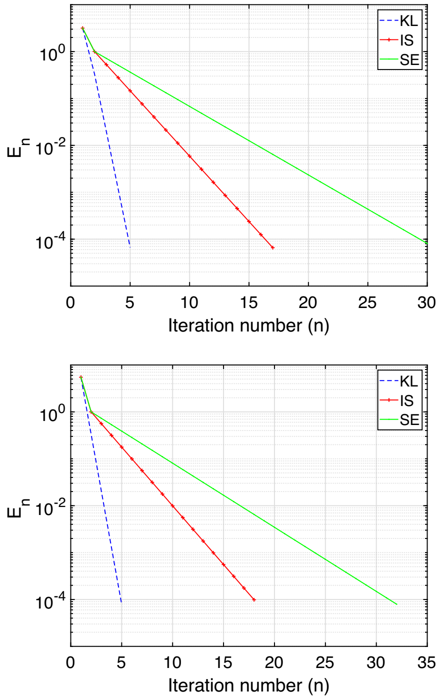

Fig. 1 Example 4.1 using Algorithm 1, top left: $m = 10$; top right: $m = 30$; bottom left: $m = 50$; bottom right: $m = 100$

Table 2 Numerical results of Algorithm 2 for Example 4.1

	KL	IS	SE
$m = 10$			
Iter.	9	15	18
Time (s)	0.0087	0.0118	0.0412
$m = 30$			
Iter.	8	16	19
Time (s)	0.0199	0.0202	0.0263
$m = 50$			
Iter.	9	17	19
Time (s)	0.0193	0.0238	0.0344
$m = 100$			
Iter.	9	17	20
Time (s)	0.0526	0.0770	0.0874

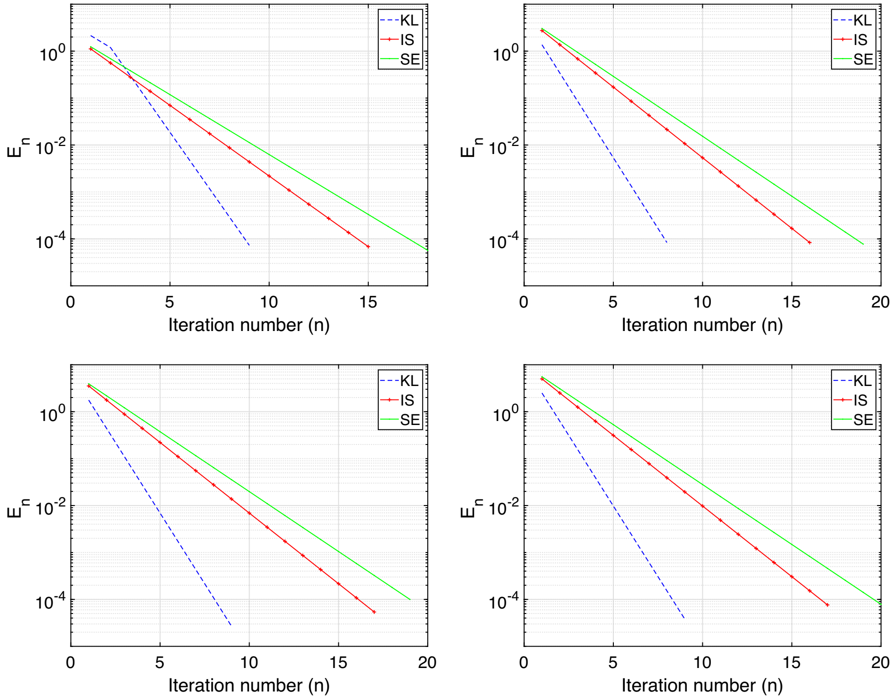

Fig. 2 Example 4.1 using Algorithm 2, top left: $m = 10$; top right: $m = 30$; bottom left: $m = 50$; bottom right: $m = 100$

Table 3 Numerical results for Example 4.2

	Algorithm 1	Algorithm 2	TEGM	SEGM
$m = 10$				
Iter.	6	17	26	30
Time (s)	0.0144	0.0494	0.1225	0.1682
$m = 30$				
Iter.	6	18	25	32
Time (s)	0.0201	0.0561	0.1615	0.2912
$m = 50$				
Iter.	6	18	19	33
Time (s)	0.0236	0.0703	0.1590	0.3925
$m = 100$				
Iter.	7	19	27	34
Time (s)	0.0882	0.1210	0.2662	1.1033

4.2 Comparison of Algorithms 1 and 2 with other algorithms

In this subsection, we compare the proposed algorithms to TEGM (Algorithm (1.7)) and SEGM (Algorithm (1.9)). In what follows, let us define $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in H$.

Example 4.2 Let A, Q, M, p and α be the same as in Example 4.1. The feasible set is $C = \{x \in \mathbb{R}^m : Bx \leq b\}$, where B is an $l \times m$ matrix and b is a nonnegative vector in $\mathbb{R}^l$ ($l = 10$). In this experiment, we perform the numerical tests of Algorithm 1, Algorithm 2, TEGM (Algorithm (1.7)) and SEGM (Algorithm (1.9)) with $m = 10, 30, 50, 100$. For TEGM and SEGM, we take $\gamma = 4$, $l = 0.59$, $\mu = 0.63$. For Algorithms 1 and 2, we take $\lambda_1 = 0.35$ and $\mu = 0.36$. The starting point is generated randomly in $\mathbb{R}^m$ and the stopping criterion is defined by $E_n = \|x_{n+1} - x_n\| < 10^{-4}$. The numerical results of all methods have been reported in the Table 3 and Fig. 3.

Example 4.3 In this example, we take $H = L_2([0, 1])$ with the norm $\|x\|_2 = \left(\int_0^1 |x(t)|^2 dt\right)^{1/2}$ and the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ for all $x, y \in L_2([0, 1])$. The feasible set is $C = \{x \in H : \|x\| \leq 1\}$. Define an integral operator $A : C \to H$ by

$$Ax(t) = \int_0^1 \Big(x(t) - f(t, s)g(x(s))\Big)ds - h(t), \;\; x \in C \;\text{ and }\; t \in [0, 1],$$

where

$$f(t, s) = \frac{2tse^{t+s}}{e\sqrt{e^2 - 1}}, \;\; g(x) = \cos x \;\text{ and }\; h(t) = \frac{2te^t}{e\sqrt{e^2 - 1}}.$$

It was shown that A is monotone (hence it is pseudo-monotone) (see [16]). Moreover, A is Lipschitz continuous. The solution set of the corresponding variational inequality

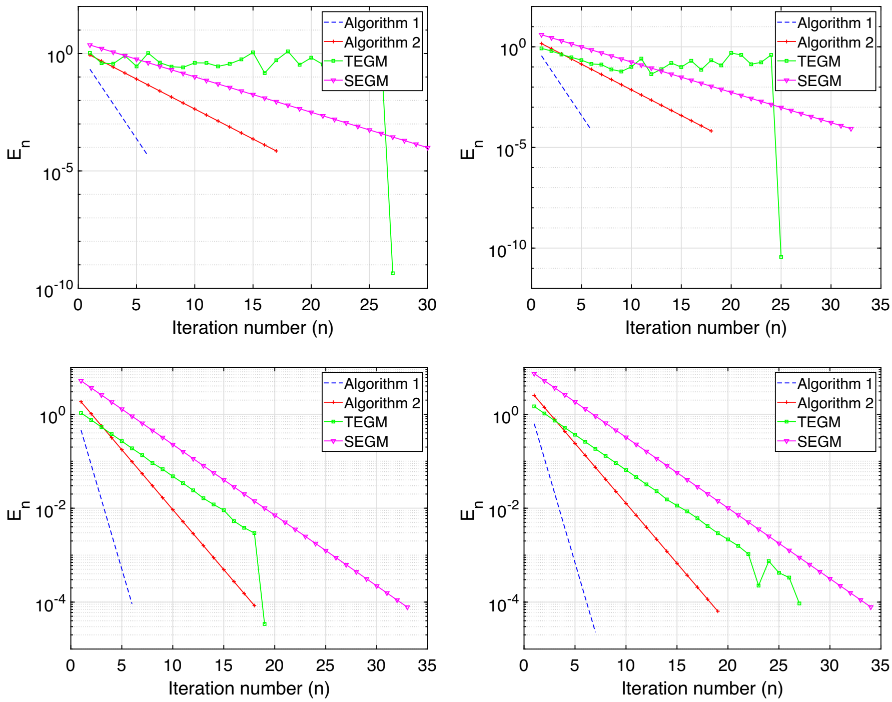

Fig. 3 Example 4.2, top left: $m = 10$; top right: $m = 30$; bottom left: $m = 50$; bottom right: $m = 100$

Table 4 Numerical results for Example 4.3

	Algorithm 1	Algorithm 2	TEGM	SEGM
Case I				
Iter.	13	10	39	24
Time (s)	6.4703	4.1315	15.9806	12.8645
Case II				
Iter.	13	10	39	24
Time (s)	6.3816	9.0463	19.7826	18.4234
Case III				
Iter.	12	9	35	23
Time (s)	4.0704	2.8576	10.7016	6.1203
Case IV				
Iter.	10	8	29	20
Time (s)	4.7375	3.1118	7.3758	6.6279

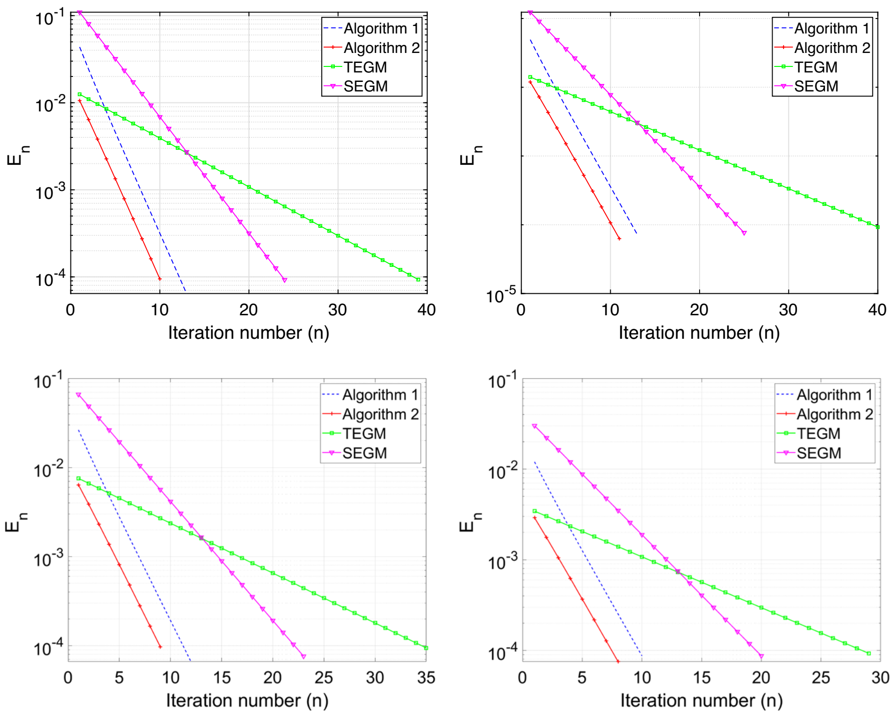

Fig. 4 Example 4.3, top left: Case I; top right: Case II; bottom left: Case III; bottom right: Case IV

problem is $VI(C, A) = \{0\}$. In this experiment, we perform the numerical tests of Algorithm 1, Algorithm 2, TEGM (Algorithm (1.7)) and SEGM (Algorithm (1.9)) with four different starting points chosen as follows:

- Case I: $x_1 = \sin(4t)$,
- Case II: $x_1 = \frac{2}{9}te^{2t}$,
- Case III: $x_1 = t^3 + 2t - 1$,
- Case IV: $x_1 = \frac{\cos(2t)}{4}$.

In Algorithm 1, Algorithm 2, we take $\lambda_1 = 0.58$ and $\mu = 0.36$. In (1.7) and (1.9), we take $\gamma = 2$, $l = 0.34$ and $\mu = 0.5$. The stopping criterion is defined by $E_n = \|x_n - 0\| < 10^{-4}$. The numerical results of all methods have been reported in the Table 4 and Fig. 4.

5 Conclusions

In this paper, we proposed two new Bregman projection methods based on Tseng's extragradient method for approximating a solution of variational inequality problem involving pseudo-monotone and Lipschitz continuous operators in a Hilbert space. The weak and strong convergence theorems are proved without any requirement of the knowledge of the Lipschitz constant of the cost operator. Finally, several numerical experiments with various functions of the Bregman distance have been performed to illustrate the convergence of the proposed algorithms including the comparisons to some known algorithms.

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Declaration

Conflict of interest There is no conflicts of interest.

References

1. Anh, P.K., Thong, D.V., Vinh, N.T.: Improved inertial extragradient methods for solving pseudomonotone variational inequalities. Optimization (2020). https://doi.org/10.1080/02331934.2020.1808644
2. Antipin, A.S.: On a method for convex programs using a symmetrical modification of the Lagrange function. Ekon. Mat. Metody **12**, 1164–1173 (1976)
3. Baiocchi, C., Capelo, A.: Variational and Quasi-variational Inequalities. Wiley, New York (1984)
4. Barbu, V., Precupanu, T.: Convexity and Optimization in Banach Spaces. Springer, Dordrecht (2010)
5. Bnouhachem, A., Noor, M.A., Hao, Z.: Some new extragradient iterative methods for variational inequalities. Nonlinear Anal. **70**, 1321–1329 (2009)
6. Beck, A.: First-Order Methods in Optimization. Society for Industrial and Applied Mathematics, Philadelphia (2017)
7. Bot, R.I., Csetnek, E.R., Vuong, P.T.: The forward–backward–forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces. Eur. J. Oper. Res. **287**, 49–60 (2020)
8. Bregman, L.M.: The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. **7**, 200–217 (1967)
9. Butnariu, D., Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. 2006, pp. 1–39 (Art. ID 84919) (2006)
10. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
11. Cottle, R.W., Yao, J.C.: Pseudo-monotone complementarity problems in Hilbert space. J. Optim. Theory Appl. **75**, 281–295 (1992)
12. Dafermos, S.: Exchange price equilibria and variational inequalities. Math. Program. **46**, 391–402 (1990)
13. Gibali, A.: A new Bregman projection method for solving variational inequalities in Hilbert spaces. Pure Appl. Funct. Anal. **3**, 403–415 (2018)
14. Hieu, D.V., Cholamjiak, P.: Modified extragradient method with Bregman distance for variational inequalities. Appl. Anal. (2020). https://doi.org/10.1080/00036811.2020.1757078
15. Hieu, D.V., Reich, S.: Two Bregman projection methods for solving variational inequalities. Optimization (2020). https://doi.org/10.1080/02331934.2020.1836634
16. Hieu, D.V., Anh, P.K., Muu, L.D.: Modified hybrid projection methods for finding common solutions to variational inequality problems. Comput. Optim. Appl. **66**, 75–96 (2017)
17. Iusem, A.N., Svaiter, B.F.: A variant of Korpelevich's method for variational inequalities with a new search strategy. Optimization **42**, 309–321 (1997)
18. Jolaoso, L.O., Aphane, M.: Weak and strong convergence Bregman extragradient schemes for solving pseudo-monotone and non-Lipschitz variational inequalities. J. Inequ. Appl. **2020**, 195 (2020)
19. Jolaoso, L.O., Aphane, M., Khan, S.H.: Two Bregman projection methods for solving variational inequality problems in Hilbert spaces with applications to signal processing. Symmetry **2020**, 12 (2007). https://doi.org/10.3390/sym12122007
20. Khanh, P.D., Vuong, P.T.: Modified projection method for strongly pseudo-monotone variational inequalities. J. Glob. Optim. **58**, 341–350 (2014)

21. Khanh, P.Q., Thong, D.V., Vinh, N.T.: Versions of the subgradient extragradient method for pseudomonotone variational inequalities. Acta Appl. Math. **170**, 319–345 (2020)
22. Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity. SIAM, Philadelphia (1988)
23. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
24. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Matecon **12**, 747–756 (1976)
25. Liu, L., Qin, X.: Strong convergence of an extragradient-like algorithm involving pseudo-monotone mappings. Numer. Algorithms **83**, 1577–1590 (2020)
26. Nakajo, K., Takahashi, W.: Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J. Math. Anal. Appl. **279**, 372–379 (2003)
27. Noor, M.A.: New approximation schemes for general variational inequalities. J. Math. Anal. Appl. **251**, 217–229 (2000)
28. Noor, M.A.: Projection–splitting algorithms for general mixed variational inequalities. J. Comput. Anal. Appl. **4**, 47–61 (2002)
29. Noor, M.A.: Variational Inequalities and Applications, Lecture Notes. Mathematics Department, COMSATS Institute of Information Technology, Islamabad (2007)
30. Noor, M.A., Noor, K.I.: Self-adaptive projection algorithms for general variational inequalities. Appl. Math. Comput. **151**, 659–670 (2004)
31. Noor, M.A., Noor, K.I., Rassias, T.M.: Some aspects of variational inequalities. J. Comput. Appl. Math. **47**, 285–312 (1993)
32. Noor, M.A., Huang, Z.: Wiener–Hopf equation technique for variational inequalities and nonexpansive mappings. Appl. Math. Comput. **191**, 504–510 (2007)
33. Popov, L.D.: A modification of the Arrow–Hurwicz method for searching for saddle points. Mat. Zametki **28**, 777–784 (1980)
34. Reich, S., Sabach, S.: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. **10**, 471–485 (2009)
35. Reich, S., Thong, D.V., Dong, Q.L., Xiao, H.L., Dung, V.T.: New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings. Numer. Algorithms (2020). https://doi.org/10.1007/s11075-020-00977-8
36. Stampacchia, G.: Formes bilinéaires coercitives sur les ensembles convexes. C. R. Acad. Sci. Paris **258**, 4413–4416 (1964)
37. Thong, D.V., Gibali, A.: Two strong convergence subgradient extragradient methods for solving variational inequalities in Hilbert spaces. Jpn. J. Ind. Appl. Math. **36**, 299–321 (2019)
38. Thong, D.V., Hieu, D.V.: Weak and strong convergence theorems for variational inequality problems. Numer. Algorithms **78**, 1045–1060 (2018)
39. Thong, D.V., Vuong, P.T.: Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities. Optimization **68**, 2207–2226 (2019)
40. Thong, D.V., Shehu, Y., Iyiola, O.S., Thang, H.V.: New hybrid projection methods for variational inequalities involving pseudomonotone mappings. Optim. Eng. **22**, 363–386 (2021)
41. Thong, D.V., Yang, J., Cho, Y.J., Rassias, T.M.: Explicit extragradient-like method with adaptive stepsizes for pseudomonotone variational inequalities. Optim. Lett. (2021). https://doi.org/10.1007/s11590-020-01678-w
42. Thong, D.V., Vinh, N.T., Hieu, D.V.: Accelerated hybrid and shrinking projection methods for variational inequality problems. Optimization **68**, 981–998 (2019)
43. Tseng, P.: A modified forward–backward splitting method for maximal monotone mapping. SIAM J. Control. Optim. **38**, 431–446 (2000)
44. Yang, J., Liu, H.: Strong convergence result for solving monotone variational inequalities in Hilbert space. Numer. Algorithms **80**, 741–752 (2019)
45. Yang, J., Liu, H., Li, G.: Convergence of a subgradient extragradient algorithm for solving monotone variational inequalities. Numer. Algorithms **84**, 389–405 (2020)
46. Yang, J., Liu, H., Liu, Z.: Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization **67**, 2247–2258 (2018)
47. Yao, Y., Noor, M.A.: On viscosity iterative methods for variational inequalities. J. Math. Anal. Appl. **325**, 776–787 (2007)

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Research Article

Unsteady MHD Flow for Fractional Casson Channel Fluid in a Porous Medium: An Application of the Caputo-Fabrizio Time-Fractional Derivative

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Theoretically, this work describes the exact solutions of fractional Casson fluid through a channel under the effect of MHD and porous medium. The unsteady fluid motion of the bottom plate, which is confined by parallel but perpendicular sidewalls, supports the flow. By introducing the dimensionless parameters and variables, the momentum equation, as well as the initial and boundary conditions, has been transformed to a dimensionless form. A mix of Laplace and Fourier transformations is used to get the exact solution for the momentum equation. The constitutive equations for Caputo-Fabrizio's time-fractional derivative are also incorporated for recovering the exact solutions of the flow problem under consideration. After recovering the exact solutions for flow characteristics, three different cases at the surface of the bottom plate are discussed, by addressing the limiting cases under the influence of the side walls. Moreover, these solutions are captured graphically, and the effects of the Reynolds number Re, fractional parameter α, effective permeability K_{eff}, and dimensionless parameter for Casson fluid β on the fluid's motion are observed.

1. Introduction

Fractional calculus plays a critical part in the solving of complicated engineering issues. Because of its significance, a fractional model solution for flow issues is preferred by many scientists and researchers. A fraction model correctly depicts the motion of a flow issue when compared to ordinary differential equations (ODEs). It recovers an ODE's solution, which explains minute flow system fluctuations. Even for Newtonian fluids, the older scientists' operators including the Caputo operator employed a solitary kernel that resulted in complicated series solutions. In 2015, Caputo and Fabrizio [1] suggested a new fractional operator that may be employed in simple ways to solve this problem. Following that, a lot of scholars have applied this notion to numerous sorts of flows using varied geometries. Alshabanat et al. [2] proposed a new fractional derivative utilizing a nonsingular form kernel of exponential and trigonometric functions. Singh et al. [3] investigated a fractional epidemiological model for viral determination in the computer utilizing fractional derivatives and numerically solved the modeled issue using the iterative technique. Shah and Khan [4] have discussed heat transmission for an oscillating second-grade fluid upon a vertical surface by employing the Caputo-Fabrizio derivative. In this work, the authors have determined the exact solutions for flow and thermal characteristics by applying Laplace transform. The authors of this investigation have also carried out a comparative study for the time derivative of fractional and integral order both for Newtonian and second-grade fluids and have highlighted that fractional parameter enhanced the flow characteristics due to augmented velocities of fractional fluid.

Researchers have been studying mathematical models for non-Newtonian fluids because of the growing trend in technological and industrial applications. Because of their relevance at the industrial level, research into these fluids is desirable. Non-Newtonian fluids, for example, lubricant production for a variety of vehicles, spinning of metal, metal extrusion, removing nonmetallic inclusions from molten metal, food, shoe manufacturing (the shoe must be filled with a non-Newtonian fluid to protect the feet from damage), and industries that deal with medicine and coolant, have a wide range of uses. Casson fluid is known as Non-Newtonian fluid because of its rheological features. Casson's model, introduced in 1959, was shown to be extremely flexible and to best reflect the curves of silicon suspension, among other things [5]. Hussain et al. [6] used the shooting method to find the numerical solution of a Casson fluid by changing viscosity flows near a shrinking/extending sheet by slip effects in MHD stagnation point flow. Casson nanofluid flows hydromagnetically through a porous stretched cylinder under Newtonian heat and mass conditions which were discussed by Naqvi et al. [7]. Rao et al. [8] found the exact solution of Casson fluid near a plate which is infinite, exponentially accelerated, and vertical with the effect of MHD and porosity. An overview of numerical approaches for heat and mass transfer in Casson fluids is presented by Verma and Mondal [9]. Sheikh et al. [10] obtained exact solutions of free convection MHD flow of Casson fluid in a channel by using Laplace transform. A fractional model with the Mittag-Leffler memory for generalized Casson MHD fluid by Newtonian heating was discussed by Tassaddiq et al. [11]. Goud et al. [12] calculated the exact solution of natural convection MHD flow of Casson fluid near a perpendicular plate through a porous medium by finite element method. The behavior of a non-Newtonian micropolar-Casson fluid pulsatile flow in a restricted channel influenced by Lorentz force according to Darcy's law is investigated by Ali et al. [13].

The flow in the channels is also more important. Because of its relevance, numerous academics have been drawn to explore channel flows during the previous few decades. Using the Dufour effect, Jha and Ajibade [14] examined heat and mass transfer for free convective fluid flow along a vertical channel. Free convection of transient flow on a flat surface was studied by Ingham [15]. The authors of this study focused on a vertically oriented flat plate. Free convective flow with MHD effect on a flat plate was examined by Raptis and Singh [16]. The authors employed an accelerated vertical plate in this investigation. Fluid flow near an exponential plate has been explored by Singh and Kumar [17]. MHD fluid flow across a flat plate was studied by Khan et al. [18]. The authors of this study looked at the effect of sidewalls on fluid flow. Haq et al. [19] explored the flow of MHD fluid on a porous sheet. Fetecau [20] has solved the fluid flow via a pipe analytically. The author used the Steklov expansion theorem to find the exact answer in this study. Furthermore, the same flow has been explored under the impact of side walls in this work by addressing the limiting instances. Using the Fourier transform, Fetecău and Zierep [21] examined a set of accurate solutions for the second-grade fluid modeled issue. An abrupt jerk was applied to the fluid in this experiment to get fluid motion.

This work investigates a time-dependent fractional Casson fluid through a channel with porosity and MHD effect; the flow is caused by the bottom plate's unstable motion, which is confined by sidewalls that are parallel to one another but normal to the bottom plate. For the aim of creating a dimensionless form of governing equations, a group of nondimensional variables has been used in the momentum equation and applied boundary conditions. The exact solutions for the momentum equation were then obtained using integral transforms [22, 23] such as Laplace, finite Fourier, and Fourier transforms, as well as the Caputo-Fabrizio fractional derivative. These ideas were then addressed for various bottom plate instances. We utilized Mathcad 15 to explain the graphical results for the modeled issue after recovering the exact answers for various scenarios. The impact of different parameters involved in the solution of the flow problem has been discussed upon flow characteristics.

2. Physical Description of Problem

Assume that a fractional unsteady Casson fluid flows on an infinite plate. The plate is limited by two sidewalls that are parallel to each other but normal to the bottom plate and are separated by a distance of d. The flow is caused by the bottom plate because when $t = 0$, both the plate and the fluid are at rest, and the bottom plate begins to move at an unsteady velocity $U_0 f(t)$, which satisfies the condition $f(t) = 0$ at $t = 0$ and piecewise continuous and exponential order at infinity. Because the motion is unidirectional, the equation describes the velocity of the flow system $V = u(y, z, t)i$ such that $v = w = 0$.

Assumed flow system's governing equations [24] are as follows:

$$\rho \frac{\partial u}{\partial t} = \left(1 + \frac{1}{\xi}\right)\left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) - \left(\sigma B_o^2 + \frac{\mu\varphi}{k}\right)u, \quad y > 0, \quad 0 \le z \le d, \tag{1}$$

where

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}, \qquad \tau_{xz} = \mu \frac{\partial u}{\partial z}. \tag{2}$$

The initial and boundary conditions for the problem are as follows:

$$\begin{aligned}
&\tau_{xy}(y,z,0)=\tau_{xz}(y,z,0)=0,\\
&u(y,z,0)=0,\\
&u(0,z,t)=U_0 f(t),\\
&u(y,0,t)=u(y,d,t)=0,
\end{aligned}\tag{3}$$

although

$$u(y,z,t)\longrightarrow 0,\ \frac{\partial u(y,z,t)}{\partial y}\longrightarrow 0,\quad \text{for}\quad y\longrightarrow\infty,\tag{4}$$

where μ represents the dynamic viscosity and ρ is the density. Also, $f(t)=0$ at $t=0$ and $f(t)\neq 0$ for $t>0$.

Consider the collection of dimensionless variables to transform the governing equation and its initial boundary conditions for the postulated flow problem into nondimensional form.

$$\begin{aligned}
y^* &= \frac{y}{d_0},\\
z^* &= \frac{z}{d_0},\\
\left(\tau_{xy}^*,\tau_{xz}^*\right) &= \frac{(\tau_{xy},\tau_{xz})}{\rho {U_0}^2},\\
u^* &= \frac{u}{U_0},\\
t^* &= \frac{tU_0}{d_0},\\
L^* &= \frac{d}{d_0}.
\end{aligned}\tag{5}$$

By plugging Equation (5) into Equations (1) and (2), we get the following:

$$\frac{\partial u}{\partial t}=\frac{1}{\beta}\left(\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)-K_{\text{eff}}u,\quad y>0,\quad 0\le z\le L,\tag{6}$$

$$\begin{aligned}
\tau_{xy}&=\frac{1}{\text{Re}}\frac{\partial u}{\partial y},\\
\tau_{xz}&=\frac{1}{\text{Re}}\frac{\partial u}{\partial z}.
\end{aligned}\tag{7}$$

Here, $K_{\text{eff}}=M+(1/K)$ is the effective permeability; moreover, $1/K=\nu\varphi d_0/kU_0$ is the porous medium inverse permeability, and $M=\sigma B_0^2 d_0/\rho U_0$ is the magnetic parameter.

In a dimensionless form, the initial and boundary conditions are as follows:

$$\begin{aligned}
&u(y,z,t)=0,\quad t=0,\\
&u(y,z,t)=f\left(\frac{d_0 t}{U_0}\right)=f(t),\quad y=0,\\
&u(y,0,t)=u(y,L,t)=0,\\
&u(y,z,t)\longrightarrow 0,\\
&\frac{\partial u(y,z,t)}{\partial y}\longrightarrow 0,\quad y\longrightarrow\infty.
\end{aligned}\tag{8}$$

The length characteristics d_0, Reynolds number $\text{Re}=U_0 d_0/\nu$, and dimensionless Casson fluid parameters are depicted above.

In a generalized form, the fractional constitutive equations are written as follows:

$$\begin{aligned}
\tau_{xy}&=\frac{1}{\text{Re}}\,{}^{CF}D_t^{\alpha}\frac{\partial u}{\partial y},\\
\tau_{xz}&=\frac{1}{\text{Re}}\,{}^{CF}D_t^{\alpha}\frac{\partial u}{\partial z}.
\end{aligned}\tag{9}$$

3. Problem Solution

Using the Laplace transform to solve Equations (6) and (7) and incorporating Equation (9), we have the following:

$$\begin{aligned}
s\bar{u}&=\frac{1}{\beta}\left(\frac{\partial\bar{\tau}_{xy}}{\partial y}+\frac{\partial\bar{\tau}_{xz}}{\partial z}\right)-K_{\text{eff}}\bar{u},\\
\bar{\tau}_{xy}&=\frac{1}{\text{Re}}\frac{s}{(1-\alpha)s+\alpha}\frac{\partial\bar{u}}{\partial y},\\
\bar{\tau}_{xz}&=\frac{1}{\text{Re}}\frac{s}{(1-\alpha)s+\alpha}\frac{\partial\bar{u}}{\partial z}.
\end{aligned}\tag{10}$$

By simplifying the above equations, we have the following:

$$\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2}=\frac{\text{Re}\,\beta(s+K_{\text{eff}})((1-\alpha)s+\alpha)}{s}\bar{u}.\tag{11}$$

After using the Laplace transformation, the initial and boundary conditions were reduced to

$$\begin{aligned}
&\bar{u}(y,z,s)=0,\quad s=0,\\
&\bar{u}(y,z,s)=\bar{F}(s),\quad y=0,\\
&\bar{u}(y,z,s)=\bar{u}(y,z,s)=0,\quad z=0,\quad z=L,\\
&\bar{u}(y,z,s)\longrightarrow 0,\\
&\frac{\partial\bar{u}(y,z,s)}{\partial y}\longrightarrow 0,\quad y\longrightarrow\infty.
\end{aligned}\tag{12}$$

After that, multiply Equation (11) by $\sqrt{2/\pi}\int_0^\infty\int_0^L \sin(\psi_n z)\sin(y\chi)dzdy$ and using Equation (12), in which $\zeta_n = n\pi/L$ and the resulting equation was rewritten as follows:

$$\bar{u}_{sn}(\chi,s)=\sqrt{\frac{2}{\pi}}\left(\frac{1-(-1)^n}{\zeta_n}\right)\left[\frac{\chi sF(s)}{\text{Re}\,\beta((1-\alpha)s+\alpha)(s+K_{\text{eff}})+s\left(\chi^2+\zeta_n^2\right)}\right]. \tag{13}$$

Rewrite Equation (13) in the following form:

$$\begin{aligned}\bar{u}_{sn}(\chi,s)&=\sqrt{\frac{2}{\pi}}\left(\frac{1-(-1)^n}{\zeta_n}\right)\left[\frac{\chi F(s)}{\chi^2+\zeta_n^2}\right]\\&-\sqrt{\frac{2}{\pi}}\left(\frac{1-(-1)^n}{\zeta_n}\right)\frac{\chi F(s)}{\chi^2+\zeta_n^2}\\&\cdot\left[\frac{((1-\alpha)s+\alpha)(s+K_{\text{eff}})}{((1-\alpha)s+\alpha)(s+K_{\text{eff}})+\left(s\left(\chi^2+\zeta_n^2\right)/\text{Re}\,\beta\right)}\right].\end{aligned} \tag{14}$$

Making use of Laplace inverse transform along with convolution theorem, we have the following equation:

$$\begin{aligned}\bar{u}_{sn}(\chi,t)&=\sqrt{\frac{2}{\pi}}\left(\frac{1-(-1)^n}{\zeta_n}\right)\left[\frac{\chi f(t)}{\chi^2+\zeta_n^2}\right]\\&+\sqrt{\frac{2}{\pi}}\left(\frac{1-(-1)^n}{\zeta_n}\right)\left(f''(t)+F_1f'(t)+F_2f(t)\right)\\&\cdot\frac{\chi}{2A_{1n}(\chi)\left(\chi^2+\zeta_n^2\right)}*\left(e^{-D_{1n}(\chi)}-e^{-D_{2n}(\chi)}\right).\end{aligned} \tag{15}$$

In Equation (15), we have the following:

$$\begin{aligned}F_1&=\frac{(1-\alpha)K_{\text{eff}}+\alpha}{(1-\alpha)},\\F_2&=\frac{\alpha K_{\text{eff}}}{(1-\alpha)},\\D_{1n}(\chi)&=\frac{C_{1n}(\chi)}{2}+A_{1n}(\chi),\\D_{2n}(\chi)&=\frac{C_{1n}(\chi)}{2}-A_{1n}(\chi),\\A_{1n}(\chi)&=\sqrt{\left(\frac{{C_{1n}}^2(\chi)}{4}-E_2\right)},\\C_{1n}(\chi)&=\frac{\text{Re}\,\beta(1-\alpha)K_{\text{eff}}+\text{Re}\,\beta\alpha+\chi^2+\zeta_n^2}{\text{Re}\,\beta(1-\alpha)}.\end{aligned} \tag{16}$$

After using Fourier inversion methods to solve Equation (15) and simplifying the resulting equation,

$$\begin{aligned}u(y,z,t)&=\frac{4f(t)}{L}\sum_{n=1}^{\infty}\frac{e^{-\zeta_m y}\sin(\zeta_m z)}{\zeta_m}\\&+\frac{4}{\pi L}\left(f''(t)+F_1f'(t)+F_2f(t)\right)\sum_{n=1}^{\infty}\frac{\sin(\zeta_m z)}{\zeta_m}\\&*\int_0^\infty\frac{\chi\left(e^{-D_{11m}(\chi)}-e^{-D_{22m}(\chi)}\right)\sin(y\xi)}{A_{11m}(\chi)\left(\chi^2+\zeta_m^2\right)}d\chi.\end{aligned} \tag{17}$$

In Equation (17), $m = 2n-1$, while

$$\begin{aligned}A_{11m}(\chi)&=\sqrt{\left(\frac{{C_{11m}}^2(\chi)}{4}-F_2\right)},\\C_{11m}(\chi)&=\frac{\text{Re}\,\beta(1-\alpha)K_{\text{eff}}+\text{Re}\,\beta\alpha+\chi^2+\zeta_m^2}{\text{Re}\,\beta(1-\alpha)},\\D_{11m}(\chi)&=\frac{C_{11m}(\chi)}{2}+A_{11m}(\chi),\\D_{22m}(\chi)&=\frac{C_{11m}(\chi)}{2}-A_{11m}(\chi),\\A_{11m}(\chi)&=\sqrt{\left(\frac{{C_{11m}}^2(\chi)}{4}-F_2\right)}.\end{aligned} \tag{18}$$

Take $L = 2h$ by translating system of the coordinate axis by putting $z = z^* + h$ and $\xi_m = (2n-1/2h)\pi$ with $\xi_m = \zeta_m$ in Equation (17); after that, we have the following:

$$\begin{aligned}u(y,z,t)&=\frac{2f(t)}{h}\sum_{n=1}^{\infty}\frac{e^{-\xi_m y}(-1)^{n+1}\cos(\xi_m z)}{\xi_m}\\&+\frac{2}{\pi h}\left(f''(t)+F_1f'(t)+F_2f(t)\right)\\&\times\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m}\\&*\int_0^\infty\frac{\chi\left(e^{-D_{1m}(\chi)}-e^{-D_{2m}(\chi)}\right)\sin(y\chi)}{A_{1m}(\chi)\left(\chi^2+\xi_m^2\right)}d\chi,\end{aligned} \tag{19}$$

where

$$\begin{aligned}A_{1m}(\chi)&=\sqrt{\left(\frac{{C_{1m}}^2(\chi)}{4}-F_2\right)},\\C_{1m}(\chi)&=\frac{\text{Re}\,\beta(1-\alpha)K_{\text{eff}}+\text{Re}\,\beta\alpha+\chi^2+\xi_m^2}{\text{Re}\,\beta(1-\alpha)},\\D_{11m}(\chi)&=\frac{C_{11m}(\chi)}{2}+A_{11m}(\chi),\\D_{22m}(\chi)&=\frac{C_{11m}(\chi)}{2}-A_{11m}(\chi).\end{aligned} \tag{20}$$

4. Special Cases

4.1. $h \longrightarrow \infty$ Means That Distance among the Sidewalls Will Be Maximized. When the distance between the sidewalls of a flow system is increased, the flow is no longer impacted by these sidewalls and Equation (19) is reduced to the following:

$$u(y,t) = f(t) + \frac{1}{\pi}\left(f''(t) + F_1 f'(t) + F_2 f(t)\right) * \int_0^\infty \frac{\left(e^{-L_1(\chi)t} - e^{-L_2(\chi)t}\right)\sin(y\chi)}{\chi c_1(\chi)} d\chi. \tag{21}$$

In Equation (21),

$$A_2(\chi) = \sqrt{\left(\frac{C_2{}^2(\chi)}{4} - F_2\right)},$$

$$C_2(\chi) = \frac{\operatorname{Re}\beta(1-\alpha)K_{\text{eff}} + \operatorname{Re}\beta\alpha + \chi^2}{\operatorname{Re}\beta(1-\alpha)},$$

$$L_1(\chi) = \frac{C_2(\chi)}{2} + A_2(\chi),$$

$$L_2(\chi) = \frac{C_2(\chi)}{2} - A_2(\chi). \tag{22}$$

4.2. Stokes' First Problem When $f(t) = H(t)$. By choosing $f(t) = H(t)$in Equation (19) (where $H(t)$is Heaviside's unit step function), we will get the following:

$$u(y,z,t) = \frac{2}{h}H(t)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-\xi_m y}\cos(\xi_m z)}{\xi_m} + \frac{2}{h\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m} \times \int_0^\infty \frac{\chi\left[D_{2m}(\chi)\left(D_{1m}(\chi)F_1 - D_{1m}^2(\chi) - F_2\right)e^{-D_{1m}(\chi)t} - D_{1m}(\chi)\left(D_{2m}(\chi)F_1 - D_{2m}^2(\chi) - F_2\right)e^{-D_{2m}(\chi)t}\right]\sin(y\chi)}{D_{1m}(\chi)D_{2m}(\chi)A_{1m}(\chi)\left(\chi^2 + \xi_m^2\right)} d\chi. \tag{23}$$

If the side walls are ignored, Equation (23) becomes the following:

$$u(y,t) = H(t) + \frac{1}{\pi}\int_0^\infty \frac{\begin{pmatrix} L_2(\chi)\left(L_1(\chi)F_1 - L^2{}_1(\chi) - F_2\right)e^{-L_1(\chi)t} \\ -L_1(\chi)\left(L_2(\chi)F_1 - L^2{}_2(\chi) - F_2\right)e^{-L_2(\chi)t}\end{pmatrix}}{\chi L_1(\chi)L_2(\chi)c_1(\chi)} \cdot \sin(y\chi) d\chi. \tag{24}$$

4.3. If $f(t) = t^a$. By putting this case in Equation (19), after simplification, we will obtain the following:

$$u(y,z,t) = \frac{2}{h}t^a\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-\xi_m y}\cos(\xi_m z)}{\xi_m} + \frac{2}{h\pi}\left(\frac{d^2(t^a)}{dt^2} + F_1\frac{d(t^a)}{dt} + F_2 t^a\right) \times \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m} * \int_0^\infty \frac{\chi\left(e^{-D_{1m}(\chi)t} - e^{-D_{2m}(\chi)t}\right)\sin(y\chi)}{c_{1m}(\chi)\left(\chi^2 + \xi_m{}^2\right)} d\chi. \tag{25}$$

The solution for an accelerating fractional Casson fluid is given by Equation (25). If the bottom plate is constantly accelerated, then $f(t) = t$; we get from Equation (19) that we obtain after simplification.

$$u(y,z,t) = \frac{2}{h}t\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-\xi_m y}\cos(\xi_m z)}{\xi_m} + \frac{2}{h\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m} \times \int_0^\infty \frac{\chi\sin(y\chi)}{c_{1m}(\chi)\left(\chi^2 + \xi_m{}^2\right)} \cdot \begin{bmatrix} \dfrac{F_1}{D_{1m}(\chi)D_{2m}(\chi)}\left(D_{1m}(\chi)e^{-D_{2m}(\chi)t} - D_{2m}(\chi)e^{-D_{1m}(\chi)t}\right) + \dfrac{F_2}{D^2{}_{1m}(\chi)D^2{}_{2m}(\chi)} \\ \left(-2c_{1m}(\chi)(D_{1m}(\chi)D_{2m}(\chi)t - A_{3m}(\chi)) + D^2{}_{2m}(\chi)e^{-D_{1m}(\chi)t} - D^2{}_{1m}(\chi)e^{-D_{2m}(\chi)t}\right)\end{bmatrix} d\chi. \tag{26}$$

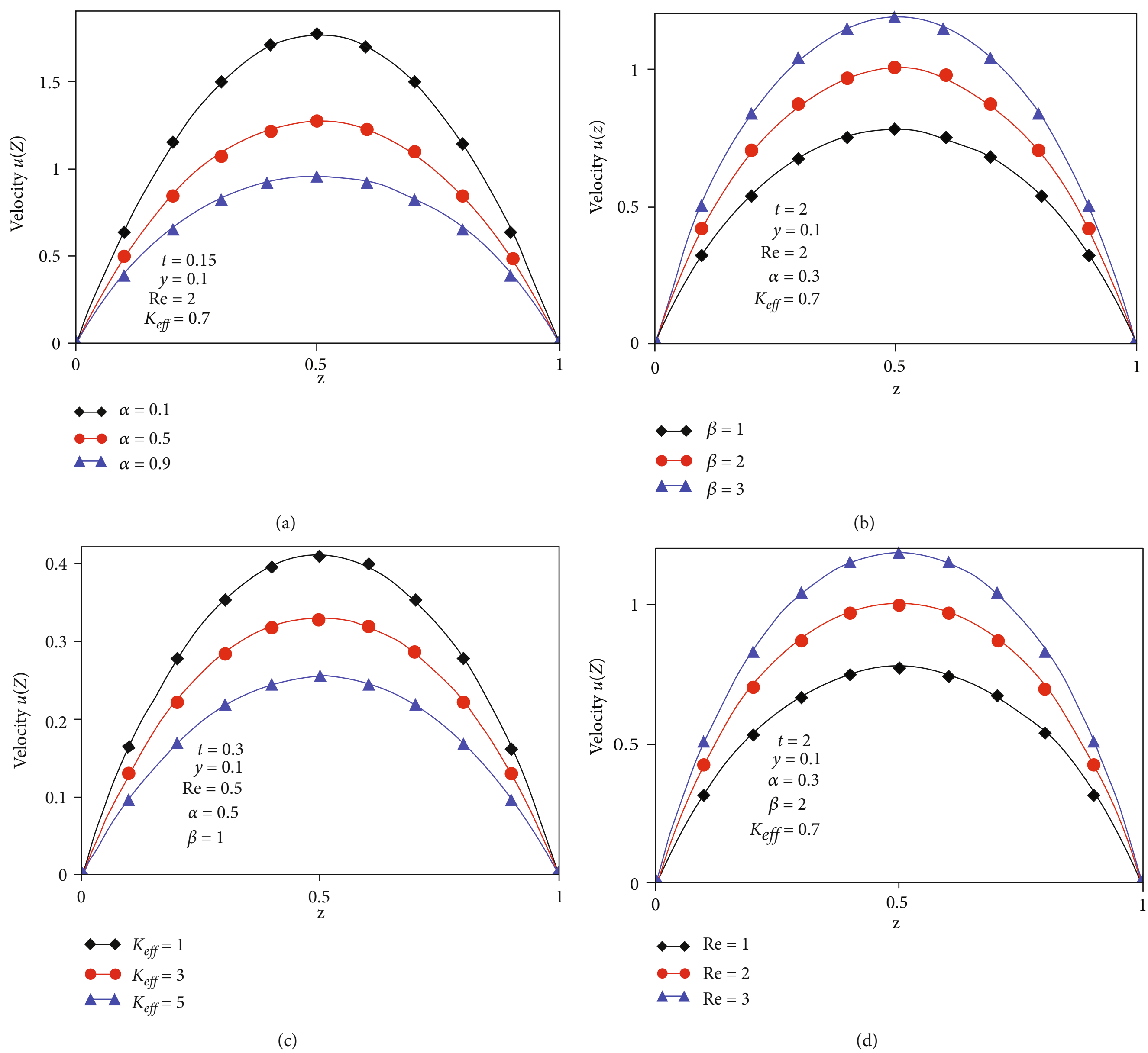

FIGURE 1: Flow characteristics for the impulsive type motion of fractional fluid when $z \in [0, 1]$.

Again, by neglecting the effects of sidewalls that are $h \longrightarrow \infty$ on fractional fluid flow, the velocity field takes the following form.

$$u(y,t) = t + \frac{1}{\pi}\int_0^\infty \frac{\sin(y\chi)}{\chi c_1(\chi)} \cdot \left[\begin{array}{c} \dfrac{F_1}{L_1(\chi)L_2(\chi)}\left(L_1(\chi)e^{-L_2(\chi)t} - L_2(\chi)e^{-L_1(\chi)t}\right) + \dfrac{F_2}{L^2{}_1(\chi)L^2{}_2(\chi)} \\ \left(-2c_1(\chi)(L_1(\chi)L_2(\chi)t - A_3(\chi)) + L^2{}_2(\chi)e^{-L_1(\chi)t} - L^2{}_1(\chi)e^{-L_2(\chi)t}\right) \end{array} \right] d\xi. \tag{27}$$

4.4. $f(t) = H(t)e^{i\omega t}$.

After the oscillation of the bottom plate, the fractional Casson fluid has oscillated with velocities $u_s(y,z,t)$ and $u_c(y,z,t)$,, respectively, corresponding to sine and cosine oscillations. This case gives the following:

$$\begin{aligned} u_{cs}(y,z,t) &= \frac{2}{h}\cos(\omega t)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-\xi_m y}\cos(\xi_m z)}{\xi_m} \\ &+ \frac{2}{h\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m}\int_0^\infty \frac{\chi\sin(y\chi)}{A_{1m}(\chi)\left(\chi^2+\xi_m^2\right)} \\ &\times \left(\begin{array}{c} \left\{\left(D_{1m}(\chi)D_{2m}(\chi)-\omega^2\right)\left(\omega^2-F_2\right) - F_1\omega^2 C_{1m}(\chi)\right\}\cos(\omega t) \\ +\left\{C_{1m}(\chi)\left(\omega^2-F_2\right) + F_1 D_{1m}(\chi)D_{2m}(\chi) - F_1\omega^2\right\}\sin(\omega t) \end{array} \right) \\ &\times \frac{-2A_{1m}(\chi)d\chi}{\left(D_{1m}^2(\chi)+\omega^2\right)\left(D_{2m}^2(\chi)+\omega^2\right)}, \end{aligned}$$

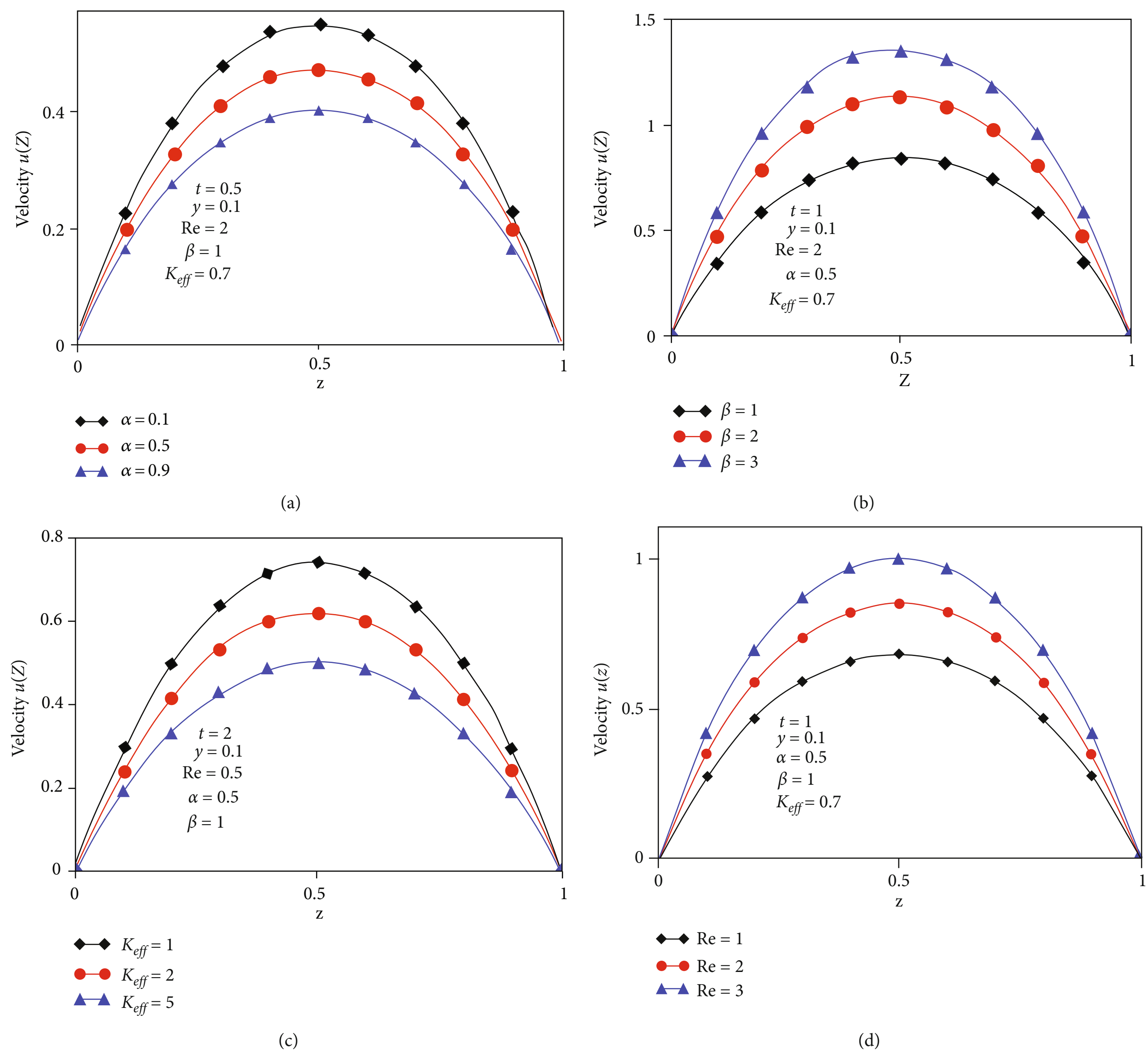

FIGURE 2: Flow characteristics for constant acceleration of fractional fluid when $z \in [0, 1]$.

$$u_{ct}(y,z,t) = \frac{2}{h\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m}\int_0^{\infty}\frac{\chi\sin(y\chi)}{A_{1m}(\chi)\left(\chi^2+\xi_m^2\right)} \times\left[\frac{D_{1m}(\chi)\left(F_2-\omega^2\right)+F_1\omega^2}{D_{1m}^2(\chi)+\omega^2}e^{-D_{1m}(\chi)t} + \frac{D_{2m}(\chi)\left(F_2-\omega^2\right)-F_1\omega^2}{D_{2m}^2(\chi)+\omega^2}e^{-D_{2m}(\chi)t}\right]d\chi,$$

$$u_{ss}(y,z,t) = \frac{2}{h}\sin(\omega t)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-\xi_m y}\cos(\xi_m z)}{\xi_m} + \frac{2}{h\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m}\int_0^{\infty}\frac{\chi\sin(y\chi)}{A_{1m}(\chi)\left(\chi^2+\xi_m^2\right)} \times\left(\begin{array}{l}\left\{\left(D_{1m}(\chi)D_{2m}(\chi)-\omega^2\right)\left(\omega^2-F_2\right)-F_1\omega^2C_{1m}(\chi)\right\}\sin(\omega t)\\ +\left\{F_1\omega\left(\omega^2-D_{1m}(\chi)D_{2m}(\chi)\right)-\omega C_{1m}(\chi)\left(\omega^2+F_2\right)\right\}\cos(\omega t)\end{array}\right) \times\frac{-2A_{1m}(\chi)d\chi}{\left(D_{1m}^2(\chi)+\omega^2\right)\left(D_{2m}^2(\chi)+\omega^2\right)},$$

$$u_{st}(y,z,t) = \frac{2}{h\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(\xi_m z)}{\xi_m}\int_0^{\infty}\frac{\chi\sin(y\chi)}{A_{1m}(\chi)\left(\chi^2+\xi_m^2\right)} \times\left[\frac{\omega\left(\omega^2+F_1D_{1m}(\chi)-F_2\right)}{D_{1m}^2(\chi)+\omega^2}e^{-D_{1m}(\chi)t} - \frac{\omega\left(\omega^2+F_1D_{2m}(\chi)-F_2\right)}{D_{2m}^2(\chi)+\omega^2}e^{-D_{2m}(\chi)t}\right]d\chi. \tag{28}$$

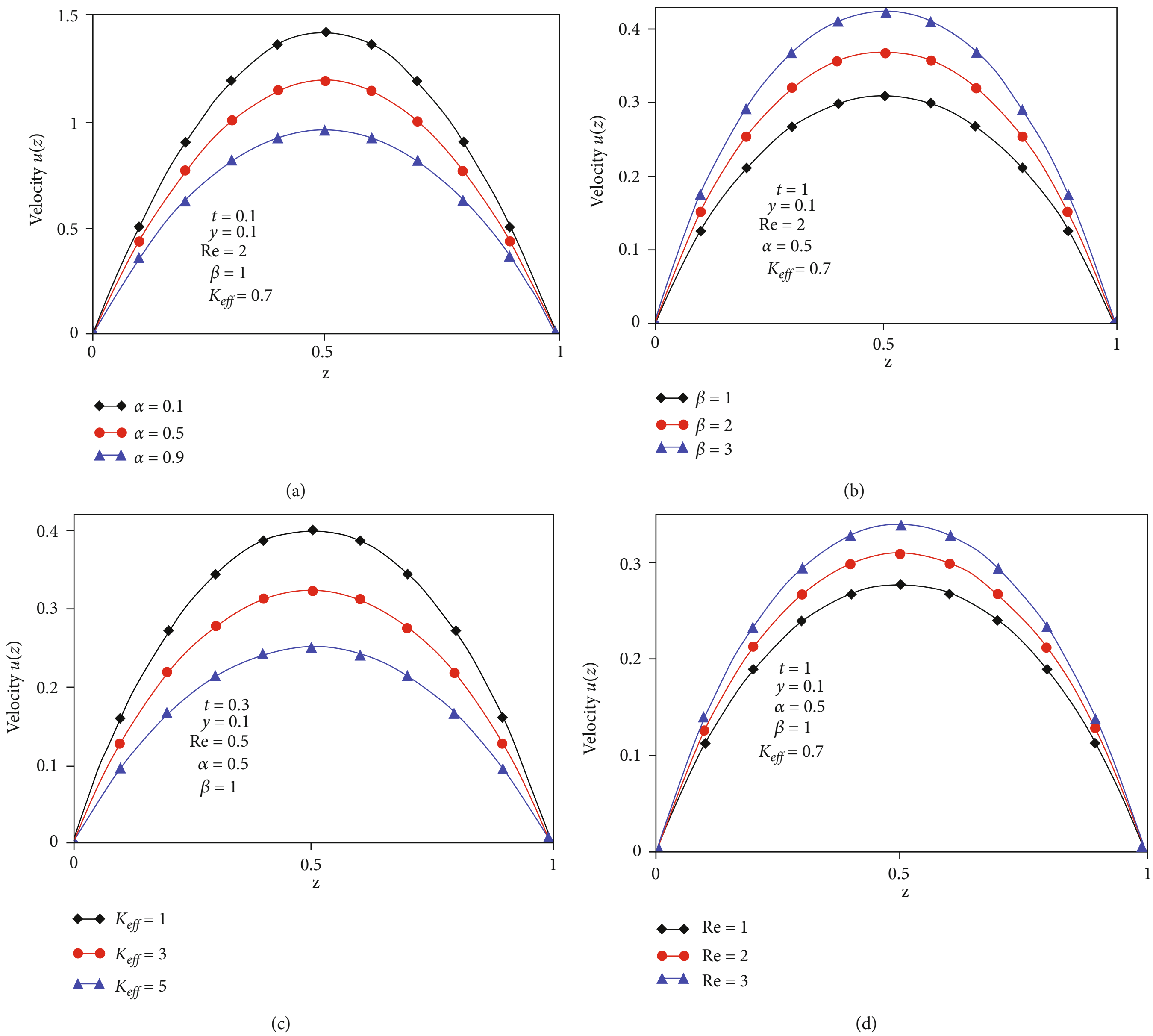

FIGURE 3: Flow characteristics for cosine oscillations of fractional fluid when $z \in [0, 1]$.

Now, if we neglect the effects of sidewalls that are $h \longrightarrow \infty$ on fractional fluid flow as a limiting case, then the velocity fields for such fluid flow caused by an oscillating bottom plate are as follows.

$$u_{cs}(y,z,t) = \cos(\omega t) + \frac{1}{\pi}\int_0^\infty \frac{\sin(y\chi)}{\chi A_2(\chi)} \cdot \begin{pmatrix} \left\{\left(L_1(\chi)L_2(\chi)-\omega^2\right)\left(\omega^2 - F_2\right) - F_1\omega^2 C_2(\chi)\right\}\cos(\omega t) \\ +\left\{C_2(\chi)\left(\omega^2 - F_2\right) + F_1 L_1(\chi)L_2(\chi) - F_1\omega^2\right\}\sin(\omega t) \end{pmatrix} \times \frac{-2A_2(\chi)d\chi}{\left(L_1^2(\chi)+\omega^2\right)\left(L_2^2(\chi)+\omega^2\right)},$$

$$u_{ct}(y,z,t) = \frac{1}{\pi}\int_0^\infty \frac{\sin(y\chi)}{\chi A_2(\chi)}\left[\frac{L_1(\chi)\left(F_2-\omega^2\right)+F_1\omega^2}{L_1^2(\chi)+\omega^2}e^{-L_1(\chi)t} + \frac{L_2(\chi)\left(F_2-\omega^2\right)-F_1\omega^2}{L_2^2(\chi)+\omega^2}e^{-L_2(\chi)t}\right]d\chi,$$

$$u_{ss}(y,z,t) = \sin(\omega t) + \frac{1}{\pi}\int_0^\infty \frac{\sin(y\chi)}{\chi A_2(\chi)} \cdot \begin{pmatrix} \left\{\left(L_1(\chi)L_2(\chi)-\omega^2\right)\left(\omega^2 - F_2\right) - E_1\omega^2 C_2(\chi)\right\}\sin(\omega t) \\ +\left\{F_1\omega\left(\omega^2 - L_1(\chi)L_2(\chi)\right) - \omega C_2(\chi)\left(\omega^2 + F_2\right)\right\}\cos(\omega t) \end{pmatrix} \times \frac{-2A_2(\chi)d\chi}{\left(L_1^2(\chi)+\omega^2\right)\left(L_2^2(\chi)+\omega^2\right)},$$

$$u_{st}(y,z,t) = \frac{1}{\pi}\int_0^\infty \frac{\sin(y\chi)}{\chi A_2(\chi)}\left[\frac{\omega\left(\omega^2+F_1L_1(\chi)-F_2\right)}{L_1^2(\chi)+\omega^2}e^{-L_1(\chi)t} - \frac{\omega\left(\omega^2+F_1L_2(\chi)-F_2\right)}{L_2^2(\chi)+\omega^2}e^{-L_2(\chi)t}\right]d\chi. \tag{29}$$

5. Numerical Results and Discussions

With the influence of MHD and porosity, three distinct forms of motion have been examined for fractional Casson fluid in a

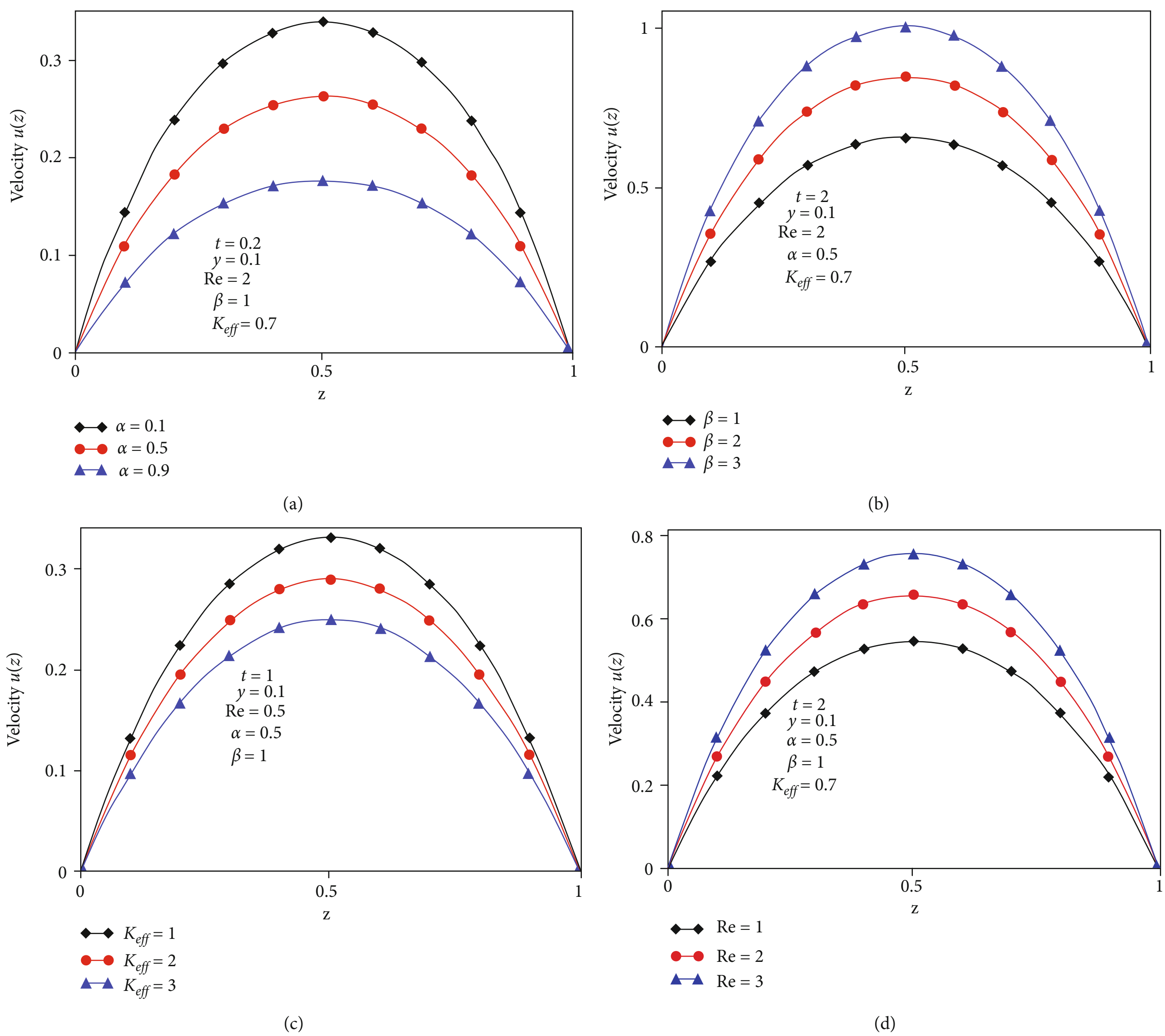

FIGURE 4: Flow characteristics for sine oscillations of fractional fluid when $z \in [0, 1]$.

channel passing from a horizontal, unsteadily, and infinite moving plate that is confined by two parallel side walls separated by a distance d. For the objective of obtaining an exact solution, joint Laplace and Fourier transforms were used. For the problem under examination, many situations at the bottom plate have been studied. The same flow has been studied under the impact of side walls by addressing the limiting instances. After determining general solutions for any arbitrary flow, solution for impulsive, solution for constantly accelerated, and solution for the sinusoidal fluid motion of the bottom plate were retrieved.

Flow characteristics for Stokes' first issue are depicted in Figures 1(a)–1(d). It can be seen from these graphs that velocity decreases as the fractional parameter (α) increases because when the fractional parameter is increased, the fluid velocity decreases. An opposite impact is also observed for the dimensionless parameter for Casson fluid (β) upon flow characteristics. It is also worth noting that velocity is a decreasing function of effective permeability (K_{eff}), in these graphs. On the other hand, Reynolds number (Re) has the opposite effect on flow properties. The velocity of the fluid rises as the Reynolds number increases.

Figures 2(a)–2(d) show velocity profiles for fractional fluid flow with a ramping type. Figure 2(a) indicates that when $K_{\mathrm{eff}} = 0.7$, $\beta = 1$, and $\mathrm{Re} = 2$ then with decreasing values of fractional parameter (α), the characteristics of the flow are also improved. The velocity profile rises with increasing Casson fluid values, as seen in Figure 2(b). In Figure 2(c), the velocity decreases as the effective permeability increases. Similarly, when the Reynolds number (Re) increases, the flow properties improve. As a result, the velocity profile is a decreasing function of the fractional and effective permeability parameters.

The effects of cosine oscillations on flow characteristics are depicted in Figures 3(a)–3(d). In the instance of cosine oscillations, we can see that velocity decreases as the fractional parameter (α) and effective permeability (K_{eff}) increase, and that flow profiles decrease as these two components increase which is observed from Figures 3(a) and 3(c). The flow profile is rising with greater values of the dimensionless parameter for Casson fluid parameter (β) and Reynolds number (Re) as seen in Figures 3(b) and 3(d). The effect of sine oscillations on flow characteristics is seen in Figures 4(a)–4(d). For sine oscillations on flow characteristics, a similar effect has been seen for various values of significant parameters.

6. Conclusions

The precise solutions for a time-dependent fractional Casson fluid traversing a channel under MHD and porosity effects are described in this paper. The flow is caused by the bottom plate's unstable motion, which is restricted by parallel but perpendicular sidewalls. By using an appropriate set of dimensionless variables, the momentum equation, as well as the initial and boundary conditions, has been changed to a dimensionless form. A mix of Laplace and Fourier transformations is used to get the exact solution of the momentum equation. For the problem under examination, many situations at the bottom plate have been studied. The same flow has been studied under the impact of side walls by addressing the limiting instances. After determining general solutions for any arbitrary flow, solutions for impulsive, solutions for constant accelerated, and solutions for the sinusoidal fluid motion of the bottom plate were retrieved. These precise solutions are also visually depicted in support of our effort and then theoretically presented. The following points are emphasized after a thorough evaluation of the work.

(i) In this study, it was discovered that Reynolds number has a significant influence on flow characteristics since the velocity of the fluid increases as the Reynolds number grows in all limiting/special instances

(ii) Whether the bottom plate is at rest or moving by constant acceleration, the velocity of the fluid decreases as the fractional parameter and effective permeability increase for long periods of time

(iii) The velocity of cosine/sine oscillations increases with the rising Reynolds number and Casson parameter, since flow profiles increase with rising values of these two parameters, as seen in this work. The flow profile, on the other hand, decreases when the fractional parameter and effective permeability increase

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

[1] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–13, 2015.

[2] A. Alshabanat, M. Jleli, S. Kumar, and B. Samet, "Generalization of Caputo-Fabrizio fractional derivative and applications to electrical circuits," *Frontiers in Physics*, vol. 8, p. 64, 2020.

[3] J. Singh, D. Kumar, Z. Hammouch, and A. Atangana, "A fractional epidemiological model for computer viruses pertaining to a new fractional derivative," *Applied Mathematics and Computation*, vol. 316, pp. 504–515, 2018.

[4] N. A. Shah and I. Khan, "Heat transfer analysis in a second grade fluid over and oscillating vertical plate using fractional Caputo–Fabrizio derivatives," *The European Physical Journal C*, vol. 76, no. 7, pp. 1–11, 2016.

[5] M. Wahiduzzaman, M. T. Islam, P. Sultana, and M. Afikuzzaman, "MHD Couette flow of a Casson fluid between parallel porous plates," *Progress in Nonlinear Dynamics and Chaos*, vol. 2, no. 2, pp. 51–60, 2014.

[6] A. Hussain, S. Afzal, R. Rizwana, and M. Y. Malik, "MHD stagnation point flow of a Casson fluid with variable viscosity flowing past an extending/shrinking sheet with slip effects," *Physica A*, vol. 553, p. 124080, 2020.

[7] S. M. R. S. Naqvi, T. Muhammad, and M. Asma, "Hydromagnetic flow of Casson nanofluid over a porous stretching cylinder with Newtonian heat and mass conditions," *Physica A*, vol. 550, p. 123988, 2020.

[8] S. R. Rao, G. Vidyasagar, and G. V. S. R. Deekshitulu, "Unsteady MHD free convection Casson fluid flow past an exponentially accelerated infinite vertical porous plate through porous medium in the presence of radiation absorption with heat generation/absorption," *Materials Today: Proceedings*, vol. 42, pp. 1608–1616, 2021.

[9] V. R. Verma and S. Mondal, "A brief review of numerical methods for heat and mass transfer of Casson fluids," *Partial Differential Equations in Applied Mathematics*, vol. 3, article 100034, 2021.

[10] N. A. Sheikh, D. L. C. Ching, I. Khan, D. Kumar, and K. S. Nisar, "A new model of fractional Casson fluid based on generalized Fick's and Fourier's laws together with heat and mass transfer," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 2865–2876, 2020.

[11] A. Tassaddiq, I. Khan, K. S. Nisar, and J. Singh, "MHD flow of a generalized Casson fluid with Newtonian heating: a fractional model with Mittag-Leffler memory," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3049–3059, 2020.

[12] B. S. Goud, P. P. Kumar, and B. S. Malga, "Effect of heat source on an unsteady MHD free convection flow of Casson fluid past

a vertical oscillating plate in porous medium using finite element analysis," *Partial Differential Equations in Applied Mathematics*, vol. 2, article 100015, 2020.

[13] A. Ali, M. Umar, Z. Bukhari, and Z. Abbas, "Pulsating flow of a micropolar-Casson fluid through a constricted channel influenced by a magnetic field and Darcian porous medium: a numerical study," *Results Phys*, vol. 19, p. 103544, 2020.

[14] B. K. Jha and A. O. Ajibade, "Free convection heat and mass transfer flow in a vertical channel with the Dufour effect," *Journal of Mechanical Engineering*, vol. 224, no. 2, pp. 91–101, 2010.

[15] D. B. Ingham, "Transient free convection on an isothermal vertical flat plate," *International Journal of Heat and Mass Transfer*, vol. 21, no. 1, pp. 67–69, 1978.

[16] A. Raptis and A. K. Singh, "MHD free convection flow past an accelerated vertical plate," *International Communications in Heat and Mass*, vol. 10, no. 4, pp. 313–321, 1983.

[17] A. K. Singh and N. Kumar, "Free-convection flow past an exponentially accelerated vertical plate," *Astrophysics and Space Science*, vol. 98, no. 2, pp. 245–248, 1984.

[18] M. Khan, R. Malik, and A. Anjum, "Analytic solutions for MHD flows of an Oldroyd-B fluid between two sidewalls perpendicular to the plate," *Chemical Engineering Communications*, vol. 198, no. 11, pp. 1415–1434, 2011.

[19] S. U. Haq, A. U. Rahman, I. Khan, F. Ali, and S. I. A. Shah, "The impact of side walls on the MHD flow of a second-grade fluid through a porous medium," *Neural Computing and Applications*, vol. 30, no. 4, pp. 1103–1109, 2018.

[20] C. Fetecau, "Analytical solutions for non-Newtonian fluid flows in pipe-like domains," *International Journal of Non-Linear Mechanics*, vol. 39, no. 2, pp. 225–231, 2004.

[21] C. Fetecău and J. Zierep, "On a class of exact solutions of the equations of motion of a second grade fluid," *Acta Mechanica*, vol. 150, no. 1-2, pp. 135–138, 2001.

[22] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Academic Press, Seventh edition, 2007.

[23] L. Debnath and D. Bhatta, *Integral transforms and their applications*, Chapman & Hall/CRC Taylor & Francis Group, Second edition, 2006.

[24] X. H. Zhang, R. Shah, S. Saleem, N. A. Shah, Z. A. Khan, and J. D. Chung, "Natural convection flow maxwell fluids with generalized thermal transport and newtonian heating," *Case Studies in Thermal Engineering*, vol. 27, p. 101226, 2021.

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Research Article

Solving Fractional-Order Diffusion Equations in a Plasma and Fluids via a Novel Transform

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Motivated by the importance of diffusion equations in many physical situations in general and in plasma physics in particular, therefore, in this study, we try to find some novel solutions to fractional-order diffusion equations to explain many of the ambiguities about the phenomena in plasma physics and many other fields. In this article, we implement two well-known analytical methods for the solution of diffusion equations. We suggest the modified form of homotopy perturbation method and Adomian decomposition methods using Jafari-Yang transform. Furthermore, illustrative examples are introduced to show the accuracy of the proposed methods. It is observed that the proposed method solution has the desire rate of convergence toward the exact solution. The suggested method's main advantage is less number of calculations. The proposed methods give series form solution which converges quickly towards the exact solution. To show the reliability of the proposed method, we present some graphical representations of the exact and analytical results, which are in strong agreement with each other. The results we showed through graphs and tables for different fractional-order confirm that the results converge towards exact solution as the fractional-order tends towards integer-order. Moreover, it can solve physical problems having fractional order in different areas of applied sciences. Also, the proposed method helps many plasma physicists in modeling several nonlinear structures such as solitons, shocks, and rogue waves in different plasma systems.

1. Introduction

The integer-order differentiation operators are used to study local phenomena, whereas fractional-order operators are used to studying nonlocal phenomena [1]. The mathematical groundwork for fractional-order derivatives was laid by the collective struggles of various mathematicians, such as Riemann, Liouville, Caputo, Podlubny, Miller, and Ross. Afterward, numerous mathematicians dedicated their efforts to this area. Fractional calculus (FC) can be described as very successful in many phenomena in applied sciences, fluid mechanics, physics of plasmas [2, 3], and other biology utilising mathematical tools of FC. [4, 5]. Other numerous applications of FC in the field of science and technology are related to solid mechanics [6, 7], anomalous transport [8], continuum and statistical mechanics [9], economics

[10], relaxation electrochemistry [11], diffusion procedures [12], complex networks [13, 14], and optimal control problems [15, 16].

Fractional differential equations (FDEs) have gotten a lot of attention from researchers in the last decade due to their ability to improve real-world challenges in different areas of engineering and physics. Mathematicians used various methods described based on the above applications to solve different important fractional-order differential equations (FDEs), mainly partial differential equations of fractional-orders (FPDEs). FPDEs are fundamental mathematical approaches that can be utilized to model different physical models more accurately than integer-order models. Nonlinear FPDEs define various phenomena in engineering, plasma physics, and applied sciences. Especially, nonlinear FPDEs are the preeminent tools to be used in various areas, for example, electrochemistry [17], mathematical social dynamics [18, 19], signal processing [20], informatics [21], traffic model [22], theory of solitons in plasma physics [23], biology [24], and much more [25–28]. Moreover, many authors reduced the fluid plasma equations to FDEs for studying the impact of derivative time-frational on the profile of nonlinear structures in a plasma [2, 3]. For instance, El-Wakil et al. [2] reduced the basic equations of a collisionaless unmagnetized nonthermal plasma having cold inertial electrons and inertialess nonthermal electrons as well as stationary positive ions to a normal KdV equation. After that, the authors use a suitable transformation to convert the normal KdV equation to a time-fractional KdV equation in order to investigate the time-fractional on the profile of electron-acoustic (EA) solitons. Furthermore, the basic equations of an ultrarelativistic plasma were reduced to the normal cylindrical Kadomtsev–Petviashvili (CKP) and cylindrical modified KP (CmKP) equations using a reductive perturbation techniques [3]. Posteriorly, the mentioned equations were converted into space-time fractional CKP and CmKP equations using one of the proper transformations in order to study the influence of space-time fractional domain of the characteristics of the ion-acoustic waves (IAWs) in the ultrarelativistic plasma. The authors made a comparison between the integer- and fractional-order models and found that the fractional-order model gives description to the IAWs in the ultrarelativistic plasmas better than the integer-order model [3]. In literature, different methods are implemented for solving FDEs, such as Iterative Laplace Transform method (ILTM) [29, 30], Approximate-analytical method (AAM) [31], Homotopy Analysis method (HAM) [32], Variational Iteration method (VIM) [33], Elzaki Transform Decomposition method (ETDM) [34], the Differential Transformation method (DTM) [35], and the homotopy perturbation method (HPM) [36, 37].

In literature, there is lot of transformations [38–40], but in this article, we implement the Homotopy Perturbation Jafari-Yang Transform Method (HPYTM) and Jafari-Yang transform decomposition method (YTDM) for the analysis of fractional-order diffusion equations. Xiao-Jun Jafari-Yang introduce the Jafari-Yang transformation and applied for the analysis of different differential problems with constant coefficients, while the Adomian decomposition method [41, 42], on the other hand, is a renowned method to solve linear and nonlinear and nonhomogeneous and homogeneous differential, integro, ordinary, and partial differential equations. It gives analysis in the form of series that converges towards the exact solutions quickly. In 1998, He introduced the homotopy perturbation technique [43, 44]. Later, the nonlinear nonhomogeneous partial differential equations are solved using the HPM (homotopy perturbation method), a semianalytical technique [45–48]. The solution is assumed to be the sum of an infinite sequence that converges rapidly to the exact results. This approach was investigated to analyze both linear and nonlinear problems. In the current work, we proposed a novel approximate analytical method known as (HPYTM). The newly developed technique is the combination of Jafari-Yang transform and HPM. It is investigated that the present methods are very effective in finding fractional diffusion equations analytical solution. The fractional problem results using the proposed methods are also devoted to the fractional view analysis of the problems. It is confirmed that the current techniques can be modified to solve other fractional partial differential equations.

A type of PDE that expresses the phenomenon of atoms or molecule's movement from a region of higher concentration to an area of lower concentration is known as diffusion equations. Adolf Fick, a physiologist, was the first to present Fick's law of diffusion. Fick's law was then transformed into the diffusion equation. Scholar modified diffusion equations such as slow diffusion and the hybrid classical wave equation by generalizing the classical diffusion law [49]. Several implementations of diffusion equation are phase transition, electromagnetism, filtration, biochemistry, geochemistry, dynamics of biological groups, cosmology, plasma physics, and acoustics [50]. There are many investigations related to the Diffusion-type equation and its applications in plasma physics and fluids [51–53]. Motivated by these investigation, in this article, we implement YTDM and HPYTM for solving diffusion equations of the form.

(1) Fractional-order diffusion equation in one dimension as

$$\frac{\partial^\delta \nu}{\partial \tau^\delta} = \frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu \, 0 < \delta \le 1, \tau \ge 0 \tag{1}$$

having initial values

$$\nu(\psi, 0) = g(\psi) \tag{2}$$

(2) Fractional-order diffusion equation in two dimension as

$$\frac{\partial^\delta \nu}{\partial \tau^\delta} = \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} 0 < \delta \le 1, \tau \ge 0 \tag{3}$$

having initial values

$$\nu(\psi,\phi,0)=g(\psi,\phi) \tag{4}$$

(3) Fractional-order diffusion equation in three dimension as

$$\frac{\partial^\delta \nu}{\partial\tau^\delta}=\frac{\partial^2\nu}{\partial\psi^2}+\frac{\partial^2\nu}{\partial\phi^2}+\frac{\partial^2\nu}{\partial\varphi^2}0<\delta\le 1,\tau\ge 0 \tag{5}$$

having initial values

$$\nu(\psi,\phi,\varphi,0)=g(\psi,\phi,\varphi) \tag{6}$$

2. Preliminaries

We covered several fundamental fractional calculus definitions as well as Laplace transform theory properties in this section.

Definition 1. In Caputo manner, the fractional derivative is given as [54].

$$D^\sigma_\varphi \nu(\psi,\varphi)=\frac{1}{\Gamma(k-\sigma)}\int_0^\varphi(\varphi-\rho)^{k-\sigma-1}\nu^{(k)}(\psi,\rho)d\rho,k-1<\sigma\le k,k\in N. \tag{7}$$

Definition 2. Yang and Jafari introduced the Jafari-Yang Laplace transform in 2018. $Y(.)$ determines the Jafari-Yang transform for a function $\xi(\tau)$ and is given as [55].

$$Y\{\nu(\tau)\}=T(u)=\int_0^\infty e^{-\tau/u}\nu(\tau)d\tau,\tau>0,u\in(-\tau_1,\tau_2). \tag{8}$$

The inverse transform is given as

$$Y^{-1}\{T(u)\}=\nu(\tau). \tag{9}$$

Definition 3. For derivative of order n, the Jafari-Yang transform is determined as [55].

$$Y\{\nu^n(\tau)\}=\frac{T(u)}{u^n}-\sum_{k=0}^{n-1}\frac{u^k(0)}{\nu^{k-n-1}},\forall n=1,2,3,\cdots. \tag{10}$$

Definition 4. For fractional-order derivatives, the Jafari-Yang transform is given as [55].

$$T\{u^\sigma(\tau)\}=\frac{T(\nu)}{u^\sigma}-\sum_{k=0}^{n-1}\frac{\nu^k(0)}{u^{k-(\sigma+1)}},0<\sigma\le n. \tag{11}$$

Definition 5. The Mittag-Leffler function, a generalisation of the exponential function, is as [54].

$$E_\sigma(\varphi)=\sum_{q=0}^\infty\frac{\varphi^q}{\Gamma(\sigma q+1)}(\sigma\in C,\mathrm{Re}(\sigma)>0). \tag{12}$$

Equation (12) further generalization is of the form

$$E_{\sigma,\beta}(\varphi)=\sum_{q=0}^\infty\frac{\varphi^q}{\Gamma(\sigma q+\beta)};(\sigma,\beta\in C,\mathrm{Re}(\sigma)>0),\mathrm{Re}(\beta)>0). \tag{13}$$

3. Homotopy Perturbation Jafari-Yang Transform Method

To explain the basic ideas of this approach the following equation is considered:

$$^\sigma_\tau\nu(\psi,\tau)=L[\psi]\nu(\psi,\tau)+N[\psi]\nu(\psi,\tau),0<\sigma\le 2, \tag{14}$$

with some initial sources

$$\nu(\psi,0)=\xi(\psi),\ \frac{\partial}{\partial\tau}\nu(\psi,0)=\zeta(\psi), \tag{15}$$

where $D^\sigma_\tau=\partial^\sigma/\partial\tau^\sigma$ Caputo‘s derivative, and $L[\psi]$ and $N[\psi]$ are the linear and nonlinear operators respectively.

Implement Jafari-Yang transform to (14), we have

$$[D^\sigma_\tau\nu(\psi,\tau)]=Y[L[\psi]\nu(\psi,\tau)+N[\psi]\nu(\psi,\tau)], \tag{16}$$

$$\frac{1}{u^\sigma}\left\{T(u)-u\nu(0)-u^2\nu'(0)\right\}=Y[L[\psi]\nu(\psi,\tau)+N[\psi]\nu(\psi,\tau)]. \tag{17}$$

Equation (17) implies that

$$T(\nu)=u\nu(0)+u^2\nu'(0)+u^\sigma Y[L[\psi]\nu(\psi,\tau)+N[\psi]\nu(\psi,\tau)]. \tag{18}$$

We now have by using the inverse Jafari-Yang transform

$$\nu(\psi,\tau)=\nu(0)+\nu'(0)+Y^{-1}[u^\sigma Y[L[\psi]\nu(\psi,\tau)+N[\psi]\nu(\psi,\tau)]]. \tag{19}$$

Now, perturbation technique having parameter ε is given as

$$\nu(\psi,\tau)=\sum_{k=0}^\infty\varepsilon^k\nu_k(\psi,\tau), \tag{20}$$

where perturbation parameter is ε and $\varepsilon\in[0,1]$.

The decomposition of nonlinear terms is defined as

$$N[\psi]\nu(\psi,\tau)=\sum_{k=0}^\infty\varepsilon^k H_n(\nu), \tag{21}$$

where H_n are of the form $\nu_0,\nu_1,\nu_2,\cdots,\nu_n$, and can be determined as

$$H_n(\nu_0, \nu_1, \cdots, \nu_n) = \frac{1}{\gamma(n+1)} D_\varepsilon^k \left[N\left(\sum_{k=0}^{\infty} \varepsilon^i \nu_i \right) \right]_{\varepsilon=0}, \quad (22)$$

where $D_\varepsilon^k = \partial^k / \partial \varepsilon^k$.

Using (21) and (22) in (19) and making the homotopy, we get

$$\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) = \nu(0) + \nu'(0) + \varepsilon \times \left(Y^{-1} \left[u^\sigma Y \left\{ L \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) + \sum_{k=0}^{\infty} \varepsilon^k H_k(\nu) \right\} \right] \right). \quad (23)$$

When the coefficient of ε on both sides is compared, we get

$$\begin{aligned} \varepsilon^0 &: \nu_0(\psi, \tau) = \nu(0) + \nu'(0), \\ \varepsilon^1 &: \nu_1(\psi, \tau) = Y^{-1}[u^\sigma Y(L[\psi]\nu_0(\psi, \tau) + H_0(\nu))], \\ \varepsilon^2 &: \nu_2(\psi, \tau) = Y^{-1}[u^\sigma Y(L[\psi]\nu_1(\psi, \tau) + H_1(\nu))], \\ &\vdots \\ \varepsilon^k &: \nu_n(\psi, \tau) = Y^{-1}[u^\sigma Y(L[\psi]\nu_{k-1}(\psi, \tau) + H_{k-1}(\nu))]. \end{aligned} \quad (24)$$

As a result, we can quickly determine component $\nu_k(\psi, \tau)$, which leads us to the convergent series. We get by taking $\varepsilon \longrightarrow 1$,

$$\nu(\psi, \tau) = \lim_{M \longrightarrow \infty} \sum_{k=1}^{M} \nu_k(\psi, \tau). \quad (25)$$

The result is in the form of a series that rapidly converges to the exact solution of the problem.

4. Idea of YTDM

Consider the fractional order partial differential equation

$$D_\tau^\sigma \nu(\psi, \tau) = \bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi), 0 < \sigma \le 2, \quad (26)$$

with some initial sources

$$\nu(\psi, 0) = \xi(\psi), \frac{\partial}{\partial \tau}\nu(\psi, 0) = \zeta(\psi), \quad (27)$$

where $D_\tau^\sigma = \partial^\sigma / \partial \tau^\sigma$ denotes the derivative having fractional-order σ in Caputo manner, and $\bar{\mathscr{G}}_1$ and $\mathscr{N}_1$ denote linear and nonlinear functions.

On taking Jafari-Yang transformation of (26), we get

$$Y[D_\tau^\sigma \nu(\psi, \tau)] = Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right]. \quad (28)$$

By using Jafari-Yang transform property of differentiation, we have

$$\frac{1}{u^\sigma}\left\{T(u) - u\nu(0) - u^2\nu'(0)\right\} = Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right]. \quad (29)$$

Equation (29) implies that

$$T(\nu) = u\nu(0) + u^2\nu'(0) + u^\sigma Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right]. \quad (30)$$

We now have by using the inverse Jafari-Yang transform

$$\nu(\psi, \tau) = \nu(0) + \nu'(0) + Y^{-1}\left[u^\sigma Y\left[\bar{\mathscr{G}}_1(\nu, \varphi) + \mathscr{N}_1(\nu, \varphi)\right].\right. \quad (31)$$

The infinite series of $\nu(\psi, \tau)$

$$\nu(\psi, \tau) = \sum_{k=0}^{\infty} \nu_k(\psi, \tau). \quad (32)$$

The nonlinear term decomposition of $\mathscr{N}_1$ by Adomian polynomials is expressed as

$$\mathscr{N}_1(\nu, \varphi) = \sum_{k=0}^{\infty} \mathscr{A}_k. \quad (33)$$

All nonlinear terms can be denoted by means of Adomian polynomials as

$$\mathscr{A}_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial \ell^k}\left\{\mathscr{N}_1\left(\sum_{j=0}^{\infty} \ell^j \nu_j, \sum_{j=0}^{\infty} \ell^j \varphi_j\right)\right\}\right]_{\ell=0}, \quad (34)$$

putting Equations (32) and (34) into (31), gives

$$\sum_{k=0}^{\infty} \nu_k(\psi, \tau) = \nu(0) + \nu'(0) + Y^{-1}u^\sigma \cdot \left[Y\left\{\bar{\mathscr{G}}_1\left(\sum_{k=0}^{\infty} \nu_k, \sum_{k=0}^{\infty} \varphi_k\right) + \sum_{k=0}^{\infty} \mathscr{A}_k\right\}\right]. \quad (35)$$

The following terms are described as

$$\begin{aligned} \nu_0(\psi, \tau) &= \nu(0) + \tau\nu'(0), \\ \nu_1(\psi, \tau) &= Y^-\left[u^\sigma Y^+\left\{\bar{\mathscr{G}}_1(\nu_0, \varphi_0) + \mathscr{A}_0\right\}\right]. \end{aligned} \quad (36)$$

The general for $k \ge 1$ is determined as

$$\nu_{k+1}(\psi, \tau) = Y^-\left[u^\sigma Y^+\left\{\bar{\mathscr{G}}_1(\nu_k, \varphi_k) + \mathscr{A}_k\right\}\right]. \quad (37)$$

5. Applications

The solutions to various fractional order diffusion equations are obtained by implementing HPYTM and YTDM in this section.

Example 1. Consider one-dimension fractional-order diffusion equation [54]

$$\frac{\partial^\sigma \nu}{\partial \tau^\sigma} = \frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu \, 0 < \sigma \leq 1, \tau > 0, \quad (38)$$

with initial source

$$\nu(\psi, 0) = e^{\psi}. \quad (39)$$

On taking Jafari-Yang transformation of Equation (38), we get

$$Y\left(\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right) = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right). \quad (40)$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right), \quad (41)$$

$$T(u) = u\nu(0) + u^\sigma Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right). \quad (42)$$

By applying inverse Jafari-Yang transform to Equation (42)

$$\begin{aligned} \nu(\psi, \tau) &= \nu(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right)\right\}\right], \\ \nu(\psi, \tau) &= e^{\psi} + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} - \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial^2 \nu}{\partial \psi^2} - \nu^2 + \nu\right)\right\}\right]. \end{aligned} \quad (43)$$

Now, applying the abovementioned homotopy perturbation technique as in (23), we obtain

$$\begin{aligned} &\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) \\ &= e^{\psi} + \varepsilon\left(Y^{-1}\left[u^\sigma Y\left[\left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau)\right)_{\psi\psi}\right.\right.\right. \\ &\quad - \left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau)\right)_{\psi} + \left(\sum_{k=0}^{\infty} \varepsilon^k H_k(\nu)\right) \\ &\quad \left.\left.\left. + \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau)\right]\right]\right). \end{aligned} \quad (44)$$

where $H_k(\nu)$ are He's polynomial that represents the nonlinear terms. He's polynomials first few components are given by

$$\begin{aligned} H_0(\nu) &= \nu_0(\nu_0)_{\psi\psi} - \nu_{0,}^2 \\ H_1(\nu) &= \left(\nu_1(\nu_0)_{\psi\psi} + \nu_0(\nu_1)_{\psi\psi}\right) - (2\nu_0\nu_1), \\ H_2(\nu) &= \left(\nu_2(\nu_0)_{\psi\psi} + \nu_1(\nu_1)_{\psi\psi} + \nu_0(\nu_2)_{\psi\psi}\right) \\ &\quad - \left(2\nu_0\nu_2 + (\nu_2)^2\right). \\ &\vdots \end{aligned} \quad (45)$$

Comparing the same power coefficient of ε, we obtain

$$\begin{aligned} \varepsilon^0 &: \nu_0(\psi, \tau) = e^{\psi}, \\ \varepsilon^1 &: \nu_1(\psi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_0}{\partial \psi^2} - \frac{\partial \nu_0}{\partial \psi} + \nu_0 \frac{\partial^2 \nu_0}{\partial \psi^2} - \nu_0^2 + \nu_0\right]\right) = e^{\psi} \frac{\tau^\sigma}{\Gamma(\sigma+1)}, \\ \varepsilon^2 &: \nu_2(\psi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_1}{\partial \psi^2} - \frac{\partial \nu_1}{\partial \psi} + \nu_1 \frac{\partial^2 \nu_0}{\partial \psi^2} + \nu_0 \frac{\partial^2 \nu_1}{\partial \psi^2} - 2\nu_0\nu_1 + \nu_1\right]\right) = e^{\psi} \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}, \\ \varepsilon^3 &: \nu_3(\psi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_2}{\partial \psi^2} - \frac{\partial \nu_2}{\partial \psi} + \nu_2 \frac{\partial^2 \nu_0}{\partial \psi^2} + \nu_1 \frac{\partial^2 \nu_1}{\partial \psi^2} + \nu_0 \frac{\partial^2 \nu_2}{\partial \psi^2} - 2\nu_0\nu_2 - (\nu_2)^2 + \nu_2\right]\right) \\ \varepsilon^3 &: \nu_3(\psi, \tau) = e^{\psi} \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \\ &\vdots \end{aligned} \quad (46)$$

By taking $\varepsilon \longrightarrow 1$, we obtain the convergence series type result is given as

$$\begin{aligned}\nu(\psi,\tau)&=\nu_0+\nu_1+\nu_2+\nu_3+\nu_4+\cdots\\&=e^{\psi}+e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\cdots,\\\nu(\psi,\tau)&=e^{\psi}\left(1+\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\cdots\right),\\\nu(\psi,\tau)&=e^{\psi}\sum_{k=0}^{\infty}\frac{(t^{\sigma})^k}{\Gamma(k\sigma+1)}=e^{\psi}E_{\sigma}(\tau^{\sigma}).\end{aligned}\tag{47}$$

When $\sigma=1$, the HPYTM solution is

$$\nu(\psi,\tau)=e^{\psi}\sum_{k=0}^{\infty}\frac{(t)^k}{k!},\tag{48}$$

The analytical results by YTDM.

On taking Jafari-Yang transformation of Equation (38), we obtain

$$Y\left\{\frac{\partial^{\sigma}\nu}{\partial\tau^{\sigma}}\right\}=Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right].\tag{49}$$

Using the Jafari-Yang transform the differential property, we get

$$\frac{1}{u^{\sigma}}\{T(u)-u\nu(0)\}=Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right],\tag{50}$$

$$T(u)=u\nu(0)+u^{\sigma}Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right].\tag{51}$$

By inverse Jafari-Yang transform of Equation (51)

$$\begin{aligned}\nu(\psi,\tau)&=\nu(0)+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right)\right\}\right],\\\nu(\psi,\tau)&=e^{\psi}+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\nu\frac{\partial^2\nu}{\partial\psi^2}-\nu^2+\nu\right)\right\}\right],\end{aligned}\tag{52}$$

Assuming the unknown $\nu(\psi,\tau)$ function has the following infinite series form solution:

$$\nu(\psi,\tau)=\sum_{k=0}^{\infty}\nu_k(\psi,\tau).\tag{53}$$

The Adomian polynomials $\nu(\nu)_{\psi\psi}=\sum_{k=0}^{\infty}\mathscr{A}_k$ and $\nu^2=\sum_{k=0}^{\infty}\mathscr{B}_k$, as well as the nonlinear term, have been described. Applying specific function, Equation (52) may be determined as

$$\begin{aligned}\sum_{k=0}^{\infty}\nu_k(\psi,\tau)&=\nu(\psi,0)+Y^{-}\left[u^{\sigma}Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\sum_{k=0}^{\infty}\mathscr{A}_k-\sum_{k=0}^{\infty}\mathscr{B}_k+\nu\right]\right],\\\sum_{k=0}^{\infty}\nu_k(\psi,\tau)&=e^{\psi}+Y^{-}\left[u^{\sigma}Y\left[\frac{\partial^2\nu}{\partial\psi^2}-\frac{\partial\nu}{\partial\psi}+\sum_{k=0}^{\infty}\mathscr{A}_k-\sum_{k=0}^{\infty}\mathscr{B}_k+\nu\right]\right].\end{aligned}\tag{54}$$

According to Equation (34), the nonlinear function can be find with the help of Adomian polynomials is given as

$$\begin{aligned}&\mathscr{A}_0=\nu_0(\nu_0)_{\psi\psi},\mathscr{A}_1=\nu_1(\nu_0)_{\psi\psi}+\nu_0(\nu_1)_{\psi\psi},\\&\mathscr{A}_2=\nu_2(\nu_0)_{\psi\psi}+\nu_1(\nu_1)_{\psi\psi}+\nu_0(\nu_2)_{\psi\psi},\\&\mathscr{B}_0=\nu_0^2,\mathscr{B}_1=2\nu_0\nu_1,\mathscr{B}_2=2\nu_0\nu_2+(\nu_2)^2.\end{aligned}\tag{55}$$

Thus, on comparing both sides of Equation (54)

$$\nu_0(\psi,\tau)=e^{\psi},1\tag{56}$$

For $k=0$

$$\nu_1(\psi,\phi,\varphi,\tau)=-3\sin\psi\sin\phi\sin\varphi\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}.\tag{57}$$

For $k=1$

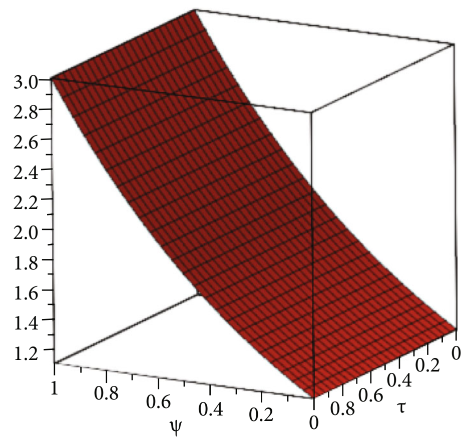
$$\nu_2(\psi,\tau)=e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}.\tag{58}$$

For $k=2$

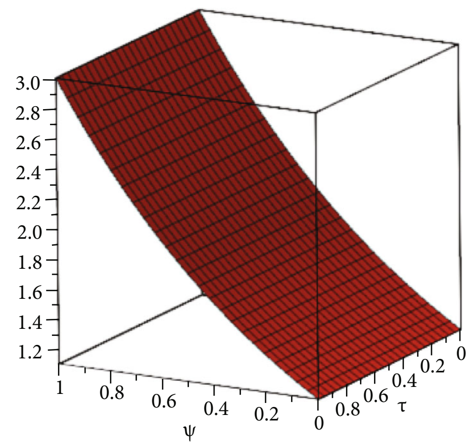
$$\nu_3(\psi,\tau)=e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}.\tag{59}$$

The remaining YTDM solution elements ρ_k for $(k\geq 2)$ are similarly simple to get. As a result, the solution in series form is as

$$\begin{aligned}\nu(\psi,\tau)&=\sum_{k=0}^{\infty}\nu_k(\psi,\tau)=\nu_0(\psi,\tau)+\nu_1(\psi,\tau)\\&\quad+\nu_2(\psi,\tau)+\nu_3(\psi,\tau)+\cdots\nu(\psi,\tau)\\&=e^{\psi}+e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}\\&\quad+e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\cdots.\end{aligned}\tag{60}$$



(a)



(b)

Figure 1: Example 1 solution graph. (a) Exact solution and (b) analytical solution at $\sigma = 1$.

When $\sigma = 1$, the solution by YTDM is

$$\nu(\psi, \tau) = e^{\psi} \sum_{k=0}^{\infty} \frac{(t)^k}{k!}. \tag{61}$$

In closed form, the exact solution is

$$\nu(\psi, \tau) = e^{(\psi+\tau)}. \tag{62}$$

The exact solution shown in Figures 1(a) and 1(b) shows HPYTM and YTDM solution at $\sigma = 1$ and $0 \le \psi \ge 1$. Figure 2(a) shows the solution graph for different values of $\sigma = 1,0.9,0.8,0.7$ and $0 \le \psi \ge 1$ of Example 1 and Figure 2(b), respectively, at $\tau \in [0, 1]$ and $0 \le \psi \ge 1$ while Figure 2(c) shows the error graph. Also, Table 1 shows the comparison of the exact solution and our methods solution with the aid of absolute error at various fractional order. From the figures and table, it is clear that HPYTM and YTDM solution shows strong contact with the exact solutions of the problem.

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Example 2. Consider one-dimension fractional-order gas dynamic equation [56]

$$\frac{\partial^\sigma \nu}{\partial \tau^\sigma} + \nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu = 0\, 0 < \sigma \le 1, \tau > 0, \tag{63}$$

with initial source

$$\nu(\psi, 0) = e^{-\psi}. \tag{64}$$

On taking Jafari-Yang transformation of Equation (63), we get

$$Y\left(\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right) = Y\left(-\nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right). \tag{65}$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left(-\nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right), \tag{66}$$

$$T(u) = u\nu(0) + u^\sigma Y\left(-\nu \frac{\partial \nu}{\partial \psi} + \nu^2 - \nu\right). \tag{67}$$

By applying inverse Jafari-Yang transform to Equation (67)

$$\begin{aligned} \nu(\psi, \tau) &= \nu(0) - Y^{-1}\left[u^\sigma\left\{Y\left(\nu \frac{\partial \nu}{\partial \psi} - \nu^2 + \nu\right)\right\}\right], \\ \nu(\psi, \tau) &= e^{-\psi} - Y^{-1}\left[u^\sigma\left\{Y\left(\nu \frac{\partial \nu}{\partial \psi} - \nu^2 + \nu\right)\right\}\right]. \end{aligned} \tag{68}$$

Now, applying the abovementioned homotopy perturbation technique as in (23), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau) = e^{-\psi} - \varepsilon \Bigg(Y^{-1}\Bigg[u^\sigma Y\Bigg(\Bigg(\sum_{k=0}^{\infty} \varepsilon^k H_k(\nu)\Bigg) \\ + \sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \tau)\Bigg]\Bigg]\Bigg). \end{aligned} \tag{69}$$

where $H_k(\nu)$ are He's polynomial that represents the nonlinear terms. He's polynomials first few components are given by

$$\begin{aligned} H_0(\nu) &= \nu_0(\nu_0)_\psi - \nu_0^2, \\ H_1(\nu) &= \left(\nu_1(\nu_0)_\psi + \nu_0(\nu_1)_\psi\right) - (2\nu_0\nu_1), \\ H_2(\nu) &= \left(\nu_2(\nu_0)_\psi + \nu_1(\nu_1)_\psi + \nu_0(\nu_2)_\psi\right) - \left(2\nu_0\nu_2 + (\nu_2)^2\right). \\ &\vdots \end{aligned} \tag{70}$$

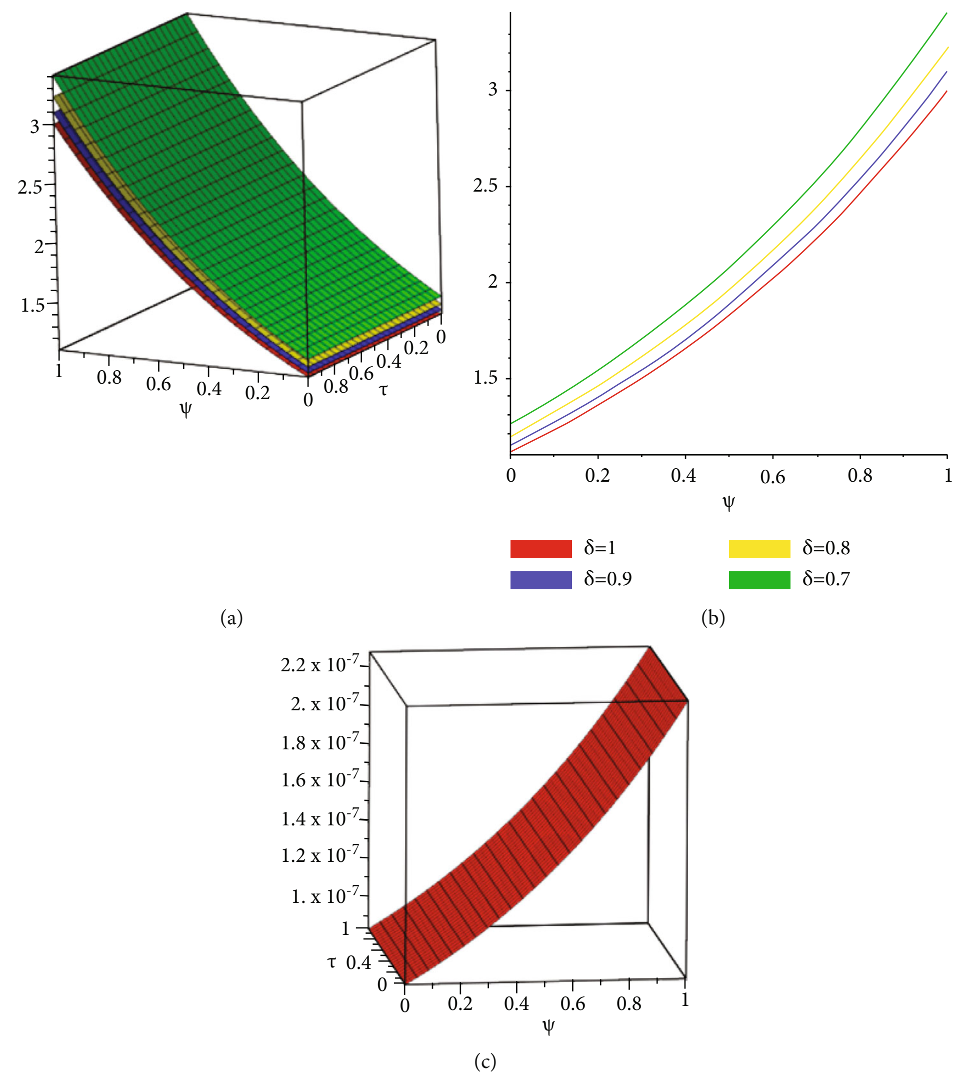

Figure 2: Example 1. (a) Analytical solution at various fractional orders of σ (b) $\phi = 0.5$ and (c) error.

Table 1: $v(\psi, \tau)$ comparison of exact solution, our methods' solution, and Absolute Error (AE) of Example 1.

$\tau = 0.01$ ψ	Absolute error $\sigma = 0.8$	Absolute error $\sigma = 0.9$	Exact solution $\sigma = 1$	Our methods' solution $\sigma = 1$	Absolute error $\sigma = 1$
0	5.2118400000E-04	5.0962000000E-05	1.0100501670000	1.0100501670000	0.00000000E+00
0.1	5.7599800000E-04	5.6322000000E-05	1.1162780700000	1.1162780700000	0.00000000E+00
0.2	6.3657500000E-04	6.2245000000E-05	1.2336780600000	1.2336780600000	0.00000000E+00
0.3	7.0352500000E-04	6.8792000000E-05	1.3634251140000	1.3634251140000	0.00000000E+00
0.4	7.7751600000E-04	7.6027000000E-05	1.5068177850000	1.5068177850000	0.00000000E+00
0.5	8.5928700000E-04	8.4022000000E-05	1.6652911950000	1.6652911950000	0.00000000E+00
0.6	9.4965800000E-04	9.2858000000E-05	1.8404313990000	1.8404313980000	1.00000000E-09
0.7	1.0495350000E-03	1.0262400000E-04	2.0339912590000	2.0339912580000	1.00000000E-09
0.8	1.1599150000E-03	1.1341700000E-04	2.2479079870000	2.2479079860000	1.00000000E-09
0.9	1.2819060000E-03	1.2534600000E-04	2.4843225330000	2.4843225330000	0.00000000E+00
1.0	1.4167240000E-03	1.3852800000E-04	2.7456010150000	2.7456010140000	1.00000000E-09

Comparing the same power coefficient of ε, we obtain

$$\begin{aligned}
\varepsilon^0 &: v_0(\psi,\tau) = e^{-\psi},\\
\varepsilon^1 &: v_1(\psi,\tau) = Y^{-1}\left(u^\sigma Y\left[v_0\frac{\partial v_0}{\partial\psi} - v_0^2 + v_0\right]\right)\\
&= e^{-\psi}\frac{\tau^\sigma}{\Gamma(\sigma+1)},\\
\varepsilon^2 &: v_2(\psi,\tau) = Y^{-1}\left(u^\sigma Y\left[v_1\frac{\partial v_0}{\partial\psi} + v_0\frac{\partial v_1}{\partial\psi}\right.\right.\\
&\quad \left.\left. - 2v_0v_1 + v_1\right]\right) = e^{-\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)},\\
\varepsilon^3 &: v_3(\psi,\tau) = Y^{-1}\left(u^\sigma Y\left[v_2\frac{\partial v_0}{\partial\psi} + v_1\frac{\partial v_1}{\partial\psi}\right.\right.\\
&\quad \left.\left. + v_0\frac{\partial v_2}{\partial\psi} - 2v_0v_2 - (v_2)^2v_2\right]\right),\\
\varepsilon^3 &: v_3(\psi,\tau) = e^{-\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}\\
&\vdots
\end{aligned} \tag{71}$$

By taking $\varepsilon \longrightarrow 1$, we obtain the convergence series type result is given as

$$\begin{aligned}
v(\psi,\tau) &= v_0 + v_1 + v_2 + v_3 + v_4 + \cdots\\
&= e^{-\psi} + e^{-\psi}\frac{\tau^\sigma}{\Gamma(\sigma+1)} + e^{-\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}\\
&\quad + e^{-\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots v(\psi,\tau)\\
&= e^{-\psi}\left(1 + \frac{\tau^\sigma}{\Gamma(\sigma+1)} + \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots\right),\\
v(\psi,\tau) &= e^{-\psi}\sum_{k=0}^{\infty}\frac{(t^\sigma)^k}{\Gamma(k\sigma+1)} = e^{\psi}E_\sigma(\tau^\sigma).
\end{aligned} \tag{72}$$

When $\sigma = 1$, the HPYTM solution is

$$v(\psi,\tau) = \exp^{-\psi}\sum_{k=0}^{\infty}\frac{(t)^k}{k!}. \tag{73}$$

The analytical results by YTDM

On taking Jafari-Yang transformation of Equation (63), we obtain

$$Y\left\{\frac{\partial^\sigma v}{\partial\tau^\sigma}\right\} = Y\left[-v\frac{\partial v}{\partial\psi} + v^2 - v\right], \tag{74}$$

using the Jafari-Yang transform the differential property, we get

$$\frac{1}{u^\sigma}\{T(u) - uv(0)\} = Y\left[-v\frac{\partial v}{\partial\psi} + v^2 - v\right], \tag{75}$$

$$T(u) = uv(0) + u^\sigma Y\left[-v\frac{\partial v}{\partial\psi} + v^2 - v\right]. \tag{76}$$

By inverse Jafari-Yang transform of Equation (76)

$$\begin{aligned}
v(\psi,\tau) &= v(0) - Y^{-1}\left[u^\sigma\left\{Y\left(v\frac{\partial v}{\partial\psi} - v^2 + v\right)\right\}\right],\\
v(\psi,\tau) &= e^{-\psi} - Y^{-1}\left[u^\sigma\left\{Y\left(v\frac{\partial v}{\partial\psi} - v^2 + v\right)\right\}\right].
\end{aligned} \tag{77}$$

Assuming the unknown $v(\psi,\tau)$ function has the following infinite series form solution:

$$v(\psi,\tau) = \sum_{k=0}^{\infty} v_k(\psi,\tau). \tag{78}$$

The Adomian polynomials $v(v)_{\psi\psi} = \sum_{k=0}^{\infty}\mathscr{A}_k$ and $v^2 = \sum_{k=0}^{\infty}\mathscr{B}_k$, as well as the non-linear term, have been described. Applying specific function, Equation (77) may be determined as

$$\begin{aligned}
\sum_{k=0}^{\infty} v_k(\psi,\tau) &= v(\psi,0) - Y^{-}\left[u^\sigma Y\left[\sum_{k=0}^{\infty}\mathscr{A}_k - \sum_{k=0}^{\infty}\mathscr{B}_k + v\right]\right],\\
\sum_{k=0}^{\infty} v_k(\psi,\tau) &= e^{-\psi} - Y^{-}\left[u^\sigma Y\left[\sum_{k=0}^{\infty}\mathscr{A}_k - \sum_{k=0}^{\infty}\mathscr{B}_k + v\right]\right].
\end{aligned} \tag{79}$$

According to Equation (34), the nonlinear function can be found with the help of Adomian polynomials is given as

$$\begin{aligned}
&\mathscr{A}_0 = v_0(v_0)_\psi, \mathscr{A}_1 = v_1(v_0)_{\psi\psi} + v_0(v_1)_\psi,\\
&\mathscr{A}_2 = v_2(v_0)_\psi + v_1(v_1)_\psi + v_0(v_2)_\psi,\\
&\mathscr{B}_0 = v_0^2, \mathscr{B}_1 = 2v_0v_1, \mathscr{B}_2 = 2v_0v_2 + (v_2)^2.
\end{aligned} \tag{80}$$

Thus, on comparing both sides of Equation (79)

$$v_0(\psi,\tau) = e^{-\psi}. \tag{81}$$

For $k = 0$

$$v_1(\psi,\tau) = e^{-\psi}\frac{\tau^\sigma}{\Gamma(\sigma+1)} \tag{82}$$

For $k = 1$

$$v_2(\psi,\tau) = e^{-\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}. \tag{83}$$

For $k = 2$

$$v_3(\psi,\tau) = e^{-\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \tag{84}$$

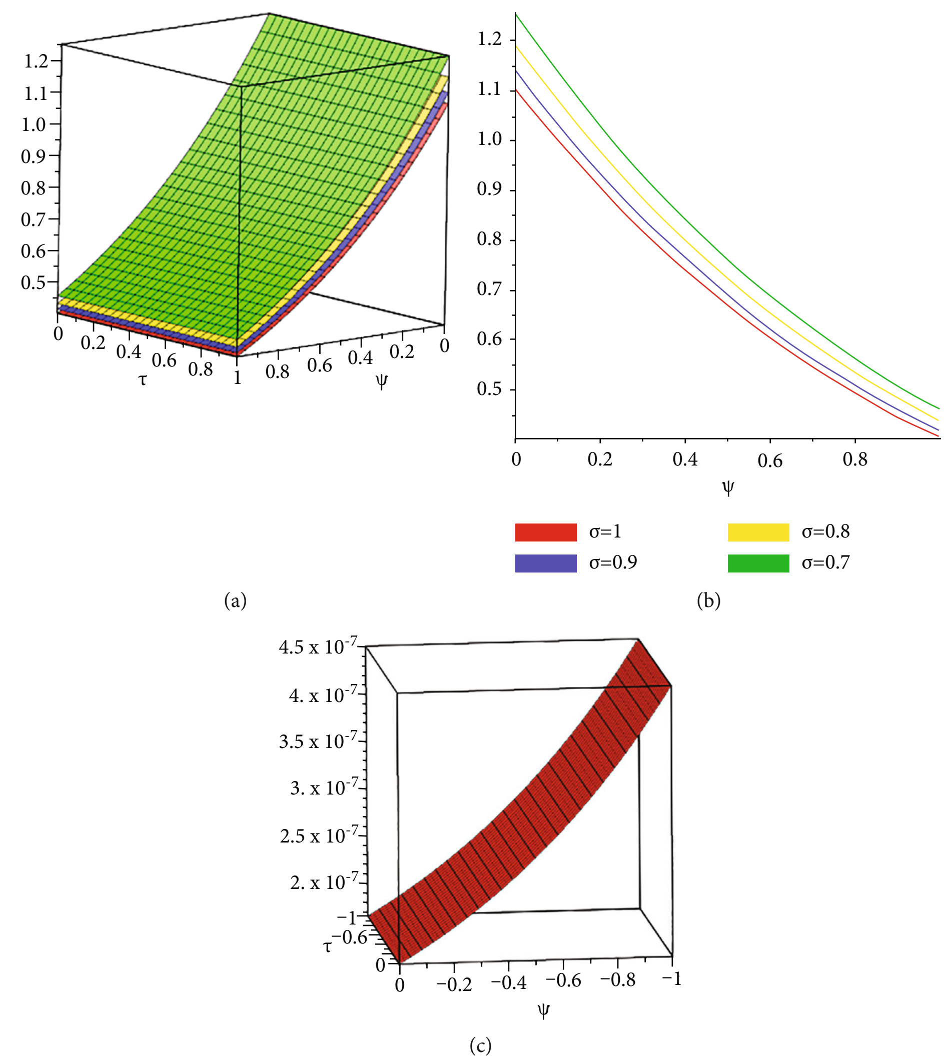

FIGURE 3: Example 2. (a) Analytical solution at various fractional orders of σ (b) $\phi = 0.5$ and (c) error.

The remaining YTDM solution elements ρ_k for $(k \geq 3)$ are similarly simple to get. As a result, the solution in series form is as

$$\begin{aligned} \nu(\psi, \tau) &= \sum_{k=0}^{\infty} \nu_k(\psi, \tau) = \nu_0(\psi, \tau) + \nu_1(\psi, \tau) \\ &\quad + \nu_2(\psi, \tau) + \nu_3(\psi, \tau) \\ \nu(\psi, \tau) &= e^{-\psi} + e^{-\psi} \frac{\tau^{\sigma}}{\Gamma(\sigma+1)} + \exp^{-\psi} \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} \\ &\quad + e^{-\psi} \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \cdots. \end{aligned} \tag{85}$$

When $\sigma = 1$, the solution by YTDM is

$$\nu(\psi, \tau) = e^{-\psi} \sum_{k=0}^{\infty} \frac{(t)^k}{k!}. \tag{86}$$

In closed form, the exact solution is

$$\nu(\psi, \tau) = e^{(-\psi+\tau)}. \tag{87}$$

Figure 3(a) shows the solution graph for different values of $\sigma = 1,\ 0.9,\ 0.8,\ 0.7$ and $0 \leq \psi, \phi \geq 1$ of Example 2 and Figure 3(b), respectively, at $\phi = 0.5, \tau \in [0, 1]$ and $0 \leq \psi \geq 1$ while Figure 3(c) shows the error graph. It is verified from the figures that HPYTM and YTDM solution is closely related with the exact solution.

Example. 3. Consider two-dimension fractional-order diffusion equation [54]

$$\frac{\partial^{\sigma} \nu}{\partial \tau^{\sigma}} = \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} 0 < \sigma \leq 1, \tau \geq 0, \tag{88}$$

with initial source

$$\nu(\psi, \phi, 0) = (1 - \phi) e^{\psi}. \tag{89}$$

Taking Jafari-Yang transformation to Equation (88), we get

$$Y\left(\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right) = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right), \tag{90}$$

We get by using the Jafari-Yang transform's differential property

$$\begin{aligned}\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} &= Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right),\\ T(u) &= u\nu(0) + u^\sigma Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right).\end{aligned} \tag{91}$$

By applying inverse Jafari-Yang transform to Equation (91)

$$\begin{aligned}\nu(\psi, \phi, \tau) &= \nu(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right)\right\}\right],\\ \nu(\psi, \phi, \tau) &= (1-\phi)e^\psi + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right)\right\}\right].\end{aligned} \tag{92}$$

Now, using the abovementioned homotopy perturbation technique as in (23), we get

$$\begin{aligned}&\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \tau)\\ &\quad = (1-\phi)e^\psi + \varepsilon\left(Y^{-1}\left[u^\sigma Y\left[\left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \tau)\right)_{\psi\psi}\right.\right.\right.\\ &\qquad \left.\left.\left. + \left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \tau)\right)_{\phi\phi}\right]\right]\right).\end{aligned} \tag{93}$$

Comparing the same power coefficient of ε, we obtain

$$\begin{aligned}\varepsilon^0 &: \nu_0(\psi, \phi, \tau) = (1-\phi)e^\psi,\\ \varepsilon^1 &: \nu_1(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_0}{\partial \psi^2} + \frac{\partial^2 \nu_0}{\partial \phi^2}\right]\right) = (1-\phi)e^\psi \frac{\tau^\sigma}{\Gamma(\sigma+1)},\\ \varepsilon^2 &: \nu_2(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_1}{\partial \psi^2} + \frac{\partial^2 \nu_1}{\partial \phi^2}\right]\right) = (1-\phi)e^\psi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)},\\ \varepsilon^3 &: \nu_3(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_2}{\partial \psi^2} + \frac{\partial^2 \nu_2}{\partial \phi^2}\right]\right) = (1-\phi)e^\psi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)},\\ \varepsilon^4 &: \nu_4(\psi, \phi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_3}{\partial \psi^2} + \frac{\partial^2 \nu_3}{\partial \phi^2}\right]\right) = (1-\phi)e^\psi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}.\\ &\vdots\end{aligned} \tag{94}$$

By taking $\varepsilon \longrightarrow 1$, we obtain convergence series type result is

$$\begin{aligned}\nu(\psi, \phi, \tau) &= \nu_0 + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \cdots\\ &= (1-\phi)e^\psi + (1-\phi)e^\psi \frac{\tau^\sigma}{\Gamma(\sigma+1)} + (1-\phi)e^\psi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} + (1-\phi)e^\psi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}\\ &\quad + (1-\phi)e^\psi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} + \cdots \nu(\psi, \phi, \tau)\\ &= (1-\phi)e^\psi\left(1 + \frac{\tau^\sigma}{\Gamma(\sigma+1)} + \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} + \cdots\right),\\ \nu(\psi, \phi, \tau) &= (1-\phi)e^\psi \sum_{k=0}^{\infty} \frac{(t^\sigma)^k}{\Gamma(k\sigma+1)} = (1-\phi)e^\psi E_\sigma(t^\sigma).\end{aligned} \tag{95}$$

When $\sigma = 1$, the HPYTM solution is

$$\nu(\psi, \phi, \tau) = (1-\phi)e^\psi \sum_{k=0}^{\infty} \frac{(t)^k}{k!}. \tag{96}$$

The analytical results by YTDM

Taking Jafari-Yang transformation of Equation (88), we obtain

$$Y\left\{\frac{\partial^\sigma \nu}{\partial \tau^\sigma}\right\} = Y\left[\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right]. \tag{97}$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left[\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right], \tag{98}$$

$$T(u) = u\nu(0) + u^\sigma Y\left[\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2}\right]. \tag{99}$$

By inverse Jafari-Yang transform of Equation (99)

$$\nu(\psi,\phi,\tau)=\nu(0)+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2\nu}{\partial\psi^2}+\frac{\partial^2\nu}{\partial\phi^2}\right)\right\}\right],$$
$$\nu(\psi,\phi,\tau)=(1-\phi)e^{\psi}+Y^{-1}\left[u^{\sigma}\left\{Y\left(\frac{\partial^2\nu}{\partial\psi^2}+\frac{\partial^2\nu}{\partial\phi^2}\right)\right\}\right]. \tag{100}$$

Assuming the unknown $\nu(\psi,\phi,\tau)$ function has the following infinite series form solution:

$$\nu(\psi,\phi,\tau)=\sum_{k=0}^{\infty}\nu_k(\psi,\phi,\tau),$$
$$\sum_{k=0}^{\infty}\nu_k(\psi,\phi,\tau)=e^{\psi}+Y^{-}\left[u^{\sigma}Y\left[\frac{\partial^2\nu}{\partial\psi^2}+\frac{\partial^2\nu}{\partial\phi^2}\right]\right]. \tag{101}$$

Thus, on comparing both sides of Equation (101)

$$\nu_0(\psi,\phi,\tau)=(1-\phi)e^{\psi}, \tag{102}$$

For $k=0$

$$\nu_1(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}. \tag{103}$$

For $k=1$

$$\nu_2(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}. \tag{104}$$

For $k=2$

$$\nu_3(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \tag{105}$$

For $k=3$

$$\nu_4(\psi,\phi,\tau)=(1-\phi)e^{\psi}\frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}. \tag{106}$$

The remaining YTDM solution elements ρ_k for $(k\geq 3)$ are similarly simple to get. As a result, the solution in series form is as

$$\nu(\psi,\phi,\tau)=\sum_{k=0}^{\infty}\nu_k(\psi,\phi,\tau)=\nu_0(\psi,\phi,\tau)+\nu_1(\psi,\phi,\tau)+\nu_2(\psi,\phi,\tau)+\nu_3(\psi,\phi,\tau)+\nu_4(\psi,\phi,\tau)+\cdots$$
$$\nu(\psi,\phi,\tau)=(1-\phi)e^{\psi}+(1-\phi)e^{\psi}\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+(1-\phi)e^{\psi}\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+(1-\phi)e^{\psi}\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+(1-\phi)e^{\psi}\frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}+\cdots$$
$$\nu(\psi,\phi,\tau)=(1-\phi)e^{\psi}\left(1+\frac{\tau^{\sigma}}{\Gamma(\sigma+1)}+\frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}+\frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}+\frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}+\cdots\right),$$
$$\nu(\psi,\phi,\tau)=(1-\phi)e^{\psi}\sum_{k=0}^{\infty}\frac{(t^{\sigma})^k}{\Gamma(k\sigma+1)}=(1-\phi)e^{\psi}E_{\sigma}(t^{\sigma}). \tag{107}$$

When $\sigma=1$, the solution by YTDM is

$$\nu(\psi,\phi,\tau)=(1-\phi)e^{\psi}\sum_{k=0}^{\infty}\frac{(t)^k}{k!}. \tag{108}$$

In closed form, the exact solution is:

$$\nu(\psi,\phi,\tau)=(1-\phi)e^{(\psi+\tau)}. \tag{109}$$

The exact solution shown in Figures 4(a) and 4(b) shows HPYTM and YTDM solutions at $\sigma=1$ and $0\leq\psi,\phi\geq 1$. Figure 5(a) shows the error graph for different values of $\sigma=1,0.8,0.6,0.5$ and $0\leq\psi,\phi\geq 1$ of Example 3 and Figure 5(d) respectively, at $\phi=0.5,\tau\in[0,1]$ and $0\leq\psi\geq 1$ while Figure 5(e) shows the error graph. From the figures and Table 2, it is clear that HPYTM and YTDM solution shows strong agreement with the exact solutions of the problem.

Example 4. Consider three-dimension fractional-order diffusion equation [54]

$$\frac{\partial^{\sigma}\nu}{\partial\tau^{\sigma}}=\frac{\partial^2\nu}{\partial\psi^2}+\frac{\partial^2\nu}{\partial\phi^2}+\frac{\partial^2\nu}{\partial\varphi^2},\ 0<\sigma\leq 1,\ \tau\geq 0, \tag{110}$$

with initial source

$$\nu(\psi,\phi,\varphi,0)=\sin\psi\sin\phi\sin\varphi. \tag{111}$$

Taking Jafari-Yang transformation of Equation (110), we get

$$Y\left(\frac{\partial^{\sigma}\nu}{\partial\tau^{\sigma}}\right)=Y\left(\frac{\partial^2\nu}{\partial\psi^2}+\frac{\partial^2\nu}{\partial\phi^2}+\frac{\partial^2\nu}{\partial\varphi^2}\right). \tag{112}$$

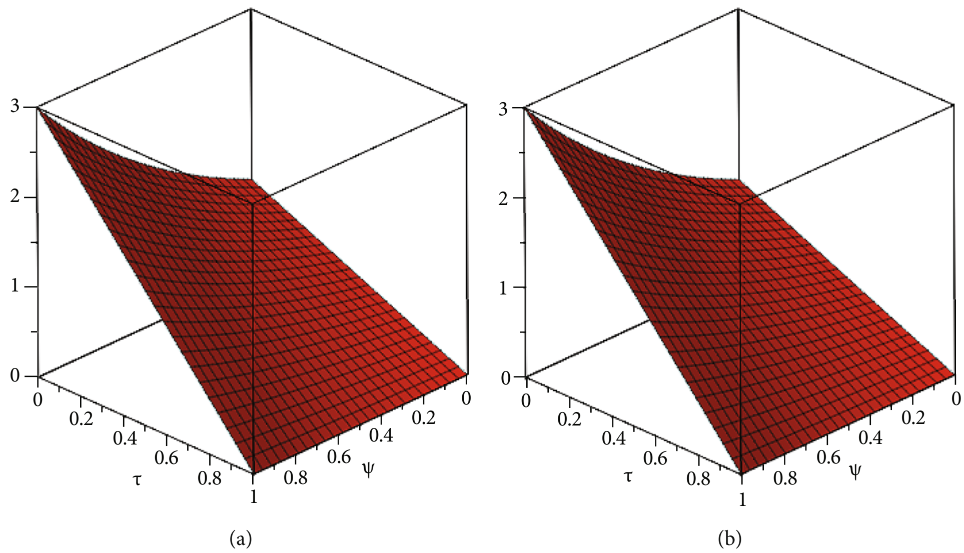

Figure 4: Example 3solufiguretion graph. (a) Exact solution and (b) analytical solution at $\sigma = 1$.

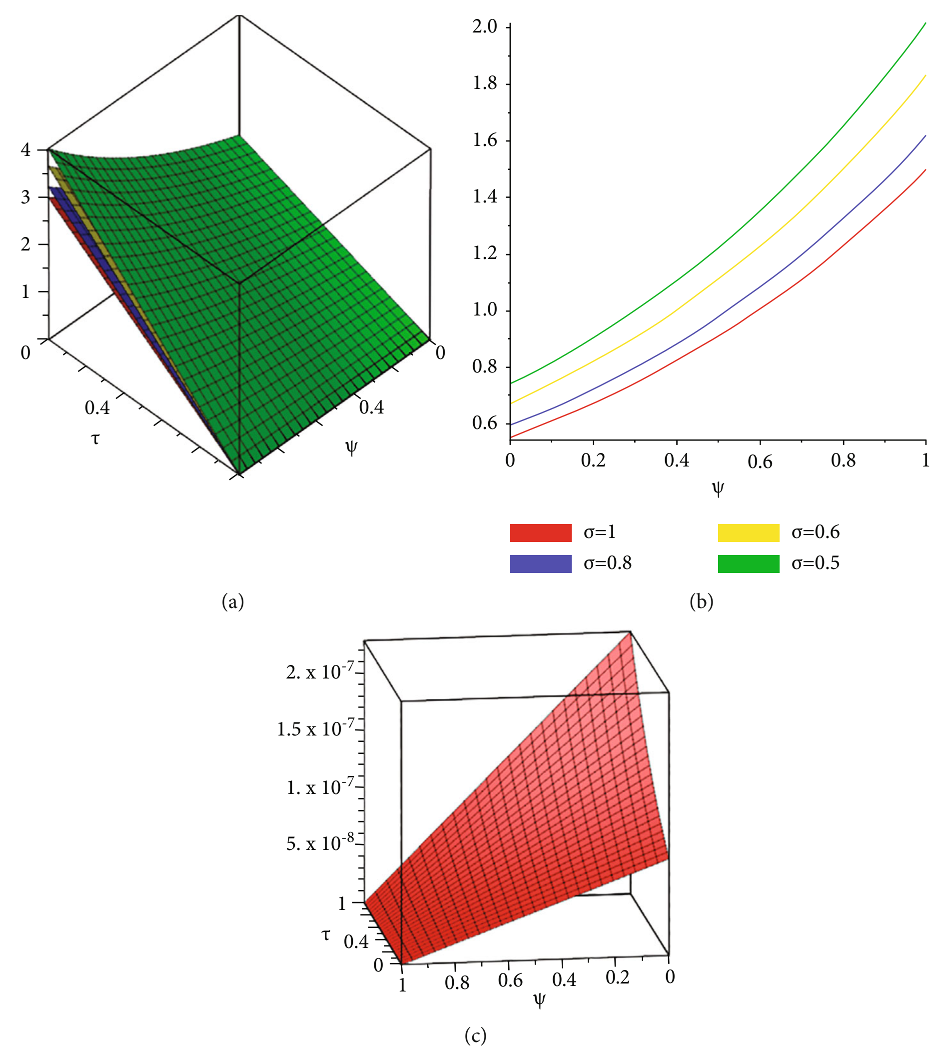

Figure 5: Example 3. (a) Analytical solution at various fractional orders of σ (b) $\phi = 0.5$ and (c) error.

Table 2: $\nu(\psi, \phi, \tau)$ comparison of exact solution, our methods' solution, and Absolute Error (AE) of Example 3.

$\tau = 0.01$ ψ	Exact solution $\sigma = 1$	Our methods' solution $\sigma = 1$	AE of our methods $\sigma = 1$	AE of our methods $\sigma = 0.9$	AE of our methods $\sigma = 0.8$
0	0.505025083500000	0.505025083500000	0.0000000000E+00	2.5480900000E-05	2.6059160000E-04
0.1	0.558139035000000	0.558139035100000	1.0000000000E-10	2.8160900000E-05	2.8799840000E-04
0.2	0.616839030000000	0.616839029800000	2.0000000000E-10	3.1122300000E-05	3.1828710000E-04
0.3	0.681712557000000	0.681712557200000	2.0000000000E-10	3.4395800000E-05	3.5176210000E-04
0.4	0.753408892500000	0.753408892700000	2.0000000000E-10	3.8013200000E-05	3.8875720000E-04
0.5	0.832645597500000	0.832645597600000	1.0000000000E-10	4.2011000000E-05	4.2964300000E-04
0.6	0.920215699500000	0.920215699100000	4.0000000000E-10	4.6428800000E-05	4.7482850000E-04
0.7	1.016995630000000	1.016995629000000	1.0000000000E-09	5.1311000000E-05	5.2476600000E-04
0.8	1.123953994000000	1.123953993000000	1.0000000000E-09	5.6708000000E-05	5.7995600000E-04
0.9	1.242161266000000	1.242161267000000	1.0000000000E-09	6.2673000000E-05	6.4095200000E-04
1.0	1.372800508000000	1.372800507000000	1.0000000000E-09	6.9263000000E-05	7.0836100000E-04

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - u\nu(0)\} = Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} + \frac{\partial^2 \nu}{\partial \varphi^2}\right), \quad (113)$$

$$T(u) = u\nu(0) + u^\sigma Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} + \frac{\partial^2 \nu}{\partial \varphi^2}\right), \quad (114)$$

By applying inverse Jafari-Yang transform to Equation (113)

$$\begin{aligned} \nu(\psi, \phi, \varphi, \tau) &= \nu(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} + \frac{\partial^2 \nu}{\partial \varphi^2}\right)\right\}\right], \\ \nu(\psi, \phi, \varphi, \tau) &= \sin\psi \sin\phi \sin\varphi + Y^{-1} \\ &\quad \cdot \left[u^\sigma\left\{Y\left(\frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \phi^2} + \frac{\partial^2 \nu}{\partial \varphi^2}\right)\right\}\right]. \end{aligned} \quad (115)$$

Now, using the abovementioned homotopy perturbation technique as in (23), we obtain

$$\begin{aligned} &\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \varphi, \tau) \\ &\quad = \sin\psi \sin\phi \sin\varphi \\ &\qquad + \varepsilon\left(Y^{-1}\left[u^\sigma Y\left[\left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \varphi, \tau)\right)_{\psi\psi}\right.\right.\right. \\ &\qquad + \left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \varphi, \tau)\right)_{\phi\phi} \\ &\qquad \left.\left.\left. + \left(\sum_{k=0}^{\infty} \varepsilon^k \nu_k(\psi, \phi, \varphi, \tau)\right)_{\varphi\varphi}\right]\right]\right). \end{aligned} \quad (116)$$

Comparing the same power coefficient of ε, we get

$$\begin{aligned} \varepsilon^0 &: \nu_0(\psi, \phi, \varphi, \tau) = \sin\psi \sin\phi \sin\varphi, \\ \varepsilon^1 &: \nu_1(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_0}{\partial \psi^2} + \frac{\partial^2 \nu_0}{\partial \phi^2}\right]\right) \\ &\qquad = -3 \sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)}, \\ \varepsilon^2 &: \nu_2(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_1}{\partial \psi^2} + \frac{\partial^2 \nu_1}{\partial \phi^2}\right]\right) \\ &\qquad = (-3)^2 \sin\psi \mathrm{ssin}\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}, \\ \varepsilon^3 &: \nu_3(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_2}{\partial \psi^2} + \frac{\partial^2 \nu_2}{\partial \phi^2}\right]\right) \\ &\qquad = (-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}, \\ \varepsilon^4 &: \nu_4(\psi, \phi, \varphi, \tau) = Y^{-1}\left(u^\sigma Y\left[\frac{\partial^2 \nu_3}{\partial \psi^2} + \frac{\partial^2 \nu_3}{\partial \phi^2}\right]\right) \\ &\qquad = (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}. \\ &\vdots \end{aligned} \quad (117)$$

By taking $\varepsilon \longrightarrow 1$, we obtain the convergence series type result is given as

$$\begin{aligned} \nu(\psi, \phi, \varphi, \tau) &= \nu_0 + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \cdots = \sin\psi \sin\phi \sin\varphi \\ &\quad - 3 \sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)} \\ &\quad + (-3)^2 \sin\psi \sin\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} \end{aligned}$$

$$+(-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)} + \cdots v(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi \cdot \left(1 - \frac{3\tau^\sigma}{\Gamma(\sigma+1)} + \frac{(-3\tau^\sigma)^2}{\Gamma(2\sigma+1)} + \frac{(-3\tau^\sigma)^3}{\Gamma(3\sigma+1)} + \frac{(-3\tau^\sigma)^4}{\Gamma(4\sigma+1)} + \cdots\right). \quad (118)$$

When $\sigma = 1$, then the HPYTM solution in a closed form:

$$v(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi \left(1 - 3\tau + \frac{(-3\tau)^2}{2!} + \frac{(-3\tau)^3}{3!} + \frac{(-3\tau)^4}{4!} + \cdots,\right). \quad (119)$$

The analytical results by YTDM

Taking Jafari-Yang transformation of Equation (110), we obtain

$$Y\left\{\frac{\partial^\sigma v}{\partial \tau^\sigma}\right\} = Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right]. \quad (120)$$

We get by using the Jafari-Yang transform's differential property

$$\frac{1}{u^\sigma}\{T(u) - uv(0)\} = Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right], \quad (121)$$

$$T(u) = uv(0) + u^\sigma Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right], \quad (122)$$

by inverse Jafari-Yang transform of Equation (121)

$$v(\psi,\phi,\varphi,\tau) = v(0) + Y^{-1}\left[u^\sigma\left\{Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right)\right\}\right],$$
$$v(\psi,\phi,\varphi,\tau) = \exp(\psi+\phi) + Y^{-1} \cdot \left[u^\sigma\left\{Y\left(\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right)\right\}\right]. \quad (123)$$

Assuming the unknown $v(\psi,\phi,\varphi,\tau)$ function has the following infinite series form solution:

$$v(\psi,\phi,\varphi,\tau) = \sum_{k=0}^{\infty} v_k(\psi,\phi,\varphi,\tau), \quad (124)$$

$$\sum_{k=0}^{\infty} v_k(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi + Y^{-} \cdot \left[u^\sigma Y\left[\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial \varphi^2}\right]\right]. \quad (125)$$

Thus, on comparing both sides of Equation (124)

$$v_0(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi. \quad (126)$$

For $k = 0$

$$v_1(\psi,\phi,\varphi,\tau) = -3 \sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)}, \quad (127)$$

For $k = 1$

$$v_2(\psi,\phi,\varphi,\tau) = (-3)^2 \sin\psi \sin\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)}. \quad (128)$$

For $k = 2$

$$v_3(\psi,\phi,\varphi,\tau) = (-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)}. \quad (129)$$

For $k = 3$

$$v_4(\psi,\phi,\varphi,\tau) = (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}. \quad (130)$$

The remaining YTDM solution elements ρ_k for $(k \geq 3)$ are similarly simple to get. As a result, the solution in series form is as

$$v(\psi,\phi,\varphi,\tau) = v_0(\psi,\phi,\varphi,\tau) + v_1(\psi,\phi,\varphi,\tau) + v_2(\psi,\phi,\varphi,\tau) + v_3(\psi,\phi,\varphi,\tau) + \cdots v(\psi,\phi,\varphi,\tau)$$
$$= \sin\psi \sin\phi \sin\varphi - 3\sin\psi \sin\phi \sin\varphi \frac{\tau^\sigma}{\Gamma(\sigma+1)} + (-3)^2 \sin\psi \sin\phi \sin\varphi \frac{\tau^{2\sigma}}{\Gamma(2\sigma+1)} + (-3)^3 \sin\psi \sin\phi \sin\varphi \frac{\tau^{3\sigma}}{\Gamma(3\sigma+1)} + (-3)^4 \sin\psi \sin\phi \sin\varphi \frac{\tau^{4\sigma}}{\Gamma(4\sigma+1)}$$
$$+ \cdots v(\psi,\phi,\varphi,\tau) = \sin\psi \sin\phi \sin\varphi \cdot \left(1 - \frac{3\tau^\sigma}{\Gamma(\sigma+1)} + \frac{(-3\tau^\sigma)^2}{\Gamma(2\sigma+1)} + \frac{(-3\tau^\sigma)^3}{\Gamma(3\sigma+1)} + \frac{(-3\tau^\sigma)^4}{\Gamma(4\sigma+1)} + \cdots\right). \quad (131)$$

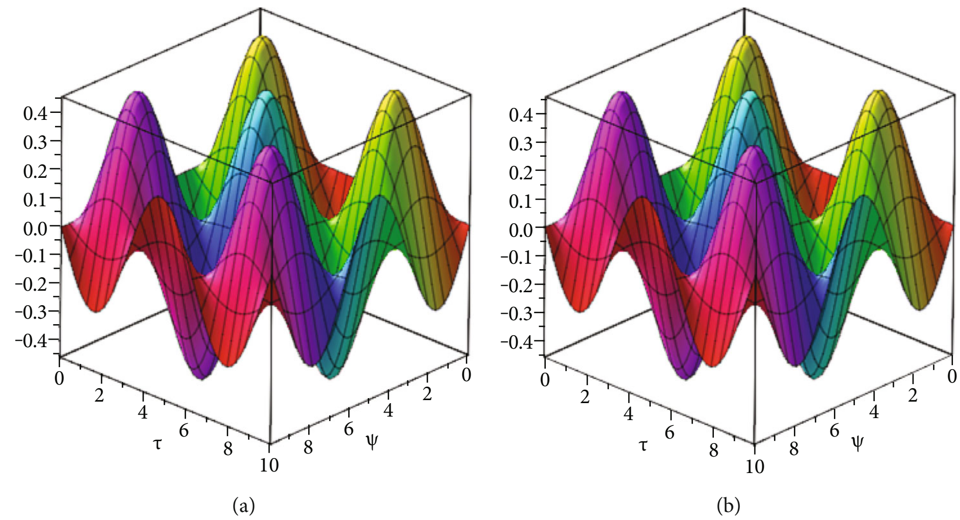

Figure 6: Example 4 solution graph. (a) Exact solution. (b) Analytical solution at $\sigma = 1$.

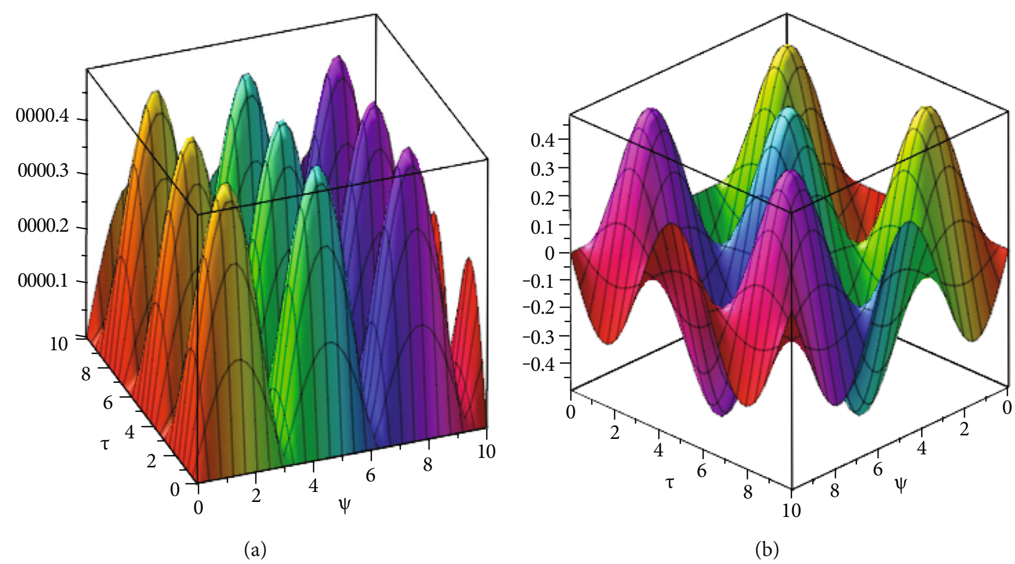

Figure 7: (a) Error graph and solution graph (b) at $\sigma = 0.5$ of Example 4.

Table 3: $v(\psi, \phi, ,\varphi, \tau)$ comparison of exact solution, our methods' solution, and Absolute Error (AE) of Example 4.

$\tau = 0.01$ ψ	Exact solution $\sigma = 1$	Our methods' solution $\sigma = 1$	AE of our methods $\sigma = 1$	AE of our methods $\sigma = 0.9$	AE of our methods $\sigma = 0.8$
0	0.00000000000000	0.00000000000000	0.000000000E+00	0.000000000E+00	0.000000000E+00
0.1	0.039084756460000	0.039084756470000	8.4147098480E-12	5.8916348320E-06	6.0179783350E-05
0.2	0.077778990960000	0.077778990980000	1.6829419700E-11	1.1724408770E-05	1.1975827680E-04
0.3	0.115696083500000	0.115696083500000	0.0000000000E+00	1.7440075190E-05	1.7814016480E-04
0.4	0.152457179000000	0.152457179000000	0.0000000000E+00	2.2981414070E-05	2.3474221550E-04
0.5	0.187694972800000	0.187694972800000	8.4147098480E-11	2.8293115510E-05	2.8899866580E-04
0.6	0.221057380400000	0.221057380400000	0.0000000000E+00	3.3322251000E-05	3.4036760840E-04
0.7	0.252211055800000	0.252211055900000	8.4147098480E-11	3.8018332270E-05	3.8833574600E-04
0.8	0.280844721700000	0.280844721700000	8.4147098480E-11	4.2334573540E-05	4.3242369270E-04
0.9	0.306672279900000	0.306672280000000	8.4147098480E-11	4.6227807340E-05	4.7219102240E-04
1.0	0.329435670100000	0.329435670200000	8.4147098480E-11	4.9659157730E-05	5.0724039260E-04

When $\sigma = 1$, then the closed form solution by YTDM

$$\nu(\psi, \phi, \varphi, \tau) = \sin \psi \sin \phi \sin \varphi \left(1 - 3\tau + \frac{(-3\tau)^2}{2!} + \frac{(-3\tau)^3}{3!} + \frac{(-3\tau)^4}{4!} + \cdots \right). \tag{132}$$

In closed form, the exact solution is

$$\nu(\psi, \phi, \varphi, \tau) = \exp^{-3\tau} \sin \psi \sin \phi \sin \varphi. \tag{133}$$

In Figures 6(a) and 6(b), we consider fixed order $\sigma = 1$ for piecewise approximation values of ψ, ϕ in the domain $0 \leq \psi, \phi \geq 10$ and $=1$. Figure 7(a) shows error graph and Figure 7(b) represents HPYTM and YTDM solution at $\sigma = 0.6$ of Example 4. It is verified from the Figures 6(a) and 6(b) and Table 3 that HPYTM and YTDM solution is closely related with the exact solution.

6. Conclusion

In the present article, different analytical techniques are used to show the fractional view analysis of diffusion equations. In Caputo, manner fractional derivative is considered. The suggested techniques are tested to solve fractional-order diffusion equations. It is observed that the suggested techniques are the best tool for investigating fractional partial differential equations. The close relation between the exact and analytical results is confirmed by the plotted graphs. The given methods give series form solution which have higher convergence rate towards the exact results. It is also shown that both methods give same solution for the proposed problems. Finally, both proposed methods are very methodical and efficient and may be used to investigate nonlinear physical problems related to physics of plasmas such as modeling nonlinear unmodulated and modulated structures. Moreover, the obtained results/solutions can be useful in investigating the diffusion characteristics of some plasmas and fluids.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

[1] J. H. He, “Nonlinear oscillation with fractional derivative and its applications,” in *International conference on vibrating engineering*, pp. 288–291, Dalian, China, 1998.

[2] S. A. El-Wakil, E. M. Abulwafa, E. K. El-Shewy, and A. A. Mahmoud, “Time-fractional KdV equation for plasma of two different temperature electrons and stationary ion,” *Physics of Plasmas*, vol. 18, no. 9, article 092116, 2011.

[3] Q. Liu and L. Chen, “Time-space fractional model for complex cylindrical ion-acoustic waves in Ultrarelativistic plasmas,” *Complexity*, vol. 2020, Article ID 9075823, 16 pages, 2020.

[4] A. Atangana and J. F. Gmez-Aguilar, “Decolonisation of fractional calculus rules: breaking commutativity and associativity to capture more natural phenomena,” *The European Physical Journal Plus*, vol. 133, no. 4, p. 166, 2018.

[5] R. Almeida, N. R. Bastos, and M. T. T. Monteiro, “Modeling some real phenomena by fractional differential equations,” *Mathematical Methods in the Applied Sciences*, vol. 39, no. 16, pp. 4846–4855, 2016.

[6] Y. A. Rossikhin and M. V. Shitikova, “Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results,” *Applied Mechanics Reviews*, vol. 63, no. 1, 2010.

[7] Y. A. Rossikhin, “Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids,” *Applied Mechanics Reviews*, vol. 63, no. 1, 2010.

[8] R. Klages, G. Radons, and I. M. Sokolov, Eds., *Anomalous Transport: Foundations and Applications*, John Wiley and Sons, 2008.

[9] A. Carpinteri and F. Mainardi, Eds., *Fractals and Fractional Calculus in Continuum Mechanics*, vol. 378, Springer, 2014.

[10] V. E. Tarasov, “On history of mathematical economics: application of fractional calculus,” *Mathematics*, vol. 7, no. 6, p. 509, 2019.

[11] V. Martynyuk and M. Ortigueira, “Fractional model of an electrochemical capacitor,” *Signal Processing*, vol. 107, pp. 355–360, 2015.

[12] Y. Zhang, H. Sun, H. H. Stowell, M. Zayernouri, and S. E. Hansen, “A review of applications of fractional calculus in earth system dynamics,” *Chaos, Solitons and Fractals*, vol. 102, p. 29, 2017.

[13] M. Turalska and B. J. West, “Fractional dynamics of individuals in complex networks,” *Frontiers in Physics*, vol. 6, p. 110, 2018.

[14] H. L. Li, J. Cao, H. Jiang, and A. Alsaedi, “Finite-time synchronization and parameter identification of uncertain fractional-order complex networks,” *Physica A: Statistical Mechanics and its Applications*, vol. 533, article 122027, 2019.

[15] F. Mohammadi, L. Moradi, D. Baleanu, and A. Jajarmi, “A hybrid functions numerical scheme for fractional optimal control problems: application to nonanalytic dynamic systems,” *Journal of Vibration and Control*, vol. 24, no. 21, pp. 107754631774176–107754631775043, 2017.

[16] A. Jajarmi and D. Baleanu, “On the fractional optimal control problems with a general derivative operator,” *Asian Journal of Control*, vol. 23, no. 2, pp. 1062–1071, 2021.

[17] K. B. Oldham, “Fractional differential equations in electrochemistry,” *Advances in Engineering Software*, vol. 41, no. 1, pp. 9–12, 2010.

[18] N. K. Vitanov, Z. I. Dimitrova, and M. Ausloos, "Verhulst-Lotka-Volterra (VLV) model of ideological struggle," *Physica A*, vol. 389, no. 21, pp. 4970–4980, 2010.

[19] N. K. Vitanov and S. Panchev, "Generalization of the model of conflict between two armed groups," *Comptes Rendus de l'Académie Bulgare des Sciences*, vol. 61, pp. 1121–1126, 2008.

[20] Y. Povstenko, "Signaling problem for time-fractional diffusion-wave equation in a half-space in the case of angular symmetry," *Nonlinear Dynamics*, vol. 59, no. 4, pp. 593–605, 2010.

[21] B. Mandelbrot, "Some noises withI/fspectrum, a bridge between direct current and white noise," *IEEE Transactions on Information Theory*, vol. 13, no. 2, pp. 289–298, 1967.

[22] J. H. He, "Some applications of nonlinear fractional differential equations and their approximations," *Bulletin of Science and Technology*, vol. 15, no. 2, pp. 86–90, 1999.

[23] M. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, England, 2010.

[24] A. C. Scott, *Neuroscience: A Mathematical Primer*, Springer, New York, NY, USA, 2002.

[25] H. Singh and A. M. Wazwaz, "Computational method for reaction diffusion-model arising in a spherical catalyst," *International Journal of Applied and Computational Mathematics*, vol. 7, no. 3, pp. 1–11, 2021.

[26] R. P. Agarwal, F. Mofarreh, R. Shah, W. Luangboon, and K. Nonlaopon, "An analytical technique, based on natural transform to solve fractional-order parabolic equations," *Entropy*, vol. 23, no. 8, p. 1086, 2021.

[27] H. Singh, H. M. Srivastava, and D. Kumar, "A reliable numerical algorithm for the fractional vibration equation," *Chaos, Solitons & Fractals*, vol. 103, pp. 131–138, 2017.

[28] N. H. Aljahdaly, R. P. Agarwal, R. Shah, and T. Botmart, "Analysis of the time fractional-order coupled burgers equations with non-singular kernel operators," *Mathematics*, vol. 9, no. 18, p. 2326, 2021.

[29] H. Khan, A. Khan, M. Al-Qurashi, R. Shah, and D. Baleanu, "Modified modelling for heat like equations within Caputo operator," *Energies*, vol. 13, no. 8, p. 2002, 2020.

[30] H. Jafari, M. Nazari, D. Baleanu, and C. M. Khalique, "A new approach for solving a system of fractional partial differential equations," *Computers & Mathematics with Applications*, vol. 66, no. 5, pp. 838–843, 2013.

[31] S. H. Mirmoradia, I. Hosseinpoura, S. Ghanbarpour, and A. Barari, "Application of an approximate analytical method to nonlinear Troeschs problem," *Applied Mathematical Sciences*, vol. 3, no. 32, pp. 1579–1585, 2009.

[32] S. Liao, "On the homotopy analysis method for nonlinear problems," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 499–513, 2004.

[33] J. Lu, "An analytical approach to the Fornberg-Whitham type equations by using the variational iteration method," *Computers & Mathematcs with Applications*, vol. 61, no. 8, pp. 2010–2013, 2011.

[34] H. Khan, A. Khan, P. Kumam, D. Baleanu, and M. Arif, "An approximate analytical solution of the Navier–Stokes equations within Caputo operator and Elzaki transform decomposition method," *Adv. Difference Equ.*, vol. 2020, no. 1, 2020.

[35] D. H. Shou and J. H. He, "Beyond Adomian method: the variational iteration method for solving heat-like and wave-like equations with variable coefficients," *Physics Letters A*, vol. 372, no. 3, pp. 233–237, 2008.

[36] Y. Qin, A. Khan, I. Ali et al., "An efficient analytical approach for the solution of certain fractional-order dynamical systems," *Energies*, vol. 13, no. 11, p. 2725, 2020.

[37] N. Iqbal, A. Akgül, R. Shah, A. Bariq, M. Mossa Al-Sawalha, and A. Ali, "On solutions of fractional-order gas dynamics equation by effective techniques," *Journal of Function Spaces*, vol. 2022, p. 3341754, 2022.

[38] H. Jafari, "A new general integral transform for solving integral equations," *Journal of Advanced Research*, vol. 32, pp. 133–138, 2021.

[39] M. Meddahi, H. Jafari, and X. J. Yang, "Towards new general double integral transform and its applications to differential equations," *Mathematical Methods in the Applied Sciences*, vol. 45, no. 4, pp. 1916–1933, 2022.

[40] M. Meddahi, H. Jafari, and M. N. Ncube, "New general integral transform via Atangana-Baleanu derivatives," *Adv. Difference Equ.*, vol. 2021, no. 1, 2021.

[41] G. Adomian, "Solution of physical problems by decomposition," *Computers and Mathematics with Applications*, vol. 27, no. 9-10, pp. 145–154, 1994.

[42] G. Adomian, "A review of the decomposition method in applied mathematics," *Journal of Mathematical Analysis and Applications*, vol. 135, article 501544, 1988.

[43] J. H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.

[44] J. H. He, "A coupling method of a homotopy technique and a perturbation technique for non-linear problems," *International Journal of Non-Linear Mechanics*, vol. 35, no. 1, pp. 37–43, 2000.

[45] J. H. He, "Application of homotopy perturbation method to nonlinear wave equations," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 695–700, 2005.

[46] J. H. He, "Limit cycle and bifurcation of nonlinear problems," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 827–833, 2005.

[47] J. H. He, "Homotopy perturbation method for solving boundary value problems," *Physics Letters A*, vol. 350, no. 1-2, pp. 87-88, 2006.

[48] J. H. He, *Non-perturbative Methods for Strongly Nonlinear Problems, [Ph.D. thesis]*, de-Verlag im Internet GmbH, 2006.

[49] H. Jafari, *Iterative Methods for Solving System of Fractional Differential Equations, [Ph.D. thesis]*, Pune University, Pune City, India, 2006.

[50] K. Shah, H. Khalil, and R. A. Khan, "Analytical solutions of fractional order diffusion equations by natural transform method," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 42, no. 3, pp. 1479–1490, 2018.

[51] S. A. El-Tantawy, A. H. Salas, and Wedad Albalawi, "New localized and periodic solutions to a Korteweg-de Vries equation with power law nonlinearity: applications to some plasma models," *Symmetry*, vol. 14, no. 2, p. 197, 2020.

[52] S. A. El-Tantawy, R. A. Alharbey, and A. H. Salas, "Novel approximate analytical and numerical cylindrical rogue wave and breathers solutions: An application to electronegative plasma," *Chaos, Solitons & Fractals*, vol. 155, p. 111776, 2022.

[53] N. A. Shah, H. A. Alyousef, S. A. El-Tantawy, R. Shah, and J. D. Chung, "Analytical investigation of fractional-order Korteweg-De-Vries-Type equations under Atangana-Baleanu-

Caputo operator: modeling nonlinear waves in a plasma and fluid," *Symmetry*, vol. 14, no. 4, p. 739, 2022.

[54] H. Liu, H. Khan, S. Mustafa, L. Mou, and D. Baleanu, "Fractional-order investigation of diffusion equations via analytical Approach," *Physics*, vol. 8, p. 673, 2021.

[55] M. K. U. Dattu, "New integral transform: fundamental properties, investigations and applications," *Journal for Advanced Research in Applied Sciences*, vol. 5, no. 4, pp. 534–539, 2018.

[56] B. K. Singh and P. Kumar, "Numerical Computation for Time-Fractional Gas Dynamics Equations by Fractional Reduced Differential," *Journal of Mathematics and System Science*, vol. 6, no. 6, pp. 248–259, 2016.

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Research Article

A Novel Multicriteria Decision-Making Approach for Einstein Weighted Average Operator under Pythagorean Fuzzy Hypersoft Environment

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The experts used the Pythagorean fuzzy hypersoft set (PFHSS) in their research to discourse ambiguous and vague information in decision-making processes. The aggregation operator (AO) plays a prominent part in the sensitivity of the two forefront loops and eliminates anxiety from that perception. The PFHSS is the most influential and operative extension of the Pythagorean fuzzy soft set (PFSS), which handles the subparameterized values of alternatives. It is also a generalized form of Intuitionistic fuzzy hypersoft set (IFHSS) that provides better and more accurate assessments in the decision-making (DM) process. In this work, we present some operational laws for Pythagorean fuzzy hypersoft numbers (PFHSNs) and then formulate Pythagorean fuzzy hypersoft Einstein weighted average (PFHSEWA) operator based on developed operational laws. We discuss essential features such as idempotency, boundedness, and homogeneity for the proposed PFHSEWA operator. Furthermore, a DM approach has been developed based on the built-in operator to address multicriteria decision-making (MCDM) issues. A numerical case study of decision-making problems in real-life agricultural farming is considered to validate the settled technique's dominance and applicability. The consequences display that the planned model is more operative and consistent to handle inexact data based on PFHSS.

1. Introduction

In farming history, the subjugation of vegetations, wildlife, and the manufacturing and propagation methods used for high-yielding cultivation have been recorded. Farming started independently in numerous parts of the world, including a wide range of taxa. By 8000 BC, farming along the Nile was widely known. Around this time, farming developed autonomously in the Far East, most likely in China, and the main crop was rice instead of wheat. Modern agricultural practices result from an excessive water supply, extensive deforestation, and reduced soil fertility. Since there is lacking water to endure farming, it is compulsory to reexamine how to use essential water, land, and environmental resources to raise crop vintages. Highlighting the importance of the ecosystem, considering the balances among the atmosphere and livings, and balancing the privileges and benefits of a range of manipulators may explain. The discriminations rising from these steps need to be addressed, such as the redeployment of water resources from the poor to the rich and clearing land to make room for more profitable farmland. Scientific development supports farmers with apparatuses and facilities to help them become more affluent. Maintenance farming is agricultural expertise that evades land loss due to deforestation, decreases water contamination, and rallies carbon impounding. It is a sample of scientific invention. Agriculture has not been a simple task to meet the rising mandate for nutrition and now requires more analysis and expertise. Statisticians, hydrologists, and agriculturalists met in California to progress a plan to diminish crop water ingesting while still generating profits for farmers and meeting market demand. Scientific representations use information, including plant growing features and water supplies, to regulate which yields and zones should not be planted. Farmers are gratified with the sensible use of their tools, while mathematicians work with professional specialists.

MCDM is considered the most appropriate technique for finding the most acceptable alternative from all possible options, following criteria or attributes. In real-life circumstances, most decisions are taken when the objectives and limitations are usually indefinite or ambiguous. To overcome such ambiguities and anxieties, Zadeh offered the idea of the fuzzy set (FS) [1], a prevailing tool to handle the obscurities and uncertainties in DM. Such a set allocates to all objects a membership value ranging from 0 to 1. Experts mainly consider membership and a nonmembership value in the DM process that FS cannot handle. Atanassov [2] introduced the generalization of the FS, the idea of the intuitionistic fuzzy set (IFS) to overcome the limitation mentioned above. Wang and Liu [3] presented numerous operations on IFS, such as Einstein product and Einstein sum, and constructed two aggregation operators (AOs). They also discussed some essential properties of these operators and utilized their proposed AO to resolve multi-attribute decision making (MADM) for the IFS information. Atanassov [4] presented a generalized form of IFS in the light of ordinary interval values, called interval-valued intuitionistic fuzzy set (IVIFS). As a generalization of the IFS and IVIFS, Garg and Kaur [5] extended the concept of IFS and presented a novel idea of the cubic intuitionistic fuzzy set (CIFS). CIFS is a successful tool representing incomplete data by embedding IFS and IVIFS. They also discussed several desirable properties of CIFS.

The models mentioned above have been well-recognized by the specialists. Still, the existing IFS cannot handle the inappropriate and vague data because it is considered to envision the linear inequality between the membership and nonmembership grades. For example, if decision-makers choose membership and nonmembership values 0.7 and 0.6, respectively, then $0.7 + 0.6 \geq 1$. The IFS mentioned above theory cannot be applied to these data. To resolve the limitation described above, Yager [6] presented the idea of the Pythagorean fuzzy set (PFS) by amending the basic condition $a + b \leq 1$ to $a^2 + b^2 \leq 1$ and developed some results associated with score function and accuracy function. Ejegwa [7] extended the notion of PFS and presented a decision-making technique. Rahman et al. [8] formed the Einstein weighted geometric operator for PFS and presented a multiattribute group decision-making (MAGDM) methodology utilizing the proposed operator. Zhang and Xu [9] developed some basic operational laws and prolonged the technique for order preference by similarity to ideal solution (TOPSIS) method to resolve MCDM complications for PFS information. Pythagorean fuzzy power AOs along with essential characteristics were introduced by Wei and Lu [10]. They also recommended a DM technique to resolve MADM difficulties based on presented operators. Wang and Li [11] offered the interaction operational laws for PFNs and developed power Bonferroni mean operators under the PFS environment. They also discussed some definite cases of developed operators and discussed their basic characteristics. IIbahar et al. [12] offered the Pythagorean fuzzy proportional risk assessment technique to assess the professional health risk. Zhang [13] proposed a novel decision-making (DM) approach based on similarity measures to resolve multicriteria group decision-making (MCGDM) difficulties for the PFS.

Peng and Yang [14] introduced the division and subtraction operations for Pythagorean fuzzy numbers (PFNs), proved their basic properties, and presented a superiority and inferiority ranking approach under the PFS to overcome the MAGDM difficulties. Garg [15] introduced operational laws based on Einstein norms for PFNs, proposed generalized Pythagorean fuzzy Einstein average AOs, and then utilized these operators for DM. Garg [16] presented the generalized geometric AOs and established an MCDM approach based on developed operators. Garg [17] introduced logarithmic operational laws for the PFS and constructed various weighted operators based on presented logarithm operational laws. Gao et al. [18] developed numerous interaction aggregation operators under the PFS setting. Wang et al. [19] offered the interactive Hamacher operations for the PFS and settled a DM method to solve MCDM difficulties. Zulqarnain and Dayan [20] utilized the fuzzy TOPSIS to select the best alternative.

Peng and Yuan [21] explored some new inequalities of the Pythagorean fuzzy weighted average (PFWA) operator. They introduced some point operators under the PFS environment. They combined the Pythagorean fuzzy point operators with the generalized PFWA operator, developed a novel operator, and established a MADM methodology based on developed operators. Wang and Garg [22] presented the Archimedean-based interactive AOs for PFS and developed an algorithm to solve MADM problems. Rahman et al. [23] defined the interval-valued weighted AOs for interval-valued PFNs. They utilized the proposed operators to resolve the MADM issues under the interval-valued PFS. Wang and Li [24] used the interval-valued PFS, presented some novel PFS operators, and offered a DM approach to resolve the MCGDM complications. Arora and Garg [25] presented basic operational laws and suggested several selected AOs for linguistic IFS. To examine the ranking of normal IFS and IVIFS, Garg [26] gave novel algorithms for solving the MADM problems. Ma and Xu [27] modified the existing score function and accuracy function for PFNs and defined novel Pythagorean fuzzy weighted geometric and Pythagorean fuzzy weighted averaging operators.

All the methods mentioned above have too many applications in many fields. However, due to their inefficiency, these methods have many limitations in terms of parameterization tools. Presenting the solution of this sort of obscurity and ambiguity, Molodtsov [28] introduced the basic notions of soft sets (SSs) and debated some elementary operations with their possessions. Maji et al. [29] prolonged the idea of SS. They defined several basic operations, and binary operations for Maji et al. [30] further applied the SS theory to solve the DM problems using rough mathematics. Moreover, Maji et al. [31] combined two prevailing notions, such as FS and SS, and developed the idea of FSS, which is a more robust and reliable tool. They also presented basic operations and established and applied this concept in the study by Maji et al. [32] who demonstrated the intuitionistic fuzzy soft set (IFSS) theory and offered some basic operations with their essential properties. Deli and Çağman [33] developed the intuitionistic fuzzy parameterized soft sets and DM methodology properties. Later on, Garg and Arora [34] presented Maclaurin symmetric mean operator for dual hesitant fuzzy soft numbers. Arora and Garg [35] developed the correlation coefficients and introduced an MCDM technique based on the generated correlation coefficients to measure the affiliation of two IFSS. In 2018, Garg and Arora [36] proposed generalized Maclaurin symmetric mean AOs based on Archimedean t-norm under the IFS environment. Garg and Arora [37] developed the TOPSIS concept and presented correlation measures based on previously constructed correlations. Wang and Liu [38] introduced the Maclaurin symmetric mean AOs based on Schweizer–Sklar operations for IFS and established the MAGDM technique to solve DM issues. Liu and Wang [39] presented the Bonferroni mean AOs for q-rung orthopair fuzzy sets and settled a MADM approach to solving DM complications.

Nowadays, the conception and application consequences of soft sets and the earlier-mentioned several research developments are evolving speedily. Peng et al. [40] developed the concept of PFSS by merging two existing models, PFS and SS. They also discussed some fundamental operations with their basic properties. Athira et al. [41] established entropy measures for the PFSS. They also offered Euclidean distance and hamming distance for the PFSS and utilized their methods for DM [42]. Naeem et al. [43] developed the TOPSIS and VIKOR methods for PFSS and presented an approach for the stock exchange investment problem. Zulqarnain et al. [44] introduced the AOs under the PFSS environment and presented an application for the green supplier chain management. Zulqarnain et al. [45] developed the interaction AOs for PFSS and constructed a DM technique to resolve the MCDM problems. Zulqarnain et al. [46-47] formed the Einstein weighted average and geometric AOs for PFSS. They also proposed the MAGDM techniques using their developed operators for sustainable supplier selection and a business to finance money. Siddique et al. [48] proposed a novel DM technique for PFSS using a score matrix. Zulqarnain et al. [49] introduced the TOPSIS method for PFSS based on the correlation coefficient.

Samarandche [50] proposed the idea of the hypersoft set (HSS), which penetrates multiple subattributes in the parameter function f, which is a characteristic of the Cartesian product with the n attribute. Compared with SS and other existing concepts, Samarandche HSS is the most suitable theory which handles the multiple subattributes of the considered parameters. Several HSS extensions and their decision-making methods have been proposed. Several researchers developed different hybrid structures HSS and presented several AOs with their DM techniques [51-60]. Deli [61] introduced several hybrid structures for other extensions by merging neutrosophic sets and HSS. PFHSS is a hybrid intellectual structure of PFSS. The AOs stated formerly are based on the elementary algebraic product and algebra sum, which is not the only operation that can model the intersection and union of PFHSS. Similarly, Einstein operations contain Einstein product and Einstein sum, an excellent alternate to algebraic product and algebra sum. Moreover, there appears to be some study on aggregation techniques using Einstein operations on PFHSS. Wang and Liu [62] proposed the average AOs under the IFS setting and constructed the MADM approach under their considered environment. Liu and Wang [63] developed a MADM method based on interaction Einstein AOs under the IFS setting.

An enhanced sorting approach fascinates investigators to crack baffling and inadequate information. Rendering to the investigation outcomes, PFHSS plays a vital role in DM by collecting numerous sources into a single value. According to the most generally known knowledge, the emergence of PFSS and hypersoft set (HSS) hybridization has not been combined with the PFSS background. PFHSS is a hybrid intellectual structure of PFSS. So, to encourage the modern exploration of PFHSS, we will state AOs based on rough data. The main intentions of the current study are given as follows:

(i) The PFHSS efficiently deals with the complex apprehensions seeing the multi-sub-attributes of the DM method's considered factors. To reserve this value in attention, we prolong Einstein operational laws for PFHSS and establish the Einstein AOs for PFHSS.

(ii) The Einstein AOs for PFHSS are well-known attractive evaluation AOs. It has been detected that the prevailing AOs feature is insensitive to scratch the exact outcome over the DM method in some states. To overcome these particular obstacles, these AOs need to be reviewed. We determine inventive Einstein operational laws for Pythagorean fuzzy hypersoft numbers (PFHSNs).

(iii) Pythagorean fuzzy hypersoft Einstein weighted average and geometric operators have presented their essential properties expending advanced Einstein operational laws.

(iv) An innovative procedure was established on the intended operators to resolve the DM problem.

(v) Real-life agricultural farming is deliberated to endorse the developed method's supremacy and applicability. The significances show that the prearranged model is more operational and reliable to grip indefinite facts.

This study is systematized as follows. Basic knowledge of some important notions like SS, HSS, IFHSS, PFHSS, and Einstein norms are deliberated in section 2. Section 3 demarcated some basic operational laws for PFHSNs based on Einstein norms and established the PFHSEWA operator. Also, some dynamic properties of the planned operator have been debated in the same section. Section 4 also uses the agricultural example to explain several agricultural problems. The algorithm given in this section shows that it is realistic and appropriate. In Section 5, a comparison with some standing approaches is provided.

2. Preliminaries

This section remembers some fundamental notions such as soft set (SS), HSS, IFHSS, and PFHSS.

Definition 1 (see [28]). Let X and $\mathbb{N}$ be the universe of discourse and set of attributes, respectively. Let $\mathrm{P}(X)$ be the power set of X and $\mathscr{A}\subseteq\mathbb{N}$. A pair $(\Omega, \mathscr{A})$ is called a SS over X, and its mapping is expressed as follows:

$$\Omega : \mathscr{A} \longrightarrow \mathrm{P}(X). \tag{1}$$

Also, it can be defined as follows:

$$(\Omega, \boldsymbol{\mathscr{A}}) = \{\Omega(e) \in \boldsymbol{\mathscr{P}}(X) \colon e \in \mathbb{N}, \Omega(e) = \varnothing \text{if } e \notin \boldsymbol{\mathscr{A}}\}. \tag{2}$$

Definition 2 (see [50]). Let X be a universe of discourse and $\mathrm{P}(X)$ be a power set of X and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$, and K_i represented the set of attributes and their corresponding subattributes such as $K_i \cap K_j = \varnothing$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \dddot{\mathscr{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of subattributes, where $1 \leq h \leq \alpha, 1 \leq k \leq \beta$, $1 \leq l \leq \gamma$, and $\alpha\beta\gamma \in N$, Then, the pair $(\Omega, K_1 \times K_2 \times K_3 \times \ldots \times K_n) = (\Omega\dddot{\mathscr{A}})$ is known as HSS defined as follows:

$$\Omega\colon K_1 \times K_2 \times K_3 \times \ldots \times K_n = \dddot{\mathscr{A}} \longrightarrow \mathrm{P}(X). \tag{3}$$

It is also defined as

$$(\Omega, \dddot{\mathscr{A}}) = \left\{\check{d}, \Omega_{\dddot{\mathscr{A}}}(\check{d})\colon \check{d} \in \dddot{\mathscr{A}}, \Omega_{\dddot{\mathscr{A}}}(\check{d}) \in \mathrm{P}(X)\right\}. \tag{4}$$

Definition 3 (see [50]). Let X be a universe of discourse and $\mathrm{P}(X)$ be a power set of X and $k = \{k_1, k_2, k_3, \ldots k_n\}$, $(n \geq 1)$, and K_i represented the set of attributes and their corresponding subattributes such as $K_i \cap K_j = \varnothing$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \dddot{\mathscr{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of subattributes, where $1 \leq h \leq \alpha, 1 \leq k \leq \beta$and $1 \leq l \leq \gamma$, and $\alpha\beta\gamma \in N$. And, IFS^X expresses the intuitionistic fuzzy power set over X. Then, the pair $(\Omega, K_1 \times K_2 \times K_3 \times \ldots \times K_n) = (\Omega\dddot{\mathscr{A}})$ is known as IFHSS defined as follows:

$$\Omega\colon K_1 \times K_2 \times K_3 \times \ldots \times K_n = \dddot{\mathscr{A}} \longrightarrow IFS^X. \tag{5}$$

It is also defined as

$$(\Omega, \dddot{\mathscr{A}}) = \left\{\left(\check{d}, \Omega_{\dddot{\mathscr{A}}}(\check{d})\right)\colon \check{d} \in \dddot{\mathscr{A}}, \Omega_{\dddot{\mathscr{A}}}(\check{d}) \in IFS^X\right\}, \tag{6}$$

where $\Omega_{\dddot{\mathscr{A}}}(\check{d}) = \left\{\left\langle \delta, a_{\Omega(\check{d})}(\delta), b_{\Omega(\check{d})}(\delta)\right\rangle\colon \delta \in X\right\}$, where $a_{\Omega(\check{d})}(\delta)$ and $b_{\Omega(\check{d})}(\delta)$ signify the membership value (Mem) and nonmembership value (NMem) of the subattributes:

$a_{\Omega(\check{d})}(\delta), b_{\Omega(\check{d})}(\delta), \in [0, 1]$, and $0 \leq a_{\Omega(\check{d})}(\delta) + b_{\Omega(\check{d})}(\delta) \leq 1$.

Definition 4 (see [53]). Let X be a universe of discourse and $\mathrm{P}(X)$ be a power set of X and $k = \{k_1, k_2, k_3, \ldots k_n\}$, $(n \geq 1)$, and K_i represented the set of attributes and their corresponding subattributes such as $K_i \cap K_j = \varnothing$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \dddot{\mathscr{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of subattributes, where $1 \leq h \leq \alpha, 1 \leq k \leq \beta$, $1 \leq l \leq \gamma$ and $\alpha\beta\gamma \in N$. And, PFS^X expresses the Pythagorean fuzzy power set over X. Then, the pair $(\Omega, K_1 \times K_2 \times K_3 \times \ldots \times K_n) = (\Omega\dddot{\mathscr{A}})$ is known as PFHSS defined as follows:

$$\Omega\colon K_1 \times K_2 \times K_3 \times \ldots \times K_n = \dddot{\mathscr{A}} \longrightarrow PFS^X. \tag{7}$$

It is also defined as

$$(\Omega, \dddot{\mathscr{A}}) = \left\{\left(\check{d}, \Omega_{\dddot{\mathscr{A}}}(\check{d})\right)\colon \check{d} \in \dddot{\mathscr{A}}, \Omega_{\dddot{\mathscr{A}}}(\check{d}) \in PFS^X\right\}, \tag{8}$$

where $\Omega_{\dddot{\mathscr{A}}}(\check{d}) = \left\{\delta, a_{\Omega(\check{d})}(\delta), b_{\Omega(\check{d})}(\delta)\colon \delta \in \mathscr{U}\right\}$, where $a_{\Omega(\check{d})}(\delta)$ and $b_{\Omega(\check{d})}(\delta)$ signify the Mem and NMem values of the attributes:

$a_{\Omega(\check{d})}(\delta), b_{\Omega(\check{d})}(\delta), \in [0, 1]$, and $0 \leq (a_{\Omega(\check{d})}(\delta))^2 + (b_{\Omega(\check{d})}(\delta))^2 \leq 1$.

A Pythagorean fuzzy hypersoft number (PFHSN) can be stated as $\Omega = \left\{(a_{\Omega(\check{d})}(\delta), b_{\Omega(\check{d})}(\delta))\right\}$, where $0 \leq (a_{\Omega(\check{d})}(\delta))^2 + (b_{\Omega(\check{d})}(\delta))^2 \leq 1$.

Remark 1. If $(a_{\Omega(\check{d})}(\delta))^2 + (b_{\Omega(\check{d})}(\delta))^2$ and $a_{\Omega(\check{d})}(\delta) + b_{\Omega(\check{d})}(\delta) \leq 1$ both are holds, then PFHSS was reduced to IFHSS [58].

For readers' suitability, the PFHSN $\Omega_{\delta_i}(\check{d}_j) = \left\{(a_{\Omega(\check{d}_j)}(\delta_i), \mathrm{b}_{\Omega(\check{d}_j)}(\delta_i)) | \delta_i \in \mathscr{U}\right\}$ can be written as $\mathfrak{J}_{\check{d}_{ij}} = a_{\Omega(\check{d}_{ij})}, b_{\Omega(\check{d}_{ij})}$. The score function [65] for $\mathfrak{J}_{\check{d}_{ij}}$ is expressed as follows:

$$\mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) = a^2_{\Omega\left(\check{d}_{ij}\right)} - b^2_{\Omega\left(\check{d}_{ij}\right)} \mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) \in [-1, 1], \tag{9}$$

However, in some cases, the above-defined score function cannot handle the scenario. For example, if we consider two PFHSNs, such as $\mathfrak{J}_{\check{d}_{11}} = .4, .7$ and $\mathfrak{J}_{\check{d}_{12}} = .5, .8$. The score function cannot deliver relevant results to subtract the PFHSNs. So, in such situations, it is tough to achieve the most suitable alternative $\mathbb{S}(\mathfrak{J}_{\check{d}_{11}}) = .3 = \mathbb{S}(\mathfrak{J}_{\check{d}_{12}})$. To intimidated such problems, the accuracy function [65] had been developed.

$$H\left(\mathfrak{J}_{\check{d}_{ij}}\right)a^2_{\Omega(\check{d}_{ij})} + b^2_{\Omega(\check{d}_{ij})}H\left(\mathfrak{J}_{\check{d}_{ij}}\right) \in [0,1]. \tag{10}$$

The following comparison laws have been projected to compute two PFHSNs $\mathfrak{J}_{\check{d}_{ij}}$ and $\mathfrak{T}_{\check{d}_{ij}}$:

(1) If $\mathbb{S}(\mathfrak{J}_{\check{d}_{ij}}) > \mathbb{S}(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$

(2) If $\mathbb{S}(\mathfrak{J}_{\check{d}_{ij}}) = \mathbb{S}(\mathfrak{T}_{\check{d}_{ij}})$, then

(3) If $H(\mathfrak{J}_{\check{d}_{ij}}) > H(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$

(4) If $H(\mathfrak{J}_{\check{d}_{ij}}) = H(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{T}_{\check{d}_{ij}}$

Definition 5. Einstein sum $\oplus_\varepsilon$ and Einstein product $\oplus_\varepsilon$ are good alternatives of algebraic t-norm and t-conorm, respectively, given as follows:

$$a \oplus_\varepsilon b = \frac{a+b}{1+(a.b)} \text{ and } a \otimes_\varepsilon b = \frac{a.b}{1+(1-a).(1-b)}, \tag{11}$$

$\forall (a,b) \in [0,1]^2$.

Under the Pythagorean fuzzy environment, Einstein sum $\oplus_\varepsilon$ and Einstein product $\oplus_\varepsilon$ are defined as follows:

$$a \oplus_\varepsilon b = \sqrt{\frac{a^2+b^2}{1+\left(a^2.b^2\right)}},$$

$$a \otimes_\varepsilon b = \frac{a.b}{\sqrt{1+\left(1-a^2\right).\left(1-b^2\right)}}, \tag{12}$$

$$\forall (a,b) \in [0,1]^2,$$

where $a\oplus_\varepsilon b$ and $a\oplus_\varepsilon b$ are known as t-norm and t-conorm, respectively, satisfying the bounded, monotonicity, commutativity, and associativity properties.

3. Einstein Weighted Aggregation Operators for Pythagorean Fuzzy Hypersoft Set

This section will introduce a novel Einstein weighted AO such as the PFHSEWA operator for PFHSNs with essential properties.

3.1. Operational Laws for PFHSNs

Definition 6. Let $\mathfrak{J}_{\check{d}_k} = (a_{\check{d}_k}, b_{\check{d}_k})$, $\mathfrak{J}_{\check{d}_{11}} = (a_{\check{d}_{11}}, b_{\check{d}_{11}})$, and $\mathfrak{J}_{\check{d}_{12}} = (a_{\check{d}_{12}}, b_{\check{d}_{12}})$ represent the PFHSNs and ∂ is a positive real number. Then, operational laws for PFHSNs based on Einstein norms can be expressed as follows:

1 $\mathfrak{J}_{\check{d}_{11}} \oplus_\varepsilon \mathfrak{J}_{\check{d}_{12}} = \left(\sqrt{(1+a^2_{\check{d}_{12}}) - (1-a^2_{\check{d}_{12}})} / \sqrt{(1+a^2_{\check{d}_{12}}) + (1-a^2_{\check{d}_{12}})}\right), \left(\sqrt{2b^2_{\check{d}_{12}}} / \sqrt{(2-b^2_{\check{d}_{12}}) + b^2_{\check{d}_{12}}}\right)$

$$\mathfrak{J}_{\check{d}_{11}} \otimes_\varepsilon \mathfrak{J}_{\check{d}_{12}} = \left(\frac{\sqrt{2\alpha^2_{\check{d}_{12}}}}{\sqrt{\left(2-\alpha^2_{\check{d}_{12}}\right)+\alpha^2_{\check{d}_{12}}}}, \frac{\sqrt{\left(1+b^2_{\check{d}_{12}}\right)-\left(1-b^2_{\check{d}_{12}}\right)}}{\sqrt{\left(1+b^2_{\check{d}_{12}}\right)+\left(1-b^2_{\check{d}_{12}}\right)}}\right) \tag{13}$$

2 $\partial\mathfrak{J}_{\check{d}_k} = \left(\sqrt{(1+a^2_{\check{d}_k})^\partial - (1-a^2_{\check{d}_k})^\partial} / \sqrt{(1+a^2_{\check{d}_k})^\partial + (1-a^2_{\check{d}_k})^\partial}\right), \left(\sqrt{2(b^2_{\check{d}_k})^\partial} / \sqrt{(2-b^2_{\check{d}_k})^\partial + (b^2_{\check{d}_k})^\partial}\right)$

3 $\mathfrak{J}^\partial_{\check{d}_k} = \sqrt{2(a)^\partial} / \sqrt{(2-a^2_{\check{d}_k})^\partial + (a^2_{\check{d}_k})^\partial}, \left(\sqrt{(1+b^2_{\check{d}_k})^\partial - (1-b^2_{\check{d}_k})^\partial} / \sqrt{(1+b^2_{\check{d}_k})^\partial + (1-b^2_{\check{d}_k})^\partial}\right)$

Definition 7. Let $\mathfrak{J}_{\check{d}_{ij}} = (a_{\check{d}_{ij}}, b_{\check{d}_{ij}})$ be a collection of PFHSNs, then the PFHSEWA operator is defined as follows:

$$PFHSEWA = \left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) \oplus^m_{\varepsilon j=1} \lambda_j \left(\oplus^n_{\varepsilon i=1} \theta_i \mathfrak{J}_{\check{d}_{ij}}\right), \tag{14}$$

where $(i = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, m)$, and θ_i and λ_j represent the weighted vectors such that $\theta_i > 0$, $\sum^n_{i=1} \theta_i = 1$, and $\lambda_j > 0$ and $\sum^n_{j=1} \lambda_j = 1$.

Theorem 1. *Let $\mathfrak{J}_{\check{d}_{ij}} = (a_{\check{d}_{ij}}, b_{\check{d}_{ij}})$ be a collection of PFHSNs, then the aggregated value attained by equation (3) given as*

$$PFHSEWA\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \oplus^m_{j=1} \lambda_j \left(\oplus^n_{i=1} \theta_i \mathfrak{J}_{\check{d}_{ij}}\right),$$

$$= \frac{\sqrt{\prod^m_{j=1}\left(\prod^n_{i=1}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} - \prod^m_{j=1}\left(\prod^n_{i=1}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod^m_{j=1}\left(\prod^n_{i=1}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod^m_{j=1}\left(\prod^n_{i=1}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod^m_{j=1}\left(\prod^n_{i=1}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod^m_{j=1}\left(\prod^n_{i=1}\left(2-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod^m_{j=1}\left(\prod^n_{i=1}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \tag{15}$$

where $(i = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, m)$ *and* θ_i *and* λ_j *represent the weight vectors such that* $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$, *and* $\lambda_j > 0$, $\sum_{j=1}^{n} \lambda_j = 1$.

Proof. We will prove it by using mathematical induction.

For $n = 1$, we get $\theta_i = 1$

$$PFHSEWA = \left(\mathfrak{J}_{\breve{d}_{11}}, \mathfrak{J}_{\breve{d}_{12}}, \ldots, \mathfrak{J}_{\breve{d}_{nm}}\right) = \oplus_{j=1}^{m} \lambda_j \mathfrak{J}_{\breve{d}_{1j}}$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(1 + a^2_{\breve{d}_{1j}}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(1 - a^2_{\breve{d}_{1j}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(1 + a^2_{\breve{d}_{1j}}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(1 - a^2_{\breve{d}_{1j}}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(b^2_{\breve{d}_{1j}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(2 - b^2_{\breve{d}_{1j}}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(b^2_{\breve{d}_{1j}}\right)^{\lambda_j}}} \right\rangle,$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 + a^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 - a^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 + a^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 - a^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(2 - b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle. \tag{16}$$

For $m = 1$, we get $\lambda_j = 1$.

$$PFHSEWA\left(\mathfrak{J}_{\breve{d}_{11}}, \mathfrak{J}_{\breve{d}_{12}}, \ldots, \mathfrak{J}_{\breve{d}_{nm}}\right) = \oplus_{i=1}^{n} \theta_i \mathfrak{J}_{\breve{d}_{i1}}$$
$$= \left\langle \frac{\sqrt{\prod_{i=1}^{n}\left(1 + a^2_{\breve{d}_{i1}}\right)^{\theta_i} - \prod_{i=1}^{n}\left(1 - a^2_{\breve{d}_{i1}}\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(1 + a^2_{\breve{d}_{i1}}\right)^{\theta_i} + \prod_{i=1}^{n}\left(1 - a^2_{\breve{d}_{i1}}\right)^{\theta_i}}}, \frac{\sqrt{2\prod_{i=1}^{n}\left(b^2_{\breve{d}_{i1}}\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(2 - b^2_{\breve{d}_{i1}}\right)^{\theta_i} + \prod_{i=1}^{n}\left(b^2_{\breve{d}_{i1}}\right)^{\theta_i}}} \right\rangle$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 + \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 - \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 + \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 - \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(2 - b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle. \tag{17}$$

So, equation (4) is true for $n = 1$ and $m = 1$.

Suppose that equation holds for $n = \delta_2$, $m = \delta_1 + 1$ and for $n = \delta_2 + 1$, $m = \delta_1$,

$$\oplus_{j=1}^{\delta_1+1} \lambda_j\left(\oplus_{i=1}^{\delta_2} \theta_i \mathfrak{J}_{\breve{d}_{ij}}\right) = \left\langle \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 + \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 - \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 + \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 - \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(b^2_{\breve{d}_{a_{ij}}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2 - b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle,$$
$$\oplus_{j=1}^{\delta_1} \lambda_j\left(\oplus_{i=1}^{\delta_2+1} \theta_i \mathfrak{J}_{\breve{d}_{ij}}\right) = \left\langle \frac{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 + \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 - \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 + \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 - \alpha^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(2 - b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(b^2_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle. \tag{18}$$

Now, we prove the equation for $m = \delta_1 + 1$ and $= n = \delta_2 + 1$:

$$
\begin{aligned}
&\oplus_{j=1}^{\delta_1+1}\lambda_j\left(\oplus_{i=1}^{\delta_2+1}\theta_i\mathfrak{I}_{\check{d}_{ij}}\right)=\oplus_{j=1}^{\delta_1+1}\lambda_j\left(\oplus_{i=1}^{\delta_2}\theta_i\mathfrak{I}_{\check{d}_{ij}}\oplus\theta_{i+1}\mathfrak{I}_{\check{d}_{(\delta_2+1)j}}\right)\\
&=\left(\oplus_{j=1}^{\delta_1+1}\oplus_{i=1}^{\delta_2}\theta_i\lambda_j\,\mathfrak{I}_{\check{d}_{ij}}\right)\left(\oplus_{j=1}^{\delta_1+1}\lambda_j\theta_{i+1}\mathfrak{I}_{\check{d}_{(\delta_2+1)j}}\right)\\
&=\left\langle\frac{\sqrt{\prod_{j=2}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}\oplus\frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha_{\check{d}_{(\delta_2+1)j}}^2\right)^{\theta\delta_2+1}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha_{\check{d}_{(\delta_2+1)j}}^2\right)^{\theta\delta_2+1}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha_{\check{d}_{(\delta_2+1)j}}^2\right)^{\theta\delta_2+1}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha_{\check{d}_{(\delta_2+1)j}}^2\right)^{\theta\delta_2+1}\right)^{\lambda_j}}}\right.\\
&\qquad\left.\frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2+b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\oplus\frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\left(b_{\check{d}_{(\delta_2+1)j}}^2\right)^{\theta\delta_2+1}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(2-b_{\check{d}_{(\delta_2+1)j}}^2\right)^{\theta\delta_2+1}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\left(b_{\check{d}_{(\delta_2+1)j}}^2\right)^{\theta\delta_2+1}\right)^{\lambda_j}}}\right\rangle\\
&=\left\langle\frac{\sqrt[r]{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\left(\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha_{\check{d}_{ij}}^2\right)\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}},\frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\left(\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(2-b_{\check{d}_{ij}}^2\right)\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\right\rangle\\
&=\oplus_{j=1}^{\delta_1+1}\lambda_j\left(\oplus_{i=1}^{\delta_2+1}\theta_i\mathfrak{I}_{\check{d}_{ij}}\right).
\end{aligned}\tag{19}
$$

So, it is true for $m=\delta_1+1$ and $n=\delta_2+1$. □

3.2. Example. Let $\mathscr{R}=\{\mathscr{R}_1,\mathscr{R}_2,\mathscr{R}_3,\mathscr{R}_4\}$ be a set of experts with the given weight vector $\theta_i=(0.1,0.3,0.3,0.3)^T$. The team of experts is going to describe the attractiveness of a house under-considered set of attributes $A=\{d_1=$ lawn, $d_2=$ security system$\}$ with their corresponding sub-attributes Lawn$d_1=\{d_{11}=$ with grass, $d_{12}=$ without grass$\}$ and security system$d_2=\{d_{21}=$ guards, $d_{22}=$ cameras$\}$. Let $\dddot{\mathscr{A}}=d_1\times d_2$ be a set of subattributes $\dddot{\mathscr{A}}=d_1\times d_2=\{d_{11},d_{12}\}\times\{d_{21},d_{22}\}=\{(d_{11},d_{21}),(d_{11},d_{22}),(d_{12},d_{21}),(d_{12},d_{22})\}$.

$\dddot{\mathscr{A}}=\{\check{d}_1,\check{d}_2,\check{d}_3,\check{d}_4\}$ represents the set subattributes with weights with weight vector $\lambda_j=(0.2,0.2,0.2,0.4)^T$. The supposed rating values for all attributes in the form of PFSNs $(\mathscr{H},\dddot{\mathscr{A}})=(a_{ij},b_{ij})_{4\times4}$ are given as follows:

$$
(\mathscr{H},\dddot{\mathscr{A}})=\begin{matrix}(0.5,0.8)&(0.7,0.5)&(0.4,0.6)&(0.7,0.4)\\(0.5,0.6)&(0.9,0.1)&(0.3,0.7)&(0.4,0.5)\\(0.4,0.8)&(0.7,0.5)&(0.4,0.6)&(0.3,0.5)\\(0.3,0.7)&(0.6,0.5)&(0.5,0.4)&(0.5,0.7)\end{matrix}\tag{20}
$$

As we know that

$$
PFHSEWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{nm}}\right)=\left\langle\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}},\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\right\rangle,\tag{21}
$$

$$
PFHSEWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},..,\mathfrak{I}_{\check{d}_{44}}\right)=\left\langle\begin{array}{c}\dfrac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{4}\left(\prod_{i=1}^{n}\left(1-\alpha_{\check{d}_{ij}}^4\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\alpha_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}},\\ \dfrac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(b_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\end{array}\right\rangle
$$

$$
\begin{aligned}
&= \left\langle \frac{\sqrt{\begin{array}{c}\{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2}\\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4}\\ -\\ [\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\\ \{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}}{\sqrt{\begin{array}{c}\{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2}\\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4}\\ +\\ [\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\\ \{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}}, \right. \\
&\left. \frac{\sqrt{2\left[\begin{array}{c}(\{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4})\end{array}\right]}}{\sqrt{\begin{array}{c}\{(1.36)^{0.1}(1.64)^{0.3}(1.36)^{0.3}(1.51)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.99)^{0.3}(1.75)^{0.3}(1.75)^{0.3}\}^{0.2}\\ \{(1.64)^{0.1}(1.51)^{0.3}(1.64)^{0.3}(1.84)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.75)^{0.3}(1.75)^{0.3}(1.51)^{0.3}\}^{0.4}\\ +\\ \{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4}\end{array}}} \right\rangle \\
&= \left\langle \frac{\sqrt{\begin{array}{c}(1.0324)(1.0897)(1.0309)(1.0734)-\\ [(0.9616)(0.8350)(0.9638)(0.9105)]\end{array}}}{\sqrt{\begin{array}{c}(1.0324)(1.0897)(1.0309)(1.0734)+\\ [(0.9616)(0.8350)(0.9638)(0.9105)]\end{array}}}, \frac{\sqrt{2[(0.8695)(0.6247)(0.7909)(0.6116)]}}{\sqrt{\begin{array}{c}(1.0822)(1.1270)(1.1061)(1.2313)+\\ (0.8695)(0.6247)(0.7909)(0.6116)\end{array}}} \right\rangle \\
&= 0.5263, 0.5225.
\end{aligned} \tag{22}
$$

Lemma 1. *Let* $\mathfrak{I}_{\check{d}_{ij}} = a_{\check{d}_{ij}}, b_{\check{d}_{ij}}$, *where* $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$, *and* $\lambda_j > 0$, $\sum_{j=1}^{n} \lambda_j = 1$, *then*

$$\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{I}_{\check{d}_{ij}} \right)^{\theta_i} \right)^{\lambda_j} = \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} \theta_i \mathfrak{I}_{\check{d}_{ij}}. \tag{23}$$

Theorem 2. *Let* $\mathfrak{I}_{\check{d}_{ij}} = a_{\check{d}_{ij}}, b_{\check{d}_{ij}}$ *be a collection of PFHSNs, then*

$$PFHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \geq PFHSEWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right), \tag{24}$$

where $(i = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, m)$, *and* θ_i *and* λ_j *represent the weight vectors such as* $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$, *and* $\lambda_j > 0$, $\sum_{j=1}^{n} \lambda_j = 1$.

Proof. As we know that

$$\begin{aligned}
&\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1+a_{\breve{d}_{ij}}^{2}\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1-a_{\breve{d}_{ij}}^{2}\right)}\\
&\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1+a_{\breve{d}_{ij}}^{2}\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1-a_{\breve{d}_{ij}}^{2}\right)} = \sqrt{2}\\
&\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+a_{\breve{d}_{ij}}^{2}\right)^{\theta_i)}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{2}\\
&\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+a_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+a_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}},
\end{aligned} \tag{25}$$

again

$$\begin{aligned}
&\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(2-b_{\breve{d}_{ij}}^{2}\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(b_{\breve{d}_{ij}}^{2}\right)}\\
&\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(2-b_{\breve{d}_{ij}}^{2}\right)+\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(b_{\breve{d}_{ij}}^{2}\right)} \leq \sqrt{2}\\
&\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{2}\\
&\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\breve{d}_{ij}}^{2}\right)^{\theta_i}\right)\Bigg)^{\lambda_j}}} \geq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\breve{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}.
\end{aligned} \tag{26}$$

Let $PFHSWA(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}}) = \mathfrak{F}_{\breve{d}_k} = (a_{\mathfrak{F}_{\breve{d}_k}}, b_{\mathfrak{F}_{\breve{d}_k}})$ and PFHSEWA $(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}}) = \mathfrak{F}^{\varepsilon}_{\breve{d}_k} = (a_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}}, b_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}})$. Then, inequalities (A) and (B) can be transformed into the forms $a_{\mathfrak{F}_{\breve{d}_k}} \geq a_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}}$ and $b_{\mathfrak{F}_{\breve{d}_k}} \leq b_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}}$ respectively. So, $S(\mathfrak{F}_{\breve{d}_k}) = a^2_{\mathfrak{F}_{\breve{d}_k}} - b^2_{\mathfrak{F}_{\breve{d}_k}} \geq a^2_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}} - b^2_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}} = S(\mathscr{H}^{\varepsilon})$. Hence, $S(\mathfrak{F}_{\breve{d}_k}) \geq S(\mathfrak{F}^{\varepsilon}_{\breve{d}_k})$.

If $SS(\mathfrak{F}_{\breve{d}_k}) > S(\mathfrak{F}^{\varepsilon}_{\breve{d}_k})$, then $PFHSWA(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}}) > PFHSEWA(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}})$ (C).

If $S(\mathfrak{F}_{\breve{d}_k}) = S(\mathfrak{F}^{\varepsilon}_{\breve{d}_k})$, then $S(\mathfrak{F}_{\breve{d}_k}) = a^2_{\mathfrak{F}_{\breve{d}_k}} - b^2_{\mathfrak{F}_{\breve{d}_k}} = a^2_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}} - b^2_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}} = S(\mathfrak{F}^{\varepsilon}_{\breve{d}_k})$.

So, $a_{\mathfrak{F}_{\breve{d}_k}} = a_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}}$ and $b_{\mathfrak{F}_{\breve{d}_k}} = b_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}}$; then, by accuracy function, $A(\mathfrak{F}_{\breve{d}_k}) = a^2_{\mathfrak{F}_{\breve{d}_k}} + b^2_{\mathfrak{F}_{\breve{d}_k}} = a^2_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}} + b^2_{\mathfrak{F}^{\varepsilon}_{\breve{d}_k}} = A(\mathfrak{F}^{\varepsilon}_{\breve{d}_k})$. Thus, $PFHSWA(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}}) > PFHSEWA(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}})$(D).

From (C) and (D), we get

$PFHSWA(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}}) > PFHSEWA(\mathfrak{F}_{\breve{d}_{11}}, \mathfrak{F}_{\breve{d}_{12}}, \ldots, \mathfrak{F}_{\breve{d}_{nm}})$. □

3.3. Example. Using the data given in example 3.1,

$$\begin{aligned}
&PFHSWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{44}}\right)=\left\langle\sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}},\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(b_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right\rangle\\
&PFHSWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{44}}\right)=\left\langle\begin{matrix}\sqrt{1-\left[\begin{matrix}\left[\left\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\right\}^{0.2}\left\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\right\}^{0.2}\right.\\\left.\left\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\right\}^{0.2}\left\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\right\}^{0.4}\right]\end{matrix}\right]},\\\left(\begin{matrix}\left\{(0.8)^{0.1}(0.6)^{0.3}(0.8)^{0.3}(0.7)^{0.3}\right\}^{0.2}\left\{(0.5)^{0.1}(0.1)^{0.3}(0.5)^{0.3}(0.5)^{0.3}\right\}^{0.2}\\\left\{(0.6)^{0.1}(0.7)^{0.3}(0.6)^{0.3}(0.4)^{0.3}\right\}^{0.2}\left\{(0.4)^{0.1}(0.5)^{0.3}(0.5)^{0.3}(0.7)^{0.3}\right\}^{0.4}\end{matrix}\right)\end{matrix}\right\rangle\\
&\qquad=\left\langle\sqrt{1-[(0.9616)(0.8350)(0.9638)(0.9105)]},((0.9324)(0.7904)(0.8893)(0.7820))\right\rangle\\
&\qquad=\langle 0.5404,0.5125\rangle.
\end{aligned}\tag{27}$$

Hence, from examples 3.1 and 3.2, it is proved that $PFHSWA(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{nm}})>PFHSEWA(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{nm}})$.

3.4. Properties of PFHSEWA Operator

3.4.1. Idempotency. If $\mathfrak{I}_{\check{d}_{ij}}=\mathfrak{I}_{\check{d}_k}=(a_{\check{d}_{ij}},b_{\check{d}_{ij}})$ $\forall i,j$, then PFHSEWA $(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{nm}})=\mathfrak{I}_{\check{d}_k}$.

Proof. We know that

$$\begin{aligned}
&PFHSEWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{nm}}\right)\\
&=\left\langle\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}},\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\check{d}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}\right\rangle\\
&=\left\langle\frac{\sqrt{\left(\left(1+\alpha_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}-\left(\left(1-\alpha_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(1+\alpha_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}+\left(\left(1-\alpha_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}},\frac{\sqrt{2\left(\left(b_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(2-b_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}+\left(\left(b_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}\right\rangle\\
&=\frac{\sqrt{\left(1+\alpha_{\check{d}_{ij}}^{2}\right)-\left(1-\alpha_{\check{d}_{ij}}^{2}\right)}}{\sqrt{\left(1+\alpha_{\check{d}_{ij}}^{2}\right)+\left(1-\alpha_{\check{d}_{ij}}^{2}\right)}},\frac{\sqrt{2b_{\check{d}_{ij}}^{2}}}{\sqrt{\left(2-b_{\check{d}_{ij}}^{2}\right)+\left(b_{\check{d}_{ij}}^{2}\right)}}\\
&=a_{\check{d}_{ij}},b_{\check{d}_{ij}}\\
&=\mathfrak{I}_{\check{d}_k}.
\end{aligned}\tag{28}$$

□

3.4.2. Boundedness. Let $\mathfrak{I}_{\check{d}_{ij}}=(a_{\check{d}_{ij}},b_{\check{d}_{ij}})$ be a Collection PFHSNs and $\mathfrak{I}_{\min}=\min(\mathfrak{I}_{\check{d}_{ij}})\mathfrak{I}_{\max}=\max(\mathfrak{I}_{\check{d}_{ij}})$. Then, $\mathfrak{I}_{\min}\le PFHSEWA\le(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\ldots,\mathfrak{I}_{\check{d}_{nm}})\le\mathfrak{I}_{\max}$.

Proof. Let $g(y)=\sqrt{1-y^2/1+y^2}, y\in 0,1]]$, then $(d/dy)(g(y))=(-2y/(1+y^2)^2)\sqrt{1+y^2/1-y^2}<0$ which shows that $g(y)$ is decreasing function on $0,1]]$. So,

$a_{\min} \le a_{ij} \le a_{\max}, \forall i, j$. Hence, $g(a_{\max}) \le g(a_{\check{d}_{ij}}) \le g(a_{\min})$, $\forall i, j$

$\Rightarrow \sqrt{1 - a_{\max}^2 / 1 + a_{\max}^2} \le \sqrt{1 - a_{\check{d}_{ij}}^2 / 1 + a_{\check{d}_{ij}}^2} \le \sqrt{1 - a_{\min}^2 / 1 + a_{\min}^2}$, $(i = 1, 2, \ldots, n)$ and $(j = 1, 2, \ldots, m)$.

Let θ_i and λ_j represent the weight vectors such as $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$, and $\lambda_j > 0$, $\sum_{j=1}^{n} \lambda_j = 1$. We have

$$
\begin{aligned}
&\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\max}^2/1 + a_{\max}^2\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\min}^2/1 + a_{\min}^2\right)^{\theta_i}\right)^{\lambda_j}}\\
&\Leftrightarrow \sqrt{\left(\left(1 - a_{\max}^2/1 + a_{\max}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\left(\left(1 - a_{\min}^2/1 + a_{\min}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}\\
&\Leftrightarrow \sqrt{1 + \left(\frac{1 - a_{\max}^2}{1 + a_{\max}^2}\right)} \le \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{1 + \left(\frac{1 - a_{\min}^2}{1 + a_{\min}^2}\right)}\\
&\Leftrightarrow \sqrt{\frac{2}{1 + a_{\max}^2}} \le \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\frac{2}{1 + a_{\min}^2}}\\
&\Leftrightarrow \sqrt{\frac{1 + a_{\min}^2}{2}} \le \left(\frac{1}{\sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\right) \le \sqrt{\frac{1 + a_{\max}^2}{2}}\\
&\Leftrightarrow \sqrt{1 + a_{\min}^2} \le \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}} \le \sqrt{1 + a_{\max}^2}\\
&\Leftrightarrow \sqrt{1 + a_{\min}^2 - 1} \le \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}} - 1} \le \sqrt{1 + a_{\max}^2 - 1}\\
&\Leftrightarrow \sqrt{a_{\min}^2} \le \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}} - 1} \le \sqrt{a_{\max}^2}\\
&\Leftrightarrow a_{\min} \le \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2/1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}} - 1} \le a_{\max}
\end{aligned}
\tag{29}
$$

$$
a_{\min} \le \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{\check{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}} \le a_{\max}.
$$

Let $f(x)=\sqrt{2-x^2/x^2}$, $x\in 0,1]]$, then $(d/dx)(f(x))=(-2/x^3)\sqrt{x^2/2-x^2}<0$. So, $f(x)$ is decreasing function on $0,1]]$. Since $b_{\min}\leq b_{ij}\leq b_{\max}$, $\forall i,j$, then $f(b_{\max})\leq f(b_{ij})\leq f(b_{\min})$. So, $\sqrt{(2-b_{\max}^2/b_{\max}^2)}\leq\sqrt{2-b_{\breve{d}_{ij}}^2/b_{\breve{d}_{ij}}^2}\leq\sqrt{(2-b_{\min}^2/b_{\min}^2)}$, $(i=1,2,\ldots,n)$ and $(j=1,2,\ldots,m)$. Let θ_i and λ_j represent the weight vectors such as $\theta_i>0$, $\sum_{i=1}^n\theta_i=1$, and $\lambda_j>0$, $\sum_{j=1}^n\lambda_j=1$. We have

$$
\begin{aligned}
&\Leftrightarrow\sqrt{\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\max}^2/b_{\max}^2\right)^{\theta_i}\right)^{\lambda_j}}\leq\sqrt{\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\breve{d}_{ij}}^2/b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}\leq\sqrt{\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\min}^2/b_{\min}^2\right)^{\theta_i}\right)^{\lambda_j}}\\
&\Leftrightarrow\sqrt{\left(\left(2-b_{\max}^2/b_{\max}^2\right)^{\sum_{i=1}^n\theta_i}\right)^{\sum_{j=1}^m\lambda_j}}\leq\sqrt{\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\breve{d}_{ij}}^2/b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}\leq\sqrt{\left(\left(2-b_{min}^2/b_{min}^2\right)^{\sum_{i=1}^n\theta_i}\right)^{\sum_{j=1}^m\lambda_j}}\\
&\Leftrightarrow\sqrt{1+\frac{2-b_{\max}^2}{b_{\max}^2}}\leq\sqrt{1+\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\breve{d}_{ij}}^2/b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}\leq\sqrt{1+\frac{2-b_{\min}^2}{b_{\min}^2}}\\
&\Leftrightarrow\sqrt{\frac{2}{b_{\max}^2}}\leq\sqrt{1+\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\breve{d}_{ij}}^2/b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}\leq\sqrt{\frac{2}{b_{\min}^2}}\\
&\Leftrightarrow\sqrt{\frac{b_{\min}^2}{2}}\leq\frac{1}{\sqrt{1+\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\breve{d}_{ij}}^2/b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\leq\sqrt{b_{\max}^2/2}\\
&\Leftrightarrow b_{\min}\leq\sqrt{\frac{2}{1+\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\breve{d}_{ij}}^2/b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\leq b_{\max}\\
&b_{\min}\leq\frac{\sqrt{2\prod_{j=1}^m\left(\prod_{i=1}^n\left(b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^m\left(\prod_{i=1}^n\left(2-b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^m\left(\prod_{i=1}^n\left(b_{\breve{d}_{ij}}^2\right)^{\theta_i}\right)^{\lambda_j}}}\leq b_{\max}.
\end{aligned}
\tag{30}
$$

Let $PFHSEWA(\mathfrak{I}_{\breve{d}_{11}},\mathfrak{I}_{\breve{d}_{12}},\ldots,\mathfrak{I}_{\breve{d}_{nm}})=\mathfrak{I}_{\breve{d}_k}$. Then, inequalities (E) and (F) can be written as $a_{\min}\leq a\leq a_{\max}$ and $b_{\min}\leq b\leq b_{\max}$. Thus, S $(\mathfrak{I}_{\breve{d}_k})=a^2-b^2\leq a_{\max}^2-b_{\min}^2=S(\mathfrak{I}_{\breve{d}_k^{\max}})$ and S $(S(\mathfrak{I}_{\breve{d}_k})=a^2-b^2\geq a_{\min}^2-b_{\max}^2=S(\mathfrak{I}_{\breve{d}_k^{\min}})$.

If $S(\mathfrak{I}_{\breve{d}_k})<S(\mathfrak{I}_{\breve{d}_k^{\max}})$ and $S(\mathfrak{I}_{\breve{d}_k})>S(\mathfrak{I}_{\breve{d}_k^{\min}})$, then we have

$$
\mathfrak{I}_{\breve{d}_k^{\min}}<PFHSEWA\left(\mathfrak{I}_{\breve{d}_{11}},\mathfrak{I}_{\breve{d}_{12}},\ldots,\mathfrak{I}_{\breve{d}_{nm}}\right)<\mathfrak{I}_{\breve{d}_k^{\min}},
\tag{31}
$$

If $S(\mathfrak{I}_{\breve{d}_k})=S(\mathfrak{I}_{\breve{d}_k^{\max}})$, then we have $a^2=a_{\max}^2$ and $b^2=b_{\max}^2$. Thus, $S(\mathfrak{I}_{\breve{d}_k})=a^2-b^2==a_{\max}^2-b_{\max}^2=S(\mathfrak{I}_{\breve{d}_k^{\max}})$. Therefore,

$$
PFHSEWA\left(\mathfrak{I}_{\breve{d}_{11}},\mathfrak{I}_{\breve{d}_{12}},\ldots,\mathfrak{I}_{\breve{d}_{nm}}\right)=\mathfrak{I}_{\breve{d}_k^{\max}}.
\tag{32}
$$

If S $(\mathfrak{I}_{\breve{d}_k})=S(\mathfrak{I}_{\breve{d}_k\min})$, then we have $a^2-b^2=a_{\min}^2-b_{\min}^2\Rightarrow a^2=a_{\min}^2$ and $b^2=b_{\min}^2$.

Thus, $A(\mathfrak{I}_{\breve{d}_k})=a^2+b^2=a_{\min}^2+b_{\min}^2=A(\mathfrak{I}_{\breve{d}_k^{\min}})$. So,

$$
PFHSEWA\left(\mathfrak{I}_{\breve{d}_{11}},\mathfrak{I}_{\breve{d}_{12}},\ldots,\mathfrak{I}_{\breve{d}_{nm}}\right)=\mathfrak{I}_{\breve{d}_k^{\min}}.
\tag{33}
$$

$$
\mathfrak{I}_{\breve{d}_k^{\min}}\leq PFHSEWA(\mathfrak{I}_{\breve{d}_{11}},\mathfrak{I}_{\breve{d}_{12}},\ldots,\mathfrak{I}_{\breve{d}_{nm}})\leq\mathfrak{I}_{\breve{d}_k^{\max}}.
$$

□

3.4.3. Homogeneity. Prove that $PFHSEWA(\mathfrak{I}_{\breve{d}_{11}},\mathfrak{I}_{\breve{d}_{12}},\ldots,\mathfrak{I}_{\breve{d}_{nm}})=\partial P\ FHSEWA(\mathfrak{I}_{\breve{d}_{11}},\mathfrak{I}_{\breve{d}_{12}},\ldots,\mathfrak{I}_{\breve{d}_{nm}})$ for $\partial>0$.

Proof. Let $\mathfrak{J}_{\check{d}_{ij}}$ be a PFHSN and ∂ is a positive number, then by $\partial\mathfrak{J}_{\check{d}_{ij}} = (\sqrt{(1 + a^2_{\check{d}_{ij}})^{\partial} - (1 - a^2_{\check{d}_{ij}})^{\partial}} / \sqrt{(1 + a^2_{\check{d}_{ij}})^{\partial} + (1 - a^2_{\check{d}_{ij}})^{\partial}}), \sqrt{2(b^2_{\check{d}_{ij}})^{\partial}} / \sqrt{(2 - b^2_{\check{d}_{ij}})^{\partial} + (b^2_{\check{d}_{ij}})^{\partial}}$. So,

$$
\begin{aligned}
&PFHSEWA\left(\partial\mathfrak{J}_{\check{d}_{11}}, \partial\mathfrak{J}_{\check{d}_{12}} \partial\mathfrak{J}_{\check{d}_{nm}}\right)\\
&= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle\\
&= \left\langle \frac{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial} - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial} + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}, \frac{\sqrt{\left(2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial} + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}} \right\rangle\\
&= PFHSEWA\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}} \mathfrak{J}_{\check{d}_{nm}}\right).
\end{aligned}
\tag{34}
$$

□

4. Novel Multicriteria Decision-Making Approach

This section has developed a DM approach for solving MCDM problems based on the proposed PFHSEWA operator and numerical examples.

4.1. Proposed Approach. Consider $\mathfrak{H} = \{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \ldots, \mathfrak{H}^s\}$ be a set of alternatives and $O = \{O_1, O_2, O_3, \ldots, O_r\}$ be a set of experts. The weights of experts are given as $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)^T$ such that $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$. Let L= $\{d_1, d_2, \ldots, d_m\}$ express the set of attributes with their corresponding multi-sub-attributes such as $\mathfrak{L}' = \{(d_{1\rho} \times d_{2\rho} \times \ldots \times d_{m\rho})$ for all $\rho \in \{1, 2, \ldots, t\}\}$ with weights $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)^T$ such that $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$, and can be stated as $\mathfrak{L}' = \{\check{d}_{\partial}: \partial \in \{1, 2, \ldots, m\}\}$. The group of experts $\{\kappa^i, i = 1, 2, \ldots, n\}$ assess the alternatives $\{\mathfrak{H}^{(z)}, z = 1, 2, \ldots S\}$ under the chosen subattributes $\{\check{d}_{\partial}, \partial = 1, 2, \ldots, k\}$ in the form of PFHSNs such as $(\mathfrak{H}^{(z)}_{\check{d}_{\partial}})_{n\times m} = (\alpha_{\check{d}_{ij}}, b_{\check{d}_{ij}})_{n*m}$, where $0 \le \alpha_{\check{d}_{ij}}, b_{\check{d}_{ij}} \le 1$ and $0 \le (\alpha_{\check{d}_{ij}})^2 + (b_{\check{d}_{ij}})^2 \le 1$ for all i, k. The experts provide their opinion in the form of PFHSNs $\mathscr{L}_k$ for each alternative and present the step-wise algorithm to obtain the most suitable alternative.

Step 1. Obtain decision matrices $F = (\mathfrak{J}_{\check{d}_{ij}})_{n*m}$ in the form of PFHSNs for alternatives relative to attributes.

$$
\left(\mathfrak{H}^{(z)}_{\check{d}_{ik}}, \mathfrak{L}'\right)_{n\times\partial} = \begin{array}{c} O_1 \\ O_2 \\ \vdots \\ O_n \end{array} \begin{pmatrix} \left(\alpha^{(z)}_{\check{d}_{11}}, b^{(z)}_{\check{d}_{11}}\right) & \left(\alpha^{(z)}_{\check{d}_{12}}, b^{(z)}_{\check{d}_{12}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{1\partial}}, b^{(z)}_{\check{d}_{1\partial}}\right) \\ \left(\alpha^{(z)}_{\check{d}_{21}}, b^{(z)}_{\check{d}_{21}}\right) & \left(\alpha^{(z)}_{\check{d}_{22}}, b^{(z)}_{\check{d}_{22}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{2\partial}}, b^{(z)}_{\check{d}2\partial}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\alpha^{(z)}_{\check{d}_{n1}}, b^{(z)}_{\check{d}_{n1}}\right) & \left(\alpha^{(z)}_{\check{d}_{n2}}, b^{(z)}_{\check{d}_{n2}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{n\partial}}, b^{(z)}_{\check{d}_{n\partial}}\right) \end{pmatrix}.
\tag{35}
$$

Step 2. Use the normalization formula to normalize the decision matrix and convert the rating value of the cost type parameter to the benefit type parameter.

$$
M_{\check{d}_{ij}} = \begin{cases} \mathfrak{J}^{c}_{\check{d}_{ij}} = \left(b_{\check{d}_{ij}}, \alpha_{\check{d}_{ij}}\right) \text{costtypeparameter}, \\ \mathfrak{J}_{\check{d}_{ij}} = \left(\alpha_{\check{d}_{ij}}, b_{\check{d}_{ij}}\right) \text{benefittypeparameter}. \end{cases}
\tag{36}
$$

Step 3. Use the settled PFHSEWA operator to collect the PFHSNs $\mathfrak{J}_{\check{d}_{ij}}$ for each alternative $\mathfrak{H} = \{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \ldots, \mathfrak{H}^s\}$ into the decision matrix $\mathscr{L}_k$.

Step 4. Use equation (1) to calculate the scores for all alternatives.

Step 5. Choose the alternative with the highest score.

Step 6. Rank the alternatives.

The graphical demonstration of the planned model is given in Figure 1.

4.2. Numerical Example. In this section, a practical MCDM problem comprises decisive adequate agricultural models in numerous kinds of farming to confirm that the conventional approach is pertinent and reasonable.

4.2.1. Case Study. Green agriculture claims sustainable growth ideas to farming, such as confirming food production and fiber while preserving financial and societal constrictions to ensure the long-term viability of production. For example, sustainable agriculture diminishes the practice of pesticides that are harmful to the health of agriculturalists and customers. Accuracy farming and intellectual farming

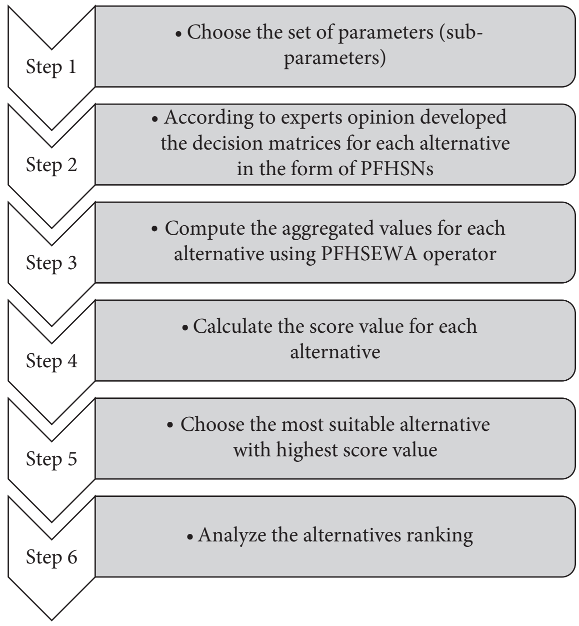

Figure 1: Flowchart of the proposed model.

are the core mechanisms of sustainable farming. Increasing crops and raising livestock are agricultural professions or businesses. Farming comprises raising animals and growing crops, which deliver nutrition and raw ingredients. Agriculture was initiated approximately 5,000 years earlier, but the particular time and source are indefinite. Agriculture is a technique of lifecycle, not just a profession. We are all farmers, and we like farming no matter whether we are at home or in the fields. This love of gardening must be a lifelong practice, regardless of age. Due to this land devastation, food prices will skyrocket, and we will have to pay more for daily food needs. Farmers must focus on increasing production through agricultural robots to get out of this situation. The use of robots in farming is an illustration of inspiration beyond origination. Agriculture, as an industry, will grow into a high-tech sector in the new era. Agribots or agri-robots are other terms for agricultural robots [64]. Five key alternatives are interrelated to sustainable agriculture such as good crop production ($\mathfrak{H}^1$), environmental protection ($\mathfrak{H}^2$), natural resources availability ($\mathfrak{H}^3$), food security and productivity ($\mathfrak{H}^4$), and availability of machines ($\mathfrak{H}^5$). In addition, the abovementioned five alternatives are evaluated using four parameters. The attribute of robotic agriculture is given as follows: $\mathfrak{L}\{d_1$ = Qualityproduction, d_2 = Completionofatime– consumingproject, d_3 = Consistentroleincompletingaproject, d_4 = Limitingtheneedformanuallabor$\}$. The corresponding subattributes of the considered parameters are Qualityproduction = d_1 = $\{d_{11}$ = High – qualityproduction, d_{12} = Low – qualityproduction$\}\{d_{11}$ = High– qualityproduction, d_{12} = Low – qualityproduction$\}$, Completionofatime – consumingproject = d_2 = $\{d_{21}$ = Shortterm, d_{22} = Longterm$\}$= $\{d_{21}$ = Shortterm, d_{22} = Long term $\}$, Consistentroleincompletingaproject = d_3 = $\{d_{31}$ = Projectbudgetingandforecasting, d_{32} = Developingariskmanagementplan$\}$= $\{d_{31}$ = Projectbudgetingandforecasting, d_{32} = Developingariskmanagementplan$\}$, Limiting the need for manual labor = d_4 = $\{d_{41}$ = Limiting the need for manual labor$\}$ = $\{d_{41}$ = Limiting the need for manual labor$\}$. Let $\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4$ be a set of subattributes. $\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} \times \{d_{31}, d_{32}\} \times \{d_{41}\} = \{ (d_{11}, d_{21}, d_{31}, d_{41}), (d_{11}, d_{21}, d_{32}, d_{41}), (d_{11}, d_{22}, d_{31}, d_{41}), (d_{11}, d_{22}, d_{32}, d_{41}), (d_{12}, d_{21}, d_{31}, d_{41}), (d_{12}, d_{21}, d_{32}, d_{41}), (d_{12}, d_{22}, d_{31}, d_{41}), (d_{12}, d_{22}, d_{32}, d_{41})\}$, $\mathfrak{L}' = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4, \check{d}_5, \check{d}_6, \check{d}_7, \check{d}_8\}$ be a set of all sub-attributes with weights $(0.12, 0.18, 0.1, 0.15, 0.05, 0.22, 0.08, 0.1)^T$. Let $\{O_1, O_2, O_3\}$ be a set of three experts with weights $(0.143, 0.514, 0.343)^T$. To judge the optimum alternative, specialists provide their preferences in the form of PFHSNs.

4.2.2. PFHSEWA Operator

Step 1. According to the expert's opinion, Pythagorean fuzzy hypersoft decision matrices for all alternatives are given in Tables 1–5.

Step 2. There is no need to normalize because all parameters are the same type.

Step 3. Apply the proposed PFHSEWA operator to the obtained data (Tables 1–5), and then we get the opinions of decision-makers on alternatives in the form of PFHSN, for example,

Table 1: PFHS decision matrix for $\mathfrak{H}^1$.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
O_1	(0.3,0.8)	(0.7 0.3)	(0.6,0.7)	(0.5,0.4)	(0.2,0.4)	(0.4,0.6)	(0.5,0.8)	(0.9,0.3)
O_2	(0.7,0.6)	(0.3,0.4)	(0.6,0.5)	(0.3,0.9)	(0.5,0.4)	(0.4,0.6)	(0.7,0.5)	(0.4,0.8)
O_3	(0.5,0.7)	(0.8,0.5)	(0.7,0.4)	(0.4,0.3)	(0.4,0.9)	(0.2,0.4)	(0.8,0.4)	(0.7,0.5)

Table 2: PFHS decision matrix for $\mathfrak{H}^2$.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
O_1	(0.6,0.7)	(0.4,0.6)	(0.3,0.4)	(0.9,0.2)	(0.3,0.8)	(0.2,0.4)	(0.7,0.5)	(0.4,0.5)
O_2	(0.8,0.5)	(0.7,0.4)	(0.9,0.2)	(0.7,0.4)	(0.4,0.5)	(0.9,0.3)	(0.2,0.7)	(0.3,0.8)
O_3	(0.8,0.5)	(0.7,0.4)	(0.8,0.5)	(0.5,0.2)	(0.5,0.7)	(0.7,0.5)	(0.7,0.6)	(0.6,0.4)

Table 3: PFHS decision matrix for $\mathfrak{H}^3$.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
O_1	(0.7,0.3)	(0.2,0.5)	(0.1,0.6)	(0.3,0.4)	(0.4 0.6)	(0.8,0.4)	(0.6,0.7)	(0.2,0.5)
O_2	(0.3,0.7)	(0.4,0.5)	(0.4,0.8)	(0.3,0.4)	(0.6,0.7)	(0.3,0.4)	(0.9,0.2)	(0.7,0.2)
O_3	(0.6,0.8)	(0.4,0.5)	(0.6,0.5)	(0.6,0.4)	(0.7,0.5)	(0.8,0.4)	(0.5,0.8)	(0.4 0.5)

Table 4: PFHS decision matrix for $\mathfrak{H}^4$.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
O_1	(0.8,0.4)	(0.2,0.9)	(0.2,0.4)	(0.4,0.6)	(0.6,0.5)	(0.5,0.6)	(0.4,0.5)	(0.8,0.3)
O_2	(0.5,0.4)	(0.7,0.6)	(0.9,0.3)	(0.8,0.5)	(0.9,0.2)	(0.2,0.4)	(0.4,0.6)	(0.6,0.5)
O_3	(0.5,0.7)	(0.9,0.3)	(0.3,0.5)	(0.5,0.7)	(0.3,0.5)	(0.8,0.5)	(0.7,0.5)	(0.2,0.5)

Table 5: PFHS decision matrix for $\mathfrak{H}^5$.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
O_1	(0.5,0.7)	(0.8,0.5)	(0.7,0.4)	(0.4,0.3)	(0.4,0.9)	(0.2,0.4)	(0.8,0.4)	(0.7,0.5)
O_2	(0.8,0.5)	(0.7,0.4)	(0.8,0.5)	(0.5,0.2)	(0.5,0.7)	(0.7,0.5)	(0.7,0.6)	(0.6,0.4)
O_3	(0.5,0.4)	(0.4,0.8)	(0.5,0.6)	(0.3,0.4)	(0.7,0.6)	(0.7,0.5)	(0.4,0.9)	(0.5,0.2)

$$
\begin{aligned}
&PFHSEWA\left(\mathfrak{J}_{\tilde{a}_{11}}, \mathfrak{J}_{\tilde{a}_{12}} \mathfrak{J}_{\tilde{a}_{mn}}\right)\\
&= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{\tilde{a}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{\tilde{a}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\tilde{a}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{\tilde{a}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{\tilde{a}_{ij}}^{2}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle
\end{aligned} \tag{37}
$$

$$
\mathscr{L}_2 = \left\langle \frac{\sqrt{\begin{array}{c}
\left\{(1.36)^{0.12}(1.16)^{0.18}(1.09)^{0.1}(1.81)^{0.15}(1.09)^{0.05}(1.04)^{0.22}(1.49)^{0.08}(1.16)0.1\right\}^{0.143}\\
\left\{(1.64)^{0.12}(1.49)^{0.18}(1.81)^{0.1}(1.49)^{0.15}(1.16)^{0.05}(1.81)^{0.22}(1.04)^{0.08}(1.09)0.1\right\}^{0.514}\\
\left\{(1.64)^{0.12}(1.49)^{0.18}(1.64)^{0.1}(1.25)^{0.15}(1.25)^{0.05}(1.49)^{0.22}(1.49)^{0.08}(1.36)0.1\right\}^{0.343}\\
-\\
\left\{(0.64)^{0.12}(0.84)^{0.18}(0.91)^{0.1}(0.19)^{0.15}(0.91)^{0.05}(0.96)^{0.22}(0.51)^{0.08}(0.84)^{0.1}\right\}^{0.143}\\
\left\{(0.36)^{0.12}(0.51)^{0.18}(0.19)^{0.1}(0.51)^{0.15}(0.84)^{0.05}(0.19)^{0.22}(0.96)^{0.08}(0.91)^{0.1}\right\}^{0.514}\\
\left\{(0.36)^{0.12}(0.51)^{0.18}(0.36)^{0.1}(0.75)^{0.15}(0.75)^{0.05}(0.51)^{0.22}(0.51)^{0.08}(0.64)^{0.1}\right\}^{0.343}
\end{array}}}{\sqrt{\begin{array}{c}
\left\{(1.36)^{0.12}(1.16)^{0.18}(1.09)^{0.1}(1.81)^{0.15}(1.09)^{0.05}(1.04)^{0.22}(1.49)^{0.08}(1.16)^{0.1}\right\}^{0.143}\\
\left\{(1.64)^{0.12}(1.49)^{0.18}(1.81)^{0.1}(1.49)^{0.15}(1.16)^{0.05}(1.81)^{0.22}(1.04)^{0.08}(1.09)^{0.1}\right\}^{0.514}\\
\left\{(1.64)^{0.12}(1.49)^{0.18}(1.64)^{0.1}(1.25)^{0.15}(1.25)^{0.05}(1.49)^{0.22}(1.49)^{0.08}(1.36)0.1\right\}^{0.343}\\
+\\
\left\{(0.64)^{0.12}(0.84)^{0.18}(0.91)^{0.1}(0.19)^{0.15}(0.91)^{0.05}(0.96)^{0.22}(0.51)^{0.08}(0.84)^{0.1}\right\}^{0.143}\\
\left\{(0.36)^{0.12}(0.51)^{0.18}(0.91)^{0.1}(0.51)^{0.15}(0.84)^{0.05}(0.91)^{0.22}(0.96)^{0.08}(0.91)^{0.1}\right\}^{0.514}\\
\left\{(0.36)^{0.12}(0.51)^{0.18}(0.36)^{0.1}(0.75)^{0.15}(0.75)^{0.05}(0.51)^{0.22}(0.51)^{0.08}(0.64)^{0.1}\right\}^{0.343}
\end{array}}}, \frac{\sqrt{2\left[\begin{array}{c}
\left\{(0.49)^{0.12}(0.36)^{0.18}(0.16)^{0.1}(0.04)^{0.15}(0.64)^{0.05}(0.16)^{0.22}(0.25)^{0.08}(0.25)^{0.1}\right\}^{0.143}\\
\left\{(0.25)^{0.12}(0.16)^{0.18}(0.04)^{0.1}(0.16)^{0.15}(0.25)^{0.05}(0.09)^{0.22}(0.49)^{0.08}(0.64)^{0.1}\right\}^{0.514}\\
\left\{(0.25)^{0.12}(0.16)^{0.18}(0.25)^{0.1}(0.04)^{0.15}(0.49)^{0.05}(0.25)^{0.22}(0.36)^{0.08}(0.16)^{0.1}\right\}^{0.343}
\end{array}\right]}}{\sqrt{\begin{array}{c}
\left\{(1.51)^{0.12}(1.64)^{0.18}(1.84)^{0.1}(1.96)^{0.15}(1.36)^{0.05}(1.84)^{0.22}(1.75)^{0.08}(1.75)^{0.1}\right\}^{0.143}\\
\left\{(1.75)^{0.12}(1.84)^{0.18}(1.96)^{0.1}(1.84)^{0.15}(1.75)^{0.05}(1.91)^{0.22}(1.51)^{0.08}(1.36)^{0.1}\right\}^{0.514}\\
\left\{(1.75)^{0.12}(1.84)^{0.18}(1.75)^{0.1}(1.96)^{0.15}(1.51)^{0.05}(1.75)^{0.22}(1.64)^{0.08}(1.84)^{0.1}\right\}^{0.343}\\
+\\
\left\{(0.49)^{0.12}(0.36)^{0.18}(0.16)^{0.1}(0.04)^{0.15}(0.64)^{0.05}(0.16)^{0.22}(0.25)^{0.08}(0.25)^{0.1}\right\}^{0.143}\\
\left\{(0.25)^{0.12}(0.16)^{0.18}(0.04)^{0.1}(0.16)^{0.15}(0.25)^{0.05}(0.09)^{0.22}(0.49)^{0.08}(0.64)^{0.1}\right\}^{0.514}\\
\left\{(0.25)^{0.12}(0.16)^{0.18}(0.25)^{0.1}(0.04)^{0.15}(0.49)^{0.05}(0.25)^{0.22}(0.36)^{0.08}(0.16)^{0.1}\right\}^{0.343}
\end{array}}} \right\rangle \tag{38}
$$

$$\langle 0.7105, 0.4250\rangle$$

$$\mathscr{L}_3 = \left\langle \frac{\sqrt{\begin{array}{c} \left\{(1.49)^{0.12}(1.04)^{0.18}(1.01)^{0.1}(1.09)^{0.15}(1.16)0.05(1.64)0.22(1.36)0.08(1.04)0.1\right\}0.143 \\ \left\{(1.09)^{0.12}(1.16)^{0.18}(1.16)^{0.1}(1.09)^{0.15}(1.36)0.05(1.09)0.22(1.81)0.08(1.49)0.1\right\}0.514 \\ \left\{(1.36)^{0.12}(1.16)^{0.18}(1.36)^{0.1}(1.36)^{0.15}(1.49)0.05(1.64)0.22(1.25)0.08(1.16)0.1\right\}0.343 \\ - \\ \left\{(0.51)^{0.12}(0.96)^{0.18}(0.01)^{0.1}(0.91)^{0.15}(0.84)^{0.05}(0.36)^{0.22}(0.64)^{0.08}(0.96)^{0.1}\right\}^{0.143} \\ \left\{(0.91)^{0.12}(0.84)^{0.18}(0.84)^{0.1}(0.91)^{0.15}(0.64)^{0.05}(0.91)^{0.22}(0.19)^{0.08}(0.51)^{0.1}\right\}^{0.514} \\ \left\{(0.64)^{0.12}(0.84)^{0.18}(0.64)^{0.1}(0.64)^{0.15}(0.51)^{0.05}(0.36)^{0.22}(0.75)^{0.08}(0.84)^{0.1}\right\}^{0.343} \end{array}}}{\sqrt{\begin{array}{c} \left\{(1.49)^{0.12}(1.04)^{0.18}(1.01)^{0.1}(1.09)^{0.15}(1.16)^{0.05}(1.64)^{0.22}(1.36)^{0.08}(1.04)^{0.1}\right\}^{0.143} \\ \left\{(1.09)^{0.12}(1.16)^{0.18}(1.16)^{0.1}(1.09)^{0.15}(1.36)^{0.05}(1.09)^{0.22}(1.81)^{0.08}(1.49)^{0.1}\right\}^{0.514} \\ \left\{(1.36)^{0.12}(1.16)^{0.18}(1.36)^{0.1}(1.36)^{0.15}(1.49)^{0.05}(1.64)^{0.22}(1.25)^{0.08}(1.16)^{0.1}\right\}^{0.343} \\ + \\ \left\{(0.51)^{0.12}(0.96)^{0.18}(0.01)^{0.1}(0.91)^{0.15}(0.84)^{0.05}(0.36)^{0.22}(0.64)^{0.08}(0.96)^{0.1}\right\}^{0.143} \\ \left\{(0.91)^{0.12}(0.84)^{0.18}(0.84)^{0.1}(0.91)^{0.15}(0.64)^{0.05}(0.91)^{0.22}(0.19)^{0.08}(0.51)^{0.1}\right\}^{0.514} \\ \left\{(0.64)^{0.12}(0.84)^{0.18}(0.64)^{0.1}(0.64)^{0.15}(0.51)^{0.05}(0.36)^{0.22}(0.75)^{0.08}(0.84)^{0.1}\right\}^{0.343} \end{array}}}, \frac{\sqrt{2\left[\begin{array}{c} \left\{(0.09)^{0.12}(0.25)^{0.18}(0.36)^{0.1}(0.16)0.15(0.36)^{0.05}(0.16)^{0.22}(0.49)^{0.08}(0.25)^{0.1}\right\}^{0.143} \\ \left\{\left\{(0.49)^{0.12}(0.25)^{0.18}(0.64)^{0.1}(0.16)0.15(0.49)^{0.05}(0.16)^{0.22}(0.04)^{0.08}(0.04)^{0.1}\right\}^{0.514}\right\} \\ \left\{(0.64)^{0.12}(0.25)^{0.18}(0.25)^{0.1}(0.16)0.15(0.25)^{0.05}(0.16)^{0.22}(0.64)^{0.08}(0.25)^{0.1}\right\}^{0.343} \end{array}\right]}}{\sqrt{\begin{array}{c} \left\{(1.91)^{0.12}(1.75)^{0.18}(1.64)^{0.1}(1.84)0.15(1.64)^{0.05}(1.84)^{0.22}(1.51)^{0.08}(1.75)^{0.1}\right\}^{0.143} \\ \left\{(1.51)^{0.12}(1.75)^{0.18}(1.36)^{0.1}(1.84)0.15(1.51)^{0.05}(1.84)^{0.22}(1.96)^{0.08}(1.96)^{0.1}\right\}^{0.514} \\ \left\{(1.36)^{0.12}(1.75)^{0.18}(1.75)^{0.1}(1.84)0.15(1.75)^{0.05}(1.84)^{0.22}(1.36)^{0.08}(1.75)^{0.1}\right\}^{0.343} \\ + \\ \left\{(0.09)^{0.12}(0.25)^{0.18}(0.36)^{0.1}(0.16)^{0.15}(0.36)^{0.05}(0.16)^{0.22}(0.49)^{0.08}(0.25)^{0.1}\right\}^{0.143} \\ \left\{(0.49)^{0.12}(0.25)^{0.18}(0.64)^{0.1}(0.16)^{0.15}(0.49)^{0.05}(0.16)^{0.22}(0.04)^{0.08}(0.04)^{0.1}\right\}^{0.514} \\ \left\{(0.64)^{0.12}(0.25)^{0.18}(0.25)^{0.1}(0.16)^{0.15}(0.25)^{0.05}(0.16)^{0.22}(0.64)^{0.08}(0.25)^{0.1}\right\}0.343 \end{array}}} \right\rangle \quad (39)$$

$$\langle 0.5834, 0.4680 \rangle$$

$$\mathscr{L}_4 = \left\langle \frac{\sqrt{\begin{array}{c} \left\{(1.64)^{0.12}(1.04)^{0.18}(1.04)^{0.1}(1.16)^{0.15}(1.36)^{0.05}(1.25)^{0.22}(1.16)^{0.08}(1.64)^{0.1}\right\}^{0.143} \\ \left\{(1.25)^{0.12}(1.49)^{0.18}(1.81)^{0.1}(1.64)^{0.15}(1.81)^{0.05}(1.04)^{0.22}(1.16)^{0.08}(1.36)^{0.1}\right\}^{0.514} \\ \left\{(1.25)^{0.12}(1.81)^{0.18}(1.09)^{0.1}(1.25)^{0.15}(1.09)^{0.05}(1.64)^{0.22}(1.49)^{0.08}(1.04)^{0.1}\right\}^{0.343} \\ - \\ \left\{(0.36)^{0.12}(0.96)^{0.18}(0.96)^{0.1}(0.84)^{0.15}(0.64)0.05(0.75)^{0.22}(0.84)^{0.08}(0.36)^{0.1}\right\}^{0.143} \\ \left\{(0.75)^{0.12}(0.51)^{0.18}(0.19)^{0.1}(0.36)^{0.15}(0.19)0.05(0.96)^{0.22}(0.84)^{0.08}(0.64)^{0.1}\right\}^{0.514} \\ \left\{(0.75)^{0.12}(0.19)^{0.18}(0.91)^{0.1}(0.75)^{0.15}(0.91)0.05(0.36)^{0.22}(0.51)^{0.08}(0.96)^{0.1}\right\}^{0.343} \end{array}}}{\sqrt{\begin{array}{c} \left\{(1.64)^{0.12}(1.04)^{0.18}(1.04)^{0.1}(1.16)^{0.15}(1.36)^{0.05}(1.25)^{0.22}(1.16)^{0.08}(1.64)^{0.1}\right\}^{0.143} \\ \left\{(1.25)^{0.12}(1.49)^{0.18}(1.81)^{0.1}(1.64)^{0.15}(1.81)^{0.05}(1.04)^{0.22}(1.16)^{0.08}(1.36)^{0.1}\right\}^{0.514} \\ \left\{(1.25)^{0.12}(1.81)^{0.18}(1.09)^{0.1}(1.25)^{0.15}(1.09)^{0.05}(1.64)^{0.22}(1.49)^{0.08}(1.04)^{0.1}\right\}^{0.343} \\ + \\ \left\{(0.36)^{0.12}(0.96)^{0.18}(0.96)^{0.1}(0.84)^{0.15}(0.64)^{0.05}(0.75)^{0.22}(0.84)^{0.08}(0.36)^{0.1}\right\}^{0.143} \\ \left\{(0.75)^{0.12}(0.51)^{0.18}(0.19)^{0.1}(0.36)^{0.15}(0.19)^{0.05}(0.96)^{0.22}(0.84)^{0.08}(0.64)^{0.1}\right\}^{0.514} \\ \left\{(0.75)^{0.12}(0.19)^{0.18}(0.91)^{0.1}(0.75)^{0.15}(0.91)^{0.05}(0.36)^{0.22}(0.51)^{0.08}(0.96)^{0.1}\right\}^{0.343} \end{array}}}, \frac{\sqrt{2\left[\begin{array}{c} \left\{(0.16)^{0.12}(0.81)^{0.18}(0.16)^{0.1}(0.36)^{0.15}(0.25)^{0.05}(0.36)^{0.22}(0.25)^{0.08}(0.09)^{0.1}\right\}^{0.143} \\ \left\{(0.16)^{0.12}(0.36)^{0.18}(0.09)^{0.1}(0.25)^{0.15}(0.04)^{0.05}(0.16)^{0.22}(0.36)^{0.08}(0.25)^{0.1}\right\}^{0.514} \\ \left\{(0.49)^{0.12}(0.09)^{0.18}(0.25)^{0.1}(0.49)^{0.15}(0.25)^{0.05}(0.25)^{0.22}(0.25)^{0.08}(0.25)^{0.1}\right\}^{0.343} \end{array}\right]}}{\sqrt{\begin{array}{c} (1.84)^{0.12}(1.19)^{0.18}(1.84)^{0.1}(1.64)^{0.15}(1.75)^{0.05}(1.64)^{0.22}(1.75)^{0.08}(1.91)^{0.1 0.143\}} \\ \left\{(1.84)^{0.12}(1.64)^{0.18}(1.91)^{0.1}(1.75)^{0.15}(1.96)^{0.05}(1.84)^{0.22}(1.64)^{0.08}(1.75)^{0.1}\right\}^{0.514} \\ \left\{(1.51)^{0.12}(1.91)^{0.18}(1.75)^{0.1}(1.51)^{0.15}(1.75)^{0.05}(1.75)^{0.22}(1.75)^{0.08}(1.75)^{0.1}\right\}^{0.343} \\ + \\ \left\{(0.16)^{0.12}(0.81)^{0.18}(0.16)^{0.1}(0.36)^{0.15}(0.25)^{0.05}(0.36)^{0.22}(0.25)^{0.08}(0.09)^{0.1}\right\}^{0.143} \\ \left\{(0.16)^{0.12}(0.36)^{0.18}(0.09)^{0.1}(0.25)^{0.15}(0.04)^{0.05}(0.16)^{0.22}(0.36)^{0.08}(0.25)^{0.1}\right\}^{0.514} \\ \left\{(0.49)^{0.12}(0.09)^{0.18}(0.25)^{0.1}(0.49)^{0.15}(0.25)^{0.05}(0.25)^{0.22}(0.25)^{0.08}(0.25)^{0.1}\right\}^{0.343} \end{array}}} \right\rangle \tag{40}$$

$$\langle 0.6521, 0.4253 \rangle$$

$$
\mathscr{L}_5=\left\langle \frac{\sqrt{\begin{array}{c}
\left\{(1.25)^{0.12}(1.64)0.18(1.49)^{0.1}(1.16)^{0.15}(1.16)^{0.05}(1.04)^{0.22}(1.64)^{0.08}(1.49)^{0.1}\right\}^{0.143}\\
\left\{(1.64)^{0.12}(1.49)0.18(1.64)^{0.1}(1.25)^{0.15}(1.25)^{0.05}(1.49)^{0.22}(1.49)^{0.08}(1.36)^{0.1}\right\}^{0.514}\\
\left\{(1.25)^{0.12}(1.16)0.18(1.25)^{0.1}(1.09)^{0.15}(1.49)^{0.05}(1.49)^{0.22}(1.16)^{0.08}(1.25)^{0.1}\right\}^{0.343}\\
-\\
\left\{(1.25)^{0.12}(1.16)^{0.18}(1.25)^{0.1}(1.09)^{0.15}(1.49)^{0.05}(1.49)^{0.22}(1.16)^{0.08}(1.25)^{0.1}\right\}^{0.343}\\
\left\{(0.36)^{0.12}(0.51)^{0.18}(0.36)^{0.1}(0.75)^{0.15}(0.75)^{0.05}(0.51)^{0.22}(0.51)^{0.08}(0.64)^{0.1}\right\}^{0.514}\\
\left\{(0.75)^{0.12}(0.84)^{0.18}(0.75)0.1(0.91)^{0.15}(0.51)^{0.05}(0.51)^{0.22}(0.84)^{0.08}(0.75)^{0.1}\right\}^{0.343}
\end{array}}}{\sqrt{\begin{array}{c}
\left\{(1.25)^{0.12}(1.64)^{0.18}(1.49)^{0.1}(1.16)^{0.15}(1.16)^{0.05}(1.04)^{0.22}(1.64)^{0.08}(1.49)^{0.1}\right\}^{0.143}\\
\left\{(1.64)^{0.12}(1.49)^{0.18}(1.64)^{0.1}(1.25)^{0.15}(1.25)^{0.05}(1.49)^{0.22}(1.49)^{0.08}(1.36)^{0.1}\right\}^{0.514}\\
\left\{(1.25)^{0.12}(1.16)^{0.18}(1.25)^{0.1}(1.09)^{0.15}(1.49)^{0.05}(1.49)^{0.22}(1.16)^{0.08}(1.25)^{0.1}\right\}^{0.343}\\
+\\
\left\{(0.75)^{0.12}(0.36)^{0.18}(0.51)^{0.1}(0.84)^{0.15}(0.84)^{0.05}(0.96)^{0.22}(0.36)^{0.08}(0.51)^{0.1}\right\}^{0.143}\\
\left\{(0.36)^{0.12}(0.51)^{0.18}(0.36)^{0.1}(0.75)^{0.15}(0.75)^{0.05}(0.51)^{0.22}(0.51)^{0.08}(0.64)^{0.1}\right\}^{0.514}\\
\left\{(0.75)^{0.12}(0.84)^{0.18}(0.75)^{0.1}(0.91)^{0.15}(0.51)^{0.05}(0.51)^{0.22}(0.84)^{0.08}(0.75)^{0.1}\right\}^{0.343}
\end{array}}}, \frac{\sqrt{2\left[\begin{array}{c}
\left\{(0.49)^{0.12}(0.25)^{0.18}(0.16)^{0.1}(0.09)^{0.15}(0.81)^{0.05}(0.16)^{0.22}(0.16)^{0.08}(0.25)^{0.1}\right\}^{0.143}\\
\left\{(0.25)^{0.12}(0.16)^{0.18}(0.25)^{0.1}(0.04)^{0.15}(0.49)^{0.05}(0.25)^{0.22}(0.36)^{0.08}(0.16)^{0.1}\right\}^{0.514}\\
\left\{(0.16)^{0.12}(0.64)^{0.18}(0.36)^{0.1}(0.16)^{0.15}(0.36)^{0.05}(0.25)^{0.22}(0.81)^{0.08}(0.04)^{0.1}\right\}^{0.343}
\end{array}\right]}}{\sqrt{\begin{array}{c}
\left\{(1.51)^{0.12}(1.75)^{0.18}(1.84)^{0.1}(1.91)^{0.15}(1.19)^{0.05}(1.84)^{0.22}(1.84)^{0.08}(1.75)^{0.1}\right\}^{0.143}\\
\left\{(1.75)^{0.12}(1.84)^{0.18}(1.75)^{0.1}(1.96)^{0.15}(1.51)^{0.05}(1.75)^{0.22}(1.64)^{0.08}(1.84)^{0.1}\right\}^{0.514}\\
\left\{(1.84)^{0.12}(1.36)^{0.18}(1.64)^{0.1}(1.84)^{0.15}(1.64)^{0.05}(1.75)^{0.22}(1.19)^{0.08}(1.96)^{0.1}\right\}^{0.343}\\
+\\
\left\{(0.49)^{0.12}(0.25)^{0.18}(0.16)^{0.1}(0.09)^{0.15}(0.81)^{0.05}(0.16)^{0.22}(0.16)^{0.08}(0.25)^{0.1}\right\}^{0.143}\\
\left\{(0.25)^{0.12}(0.16)^{0.18}(0.25)^{0.1}(0.04)^{0.15}(0.49)^{0.05}(0.25)^{0.22}(0.36)^{0.08}(0.16)^{0.1}\right\}^{0.514}\\
\left\{(0.16)^{0.12}(0.64)^{0.18}(0.36)^{0.1}(0.16)^{0.15}(0.36)^{0.05}(0.25)^{0.22}(0.81)^{0.08}(0.04)^{0.1}\right\}^{0.343}
\end{array}}} \right\rangle
$$

$$\langle 0.6260, 0.4583\rangle$$

$$\mathscr{L}_1 = \left\langle \frac{\sqrt{\begin{array}{c} \left\{(1.09)^{0.12}(1.49)^{0.18}(1.36)^{0.1}(1.25)^{0.15}(1.04)^{0.05}(1.16)^{0.22}(1.25)^{0.08}(1.81)^{0.1}\right\}^{0.143} \\ \left\{(1.49)^{0.12}(1.09)^{0.18}(1.36)^{0.1}(1.09)^{0.15)}(1.25)^{0.05}(1.16)^{0.22}(1.49)^{0.08}(1.09)^{0.1}\right\}^{0.514} \\ \left\{(1.25)^{0.12}(1.64)^{0.18}(1.49)^{0.1}(1.16)^{0.15}(1.16)^{0.05}(1.04)^{0.22}(1.64)^{0.08}(1.49)^{0.1}\right\}^{0.343} \\ - \\ \left\{(0.91)^{0.12}(0.51)^{0.18}(0.64)^{0.1}(0.75)^{0.15}(0.96)^{0.05}(0.84)^{0.22}(0.75)^{0.08}(0.19)^{0.1}\right\}^{0.143} \\ \left\{(0.51)^{0.12}(0.91)^{0.18}(0.64)^{0.1}(0.91)^{0.15}(0.75)^{0.05}(0.84)^{0.22}(0.51)^{0.08}(0.91)^{0.1}\right\}^{0.514} \\ \left\{(0.75)^{0.12}(0.36)^{0.18}(0.51)^{0.1}(0.84)^{0.15}(0.84)^{0.05}(0.96)^{0.22}(0.36)^{0.08}(0.51)^{0.1}\right\}^{0.343} \end{array}}}{\sqrt{\begin{array}{c} \left\{(1.09)^{0.12}(1.49)^{0.18}(1.36)^{0.1}(1.25)^{0.15}(1.04)^{0.05}(1.16)^{0.22}(1.25)^{0.08}(1.81)^{0.1}\right\}^{0.143} \\ \left\{(1.49)^{0.12}(1.09)^{0.18}(1.36)^{0.1}(1.09)^{0.15}(1.25)^{0.05}(1.16)^{0.22}(1.49)^{0.08}(1.09)^{0.1}\right\}^{0.514} \\ \left\{(1.25)^{0.12}(1.64)^{0.18}(1.49)^{0.1}(1.16)^{0.15}(1.16)^{0.05}(1.04)^{0.22}(1.64)^{0.08}(1.49)^{0.1}\right\}^{0.343} \\ + \\ \left\{(0.91)^{0.12}(0.51)^{0.18}(0.64)^{0.1}(0.75)^{0.15}(0.96)^{0.05}(0.84)^{0.22}(0.75)^{0.08}(0.19)^{0.1}\right\}^{0.143} \\ \left\{(0.51)^{0.12}(0.91)^{0.18}(0.64)^{0.1}(0.91)^{0.15}(0.75)^{0.05}(0.84)^{0.22}(0.51)^{0.08}(0.91)^{0.1}\right\}^{0.514} \\ \left\{(0.75)^{0.12}(0.36)^{0.18}(0.51)^{0.1}(0.84)^{0.15}(0.84)^{0.05}(0.96)^{0.22}(0.36)^{0.08}(0.51)^{0.1}\right\}^{0.343} \end{array}}}, \frac{\sqrt{2\left[\begin{array}{c} \left\{(0.64)^{0.12}(0.09)^{0.18}(0.49)^{0.1}(0.16)^{0.15}(0.16)^{0.05}(0.36)^{0.22}(0.64)^{0.08}(0.09)^{0.1}\right\}^{0.143} \\ \left\{(0.36)^{0.12}(0.16)^{0.18}(0.25)^{0.1}(0.81)^{0.15}(0.16)^{0.05}(0.36)^{0.22}(0.25)^{0.08}(0.64)^{0.1}\right\}^{0.514} \\ \left\{(0.49)^{0.12}(0.25)^{0.18}(0.16)^{0.1}(0.09)^{0.15}(0.81)^{0.05}(0.16)^{0.22}(0.16)^{0.08}(0.25)^{0.1}\right\}^{0.343} \end{array}\right]}}{\sqrt{\begin{array}{c} \left\{(1.36)^{0.12}(1.91)^{0.18}(1.51)^{0.1}(1.84)^{0.15}(1.84)^{0.05}(1.64)^{0.22}(1.36)^{0.08}(1.91)^{0.1}\right\}^{0.143} \\ \left\{(1.64)^{0.12}(1.84)^{0.18}(1.75)^{0.1}(1.19)^{0.15}(1.84)^{0.05}(1.64)^{0.22}(1.75)^{0.08}(1.36)^{0.1}\right\}^{0.514} \\ \left\{(1.51)^{0.12}(1.75)^{0.18}(1.84)^{0.1}(1.91)^{0.15}(1.19)^{0.05}(1.84)^{0.22}(1.84)^{0.08}(1.75)^{0.1}\right\}^{0.343} \\ + \\ \left\{(0.64)^{0.12}(0.09)^{0.18}(0.49)^{0.1}(0.16)^{0.15}(0.16)^{0.05}(0.36)^{0.22}(0.64)^{0.08}(0.09)^{0.1}\right\}^{0.143} \\ \left\{(0.36)^{0.12}(0.16)^{0.18}(0.25)^{0.1}(0.81)^{0.15}(0.16)^{0.05}(0.36)^{0.22}(0.25)^{0.08}(0.64)^{0.1}\right\}^{0.514} \\ \left\{(0.49)^{0.12}(0.25)^{0.18}(0.16)^{0.1}(0.09)^{0.15}(0.81)^{0.05}(0.16)^{0.22}(0.16)^{0.08}(0.25)^{0.1}\right\}^{0.343} \end{array}}} \right\rangle \tag{41}$$

$\langle 0.5387, 0.5299 \rangle$

Step 4. Use equation (1) $\mathbb{S} = a^2_{\mathscr{F}(\breve{d}_{ij})} - b^2_{\mathscr{F}(\breve{d}_{ij})}$ to compute the score values for all alternatives.

$\mathbb{S}(\mathscr{H}_1) = 0.0088$, $\mathbb{S}(\mathscr{H}_2) = 0.2855$, $\mathbb{S}(\mathscr{H}_3) = 0.1154$, $\mathbb{S}(\mathscr{H}_4) = 0.2268$, and $\mathbb{S}(\mathscr{H}_5) = 0.1677$.

Step 5. After calculation, we get the ranking of alternatives $\mathbb{S}(\mathscr{H}_2) > \mathbb{S}(\mathscr{H}_4) > \mathbb{S}(\mathscr{H}_5) > \mathbb{S}(\mathscr{H}_3) > \mathbb{S}(\mathscr{H}_1)$. So, $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1$.

Hence, the best alternative is $\mathfrak{H}^2$.

Table 6: Feature analysis of different models with a proposed model.

	Fuzzy information	Aggregated parameters information	Einstein aggregated parameters information	Multi-sub-attributes information of each attribute
IFWA [66]	✓	×	×	×
IFEWA [62]	✓	✓	✓	×
IFSWA [35]	✓	✓	×	×
IFHSWA [58]	✓	✓	×	✓
PFSWA [44]	✓	✓	×	×
PFEWA [27]	✓	✓	✓	×
PFSEOWA [46]	✓	✓	✓	×
PFHSWA [65]	✓	✓	×	✓
Proposed operator	✓	✓	✓	✓

Table 7: Comparison of proposed operators with some existing operators.

Approach	H^1	H^2	H^3	H^4	H^5	Alternatives ranking
PFWA operator	0.0039	0.0644	0.0433	−0.0179	−0.0376	$\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^1 > \mathfrak{H}^4 > \mathfrak{H}^5$
PFEWA operator	−0.3306	0.5957	0 .1383	−0.1661	0.1092	$\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^5 > \mathfrak{H}^4 > \mathfrak{H}^1$
PFSWA operator	0.0293	0.0938	0.0783	0.0694	0.0369	$\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^1$
PFHSWA operator	0.1975	0.3513	0.2632	0.2297	0.1204	$\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^4 > \mathfrak{H}^1 > \mathfrak{H}^5$
PFHSEWA operator	0.0088	0.2855	0.1154	0.2268	0.1677	$\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1$

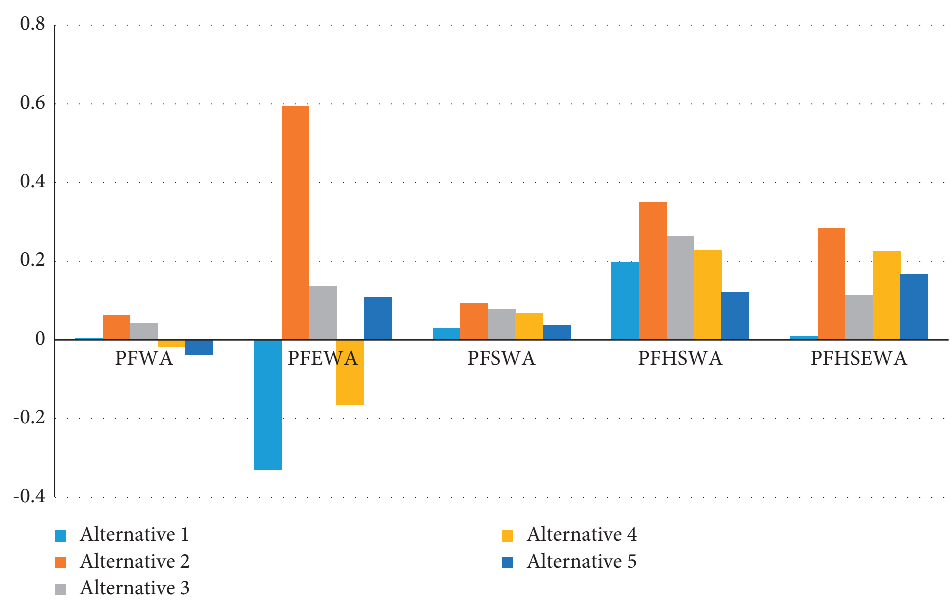

Figure 2: Graphical results of comparative studies.

5. Comparative Studies and Supremacy of the Proposed Model

To demonstrate the efficiency of the anticipated approach, some existing techniques under the PFS, PFSS, and proposed PFHSS model were compared.

5.1. Supremacy of the Proposed Method. The deliberate approach is proficient and convincing; we have constructed a pioneering MCDM model under the PFHSS setting over the PFHSEWA operator. Our advanced model is extra brilliant than prevailing techniques and can convey the utmost subtle connotations in MCDM obstacles. The collective model is flexible and conversant, adjusting to potential instability, commitment, and production. Different models have exhaustive ranking processes, so there is an instantaneous variance among the positions of the offered method to be realistic conferring to their conventions. This systematic exploration and estimation determine that outcomes gained from present techniques are volatile equated to hybrid structures. It is informal to syndicate inadequate and indeterminate facts in DM methods. Hence, our deliberated methodology will be more capable, authoritative, superior, and better than various hybrid-structured FS. Table 6 presents the supremacy analysis of the anticipated technique and some standing models.

5.2. Comparative Studies. To validate the usefulness of the projected technique, we compare the obtained results with some existing techniques under the environment of PFS and PFSS. A summary of all numerical and graphical outcomes is given in Table 7 and Figure 2. Firstly, we present a comparison with methods proposed by Siddique et al. [65] and Zulqarnain et al. [44]. Their proposed AOs are based on algebraic norms, while the proposed operators in this work are based on Einstein norms. Secondly, we compare the PFEWA operator proposed by Garg [15]. He developed the DM technique for PFNs by utilizing Einstein norms that cannot accommodate the parametrized values of the alternatives. On the other hand, our established approach competently deals with parametrized values of the alternatives and delivers better information than existing techniques. This work recommends innovative Einstein AO, such as PFHSEWA, to integrate the evaluation materials and then use the score function to calculate the substitute score. Therefore, it is inevitable that, based on the above facts, the plan operator in this work is more influential, consistent, and effective. The graphical ranking order of the alternatives of our proposed model with existing models is given in Figure 2.

6. Conclusion

Mathematical validation in agri-farming developments feats all resources while integrating objectives under economic, superior, and protection boundaries. Studies must be delimited for the most acceptable decision, accessing judgment requirements. In genuine DM, the assessment of alternative details carried by the expert is regularly incorrect, rough, and impetuous, so PFHSNs can be used to comport this indeterminate information. The core goal of this research is to use Einstein's norms to develop some operational laws for PFHSS. Then, a new operator, such as PFHSEWA, was developed according to the designed operational laws. In addition, some basic properties are proposed, such as the idempotence, homogeneity, and boundedness of the developed PFHSEWA operator. Furthermore, a DM approach has been designed to address MCDM problems based on endorsed operators. To certify the robustness of the settled approach, we provide an inclusive mathematical illustration for selecting the best agricultural robots in agri-farming. A comparative analysis with some current methods is presented. Finally, based on the outcomes attained, it is determined that the technique projected in this research is the most practical and effective way to solve the problem of MCDM. Future research focuses on developing more decision-making methods in the PFHSS environment using other operators, such as Einstein's hybrid geometric and Einstein's hybrid average operator.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Inforation and Control*, vol. 8, pp. 338–353, 1965.
[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
[3] W. Wang and X. Liu, "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations," *International Journal of Intelligent Systems*, vol. 26, no. 11, pp. 1049–1075, 2011.
[4] K. T. Atanassov, "Interval valued intuitionistic fuzzy sets," in *Intuitionistic Fuzzy Sets*, pp. 139–177, Physica, Heidelberg, Germany, 1999.
[5] H. Garg and G. Kaur, "Cubic intuitionistic fuzzy sets and their fundamental properties," *Journal of Multiple-Valued Logic and Soft Computing*, vol. 33, no. 6, 2019.
[6] R. R. Yager, "Pythagorean membership grades in multi-criteria decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2013.
[7] P. A. Ejegwa, "Pythagorean fuzzy set and its application in career placements based on academic performance using max-min-max composition," *Complex & Intelligent Systems*, vol. 5, no. 2, pp. 165–175, 2019.
[8] K. Rahman, S. Abdullah, R. Ahmed, and M. Ullah, "Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 33, no. 1, pp. 635–647, 2017.
[9] X. Zhang and Z. Xu, "Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 29, no. 12, pp. 1061–1078, 2014.
[10] G. Wei and M. Lu, "Pythagorean fuzzy power aggregation operators in multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 33, no. 1, pp. 169–186, 2018.
[11] L. Wang and N. Li, "Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 35, no. 1, pp. 150–183, 2020.
[12] E. Ilbahar, A. Karaşan, S. Cebi, and C. Kahraman, "A novel approach to risk assessment for occupational health and safety using Pythagorean fuzzy AHP & fuzzy inference system," *Safety Science*, vol. 103, pp. 124–136, 2018.
[13] X. Zhang, "A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making," *International Journal of Intelligent Systems*, vol. 31, no. 6, pp. 593–611, 2016.
[14] X. Peng and Y. Yang, "Some results for Pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 30, no. 11, pp. 1133–1160, 2015.
[15] H. Garg, "A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to

decision making," *International Journal of Intelligent Systems*, vol. 31, no. 9, pp. 886–920, 2016.

[16] H. Garg, "Generalized pythagorean fuzzy geometric aggregation operators using einsteint-norm andt-conorm for multicriteria decision-making process," *International Journal of Intelligent Systems*, vol. 32, no. 6, pp. 597–630, 2017.

[17] H. Garg, "New logarithmic operational laws and their aggregation operators for Pythagorean fuzzy set and their applications," *International Journal of Intelligent Systems*, vol. 34, no. 1, pp. 82–106, 2019.

[18] H. Gao, M. Lu, G. Wei, and Y. Wei, "Some novel Pythagorean fuzzy interaction aggregation operators in multiple attribute decision making," *Fundamenta Informaticae*, vol. 159, no. 4, pp. 385–428, 2018.

[19] L. Wang, H. Garg, and N. Li, "Pythagorean fuzzy interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight," *Soft Computing*, vol. 25, no. 2, pp. 973–993, 2021.

[20] M. Zulqarnain and F. Dayan, "Choose best criteria for decision making via fuzzy topsis method," *Mathematics and Computer Science*, vol. 2, no. 6, pp. 113–119, 2017.

[21] X. Peng and H. Yuan, "Fundamental properties of Pythagorean fuzzy aggregation operators," *Fundamenta Informaticae*, vol. 147, no. 4, pp. 415–446, 2016.

[22] L. Wang and H. Garg, "Algorithm for multiple attribute decision-making with interactive archimedean norm operations under pythagorean fuzzy uncertainty," *International Journal of Computational Intelligence Systems*, vol. 14, no. 1, pp. 503–527, 2021.

[23] K. Rahman, S. Abdullah, M. Shakeel, M. S. Ali Khan, and M. Ullah, "Interval-valued Pythagorean fuzzy geometric aggregation operators and their application to group decision making problem," *Cogent Mathematics*, vol. 4, no. 1, Article ID 1338638, 2017.

[24] L. Wang and N. Li, "Continuous interval-valued Pythagorean fuzzy aggregation operators for multiple attribute group decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 6, pp. 6245–6263, 2019.

[25] R. Arora and H. Garg, "Group decision-making method based on prioritized linguistic intuitionistic fuzzy aggregation operators and their fundamental properties," *Computational and Applied Mathematics*, vol. 38, no. 2, pp. 1–32, 2019.

[26] H. Garg, "New ranking method for normal intuitionistic sets under crisp, interval environments and its applications to multiple attribute decision making process," *Complex & Intelligent Systems*, vol. 6, no. 3, pp. 559–571, 2020.

[27] Z. Ma and Z. Xu, "Symmetric pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems," *International Journal of Intelligent Systems*, vol. 31, no. 12, pp. 1198–1219, 2016.

[28] D. Molodtsov, "Soft set theory—first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[29] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[30] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[31] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, pp. 589–602, 2001.

[32] P. K. Maji, R. Biswas, and A. R. Roy, "Intuitionistic fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, pp. 677–692, 2001.

[33] I. Deli and N. Çağman, "Intuitionistic fuzzy parameterized soft set theory and its decision making," *Applied Soft Computing*, vol. 28, pp. 109–113, 2015.

[34] H. Garg and R. Arora, "Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set," *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, no. 1, pp. 375–410, 2020.

[35] R. Arora and H. Garg, "A robust aggregation operators for multi-criteria decision-making with intuitionistic fuzzy soft set environment," *Scientia Iranica*, vol. 25, no. 2, pp. 931–942, 2018.

[36] H. Garg and R. Arora, "Generalized Maclaurin symmetric mean aggregation operators based on Archimedean t-norm of the intuitionistic fuzzy soft set information," *Artificial Intelligence Review*, vol. 54, no. 4, pp. 3173–3213, 2021.

[37] H. Garg, R. Arora, and R. Arora, "TOPSIS method based on correlation coefficient for solving decision-making problems with intuitionistic fuzzy soft set information," *AIMS Mathematics*, vol. 5, no. 4, pp. 2944–2966, 2020.

[38] P. Wang and P. Liu, "Some Maclaurin symmetric mean aggregation operators based on Schweizer-Sklar operations for intuitionistic fuzzy numbers and their application to decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 4, pp. 3801–3824, 2019.

[39] P. Liu and P. Wang, "Multiple-attribute decision-making based on Archimedean Bonferroni operators of q-rung orthopair fuzzy numbers," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 5, pp. 834–848, 2018.

[40] X. D. Peng, Y. Yang, J. Song, and Y. Jiang, "Pythagorean fuzzy soft set and its application," *Computer Engineering*, vol. 41, no. 7, pp. 224–229, 2015.

[41] T. M. Athira, S. J. John, S. Jacob John, and H. Garg, "A novel entropy measure of Pythagorean fuzzy soft sets," *AIMS Mathematics*, vol. 5, no. 2, pp. 1050–1061, 2020.

[42] T. M. Athira, S. J. John, and H. Garg, "Entropy and distance measures of Pythagorean fuzzy soft sets and their applications," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 3, pp. 4071–4084, 2019.

[43] K. Naeem, M. Riaz, X. Peng, and D. Afzal, "Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 5, pp. 6937–6957, 2019.

[44] R. M. Zulqarnain, X. L. Xin, H. Garg, and W. A. Khan, "Aggregation operators of Pythagorean fuzzy soft sets with their application for green supplier chain management," *Journal of Intelligent and Fuzzy Systems*, vol. 40, no. 3, pp. 5545–5563, 2021.

[45] R. M. Zulqarnain, X. L. Xin, H. Garg, and R. Ali, "Interaction aggregation operators to solve multi criteria decision making problem under pythagorean fuzzy soft environment," *Journal of Intelligent and Fuzzy Systems*, vol. 41, no. 1, pp. 1151–1171, 2021.

[46] R. M. Zulqarnain, I. Siddique, S. Ahmad et al., "Pythagorean fuzzy soft Einstein ordered weighted average operator in sustainable supplier selection problem," *Mathematical Problems in Engineering*, vol. 2021, 2021.

[47] R. M. Zulqarnain, I. Siddique, and S. EI-Morsy, "Einstein-ordered weighted geometric operator for pythagorean fuzzy soft set with its application to solve magdm problem," *Mathematical Problems in Engineering*, vol. 2022, Article ID 5199427, 14 pages, 2022.

[48] I. Siddique, R. M. Zulqarnain, R. Ali, A. Alburaikan, A. Iampan, and A. E. W. Khalifa, "A decision-making approach based on score matrix for pythagorean fuzzy soft set,"

Computational Intelligence and Neuroscience, vol. 2021, Article ID 5447422, 16 pages, 2021.

[49] R. M. Zulqarnain, X. L. Xin, I. Siddique, W. Asghar Khan, and M. A. Yousif, "TOPSIS method based on correlation coefficient under pythagorean fuzzy soft environment and its application towards green supply chain management," *Sustainability*, vol. 13, no. 4, p. 1642, 2021.

[50] F. Smarandache, "Extension of soft set to hypersoft set, and then to plithogenic hypersoft set," *Neutrosophic Sets and Systems*, vol. 22, no. 1, pp. 168–170, 2018.

[51] R. M. Zulqarnain, X. L. Xin, M. Saqlain, and F. Smarandache, "Generalized aggregate operators on neutrosophic hypersoft set," *Neutrosophic Sets and Systems*, vol. 36, pp. 271–281, 2020.

[52] A. U. Rahman, M. Saeed, M. Saeed, H. A. E.-W. Khalifa, and W. A. Afifi, "Decision making algorithmic techniques based on aggregation operations and similarity measures of possibility intuitionistic fuzzy hypersoft sets," *AIMS Mathematics*, vol. 7, no. 3, pp. 3866–3895, 2022.

[53] R. M. Zulqarnain, X. L. Xin, and M. Saeed, *A Development of Pythagorean Fuzzy Hypersoft Set with Basic Operations and Decision-Making Approach Based on the Correlation Coefficient, Theory and Application of Hypersoft Set*, Pons Publishing House, Brussels, Belgium, 2021.

[54] A. Ur Rahman, M. Saeed, S. Alodhaibi, and H. Abd El-Wahed Khalifa, "Decision making algorithmic approaches based on parameterization of neutrosophic set under hypersoft set environment with fuzzy, intuitionistic fuzzy and neutrosophic settings," *Computer Modeling in Engineering and Sciences*, vol. 128, no. 2, pp. 743–777, 2021.

[55] A. Samad, R. M. Zulqarnain, E. Sermutlu et al., "Selection of an effective hand sanitizer to reduce COVID-19 effects and extension of TOPSIS technique based on correlation coefficient under neutrosophic hypersoft set," *Complexity*, vol. 2021, Article ID 5531830, 22 pages, 2021.

[56] R. M. Zulqarnain, I. Siddique, R. Ali, F. Jarad, A. Samad, and T. Abdeljawad, "Neutrosophic hypersoft matrices with application to solve multiattributive decision-making problems," *Complexity*, vol. 2021, Article ID 5589874, 17 pages, 2021.

[57] M. Saeed, M. Ahsan, A. Ur Rahman, M. H. Saeed, and A. Mehmood, "An application of neutrosophic hypersoft mapping to diagnose brain tumor and propose appropriate treatment," *Journal of Intelligent and Fuzzy Systems*, no. Preprint, pp. 1–23, 2021.

[58] R. M. Zulqarnain, X. L. Xin, and M. Saeed, "Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision making problem," *AIMS Mathematics*, vol. 6, no. 3, pp. 2732–2755, 2020.

[59] R. M. Zulqarnain, I. Siddique, R. Ali, D. Pamucar, D. Marinkovic, and D. Bozanic, "Robust aggregation operators for intuitionistic fuzzy hypersoft set with their application to solve MCDM problem," *Entropy*, vol. 23, no. 6, p. 688, 2021.

[60] R. M. Zulqarnain, I. Siddique, F. Jarad, R. Ali, and T. Abdeljawad, "Development of TOPSIS technique under pythagorean fuzzy hypersoft environment based on correlation coefficient and its application towards the selection of antivirus mask in COVID-19 pandemic," *Complexity*, vol. 2021, Article ID 6634991, 27 pages, 2021.

[61] I. Deli, *Hybrid Set Structures under Uncertainly Parameterized Hypersoft Sets, Theory and Applications. Theory and Application of Hypersoft Set*, Pons Publishing House, Brussel, Belgium, 2021.

[62] W. Wang and X. Liu, "Intuitionistic fuzzy information aggregation using Einstein operations," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 5, pp. 923–938, 2012.

[63] P. Liu and P. Wang, "Multiple attribute group decision making method based on intuitionistic fuzzy Einstein interactive operations," *International Journal of Fuzzy Systems*, vol. 22, no. 3, pp. 790–809, 2020.

[64] M. Riaz, M. T. Hamid, D. Afzal, D. Pamucar, and Y.-M. Chu, "Multi-criteria decision making in robotic agri-farming with q-rung orthopair m-polar fuzzy sets," *PLoS One*, vol. 16, no. 2, Article ID e0246485, 2021.

[65] I. Siddique, R. M. Zulqarnain, R. Ali, F. Jarad, and A. Iampan, "Multicriteria decision-making approach for aggregation operators of pythagorean fuzzy hypersoft sets," *Computational Intelligence and Neuroscience*, vol. 2021, Article ID 2036506, 19 pages, 2021.

[66] Z. Zeshui Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 6, pp. 1179–1187, 2007.

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Two New Inertial Algorithms for Solving Variational Inequalities in Reflexive Banach Spaces

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Two New Inertial Algorithms for Solving Variational Inequalities in Reflexive Banach Spaces

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ABSTRACT
The purpose of this paper is to introduce and analyze two inertial algorithms with self-adaptive stepsizes for solving variational inequalities in reflexive Banach spaces. Our algorithms are based on inertial hybrid and shrinking projection methods. Knowledge of the Lipschitz constant of the cost operator is not required. Under appropriate conditions, the strong convergence of the algorithms is established. We also present several numerical experiments which bring out the efficiency and the advantages of the proposed algorithms. Our work provides extensions of many known results from Hilbert spaces to reflexive Banach spaces.

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1. Introduction

Let E be a real Banach space with norm $||\cdot||$ and let E^* be the dual of E. We denote by $\langle x,f\rangle$ the value of $f\in E^*$ at $x\in E$, that is, $\langle x,f\rangle = f(x)$. Let C be a nonempty, closed and convex subset of E and $A : C \to E^*$ be a continuous mapping. The *variational inequality problem* (VIP) is to find a point $z\in C$ such that

$$\langle x-z, Az\rangle \geq 0 \quad \forall x\in C. \tag{1.1}$$

The solution set of VIP is denoted by $VI(C, A)$. It is well known that the VIP is a general problem formulation that encompasses a plethora of mathematical problems, including systems of nonlinear equations, optimization

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problems, equilibrium problems, complementarity problems, and problems in game theory and in fixed point theory.

A classical method for solving the VIP in a Hilbert space H is the *projected gradient method*, which is defined by the following scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda A x_n), \end{cases} \tag{1.2}$$

where P_C is the projection operator onto the closed convex subset C of H and $\lambda > 0$ is a suitable stepsize. The sequences generated by this method converge weakly to a solution of the VIP if one of the following assumptions, $(a1)$ or $(a2)$, holds:

$(a1)$ A is α-strongly monotone and L-Lipschitz continuous, and $\lambda \in \left(0, \frac{2\alpha}{L^2}\right)$;
$(a2)$ A is γ-inverse strongly monotone and $\lambda \in (0, 2\gamma)$,

where $\gamma > 0$ and $L > 0$ are the strong monotonicity and Lipschitz constants, respectively, and $\alpha > 0$ is the inverse strong monotonicity constant of A. We remark that assumptions $(a1)$ and $(a2)$ are rather strong. However, when the strong monotonicity assumption is not satisfied, the gradient projection method may diverge (see [1]). To overcome this drawback, Korpelevich [2] introduced the following so-called *extragradient method* for solving the VIP in a finite-dimensional Euclidean space $\mathbb{R}^m$:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{1.3}$$

where $C \subset \mathbb{R}^m$ is a nonempty, closed and convex set, $A : C \to \mathbb{R}^m$ is monotone and L-Lipschitz continuous, and $\lambda \in \left(0, \frac{1}{L}\right)$. She showed that if *VI*(C, A) is nonempty, then the sequence $\{x_n\}$ generated by (1.3) converges to a point in *VI*(C, A). The idea of the extragradient method was successfully generalized and extended not only for Euclidean but also for infinite-dimensional Hilbert spaces. As a matter of fact, in recent years, the extragradient method has been studied by many authors, who have improved it in various ways (see, for example, [3–6]). Note that the extragradient method is based on two projections onto the feasible set C, that is, two projections onto C have to be computed in each iterative step. Therefore, in some cases, when the structure of the set C is not given explicitly or is complicated, the projections onto C might be difficult to compute and the implementation of the extragradient algorithm becomes less practical.

In order to improve upon the extragradient method, Tseng [6] proposed a modification of it, which later became known as *Tseng's extragradient method*. This algorithm is of the following form:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(Ay_n - Ax_n), \end{cases} \tag{1.4}$$

where $C \subset H$ is a nonempty, closed and convex set, $A : C \to H$ is monotone and L-Lipschitz continuous, and $\lambda \in \left(0, \frac{1}{L}\right)$. He then established the weak convergence of his method. Note that in Tseng's extragradient method only one projection is calculated at each iterative step. Thus it is simpler than the original extragradient method. For these reasons, the Tseng extragradient method has received considerable attention with the aim of obtaining weak and strong convergence theorems for it and for related methods (see, for example, [7–16]).

We also recall that in optimization theory, the inertial technique has been intensively used for accelerating the convergence rate of algorithms (see, for instance, [17–20]). This technique originates in an implicit time discretization method (the heavy ball method) of second-order dynamical systems. In [17], Alvarez and Attouch employed the idea of the heavy ball method in order to construct the following so-called *inertial proximal point algorithm* for finding a zero point of a maximal monotone operator $A : H \to 2^H$:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})), \end{cases} \tag{1.5}$$

where $J^A_{\lambda_n}$ is the resolvent of A and the term $x_n + \theta_n(x_n - x_{n-1})$ is said to be the *inertial term*. They proved that if the sequence $\{\lambda_n\}$ is increasing and $\theta_n \in [0, 1)$ are chosen so that $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}||^2 < \infty$, then the sequence $\{x_n\}$ generated by (1.5) converges weakly to a zero point of A.

Applying the technique in (1.5), Thong et al. [15] proposed an inertial algorithm for solving the VIP in a real Hilbert space by combining Tseng's extragradient algorithm and the hybrid projection algorithm introduced by Nakajo and Takahashi [21]. Given two real sequences $\{\theta_n\}$ and $\{\alpha_n\}$ such that $-\theta \leq \theta_n \leq \theta$ for some $\theta > 0$ and $0 \leq \alpha_n \leq \alpha < 1$, this algorithm is of the following form:

$$\begin{cases} x_0, x_1 \in C, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda A u_n), \\ z_n = \alpha_n u_n + (1 - \alpha_n)(y_n - \lambda(Ay_n - Au_n)), \\ C_n = \{w \in H : ||z_n - w|| \leq ||u_n - w||\}, \\ Q_n = \{w \in H : \langle w - x_n, x_1 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1). \end{cases} \tag{1.6}$$

In addition, they also proposed another inertial algorithm for solving the VIP. This algorithm is based on the shrinking projection algorithm introduced by Takahashi et al. [22]. It is of following form:

$$\begin{cases} C_1 = C, \\ x_0, x_1 \in C, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda A u_n), \\ z_n = \alpha_n u_n + (1-\alpha_n)(y_n - \lambda(Ay_n - Au_n)), \\ C_{n+1} = \{w \in H : ||z_n - w|| \leq ||u_n - w||\}, \\ x_{n+1} = P_{C_{n+1}}(x_1). \end{cases} \tag{1.7}$$

They proved that the sequences $\{x_n\}$ generated by (1.6) and (1.7) converge strongly to a point in $VI(C, A)$ provided $\lambda \in \left(0, \frac{1}{L}\right)$.

It is definitely of interest to extend Tseng's result to certain Banach spaces. Indeed, very recently Shehu [23] has proposed the following algorithm for approximating a solution to the VIP in a 2-uniformly convex and 2-uniformly smooth Banach space E:

$$\begin{cases} x_1 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \end{cases} \tag{1.8}$$

where $A : E \to E^*$ is monotone and L-Lipschitz continuous, Π_C is the generalized projection from E onto C, J is the normalized duality mapping on E and the sequence of stepsizes satisfies the following inequality:

$$0 < a \leq \lambda_n \leq b < \frac{1}{\sqrt{2\mu\kappa}L},$$

where $\mu > 0$ is the 2-uniform convexity constant of E and $\kappa > 0$ is its 2-uniform smoothness constant. He proved that the sequence $\{x_n\}$ generated by (1.8) converges weakly to a point in $VI(C, A)$ when J is assumed to be weakly sequentially continuous. It is, however, known that this assumption is very severe; it is not even satisfied by the function spaces $L_p([a, b])$ and W_p^m $(m \leq p, p \neq 2)$. Moreover, the choice of the sequence of stepsizes depends on the Lipschitz constant of the mapping. This is quite restrictive in applications. At the same time, he has also proposed a variant of (1.8), with a linesearch, for solving the VIP. Given $\gamma > 0$, $l \in (0, 1)$ and $\theta \in \left(0, \frac{1}{\sqrt{2\mu\kappa}}\right)$, this variant is also of the following form:

$$\begin{cases} x_1 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \end{cases} \tag{1.9}$$

Shehu [23] has proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to a point in $VI(C, A)$ when J is assumed to be weakly sequentially continuous and λ_n is chosen to be the largest

$$\lambda \in \{\gamma, \gamma l, \gamma l^2, \ldots\} \quad \text{satisfying} \quad \lambda||Ax_n - Ay_n|| \leq \theta||x_n - y_n||.$$

Like any algorithm with a linesearch, this algorithm needs an inner loop with a stopping criterion at each iterative step.

Motivated and inspired by the works of Attouch and Alvarez [17], Shehu [23], and Thong et al. [15], we introduce in this paper two inertial extragradient algorithms of Tseng's type with adaptive stepsizes for solving variational inequalities in the framework of reflexive Banach spaces. In our first algorithm we use the hybrid projection method and in the second one we employ the shrinking projection method. We establish two strong convergence theorems without assuming the weak sequential continuity of any mapping. The main advantages of both algorithms are that the sequence of stepsizes is chosen without the knowledge of the Lipschitz constant of the corresponding operator and without any linesearch procedure. This approach allows the algorithms to be computed more easily. Our paper is organized as follows: In Section 2 we collect some preliminary definitions and results which are needed for our subsequent analysis. The strong convergence theorems for the algorithms we propose are established in Section 3. Several numerical results, as well as comparisons with other algorithms, are presented in Section 4.

2. Preliminaries

Throughout this paper, we let E be a real reflexive Banach space and E^* be its dual. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and the weak convergence of the sequence $\{x_n\} \subset E$ to $x \in E,$ respectively. The weak ω-limit set of the sequence $\{x_n\}$ is defined by

$$\omega_w(x_n) := \{x \in E : x_{n_k} \rightharpoonup x \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}.$$

Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function and let its domain be defined by $\text{dom} f := \{x \in E : f(x) < \infty\}.$ The *subdifferential* of f is defined by

$$\partial f(x) := \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) \ \ \forall y \in E\}, \ \ x \in E.$$

The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, \infty]$ defined by

$$f^*(x^*) := \sup_{x \in E}\{\langle x, x^* \rangle - f(x)\}.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x, x^* \rangle$ (see [24, Theorem 23.5]).

For any $x \in \text{int}(\text{dom}\, f)$ and $y \in E,$ the *directional derivative* of f at x in the direction $y \in E$ is given by

$$f'(x,y) := \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteaux differentiable* at x if the limit as $t \to 0$ in (2.1) exists for each y. In this case, the *gradient* of f at x is the linear function $\nabla f(x) : E \to E^*$ defined by $\langle y, \nabla f(x)\rangle = f'(x,y)$ for each $y \in E$. For more details regarding the gradient, we recommend [25, Remark 3.32]. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. It is known that if f is continuous at x and $\partial f(x)$ is single-valued, then f is Gâteaux differentiable at x and $\nabla f(x) = \partial f(x)$ (see [26, Proposition 2.40]). The function f is said to be *Fréchet differentiable* at x if the limit (2.1) is attained uniformly for $||y|| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (2.1) is attained uniformly for $x \in C$ and $||y|| = 1$. It is also known that every Fréchet differentiable function is Gâteaux differentiable and if f is Fréchet differentiable, then it is continuous. However, if f is Gâteaux differentiable, then it is not necessarily continuous (see [27, p. 142]).

Proposition 2.1. ([28, Proposition 1]) *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 2.2. ([29, p. 25]) *The function $f : E \to (-\infty, \infty]$ is said to be Legendre if it satisfies the following two conditions:*

(I.1) $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f)$;
(I.2) $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$.

Remark 2.3. In a reflexive Banach space, we always have $(\partial f)^{-1} = \partial f^*$ (see [30, p. 83]). This fact, when combined with conditions (L1) and (L2), implies the following two facts:

i. ∇f is a bijection from $\operatorname{int}(\operatorname{dom} f)$ into $\operatorname{int}(\operatorname{dom} f^*)$ satisfying $\nabla f = (\nabla f^*)^{-1}$ (see [31, Theorem 5.10]);
ii. $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f)$ (see [32, p. 123]),

where $\operatorname{ran}\nabla f$ denotes the range of ∇f. Also, conditions (L1) and (L2), in conjunction with [31, Theorem 5.4], imply that the functions f and f^* are essentially strictly convex on the interior of their respective domains.

Definition 2.4. ([33]) Let $f : E \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function on $\mathrm{int}(\mathrm{dom} f)$. The bifunction $D_f : \mathrm{dom} f \times \mathrm{int}(\mathrm{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$$

is called the *Bregman distance* with respect to *f*.

In general, the Bregman distance is not a metric because it is not symmetric and it does not satisfy the triangle inequality. Note that $D_f(x, x) = 0$, but $D_f(x, y) = 0$ may not imply that $x = y$.

Example 2.5. If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}||x||^2$ for all $x \in E$, then $\nabla f(x) = Jx$, where J is the normalized duality mapping defined by $Jx := \{j \in E^* : \langle x, j\rangle = ||x||^2 = ||j||^2\}$. Then $D_f(x, y) = \frac{1}{2}\phi(x, y)$, where ϕ is the Lyapunov functional which was studied in [34, 35]. For a 2-uniformly convex and smooth Banach space E, this Lyapunov functional satisfies the following inequality:

$$\phi(x, y) \geq c||x - y||^2, \tag{2.2}$$

where $c > 0$ is the 2-uniform convexity constant of E (see [36, Lemma 2.3]). It is well known that $\phi(x, y) = ||x - y||^2$ and $c = 1$ whenever E is a Hilbert space (Figure 1).

For more examples of Bregman distances, we recommend [37–41].

The Bregman distance has the following two important properties which are called the *two-point identity* and the *three-point identity*, respectively: for any $x, y \in \mathrm{int}(\mathrm{dom} f)$, we have

$$D_f(x, y) + D_f(y, x) = \langle x - y, \nabla f(x) - \nabla f(y)\rangle, \tag{2.3}$$

and for any $x \in \mathrm{dom} f$ and $y, z \in \mathrm{int}(\mathrm{dom} f)$, we have

$$D_f(x, y) = D_f(x, z) - D_f(y, z) + \langle x - y, \nabla f(z) - \nabla f(y)\rangle. \tag{2.4}$$

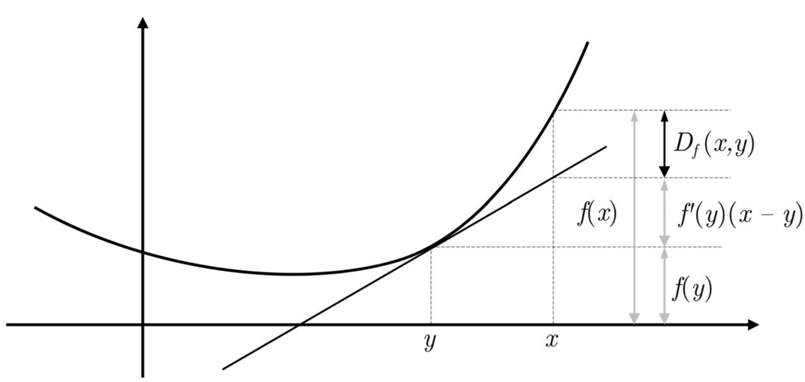

Figure 1. Bregman distance with respect to *f*.

The *modulus of total convexity* of f at $x \in \operatorname{int}(\operatorname{dom} f)$ is the function $v_f : \operatorname{int}(\operatorname{dom} f) \times [0, \infty) \to [0, \infty]$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \operatorname{dom} f, ||y - x|| = t\}.$$

The function f is called *totally convex* at x if $v_f(x, t) > 0$ whenever $t > 0$. It is called *totally convex* if it is totally convex at any point $x \in \operatorname{int}(\operatorname{dom} f)$. The *modulus of total convexity* of the function f on the set X is the function $v_f : \operatorname{int}(\operatorname{dom} f) \times [0, \infty) \to [0, \infty]$ defined by

$$v_f(X, t) := \inf\{v_f(x, t) : x \in X \cap \operatorname{dom} f\}.$$

The function f is said to be *totally convex on bounded sets* of E if $v_f(X, t) > 0$ for any nonempty bounded subset X of E and $t > 0$.

Let $B_r := \{x \in E : ||x|| \leq r\}$ for all $r > 0$. Then a function $f : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets* of E if $\rho_r(t) > 0$ for all r, $t > 0$, where $\rho_r : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_r(t) = \inf_{x, y \in B_r, ||x-y||=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$. It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [42, Theorem 2.10]).

A Gâteaux differentiable function f is said to be *σ-strongly convex* if there exists a constant $\sigma > 0$ such that

$$f(x) \geq f(y) + \langle x - y, \nabla f(y) \rangle + \frac{\sigma}{2} ||x - y||^2 \quad \forall x \in \operatorname{dom} f, \;\; y \in \operatorname{int}(\operatorname{dom} f).$$

It is clear that if f is a σ-strongly convex function then it is a uniformly convex function. From the definition of Bregman distance, it then follows that

$$D_f(x, y) \geq \frac{\sigma}{2} ||x - y||^2. \tag{2.5}$$

The *Bregman projection* with respect to f of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed and convex set $C \subset \operatorname{dom} f$ is the minimizer over C defined by

$$\Pi_C^f(x) := \operatorname{argmin}\{D_f(y, x) : y \in C\}.$$

If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}||x||^2$ for all $x \in E$, then Π_C^f coincides with the generalized projection Π_C (see [43, Definition 7.2]). If E is a Hilbert space, then Π_C^f coincides the metric projection P_C.

Lemma 2.6. ([42, Corollary 4.4]) *Suppose that f is Gâteaux differentiable and totally convex on $\operatorname{int}(\operatorname{dom} f)$. Let $x \in \operatorname{int}(\operatorname{dom} f)$ and let C be a* *nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$*. If* $z \in C$*, then the following statements are equivalent:*

i. $z = \Pi_C^f(x)$ *is the Bregman projection of* x *onto* C *with respect to* f*;*
ii. z *is the unique solution of the following variational inequality:*

$$\langle y - z, \nabla f(x) - \nabla f(z) \rangle \leq 0 \quad \forall y \in C;$$

iii. z *is the unique solution of the following inequality:*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \quad \forall y \in C.$$

Lemma 2.7. ([29, Lemma 3.2]) *Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function,* $x \in E$ *and let* C *be a nonempty, closed and convex subset of* E*. Suppose that the sequence* $\{x_n\}$ *is bounded and that any weak subsequential limit of* $\{x_n\}$ *belongs to* C*. If* $D_f(x_n, x) \leq D_f(\Pi_C^f(x), x)$ *for all* $n \in \mathbb{N}$*, then* $\{x_n\}$ *converges strongly to* $\Pi_C^f(x)$*.*

Lemma 2.8. ([29, Lemma 3.1]) *Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. Suppose that* $x \in E$*. If* $\{D_f(x_n, x)\}$ *is bounded, then the sequence* $\{x_n\}$ *is bounded.*

Lemma 2.9. ([44, Proposition 2.2]) *If* $x \in \operatorname{int}(\operatorname{dom} f)$*, then the following statements are equivalent:*

i. *The function* f *is totally convex at* x*;*
ii. *For any sequence* $\{y_n\} \subset \operatorname{dom} f$*,*

$$\lim_{n\to\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x|| = 0.$$

Recall that the function f is called *sequentially consistent* [42, p. 9], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\operatorname{dom} f$ and $\operatorname{int}(\operatorname{dom} f)$, respectively, such that the first one is bounded and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.10. ([45, Lemma 2.1.2]) *The function* $f : E \to (-\infty, \infty]$ *is sequentially consistent if and only if it is totally convex on bounded subsets of* E*.*

It follows from Lemma 2.10 that if we assume, in addition, that f is a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subsets of E, then for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\operatorname{dom} f$ and $\operatorname{int}(\operatorname{dom} f)$, respectively, such that the first one is bounded, we have

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Rightarrow \lim_{n\to\infty} ||x_n - y_n|| = 0 \Rightarrow \lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(y_n)|| = 0.$$

Definition 2.11. Let C be a nonempty subset of E. A mapping $A : C \to E^*$ is said to be:

1. *monotone* if $\langle x - y, Ax - Ay \rangle \geq 0$ for all $x, y \in C$;
2. *Lipschitz continuous* if there exists a constant $L > 0$ such that $||Ax - Ay|| \leq L||x - y||$ for all $x, y \in C$;
3. *hemicontinuous* if for each $x, y \in C$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) := A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*.

Lemma 2.12. [36, Proposition 2.6] *Let C be a nonempty, closed and convex subset of E, and A be a monotone and hemicontinuous mapping of C into E^*. Then VI(C, A) is closed and convex.*

Lemma 2.13. [46, Lemma 7.1.7] *Let C be a nonempty and convex subset of a Banach space E, and let A be a hemicontinuous mapping of C into E^*. Let z be an element of C such that*

$$\langle x - z, Ax \rangle \geq 0 \quad \forall x \in C.$$

Then z is a solution to the VIP.

3. Main results

In this section we propose two new inertial Tseng extragradient algorithms for solving monotone variational inequality problems in reflexive Banach spaces. In order to estabish convergence theorems for the proposed algorithms, we make the following assumption:

Assumption 3.1.

i. *The feasible set C is a nonempty, closed and convex subset of a real reflexive Banach space E.*
ii. *The function $f : E \to \mathbb{R}$ is σ-strongly convex and Legendre, and is bounded, uniformly Fréchet differentiable on bounded subsets of E.*
iii. *The mapping $A : E \to E^*$ is monotone and Lipschitz continuous with a constant $L > 0$.*
iv. *The solution set of the VIP is nonempty, that is, $\Omega := VI(C, A) \neq \emptyset$.*

Our first algorithm is based on the concept of the so-called *hybrid projection method.*

Algorithm 1: Inertial hybrid Tseng extragradient algorithm

Given $\lambda_1 > 0, -\theta \leq \theta_n \leq \theta$ for some $\theta > 0$ and $\mu \in (0, \sigma)$, where σ is the constant which appears in (2.5), let $x_0, x_1 \in E$ be arbitrary. Set $n := 1$.

Step 1. Compute

$$u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))).$$

Step 2. Compute

$$y_n = \Pi_C^f(\nabla f^*(\nabla f(u_n) - \lambda_n A u_n)).$$

If $y_n = u_n$, then stop and y_n is a solution of the VIP. Otherwise,

Step 3. Compute

$$z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Au_n)).$$

Step 4. Construct two half-spaces

$$\begin{cases} C_n = \{w \in E : D_f(w, z_n) \leq D_f(w, u_n) - \xi_n\}, \\ Q_n = \{w \in E : \langle w - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \leq 0\} \end{cases}$$

and compute

$$x_{n+1} = \Pi^f_{C_n \cap Q_n}(x_1),$$

where the stepsize is adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu||u_n - y_n||}{||Au_n - Ay_n||}, \lambda_n\right\} & \text{if } Au_n - Ay_n \neq 0, \\ \lambda_n & \text{otherwise} \end{cases} \tag{3.1}$$

and

$$\xi_n := \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, u_n) + \left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n). \tag{3.2}$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 3.2. *The sequence $\{\lambda_n\}$ generated by (3.1) is decreasingand*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\frac{\mu}{L}, \lambda_1\right\}.$$

Proof. It obviously follows from (3.1) that $\lambda_{n+1} \leq \lambda_n$ for all $n \geq 1$. Since A is Lipschitz continuous with constant $L > 0$, in the case where $Au_n - Ay_n \neq 0$, we have

$$\frac{\mu||u_n - y_n||}{||Au_n - Ay_n||} \geq \frac{\mu||u_n - y_n||}{L||u_n - y_n||} = \frac{\mu}{L}.$$

Clearly,

$$\lambda_{n+1} \geq \min\left\{\frac{\mu}{L}, \lambda_n\right\}.$$

Using induction, we immediately obtain that the sequence $\{\lambda_n\}$ is bounded from below by $\min\{\frac{\mu}{L}, \lambda_1\}$. Thus there exists $\lambda := \lim_{n\to\infty} \lambda_n \geq \min\{\frac{\mu}{L}, \lambda_1\}$. □

Lemma 3.3. *Suppose that Assumption 3.1 is satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then for each $n \geq 1$, we have*

$$D_f(p, z_n) \leq D_f(p, u_n) - \xi_n \quad \forall p \in \Omega. \tag{3.3}$$

Proof. Take $p \in \Omega$. By the definition of the Bregman distance, we have

$$\begin{aligned}
D_f(p, z_n) &= D_f(p, \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Au_n))) \\
&= f(p) - f(z_n) - \langle p - z_n, \nabla f(y_n) - \lambda_n(Ay_n - Au_n)\rangle \\
&= f(p) - f(z_n) - \langle p - z_n, \nabla f(y_n)\rangle + \lambda_n\langle p - z_n, Ay_n - Au_n\rangle \\
&= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle + \langle p - y_n, \nabla f(y_n)\rangle + f(y_n) - f(z_n) - \langle p - z_n, \nabla f(y_n)\rangle \\
&\quad + \lambda_n\langle p - z_n, Ay_n - Au_n\rangle \\
&= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle - f(z_n) + f(y_n) + \langle z_n - y_n, \nabla f(y_n)\rangle + \lambda_n\langle p - z_n, Ay_n - Au_n\rangle \\
&= D_f(p, y_n) - D_f(z_n, y_n) + \lambda_n\langle p - z_n, Ay_n - Au_n\rangle.
\end{aligned} \tag{3.4}$$

Using (2.4), we see that

$$D_f(p, y_n) = D_f(p, u_n) - D_f(y_n, u_n) + \langle p - y_n, \nabla f(u_n) - \nabla f(y_n)\rangle. \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$\begin{aligned}
D_f(p, z_n) = D_f(p, u_n) - D_f(y_n, u_n) - D_f(z_n, y_n) + \langle p - y_n, \nabla f(u_n) - \nabla f(y_n)\rangle \\
+ \lambda_n\langle p - z_n, Ay_n - Au_n\rangle.
\end{aligned} \tag{3.6}$$

Note that $y_n = \Pi_C^f(\nabla f^*(\nabla f(u_n) - \lambda_n Au_n))$, by Lemma 2.6, we have

$$\langle p - y_n, \nabla f(u_n) - \lambda_n Au_n - \nabla f(y_n)\rangle \leq 0.$$

This implies that

$$\langle p - y_n, \nabla f(u_n) - \nabla f(y_n)\rangle \leq \lambda_n\langle p - y_n, Au_n\rangle. \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$\begin{aligned}D_f(p,z_n) &\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle p-y_n,Au_n\rangle+\lambda_n\langle p-z_n,Ay_n-Au_n\rangle\\&=D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle p-y_n,Au_n\rangle+\lambda_n\langle p-z_n,Ay_n\rangle-\lambda_n\langle p-z_n,Au_n\rangle\\&=D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle z_n-y_n,Au_n\rangle+\lambda_n\langle p-z_n,Ay_n\rangle\\&=D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle z_n-y_n,Au_n\rangle-\lambda_n\langle y_n-p,Ay_n\rangle+\lambda_n\langle y_n-z_n,Ay_n\rangle\\&=D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle z_n-y_n,Au_n-Ay_n\rangle-\lambda_n\langle y_n-p,Ay_n\rangle.\end{aligned}\tag{3.8}$$

Using now the fact that $\langle y_n-p,Ap\rangle\ge 0$ and the monotonicity of A, we obtain $\langle y_n-p,Ay_n\rangle\ge 0$. Using (3.1), we observe that

$$\lambda_{n+1}=\min\left\{\frac{\mu||u_n-y_n||}{||Au_n\ -\ Ay_n||},\lambda_n\right\}\le\frac{\mu||u_n-y_n||}{||Au_n\ -\ Ay_n||},$$

which implies that

$$||Au_n-Ay_n||\le\frac{\mu}{\lambda_{n+1}}||u_n-y_n||.\tag{3.9}$$

It now follows from (3.8), (3.9) and (2.5) that

$$\begin{aligned}D_f(p,z_n)&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n\langle z_n-y_n,Au_n-Ay_n\rangle\\&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\lambda_n||z_n-y_n||||Au_n-Ay_n||\\&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\mu\frac{\lambda_n}{\lambda_{n+1}}||z_n-y_n||||u_n-y_n||\\&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\frac{\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}||z_n-y_n||^2+\frac{\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}||u_n-y_n||^2\\&\le D_f(p,u_n)-D_f(y_n,u_n)-D_f(z_n,y_n)+\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}D_f(z_n,y_n)+\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}D_f(y_n,u_n)\\&=D_f(p,u_n)-\left(1-\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,u_n)-\left(1-\frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(z_n,y_n).\end{aligned}\tag{3.10}$$

By the definition of ξ_n, we can rewrite (3.10) as follows:

$$D_f(p,z_n)\le D_f(p,u_n)-\xi_n.$$

This completes the proof of the lemma. □

Theorem 3.4. *Suppose that Assumption 3.1 is satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $z=\Pi^f_\Omega(x_1)$.*

Proof. **Claim 1.** The set $C_n\cap Q_n$ is closed and convex for all $n\in\mathbb{N}$.

From the construction of Q_n, it is also obvious that Q_n is a closed and convex.

Appealing to the definition of C_n, we observe that $w\in C_n$ if and only if

$$D_f(w,z_n)\le D_f(w,u_n)-\xi_n,$$

an inequality which is equivalent to

$$\langle w, \nabla f(z_n) - \nabla f(u_n)\rangle \le f(z_n) - f(u_n) - \langle z_n, \nabla f(z_n)\rangle + \langle u_n, \nabla f(u_n)\rangle - \xi_n.$$

It is now obvious that C_n is both closed and convex for each $n \in \mathbb{N}$. Hence $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$, as claimed.

Claim 2. The sequence $\{x_n\}$ is well defined.

Let $p \in \Omega$. Using Lemma 3.3, we have

$$D_f(p, z_n) \le D_f(p, u_n) - \xi_n.$$

This mean that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. For $n = 1$, we have $Q_1 = E$ and hence $\Omega \subset C_1 \cap Q_1$. Suppose now that $\Omega \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. Then $x_{k+1} := \Pi^f_{C_k \cap Q_k}(x_1)$ is well defined. Using Lemma 2.6, we know that

$$\langle w - x_{k+1}, \nabla f(x_1) - \nabla f(x_{k+1})\rangle \le 0 \quad \forall w \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we obviously have

$$\langle y - x_{k+1}, \nabla f(x_1) - \nabla f(x_{k+1})\rangle \le 0 \quad \forall y \in \Omega.$$

This implies that $\Omega \subset Q_{k+1}$ and hence $\Omega \subset C_{k+1} \cap Q_{k+1}$. Applying induction, we conclude that $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$ and hence $\{x_n\}$ is well defined, as claimed.

Claim 3. The sequences $\{x_n\}$ and $\{u_n\}$ are bounded.

The definition of Q_n implies that

$$\langle y - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \le 0 \quad \forall y \in Q_n. \tag{3.11}$$

By Lemma 2.6, we have $x_n = \Pi^f_{Q_n}(x_1)$ and so

$$D_f(x_n, x_1) \le D_f(y, x_1) - D_f(y, x_n) \le D_f(y, x_1) \tag{3.12}$$

for all $y \in Q_n$. Since $\Omega \subset Q_n$, we have $D_f(x_n, x_1) \le D_f(p, x_1)$ for all $p \in \Omega$. This implies that the real sequence $\{D_f(x_n, x_1)\}$ is bounded. Applying Lemma 2.8, we see that $\{x_n\}$ is bounded and consequently so is $\{u_n\}$.

Claim 4. We have $\lim_{n\to\infty} ||y_n - u_n|| = \lim_{n\to\infty} ||z_n - y_n|| = 0$.

Since $x_{n+1} \in Q_n$, inequality (3.12) implies that $D_f(x_n, x_1) \le D_f(x_{n+1}, x_1)$. Thus the real sequence $\{D_f(x_n, x_1)\}$ is increasing and so $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Again since $x_{n+1} \in Q_n$, it follows from (2.4) and (3.11) that

$$\begin{aligned} D_f(x_{n+1}, x_n) &= D_f(x_{n+1}, x_1) - D_f(x_n, x_1) + \langle x_{n+1} - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \\ &\le D_f(x_{n+1}, x_1) - D_f(x_n, x_1) \\ &\to 0 \end{aligned}$$

as $n \to \infty$, which implies that

$$\lim_{n\to\infty} ||\nabla f(x_{n+1}) - \nabla f(x_n)|| = 0.$$

Moreover, using the definition of u_n, we see that

$$\begin{aligned}||\nabla f(u_n) - \nabla f(x_n)|| &= |\theta_n|||\nabla f(x_n) - \nabla f(x_{n-1})|| \\ &\le \theta||\nabla f(x_n) - \nabla f(x_{n-1})|| \\ &\to 0\end{aligned}$$

as $n \to \infty$. Thus we have

$$\begin{aligned}||\nabla f(x_{n+1}) - \nabla f(u_n)|| &\le ||\nabla f(x_{n+1}) - \nabla f(x_n)|| + ||\nabla f(x_n) - \nabla f(u_n)|| \\ &\to 0\end{aligned}$$

as $n \to \infty$. It follows from (2.3) that

$$\begin{aligned}D_f(x_{n+1}, u_n) &\le \langle x_{n+1} - u_n, \nabla f(x_{n+1}) - \nabla f(u_n)\rangle \\ &\le M||\nabla f(x_{n+1}) - \nabla f(u_n)|| \\ &\to 0 \end{aligned} \tag{3.13}$$

as $n \to \infty$, where $M > 0$. Since $x_{n+1} \in C_n$, we have

$$D_f(x_{n+1}, z_n) \le D_f(x_{n+1}, u_n) - \xi_n. \tag{3.14}$$

On the other hand, since $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and $\mu \in (0, \sigma)$, we have

$$\lim_{n\to\infty}\left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \frac{\mu}{\sigma} = \frac{\sigma - \mu}{\sigma} > 0.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}} > 0 \quad \forall n \ge n_0.$$

From this it follows that $\xi_n \ge 0$ for all $n \ge n_0$. Using (3.14), we have

$$D_f(x_{n+1}, z_n) \le D_f(x_{n+1}, u_n) - \xi_n \le D_f(x_{n+1}, u_n) \quad \forall n \ge n_0.$$

It follows from (3.13) that

$$\lim_{n\to\infty} D_f(x_{n+1}, z_n) = 0$$

and hence

$$\lim_{n\to\infty} ||\nabla f(x_{n+1}) - \nabla f(z_n)|| = 0.$$

Thus we have

$$\begin{aligned}||\nabla f(z_n) - \nabla f(u_n)|| &\le ||\nabla f(z_n) - \nabla f(x_{n+1})|| + ||\nabla f(x_{n+1}) - \nabla f(u_n)|| \\ &\to 0\end{aligned} \tag{3.15}$$

as $n \to \infty$. Using Lemma 3.3, we have

$$\xi_n \le D_f(p, u_n) - D_f(p, z_n). \tag{3.16}$$

By (2.4), we have

$$D_f(p, u_n) = D_f(p, z_n) - D_f(u_n, z_n) + \langle p - u_n, \nabla f(z_n) - \nabla f(u_n) \rangle.$$

This implies that

$$\begin{aligned} D_f(p, u_n) - D_f(p, z_n) &= -D_f(u_n, z_n) + \langle p - u_n, \nabla f(z_n) - \nabla f(u_n) \rangle \\ &\le \langle p - u_n, \nabla f(z_n) - \nabla f(u_n) \rangle \\ &\le K||\nabla f(z_n) - \nabla f(u_n)||, \end{aligned}$$

where $K > 0$. By (3.15), we have

$$\lim_{n\to\infty} (D_f(p, u_n) - D_f(p, z_n)) = 0. \tag{3.17}$$

Since $\xi_n \ge 0$ for all $n \ge n_0$, it follows from (3.16), (3.17) and the definition of ξ_n that $\lim_{n\to\infty} D_f(y_n, u_n) = \lim_{n\to\infty} D_f(z_n, y_n) = 0$. Thus we have

$$\lim_{n\to\infty} ||y_n - u_n|| = \lim_{n\to\infty} ||z_n - y_n|| = 0,$$

as claimed.

Claim 5. We have $\omega_w(x_n) \subset \Omega$.

By the reflexivity of E and the boundedness of $\{x_n\}$, there do exist subsequences $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$. Consider such a subsequence. Since $||u_{n_k} - x_{n_k}|| \to 0$, we also have $u_{n_k} \rightharpoonup \hat{x}$. Moreover, since $||y_{n_k} - u_{n_k}|| \to 0$ and $y_n \in C$, we have $\hat{x} \in C$. From the definition of y_{n_k} it follows that

$$\langle x - y_{n_k}, \nabla f(u_{n_k}) - \lambda_{n_k} A u_{n_k} - \nabla f(y_{n_k}) \rangle \le 0 \quad \forall x \in C.$$

The monotonicity of A implies that

$$\begin{aligned} 0 &\le \langle x - y_{n_k}, \nabla f(y_{n_k}) - \nabla f(u_{n_k}) \rangle + \lambda_{n_k} \langle x - y_{n_k}, A u_{n_k} \rangle \\ &= \langle x - y_{n_k}, \nabla f(y_{n_k}) - \nabla f(u_{n_k}) \rangle + \lambda_{n_k} \langle x - u_{n_k}, A u_{n_k} \rangle + \lambda_{n_k} \langle u_{n_k} - y_{n_k}, A u_{n_k} \rangle \\ &\le \langle x - y_{n_k}, \nabla f(y_{n_k}) - \nabla f(u_{n_k}) \rangle + \lambda_{n_k} \langle x - u_{n_k}, A x \rangle + \lambda_{n_k} \langle u_{n_k} - y_{n_k}, A u_{n_k} \rangle. \end{aligned}$$

Since $||y_{n_k} - u_{n_k}|| \to 0$ and $\lambda_{n_k} \to \lambda > 0$, it follows that

$$\langle x - \hat{x}, Ax \rangle \ge 0 \quad \forall x \in C.$$

Using Lemma 2.13, we now get $\hat{x} \in \Omega$, as required.

Claim 6. We have $x_n \to z = \Pi^f_\Omega(x_1)$.

Let $z = \Pi^f_\Omega(x_1)$. Since $x_{n+1} = \Pi^f_{C_n \cap Q_n}(x_1)$ and $\Omega \subset C_n \cap Q_n$, we obtain $D_f(x_{n+1}, x_1) \le D_f(z, x_1)$. Therefore Lemma 2.7 allows us to conclude that the sequence $\{x_n\}$ converges strongly to $z = \Pi^f_\Omega(x_1)$.

This completes the proof of the theorem. □

If we take $f(x) = \frac{1}{2}||x||^2$ for all $x \in E$ in Theorem 3.4, then we can deduce the following results.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, and let $A : E \to E^*$ be a monotone and Lipschitz continuous mapping. Assume that $\Omega \neq \emptyset$. Given $\lambda_1 > 0$, $-\theta \leq \theta_n \leq \theta$ for some $\theta > 0$ and $\mu \in (0, c)$, where c is the constant which appears in (2.2), let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0, x_1 \in E, \\ u_n = J^{-1}(Jx_n + \theta_n(Jx_n - Jx_{n-1})), \\ y_n = \Pi_C(J^{-1}(Ju_n - \lambda_n Au_n)), \\ z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Au_n)), \\ C_n = \{w \in E : \phi(w, z_n) \leq \phi(w, u_n) - \xi'_n\}, \\ Q_n = \{w \in E : \langle w - x_n, Jx_1 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_1) \quad \forall n \geq 1, \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu||u_n - y_n||}{||Au_n - Ay_n||}, \lambda_n\right\} & \text{if} \quad Au_n - Ay_n \neq 0, \\ \lambda_n & \text{otherwise} \end{cases} \tag{3.18}$$

and

$$\xi'_n := \left(1 - \frac{\mu}{c}\frac{\lambda_n}{\lambda_{n+1}}\right)\phi(y_n, u_n) + \left(1 - \frac{\mu}{c}\frac{\lambda_n}{\lambda_{n+1}}\right)\phi(z_n, y_n). \tag{3.19}$$

Then $\{x_n\}$ converges strongly to $z = \Pi_\Omega(x_1)$.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $A : H \to H$ be a monotone and Lipschitz continuous mapping. Assume that $\Omega \neq \emptyset$. Given $\lambda_1 > 0$, $-\theta \leq \theta_n \leq \theta$ for some $\theta > 0$ and $\mu \in (0, 1)$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0, x_1 \in H, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda_n Au_n), \\ z_n = y_n - \lambda_n(Ay_n - Au_n), \\ C_n = \{w \in H : ||z_n - w||^2 \leq ||u_n - w||^2 - \xi''_n\}, \\ Q_n = \{w \in H : \langle w - x_n, x_1 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1) \quad \forall n \geq 1, \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu||u_n - y_n||}{||Au_n - Ay_n||}, \lambda_n\right\} & \text{if} \quad Au_n - Ay_n \neq 0, \\ \lambda_n & \text{otherwise} \end{cases} \tag{3.20}$$

and

$$\xi''_n := \left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)||y_n-u_n||^2+\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)||z_n-y_n||^2. \tag{3.21}$$

Then $\{x_n\}$ *converges strongly to* $z=P_\Omega(x_1)$.

Next, we propose another algorithm with a different construction of the sequence $\{C_n\}$. Our second algorithm is based on the concept of the so-called *shrinking projection method.*

Algorithm 2: Inertial shrinking Tseng extragradient algorithm

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Given $\lambda_1>0$, $-\theta\le\theta_n\le\theta$ for some $\theta>0$ and $\mu\in(0,\sigma)$, where σ is the constant which appears in (2.5), let $C_1=C$ and $x_0,x_1\in E$ be arbitrary. Set $n:=1$.

Step 1. Compute

$$u_n=\nabla f^*(\nabla f(x_n)+\theta_n(\nabla f(x_n)-\nabla f(x_{n-1}))).$$

Step 2. Compute

$$y_n=\Pi^f_C(\nabla f^*(\nabla f(u_n)-\lambda_n Au_n)).$$

If $y_n=u_n$, then stop and y_n is a solution to the VIP. Otherwise,

Step 3. Compute

$$z_n=\nabla f^*(\nabla f(y_n)-\lambda_n(Ay_n-Au_n)).$$

Step 3. Construct

$$C_{n+1}=\{w\in C_n:D_f(w,z_n)\le D_f(w,u_n)-\xi_n\}$$

and compute

$$x_{n+1}=\Pi^f_{C_{n+1}}(x_1),$$

where λ_n and ξ_n are defined in (3.1) and (3.2), respectively.

Set $n:=n+1$ and go to **Step 1**.

Theorem 3.7. *Suppose that Assumption 3.1 is satisfied. Then the sequence* $\{x_n\}$ *generated by Algorithm 2 converges strongly to* $z=\Pi^f_\Omega(x_1)$.

Proof. **Claim 1.** The set C_n is closed and convex for each $n\in\mathbb{N}$.

It is obvious that $C_1=C$ is closed and convex. Suppose that C_k is closed and convex for some $k\in\mathbb{N}$. Using the construction of C_{k+1}, we see that $C_{k+1}=C_k\cap D_k$, where

$$D_k=\{w\in E:D_f(w,z_k)\le D_f(w,u_k)-\xi_k\}.$$

As we have already seen in the proof of Theorem 3.4, the set D_k is closed and convex. Hence C_{k+1} is also closed and convex. Using induction, we conclude that the set C_n is closed and convex for all $n \in \mathbb{N}$, as claimed.

Claim 2. The sequence $\{x_n\}$ is well defined.

It is clear that $\Omega \subset C = C_1$. Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. Let $p \in \Omega$. From Lemma 3.3, we deduce that

$$D_f(p, z_k) \le D_f(p, u_k) - \xi_k \quad \forall p \in \Omega.$$

This implies that $\Omega \subset D_k$. Hence $\Omega \subset C_{k+1}$. Using induction, we conclude that $\Omega \subset C_n$ for all $n \in \mathbb{N}$ and hence $\{x_n\}$ is well defined, as claimed.

Claim 3. The sequence $\{x_n\}$ is bounded.

By following the argument which was used to prove Claim 3 in the proof of Theorem 3.4, we can show that the sequence $\{x_n\}$ is indeed bounded.

Claim 4. We have $x_n \to z$ for some $z \in C$.

Since $x_n = \Pi^f_{C_n}(x_1)$ and $x_{n+1} = \Pi^f_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$, it follows from Lemma 2.6 that

$$D_f(x_n, x_1) \le D_f(x_{n+1}, x_1) - D_f(x_{n+1}, x_n) \le D_f(x_{n+1}, x_1).$$

Thus the sequence $\{D_f(x_n, x_1)\}$ is increasing. So, $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Using the construction of C_n, we see that $x_m \in C_m \subset C_n$ for any positive integer $m \ge n$. Once again, since $x_n = \Pi^f_{C_n}(x_1)$, it follows from Lemma 2.6 that

$$D_f(x_m, x_n) \le D_f(x_m, x_1) - D_f(x_n, x_1) \to 0$$

as $m, n \to \infty$, which implies that

$$\lim_{m,\, n\to\infty} ||x_m - x_n|| = 0. \tag{3.22}$$

This mean that $\{x_n\}$ is a Cauchy sequence. Therefore there exists a point $z \in C$ such that $x_n \to z$ as $n \to \infty$, as claimed.

Claim 5. We have $z \in \Omega$.

By taking $m = n + 1$ in (3.22), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0.$$

Employing the argument which was used in Claim 4 in the proof of Theorem 3.4, we arrive at

$$\lim_{n\to\infty} ||u_n - x_n|| = \lim_{n\to\infty} ||y_n - u_n|| = 0.$$

Now, following in the footsteps of Claim 4 in the proof of Theorem 3.4, we can show that $z \in \Omega$, as claimed.

Claim 6. We have $z = \Pi^f_\Omega(x_1)$.

Indeed, since $x_n = \Pi^f_{C_n}(x_1)$ and $\Omega \subset C_n$, it follows from Lemma 2.6 that

$$\langle y - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \leq 0 \quad \forall y \in \Omega. \tag{3.23}$$

Taking limits as $n \to \infty$ in (3.23) and using the uniform continuity of ∇f, we conclude that

$$\langle y - z, \nabla f(x_1) - \nabla f(z)\rangle \leq 0 \quad \forall y \in \Omega.$$

This mean that $z = \Pi^f_\Omega(x_1)$, as claimed.

This completes the proof of the theorem. □

Corollary 3.8. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, and let $A : E \to E^*$ be a monotone and Lipschitz continuous mapping. Assume that $\Omega \neq \emptyset$. Given $\lambda_1 > 0$, $-\theta \leq \theta_n \leq \theta$ for some $\theta > 0$ and $\mu \in (0, c)$, where c is the constant which appears in (2.2), let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} C_1 = C, \\ x_0, x_1 \in E, \\ u_n = J^{-1}(Jx_n + \theta_n(Jx_n - Jx_{n-1})), \\ y_n = \Pi_C(J^{-1}(Ju_n - \lambda_n Au_n)), \\ z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Au_n)), \\ C_{n+1} = \{w \in C_n : \phi(w, z_n) \leq \phi(w, u_n) - \xi'_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1) \quad \forall n \geq 1, \end{cases}$$

where λ_n and ξ'_n are defined in (3.18) and (3.19), respectively. Then $\{x_n\}$ converges strongly to $z = \Pi_\Omega(x_1)$.

Corollary 3.9. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $A : H \to H$ be a monotone and Lipschitz continuous mapping. Assume that $\Omega \neq \emptyset$. Given $\lambda_1 > 0$, $-\theta \leq \theta_n \leq \theta$ for some $\theta > 0$ and $\mu \in (0, 1)$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} C_1 = C, \\ x_0, x_1 \in H, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda_n Au_n), \\ z_n = y_n - \lambda_n(Ay_n - Au_n), \\ C_{n+1} = \{w \in C_n : ||z_n - w||^2 \leq ||u_n - w||^2 - \xi''_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_1) \quad \forall n \geq 1, \end{cases}$$

where λ_n and ξ''_n are defined in (3.20) and (3.21), respectively. Then $\{x_n\}$ converges strongly to $z = P_\Omega(x_1)$.

Table 1. Numerical results for Example 4.1.

	Number of iterations			
m	Algorithm 1	Algorithm 2	Algorithm (1.8)	Algorithm (1.9)
5	786	315	4315	10565
10	953	600	6786	10574
20	2699	2057	5584	10842

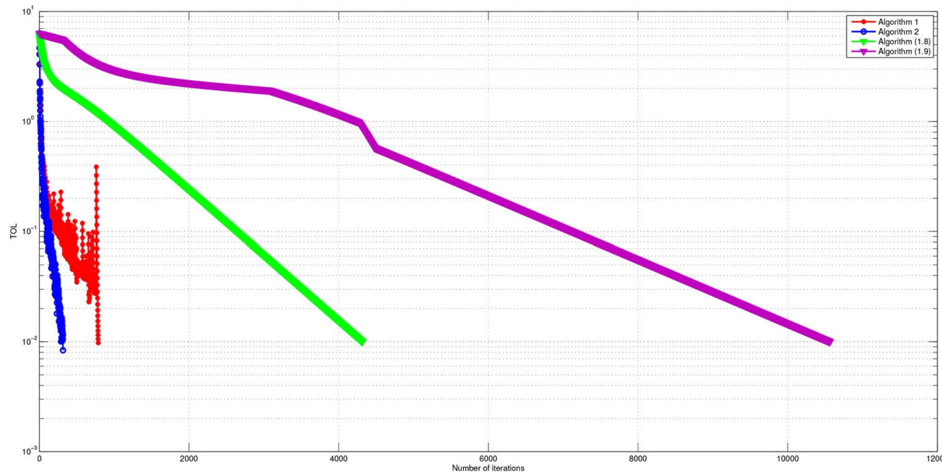

Figure 2. The behavior of TOL_n with $m = 5$.

4. Numerical examples

In this section we perform several numerical experiments in order to illustrate the numerical behavior of Algorithm 1 and Algorithm 2, and also to compare them with Algorithm (1.8) and Algorithm (1.9).

Example 4.1. We consider a problem which is taken from [47]. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be the operator defined by $Ax := Mx + q$ with $q \in \mathbb{R}^m$ and

$$M = NN^T + S + D,$$

where N is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix and D is an $m \times m$ positive definite diagonal matrix. It is clear that A is monotone and Lipschitz continuous with $L = ||M||$. The set of feasible solutions is given by $C = \{(x^{(1)}, x^{(2)}, \dots, x^{(m)})^T \in \mathbb{R}^m : -5 \le x^{(i)} \le 5\}$. All the entries of N and S are generated randomly in $[-5, 5]$, D is generated randomly in $[1, 5]$ and q is equal to the zero vector. It is obvious that $VI(C, A) = \{(0, 0, \dots, 0)^T\}$. We let $z = (0, 0, \dots, 0)^T$ and use $\mathrm{TOL}_n = ||x_n - z|| \le 10^{-2}$ as the stopping criterion. We take $\lambda_1 = 4.5$, $\mu = 0.85$, $\theta_n = 0.15$ and $x_0 = x_1$ with the coordinates generated randomly in $[2, 4]$ in order to test the convergence of Algorithm 1 and Algorithm 2. We take $\lambda_n = 0.00025$ in Algorithm (1.8) and $l = 0.5$, $\theta = 0.1$, $\gamma = 2$ in Algorithm (1.9). We perform the numerical experiments with three different choices of m ($m = 5$, $m = 10$

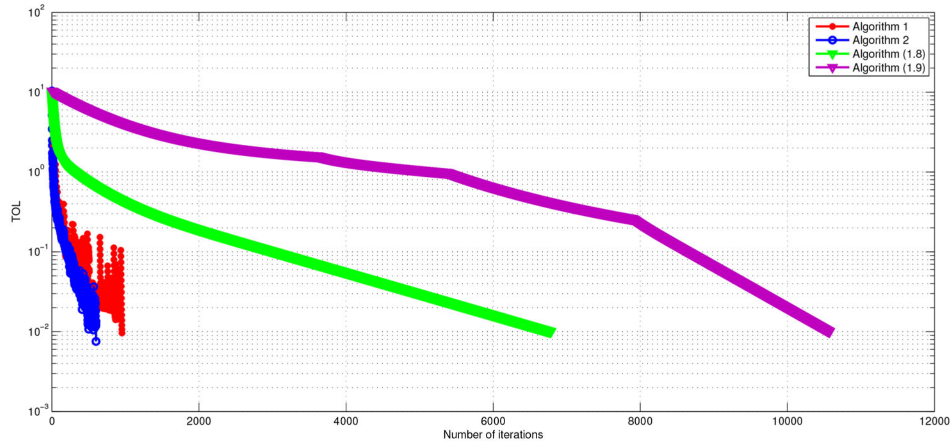

Figure 3. The behavior of TOL_n with $m=10$.

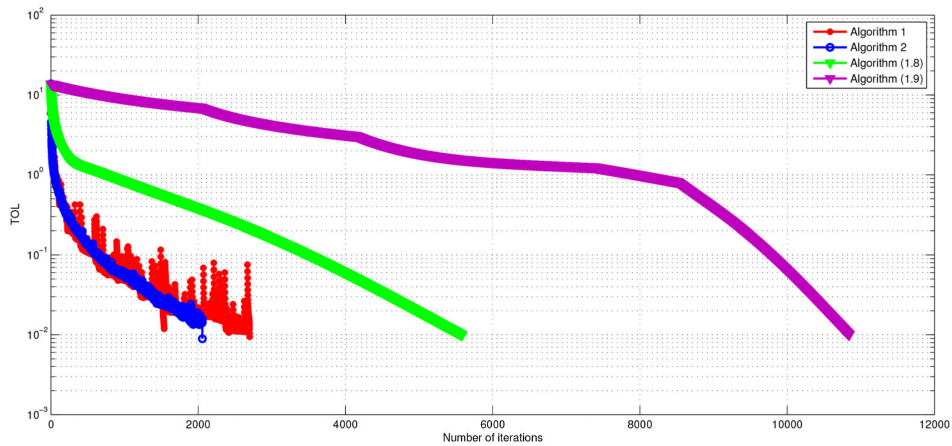

Figure 4. The behavior of TOL_n with $m=20$.

and $m=20$). The numerical results we have obtained are presented in Table 1.

The behavior of the function TOL_n in Table 1 is described in Figures 2 and 48 and in Figure 4.

Example 4.2. Let $A:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$$Ax := (ax^{(1)} + bx^{(2)} + a\sin(x^{(1)}), -bx^{(1)} + cx^{(2)} + c\sin(x^{(2)})) \text{ for all } x=(x^{(1)},x^{(2)})\in\mathbb{R}^2,$$

where a, b, c are real numbers. It is clear that A is monotone and Lipschitz continuous with $L=\sqrt{2\max\{4a^2+b^2, 4c^2+b^2\}}$ (see [49, Remark 2.1]). The set of feasible solutions is given by $C=\{(x^{(1)},x^{(2)})\in\mathbb{R}^2 : -1\le x^{(i)}\le 1\}$. It is obvious that $VI(C,A)=\{(0,0)\}$. We let $z=(0,0)$ and use $\mathrm{TOL}_n=||x_n-z||\le 10^{-5}$ as the stopping criterion. We take $\lambda_1=$

Table 2. Numerical results for Example 4.2.

	Number of iterations			
(*a*, *b*, *c*)	Algorithm 1	Algorithm 2	Algorithm (1.8)	Algorithm (1.9)
(2, 2, 2)	2333	68	5963	3057
(2, 1, 3)	4198	81	2381	2438
(2, 4, 1)	4843	120	7939	5822

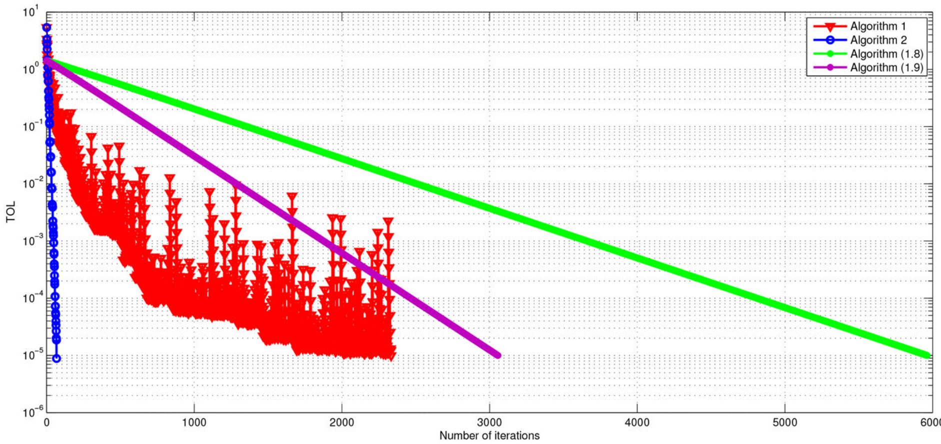

Figure 5. The behavior of TOL_n with $(a, b, c) = (2, 2, 2)$.

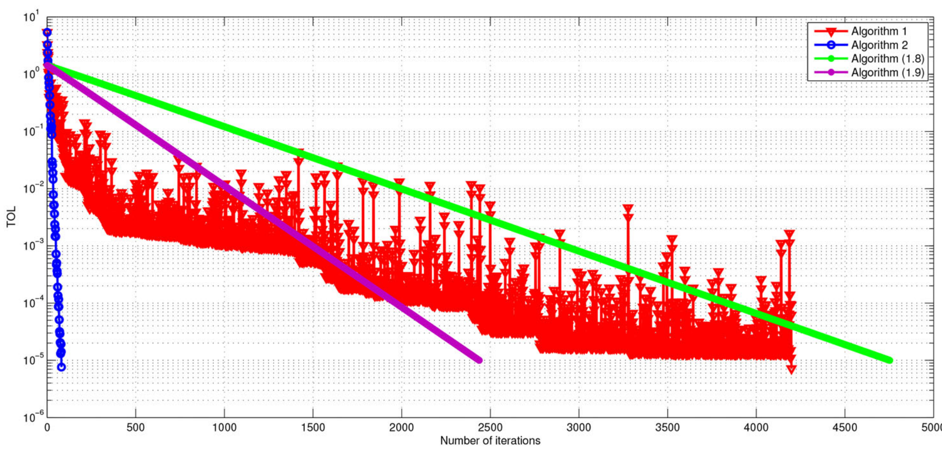

Figure 6. The behavior of TOL_n with $(a, b, c) = (2, 1, 3)$.

1.5, $\mu = 0.75$, $\theta_n = 1.75$ and $x_0 = x_1 = (2, 5)$ in both Algorithm 1 and Algorithm 2. We take $\lambda_n = 0.0005$ in Algorithm (1.8) and $l = 0.5$, $\theta = 0.005$, $\gamma = 2$ in Algorithm (1.9). We performed the numerical experiments with three different choices of a, b, c ($(a, b, c) = (2, 2, 2)$, $(a, b, c) = (2, 1, 3)$ and $(a, b, c) = (2, 4, 1)$). The numerical results are presented in Table 2.

The behavior of the function TOL_n in Table 2 is described in Figures 5–31.

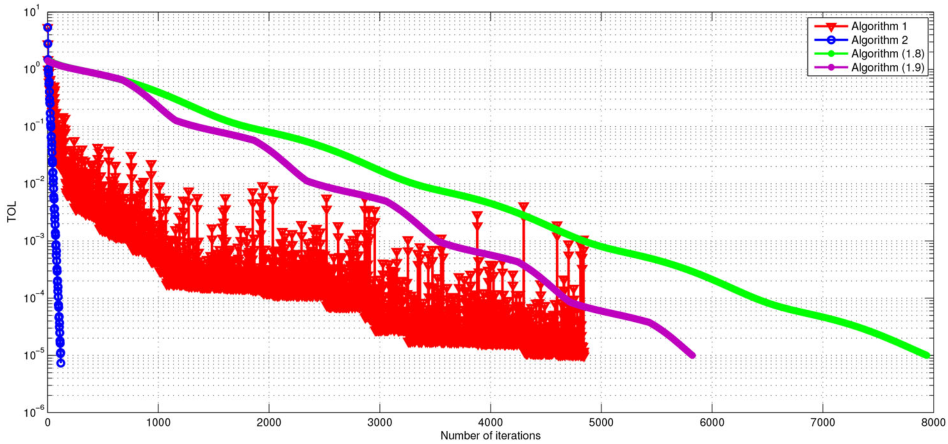

Figure 7. The behavior of TOL_n with $(a, b, c) = (2, 4, 1)$.

Finally, we apply our algorithms to solving an image restoration problem.

Example 4.3. In this example, we are given that all images have $D := M \times N$ pixels and each pixel value is known to be in the range $[0, 255]$. We set $C = [0, 255]^D$. The original image is $\bar{x}$ and the degraded image is

$$y = B\bar{x} + \xi,$$

where B is a blurring matrix and ξ is a noise term. The goal is to recover the original image $\bar{x}$ by using B and y. Next, we consider a model which produces a restored image which is obtained as a solution to the following minimization problem:

$$\min_{x \in C} \frac{1}{2} ||Bx - y||_2^2, \tag{4.1}$$

where $||x||_2$ is the Euclidean norm of x. Clearly, the minimization (4.1) can be expressed as a variational inequality problem by setting $A := B^T(Bx - y)$. It is known that the operator A in this case is monotone and Lipschitz continuous with $L = ||B^T B||$. To measure the quality of restored images, we define the signal-to-noise ratio (SNR) in decibels (dB) as follows:

$$\mathrm{SNR} := 20 \log_{10} \frac{||\bar{x}||_2}{||x - \bar{x}||_2},$$

where $\bar{x}$ is the original image, and x is a restored image. Clearly, a larger SNR value means that we have restored a better image. The initial point x_0

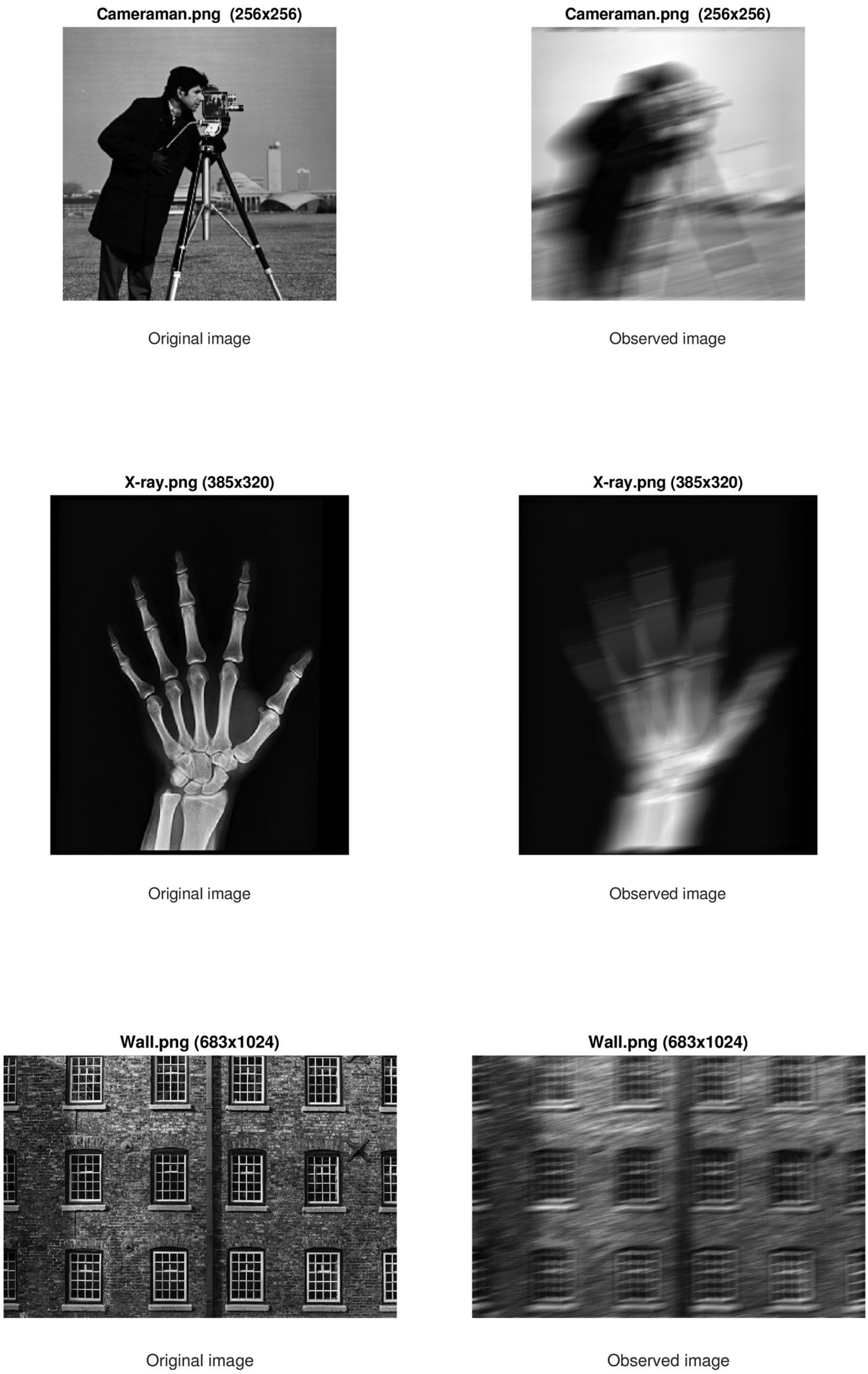

Figure 8. Left: Original images. Right: Observed images (SNR = 11.9061 dB, 6.5711 dB and 7.2420 dB, respectively). Each image is degraded by a motion blur with a motion length 45 and an angle 15.

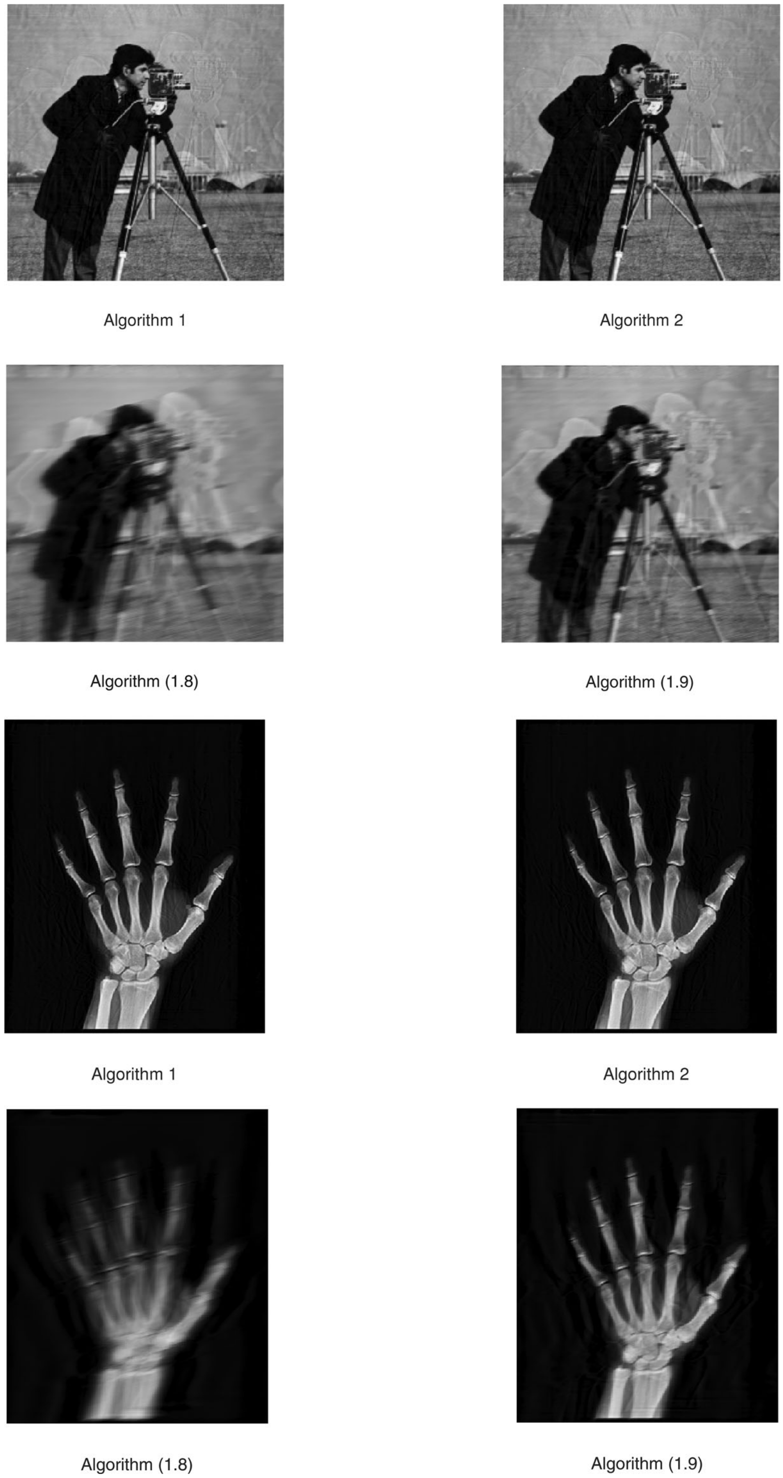

Figure 9. Comparison of recovered images by using different algorithms when $n = 50{,}000$.

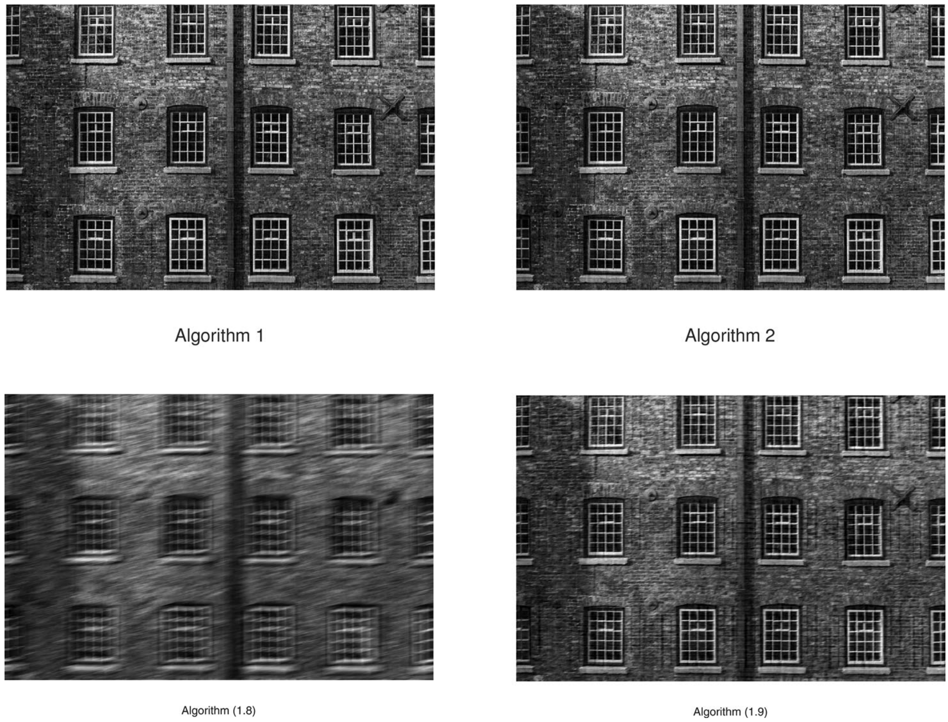

Figure 9. (Continued).

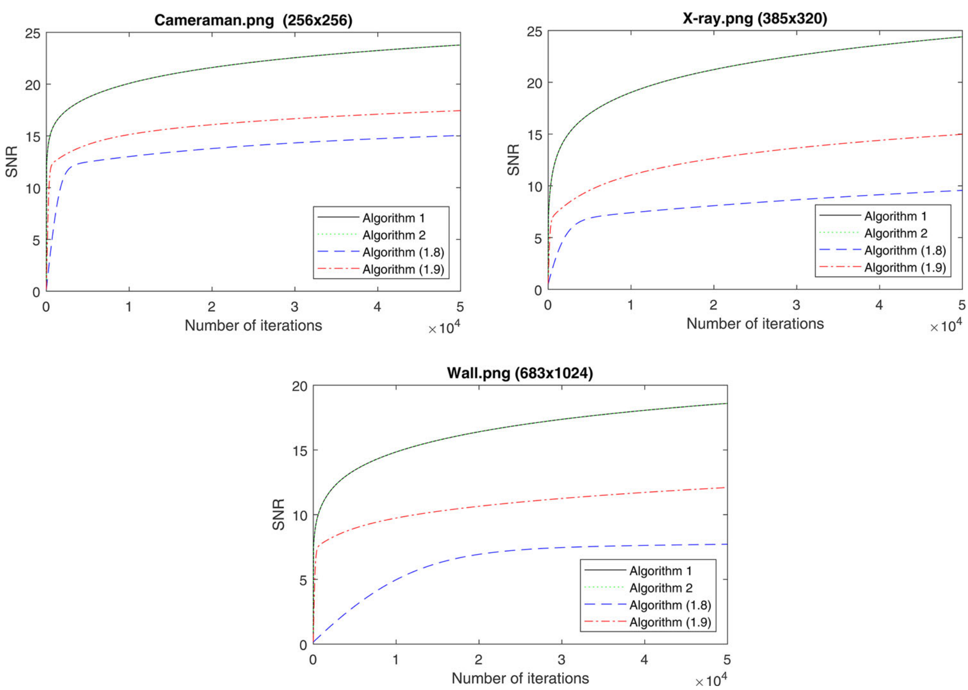

Figure 10. SNR plotting of each image.

Table 3. Numerical results for Example 4.3.

Image	n	Algorithm 1 SNR(dB)	Algorithm 2 SNR(dB)	Algorithm (1.8) SNR(dB)	Algorithm (1.9) SNR(dB)
Cameraman.png	5,000	18.7004	18.7004	12.4870	14.1609
(256 × 256)	15,000	20.9509	20.9509	13.4248	15.6889
	30,000	22.5521	22.5521	14.3251	16.6632
	50,000	23.7793	23.7793	15.0421	17.4424
X-ray.png	5,000	16.9686	16.9686	6.8758	9.5672
(385 × 320)	15,000	20.2982	20.2982	7.7609	11.9652
	30,000	22.5854	22.5854	8.6553	13.6581
	50,000	24.3939	24.3939	9.5631	14.9753
Wall.png	5,000	13.4385	13.4385	2.9318	8.9514
(683 × 1024)	15,000	15.7412	15.7412	6.2482	10.2540
	30,000	17.3672	17.3672	7.4606	11.2539
	50,000	18.6009	18.6009	7.7166	12.0998

is chosen to be $\frac{y}{2} \in \mathbb{R}^D$ and x_1 is chosen to be $2y \in \mathbb{R}^D$ (Figures 8–10 and Table 3).

We take $\lambda_1 = 0.1$, $\theta_n = 0.05$, $\mu = 0.03$ and $\sigma = 1$ in both Algorithm 1 and Algorithm 2. We take $\lambda_n = 0.9/L$ in Algorithm (1.8) and $l = 0.3$, $\theta = 0.01$ and $\gamma = 0.05$ in Algorithm (1.9).

5. Conclusions

In recent years, modifications of Tseng's extragradient algorithm have been studied intensively by many authors. Most of these modifications have been studied in Hilbert spaces and in 2-uniformly convex real Banach spaces. However, the 2-uniform convexity assumption is rather restrictive. In the present paper, we have established strong convergence theorems for new inertial hybrid and inertial shrinking Tseng extragradient algorithms for solving monotone variational inequalities in all reflexive Banach spaces without prior knowledge of the Lipschitz constant of the relevant operator. We have also performed some numerical experiments in order to illustrate the convergence of our algorithms and to compare their behavior with that of known algorithms. In a possible future project, we intend to study numerical implementations of our methods in non-Hilbert Banach spaces and in non-Euclidean settings.

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References

[1] He, B. S., Liao, L. Z. (2002). Improvements of some projection methods for monotone nonlinear variational inequalities. *J. Optim. Theory Appl.* 112(1):111–128. DOI: 10.1023/A:1013096613105.

[2] Korpelevich, G. M. (1976). The extragradient method for finding saddle points and other problems. *Matecon.* 12:747–756.

[3] Censor, Y., Gibali, A., Reich, S. (2011). The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* 148(2): 318–335. DOI: 10.1007/s10957-010-9757-3.

[4] Iusem, A. N., Svaiter, B. F. (1997). A variant of Korpelevich's method for variational inequalities with a new search strategy. *Optimization* 42(4):309–321. DOI: 10.1080/02331939708844365.

[5] Popov, L. D. (1980). A modification of the Arrow-Hurwicz method for searching for saddle points. *Mat. Zametki.* 28:777–784.

[6] Tseng, P. (2000). A modified forward-backward splitting method for maximal monotone mapping. *SIAM J. Control Optim.* 38(2):431–446. DOI: 10.1137/S0363012998338806.

[7] Anh, P. K., Thong, D. V., Vinh, N. T. (2020). Improved inertial extragradient methods for solving pseudo-monotone variational inequalities. *Optimization.* :1–24. DOI: 10.1080/02331934.2020.1808644.

[8] Chen, J., Liu, S., Chang, X. (2021). Modified Tseng's extragradient methods for variational inequality on Hadamard manifolds. *Appl. Anal.* 100(12):2627–2640. DOI: 10.1080/00036811.2019.1695783.

[9] Hammad, H. A., Ur Rehman, H., la Sen, M. D. (2020). Shrinking projection methods for accelerating relaxed inertial tseng-type algorithm with applications. *Math. Prob. Eng.* 2020:1–14. DOI: 10.1155/2020/7487383.

[10] Liu, H., Yang, J. (2020). Weak convergence of iterative methods for solving quasimonotone variational inequalities. *Comput. Optim. Appl.* 77(2):491–508. vol DOI: 10.1007/s10589-020-00217-8.

[11] Sunthrayuth, P., Jolaoso, L. O., Cholamjiak, P. (2021). New Bregman projection methods for solving pseudo-monotone variational inequality problem. *J. Appl. Math. Comput.* DOI: 10.1007/s12190-021-01581-2.

[12] Thong, D. V., Hieu, D. V. (2018). Weak and strong convergence theorems for variational inequality problems. *Numer. Algorithms* 78(4):1045–1060. DOI: 10.1007/s11075-017-0412-z.

[13] Thong, D. V., Vuong, P. T. (2019). Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities. *Optimization* 68(11):2207–2226. DOI: 10.1080/02331934.2019.1616191.

[14] Thong, D. V., Vinh, N. T., Cho, Y. J. (2020). A strong convergence theorem for Tseng's extragradient method for solving variational inequality problems. *Optim. Lett.* 14(5):1157–1175. DOI: 10.1007/s11590-019-01391-3.

[15] Thong, D. V., Vinh, N. T., Hieu, D. V. (2019). Accelerated hybrid and shrinking projection methods for variational inequality problems. *Optimization* 68(5):981–998. DOI: 10.1080/02331934.2019.1566825.

[16] Yang, J., Cholamjiak, P., Sunthrayuth, P. (2021). Modified Tseng's splitting algorithms for the sum of two monotone operators in Banach spaces. *AIMS Math.* 6(5): 4873–4900. DOI: 10.3934/math.2021286.

[17] Alvarez, F., Attouch, H. (2001). An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Var. Anal.* 9(1–2):3–11. DOI: 10.1023/A:1011253113155.

[18] Moudafi, A., Oliny, M. (2003). Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.* 155(2):447–454. DOI: 10.1016/S0377-0427(02)00906-8.

[19] Nesterov, Y. (1983). A method for solving the convex programming problem with convergence rate $O(1/k^2)$. *Dokl. Akad. Nauk SSSR*. 269:543–547.

[20] Polyak, B. T. (1964). Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* 4(5):1–17. DOI: 10.1016/0041-5553(64)90137-5.

[21] Nakajo, K., Takahashi, W. (2003). Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. *J. Math. Anal. Appl.* 279(2):372–379. DOI: 10.1016/S0022-247X(02)00458-4.

[22] Takahashi, W., Takeuchi, Y., Kubota, R. (2008). Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* 341(1):276–286. DOI: 10.1016/j.jmaa.2007.09.062.

[23] Shehu, Y. (2020). Single projection algorithm for variational inequalities in Banach spaces with applications to contact problems. *Acta Math Sci.* 40B(4):1045–1063.

[24] Rockafellar, R. T. (1970). *Convex Analysis*. Princeton, NJ: Princeton University Press.

[25] Beck, A. (2017). *First-Order Methods in Optimization*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

[26] Barbu, V., Precupanu, T. (2010). *Convexity and Optimization in Banach Spaces*. London New York: Springer Dordrecht Heidelberg.

[27] Pathak, H. K. (2018). *An Introduction to Nonlinear Analysis and Fixed Point Theory*. Springer Nature Singapore Pte Ltd.

[28] Reich, S., Sabach, S. (2009). A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. *J. Nonlinear Convex Anal.* 1:471–485.

[29] Reich, S., Sabach, S. (2010). Two strong convergence theorems for a proximal method in reflexive Banach spaces. *Numer. Funct. Anal. Optim.* 31(1):22–44. DOI: 10.1080/01630560903499852.

[30] Bonnans, J. F., Shapiro, A. (2000). *Perturbation Analysis of Optimization Problems*. New York, NY: Springer Verlag.

[31] Bauschke, H. H., Borwein, J. M., Combettes, P. L. (2001). Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. *Commun. Contemp. Math.* 03(04):615–647. DOI: 10.1142/S0219199701000524.

[32] Reich, S., Sabach, S. (2010). Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. *Nonlinear Anal. TMA*. 73(1):122–135. DOI: 10.1016/j.na.2010.03.005.

[33] Bregman, L. M. (1967). The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR Comput. Math. Math. Phys.* 7(3):200–217. DOI: 10.1016/0041-5553(67)90040-7.

[34] Alber, Y. I. (1996). Metric and generalized projection operators in Banach spaces: properties and applications. in: Kartsatos, A. G., ed. *Theory and Applications of Nonlinear Operator of Accretive and Monotone Type*. New York, NY: Marcel Dekker, pp. 15–50.

[35] Reich, S. (1996). A weak convergence theorem for the alternating method with Bregman distances. in: *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*. New York, NY: Marcel Dekker, p. 313–318.

[36] Nakajo, K. (2015). Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces. *Appl. Math. Comput*. 271:251–258. DOI: 10.1016/j.amc.2015.08.096.

[37] Bauschke, H. H., Wang, X., Ye, J., Yuan, X. (2009). Bregman distances and Chebyshev sets. *J. Approximation Theory* 159(1):3–25. DOI: 10.1016/j.jat.2008.08.014.

[38] Iyer, R., Bilmes, J. (2012). Submodular-Bregman divergences and Lovász-Bregman divergences with Applications. In: Conference on Neural Information Processing Systems.

[39] Nielsen, F., Boltz, S. (2011). The Burbea-Rao and Bhattacharyya centroids. *IEEE Trans. Inform. Theory* 57(8):5455–5466. DOI: 10.1109/TIT.2011.2159046.

[40] Nielsen, F., Nock, R. (2017). Generalizing Skew Jensen Divergences and Bregman Divergences With Comparative Convexity. *IEEE Signal Process. Lett*. 24(8): 1123–1127. DOI: 10.1109/LSP.2017.2712195.

[41] Reem, D., Reich, S., De Pierro, A. (2019). Re-examination of Bregman functions and new properties of their divergences. *Optimization*. 68(1):279–348. DOI: 10.1080/02331934.2018.1543295.

[42] Butnariu, D., Resmerita, E. (2006). Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. *Abstr. Appl. Anal*. 2006: 1–39. DOI: 10.1155/AAA/2006/84919.

[43] Alber, Y. I. (1993). Generalized projection operators in banach spaces: properties and applications. In: Proceedings of the Israel Seminar, Vol. 1 of Functional Differential Equation, p. 1–21.

[44] Resmerita, E. (2004). On total convexity, Bregman projections and stability in Banach spaces. *J. Convex Anal*. 11:1–16.

[45] Butnariu, D., Iusem, A. N. (2000). *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*. Dordrecht: Kluwer Academic Publishers.

[46] Takahashi, W. (2000). *Nonlinear Functional Analysis*. Yokohama: Yokohama Publishers.

[47] Harker, P. T., Pang, J. S. (1990). A damped-Newton method for the linear complementarity problem. In: Allgower, G., Georg, K., eds. *Computational Solution of Nonlinear Systems of Equations. Lectures in Applied Mathematics*, Vol. 26. Providence: AMS, pp. 265–284.

[48] Alber, Y., Ryazantseva, I. *Nonlinear Ill-Posed Problems of Monotone Type*, 2006th ed. Springer.

[49] Sahu, D. R., Singh, A. K. (2021). Inertial iterative algorithms for common solution of variational inequality and system of variational inequalities problems. *J. Appl. Math. Comput*. 65(1–2):351–378. DOI: 10.1007/s12190-020-01395-8.

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Weak and strong convergence results for solving monotone variational inequalities in reflexive Banach spaces

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ABSTRACT

In this paper, we introduce two modified Tseng's extragradient algorithms with a new generalized adaptive stepsize for solving monotone variational inequalities (VI) in reflexive Banach spaces. The advantage of our methods is that stepsizes do not require prior knowledge of the Lipschitz constant of the cost mapping. Based on Bregman projection-type methods, we prove weak and strong convergence of the proposed algorithms to a solution of VI. Some numerical experiments to show the efficiency of our methods including a comparison with related methods are provided.

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1. Introduction

Let E be a real Banach space with a norm $\|\cdot\|$ and E^* be a dual of E. We denote by $\langle x, f\rangle$ the value of $f \in E^*$ at $x \in E$, that is, $\langle x, f\rangle = f(x)$. Let C be a nonempty, closed and convex subset of E and $A : C \to E^*$ be a continuous mapping. The *variational inequalities* (VI) is to find a point $z \in C$ such that

$$\langle x - z, Az\rangle \geq 0 \quad \forall\, x \in C. \tag{1}$$

The solution set of VI (1) is denoted by $VI(C, A)$. Variational inequality theory is an important tool in physics, control theory, engineering, economics, management science, mathematical programming, and so on. Several iterative methods have been proposed for solving the variational inequalities. A classical method for solving the VI in a Hilbert space H is the *gradient projection method* which is given by

$$x_{n+1} = P_C(x_n - \lambda A x_n), \tag{2}$$

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where P_C is the projection operator onto the convex and closed subset C of H and $\lambda > 0$ is a suitable stepsize. However, the convergence of this method requires a strong (or inverse strong) monotonicity of A. To avoid this hypothesis, Korpelevich [1] and Antipin [2] proposed the following so-called *extragradient method* for solving VI in a finite-dimensional Euclidean space $\mathbb{R}^m$:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{3}$$

where $A : C \to \mathbb{R}^m$ is monotone and L-Lipschitz continuous and $\lambda \in (0, \frac{1}{L})$. Note that algorithm (3) involves two projections onto the set C per iteration. In order to reduce the number of evaluations of projection per iteration, Tseng [3] introduced the following method and later was known as *Tseng's extragradient method*:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n). \end{cases} \tag{4}$$

The weak convergence of this method was proved provided $\lambda \in (0, \frac{1}{L})$. It is remarkable that the Tseng's extragradient method (4) only needs to calculate one projection, which is simpler than the extragradient method. By these reasons, the Tseng's extragradient method has received great attention in various ways to obtain the weak and strong convergence of these methods. However, the convergence of (4) has been established only in Hilbert spaces (see, e.g. [4–11]). It was of great interest to extend Tseng's result to Banach spaces. Very recently, Shehu [12] first extend Tseng's result to a 2-uniformly convex Banach space E. He proposed the following algorithm:

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n (A y_n - A x_n)), \end{cases} \tag{5}$$

where $A : E \to E^*$ is monotone and L-Lipschitz continuous, Π_C is the generalized projection from E onto C, J is the normalized duality mapping on E. Also, he proposed the following algorithm which is a variant of (5) based on Halpern-type iteration:

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}(Jy_n - \lambda_n (A y_n - A x_n)), \\ x_{n+1} = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J z_n). \end{cases} \tag{6}$$

He proved weak convergence and strong convergence theorems of algorithms (5) and (6), respectively, in a 2-uniformly convex Banach space under the sequences of stepsize satisfy the following inequality:

$$0 < a \le \lambda_n \le b < \frac{1}{\sqrt{2\mu\kappa L}}, \tag{7}$$

where $\mu > 0$ is the 2-uniform convexity constant of E and $\kappa > 0$ is the 2-uniform smoothness constant of E^*. However, the 2-uniform convex Banach spaces are too restrictive for applications in L_p(or ℓ_p) with $p > 2$. Moreover, the sequence of stepsize is chosen dependently of the Lipschitz constant of the mapping. From a practical point of view the Lipschitz constant is very difficult to estimate.

Question: Can we extend and modify the Shehu's methods (5) and (6) for solving the monotone VI in more general reflexive Banach spaces which stepsizes does not require prior estimates of the Lipschitz constants?

The purpose of this paper is to give an answer to the above question. We introduce two modified Tseng's extragradient algorithms with a new generalized adaptive stepsize for solving monotone VI in the framework of reflexive Banach spaces. The stepsizes of our methods are updated over each iteration by a cheap computation. This allows the algorithms to be computed more easily without the prior knowledge the Lipschitz constant. The weak and strong convergence of the proposed methods are establish under some suitable conditions.

Our paper is organized as follows: In Section 2, we present some preliminaries which will be needed in the sequel. In Section 3, we propose two algorithms and analyze their convergence. Finally, some numerical examples are provided in Section 4.

2. Preliminaries and lemmas

Throughout this paper, let E be a real reflexive Banach space with its dual E^* and $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function with its domain is denoted by $\mathrm{dom} f := \{x \in E : f(x) < \infty\}$. We use the notations $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence and weak convergence of the sequence $\{x_n\} \subset E$ to x, respectively. We also denote by $\langle x, j\rangle$ the value of functional $j \in E^*$ at $x \in E$. The *subdifferential* of f defined by

$$\partial f(x) := \{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \le f(y) \ \forall\, y \in E\}, \quad x \in E.$$

The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, \infty]$ defined by

$$f^*(x^*) := \sup_{x\in E}\{\langle x, x^*\rangle - f(x)\}.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x, x^*\rangle$ (see [13, Theorem 23.5]).

For any $x \in \mathrm{int}(\mathrm{dom} f)$ and $y \in E$, the *directional derivative* of f at x in the direction $y \in E$ is given by

$$f'(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{8}$$

The function f is said to be *Gâteaux differentiable* at x if the limit as $t \to 0$ in (8) exists for each y. In this case, the *gradient* of f at x is the linear function $\nabla f(x) : E \to E^*$ defined by $\langle y, \nabla f(x)\rangle = f'(x, y)$ for any $y \in E$. For more details about gradient, we recommend [14, Remark 3.32]. The function f is said to be

Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \mathrm{int}(\mathrm{dom} f)$. It is known that if f is continuous at x and $\partial f(x)$ is single valued, then f is Gâteaux differentiable at x and $\nabla f(x) = \partial f(x)$ (see [15, Proposition 2.40]). The function f is said to be *Fréchet differentiable* at x if the limit (8) is attained uniformly in $\|y\| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (8) is attained uniformly for $x \in C$ and $\|y\| = 1$. We know that every Fréchet differentiable function is Gâteaux differentiable and if f is Fréchet differentiable, then it is continuous (see [16, p.142]). It is also known that if $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* (see [17, Proposition 1])

Definition 2.1: The function $f : E \to \mathbb{R}$ is said to be:

(1) *uniformly convex* with modulus ϕ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)\phi(\|x-y\|),$$

for all $x, y \in \mathrm{dom}\, f$ and $t \in (0,1)$, where ϕ is an increasing function vanishing only at 0;

(2) *strongly convex* with a constant $\sigma > 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\sigma}{2}t(1-t)\|x-y\|^2 \tag{9}$$

for all $x, y \in \mathrm{dom}\, f$ and $t \in (0,1)$.

We know that f is uniformly convex if and only if f^* is Fréchet differentiable and ∇f^* is uniformly continuous (see [18, Theorem 3.5.10]). Obviously, f is strongly convex with a constant σ if and only if it is uniformly convex with modulus $\phi(t) = \frac{\sigma}{2}t^2$ and it is also equivalent to the following inequality (see [14, Theorem 5.24]):

$$f(x) \geq f(y) + \langle x-y, \nabla f(y)\rangle + \frac{\sigma}{2}\|x-y\|^2 \tag{10}$$

for all $x \in \mathrm{dom}\, f$ and $y \in \mathrm{int}(\mathrm{dom} f)$.

Definition 2.2 ([19, p.25]): The function $f : E \to (-\infty, \infty]$ is said to be *Legendre* if it satisfies the following two conditions:

(L1) $\mathrm{int}(\mathrm{dom} f) \neq \emptyset$, f is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f)$ and $\mathrm{dom} \nabla f = \mathrm{int}(\mathrm{dom} f)$;

(L2) $\mathrm{int}(\mathrm{dom} f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f^*)$ and $\mathrm{dom} \nabla f^* = \mathrm{int}(\mathrm{dom} f^*)$.

Remark 2.3: In a reflexive Banach space, we always have $(\partial f)^{-1} = \partial f^*$ (see [20, p.83]). This fact, when combined with conditions (L1) and (L2), implies the following two facts:

(i) ∇f is a bijection from int(dom$f$) into int(domf^*) satisfying $\nabla f = (\nabla f^*)^{-1}$ (see [21, Theorem 5.10]);
(ii) ran∇f = dom∇f^* = int(domf^*) and ran∇f^* = dom∇f = int(dom$f$) (see [22, p.123]),

where ran∇f denotes the range of ∇f. Also, conditions (L1) and (L2), in conjunction with [21, Theorem 5.4], imply that the functions f and f^* are essentially strictly convex on the interior of their respective domains.

One important and interesting Legendre function is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. For more examples of Legendre functions, we recommend [21,23,24].

Definition 2.4 ([25]): Let $f : E \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function. The bifunction $D_f : \mathrm{dom}\, f \times \mathrm{int}(\mathrm{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$$

is called the *Bregman distance* with respect to f.

The geometric of Bregman distance is shown in Figure 1. In particular, if E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$ for all

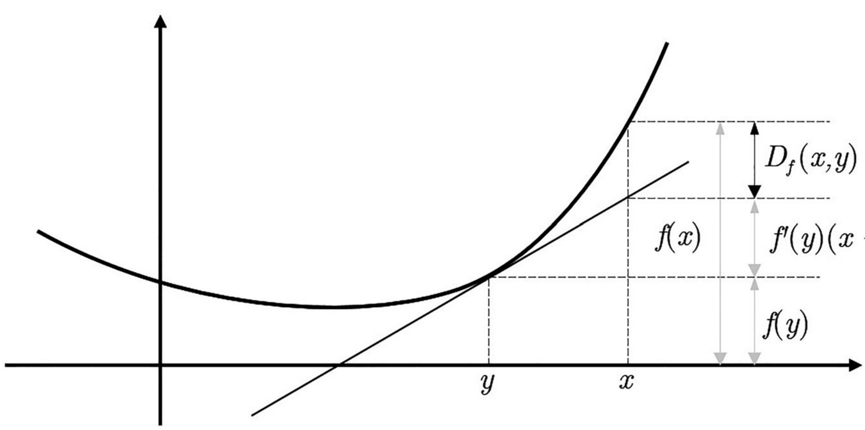

Figure 1. Bregman distance with respect to f.

$x \in E$, then $\nabla f(x) = Jx$, where J is the normalized duality mapping defined by

$$Jx := \{j \in E^* : \langle x, j\rangle = \|x\|^2 = \|j\|^2\}.$$

Then we have $D_f(x,y) = \frac{1}{2}\phi(x,y)$, where ϕ is called the *Lyapunov functional* which is defined by $\phi(x,y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ (see [26,27]). For a 2-uniformly convex and smooth Banach space E, the Lyapunov functional satisfies the following inequality:

$$\phi(x,y) \geq c\|x-y\|^2, \tag{11}$$

where $c > 0$ is the 2-uniformly convexity constant of E (see [28, Lemma 2.3]). It is well known that $\phi(x,y) = \|x-y\|^2$ and $c = 1$ whenever E is a Hilbert space.

From the definition of Bregman distance and (10), it is clear that

$$D_f(x,y) \geq \frac{\sigma}{2}\|x-y\|^2. \tag{12}$$

The following important properties follow from the definition of Bregman distance:

(i) (two-point indentity) for any $x, y \in \text{int}(\text{dom}f)$,

$$D_f(x,y) + D_f(y,x) = \langle x-y, \nabla f(x) - \nabla f(y)\rangle; \tag{13}$$

(ii) (three-point indentity) for any $x \in \text{dom}\, f$ and $y, z \in \text{int}(\text{dom}f)$,

$$D_f(x,y) = D_f(x,z) - D_f(y,z) + \langle x-y, \nabla f(z) - \nabla f(y)\rangle. \tag{14}$$

The *modulus of total convexity* of f at $x \in \text{int}(\text{dom}f)$ is the function $v_f : \text{int}(\text{dom}f) \times [0,\infty) \to [0,\infty]$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in \text{dom}\, f, \|y-x\| = t\}.$$

The function f is called *totally convex* at x if $v_f(x,t) > 0$ whenever $t > 0$. It is called *totally convex* if it is totally convex at any point $x \in \text{int}(\text{dom}f)$. The *modulus of total convexity* of the function f on the set X is the function $v_f : \text{int}(\text{dom}f) \times [0,\infty) \to [0,\infty]$ defined by

$$v_f(X,t) := \inf\{v_f(x,t) : x \in X \cap \text{dom}\, f\}.$$

The function f is said to be *totally convex on bounded sets* of E if $v_f(X,t) > 0$ for any nonempty bounded subset X of E and $t > 0$. It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [29, Theorem 2.10]).

The *Bregman projection* with respect to f of $x \in \text{int}(\text{dom}f)$ onto the nonempty, closed and convex set $C \subset \text{dom}\, f$ is the minimizer over C defined by

$$\Pi_C^f(x) := \text{argmin}\{D_f(y,x) : y \in C\}.$$

The geometric of Bregman projection is shown in Figure 2. If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then

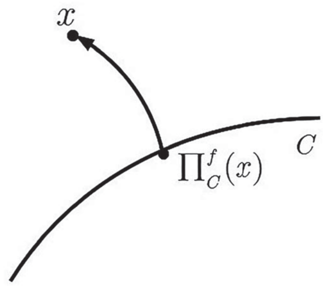

Figure 2. Bregman projection with respect to *f*.

Π^f_C coincides with the generalized projection Π_C (see [30, Definition 7.2]) and if E is a Hilbert space, then Π^f_C coincides the metric projection P_C.

Lemma 2.5 ([29, Corollary 4.4]): *Suppose that f is Gâteaux differentiable and totally convex on* int(dom$f$). *Let $x \in$* int(dom$f$) *and let C be a nonempty, closed and convex subset of* int(dom$f$). *If $z \in C$, then the following statements are equivalent:*

(i) $z = \Pi^f_C(x)$ *is the Bregman projection of x onto C with respect to f;*
(ii) *z is the unique solution of the following variational inequality:*

$$\langle y - z, \nabla f(x) - \nabla f(z) \rangle \leq 0 \quad \forall\, y \in C;$$

(iii) *z is the unique solution of the following inequality:*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \quad \forall\, y \in C.$$

Let $f : E \to \mathbb{R}$ be a Legendre function. We define the function $V_f : E \times E^* \to [0, \infty)$ associated with f by

$$V_f(x, x^*) := f(x) - \langle x, x^* \rangle + f^*(x^*) \quad \forall\, x \in E, \quad x^* \in E^*.$$

From [31, Proposition 1], we know the following properties:

(i) V_f is nonnegative and convex in the second variable;
(ii) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ $\forall x \in E,\ x^* \in E^*$;
(iii) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)$ $\forall x \in E,\ x^*, y^* \in E^*$;
(iv) $D_f(x, \nabla f^*(t\nabla f(y) + (1-t)\nabla f(z))) \leq tD_f(x, y) + (1-t)D_f(x, z)$ for all $t \in [0, 1]$ and for all $x, y, z \in E$.

Recall that the function f is called *sequentially consistent* [29, p.9], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in dom f and int(dom$f$), respectively, such that the first one is bounded and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6 ([32, Lemma 2.1.2]): *The function $f : E \to (-\infty, \infty]$ is sequentially consistent if and only if it is totally convex on bounded subsets of E.*

From Lemma 2.6, if f is additionally assumed to be Fréchet differentiable which is bounded on bounded subsets of E, then for any two sequences $\{x_n\}$ and $\{y_n\}$ in E,

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Longrightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0 \Longrightarrow \lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0.$$

Definition 2.7: Let C be a nonempty subset of E. Recall that a mapping $A : C \to E^*$ is said to be:

(1) *monotone* if $\langle x - y, Ax - Ay\rangle \geq 0$ for all $x, y \in C$;
(2) *Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$;
(3) *weakly sequentially continuous* if for any sequence $\{x_n\} \subset C$, $x_n \rightharpoonup x$ implies that $Ax_n \rightharpoonup^* Ax$;
(4) *hemicontinuous* if for each $x, y \in C$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) := A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Let $A : E \to 2^{E^*}$ be a multi-valued mapping. The domain of A is denoted by $D(A) := \{x \in E : Ax \neq \emptyset\}$ and the set of zeros of A is denoted by $A^{-1}0 := \{x \in D(A) : 0 \in Ax\}$. The graph of A is denoted by $G(A) := \{(x, y) \in E \times E^* : x \in D(A), y \in Ax\}$. A multi-valued mapping A is called *monotone* if

$$\langle x - y, u - v\rangle \geq 0 \quad \forall\ (x, u), (y, v) \in G(A).$$

A monotone mapping A on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone mapping on E. If A is maximal monotone, then $A^{-1}0$ is closed and convex.

The following lemma can be found in [33] (see also [34, Theorem 2.9]).

Lemma 2.8: *Let C be a nonempty, closed convex subset of a Banach space E. Let $A : C \to E^*$ be a monotone and hemicontinuous operator and $T : E \to 2^{E^*}$ be an operator defined as follows:*

$$Tx := \begin{cases} Ax + N_C(x) & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

where $N_C(x)$ is the normal cone for C at a point $x \in C$, that is,

$$N_C(x) := \{u \in E^* : \langle y - x, u\rangle \leq 0\ \forall\, y \in C\}.$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.9 ([35, Theorem 7.1.8]): *Let C be a nonempty, closed and convex subset of a Banach space E. Let A be a monotone and hemicontinuous mapping of C into E^*. Then $VI(C, A)$ is nonempty, closed and convex.*

Lemma 2.10 ([31, Proposition 9]): *Let $f : E \to \mathbb{R}$ be a Legendre function such that ∇f is weakly sequentially continuous. Suppose that the sequence $\{x_n\}$ is bounded and that $\lim_{n\to\infty} D_f(u, x_n)$ exists for any weak subsequential limit u of $\{x_n\}$. Then $\{x_n\}$ converges weakly to u.*

Lemma 2.11 ([36, Lemma 2.5]): *Assume that $\{a_n\}$ is a nonnegative real sequence such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 ([37, Lemma 3.1]): *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \quad \text{and} \quad a_k \le a_{m_k+1}.$$

In fact, $m_k := \max\{j \le k : a_j \le a_{j+1}\}$.

Lemma 2.13 ([38, Lemma 7]): *Assume that $\{\Gamma_n\}$ is a nonnegative real sequence such that*

$$\Gamma_{n+1} \le (1 - \delta_n)\Gamma_n + \delta_n\tau_n$$

and

$$\Gamma_{n+1} \le \Gamma_n - \eta_n + \rho_n,$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$, $\{\eta_n\}$ is a nonnegative real sequence and $\{\tau_n\}$, and $\{\rho_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(ii) $\lim_{n\to\infty} \rho_n = 0$;
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$, *implies* $\lim_{k\to\infty} \sup \tau_{n_k} \le 0$ *for any subsequence $\{n_k\}$ of $\{n\}$.*

Then $\lim_{n\to\infty} \Gamma_n = 0$.

3. Main results

In this section, we propose two Bregman projection-type methods for solving monotone VI in reflexive Banach spaces. In order to prove convergence results of our methods, we make the following assumption:

Assumption 3.1: (i) *The feasible set C is a closed and convex subset of a real reflexive Banach space E.*

(ii) *The function $f : E \to \mathbb{R}$ is σ-strongly convex, Legendre which is bounded and uniformly Fréchet differentiable on bounded subsets of E.*

(iii) *The mapping $A : E \to E^*$ is monotone and Lipschitz continuous with a constant $L > 0$.*

(iv) *The solution set of VI is nonempty, that is, $\Omega := VI(C, A) \neq \emptyset$.*

3.1. Weak convergence

In this subsection, we propose a Bregman projection-type method for solving VI in reflexive Banach spaces which is constructed based on Tseng's extragradient method. The algorithm is shown as below.

Remark 3.2: In view of Lemma 2.5, if Algorithm 1 stops in the n-th step of iterations, then y_n is a solution of VI. In what follows, we assume that the Algorithm 1 does not stop in any finite iterations and generates an infinite sequence $\{x_n\}$.

Lemma 3.3: *Let $\{\lambda_n\}$ be a sequence generated by* (15). *Then there exists $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \theta]$ such that $\lambda = \lim_{n\to\infty} \lambda_n$, where $\theta = \sum_{n=1}^{\infty} \theta_n$.*

Proof: Using the fact that A is Lipschitz-continuous with $L > 0$, in the case of $\langle x_{n+1} - y_n, Ax_n - Ay_n \rangle > 0$, we obtain

$$\frac{\epsilon_n \mu (\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\langle x_{n+1} - y_n, Ax_n - Ay_n\rangle} \geq \frac{2\epsilon_n \mu \|x_n - y_n\| \|x_{n+1} - y_n\|}{2\|x_{n+1} - y_n\| \|Ax_n - Ay_n\|}$$
$$\geq \frac{\mu \|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

The rest of the proof is same as in [7, Lemma 3.1]. ■

Remark 3.4: It is observe that the stepsize λ_n generated in Algorithm 1 is large which allowed to increase when the iteration increases. Therefore, the use of this new stepsize reduces the dependence on the initial stepsize λ_1. In particular, if $\epsilon_n = 1$ for all $n \geq 1$, then the stepsize λ_n generated in Algorithm 1 is similar to the methods in [7,39] and if $\epsilon_n = 1$ and $\theta_n = 0$ for all $n \geq 1$, then the stepsize λ_n generated in Algorithm 1 is similar to the methods in [10,40–42].

Algorithm 1:

Initialization: Choose $\lambda_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (12). Choose sequences $\{\theta_n\}$ and $\{\epsilon_n\}$ satisfy the following conditions:

(1) $\{\theta_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$;

(2) $\{\epsilon_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 1$.

Let $x_1 \in E$ be arbitrary.

Iterative steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = \Pi_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A x_n).$$

If $x_n = y_n$, then stop and y_n is a solution of VI. Otherwise,

Step 2. Compute

$$x_{n+1} = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\epsilon_n \mu \dfrac{\|x_{n+1} - y_n\|^2 + \|x_n - y_n\|^2}{2\langle x_{n+1} - y_n, Ax_n - Ay_n\rangle}, \lambda_n + \theta_n\right\} & \text{if } \langle x_{n+1} - y_n, Ax_n - Ay_n\rangle > 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{15}$$

Set $n := n + 1$ go to *Step 1*.

Lemma 3.5: *Assume that Assumption 3.1 is satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then for all $p \in \Omega$ and $n \geq 1$, we have*

$$D_f(p, x_{n+1}) \leq D_f(p, x_n) - \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) - \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(x_{n+1}, y_n).$$

Proof: Let $p \in \Omega$. Then we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n))) \\ &= f(p) - f(x_{n+1}) - \langle p - x_{n+1}, \nabla f(y_n) - \lambda_n(Ay_n - Ax_n)\rangle \end{aligned}$$

$$\begin{aligned}
&= f(p) - f(x_{n+1}) - \langle p - x_{n+1}, \nabla f(y_n)\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle + \langle p - y_n, \nabla f(y_n)\rangle \\
&\quad + f(y_n) - f(x_{n+1}) - \langle p - x_{n+1}, \nabla f(y_n)\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle - f(x_{n+1}) + f(y_n) \\
&\quad + \langle x_{n+1} - y_n, \nabla f(y_n)\rangle + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= D_f(p, y_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle. \qquad (16)
\end{aligned}$$

From the three-point identity of D_f, we have

$$D_f(p, y_n) = D_f(p, x_n) - D_f(y_n, x_n) + \langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle. \qquad (17)$$

Substituting (17) into (16), we have

$$\begin{aligned}
D_f(p, x_{n+1}) = D_f(p, x_n) &- D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\
&+ \langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \\
&+ \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle. \qquad (18)
\end{aligned}$$

Note that $y_n = \Pi_C^f \nabla f^*(\nabla f(x_n) - \lambda_n Ax_n)$, by Lemma 2.5, we have

$$\langle p - y_n, \nabla f(x_n) - \lambda_n Ax_n - \nabla f(y_n)\rangle \leq 0.$$

This implies that

$$\langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \leq \lambda_n \langle p - y_n, Ax_n\rangle. \qquad (19)$$

Substituting (19) into (18), we get

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle p - y_n, Ax_n\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n - Ax_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle p - y_n, Ax_n\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n\rangle - \lambda_n \langle p - x_{n+1}, Ax_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle x_{n+1} - y_n, Ax_n\rangle \\
&\quad + \lambda_n \langle p - x_{n+1}, Ay_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) + \lambda_n \langle x_{n+1} - y_n, Ax_n\rangle \\
&\quad - \lambda_n \langle y_n - p, Ay_n\rangle + \lambda_n \langle y_n - x_{n+1}, Ay_n\rangle \\
&= D_f(p, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\
&\quad + \lambda_n \langle x_{n+1} - y_n, Ax_n - Ay_n\rangle - \lambda_n \langle y_n - p, Ay_n\rangle. \qquad (20)
\end{aligned}$$

Since $y_n \in C$ and $p \in \Omega$, we have $\langle y_n - p, Ap\rangle \geq 0$ and the monotonicity of A implies that $\langle y_n - p, Ay_n\rangle \geq 0$. From (15), we see that

$$\begin{aligned}\lambda_{n+1} &= \min\left\{\epsilon_n\mu\frac{\|x_{n+1}-y_n\|^2+\|x_n-y_n\|^2}{2\langle x_{n+1}-y_n, Ax_n - Ay_n\rangle}, \lambda_n+\theta_n\right\}\\ &\leq \epsilon_n\mu\frac{\|x_{n+1}-y_n\|^2+\|x_n-y_n\|^2}{2\langle x_{n+1}-y_n, Ax_n - Ay_n\rangle}.\end{aligned}$$

This implies that

$$\langle x_{n+1}-y_n, Ax_n - Ay_n\rangle \leq \frac{\epsilon_n\mu}{2\lambda_{n+1}}\|x_{n+1}-y_n\|^2 + \frac{\epsilon_n\mu}{2\lambda_{n+1}}\|x_n-y_n\|^2. \tag{21}$$

Then from (20), (21) and (12), we have

$$\begin{aligned}D_f(p, x_{n+1}) &\leq D_f(p,x_n) - D_f(y_n,x_n) - D_f(x_{n+1},y_n)\\ &\quad + \lambda_n\langle x_{n+1}-y_n, Ax_n - Ay_n\rangle\\ &\leq D_f(p,x_n) - D_f(y_n,x_n) - D_f(x_{n+1},y_n)\\ &\quad + \frac{\epsilon_n\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}\|x_{n+1}-y_n\|^2 + \frac{\epsilon_n\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}\|x_n-y_n\|^2\\ &\leq D_f(p,x_n) - D_f(y_n,x_n) - D_f(x_{n+1},y_n)\\ &\quad + \frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}D_f(x_{n+1},y_n) + \frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}D_f(y_n,x_n)\\ &= D_f(p,x_n) - \left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)\\ &\quad - \left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(x_{n+1},y_n).\end{aligned}$$

Thus this lemma is proved. ■

Theorem 3.6: *Assume that Assumption 3.1 is satisfied. Suppose, in addition, that ∇f is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in Ω.*

Proof: We first prove that $\{x_n\}$ is bounded. Since $\lim_{n\to\infty}\lambda_n = \lambda > 0$, $\lim_{n\to\infty}\epsilon_n = 1$ and $\mu \in (0,\sigma)$, we have

$$\lim_{n\to\infty}\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) = 1-\frac{\mu}{\sigma} = \frac{\sigma-\mu}{\sigma} > 0.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}} > 0 \quad \forall\, n \geq n_0.$$

This implies that

$$\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)+\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(x_{n+1},y_n)\geq 0 \quad \forall\, n\geq n_0.$$

Then from Lemma 3.5, we have

$$D_f(p,x_{n+1})\leq D_f(p,x_n) \quad \forall\, n\geq n_0.$$

This show that $\lim_{n\to\infty} D_f(p,x_n)$ exists and hence $\{D_f(p,x_n)\}$ is bounded. Applying Lemma 12, we have $\{x_n\}$ is bounded and, in consequence $\{y_n\}$ is bounded. On the other hand, from Lemma 3.5, we see that

$$\begin{aligned}&\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)+\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(x_{n+1},y_n)\\&\quad\leq D_f(p,x_n)-D_f(p,x_{n+1}).\end{aligned}$$

Since $\lim_{n\to\infty} D_f(p,x_n)$ exists, there exists a nonnegative constant γ such that $\gamma=\lim_{n\to\infty} D_f(p,x_n)=\lim_{n\to\infty} D_f(p,x_{n+m})$ for all $m\in\mathbb{N}$. This implies that $\lim_{n\to\infty} D_f(y_n,x_n)=\lim_{n\to\infty} D_f(x_{n+1},y_n)=0$. Thus we have

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{22}$$

By the reflexivity of E and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\rightharpoonup z\in E$. Since $\|x_n-y_n\|\to 0$ and $y_n\in C$, we have $y_{n_k}\rightharpoonup z\in C$. We next prove $z\in\Omega$. Let T be a mapping defined as in Lemma 2.8. Then T is maximal monotone and $T^{-1}0=\Omega$. Let $(v,w)\in G(T)$. Since $w\in Tv=Av+N_C(v)$, we have $w-Av\in N_C(v)$ and hence $\langle v-x,w-Av\rangle\geq 0$ for all $x\in C$. Again, since $y_n\in C$, we have

$$\langle v-y_n,w-Av\rangle\geq 0. \tag{23}$$

From the definition of y_n and Lemma 2.5, we have $\langle v-y_n,\nabla f(x_n)-\lambda_nAx_n-\nabla f(y_n)\rangle\leq 0$ or, equivalently,

$$\left\langle v-y_n,\frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}-Ax_n\right\rangle\leq 0. \tag{24}$$

Then from (23) and (24), we have

$$\begin{aligned}\langle v-y_n,w\rangle&\geq\langle v-y_n,Av\rangle\\&\geq\langle v-y_n,Av\rangle+\left\langle v-y_n,\frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}-Ax_n\right\rangle\\&=\langle v-y_n,Av-Ax_n\rangle+\left\langle v-y_n,\frac{\nabla f(x_n)-\nabla f(y_n)}{\lambda_n}\right\rangle\\&=\langle v-y_n,Av-Ay_n\rangle+\langle v-y_n,Ay_n-Ax_n\rangle\end{aligned}$$

$$+\left\langle v - y_n, \frac{\nabla f(x_n) - \nabla f(y_n)}{\lambda_n} \right\rangle$$
$$\geq \langle v - y_n, Ay_n - Ax_n \rangle + \left\langle v - y_n, \frac{\nabla f(x_n) - \nabla f(y_n)}{\lambda_n} \right\rangle.$$

Since A is Lipschitz continuous and from (22), we have $\langle v - z, w \rangle \geq 0$. By the maximality of T, we obtain $z \in T^{-1}0$ and hence $z \in \Omega$. In summary, we have shown that $\lim_{n\to\infty} D_f(z, x_n)$ exists for any weak subsequential limit z of $\{x_n\}$. Thus by Lemma 2.10, we conclude $\{x_n\}$ converges weakly to a point in Ω. The proof is completed. ■

If E is a 2-uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have the following weak convergence result.

Corollary 3.7: *Let C be a nonempty, closed and convex subset of E, $A : E \to E^*$ be a monotone and Lipschitz continuous with a constant $L > 0$. Assume that $\Omega \neq \emptyset$. Choose $\lambda_1 > 0$ and $\mu \in (0, c)$, where c is a constant given by (11). Choose a real sequence $\{\theta_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$ and a real sequence $\{\epsilon_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 1$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)) \quad \forall\, n \geq 1, \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\epsilon_n\mu \dfrac{\|x_{n+1} - y_n\|^2 + \|x_n - y_n\|^2}{2\langle x_{n+1} - y_n, Ax_n - Ay_n\rangle}, \right. & \\ \left. \qquad \lambda_n + \theta_n \right\} & \text{if } \langle x_{n+1} - y_n, Ax_n \\ & \quad -Ay_n\rangle > 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases}$$

Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ converges weakly to a point in Ω.

3.2. Strong convergence

In this subsection, we propose another Bregman projection-type method for solving VI in reflexive Banach spaces, which is constructed based on Tseng's extragradient method and Halpern-type iteration.

Theorem 3.8: *Suppose that Assumption 3.1 is satisfied. If $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z = \Pi_{\Omega}^{f}(u)$.*

Algorithm 2:

Initialization: Choose $\lambda_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (12). Choose sequences $\{\theta_n\}$ and $\{\epsilon_n\}$ satisfy the following conditions:

(1) $\{\theta_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$;
(2) $\{\epsilon_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 1$.

Let $u, x_1 \in E$ be arbitrary.
Iterative steps: Given the current iterate x_n, calculate x_{n+1} as follows:
Step 1. Compute

$$y_n = \Pi_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A x_n).$$

If $x_n = y_n$, then stop and y_n is a solution of VI. Otherwise,
Step 2. Compute

$$z_n = \nabla f^*(\nabla f(y_n) - \lambda_n (A y_n - A x_n)).$$

Step 3. Compute

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(z_n)),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\epsilon_n \mu \dfrac{\|z_n - y_n\|^2 + \|x_n - y_n\|^2}{2\langle z_n - y_n, Ax_n - Ay_n\rangle}, \right. & \\ \left. \qquad \lambda_n + \theta_n \right\} & \text{if } \langle z_n - y_n, Ax_n \\ & \quad -Ay_n\rangle > 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ go to *Step 1*.

Proof: We first prove that $\{x_n\}$ is bounded. Using the same arguments as in the proof of Lemma 3.5, we can show that

$$\begin{aligned} D_f(p, z_n) \le D_f(p, x_n) &- \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) \\ &- \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n). \end{aligned} \tag{25}$$

Moreover, as proved in Theorem 3.6, we can deduce that

$$\left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) + \left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n) \ge 0 \quad \forall\, n \ge n_0.$$

Then from (25), we have

$$D_f(p, z_n) \leq D_f(p, x_n) \quad \forall\, n \geq n_0. \tag{26}$$

Hence

$$\begin{aligned} D_f(p, x_{n+1}) &\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, z_n) \\ &\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, x_n) \\ &\leq \max\{D_f(z, u), D_f(z, x_n)\} \\ &\ \ \vdots \\ &\leq \max\{D_f(z, u), D_f(z, x_{n_0})\}. \end{aligned}$$

This implies that $\{D_f(p, x_n)\}$ is bounded. Applying (12), we have $\{x_n\}$ is bounded and, in consequence $\{y_n\}$ and $\{z_n\}$ are bounded. Let $z = \Pi^f_\Omega(u)$. From (25), we have

$$\begin{aligned} D_f(z, x_{n+1}) &\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, z_n) \\ &\leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, x_n) \\ &\quad - (1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) \\ &\quad - (1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n). \end{aligned}$$

This implies that

$$\begin{aligned} &(1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(y_n, x_n) + (1-\alpha_n)\left(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}\right) D_f(z_n, y_n) \\ &\quad \leq \alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, z_n) - D_f(z, x_{n+1}) \\ &\quad = D_f(z, x_n) - D_f(z, x_{n+1}) + \alpha_n (D_f(z, u) - D_f(z, x_n)) \\ &\quad \leq D_f(z, x_n) - D_f(z, x_{n+1}) + \alpha_n M, \end{aligned} \tag{27}$$

where $M = \sup_{n\geq 1}\{|D_f(z, u) - D_f(z, x_n)|\}$.

We now consider the following two possible cases to prove $x_n \to z$.

Case 1. Suppose that there exists $N \in \mathbb{N}$ such that $\{D_f(z, x_n)\}$ is nonincreasing for all $n \geq N$. From this we have $\lim_{n\to\infty} D_f(z, x_n)$ exists and hence $\lim_{n\to\infty}(D_f(z, x_n) - D_f(z, x_{n+1})) = 0$. Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty}(1 - \frac{\epsilon_n \mu}{\sigma} \frac{\lambda_n}{\lambda_{n+1}}) > 0$ and from (27), we have $\lim_{n\to\infty} D_f(y_n, x_n) = \lim_{n\to\infty} D_f(z_n, y_n) = 0$. Hence

$$\lim_{n\to\infty} \|\nabla f(y_n) - \nabla f(x_n)\| = \lim_{n\to\infty} \|\nabla f(z_n) - \nabla f(y_n)\| = 0. \tag{28}$$

We also have

$$\|\nabla f(z_n) - \nabla f(x_n)\| \leq \|\nabla f(z_n) - \nabla f(y_n)\| + \|\nabla f(y_n) - \nabla f(x_n)\|$$

$$\to 0, \quad n \to \infty. \tag{29}$$

On the other hand, we observe that

$$\begin{aligned}\|\nabla f(x_{n+1}) - \nabla f(x_n)\| &\le \|\nabla f(x_{n+1}) - \nabla f(z_n)\| + \|\nabla f(z_n) - \nabla f(x_n)\| \\ &= \alpha_n \|\nabla f(u) - \nabla f(z_n)\| + \|\nabla f(z_n) - \nabla f(x_n)\|.\end{aligned}$$

Again, since $\lim_{n\to\infty} \alpha_n = 0$ and from (29), we have

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0.$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{30}$$

By the reflexivity of E and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$ as $k \to \infty$ and

$$\limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle = \lim_{k\to\infty} \langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle.$$

Since $\|\nabla f(y_n) - \nabla f(x_n)\| \to 0$, we have $\|y_n - x_n\| \to 0$. Further, by the same arguments as in the proof of Theorem 3.6, we can show that $\hat{x} \in \Omega$. Then from Lemma 2.5, we have

$$\limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle = \langle \hat{x} - z, \nabla f(u) - \nabla f(z)\rangle \le 0. \tag{31}$$

From (30) and (31), we have

$$\begin{aligned}&\limsup_{n\to\infty} \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\quad \le \limsup_{n\to\infty} \langle x_{n+1} - x_n, \nabla f(u) - \nabla f(z)\rangle \\ &\qquad + \limsup_{n\to\infty} \langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \le 0.\end{aligned} \tag{32}$$

From the properties of V_f and (26), we have

$$\begin{aligned}D_f(z, x_{n+1}) &= V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)) \\ &\le V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n) - \alpha_n(\nabla f(u) - \nabla f(z))) \\ &\quad + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &= V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(z_n)) \\ &\quad + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &= D_f(z, \nabla f^*(\alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(z_n)) \\ &\quad + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle \\ &\le \alpha_n D_f(z,z) + (1-\alpha_n) D_f(z, z_n) + \alpha_n \langle x_{n+1} - z, \nabla f(u) - \nabla f(z)\rangle\end{aligned}$$

$$\leq (1-\alpha_n)D_f(z,x_n) + \alpha_n\langle x_{n+1}-z, \nabla f(u) - \nabla f(z)\rangle. \tag{33}$$

From (32) and Lemma 2.11, we conclude $\lim_{n\to\infty} D_f(z,x_n) = 0$. Therefore, $x_n \to z$.

Case 2. Suppose that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $D_f(z,x_{n_j}) < D_f(z,x_{n_j+1})$ for all $j \in \mathbb{N}$. Then by Lemma 2.12, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$D_f(z,x_{m_k}) \leq D_f(z,x_{m_k+1}) \quad \text{and} \quad D_f(z,x_k) \leq D_f(z,x_{m_k+1}).$$

From (27), we note that

$$\begin{aligned}
&(1-\alpha_{m_k})\left(1-\frac{\epsilon_{m_k}\mu}{\sigma}\frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right)D_f(y_{m_k},x_{m_k}) \\
&\quad + (1-\alpha_{m_k})\left(1-\frac{\epsilon_{m_k}\mu}{\sigma}\frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right)D_f(z_{m_k},y_{m_k}) \\
&\leq D_f(z,x_{m_k}) - D_f(z,x_{m_k+1}) + \alpha_{m_k}M \\
&\leq \alpha_{m_k}M,
\end{aligned} \tag{34}$$

where $M > 0$. Thus we have

$$\lim_{k\to\infty} D_f(y_{m_k},x_{m_k}) = \lim_{k\to\infty} D_f(z_{m_k},y_{m_k}) = 0.$$

Following the line in the proof of Case 1, we can show that

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty}\langle x_{m_k+1}-z, \nabla f(u) - \nabla f(z)\rangle \leq 0.$$

Moreover, we can show that

$$\begin{aligned}
D_f(z,x_{m_k+1}) &\leq (1-\alpha_{m_k})D_f(z,x_{m_k}) + \alpha_{m_k}\langle x_{m_k+1}-z, \nabla f(u) - \nabla f(z)\rangle \\
&\leq (1-\alpha_{m_k})D_f(z,x_{m_k+1}) + \alpha_{m_k}\langle x_{m_k+1}-z, \nabla f(u) - \nabla f(z)\rangle.
\end{aligned}$$

This implies that

$$D_f(z,x_k) \leq D_f(z,x_{m_k+1}) \leq \langle x_{m_k+1}-z, \nabla f(u) - \nabla f(z)\rangle.$$

From this, we have $\limsup_{k\to\infty} D_f(z,x_k) = 0$. Therefore, $x_k \to z$. The proof is completed. ∎

Remark 3.9: Another proof of Theorem 3.8 can be used Lemma 2.13. As proved in Theorem 3.8 and from (25), we have

$$D_f(z,x_{n+1}) \le (1-\alpha_n)D_f(z,x_n) - (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)$$
$$- (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(z_n,y_n)$$
$$+ \alpha_n\langle x_{n+1}-z, \nabla f(u)-\nabla f(z)\rangle. \tag{35}$$

For each $n \ge 1$, let

$$\Gamma_n := D_f(z,x_n),$$
$$\tau_n := \langle x_{n+1}-z, \nabla f(u)-\nabla f(z)\rangle,$$
$$\eta_n := (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(y_n,x_n)$$
$$+ (1-\alpha_n)\left(1-\frac{\epsilon_n\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right)D_f(z_n,y_n)$$

and

$$\rho_n := \alpha_n\langle x_{n+1}-z, \nabla f(u)-\nabla f(z)\rangle.$$

Then (35) reduces to the following inequalities:

$$\Gamma_{n+1} \le (1-\alpha_n)\Gamma_n + \alpha_n\tau_n, \quad n \ge 1$$

and

$$\Gamma_{n+1} \le \Gamma_n - \eta_n + \rho_n, \quad n \ge 1.$$

Since $\lim_{n\to\infty}\alpha_n = 0$, we have $\lim_{n\to\infty}\rho_n = 0$. In order to complete the proof, it is sufficient to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \le 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. Hence $\lim_{k\to\infty}D_f(y_{n_k},x_{n_k}) = \lim_{k\to\infty}D_f(z_{n_k},y_{n_k}) = 0$ and so

$$\lim_{k\to\infty}\|\nabla f(y_{n_k})-\nabla f(x_{n_k})\| = \lim_{k\to\infty}\|\nabla f(z_{n_k})-\nabla f(y_{n_k})\| = 0.$$

Then we can show that

$$\lim_{k\to\infty}\|\nabla f(z_{n_k})-\nabla f(x_{n_k})\| = \lim_{k\to\infty}\|\nabla f(x_{n_k+1})-\nabla f(x_{n_k})\| = 0$$

and

$$\limsup_{k\to\infty}\langle x_{n_k+1}-z, \nabla f(u)-\nabla f(z)\rangle \le 0.$$

Hence $\limsup_{k\to\infty}\tau_{n_k} \le 0$. We conclude by Lemma 2.13 that $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, $x_n \to z$ as $n \to \infty$.

If E is a 2-uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have the following strong convergence result.

Corollary 3.10: *Let C be a nonempty, closed and convex subset of E, $A : E \to E^*$ be a monotone and Lipschitz continuous with a constant $L > 0$. Assume that $\Omega \neq \emptyset$. Choose $\lambda_1 > 0$ and $\mu \in (0, c)$, where c is a constant given by (11). Choose a real sequence $\{\theta_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$ and a real sequence $\{\epsilon_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 1$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u, x_1 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n) \quad \forall\, n \geq 1, \end{cases}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\epsilon_n \mu \dfrac{\|z_n - y_n\|^2 + \|x_n - y_n\|^2}{2\langle z_n - y_n, Ax_n - Ay_n\rangle}, \ \lambda_n + \theta_n\right\} & \text{if } \langle z_n - y_n, Ax_n - Ay_n\rangle > 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases}$$

If $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ converges strongly to a point in Ω.

Remark 3.11: Our results generalize and improve the main results of Shehu [12] in the following senses:

- We generalize the Shehu's results [12] from 2-uniformly convex Banach spaces to reflexive Banach spaces.
- For the structure of Banach spaces, we generalize duality mapping in Shehu's results [12] to a differentiable of Legendre-type function.
- The sequence of stepsizes of our methods is chosen without the prior knowledge of the Lipschitz constant of the mapping and without the uniform smoothness constant of E^* while the sequence of stepsizes of Sehu's results [12] requires the prior knowledge of them.

4. Numerical examples

In this section, we perform some numerical experiments to illustrate the numerical behaviour of Algorithms 1 and 2 and compare the proposed algorithms with Algorithms 5 and 6. For our experiments, we take $\theta_n = \frac{1000}{(n+1)^{1.1}}$ and $\epsilon_n =$

Table 1. Numerical results for Example 4.1.

		Algorithm 1	Algorithm 5	Algorithm 2	Algorithm 6
$m = 100$	Iter.	25	795	25	23599
	Time (s)	0.03	0.02	0.01	0.50
$m = 1000$	Iter.	22	383	22	23541
	Time (s)	1.20	2.09	0.93	73.6

$1 + \frac{1000}{(n+1)^{1.1}}$ in Algorithms 1 and 2. To terminate the algorithms, we use the condition $\|x_n - y_n\| \leq \varepsilon$ for all the algorithms. In what follows, we use 'Iter.' and 'Time (s)' to denote the number of iterations and the CPU time in seconds, respectively.

Example 4.1: We consider the HpHard problem which is taken from [43]. Let A be an operator defined by $Ax = Mx + q$ with $q \in \mathbb{R}^m$ and

$$M := NN^T + S + D,$$

where every entry of the $m \times m$ matrix N and of the $m \times m$ skew-symmetric matrix S is uniformly generated from $(-5, 5)$, and every diagonal entry of the $m \times m$ diagonal D is uniformly generated from $(0, 0.3)$ (we get M is positive definite), with every entry of q uniformly generated from $(-500, 0)$. Then A is monotone and Lipschitz continuous with $L = \|M\|$. The feasible set C is the nonnegative orthant $\mathbb{R}^m_+ := \{x = (x^{(1)}, x^{(2)}, \dots, x^{(m)})^T : x^{(i)} \geq 0\}$. We take $\lambda_n = \lambda = \frac{0.7}{\|L\|}$ in Algorithms 5 and 6, we take $\lambda_1 = \frac{0.7}{\|L\|}$ in our methods. For Algorithms 6 and 2, we take $\alpha_n = \frac{1}{n+2}$. For all tests, we take $u = x_1 = (1, 1, \dots, 1)^T$ and $\varepsilon = 10^{-3}$. For every m, as shown in Table 1, we have generated two random samples with different choice of M and q.

Example 4.2: We consider a classical problem considered in [44,45]. The feasible set C is the nonnegative orthant $\mathbb{R}^m_+$. Let $Ax := Qx$, where Q is a square matrix $m \times m$ given by the following condition:

$$q_{i,j} = \begin{cases} -1, & \text{if } j = m + 1 - i \text{ and } j > i, \\ 1, & \text{if } j = m + 1 - i \text{ and } j < i, \\ 0, & \text{otherwise.} \end{cases}$$

This is a classical example of a problem where usual gradient method does not converge. For even m, the zero vector is the solution of this problem. We take $\lambda_n = \lambda = 0.7$ in Algorithms 5 and 6, we take $\lambda_1 = 0.7$ in our methods. For Algorithms 6 and 2, we take $\alpha_n = \frac{1}{n+2}$. For all tests, we take $u = x_1 = (1, 1, \dots, 1)^T$ and $\varepsilon = 10^{-3}$. The numerical results are showed in Table 2.

Finally, we give a concrete example in a non-Euclidean distance to support our main results.

Table 2. Numerical results for Example 4.2.

		Algorithm 1	Algorithm 5	Algorithm 2	Algorithm 6
$m = 300$	Iter.	46	1424	46	1436
	Time (s)	0.006	0.172	0.006	0.125
$m = 2000$	Iter.	46	1424	46	1436
	Time (s)	0.561	16.45	0.536	17.47
$m = 5000$	Iter.	46	1424	46	1436
	Time (s)	3.592	111.7	3.537	110.4

Example 4.3: In this example, we consider the negative entropy function $f : \mathbb{R}^m \to (-\infty, \infty]$ over the nonnegative orthant $C = \mathbb{R}^m_+$ defined by

$$f(x) := \begin{cases} \sum_{i=1}^{m} x^{(i)} \ln x^{(i)} & \text{if } x \in \mathbb{R}^m_+, \\ \infty & \text{otherwise,} \end{cases}$$

In this case $0 \ln 0 = 0$. It is known that f satisfy (ii) in Assumption 3.1 (see [24, Example 2.1]) and f is 1-strongly convex [14, Remark 5.18]. The Bregman distance with respect to f defined for any $x \in \mathbb{R}^m_+$ and $y \in \mathbb{R}^m_{++}$ by

$$D_f(x,y) = \sum_{i=1}^{m} \left(x^{(i)} \ln \frac{x^{(i)}}{y^{(i)}} + y^{(i)} - x^{(i)} \right),$$

which is the so-called Kullback-Leibler divergence distance measure. Then we have the Bregman projection with respect to f,

$$\Pi^f_C(y) = \operatorname{argmin}\{D_f(x,y) : x \in \mathbb{R}^m_+\}.$$

Moreover, we have

$$\nabla f(x) = (1 + \ln x^{(1)}, 1 + \ln x^{(2)}, \ldots, 1 + \ln x^{(m)})^T \quad \text{and}$$
$$\nabla f^*(x) = (e^{x^{(1)}-1}, e^{x^{(2)}-1}, \ldots, e^{x^{(m)}-1})^T.$$

Let A be the same as in Example 4.1. We can choose $\lambda_1 > 0$ and $\mu \in (0,1)$. Then Algorithms 1 and 2 converge weakly and strongly, respectively, to a point in $VI(C, A)$.

5. Conclusions

In this work, we have proposed two modified Bregman projection-type methods with a generalized adaptive stepsize for solving monotone variational inequalities in reflexive Banach spaces. Our algorithms are motivated by the results of Shehu [12] and Bregman [25]. The weak and strong convergence of the proposed methods have been established without the prior knowledge of the Lipschitz constant of the cost mapping. Some numerical experiments have been performed to illustrate the effectiveness of the proposed algorithms.

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References

[1] Korpelevich GM. The extragradient method for finding saddle points and other problem. Ekonomika i Matematicheskie Metody. 1976;12:747–756.
[2] Antipin AS. On a method for convex programs using a symmetrical modification of the Lagrange function. Ekonomika i Matematicheskie Metody. 1976;12(6):1164–1173.
[3] Tseng P. A modified forward-backward splitting method for maximal monotone mapping. SIAM J Control Optim. 2000;38:431–446.
[4] Anh PK, Thong DV, Vinh NT. Improved inertial extragradient methods for solving pseudo-monotone variational inequalities. Optimization. 2020;71:505–528. https://doi.org/10.1080/02331934.2020.1808644.
[5] Chen J, Liu S, Chang X. Modified Tseng's extragradient methods for variational inequality on Hadamard manifolds. Appl Anal. 2019;100:2627–2640. https://doi.org/10.1080/00036811.2019.1695783.
[6] Hammad HA, Rehman H, la Sen MD. Shrinking projection methods for accelerating relaxed inertial tseng-type algorithm with applications. Math Probl Eng. 2020;2020:7487–383.
[7] Liu HW, Yang J. Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput Optim Appl. 2020;77:491–508.
[8] Thong DV, Hieu DV. Weak and strong convergence throrems for variational inequality problems. Numer Algor. 2018;78(4):1045–1060.
[9] Thong DV, Vinh NT, Hieu DV. Accelerated hybrid and shrinking projection methods for variational inequality problems. Optimization. 2019;68(5):981–998.
[10] Thong DV, Yang J, Cho YJ, et al. Explicit extragradient-likemethod with adaptive stepsizes for pseudomonotone variational inequalities. Optim Lett. 2021;15:2181–2199.
[11] Yang J, Liu H. Strong convergence result for solving monotone variational inequalities in Hilbert space. Numer Algorithms. 2019;80(3):741–752.
[12] Shehu Y. Single projection algorithm for variational inequalities in Banach spaces with application to contact problem. Acta Math Sci. 2020;40:1045–1063.
[13] Rockafellar RT. Convex analysis. Princeton: Princeton University Press; 1970.

[14] Beck A. First-order methods in optimization. Philadelphia: Society for Industrial and Applied Mathematics; 2017.
[15] Barbu V, Precupanu T. Convexity and optimization in Banach spaces. London New York: Springer Dordrecht Heidelberg; 2010.
[16] Pathak HK. An introduction to nonlinear analysis and fixed point theory. Springer Nature Singapore Pte Ltd. 2018.
[17] Reich S, Sabach S. A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J Nonlinear Convex Anal. 2009;1:471–485.
[18] Zălinescu C. Convex analysis in general vector spaces. River Edge: World Scientific, Publishing Co. Inc.; 2002.
[19] Reich S, Sabach S. Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numerical Funct Anal Optim. 2010;31(1):2–44.
[20] Bonnans JF, Shapiro A. Perturbation analysis of optimization problems. New York: Springer Verlag; 2000.
[21] Bauschke HH, Borwein JM, Combettes PL. Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun Contemp Math. 2001;3:615–647.
[22] Reich S, Sabach S. Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. TMA. 2010;73:122–135.
[23] Bauschke HH, Borwein JM. Legendre functions and the method of random Bregman projections. J Convex Anal. 1997;4:27–67.
[24] Bauschke HH, Wang X, Ye J, et al. Bregman distances and Chebyshev sets. J Approximation Theory. 2009;159:3–25.
[25] Bregman LM. The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput Math Math Phys. 1967;7:200–217.
[26] Alber YI. Metric and generalized projection operators in Banach spaces: properties and applications. In: A.G. Kartsatos editiors. Theory and applications of nonlinear operator of accretive and monotone type. New York: Marcel Dekker; 1996. p. 15–50.
[27] Reich S. A weak convergence theorem for the alternating method with Bregman distances. In: Theory and applications of nonlinear operators of accretive and monotone type. New York: Marcel Dekker; 1996. p. 313–318.
[28] Nakajo K. Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces. Appl Math Comput. 2015;271:251–258.
[29] Butnariu D, Resmerita E. Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr Appl Anal. 2006;2006:84919.
[30] Alber YI. Generalized projection operators in Banach spaces: properties and applications. In: Proceedings of the Israel Seminar, Vol. 1 of Functional Differential Equation, Ariel. 1993. p. 1–21.
[31] Martín-Márquez V, Reich S, Sabach S. Iterative methods for approximating fixed points of Bregman nonexpansive operators. Discrete Continuous Dyn Syst Ser S. 2013;6(4):1043–1063.
[32] Butnariu D, Iusem AN. Totally convex functions for fixed points computation and infinite dimensional optimization. Dordrecht: Kluwer Academic Publishers; 2000.
[33] Rockafellar RT. On the maximality of sums of nonlinear monotone operators. Trans Amer Math Soc. 1970;149:75–88.
[34] Iiduka H, Takahashi W. Weak convergence of a projection algorithm for variational inequalities in a Banach space. J Math Anal Appl. 2008;339:668–679.
[35] Takahashi W. Nonlinear functional analysis. Yokohama: Yokohama Publishers; 2000.

[36] Xu HK. Iterative algorithms for nonlinear operators. J London Math Soc. 2002;66(1): 240–256.
[37] Maingé PE. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. 2008;16:899–912.
[38] He S, Yang C. Solving the variational inequality problem defined on intersection of finite level sets. Abstr Appl Anal. 2013;2013:8.
[39] Yang J, Cholamjiak P, Sunthrayuth P. Modified Tseng's splitting algorithms for the sum of two monotone operators in Banach spaces. AIMS Math. 2021;6(5):4873–4900.
[40] Ma F. A subgradient extragradient algorithm for solving monotone variational inequalities in Banach spaces. J Inequalities Appl.. 2020;2020:26.
[41] Yang J. Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities. Appl Anal. 2021;100(5):1067–1078.
[42] Yang J, Liu HW, Liu ZX. Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization. 2018;67(12):2247–2258.
[43] Harker PT, Pang JS. A damped-Newton method for the linear complementarity problem. In: Allgower G, Georg K, editors. Computational solution of nonlinear systems of equations. Providence: AMS; 1990. p. 265–284. (Lectures in Applied Mathematics; 26).
[44] Malitsky YV. Projected reflected gradient methods for variational inequalities. SIAM J Optim. 2015;25(1):50–520.
[45] Mainge PE, Gobinddass ML. Convergence of one-step projected gradient methods for variational inequalities. J Optim Theory Appl. 2016;171:146–168.

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ORIGINAL RESEARCH

An inertial self-adaptive algorithm for the generalized split common null point problem in Hilbert spaces

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Abstract
In this paper, we propose an inertial self-adaptive algorithm for solving the generalized split common null point problem introduced by Reich and Tuyen (Optimization 69(5): 1013–1038, 2020). The strong convergence theorem of our proposed method is established in real Hilbert spaces. As applications, we apply our main result to the generalized split feasibility problem, the generalized split equilibrium problem and the generalized split minimization problem. Finally, we provide numerical experiments to show the efficiency and advantage of the proposed method.

Keywords Maximal monotone operator · Hilbert space · Strong convergence · Self adaptive method

Mathematics Subject Classification 47H09 · 47H10 · 47J25 · 47J05

1 Introduction

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator with its adjoint T^* and T. Recall the following *split feasibility problem* (SFP): find an element

$$z \in C \text{ such that } Tz \in Q. \tag{1.1}$$

The SFP was first introduced by Censor and Elfving [9] in 1994 for modeling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modeling of intensity modulated radiation therapy. Moreover, it plays an important role in medical image reconstruction and signal processing (see [6, 7]).

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Some generalizations of the SFP have been widely studied by many authors (see for instance [23, 37, 38, 42–44, 47, 48]). In particular, Byrne et al. [8] considered the following *split common null point problem* (SCNPP): find an element $z \in H_1$ such that

$$z \in A_1^{-1}0 \cap T^{-1}(A_2^{-1}0), \tag{1.2}$$

where $A_1 : H_1 \to 2^{H_1}$ and $A_2 : H_2 \to 2^{H_2}$ are set–valued operators and $T : H_1 \to H_2$ is a bounded linear operator with its adjoint operator T^* of T. It is noted that the SCNPP is a generalization of the SFP when $A_1 = \partial i_C$ and $A_2 = \partial i_Q$, where ∂i_C and ∂i_Q are subdifferential of indicator functions of C and Q, respectively.

Byrne et al. [8] proposed an efficient method for solving the SCNPP in a Hilbert space H_1. This method is defined in the following way: for any fixed $u, x_1 \in H_1$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_r^{A_1}(x_n - \tau T^*(I - Q_\mu^{A_2})Tx_n), \quad \forall n \geq 1, \tag{1.3}$$

where $J_r^{A_1} := (I + rA_1)^{-1}$ and $Q_\mu^{A_2} := (I + \mu A_2)^{-1}$ are the resolvent operators of A_1 for $r > 0$ and A_2 for $\mu > 0$, respectively, and T is the bounded linear operator with its adjoint operator T^*. They proved the strong convergence of the sequence generated by (1.3) to a solution of the SCNPP (1.2) provided the step-size $\tau \in \left(0, \frac{2}{\|T\|^2}\right)$. However, it is observed that many iterative methods suggested as above involve step-size that depends on the norm of a bounded linear operator or matrix in the finite-dimensional space. It is very difficult to calculate the norm of a bounded linear operator (see, Theorem 2.3 of [18]). To overcome this drawback, many authors have constructed the step-size that does not require the norm of a bounded linear operator. This technique is called a *self-adaptive method* introduced by López et al. [22] (see also [50]). In recent years, there have been many authors who modified some methods for solving the other split type problems such that the step-sizes do not depend on the operator norms (see for instance [15, 20, 22, 28, 29, 33, 35, 39, 46, 49, 50]).

Very recently, Reich and Tuyen [32] introduced the following new type split problem so-called *generalized split common null point problem* (GSCNPP). Let H_i, $i = 1, 2, \ldots, N$ be real Hilbert spaces and $A_i : H_i \to 2^{H_i}$, $i = 1, 2, \ldots, N$ be maximal monotone operators on H_i, respectively. Let $T_i : H_i \to H_{i+1}$, $i = 1, 2, \ldots, N-1$, be bounded linear operators. This problem is to find an element $z \in H_1$ such that

$$0 \in A_1 z, \; 0 \in A_2(T_1 z), \ldots, 0 \in A_N(T_{N-1}(T_{N-2} \cdots T_1 z)). \tag{1.4}$$

The set of solutions of problem (1.4) is denoted by

$$\Omega := A_1^{-1}0 \cap T_1^{-1}(A_2^{-1}0) \cap \ldots \cap T_1^{-1}(T_2^{-1} \ldots (T_{N-1}^{-1}(A_N^{-1}0))) \neq \emptyset. \tag{1.5}$$

It is noted that this problem is a generalization of the SFP and other split type problems. At the same time, they proposed the following iterative algorithm for solving the generalized split common null point problem (1.4) for the crucial case $N = 3$ in three Hilbert spaces H_i for $i = 1, 2, 3$: for any $u, x_1 \in H_1$ and

$$\begin{cases} y_{n,1} = x_n - \tau_1 T_1^* T_2^* (I^{H_3} - J_{r_{n,1}}^{A_3}) T_2 T_1 x_n, \\ y_{n,2} = y_{n,1} - \tau_2 T_1^* (I^{H_2} - J_{r_{n,2}}^{A_2}) T_1 y_{n,1}, \\ y_{n,3} = J_{r_{n,1}}^{A_1}(y_{n,2}), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_{n,3}, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $J_{r_i}^{A_i}$ $(i = 1, 2, 3)$ are the resolvent operators of A_i for $r_i > 0$ and $i = 1, 2, 3$, and $T : H_i \to H_{i+1}, i = 1, 2$ be bounded linear operators.

On the other hand, a technique used to speed up the convergence rate of algorithms is the inertia (see [4, 25, 30]). In 2001, Alvarez and Attouch [4] used the inertial technique to obtain an inertial proximal method for solving the problem of finding zero of a maximal monotone operator. This algorithm is of the following form:

$$x_{n+1} = J_{\lambda_n}^{A}(x_n + \theta_n(x_n - x_{n-1})), \quad \forall n \geq 1, \tag{1.7}$$

where $J_\lambda^A := (I + \lambda A)^{-1}$ is the resolvent operator of A for $\lambda > 0$. It was proved that if $\{\lambda_n\}$ is non-decreasing and $\{\theta_n\} \subset [0, 1)$ is chosen such that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, then the sequence generated by (1.7) converges weakly to a zero of A. Recently, inertial technique has been used to accelerate the convergence rate of algorithms (see for instance [3, 10, 11, 21, 27, 36, 45]).

In this paper, motivated and inspired by the works of Attouch [4], Lopez et al. [22] and Reich and Tuyen [32], we propose a new inertial self-adaptive algorithm for solving the generalized split common null point problem (1.4) in real Hilbert spaces. The step-sizes of our method do not depend on the operator norms of the bounded linear operators, which are easy to implement.

The outline of the paper is divided as follows. In Sect. 2, we give some definitions and useful lemmas which are need for our work. In Sect. 3, we prove a strong convergence theorem of the proposed method. In Sect. 4, we give some applications of the main result to the generalized split feasibility problem, the generalized split equilibrium problem and the generalized split minimization problem. Finally, some numerical experiments including comparisons with other algorithms are presented in Sect. 5.

2 Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C, then $x_n \to x$ ($x_n \rightharpoonup x$) denotes strong (weak) convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to x. Let $S : C \to C$ be a nonlinear mapping. We denote $F(S)$ by the set of fixed points of S, that is, $F(S) = \{x \in C : x = Sx\}$. A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

For a closed and convex subset C of H, there exists a unique nearest point in C, denoted by $P_C^H(x)$, such that

$$\|x - P_C^H(x)\| \leq \|x - y\|, \quad \forall x \in H \text{ and } y \in C.$$

Such P_C^H is called the *metric projection* of H onto C. It is well known that P_C^H is a nonexpansive mapping of H onto C (see [5]). Moreover, P_C^H has the following properties ([17, 40]): for each $x \in H$,

$$z = P_C^H(x) \iff \langle x - z, y - z \rangle \leq 0, \ \forall y \in C. \tag{2.1}$$

A set-valued mapping $A : H \to 2^H$ is said to be *monotone* if for each $x, y \in H$ such that

$$\langle u - v, x - y \rangle \geq 0 \text{ for } u \in Ax \text{ and } v \in Ay.$$

We denote by I^H the identity operator on H. A monotone operator A is said to be *maximal monotone* if there is no proper monotone extension of A. The *resolvent operator* of A [14], denoted by $J_r^A : H \to H$ which is defined by

$$J_\lambda^A(x) = (I + rA)^{-1}(x), \quad \forall x \in H,$$

where r is any positive number and also denote $A^{-1}0$ by the set of zeros of A, that is, $A^{-1}0 = \{x \in H : 0 \in Ax\}$. It is well known that J_r^A is a single-valued nonexpansive mapping and $F(J_r^A) = A^{-1}0$ for $r > 0$ (see [40]).

We next recall some facts which will be needed in the rest of this paper.

Lemma 2.1 ([26, 32]) *Let $A : H \to 2^H$ be a monotone operator. Then the following statements hold:*

(i) *For $0 < s \leq r$ and $x \in H$, we have*

$$\|x - J_s^A(x)\| \leq 2\|x - J_r^A(x)\|.$$

(ii) *For all $r > 0$ and $x, y \in H$, we have*

$$\langle x - y, J_r^A(x) - J_r^A(y) \rangle \geq \|J_r^A(x) - J_r^A(y)\|^2.$$

(iii) *For all $r > 0$ and $x, y \in H$, we have*

$$\langle (I^H - J_r^A)x - (I^H - J_r^A)y, x - y \rangle \geq \|(I^H - J_r^A)x - (I^H - J_r^A)y\|^2.$$

(iv) *If $A^{-1}0 \neq \emptyset$, then for all $p \in A^{-1}0$ and $x \in H$, we have*

$$\|J_r^A(x) - p\|^2 \leq \|x - p\|^2 - \|x - J_r^A(x)\|^2.$$

Remark 2.2 For every bounded sequence $\{x_n\} \subset H$, we have $\{x_n - J_r^A(x_n)\}$ is bounded. In fact, let $\{x_n\} \subset H$ be a bounded sequence and $p \in A^{-1}0$. Then we have

$$\begin{aligned}
\|x_n - J_r^A(x_n) - p\| &\leq \|x_n\| + \|J_r^A(x_n) - p\| \\
&= \|x_n\| + \|J_r^A(x_n) - J_r^A(p)\| \\
&\leq \|x_n\| + \|x_n - p\| \\
&\leq 2\|x_n\| + \|p\|.
\end{aligned}$$

Hence it is clear that $\{x_n - J_r^A(x_n)\}$ is bounded.

Lemma 2.3 ([16]) (*Demiclosed principle*) *Let C be a nonempty, closed and convex subset of a Hilbert space H and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$ implies $x = Sx$.*

Lemma 2.4 ([40]) *Let H be a real Hilbert space. Then we have*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.5 [24] *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ be nonnegative real sequences such that*

$$a_{n+1} \le (1-\delta_n)a_n + b_n + c_n, \quad n \ge 1,$$

where $\{\delta_n\}_{n\in\mathbb{N}}$ is a sequence in (0,1) and $\{b_n\}_{n\in\mathbb{N}}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $\frac{b_n}{\delta_n} \le M$ for some $M \ge 0$, then $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence.*
(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \le 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.6 [19] *Assume $\{s_n\}_{n\in\mathbb{N}}$ is a nonnegative real sequence such that*

$$s_{n+1} \le (1-\delta_n)s_n + \delta_n \lambda_n$$

and

$$s_{n+1} \le s_n - \eta_n + \sigma_n,$$

where $\{\delta_n\}_{n\in\mathbb{N}}$ is a sequence in $(0, 1)$, $\{\eta_n\}_{n\in\mathbb{N}}$ is a nonnegative real sequence and $\{\lambda_n\}_{n\in\mathbb{N}}$, and $\{\sigma_n\}_{n\in\mathbb{N}}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(ii) $\lim_{n\to\infty} \sigma_n = 0$;
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$, *implies* $\lim_{k\to\infty} \sup \lambda_{n_k} \le 0$ *for any subsequence of real sequence $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$.*

Then $\lim_{n\to\infty} s_n = 0$.

3 Main result

From now on, let H_i, $i = 1, 2, \ldots, N$ be real Hilbert spaces, $A_i : H_i \to 2^{H_i}$, $i = 1, 2, \ldots, N$ be maximal monotone operators on H_i, $J_{r_i}^{A_i}$, $i = 1, 2, \ldots, N$ be resolvent operators of A_i for $r_i > 0$ and $T_i : H_i \to H_{i+1}$, $i = 1, 2, \ldots, N-1$ be bounded linear operators. Assume that $\Omega \ne \emptyset$. In order to prove our main result, we need the following lemma.

Lemma 3.1 *For any $r_i > 0$, $i = 1, 2, \ldots, N-1$, define the following functions:*

$$h(y_{i-1}) = T_1^* T_2^* \ldots T_{N-i}^* (I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}}) T_{N-i} \ldots T_2 T_1 y_{i-1}$$

and

$$f(y_{i-1}) = \frac{1}{2} \|(I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}}) T_{N-i} \ldots T_2 T_1 y_{i-1}\|^2.$$

Then the following statements hold:

(i) *For each* $p \in \Omega$ *and* $i = 1, 2, \dots, N-1$, *we have* $\langle h(y_{i-1}), y_{i-1} - p\rangle \geq 2f(y_{i-1})$.

(ii) *For* $i = 1, 2, \dots, N-1$, *suppose that* $y_i = y_{i-1} - \tau_i h(y_{i-1})$, *where* $\tau_i = \frac{\rho f(y_{i-1})}{\|h(y_{i-1})\|^2+\theta_{i-1}}$ *with* $\theta_{i-1} > 0$ *and* $\rho > 0$. *Then for each* $p \in \Omega$, *we have*

$$\|y_i - p\|^2 \leq \|y_{i-1} - p\|^2 - \rho(4-\rho)\frac{f^2(y_{i-1})}{\|h(y_{i-1})\|^2 + \theta_{i-1}}.$$

Proof (i) Let $p \in \Omega$, then

$$\begin{aligned}
&p = J_{r_1}^{A_1}(p),\\
&T_1p = J_{r_2}^{A_2}(T_1p),\\
&T_2T_1p = J_{r_3}^{A_3}(T_2T_1p),\\
&\vdots\\
&T_{N-i}\dots T_2T_1p = J_{r_{N-i+1}}^{A_{N-i+1}}(T_{N-i}\dots T_2T_1p).
\end{aligned}$$

So we have $(I^{H_1} - J_{r_1}^{A_1})p = 0$ and $(I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}})T_{N-i}\dots T_2T_1p = 0$ for $i = 1, 2, \dots, N-1$. By Lemma 2.1 (iii), we have

$$\begin{aligned}
&\langle h(y_{i-1}), y_{i-1} - p\rangle\\
=&\langle T_1^*T_2^*\dots T_{N-i}^*(I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}})T_{N-i}\dots T_2T_1y_{i-1}, y_{i-1} - p\rangle\\
=&\langle (I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}})T_{N-i}\dots T_2T_1y_{i-1}, T_{N-i}\dots T_2T_1y_{n,i-1} - T_{N-i}\dots T_2T_1p\rangle\\
=&\langle (I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}})T_{N-i}\dots T_2T_1y_{n,i-1}\\
&- (I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}})T_{N-i}\dots T_2T_1p, T_{N-i}\dots T_2T_1y_{n,i-1} - T_{N-i}\dots T_2T_1p\rangle\\
\geq&\|(I^{H_{N-i+1}} - J_{r_{N-i+1}}^{A_{N-i+1}})T_{N-i}\dots T_2T_1y_{i-1}\|^2 = 2f(y_{i-1}).
\end{aligned}$$

(ii) Let $p \in \Omega$. From (i), we see that

$$\begin{aligned}
\|y_i - p\|^2 =&\|y_{i-1} - p - \tau_i h(y_{i-1})\|^2\\
=&\|y_{i-1} - p\|^2 - 2\tau_i\langle h(y_{i-1}), y_{i-1} - p\rangle + \tau_i^2\|h(y_{i-1})\|^2\\
\leq&\|y_{i-1} - p\|^2 - 4\tau_i f(y_{i-1}) + \tau_i^2\|h(y_{i-1})\|^2\\
=&\|y_{i-1} - p\|^2 - \frac{4\rho f^2(y_{i-1})}{\|h(y_{i-1})\|^2 + \theta_{i-1}} + \frac{\rho^2 f^2(y_{i-1})}{(\|h(y_{i-1})\|^2 + \theta_{i-1})^2}\|h(y_{i-1})\|^2\\
\leq&\|y_{i-1} - p\|^2 - \frac{4\rho f^2(y_{i-1})}{\|h(y_{i-1})\|^2 + \theta_{i-1}} + \frac{\rho^2 f^2(y_{i-1})}{\|h(y_{i-1})\|^2 + \theta_{i-1}}\\
=&\|y_{i-1} - p\|^2 - \rho(4-\rho)\frac{f^2(y_{i-1})}{\|h(y_{i-1})\|^2 + \theta_{i-1}}.
\end{aligned}$$

This completes the proof. □

We next introduce an inertial self-adaptive method for solving the generalized split common null point problem (1.4) in the crucial case where $N = 3$. The general case is proved by arguments similar method in Theorem 3.2. See Theorem 3.5 below.

For any $u, x_0, x_1 \in H_1$, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following scheme:

$$\begin{cases} w_n = x_n + \beta_n(x_n - x_{n-1}), \\ y_{n,1} = w_n - \tau_{n,1} h(w_n), \\ y_{n,2} = y_{n,1} - \tau_{n,2} h(y_{n,1}), \\ y_{n,3} = J^{A_1}_{r_{n,1}}(y_{n,2}), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_{n,3}, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset [0,\beta)$ with $\beta \in [0,1)$, $\{r_{n,i}\}_{n\in\mathbb{N}} \subset (0,\infty)$ for $i = 1,2,3$, $\{\theta_{n,j}\}_{n\in\mathbb{N}} \subset (0,\infty)$ for $j = 0, 1$, and

$$h(w_n) = T_1^* T_2^* (I^{H_3} - J^{A_3}_{r_{n,3}}) T_2 T_1 w_n \text{ and } h(y_{n,1}) = T_1^* (I^{H_2} - J^{A_2}_{r_{n,2}}) T_1 y_{n,1}. \tag{3.2}$$

Suppose the step-sizes are chosen in such a way that

$$\tau_{n,1} = \frac{\rho_n f(w_n)}{\|h(w_n)\|^2 + \theta_{n,0}} \text{ and } \tau_{n,2} = \frac{\rho_n f(y_{n,1})}{\|h(y_{n,1})\|^2 + \theta_{n,1}}, \quad 0 < \rho_n < 4, \tag{3.3}$$

where

$$f(w_n) = \frac{1}{2}\|(I^{H_3} - J^{A_3}_{r_{n,3}}) T_2 T_1 w_n\|^2 \text{ and } f(y_{n,1}) = \frac{1}{2}\|(I^{H_2} - J^{A_2}_{r_{n,2}}) T_1 y_{n,1}\|^2. \tag{3.4}$$

We now propose the strong convergence theorem of the sequence generated by the proposed method.

Theorem 3.2 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by* (3.1). *Suppose that the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$;
(C3) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$;
(C4) $\liminf_{n\to\infty} r_{n,i} > 0$ *for* $i = 1,2,3$;
(C5) $\max\left\{ \sup_n\{\theta_{n,0}\}, \sup_n\{\theta_{n,1}\}\right\} \leq K$ *with* $K > 0$.

Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $z = P^{H_1}_{\Omega}(u)$.

Proof Let $p \in \Omega$. From Lemma 3.1 (i), we have

$$\langle h(w_n), w_n - p\rangle \geq 2f(w_n) \text{ and } \langle h(y_{n,1}), y_{n,1} - p\rangle \geq 2f(y_{n,1}). \tag{3.5}$$

Also by Lemma 3.1 (ii), we have

$$\|y_{n,1} - p\|^2 \leq \|w_n - p\|^2 - \rho_n(4-\rho_n)\frac{f^2(w_n)}{\|h(w_n)\|^2 + \theta_{n,0}} \tag{3.6}$$

and

$$\|y_{n,2} - p\|^2 \leq \|y_{n,1} - p\|^2 - \rho_n(4-\rho_n)\frac{f^2(y_{n,1})}{\|h(y_{n,1})\|^2 + \theta_{n,1}}. \tag{3.7}$$

Combining (3.6) and (3.7), we immediately obtain

$$\|y_{n,2}-p\|^2 \leq \|w_n-p\|^2-\rho_n(4-\rho_n)\left[\frac{f^2(w_n)}{\|h(w_n)\|^2+\theta_{n,0}}+\frac{f^2(y_{n,1})}{\|h(y_{n,1})\|^2+\theta_{n,1}}\right]. \quad (3.8)$$

Then from Lemma 2.1 (*iv*) and (3.8), we have

$$\begin{aligned}\|y_{n,3}-p\|^2 &\leq \|J_{r_{n,1}}^{A_1}(y_{n,2})-p\|^2\\ &\leq \|y_{n,2}-p\|^2-\|J_{r_{n,1}}^{A_1}(y_{n,2})-y_{n,2}\|^2\\ &\leq \|w_n-p\|^2-\rho_n(4-\rho_n)\left[\frac{f^2(w_n)}{\|h(w_n)\|^2+\theta_{n,0}}+\frac{f^2(y_{n,1})}{\|h(y_{n,1})\|^2+\theta_{n,1}}\right]-\|J_{r_{n,1}}^{A_1}(y_{n,2})-y_{n,2}\|^2.\end{aligned} \quad (3.9)$$

This implies that

$$\|y_{n,3}-p\| \leq \|w_n-p\|. \quad (3.10)$$

Since

$$\begin{aligned}\|w_n-p\| &= \|x_n-p+\beta_n(x_n-x_{n-1})\|\\ &\leq \|x_n-p\|+\beta_n\|x_n-x_{n-1}\|,\end{aligned} \quad (3.11)$$

it follows from (3.10) and (3.11) that

$$\|y_{n,3}-p\| \leq \|x_n-p\|+\beta_n\|x_n-x_{n-1}\|. \quad (3.12)$$

Thus we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(u-p)+(1-\alpha_n)y_{n,3}\|\\ &\leq \alpha_n\|u-p\|+(1-\alpha_n)\|y_{n,3}-p\|\\ &\leq (1-\alpha_n)(\|x_n-p\|+\beta_n\|x_n-x_{n-1}\|)+\alpha_n\|u-p\|\\ &= (1-\alpha_n)\|x_n-p\|+(1-\alpha_n)\beta_n\|x_n-x_{n-1}\|+\alpha_n\|u-p\|.\end{aligned} \quad (3.13)$$

Put $\sigma_n=\frac{(1-\alpha_n)\beta_n}{\alpha_n}\|x_n-x_{n-1}\|+\|u-p\|$. We see that $\lim_{n\to\infty}\sigma_n$ exists, which implies that the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ is bounded. From Lemma 2.5, we have $\{\|x_n-p\|\}_{n\in\mathbb{N}}$ is bounded. Moreover, we have

$$\|x_n\| \leq \|x_n-p\|+\|p\| \text{ for all } n\in\mathbb{N}. \quad (3.14)$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $z=P_\Omega^{H_1}(u)$. By Lemma 2.4, we have

$$\begin{aligned}\|w_n-p\|^2 &= \|x_n-p+\beta_n(x_n-x_{n-1})\|^2\\ &\leq \|x_n-p\|^2+2\beta_n\langle x_n-x_{n-1},w_n-p\rangle\\ &\leq \|x_n-p\|^2+2\beta_n\|x_n-x_{n-1}\|\|w_n-p\|.\end{aligned} \quad (3.15)$$

Combining (3.9) and (3.15), we thus have

$$\begin{aligned}\|y_{n,3}-p\|^2 \leq &\|x_n-p\|^2+2\beta_n\|x_n-x_{n-1}\|\|w_n-p\|\\ &-\rho_n(4-\rho_n)\left[\frac{f^2(w_n)}{\|h(w_n)\|^2+\theta_{n,0}}+\frac{f^2(y_{n,1})}{\|h(y_{n,1})\|^2+\theta_{n,1}}\right]\\ &-\|J_{r_{n,1}}^{A_1}(y_{n,2})-y_{n,2}\|^2.\end{aligned} \quad (3.16)$$

It follows from Lemma 2.4 and (3.16) that

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
\leq&\|(1-\alpha_n)(y_{n,3}-z)+\alpha_n(u-z)\|^2\\
\leq&(1-\alpha_n)\|y_{n,3}-z\|^2+2\alpha_n\langle u-z,x_{n+1}-z\rangle\\
\leq&(1-\alpha_n)\Bigg(\|x_n-z\|^2+2\beta_n\|x_n-x_{n-1}\|\|w_n-z\|-\rho_n(4-\rho_n)\left[\frac{f^2(w_n)}{\|h(w_n)\|^2+\theta_{n,0}}+\frac{f^2(y_{n,1})}{\|h(y_{n,1})\|^2+\theta_{n,1}}\right]\\
&-\|J^{A_1}_{r_{n,1}}(y_{n,2})-y_{n,2}\|^2\Bigg)+2\alpha_n\langle u-z,x_{n+1}-z\rangle\\
=&(1-\alpha_n)\|x_n-z\|^2+2(1-\alpha_n)\beta_n\|x_n-x_{n-1}\|\|w_n-z\|\\
&-(1-\alpha_n)\rho_n(4-\rho_n)\left[\frac{f^2(w_n)}{\|h(w_n)\|^2+\theta_{n,0}}+\frac{f^2(y_{n,1})}{\|h(y_{n,1})\|^2+\theta_{n,1}}\right]\\
&-(1-\alpha_n)\|J^{A_1}_{r_{n,1}}(y_{n,2})-y_{n,2}\|^2+2\alpha_n\langle u-z,x_{n+1}-z\rangle.
\end{aligned} \tag{3.17}$$

For each $n\geq 1$, put $s_n=\|x_n-z\|^2$, $\lambda_n=\frac{2(1-\alpha_n)\beta_n}{\alpha_n}\|x_n-x_{n-1}\|\|w_n-z\|+2\langle u-z,x_{n+1}-z\rangle$,$\eta_n=(1-\alpha_n)\rho_n(4-\rho_n)\left[\frac{f^2(w_n)}{\|h(w_n)\|^2+\theta_{n,0}}+\frac{f^2(y_{n,1})}{\|h(y_{n,1})\|^2+\theta_{n,1}}\right]+(1-\alpha_n)\|J^{A_1}_{r_{n,1}}(y_{n,2})-y_{n,2}\|^2$ and $\sigma_n=2(1-\alpha_n)\beta_n\|x_n-x_{n-1}\|\|w_n-p\|+2\alpha_n\langle u-z,x_{n+1}-z\rangle$. Then from (3.17), we obtain thefollowing two inequalities:

$$s_{n+1}\leq(1-\alpha_n)s_n+\alpha_n\lambda_n,\quad n\geq 1 \tag{3.18}$$

and

$$s_{n+1}\leq s_n-\eta_n+\sigma_n,\quad n\geq 1. \tag{3.19}$$

It is easy to see that $(C3)$ implies $\beta_n\|x_n-x_{n-1}\|\to 0$. Thus by $(C1)$, we have $\sum_{n=1}^{\infty}\sigma_n=\infty$ and$\lim_{n\to\infty}\sigma_n=0$. In order to complete the proof, using Lemma 2.6, it is sufficient to show that from $\lim_{k\to\infty}\eta_{n_k}=0$ implies $\limsup_{k\to\infty}\lambda_{n_k}\leq 0$ for any subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$. Let $\{n_k\}_{k\in\mathbb{N}}$ be a subsequence of $\{n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty}\eta_{n_k}=0$. Thus we have

$$\|J^{A_1}_{r_{n_k,1}}(y_{n_k,2})-y_{n_k,2}\|^2\to 0,\quad \frac{f^2(y_{n_k,1})}{\|h(y_{n_k,1})\|^2_{n_k}+\theta_{n,0}}\to 0\ \text{ and }\ \frac{f^2(w_{n_k})}{\|h(w_{n_k})\|^2_{n_k}+\theta_{n,1}}\to 0. \tag{3.20}$$

Since $\{h(w_{n_k})\}_{k\in\mathbb{N}}$, $\{h(y_{n_k,1})\}_{k\in\mathbb{N}}$ are bounded (see, Remark 2.2), then there exists a positive real number M such that

$$\max\left\{\sup_n\{\|h(w_{n_k})\|^2\},\sup_n\{\|h(y_{n_k,1})\|^2\}\right\}\leq M.$$

Thus we have

$$\frac{f^2(y_{n_k,1})}{M+K}\leq\frac{f^2(y_{n_k,1})}{\|h(y_{n_k,1})\|^2_{n_k}+\theta_{n,0}}\ \text{ and }\ \frac{f^2(w_{n_k})}{M+K}\leq\frac{f^2(w_{n_k})}{\|h(w_{n_k})\|^2_{n_k}+\theta_{n,1}}. \tag{3.21}$$

Combining (3.20) and (3.21), we obtain

$$\lim_{k\to\infty}\|J^{A_1}_{r_{n_k,1}}(y_{n_k,2})-y_{n_k,2}\|=0, \tag{3.22}$$

$$\lim_{k\to\infty} f(y_{n_k,1}) = \lim_{k\to\infty} \|(I^{H_2} - J^{A_2}_{r_{n_k,2}})T_1 y_{n_k,1}\| = 0 \tag{3.23}$$

and

$$\lim_{k\to\infty} f(w_{n_k}) = \lim_{k\to\infty} \|(I^{H_3} - J^{A_3}_{r_{n,3}})T_2 T_1 w_{n_k}\| = 0. \tag{3.24}$$

From (3.22), (3.23) and (3.24), we see that

$$\|y_{n_k,3} - y_{n_k,2}\| = \|J^{A_1}_{r_{n_k,1}}(y_{n_k,2}) - y_{n_k,2}\| \to 0, \tag{3.25}$$

$$\|y_{n_k,2} - y_{n_k,1}\| = \tau_{n_k,2}\|h(y_{n_k,1})\| \le \tau_{n_k,2}\|T_1\|\|(I^{H_2} - J^{A_2}_{r_{n_k,2}})T_1 y_{n_k,1}\| \to 0 \tag{3.26}$$

and

$$\|y_{n_k,1} - w_{n_k}\| = \tau_{n_k,1}\|h(w_{n_k})\| \le \tau_{n_k,1}\|T_1\|\|T_2\|\|\|(I^{H_3} - J^{A_3}_{r_{n_k,3}})T_2 T_1 w_{n_k}\| \to 0. \tag{3.27}$$

It follows from (3.25), (3.26) and (3.27) that

$$\|y_{n_k,3} - w_{n_k}\| \to 0. \tag{3.28}$$

Since

$$\|x_{n_k+1} - y_{n_k,3}\| = \alpha_{n_k}\|u - y_{n_k,3}\| \to 0 \tag{3.29}$$

and

$$\|w_{n_k} - x_{n_k}\| = \beta_{n_k}\|x_{n_k} - x_{n_k-1}\| \to 0, \tag{3.30}$$

it follows that

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - y_{n_k,3}\| + \|y_{n_k,3} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0. \tag{3.31}$$

Since $\{x_{n_k}\}_{k\in\mathbb{N}}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_{k_j}} \rightharpoonup \hat{x}$ and

$$\limsup_{k\to\infty}\langle u - z, x_{n_k} - z\rangle = \lim_{j\to\infty}\langle u - z, x_{n_{k_j}} - z\rangle. \tag{3.32}$$

From (3.26), (3.27) and (3.30), we also have $y_{n_{k_j},2} \rightharpoonup \hat{x}$, $y_{n_{k_j},1} \rightharpoonup \hat{x}$ and $w_{n_{k_j}} \rightharpoonup \hat{x}$. Since T_1 and T_2 are bounded linear operators, we have $T_2 T_1 w_{n_{k_j}} \rightharpoonup T_2 T_1 \hat{x}$ and $T_1 y_{n_{k_j},1} \rightharpoonup T_1\hat{x}$. Since $\liminf_{n\to\infty} r_{n,i} > 0$ for $i = 1, 2, 3$, there is $r > 0$ such that $r_{n,i} \ge r$ for $i = 1, 2, 3$ and $n \ge 1$. In particular, $r_{n_{k_j},i} \ge r$ for $i = 1, 2, 3$ and $j \ge 1$. By Lemma 2.1 (i) and (3.22), we have

$$\|J^{A_1}_r(y_{n_{k_j},2}) - y_{n_{k_j},2}\| \le 2\|J^{A_1}_{r_{n_{k_j},1}}(y_{n_{k_j},2}) - y_{n_{k_j},2}\| \to 0. \tag{3.33}$$

Also, we can show that

$$\|(I^{H_2} - J^{A_2}_r)T_1 y_{n_{k_j},1}\| \to 0 \;\text{ and }\; \|(I^{H_3} - J^{A_3}_r)T_2 T_1 w_{n_{k_j}}\| \to 0. \tag{3.34}$$

Thus by Demiclosed principle (see, Lemma 2.3), we obtain $\hat{x} \in F(J_r^{A_1}) = A_1^{-1}0$, $T_1\hat{x} \in F(J_r^{A_2}) = A_2^{-1}0$ and$T_2T_1\hat{x} \in F(J_r^{A_3}) = A_3^{-1}0$. This implies that $\hat{x} \in \Omega$. So from (2.1), we obtain

$$\limsup_{k\to\infty}\langle u - z, x_{n_k} - z\rangle = \lim_{j\to\infty}\langle u - z, x_{n_{k_j}} - z\rangle \le 0. \tag{3.35}$$

By (3.31), we also have

$$\limsup_{k\to\infty}\langle u - z, x_{n_k+1} - z\rangle \le 0. \tag{3.36}$$

This together with $(C3)$ and (3.35), we have

$$\lim_{k\to\infty} \lambda_{n_k} \le 0. \tag{3.37}$$

By Lemma 2.6, we have $\lim_{n\to\infty}\|x_n - z\|^2 = 0$. Therefore $x_n \to z$ as $n \to \infty$.
This completes the proof. □

Remark 3.3 It should be pointed out that our method improves and extends the method Reich and Tuyen [32]. We use the self-adaptive technique to define new step-sizes of our method. These new step-sizes do not require the norms of bounded linear operators, while the step-sizes of method in Reich and Tuyen [32] are fixed and depend on the operator norms of the operators. Moreover, we use the inertial technique to accelerate the convergence rate of our method. Therefore our method has broader applicability and is more efficient.

Remark 3.4 We remark here that the condition $(C3)$ is easily implemented in numerical computation, since the value of $\|x_n - x_{n-1}\|$ is known before choosing β_n. In particular, the parameter β_n can be chosen as follows:

$$\beta_n = \frac{\omega_n}{\|x_n - x_{n-1}\| + c},$$

where $\{\omega_n\}$ is a positive sequence such that $\omega_n = o(\alpha_n)$ and c is any positive real number.

By using the same arguments as in the proof of Theorem 3.2, we then obtain the following general case regarding problem (1.4).

Theorem 3.5 *Let H_i, $i = 1, 2, \dots, N$ be real Hilbert spaces. Let $A_i : H_i \to 2^{H_i}$, $i = 1, 2, \dots, N$ be maximalmonotone operators on H_i, $J_{r_i}^{A_i}$ $(i = 1, 2, \dots, N)$ be resolvent operators of A_i for $r_i > 0$ and$T_i : H_i \to H_{i+1}$, $i = 1, 2, \dots, N - 1$ be bounded linear operators.*

Suppose that $\Omega \neq \emptyset$. *For any* $u, x_0, x_1 \in H_1$, *let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by the following scheme*:

$$\begin{cases} w_n = x_n + \beta_n(x_n - x_{n-1}), \\ y_{n,1} = w_n - \tau_{n,1}h(w_n), \\ y_{n,2} = y_{n,1} - \tau_{n,2}h(y_{n,1}), \\ \quad\vdots \\ y_{n,N-1} = y_{n,N-2} - \tau_{n,N-1}h(y_{n,N-2}), \\ y_{n,N} = J^{A_1}_{r_{n,1}}(y_{n,N-1}), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_{n,N}, \quad \forall n \geq 1, \end{cases} \tag{3.38}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset [0,\beta)$ *with* $\beta \in [0,1)$, $\{r_{n,i}\}_{n\in\mathbb{N}} \subset (0,\infty)$ *for* $i = 1,2,\dots,N$, $\{\theta_{n,j}\}_{n\in\mathbb{N}} \subset (0,\infty)$ *for* $j = 0,1,\dots,N-2$, *and*

$$h(y_{n,i-1}) = T_1^* T_2^* \dots T_{N-i}^*(I^{H_{N-i+1}} - J^{A_{N-i+1}}_{r_{n,N-i+1}})T_{N-i}\dots T_2T_1y_{n,i-1}, \quad i = 1,2,\dots,N-1, \tag{3.39}$$

with $y_{n,0} = w_n$. *Suppose the step-sizes are chosen in such a way that*

$$\tau_{n,i} = \frac{\rho_n f(y_{n,i-1})}{\|h(y_{n,i-1})\|^2 + \theta_{n,i-1}}, \quad i = 1,2,\dots,N-1 \text{ and } 0 < \rho_n < 4, \tag{3.40}$$

where

$$f(y_{n,i-1}) = \frac{1}{2}\|(I^{H_{N-i+1}} - J^{A_{N-i+1}}_{r_{n,N-i+1}})T_{N-i}\dots T_2T_1y_{n,i-1}\|^2, \quad i = 1,2,\dots,N-1. \tag{3.41}$$

Suppose that the following conditions are satisfied:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}\rho_n(4-\rho_n) > 0$;
(C3) $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$;
(C4) $\liminf_{n\to\infty} r_{n,i} > 0$ *for* $i = 1,2,\dots,N$;
(C5) $\max_{i=0,1,\dots,N-1}\left\{\sup_n\{\theta_{n,i}\}\right\} \leq K$.

Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $z = P^{H_1}_{\Omega}(u)$.

4 Some applications

In this section, we utilize our main result to the other split type problems.

4.1 Generalized split feasibility problem

Let H be a real Hilbert space and $g : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function. The *subdifferential* ∂g of g given by

$$\partial g(x) := \{z \in H : g(x) + \langle z, y - x\rangle \leq g(y), \ \ \forall y \in H\},$$

for all $x \in H$. It is know that the subdifferential ∂g of g is maximal monotone (see [31, 34]). Let C be a closed and convex subset of H, *the indicator function* of C given by

$$i_C(x) = \begin{cases} 0, \text{ if } x \in C; \\ \infty, \text{ if } x \notin C. \end{cases}$$

Then i_C is a proper, lower semicontinuous and convex function with its subdifferential ∂i_C is maximal monotone (see [1]). Moreover, we have

$$\partial i_C(x) := \begin{cases} N_C(x), & \text{if } x \in C; \\ \emptyset, & \text{if } x \notin C, \end{cases}$$

where N_C is the normal cone of C given by

$$N_C(x) := \{z \in H : \langle z, y - x\rangle \leq 0, \ \forall y \in C\}.$$

We define the resolvent operator $J_r^{\partial i_C}$ of ∂i_C for $r > 0$ by

$$J_r^{\partial i_C}(x) := (I^H + r\partial i_C)^{-1}(x), \ \forall x \in H.$$

So we have

$$\begin{aligned} u = J_r^{\partial i_C}(x) &\Leftrightarrow x - u \in rN_C(u) \\ &\Leftrightarrow \langle x - u, y - u\rangle \leq 0, \ \forall y \in C \\ &\Leftrightarrow u = P_C^H(x). \end{aligned}$$

Let H_i, $i = 1, 2, \ldots, N$ be real Hilbert spaces. Let C_i, $i = 1, 2, \ldots, N$ be closed and convex subsets of H_i, respectively and $T_i : H_i \to H_{i+1}$, $i = 1, 2, \ldots, N-1$ be bounded linear operators. We consider the following *generalized split feasibility problem*:

$$\text{find an element } z \in \Omega := C_1 \cap T_1^{-1}(C_2) \cap \ldots \cap T_1^{-1}(T_2^{-1} \ldots (T_{N-1}^{-1}(C_N))). \tag{4.1}$$

In fact, we can set $A_i = \partial i_{C_i}$ for $i = 1, 2, \ldots, N$ in Theorem 3.5. Then we have $J_{r_i}^{\partial i_{C_i}} = P_{C_i}^{H_i}$ and $F(J_{r_i}^{\partial i_{C_i}}) = A_i^{-1}0 = C_i$ for $r_i > 0$ and $i = 1, 2, \ldots, N$. So we obtain the following result.

Theorem 4.1 *Suppose that $\Omega \neq \emptyset$. For any $u, x_0, x_1 \in H_1$, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following scheme*:

$$\begin{cases} w_n = x_n + \beta_n(x_n - x_{n-1}), \\ y_{n,1} = w_n - \tau_{n,1}h(w_n), \\ y_{n,2} = y_{n,1} - \tau_{n,2}h(y_{n,1}), \\ \quad \vdots \\ y_{n,N-1} = y_{n,N-2} - \tau_{n,N-1}h(y_{n,N-2}), \\ y_{n,N} = P_{C_1}^{H_1}(y_{n,N-1}), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_{n,N}, \quad \forall n \geq 1. \end{cases} \tag{4.2}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, 1)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset [0, \beta)$ with $\beta \in [0, 1)$, $\{\theta_{n,i}\}_{n\in\mathbb{N}} \subset (0, \infty)$ for $i = 0, 2, \ldots, N-2$ and

$$h(y_{n,i-1}) = T_1^* T_2^* \dots T_{N-i}^* (I^{H_{N-i+1}} - P_{C_{N-i+1}}^{H_{N-i+1}}) T_{N-i} \dots T_2 T_1 y_{n,i-1}, \quad i = 1, 2, \dots, N-1, \tag{4.3}$$

with $y_{n,0} = w_n$. *Suppose the step-sizes are chosen in such a way that*

$$\tau_{n,i} = \frac{\rho_n f(y_{n,i-1})}{\|h(y_{n,i-1})\|^2 + \theta_{n,i-1}}, \quad i = 1, 2, \dots, N-1 \text{ and } 0 < \rho_n < 4, \tag{4.4}$$

where

$$f(y_{n,i-1}) = \frac{1}{2} \|(I^{H_{N-i+1}} - P_{C_{N-i+1}}^{H_{N-i+1}}) T_{N-i} \dots T_2 T_1 y_{n,i-1}\|^2, \quad i = 1, 2, \dots, N-1. \tag{4.5}$$

Suppose that the Conditions $(C1) - (C3)$ *and* $(C5)$ *in Theorem* 3.5 *hold. Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $z = P_\Omega^{H_1}(u)$.

4.2 Generalized split equilibrium problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $G : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of all real numbers. Recall that the following *equilibrium problem*: find an element

$$z \in C \text{ such that } G(z, y) \geq 0, \ \forall y \in C.$$

The solution set of equilibrium problem is denoted by $EP(G)$. For solving the equilibrium problem, let us assume that the bifunction satisfies the following properties:

(A1) $G(x, x) = 0$ for all $x \in C$;
(A2) G is monotone, that is, $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
(A3) For each $x, y, z \in C$, $\limsup_{t\to 0} G(tz + (1-t)x, y) \leq G(x, y)$;
(A4) For each $x \in C$, the function $y \mapsto G(x, y)$ is convex and lower semicontinuous.

Lemma 4.2 ([12]) *Let* $G : C \times C \to \mathbb{R}$ *be a bifunction which satisfies* $(A1) - (A4)$. *For any* $r > 0$ *and* $x \in H$, *define* $T_r^G : H \to C$ *by*

$$T_r^G(x) := \left\{ y \in C : G(y, z) + \frac{1}{r}\langle y - z, x - y\rangle \geq 0, \ \forall z \in C \right\}.$$

Then the following statements hold:

(i) T_r^G *is single-valued*;
(ii) T_r^G *is firmly nonexpansive*;
(iii) $F(T_r^G) = EP(G)$;
(iv) $EP(G)$ *is closed and convex*.

Lemma 4.3 ([41]) *Let* $G : C \times C \to \mathbb{R}$ *be a bifunction which satisfies* $(A1) - (A4)$. *Define a multivalued mapping* $A_G : H \to 2^H$ *by*

$$A_G(x) := \begin{cases} \{z \in H : G(x,y) \geq \langle z, y-x\rangle, \ \forall y \in C\}, & \text{if } x \in C; \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then the following statements hold:

(i) A_G *is maximal monotone with* $EP(G) = A_G^{-1}0$;
(ii) T_r^G *is the resolvent of* A_G, *that is*, $T_r^G = (I^H + rA_G)^{-1}$ *for* $r > 0$.

Let H_i, $i = 1,2,\dots,N$ be real Hilbert spaces. Let $G_i : H_i \times H_i \to \mathbb{R}$, $i = 1,2,\dots,N$ be bifunctions which satisfy $(A1)-(A4)$ and $T_i : H_i \to H_{i+1}$, $i = 1,2,\dots,N-1$ be bounded linear operators. We consider the following *generalized split equilibrium problem*:

$$\Omega := EP(G_1) \cap T_1^{-1}(EP(G_2)) \cap \dots \cap T_1^{-1}(T_2^{-1} \dots (T_{N-1}^{-1}(EP(G_N)))).$$

In fact, we can set $A_i = A_{G_i}$ in Theorem 3.5. Then we have $J_{r_i}^{A_{G_i}} = T_{r_i}^{G_i}$ and $EP(G_i) = A_i^{-1}0$ for $i = 1,2,\dots,N$. So we obtain the following result.

Theorem 4.4 *Suppose that* $\Omega \neq \emptyset$. *For any* $u, x_0, x_1 \in H_1$, *let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by the following scheme*:

$$\begin{cases} w_n = x_n + \beta_n(x_n - x_{n-1}), \\ y_{n,1} = w_n - \tau_{n,1} h(w_n), \\ y_{n,2} = y_{n,1} - \tau_{n,2} h(y_{n,1}), \\ \quad \vdots \\ y_{n,N-1} = y_{n,N-2} - \tau_{n,N-1} h(y_{n,N-2}), \\ y_{n,N} = T_{r_{n,1}}^{G_1}(y_{n,N-1}), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_{n,N}, \quad \forall n \geq 1. \end{cases} \tag{4.6}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset [0,\beta)$ *with* $\beta \in [0,1)$, $\{r_{n,i}\}_{n\in\mathbb{N}} \subset (0,\infty)$ *for* $i = 1,2,\dots,N$, $\{\theta_{n,j}\}_{n\in\mathbb{N}} \subset (0,\infty)$ *for* $j = 0,1,\dots,N-2$, *and*

$$h(y_{n,i-1}) = T_1^* T_2^* \dots T_{N-i}^* (I^{H_{N-i+1}} - T_{r_{n,N-i+1}}^{G_{N-i+1}}) T_{N-i} \dots T_2 T_1 y_{n,i-1}, \quad i = 1,2,\dots,N-1, \tag{4.7}$$

with $y_{n,0} = w_n$. *Suppose the step-sizes are chosen in such a way that*

$$\tau_{n,i} = \frac{\rho_n f(y_{n,i-1})}{\|h(y_{n,i-1})\|^2 + \theta_{n,i-1}}, \quad i = 1,2,\dots,N-1 \ \text{ and } \ 0 < \rho_n < 4, \tag{4.8}$$

where

$$f(y_{n,i-1}) = \frac{1}{2}\|(I^{H_{N-i+1}} - T_{r_{n,N-i+1}}^{G_{N-i+1}}) T_{N-i} \dots T_2 T_1 y_{n,i-1}\|^2, \quad i = 1,2,\dots,N-1. \tag{4.9}$$

Suppose that the Conditions $(C1)-(C5)$ *in Theorem* 3.5 *hold*. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $z = P_\Omega^{H_1}(u)$.

4.3 Generalized split minimization problem

Let H be a real Hilbert space and $g : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function. We define the set of minimizer of g by

$$\operatorname{argmin} g := \{x \in H : g(x) \le g(z), \ \ \forall z \in H\}.$$

We note that ∂g is is maximal monotone. Moreover, the zeros set of ∂g coincides with the set of minimizers of g, that is,

$$\begin{aligned} x \in (\partial g)^{-1}0 \Leftrightarrow & 0 \in \partial g(x) \\ \Leftrightarrow & g(x) \le g(z), \ \ \forall z \in H \\ \Leftrightarrow & x \in \operatorname{argmin} g. \end{aligned}$$

In this case, the resolvent of ∂g is called the *proximity operator* of g for $r > 0$ ([13]) which given by

$$\operatorname{prox}_{rg}(x) := \operatorname{argmin}_{y \in H}\left\{ g(y) + \frac{1}{2r}\|x-y\|^2 \right\}, \ \ \forall x \in H.$$

Let H_i, $i = 1,2,\dots,N$ be real Hilbert spaces. Let $g_i : H_i \to \mathbb{R} \cup \{+\infty\}$, $i = 1,2,\dots,N$ be proper, lower semicontinuous and convex functions and $T_i : H_i \to H_{i+1}$, $i = 1,2,\dots,N-1$ be bounded linear operators. We consider the following *generalized split minimization problem*:

$$\Omega := \operatorname{argmin}_{x \in H_1} g_1(x) \cap T_1^{-1}(\operatorname{argmin}_{x \in H_2} g_2(x)) \cap \dots \cap T_1^{-1}(T_2^{-1} \dots (T_{N-1}^{-1}(\operatorname{argmin}_{x \in H_N} g_N(x)))).$$

The following result is directly obtain when we set $A_i = \partial g_i$ for $i = 1,2,\dots,N$.

Theorem 4.5 *Suppose that $\Omega \ne \emptyset$. For any $u, x_0, x_1 \in H_1$, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following scheme*:

$$\begin{cases} w_n = x_n + \beta_n(x_n - x_{n-1}), \\ y_{n,1} = w_n - \tau_{n,1}h(w_n), \\ y_{n,2} = y_{n,1} - \tau_{n,2}h(y_{n,1}), \\ \quad \vdots \\ y_{n,N-1} = y_{n,N-2} - \tau_{n,N-1}h(y_{n,N-2}), \\ y_{n,N} = prox_{r_{n,1}}^{g_1}(y_{n,N-1}), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_{n,N}, \ \ \forall n \ge 1, \end{cases} \tag{4.10}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset [0,\beta)$ *with* $\beta \in [0,1)$, $\{r_{n,i}\}_{n\in\mathbb{N}} \subset (0,\infty)$ *for* $i = 1,2,\dots,N$, $\{\theta_{n,j}\}_{n\in\mathbb{N}} \subset (0,\infty)$ *for* $j = 0,1,\dots,N-2$, *and*

$$h(y_{n,i-1}) = T_1^*T_2^* \dots T_{N-i}^*(I^{H_{N-i+1}} - prox_{r_{n,N-i+1}}^{g_{N-i+1}})T_{N-i} \dots T_2T_1y_{n,i-1}, \ \ i = 1,2,\dots,N-1, \tag{4.11}$$

with $y_{n,0} = w_n$. Suppose the step-sizes are chosen in such a way that

$$\tau_{n,i} = \frac{\rho_n f(y_{n,i-1})}{\|h(y_{n,i-1})\|^2 + \theta_{n,i-1}}, \ \ i = 1,2,\dots,N-1 \ \text{ and } \ 0 < \rho_n < 4, \tag{4.12}$$

where

$$f(y_{n,i-1})=\frac{1}{2}\|(I^{H_{N-i+1}}-prox^{g_{N-i+1}}_{r_{n,N-i+1}})T_{N-i}\dots T_2T_1y_{n,i-1}\|^2,\quad i=1,2,\dots,N-1. \tag{4.13}$$

Suppose that the Conditions $(C1)-(C5)$ in Theorem 3.5 hold. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $z=P^{H_1}_{\Omega}(u)$.

5 Numerical experiments

In this section, we provide numerical experiments to show the the efficiency and advantage of the proposed method and we also compare them with the iterative method (1.6) of Reich and Tuyen [32, Theorem 4.4].

Example 5.1 We consider the generalized split feasibility problem (4.1) with $N=5$, $H_i=\mathbb{R}^{n_i}$, where $n_i=5i$, the closed and convex subsets C_i are defined by $C_i=\{x\in\mathbb{R}^{n_i}:\ \langle a_i,x\rangle\le b_i\}$, where the coordinates of a_i are randomly generated in the closed interval [3, 5], b_i are randomly generated in the closed interval [1, 2] for all $i=1,2,\dots,5$, and $T_i:\ \mathbb{R}^{n_i}\to\mathbb{R}^{n_{i+1}}$, $i=1,2,3,4$ are bounded linear operators the elements of the representing matrices which are randomly generated in the closed interval $[-5,5]$.

We now apply our iterative method (4.2) and the iterative method (1.6) of Reich and Tuyen [32] for solving (4.1). The parameters are chosen as follows:

- Iterative method (4.2): $\alpha_n=1/n$, $\rho_n=3.5$, $\theta_{n,i}=1$, $\beta_n=\frac{\alpha_n^2}{1+\|x_n-x_{n-1}\|}$.
- Iterative method (1.6) of Reich and Tuyen [32]: $\alpha_n=1/n$, $\gamma_i=1/\Pi_{j=1}^{5-i}\mathcal{M}_i$, with

$$\mathcal{M}_i=\max_{k=1,2,\dots,n_{i+1}}\Big\{\sum_{l=1}^{n_i}(t^i_{kl})^2\Big\},\ i=1,2,3,4,$$

where $(t^i_{kl})_{n_{i+1}\times n_i}$ is the matrix of the bounded linear operator T_i for all $i=1,2,3,4$. It is easy to see that $\|T_i\|^2\le\mathcal{M}_i$ for all $i=1,2,3,4$. Thus we have

$$\gamma_i=\frac{1}{\Pi_{j=1}^{5-i}\mathcal{M}_i}\le\frac{1}{\Pi_{j=1}^{5-i}\|T_i\|^2},$$

and hence the parameters γ_i satisfy the condition $\gamma_i\in\big(0,\frac{2}{\|T_1\|^2\dots\|T_{N-i}\|^2}\big)$ for all $i=1,2,\dots,4$.

When the coordinates of the initial points x_0 and u are randomly generated in the closed interval [2, 4]. We obtain the following table of numerical results.

Remark 5.2 In the above example the function TOL_n is defined by

$$\mathrm{TOL}_n:=\frac{1}{5}\Big(\|x_n-P^{\mathbb{R}^5}_{C_1}x_n\|^2+\|T_1x_n-P^{\mathbb{R}^{10}}_{C_2}T_1x_n\|^2+\|T_2T_1x_n-P^{\mathbb{R}^{15}}_{C_3}T_2T_1x_n\|^2$$
$$+\|T_3T_2T_1x_n-P^{\mathbb{R}^{20}}_{C_4}T_3T_2T_1x_n\|^2+\|T_4T_3T_2T_1x_n-P^{\mathbb{R}^{25}}_{C_5}T_4T_3T_2T_1x_n\|^2\Big)$$

Table 1 Table of numerical results for Example 5.1

	Iterative method (4.2)			Iterative method (1.6)of Reich and Tuyen		
err	TOL_n	n	Time	TOL_n	n	Time
10^{-3}	$9.996136\mathrm{e}-004$	2300	0.844	$9.999261\mathrm{e}-004$	11116	3.812
10^{-4}	$9.999425\mathrm{e}-005$	7301	2.750	$9.999761\mathrm{e}-005$	35151	12.64
10^{-5}	$9.999822\mathrm{e}-006$	24003	8.922	$9.999981\mathrm{e}-006$	111156	148.9

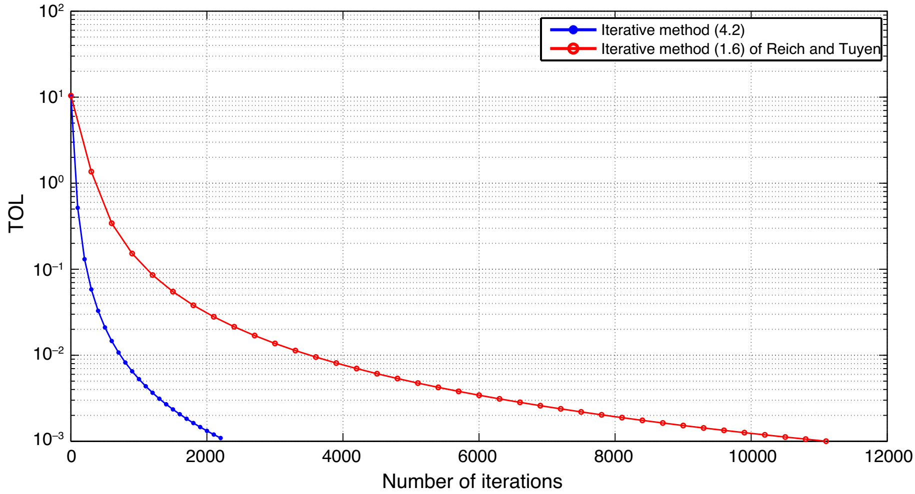

Fig. 1 The behavior of TOL_n with the stopping rule $\mathrm{TOL}_n < 10^{-3}$

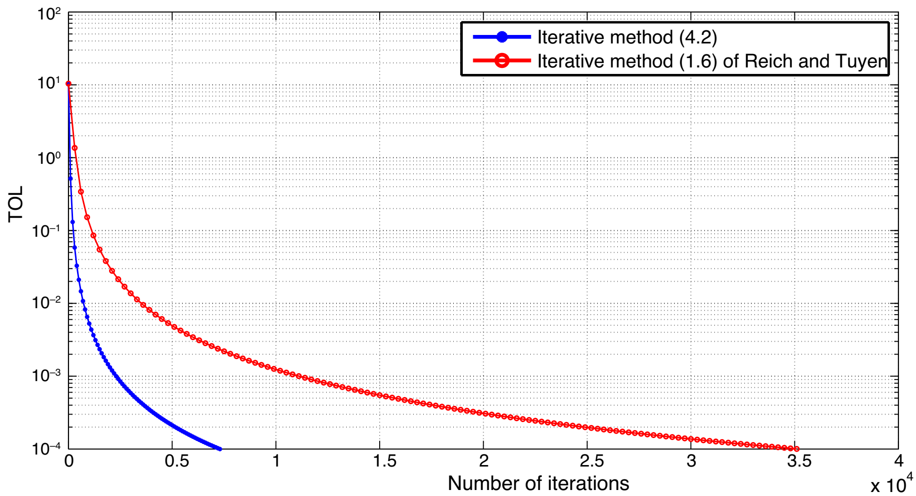

Fig. 2 The behavior of TOL_n with the stopping rule $\mathrm{TOL}_n < 10^{-4}$

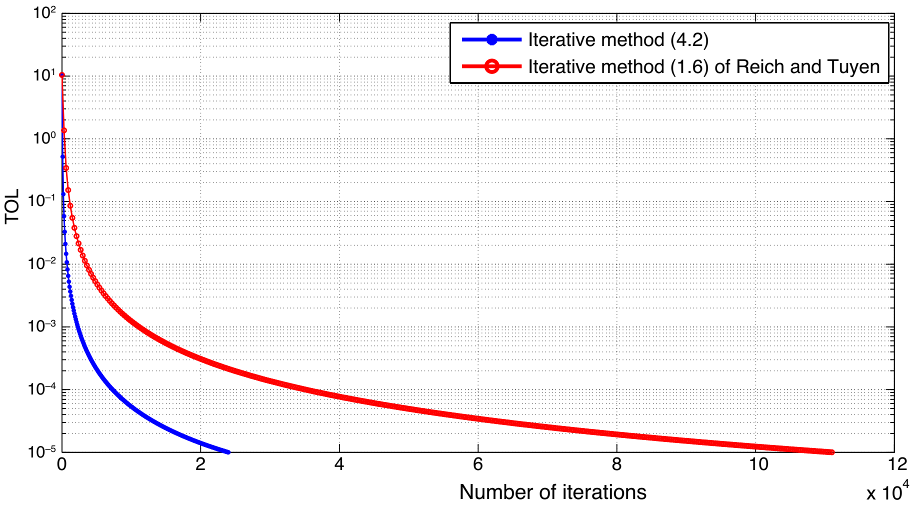

Fig. 3 The behavior of TOL_n with the stopping rule $\mathrm{TOL}_n < 10^{-5}$

for all $n \geq 1$. Note that if at the n-th step, $\mathrm{TOL}_n = 0$, then $x_n \in \Omega$, that is, x_n is a solution to this problem. Thus we use $\mathrm{TOL}_n <$ err for stopping the iterative process, where err is predetermined error.

The behaviour of the function TOL_n in Table 1 is described in the Figs. 1, 2, and 3.

Remark 5.3 From the numerical results, we see that our method has a good running effect in the sense that it requires both of a number of iterations and time less than the method of Reich and Tuyen [32].

6 Conclusions

In this work, we introduce an inertial self-adaptive algorithm for solving the generalized split common null point problem. We show that the sequence generated by the proposed method converges strongly to a solution of such a problem in the framework of Hilbert spaces. Some applications related to the obtained result are provided. Our method involves the inertial term and the step-sizes without prior knowledge of the operator norm in computation. Numerical results show that our method is more effective than the method of Reich and Tuyen [32] are provided.

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References

1. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Fixed Point Theory for Lipschitzian Type Mappings with Applications. Springer, Berlin (2009)
2. Alber, Y.I., Iusem, A.N.: Extension of subgradient techniques for nonsmooth optimization in Banach spaces. Set-Valued Anal. **9**, 315–335 (2001)
3. Alvarez, F.: Weak convergence of a relaxed and inertial hybrid projectionproximal point algorithm for maximal monotone operators in Hilbert space. SIAM J. Optim. **14**, 773–782 (2004)
4. Alvarez, F., Attouch, H.: An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Set-Valued Anal. **9**, 3–11 (2001)
5. Bruck, R.E., Reich, S.: Nonexpansive projections and resolvents of accretive operators in Banach spaces. Houston J. Math. **3**, 459–470 (1977)
6. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**, 441–453 (2002)
7. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**, 103–120 (2004)
8. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal. **13**, 759–775 (2012)
9. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in product space. Numer. Algorithms **8**, 221–239 (1994)
10. Chen, C., Ma, S., Yang, J.: A general inertial proximal point algorithm for mixed variational inequality problem. SIAM J. Optim. **25**, 2120–2142 (2015)
11. Cholamjiak, W., Cholamjiak, P., Suantai, S.: An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces. J. Fixed Point Theory Appl. **20**, 42 (2018)
12. Combettes, P.L., Hirstoaga, A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)
13. Combettes, P.L., Pesquet, J.C.: Proximal splitting methods in signal processing. Fixed Point Algorithms Inverse Probl. Sci. Eng. **49**, 185–212 (2011)
14. Deimling, K.: Nonlinear Functional Analysis. Springer, Berlin (1985)
15. Dong, Q.L., Heab, S., Zhaoab, J.: Solving the split equality problem without prior knowledge of operator norms. Optimization **64**(9), 1887–1906 (2015)
16. Gobel, K.A., Kirk, W.A.: Topics in Metric Fixed Point Theory, Cambridge Studied in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990)
17. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)
18. Hendrickx, J.M., Olshevsky, A.: Matrix p-norms are NP-hard to approximate if $P = 1, 2, \infty$. SIAM. J. Matrix Anal. Appl. **31**, 2802–2812 (2010)
19. He, S., Yang, C.: Solving the variational inequality problem defined on intersection of finite level sets. Abstract Appl. Anal. **2013**, 8 (2013)
20. Kesornprom, S., Pholasa, N., Cholamjiak, P.: On the convergence analysis of the gradient-CQ algorithms for the split feasibility problem. Numer. Algorithms **84**, 997–1017 (2020)
21. Khan, S.A., Suantai, S., Cholamjiak, W.: Shrinking projection methods involving inertial forward-backward splitting methods for inclusion problems. RACSAM **113**, 645–656 (2019)
22. López, G., Martin-Marquez, V., Wang, F., Xu, H.-K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Probl. **28**, 085004 (2012)
23. Masad, E., Reich, S.: A note on the multiple-set split convex feasibility problem in Hilbert space. J. Nonlinear Convex Anal. **3**, 357–371 (2007)
24. Maingé, P.E.: Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **325**, 469–479 (2007)
25. Maingé, P.E.: Regularized and inertial algorithms for common fixed points of nonlinear operators. J. Math. Anal. Appl. **34**, 876–887 (2008)
26. Marino, G., Xu, H.K.: Convergence of generalized proximal point algorithms. Commun. Pure Appl. Anal. **3**(4), 791–808 (2004)
27. Moudafi, A., Elisabeth, E.: An approximate inertial proximal method using enlargement of a maximal monotone operato. Int. J. Pure Appl. Math. **5**, 283–299 (2003)
28. Moudafi, A., Thakur, B.S.: Solving proximal split feasibility problems without prior knowledge of operator norms. Optim. Lett. **8**, 2099–2110 (2014)
29. Ogbuisi, F.U., Mewomo, O.T.: On split generalised mixed equilibrium problems and fixed-point problems with no prior knowledge of operator norm. J. Fixed Point Theory Appl. **19**, 2109–2128 (2017)

30. Polyak, B.T.: Some methods of speeding up the convergence of iteration methods. USSR Comput. Math. Math. Phys. **4**, 1–17 (1964)
31. Pascali, D., Sburlan, S.: Nonlinear mappings of monotone type. Sijthoff & Nordhoff International Publishers, Alphen aan den Rijn (1987)
32. Reich, S., Tuyen, T.M.: Iterative methods for solving the generalized split common null point problem in Hilbert spaces. Optimization **69**(5), 1013–1038 (2020)
33. Reich, S., Tuyen, T.M., Trang, N.M.: Parallel iterative methods for solving the split common fixed point problem in Hilbert spaces. Numer. Funct. Anal. Optim. **41**(7), 778–805 (2020)
34. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pacific J. Math. **33**, 209–216 (1970)
35. Suantai, S., Eiamniran, N., Pholasa, N., Cholamjiak, P.: Three-step projective methods for solving the split feasibility problems. Mathematics **7**(8), 712 (2019)
36. Suantai, S., Pholasa, N., Cholamjiak, P.: Relaxed CQ algorithms involving the inertial technique for multiple-sets split feasibility problems. RACSAM **113**, 1081–1099 (2019)
37. Suantai, S., Kesornprom, S., Cholamjiak, P.: Modified proximal algorithms for finding solutions of the split variational inclusions. Mathematics **7**(8), 708 (2019)
38. Suantai, S., Shehu, Y., Cholamjiak, P.: Nonlinear iterative methods for solving the split common null point problems in Banach spaces. Optim. Meth. Softw. **34**, 853–874 (2019)
39. Suantai, S., Shehu, Y., Cholamjiak, P., Iyiola, O.S.: Strong convergence of a self-adaptive method for the split feasibility problem in Banach spaces. J. Fixed Point Theory Appl. **20**, 68 (2018)
40. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
41. Takahashi, S., Takahashi, W., Toyoda, M.: Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces. J. Optim. Theory Appl. **147**(1), 27–41 (2010)
42. Tang, Y.: New inertial algorithm for solving split common null point problem in Banach spaces. J. Inequal. Appl. **2019**, 17 (2019)
43. Tang, Y.: Convergence analysis of a new iterative algorithm for solving split variational inclusion problems. J. Ind. Manage. Optim. **16**(2), 945–964 (2020)
44. Tang, Y., Gibali, A.: New self-adaptive step size algorithms for solving split variational inclusion problems and its applications. Numer. Algorithms **83**, 305–331 (2020)
45. Tian, M., Jiang, B.N.: Inertial Haugazeau's hybrid subgradient extragradient algorithm for variational inequality problems in Banach spaces. Optimization. **70**(5–6), 987–1007 (2021)
46. Tuyen, T.M., Thuy, N.T.T., Trang, N.M.: A strong convergence theorem for a parallel iterative method for solving the split common null point problem in Hilbert spaces. J. Optim. Theory Appl. **183**(1), 271–291 (2019)
47. Vinh, N.T., Cholamjiak, P., Suantai, S.: A new CQ algorithm for solving split feasibility problems in Hilbert spaces. Bull. Malays. Math. Sci. Soc. **42**, 2517–2534 (2019)
48. Wang, J., Hu, Y., Yu, C.K.W., Zhuang, X.: A family of projection gradient methods for solving the multiple-sets split feasibility problem. J. Optim. Theory Appl. **183**, 520–534 (2019)
49. Wang, F.: A new method for split common fixed-point problem without priori knowledge of operator norms. J. Fixed Point Theory Appl. **19**, 2427–2436 (2017)
50. Yang, Q.: On variable-step relaxed projection algorithm for variational inequalities. J. Math. Anal. Appl. **302**, 166–179 (2005)

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ORIGINAL ARTICLE

A New Self-Adaptive Method for the Multiple-Sets Split Common Null Point Problem in Banach Spaces

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Abstract
In this paper, we study the multiple-sets split common null point problem (MSCNPP) in Banach spaces. We introduce a new self-adaptive algorithm for solving this problem. Under suitable conditions, we prove a strong convergence theorem of the proposed algorithm. An application of the main theorem to the multiple-sets split feasibility problem in Banach spaces is also presented. Finally, we provide the numerical experiments which show the efficiency and implementation of the proposed method.

Keywords Banach space · Strong convergence · Maximal monotone · Split common null point problem

Mathematics Subject Classification (2010) 47H09 · 47H10 · 47J25 · 47J05

1 Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E and F are p-uniformly convex Banach spaces which are also uniformly smooth. Let $A_i : E \multimap E^*$, $i = 1, 2, \ldots, M$ and $B_j : F \multimap F^*$, $j = 1, 2, \ldots, N$ be set-valued mappings and $T : E \to F$ be a bounded linear operator. Consider the following so-called *multiple-sets split common null point problem* (MSCNPP): Find $z \in E$ such that

$$z \in \left(\bigcap_{i=1}^{M} A_i^{-1}0\right) \cap T^{-1}\left(\bigcap_{j=1}^{N} B_j^{-1}0\right). \tag{1.1}$$

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For simplicity we denote the solution set of (1.1) by Ω. If $M = N = 1$, then MSCNPP becomes the following well-known *split common null point problem* (SCNPP) [9]: Find $z \in E$ such that

$$z \in A^{-1}0 \cap T^{-1}(B^{-1}0). \tag{1.2}$$

Many real world problems such as in modeling intensity-modulated radiation therapy treatment planning (see [13, 15]), modeling of inverse problems arising from phase retrieval, medical image reconstruction, signal processing and in sensor networks in computerized tomography and data compression (see [8, 20]) can be mathematically modeled as problem (1.2). For example:

Example 1.1 Let H_1 and H_2 be two Hilbert spaces and $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and convex functions. If we take $A = \partial f$ and $B = \partial g$, where ∂f and ∂g are subdifferential of f and g, respectively, then SCNPP (1.2) becomes the following split minimization problem (SMP):

$$\text{Find } z \in \operatorname{argmin} f \cap T^{-1}(\operatorname{argmin} g),$$

where $\operatorname{argmin} f = \{x \in H_1 : f(x) \le f(y),\ \forall y \in H_1$ and $\operatorname{argmin} g = \{x \in H_2 : g(x) \le g(y),\ \forall y \in H_2\}$.

Example 1.2 Let C and Q be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. If we set $A = \partial i_C$ and $B = \partial i_Q$, where i_C and i_Q are indicator functions of C and Q, respectively, then SCNPP (1.2) becomes the following two-sets split feasibility problem (SFP):

$$\text{Find } z \in C \cap T^{-1}(Q). \tag{1.3}$$

The SFP has been first introduced by Censor and Elfving [13] in finite dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction.

In 2018, Raeisi et al. [29] introduced an iterative algorithm for solving multiple-sets SFP which is a generalization of two-sets SFP. They proved a strong convergence theorem of the proposed algorithm under appropriate situations in certain Banach spaces.

In order to solve the SCNPP for two maximal monotone operators A and B in the setting of Hilbert spaces H, Byrne et al. [9] introduced the following algorithm: for any fixed $u, x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_r^A (x_n - \tau T^*(I - Q_\mu^B) T x_n), \quad \forall n \ge 1, \tag{1.4}$$

where $J_r^A = (I + rA)^{-1}$ and $Q_\mu^B = (I + \mu B)^{-1}$ are the resolvent operators of A for $r > 0$ and B for $\mu > 0$, respectively. They proved that the sequence $\{x_n\}_{n \in \mathbb{N}}$ generated by (1.4) converges strongly to a solution of SCNPP (1.2) provided the stepsize $\tau \in \left(0, \frac{2}{\|T\|^2}\right)$. There have been many authors who modified this method for solving the SCNPP in several settings (see, e.g., [2, 21–23, 33, 34, 40, 42, 46–48]). However, it is observed that the stepsizes of almost of the methods depend on the norm of a bounded linear operator. It is known that the norm of a bounded linear operator or matrix in the finite dimensional space is very difficult to compute (see [24]). To overcome this difficulty, López et al. [26] (see also [51]) introduced a *self-adaptive method* for solving SFP (1.3). The advantage of this method is the stepsize does not require the prior knowledge of the norm of a bounded linear operator. It is worth to interest the self-adaptive method because we can easily compute the stepsize. In recent years, there have been many authors who studied the modified methods such that the stepsizes do not depend on the norm (see, *e.g.*, [6, 18, 43–45, 49]). The following questions arise here:

1. How to extend the iterative process (1.4) of Byrne et al. [9] to solve MSCNPP?
2. How to construct an iterative method, whose stepsizes do not require the prior knowledge of the norm of a bounded linear operator for solving MSCNPP?

To answer these questions, in this paper, we introduce a new self-adaptive algorithm and prove its strong convergence theorem for solving the multiple-sets split common null point in two Banach spaces. Our result improves and extends many existing results in literature in this direction.

Our paper is organized as follows: In Section 2, we present some preliminaries which will be needed in the sequel. The strong convergence theorem for the algorithm is established in Section 3. An application of the main result to the multiple-set split feasibility problem is also presented in Section 4. Finally, several numerical examples to illustrate our result and observe the performance of our algorithm are presented in Section 5.

2 Preliminaries

Let E be a real Banach space with its dual space E^*. Let S_E be the unit sphere of E and B_E the closed unit ball of E. The *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} \,:\, x, y \in B_E,\ \|x - y\| \geq \varepsilon\right\}.$$

Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. The space E is called *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$ and *p-uniformly convex* if there is a $c_p > 0$ such that $\delta_E(\varepsilon) \geq c_p\varepsilon^p$ for all $\varepsilon \in (0, 2]$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 \,:\, x, y \in S_E\right\}.$$

The space E is called *uniformly smooth* if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$ and *q-uniformly smooth* if there exists a $c_q > 0$ such that $\rho_E(\tau) \leq c_q t^q$ for all $t > 0$. It is observed that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [36]). Furthermore, L^p (or ℓ^p) and the Sobolev spaces are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ while Hilbert space is uniformly smooth (see [50]).

Definition 2.1 Let $p > 1$. The *generalized duality mapping* $J_p : E \multimap E^*$ is defined by

$$J_p(x) = \{x^* \in E^* :\ \langle x^*, x\rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1}\} \quad \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. In particular, $J_2 = J$ is called the *normalized duality mapping*. It is known that the generalized duality mapping J_p is one-to-one, single-valued and satisfies $J_p = (J_q)^{-1}$, where $(J_q)^{-1}$ is the generalized duality mapping of E^*. Moreover, if E is uniformly smooth then the duality mapping J_p is norm-to-norm uniformly continuous on bounded subsets of E (see [19, 36] for more details).

Definition 2.2 [7] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function. The *Bregman distance* with respect to f is defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle.$$

In the particular case if $f = \frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$, then the gradient ∇f of f is coincident with the generalized duality mapping J_p. So we have

$$\begin{aligned} D_{\frac{1}{p}\|\cdot\|^p}(x, y) = D_p(x, y) &= \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle x - y, J_p(y)\rangle \\ &= \frac{1}{p}\|x\|^p + \frac{1}{q}\|y\|^p - \langle x, J_p(y)\rangle, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 2$, then

$$D_{\frac{1}{2}\|\cdot\|^2}(x, y) = \frac{1}{2}\left(\|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2\right) = \frac{1}{2}\phi(x, y),$$

where ϕ is called *Lyapunov function* studied in [3] and [30]. Also, if E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$.

The Bregman distance has the following important properties [32]:

(1) *Two point identity*: for each $x, y \in E$,

$$D_p(x, y) + D_p(y, x) = \langle x - y, J_p(x) - J_p(y)\rangle.$$

(2) *Three point identity*: for each $x, y, z \in E$,

$$D_p(x, y) + D_p(y, z) - D_p(x, z) = \langle x - y, J_p(z) - J_p(y)\rangle.$$

For a p-uniformly convex space, the Bregman distance has the following property (see [37]):

$$\tau\|x - y\|^p \le D_p(x, y) \le \left\langle x - y, J_p^E(x) - J_p^E(y)\right\rangle, \tag{2.1}$$

where $\tau > 0$ is some fixed number.

Following [3, 14], we make use of the function $V_p : E \times E^* \to [0, \infty)$ which is given by

$$V_p(x, x^*) = \frac{\|x\|^p}{p} - \langle x, x^*\rangle + \frac{\|x^*\|^q}{q}$$

for all $x \in E$ and $x^* \in E^*$. Clearly, $V_p(x, x^*) \ge 0$ and satisfies the following properties:

$$V_p(x, x^*) = D_p(x, J_q(x^*)), \quad \forall x \in E,\ x^* \in E^*$$

and

$$V_p(x, x^*) + \langle J_q(x^*) - x, y^*\rangle \le V_p(x, x^* + y^*), \quad \forall x \in E,\ x^*, y^* \in E^*. \tag{2.2}$$

Moreover, V_p is convex in the second variable. Then for all $z \in E$, we have

$$D_p\left(z, J_q\left(\sum_{i=1}^{M} t_i J_p(x_i)\right)\right) \le \sum_{i=1}^{M} t_i D_p(z, x_i),$$

where $x_i \in E$ and $t_i \in (0, 1)$ for all $i = 1, 2, \ldots, M$ with $\sum_{i=1}^{M} t_i = 1$.

Definition 2.3 Let C be a nonempty, closed and convex subset of E. The *Bregman projection* is defined by

$$\Pi_C(x) = \operatorname*{argmin}_{y \in C} D_p(x, y), \quad x \in E.$$

We know the following properties [31]:

$$\langle z - \Pi_C(x), J_p(x) - J_p(\Pi_C(x))\rangle \leq 0, \quad \forall z \in C. \tag{2.3}$$

Let C be a nonempty subset of E and $T : C \to C$ be a mapping. We denote the fixed point set of T by $F(T) = \{x \in C : x = Tx\}$. A point $z \in C$ is called an *asymptotic fixed point* of T, if C contains a sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to z and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote $\widehat{F}(T)$ by the set of asymptotic fixed points of T. The concept of an asymptotic fixed point was introduced in [30].

Let $A : E \multimap E^*$ be a set-valued mapping. The effective domain of A is denoted by $\mathcal{D}(A) = \{x \in E : Ax \neq \emptyset\}$ and the range of A is also denoted by $\mathcal{R}(A) = \bigcup\{Ax : x \in \mathcal{D}(A)\}$. The set of null points of A is defined by $A^{-1}0 = \{x \in \mathcal{D}(A) : 0 \in Ax\}$ and also known that $A^{-1}0$ is closed and convex (see [41]).

Definition 2.4 A set-valued mapping A is said to be *monotone* if

$$\langle u - v, x - y\rangle \geq 0, \qquad \forall x, y \in \mathcal{D}(A),\ u \in Ax \text{ and } v \in Ay.$$

Next, we give some examples of monotone operator. The following classical examples played an important role for the development of monotone operator theory.

Example 2.1 ([1]) Let $G \subset \mathbb{R}^n$ be a bounded measurable domain. Define the operator $A : L^p(G) \to L^q(G)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, by the following formula

$$Ay(x) = \phi(x, |y(x)|^{p-1})|y(x)|^{p-2}y(x), \quad x \in G,$$

where the function $\phi(x, s)$ is measurable as a function on x for every $s \geq 0$ and continuous for almost all $x \in G$ as a function on s, $|\phi(x, s)| \leq M$ for all $s \geq 0$ and for almost all $x \in G$. Note that the operator A really maps $L^p(G) \to L^q(G)$ because of the inequality $|Ay| \leq M|y|^{p-1}$. It can be shown that A is a monotone mapping on $L^p(G)$.

Example 2.2 ([1]) This example is a one part from quantum mechanics. Define the operator

$$Au = -a^2\nabla^2 u + (f(x) + b)u(x) + u(x)\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy,$$

where $\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^3$, a and b are constants, $f(x) = f_0(x) + f_1(x)$, $f_0(x) \in L^\infty(\mathbb{R}^3)$, $f_1(x) \in L^2(\mathbb{R}^3)$. Let $A := L + B$, where the operator L is the linear part of A (it is the Schrödinger operator) and B is defined by the last term. It can be shown that B is a monotone mapping on $L^2(\mathbb{R}^3)$. This implies that $A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is also a monotone operator.

Definition 2.5 A monotone operator A on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.

Let E be a p-uniformly convex and uniformly smooth Banach space and $A : E \multimap E^*$ be a maximal monotone operator. We define the *resolvent* of A for $r > 0$ by

$$J_r^A(x) = (J_p + rA)^{-1}J_p(x), \quad \forall x \in E.$$

It is easy to show that $A^{-1}0 = F(J_r^A)$ for all $r > 0$. We know that J_r^A satisfies the following property [25]:

$$D_p(z, J_r^A(x)) + D_p(J_r^A(x), x) \leq D_p(z, x) \tag{2.4}$$

for all $x \in E$ and $z \in A^{-1}0$. Moreover, we have $\widehat{F}(J_r^A) = F(J_r^A)$.

For each $x \in E$ and $\mu > 0$, we define the *metric resolvent* of A for $\mu > 0$ by

$$Q_\mu^A(x) = (I + \mu J_p^{-1} A)^{-1}(x), \quad \forall x \in E.$$

It is observed that

$$0 \in J_p\left(Q_\mu^A(x) - x\right) + \mu A Q_\mu^A(x) \tag{2.5}$$

and $A^{-1}0 = F(Q_\mu^A)$ for $\mu > 0$. By (2.5), we see that

$$\frac{J_p(x - Q_\mu^A(x))}{\mu} \in A Q_\mu^A(x) \tag{2.6}$$

and

$$\frac{J_p(y - Q_\mu^A(y))}{\mu} \in A Q_\mu^A(y) \tag{2.7}$$

for all $x, y \in E$. Summing up (2.6) with (2.7) and by the monotonicity of A, we obtain

$$\langle Q_\mu^A(x) - Q_\mu^A(y), J_p(x - Q_\mu^A(x)) - J_p(y - Q_\mu^A(y))\rangle \geq 0 \tag{2.8}$$

for all $x, y \in E$. It is also known that, if $A^{-1}0 \neq \emptyset$, then

$$\langle Q_\mu^A(x) - v, J_p(x - Q_\mu^A(x))\rangle \geq 0, \tag{2.9}$$

for all $x \in E$ and $v \in A^{-1}0$ (see [4]). In fact, let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in E, then from (2.9), we have

$$\begin{aligned}\|x_n - v\|\|x_n - Q_\mu^A(x_n)\|^{p-1} &\geq \langle x_n - v, J_p(x_n - Q_\mu^A(x_n))\rangle \\ &\geq \langle x_n - Q_\mu^A(x_n), J_p(x_n - Q_\mu^A(x_n))\rangle \\ &= \|x_n - Q_\mu^A(x_n)\|^p,\end{aligned}$$

which implies that

$$\|x_n - Q_\mu^A(x_n)\| \leq \|x_n - v\|$$

for $v \in A^{-1}0$. Hence $\{x_n - Q_\mu^A(x_n)\}_{n\in\mathbb{N}}$ is bounded.

Remark 2.1 If E is a Hilbert space, then the metric resolvent operator is equivalent to the resolvent operator.

Lemma 2.1 [50] *If E is a q-uniformly smooth Banach space, then there is a constant $c_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_q(x)\rangle + c_q\|y\|^q, \quad \forall x, y \in E,$$

where c_q is called the q-uniform smoothness coefficient of E.

Lemma 2.2 [27] *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ be two nonnegative sequences such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 1,$$

with $\{\delta_n\}_{n\in\mathbb{N}} \subset (0, 1)$ and $\{b_n\}_{n\in\mathbb{N}}$ is a real sequence. If $\sum_{n=1}^\infty c_n < \infty$, $\sum_{n=1}^\infty \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 *([28]) Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a real sequence such that there is a subsequence $\{\Gamma_{n_i}\}_{i\in\mathbb{N}} \subset \{\Gamma_n\}_{n\in\mathbb{N}}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\sigma(n)\}_{n\geq n_0}$ of integers as follows:*

$$\sigma(n) = \max\{k \leq n : \ \Gamma_k < \Gamma_{k+1}\}$$

for all $n \geq n_0$ (for some n_0 large enough). Then $\{\sigma(n)\}_{n\geq n_0}$ is a nondecreasing sequence such that $\lim_{n\to\infty}\sigma(n)=\infty$, and it holds that

$$\Gamma_{\sigma(n)} \leq \Gamma_{\sigma(n)+1} \quad \textit{and} \quad \Gamma_n \leq \Gamma_{\sigma(n)+1}.$$

In what follows, we shall use the following notations:

- $x_n \to x$ means that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to x.
- $x_n \rightharpoonup x$ means that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x.

3 Main Result

Let E be a p-uniformly convex and uniformly smooth Banach space and F be a uniformly convex and smooth Banach space. Let $A_i : E \multimap E^*$, $i = 1, 2, \ldots, M$ and $B_j : F \multimap F^*$, $j = 1, 2, \ldots, N$ be maximal monotone operators. Let $J_{r_i}^{A_i}$, $i = 1, 2, \ldots, M$ be resolvent operators of A_i for $r_i > 0$ and $Q_{\mu_j}^{B_j}$, $j = 1, 2, \ldots, N$ be a metric resolvent operators of B_j for $\mu_j > 0$. Let $T : E \to F$ be a bounded linear operator with its adjoint operator T^*. We denote by J_p^E and J_p^F the generalized duality mappings of E and F, respectively, and $J_q^{E^*}$ the duality mapping of E^*, where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. In order to solve MSCNPP (1.1), we propose the following algorithm:

Algorithm 3.1 For any $u, x_1 \in E$, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following scheme:

$$\begin{cases} z_{n,1} = J_q^{E^*}\left(J_p^E(x_n) - \tau_{n,1}h(x_n)\right), \\ z_{n,2} = J_q^{E^*}\left(J_p^E(z_{n,1}) - \tau_{n,2}h(z_{n,1})\right), \\ \quad\vdots \\ z_{n,N} = J_q^{E^*}\left(J_p^E(z_{n,N-1}) - \tau_{n,N}h(z_{n,N-1})\right), \\ y_n = J_q^{E^*}\left(a_{n,0}J_p^E(x_n) + \sum_{i=1}^M a_{n,i}J_p^E\left(J_{r_i}^{A_i}z_{n,N}\right)\right), \\ x_{n+1} = J_q^{E^*}\left(\alpha_n J_p^E(u) + (1-\alpha_n)J_p^E(y_n)\right), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, 1)$, $\{a_{n,i}\}_{i=0}^M \subset (0, 1)$ and $h(z_{n,j-1}) = T^*J_p^F(I - Q_{\mu_j}^{B_j})Tz_{n,j-1}$ for $j = 1, 2, \ldots, N$ with $z_{n,0} = x_n$. Suppose the stepsizes are chosen in such a way that

$$\tau_{n,1} = \frac{\rho_n f^{p-1}(x_n)}{\|h(x_n)\|^p + \theta_n}, \quad \tau_{n,2} = \frac{\rho_n f^{p-1}(z_{n,1})}{\|h(z_{n,1})\|^p + \theta_n}, \quad \ldots, \quad \tau_{n,N} = \frac{\rho_n f^{p-1}(z_{n,N-1})}{\|h(z_{n,N-1})\|^p + \theta_n},$$

where $f(z_{n,j-1}) = \frac{1}{p}\|(I - Q_{\mu_j}^{B_j})Tz_{n,j-1}\|^p$, $\{\rho_n\}_{n\in\mathbb{N}} \subset \left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$, $\{\theta_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $(0, \infty)$ and c_q is a constant given by Lemma 2.1.

Theorem 3.1 *Assume that $\Omega \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by Algorithm 3.1 with the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty}\rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right) > 0$*;*

(C3) $\sum_{i=0}^{M} a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$.

Then $\{x_n\}_{n\in\mathbb{N}}$ *strongly converges to* $z \in \Omega$.

Proof Let $v \in \Omega$, then $v \in A_i^{-1}0$ for $i = 1, 2, \ldots, M$ and $Tv \in B_j^{-1}0$ for $j = 1, 2, \ldots, N$. From (2.9), we have

$$\begin{aligned}
&\langle z_{n,j-1} - v, h(z_{n,j-1})\rangle \\
&= \langle z_{n,j-1} - v, T^* J_p^E (I - Q_{\mu_j}^{B_j}) T z_{n,j-1}\rangle \\
&= \langle T z_{n,j-1} - Tv, J_p^E (I - Q_{\mu_j}^{B_j}) T z_{n,j-1} - J_p^E (I - Q_{\mu_j}^{B_j}) T v\rangle \\
&\geq \langle T z_{n,j-1} - Tv, J_p^E (I - Q_{\mu_j}^{B_j}) T z_{n,j-1}\rangle \\
&\quad + \langle Tv - Q_{\mu_j}^{B_j}(T z_{n,j-1}), J_p^E (I - Q_{\mu_j}^{B_j}) T z_{n,j-1}\rangle \\
&= \langle T z_{n,j-1} - Q_{\mu_j}^{B_j}(T z_{n,j-1}), J_p^E (I - Q_{\mu_j}^{B_j}) T z_{n,j-1}\rangle \\
&= \|(I - Q_{\mu_j}^{B_j}) T z_{n,j-1}\|^p = p f(z_{n,j-1}).
\end{aligned} \tag{3.1}$$

Thus by Lemma 2.1 and (3.1), we have

$$\begin{aligned}
&D_p(v, z_{n,j}) \\
&= D_p(v, J_q^{E^*}(J_p^E(z_{n,j-1}) - \tau_{n,j} h(z_{n,j-1}))) \\
&= V_p(v, J_p^E(z_{n,j-1}) - \tau_{n,j} h(z_{n,j-1})) \\
&= \frac{1}{p}\|v\|^p - \langle v, J_p^E(z_{n,j-1})\rangle + \tau_{n,j}\langle v, h(z_{n,j-1})\rangle \\
&\quad + \frac{1}{q}\|J_p^E(z_{n,j-1}) - \tau_{n,j} h(z_{n,j-1})\|^q \\
&\leq \frac{1}{p}\|v\|^p - \langle v, J_p^E(z_{n,j-1})\rangle + \tau_{n,j}\langle v, h(z_{n,j-1})\rangle \\
&\quad + \frac{1}{q}\|J_p^E(z_{n,j-1})\|^q - \tau_{n,j}\langle z_{n,j-1}, h(z_{n,j-1})\rangle + \frac{c_q \tau_{n,j}^q}{q}\|h(z_{n,j-1})\|^q \\
&= \frac{1}{p}\|v\|^p - \langle v, J_p^E(z_{n,j-1})\rangle + \frac{1}{q}\|z_{n,j-1}\|^p \\
&\quad - \tau_{n,j}\langle z_{n,j-1} - v, h(z_{n,j-1})\rangle + \frac{c_q \tau_{n,j}^q}{q}\|h(z_{n,j-1})\|^q \\
&= D_p(v, z_{n,j-1}) - \tau_{n,j} p f(z_{n,j-1}) + \frac{c_q \tau_{n,j}^q}{q}\|h(z_{n,j-1})\|^q \\
&= D_p(v, z_{n,j-1}) - \frac{\rho_n p f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} + \frac{\rho_n^q c_q}{q}\frac{f^p(z_{n,j-1})}{(\|h(z_{n,j-1})\|^p + \theta_n)^q}\|h(z_{n,j-1})\|^q \\
&\leq D_p(v, z_{n,j-1}) - \frac{\rho_n p f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} + \frac{\rho_n^q c_q}{q}\frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} \\
&= D_p(v, z_{n,j-1}) - \rho_n\left(p - \frac{\rho_n^{q-1} c_q}{q}\right)\frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n}.
\end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$
\begin{aligned}
D_p(v, z_{n,N}) &\leq D_p(v, z_{n,N-1}) - \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \frac{f^p(z_{n,N-1})}{\|h(z_{n,N-1})\|^p + \theta_n} \\
&\vdots \\
&\leq D_p(v, z_{n,1}) - \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \frac{f^p(z_{n,1})}{\|h(z_{n,1})\|^p + \theta_n} \\
&\quad - \cdots - \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \frac{f^p(z_{n,N-1})}{\|h(z_{n,N-1})\|^p + \theta_n} \\
&\leq D_p(v, x_n) - \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \frac{f^p(x_n)}{\|h(x_n)\|^p + \theta_n} \\
&\quad -\rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \frac{f^p(z_{n,1})}{\|h(z_{n,1})\|^p + \theta_n} \\
&\quad - \cdots - \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \frac{f^p(z_{n,N-1})}{\|h(z_{n,N-1})\|^p + \theta_n} \\
&= D_p(v, x_n) - \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \sum_{j=1}^{N} \frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n}.
\end{aligned}
$$

From (2.4), we have

$$
\begin{aligned}
&D_p(v, y_n) \\
&= D_p\left(v, J_q^{E^*}\left(a_{n,0} J_p^E(x_n) + \sum_{i=1}^{M} a_{n,i} J_p^E\left(J_{r_i}^{A_i}(z_{n,N})\right)\right)\right) \\
&\leq a_{n,0} D_p(v, x_n) + \sum_{i=1}^{M} a_{n,i} D_p(v, J_{r_i}^{A_i}(z_{n,N})) \\
&\leq a_{n,0} D_p(v, x_n) + \sum_{i=1}^{M} a_{n,i} D_p(v, z_{n,N}) - \sum_{i=1}^{M} a_{n,i} D_p(J_{r_i}^{A_i}(z_{n,N}), z_{n,N}) \\
&\leq a_{n,0} D_p(v, x_n) + \sum_{i=1}^{M} a_{n,i} D_p(v, x_n) \\
&\quad -\sum_{i=1}^{M} a_{n,i} \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \sum_{j=1}^{N} \frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} - \sum_{i=1}^{M} a_{n,i} D_p(J_{r_i}^{A_i}(z_{n,N}), z_{n,N}) \\
&= D_p(v, x_n) - \sum_{i=1}^{M} a_{n,i} \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) \sum_{j=1}^{N} \frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} \\
&\quad - \sum_{i=1}^{M} a_{n,i} D_p(J_{r_i}^{A_i}(z_{n,N}), z_{n,N}), \qquad (3.3)
\end{aligned}
$$

which implies that

$$D_p(v, y_n) \leq D_p(v, x_n). \tag{3.4}$$

It follows that

$$\begin{aligned} D_p(v, x_{n+1}) &= D_p\left(v, J_q^{E^*}\left(\alpha_n J_p^E(u) + (1-\alpha_n)J_p^E(y_n)\right)\right) \\ &\leq \alpha_n D_p(v, u) + (1-\alpha_n)D_p(v, y_n) \\ &\leq \alpha_n D_p(v, u) + (1-\alpha_n)D_p(v, x_n) \\ &\leq \max\{D_p(v, u), D_p(v, x_n)\} \\ &\vdots \\ &\leq \max\{D_p(v, u), D_p(v, x_1)\}. \end{aligned}$$

Hence $\{D_p(v, x_n)\}_{n\in\mathbb{N}}$ is bounded and $\{x_n\}_{n\in\mathbb{N}}$ is also bounded by (2.1). Consequently, $\{z_{n,i}\}_{n\in\mathbb{N}}$ for $i = 1, 2, \ldots, N$ is bounded. Let $z = \Pi_\Omega(u)$. From (3.3), we have

$$\begin{aligned} D_p(z, x_{n+1}) &= D_p\left(z, J_q^{E^*}\left(\alpha_n J_p^E(u) + (1-\alpha_n)J_p^E(y_n)\right)\right) \\ &\leq \alpha_n D_p(z, u) + (1-\alpha_n)D_p(z, y_n) \\ &\leq \alpha_n D_p(z, u) + (1-\alpha_n)D_p(z, x_n) \\ &\quad -(1-\alpha_n)\sum_{i=1}^{M} a_{n,i}\rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right)\sum_{j=1}^{N}\frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} \\ &\quad -(1-\alpha_n)\sum_{i=1}^{M} a_{n,i} D_p(J_{r_i}^{A_i}(z_{n,N}), z_{n,N}), \end{aligned}$$

which implies that

$$\begin{aligned} &(1-\alpha_n)\sum_{i=1}^{M} a_{n,i}\rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right)\sum_{j=1}^{N}\frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} \\ &+(1-\alpha_n)\sum_{i=1}^{M} a_{n,i} D_p(J_{r_i}^{A_i}(z_{n,N}), z_{n,N}) \\ &\leq D_p(z, x_n) - D_p(z, x_{n+1}) + \alpha_n K, \end{aligned} \tag{3.5}$$

where $K = \sup_{n\in\mathbb{N}}\{|D_p(z, u) - D_p(z, x_n)|\}$.

Now, we divide the rest of the proof into two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_p(z, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. Then we have $D_p(z, x_n) - D_p(z, x_{n+1}) \to 0$. By our assumptions, we have from (3.5) that

$$\frac{f^p(z_{n,j-1})}{\|h(z_{n,j-1})\|^p + \theta_n} \to 0 \quad \text{and} \quad D_p(J_{r_i}^{A_i}(z_{n,N}), z_{n,N}) \to 0$$

for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$. Since $\{h(z_{n,j-1})\}_{n\in\mathbb{N}}$ for $j = 1, 2, \ldots, N$ and $\{\theta_n\}_{n\in\mathbb{N}}$ are bounded, it follows that

$$\lim_{n\to\infty} f(z_{n,j-1}) = \lim_{n\to\infty}\left\|(I - Q_{\mu_j}^{B_j})Tz_{n,j-1}\right\| = 0, \quad j = 1, 2, \ldots, N. \tag{3.6}$$

Moreover, from (2.1), we have

$$\lim_{n\to\infty}\left\|J_{r_i}^{A_i}(z_{n,N})-z_{n,N}\right\|=0,\quad i=1,2,\ldots,M. \tag{3.7}$$

Since J_p^E is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty}\left\|J_p^E(J_{r_i}^{A_i}(z_{n,N}))-J_p^E(z_{n,N})\right\|=0,\quad i=1,2,\ldots,M. \tag{3.8}$$

From (3.6), we see that

$$\begin{aligned}\left\|J_p^E(z_{n,j})-J_p^E(z_{n,j-1})\right\| &= \tau_{n,j}\|h(z_{n,j-1})\|\\ &\leq \tau_{n,j}\|T^*\|\left\|(I-Q_{\mu_j}^{B_j})Tz_{n,j-1}\right\|^{p-1}\to 0\end{aligned} \tag{3.9}$$

for $j=1,2,\ldots,N$. From (3.9), we have

$$\begin{aligned}\left\|J_p^E(z_{n,N})-J_p^E(x_n)\right\| \leq{}& \left\|J_p^E(z_{n,N})-J_p^E(z_{n,N-1})\right\|+\left\|J_p^E(z_{n,N-1})-J_p^E(z_{n,N-2})\right\|\\ &+\cdots+\left\|J_p^E(z_{n,1})-J_p^E(x_n)\right\|\to 0.\end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}\left\|J_p^E(z_{n,j-1})-J_p^E(x_n)\right\| \leq{}& \left\|J_p^E(z_{n,j-1})-J_p^E(z_{n,j})\right\|+\cdots+\left\|J_p^E(z_{n,N-1})-J_p^E(z_{n,N})\right\|\\ &+\left\|J_p^E(z_{n,N})-J_p^E(x_n)\right\|\to 0.\end{aligned} \tag{3.11}$$

Since $J_q^{E^*}$ is norm-to-norm uniformly continuous on bounded subsets of E^*, it follows from (3.10) and (3.11) that

$$\lim_{n\to\infty}\|z_{n,N}-x_n\|=0$$

and

$$\lim_{n\to\infty}\|z_{n,j-1}-x_n\|=0,\quad j=1,2,\ldots,N. \tag{3.12}$$

On the other hand, from (3.8) and (3.10), we see that

$$\begin{aligned}&\left\|J_p^E(y_n)-J_p^E(z_{n,N})\right\|\\ &=\left\|a_{n,0}\left(J_p^E(x_n)-J_p^E(z_{n,N})\right)+\sum_{i=1}^M a_{n,i}\left(J_p^E\left(J_{r_i}^{A_i}(z_{n,N})\right)-J_p^E(z_{n,N})\right)\right\|\\ &\leq a_{n,0}\left\|J_p^E(x_n)-J_p^E(z_{n,N})\right\|+\sum_{i=1}^M a_{n,i}\left\|J_p^E(J_{r_i}^{A_i}(z_{n,N}))-J_p^E(z_{n,N})\right\|\\ &\to 0.\end{aligned} \tag{3.13}$$

Moreover, from (3.10) and (3.13), we have

$$\begin{aligned}\|J_p^E(y_n)-J_p^E(x_n)\| &\leq \|J_p^E(y_n)-J_p^E(z_{n,N})\|+\|J_p^E(z_{n,N})-J_p^E(x_n)\|\\ &\to 0.\end{aligned} \tag{3.14}$$

Thus, by (3.14), we have

$$\begin{aligned}\|J_p^E(x_{n+1})-J_p^E(x_n)\| &\leq \|J_p^E(x_{n+1})-J_p^E(y_n)\|+\|J_p^E(y_n)-J_p^E(x_n)\|\\ &= \alpha_n\|J_p^E(u)-J_p^E(y_n)\|+\|J_p^E(y_n)-J_p^E(x_n)\|\\ &\to 0\end{aligned}$$

and hence,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

Next, we show that

$$\limsup_{n\to\infty} \langle x_n - z, J_p^E(u) - J_p^E(z)\rangle \le 0.$$

By the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup \hat{x}$ and

$$\limsup_{n\to\infty} \langle x_n - z, J_p^E(u) - J_p^E(z)\rangle = \lim_{k\to\infty} \langle x_{n_k} - z, J_p^E(u) - J_p^E(z)\rangle.$$

Since $\|x_n - z_{n,N}\| \to 0$, then we can choose a subsequence $\{z_{n_k,N}\}_{k\in\mathbb{N}}$ of $\{z_{n,N}\}_{n\in\mathbb{N}}$ such that $z_{n_k,N} \rightharpoonup \hat{x}$. Thus by (3.7), we get $\hat{x} \in \widehat{F}(J_{r_i}^{A_i}) = F(J_{r_i}^{A_i})$ for $i = 1, 2, \ldots, M$ and so $\hat{x} \in \bigcap_{i=1}^M A_i^{-1}0$. From (2.8), we have

$$\begin{aligned}
&\langle Q_{\mu_j}^{B_j}(Tz_{n,i-1}) - Q_{\mu_j}^{B_j}(T\hat{x}), J_p^F(I - Q_{\mu_j}^{B_j})T\hat{x}\rangle \\
&\le \langle Q_{\mu_j}^{B_j}(Tz_{n,i-1}) - Q_{\mu_j}^{B_j}(T\hat{x}), J_p^F(I - Q_{\mu_j}^{B_j})Tz_{n,i-1}\rangle \\
&\le \|Q_{\mu_j}^{B_j}(Tz_{n,i-1}) - Q_{\mu_j}^{B_j}(T\hat{x})\| \|(I - Q_{\mu_j}^{B_j})Tz_{n,i-1}\|^{p-1} \\
&\quad \to 0
\end{aligned} \tag{3.16}$$

for $j = 1, 2, \ldots, N$. Moreover, we see that

$$\begin{aligned}
\|(I - Q_{\mu_j}^{B_j})T\hat{x}\|^p &= \langle (I - Q_{\mu_j}^{B_j})T\hat{x}, J_p^F(I - Q_{\mu_j}^{B_j})T\hat{x}\rangle \\
&= \langle T\hat{x} - Tz_{n_k,i-1}, J_p^F(I - Q_{\mu_j}^{B_j})T\hat{x}\rangle \\
&\quad + \langle Tz_{n_k,i-1} - Q_{\mu_j}^{B_j}(Tz_{n_k,i-1}), J_p^F(I - Q_{\mu_j}^{B_j})T\hat{x}\rangle \\
&\quad + \langle Q_{\mu_j}^{B_j}(Tz_{n,i-1}) - Q_{\mu_j}^{B_j}(T\hat{x}), J_p^F(I - Q_{\mu_j}^{B_j})T\hat{x}\rangle.
\end{aligned}$$

By (3.12) and the continuity of T, we have $Tz_{n_k,i-1} \rightharpoonup T\hat{x}$ as $k \to \infty$. Thus by (3.6) and (3.16), we have

$$\|T\hat{x} - Q_{\mu_j}^{B_j}(T\hat{x})\| = 0, \quad j = 1, 2, \ldots, N.$$

This implies that $T\hat{x} \in B_j^{-1}0$ for $j = 1, 2, \ldots, N$, i.e., $\hat{x} \in T^{-1}\left(\bigcap_{j=1}^N B_j^{-1}0\right)$ and so $\hat{x} \in \Omega$. Thus by (2.3), we have

$$\limsup_{n\to\infty} \langle x_n - z, J_p^E(u) - J_p^E(z)\rangle = \langle \hat{x} - z, J_p^E(u) - J_p^E(z)\rangle \le 0.$$

From (3.15), we also have

$$\limsup_{n\to\infty} \langle x_{n+1} - z, J_p^E(u) - J_p^E(z)\rangle \le 0. \tag{3.17}$$

Finally, we show that $\{x_n\}$ converges strongly to $z = P_\Omega(u)$. From (2.2) and (3.4), we have

$$\begin{aligned} D_p(z, x_{n+1}) &= D_p(z, J_q^{E^*}(\alpha_n J_p^E(u) + (1-\alpha_n) J_p^E(y_n))) \\ &= V_p(z, \alpha_n J_p^E(u) + (1-\alpha_n) J_p^E(y_n)) \\ &\leq V_p(z, \alpha_n J_p^E(u) + (1-\alpha_n) J_p^E(y_n) - \alpha_n (J_p^E(u) - J_p^E(z)) \\ &\quad + \alpha_n \langle x_{n+1} - z, J_p^E(u) - J_p^E(z)\rangle \\ &= V_p(z, \alpha_n J_p^E(z) + (1-\alpha_n) J_p^E(y_n)) + \alpha_n \langle x_{n+1} - z, J_p^E(u) - J_p^E(z)\rangle \\ &= \alpha_n D_p(z, z) + (1-\alpha_n) D_p(z, y_n) + \alpha_n \langle x_{n+1} - z, J_p^E(u) - J_p^E(z)\rangle \\ &\leq (1-\alpha_n) D_p(z, x_n) + \alpha_n \langle x_{n+1} - z, J_p^E(u) - J_p^E(z)\rangle. \end{aligned} \tag{3.18}$$

This together with (3.17) and (3.18), so we can conclude by Lemma 2.2 that $D_p(z, x_n) \to$ as $n \to \infty$. Therefore, $x_n \to z$.

Case 2. Put $\Gamma_n = D_p(z, x_n)$ for all $n \in \mathbb{N}$. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} \leq \Gamma_{n_{i+1}}$ for all $i \in \mathbb{N}$. Define a mapping $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\sigma(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}$$

for all $n \geq n_0$ (for some n_0 large enough). Thus by Lemma 2.3, we have σ is nondecreasing such that $\sigma(n) \to \infty$ as $n \to \infty$ and $0 \leq \Gamma_{\sigma(n)} \leq \Gamma_{\sigma(n)+1}$ for all $n \geq n_0$. It follows from (3.5) that

$$\frac{f^p(z_{\sigma(n),j-1})}{\|h(z_{\sigma(n),j-1})\|^p + \theta_{\sigma(n)}} \to 0 \quad \text{and} \quad D_p(J_{r_i}^{A_i}(z_{\sigma(n),N}), z_{\sigma(n),N}) \to 0$$

for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$. By a similar argument as in Case 1, we can show that

$$\begin{aligned} \lim_{n\to\infty} f(z_{\sigma(n),j-1}) = \lim_{n\to\infty} \|(I - Q_{\mu_j}^{B_j})Tz_{\sigma(n),j-1}\| &= 0, \quad j = 1, 2, \ldots, N, \\ \lim_{n\to\infty} \|J_{r_i}^{A_i}(z_{\sigma(n),N}) - z_{\sigma(n),N}\| &= 0, \quad i = 1, 2, \ldots, M, \end{aligned}$$

and

$$\lim_{n\to\infty} \|x_{\sigma(n)+1} - x_{\sigma(n)}\| = 0.$$

Moreover, we can show that

$$\limsup_{n\to\infty} \langle x_{\sigma(n)+1} - z, J_p^E(u) - J_p^E(z)\rangle \leq 0.$$

From (3.18), we have

$$D_p(z, x_{\sigma(n)+1}) \leq (1-\alpha_{\sigma(n)}) D_p(z, x_{\sigma(n)}) + \alpha_{\sigma(n)} \langle x_{\sigma(n)+1} - z, J_p^E(u) - J_p^E(z)\rangle.$$

By Lemma 2.2, we have $D_p(z, x_{\sigma(n)}) \to 0$. It follows that

$$\lim_{n\to\infty} D_p(z, x_{\sigma(n)+1}) = \lim_{n\to\infty} D_p(z, x_{\sigma(n)}) = 0.$$

Thus by Lemma 2.3, we have

$$D_p(z, x_n) \leq D_p(z, x_{\sigma(n)+1}) \to 0,$$

which implies that $D_p(z, x_n) \to 0$ as $n \to \infty$. Therefore, $x_n \to z$. This completes the proof. □

Remark 3.1 Some other approaches to the proof of Theorem 3.1 in the Hilbert spaces can be found in [10, Fact 2.4(i)–(ii)], [11, Corollaries 5.2(i) and 5.5(i)] or [12, Theorem 5.3(i)].

For instance, we consider MSCNPP (1.1) in case $E = H_1$ and $F = H_2$ are real Hilbert spaces. Then $p = q = 2$, $c_q = 1$ and $J_q^{E^*} = J_p^F = I$. Suppose that A_i, $i = 1, 2, \dots, M$, and B_j, $j = 1, 2, \dots, N$, are maximal monotone. Then the resolvent operators $U_i := J_{r_i}^{A_i}$, $i = 1, 2, \dots, M$, and $V_j := Q_{\mu_j}^{B_j}$, $j = 1, 2, \dots, N$, are firmly nonexpansive (see [5]). Suppose that T is a bounded linear operator with its adjoint T^* and $\Omega \neq \emptyset$. For a given bounded sequence $\{\theta_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ and a sequence $\{\lambda_n(= \frac{\rho_n}{2})\}_{n\in\mathbb{N}} \subset (0, 2)$, define a mapping S_j by

$$S_j(x) := x - \lambda_n \frac{\|(I - V_j)Tx\|^2}{\|T^*(I - V_j)Tx\|^2 + \theta_n} T^*(I - V_j)Tx, \quad \forall x \in H_1,$$

for $j = 1, 2, \dots$. Then S_j is ρ-strongly quasi-nonexpansive, where $\rho > 0$. Define an operator W_n by

$$W_n x := \alpha_{n,0}x + \sum_{i=1}^{M} \alpha_{n,i} U_i S_N S_{N-1} \dots S_1(x),$$

where $\alpha_{n,i} > 0$, $i = 1, 2, \dots, M$, and $\sum_{i=1}^M \alpha_{n,i} = 1$, $n \geq 0$. Then W_n are δ_n-strongly quasi-nonexpansive, where $\delta_n > 0$. One can also estimate bounds of δ_n. Moreover, the assumptions of Theorem 3.1 yield that the sequence $\{W_n\}$ satisfies the demiclosedness principle with $F(W_n) = \bigcap_{i=1}^M F(U_i) \cap \bigcap_{j=1}^N F(S_j) = \Omega$, $n \geq 1$ (see [10, Fact 2.4]). We can write Algorithm 3.1 in the form $x_{n+1} = \alpha_n u + (1-\alpha_n)W_n x_n$. Therefore, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $z = \Pi_\Omega(u)$.

4 Application to the Multiple-sets Split Feasibility Problem

Let E and F be p-uniformly convex and uniformly smooth Banach spaces. Let C_i, $i = 1, 2, \dots, M$ and Q_j, $j = 1, 2, \dots, N$ be nonempty, closed and convex subsets of E and F, respectively. Let $T : E \to F$ be a bounded linear operator with its adjoint T^*. Let us consider the following *multiple-sets split feasibility problem* (MSFP) ([16]): Find $z \in E$ such that

$$z \in \Omega := \left(\bigcap_{i=1}^{M} C_i\right) \cap T^{-1}\left(\bigcap_{j=1}^{N} Q_j\right). \tag{4.1}$$

This problem has broad applications in the field of medical care (see for instance, [15–17]) and in the fields of image reconstruction and signal processing (see [39]).

Let $g : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function. The subdifferential of g at x is defined as follows:

$$\partial g(x) = \{z \in E^* : g(x) + \langle y - x, z\rangle \leq g(y), \ \forall y \in E\}$$

for all $x \in E$. Let C be a closed and convex subset of E. The indicator function i_C is given by

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

It is known that i_C is a proper, semicontinuous and convex function and its subdifferentiable ∂i_C is a maximal monotone operator (see [35]). From [36], we know that

$$\partial i_C(x) = N_C(x) = \{z \in E^* : \langle y - x, z\rangle \leq 0, \ \forall y \in C\},$$

where N_C is the normal cone for C at x. Thus we can define the resolvent of ∂i_C for $r > 0$ by

$$J_r^A(x) = (J_p + r\partial i_C)^{-1} J_p(x), \quad \forall x \in E.$$

For each $x \in E$ and $r > 0$, we see that

$$\begin{aligned} z = J_r^A(x) \ &\Leftrightarrow \ J_p(x) \in J_p(z) + rN_C(z) \\ &\Leftrightarrow \ J_p(x) - J_p(z) \in rN_C(z) \\ &\Leftrightarrow \ \langle y - z, J_p(x) - J_p(z)\rangle \le 0, \quad \forall y \in C \\ &\Leftrightarrow \ z = \Pi_C(x), \end{aligned}$$

where Π_C is the Bregman projection from E onto C.

Theorem 4.1 *Assume that $\Omega \neq \emptyset$. For any $u, x_1 \in E$, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following scheme:*

$$\begin{cases} z_{n,1} &= J_q^{E^*}(J_p^E(x_n) - \tau_{n,1}h(x_n)), \\ z_{n,2} &= J_q^{E^*}(J_p^E(z_{n,1}) - \tau_{n,2}h(z_{n,1})), \\ \quad \vdots \\ z_{n,N} &= J_q^{E^*}(J_p^E(z_{n,N-1}) - \tau_{n,N}h(z_{n,N-1})), \\ y_n &= J_q^{E^*}(a_{n,0}J_p^E(x_n) + \sum_{i=1}^M a_{n,i}J_p^E(\Pi_{C_i} z_{n,N})), \\ x_{n+1} &= J_q^{E^*}(\alpha_n J_p^E(u) + (1-\alpha_n)J_p^E(y_n)), \quad \forall n \ge 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\{a_{n,i}\}_{i=0}^M \subset (0,1)$ and $h(z_{n,j-1}) = T^*J_p^F(I - \Pi_{Q_j})Tz_{n,j-1}$ for $j = 1, 2, \ldots, N$ with $z_{n,0} = x_n$. Suppose the stepsizes are chosen in such a way that

$$\tau_{n,1} = \frac{\rho_n f^{p-1}(x_n)}{\|h(x_n)\|^p + \theta_n}, \tau_{n,2} = \frac{\rho_n f^{p-1}(z_{n,1})}{\|h(z_{n,1})\|^p + \theta_n}, \ldots, \tau_{n,N} = \frac{\rho_n f^{p-1}(z_{n,N-1})}{\|h(z_{n,N-1})\|^p + \theta_n},$$

where $f(z_{n,j-1}) = \frac{1}{p}\|(I - \Pi_{Q_j})Tz_{n,j-1}\|^p$, $\{\rho_n\}_{n\in\mathbb{N}} \subset \left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$, $\{\theta_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $(0,\infty)$ and c_q is a constant given by Lemma 2.1. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \rho_n \left(p - \frac{\rho_n^{q-1} c_q}{q}\right) > 0$;

(C3) $\sum_{i=0}^M a_{n,i} = 1$ *and* $\liminf_{n\to\infty} a_{n,i} > 0$ *for* $i = 1, 2, \ldots, M$.

Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $z \in \Omega$.

Proof In this case, we set $A_i = \partial i_{C_i}$ for $i = 1, 2, \ldots, M$ and $B_j = \partial i_{Q_j}$ for $j = 1, 2, \ldots, N$ in Theorem 3.1. Then we have $J_{r_i}^{A_i} = \Pi_{C_i}$ for $i = 1, 2, \ldots, M$ and $Q_\mu^{B_j} = \Pi_{Q_j}$ for $j = 1, 2, \ldots, N$. Moreover, we also have $A_i^{-1}0 = C_i$ and $B_j^{-1}0 = Q_j$. We can conclude the desired conclusion easily from Theorem 3.1. □

5 Numerical Experiments

In this section, we propose several numerical experiments to support our main result. We perform the test examples in both finite dimensional and infinite dimensional spaces. Moreover, we compare them with some existing methods. The algorithms are implemented in MATLAB 14a running on the DESKTOP-9RLTPS0, Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz with 2.11 GHz and 8GB RAM.

Example 5.1 Consider the following *multiple-sets split minimization problem* (MSMP):

$$\text{Find } z \in \Omega := \left(\bigcap_{i=1}^{M} \operatorname{argmin} f_i\right) \cap T^{-1}\left(\bigcap_{j=1}^{N} \operatorname{argmin} g_j\right).$$

In this problem is considered here for $E = \mathbb{R}^5$ and $F = \mathbb{R}^6$ with $M = N = 2$. Let $f_i : \mathbb{R}^5 \to \mathbb{R} \cup \{+\infty\}$ and $g_i : \mathbb{R}^6 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f_i(x) = \langle A_i x, x\rangle + \langle C_i, x\rangle + K_i$$

and

$$g_i(x) = \langle B_i x, x\rangle + \langle D_i, x\rangle + L_i,$$

for all $i = 1, 2$, with

$$A_1 = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -2 & 2 & -2 & 2 & -2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} -4 & 2 & -4 & 4 & -4 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -4 & -4 & 4 & -4 & 4 & 0 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} -4 & -4 & 4 & -4 & 4 & -4 \end{pmatrix},$$

and K_i, L_i are any constants. Let $T : \mathbb{R}^5 \to \mathbb{R}^6$ be a bounded linear operator defined by

$$Tx = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

for all $x = (x_1, x_2, x_3, x_4, x_5)^\top \in \mathbb{R}^5$.

It is easy to see that f_i and g_i are proper, continuous and convex functions on $\mathbb{R}^5$ and $\mathbb{R}^6$, respectively. It is observe that $z \in \Omega$ if and only if $2A_1 z + C_1 = 0$, $2A_2 z + C_2 = 0$,

$2B_1(Tz) + D_1 = 0$ and $2B_2(Tz) + D_2 = 0$. It is easy to check that Ω in Example 5.1 is given by $\Omega = \{(3t-3, t, -2t+2, -2, 1)^\intercal : t \in \mathbb{R}\}$ and

$$z = \left(-\frac{6}{7}, \frac{5}{7}, \frac{4}{7}, -2, 1\right)^\intercal \approx (-0.8571, 0.7142, 0.5714, -2, 1)^\intercal.$$

We define the function TOL_n by

$$\mathrm{TOL}_n = \|x_n - z\|^2, \quad \forall n \geq 1$$

and we use the condition $\mathrm{TOL}_n < \varepsilon$ to stop the iterative process, where ε is a given positive real number.

Now, we compare our algorithm with Algorithm (4) of Tuyen et al. in [49]. The parameters are chosen as follows:

- Our algorithm (Algorithm 3.1): $r_i = \mu_i = 1$ for all $i = 1, 2$, $x_1 = (2, -1, -2, 4, 5)^\intercal$, $u = (1, -2, 2, 3, 8)^\intercal$, $\alpha_n = \frac{1}{n}$, $\rho_n = 2.5$, $\theta_n = 10^{-10}$ and $a_{n,i} = \frac{1}{3}$ for all $n \geq 1$ and for all $i = 0, 1, 2$.
- Algorithm (4) of Tuyen et al.: $\beta_{i,n} = 1.25$, $\lambda_{j,n} = 1.75$, $\delta_n = 0.35$, $\alpha_n = \frac{1}{n}$, for all $n \geq 1$ and $i = 1, 2$, and $x_1 = (2, -1, -2, 4, 5)^\intercal$, $f(x) = u = (1, -2, 2, 3, 8)^\intercal$.

We obtain Table 1 of numerical results.

The behavior of TOL_n in the case where $\varepsilon = 10^{-3}$ is described in Fig. 1.

Example 5.2 Let $E = \mathbb{R}^3$ and $F = \mathbb{R}^4$. We consider MSFP (4.1) with $M = 100$ and $N = 200$.

Now, let

$$C_i = [0, 1/i] \times [-1/i, 2i-1] \times [1-i, 1+i] \subset \mathbb{R}^3$$

and

$$Q_j = [1-j, j] \times [1-j/2, 1+j] \times [1-2j, j] \times [-1/j, 2j-1] \subset \mathbb{R}^4$$

for all $i = 1, 2, \ldots, 100$ and $j = 1, 2, \ldots, 200$. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a bounded linear operator defined by

$$T(x_1, x_2, x_3) = (5x_1, 8x_2, 10x_3, 4x_1)$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. It is easy to show that $\bigcap_{i=1}^{100} C_i = [0, 1/100] \times [-1/100, 1] \times [0, 2]$ and $\bigcap_{j=1}^{200} Q_j = [0, 1] \times [1/2, 2] \times [-1, 1] \times [1/200, 1]$. Hence

$$\Omega = \left[0, \frac{1}{100}\right] \times \left[\frac{1}{16}, \frac{1}{4}\right] \times \left[0, \frac{1}{10}\right].$$

Table 1 Table of numerical results for Example 5.1

Algorithms	ε	n	TOL_n	x_n	Time(s)
Algorithm 3.1					
	10^{-3}	5826	9.9977×10^{-4}	$(-0.8488, 0.6989, 0.5762, -1.9742, 0.9972)^\intercal$	0.0907
	10^{-4}	18419	9.9990×10^{-5}	$(-0.8545, 0.7094, 0.5729, -1.9918, 0.9991)^\intercal$	0.2840
	10^{-5}	58149	9.9998×10^{-6}	$(-0.8563, 0.7127, 0.5719, -1.9974, 0.9997)^\intercal$	0.9127
Tuyen et al.'s Algorithm					
	10^{-3}	6636	9.9871×10^{-4}	$(-0.8504, 0.7000, 0.5743, -1.9729, 0.9971)^\intercal$	0.1100
	10^{-4}	20976	9.9955×10^{-5}	$(-0.8550, 0.7097, 0.5723, -1.9914, 0.9990)^\intercal$	0.3455
	10^{-5}	66321	9.9988×10^{-6}	$(-0.8564, 0.7128, 0.5717, -1.9972, 0.9997)^\intercal$	1.0909

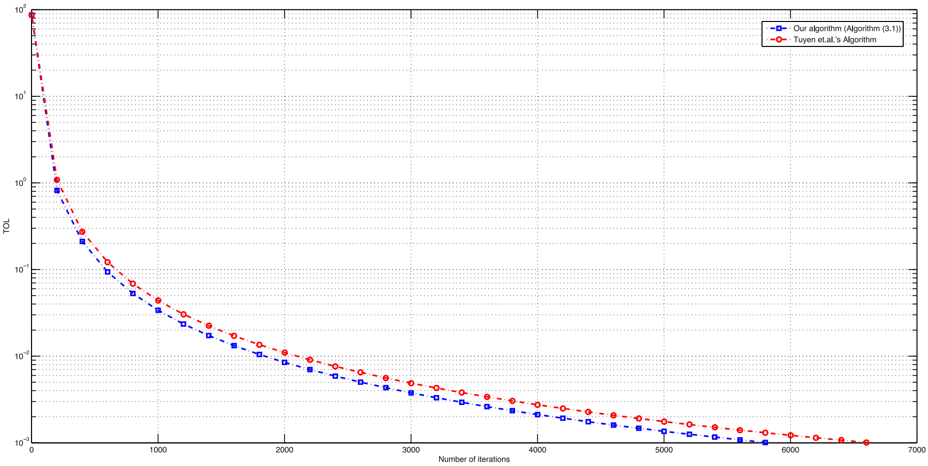

Fig. 1 The behavior of TOL_n with $\varepsilon = 10^{-3}$

We now compare our algorithm (Algorithm (4.2)) with Algorithm (27) of Tuyen et al. in [49] and Shehu's algorithm in [38]. The parameters are chosen as follows:

- Algorithm (4.2): $\alpha_n = 1/n$, $\rho_n = 3.9$, $\theta_n = 10^{-10}$ and $a_{n,i} = 1/101$ for all $n \geq 1$ and for all $i = 0, 1, \ldots, 100$, and $x_1 = (-1, 2, -3)$, $u = (1, 2, 3)$.
- Algorithm (27) of Tuyen et al.: $\beta_{i,n} = 0.5$, $\lambda_{j,n} = 0.5$, $\delta_n = 0.5$, $\alpha_n = \frac{1}{n}$, for all $n \geq 1$ and $i = 1, 2$, and $x_1 = (-1, 2, -3)^\intercal$, $f(x) = u = (1, 2, 3)^\intercal$.
- Shehu's algorithm: $\gamma = 10001$, $\lambda_j = 1$, $\beta_n = 0.25$, and $x_1 = (-1, 2, -3)$, $f(x) = u = (1, 2, 3)$.

The solution of the problem is $z = (0.01, 0.25, 0.1)$. In this example, we also use the stopping rule as in Example 5.1.

The behavior of TOL_n in Table 2 is described in Fig. 2.

Table 2 Table of numerical results for Example 5.2

Algorithms	ε	n	TOL_n	x_n	Time(s)
Algorithm (4.2)					
	10^{-3}	86	9.88767×10^{-4}	(0.03351, 0.26431, 0.11518)	0.0831
	10^{-4}	389	9.98576×10^{-5}	(0.01886, 0.25316, 0.10335)	0.3873
	10^{-5}	2439	9.99814×10^{-6}	(0.01307, 0.25050, 0.10053)	2.4414
Tuyen et al.'s Algorithm					
	10^{-3}	443	9.99097×10^{-4}	(0.01446, 0.27146, 0.12276)	0.4358
	10^{-4}	1401	9.98938×10^{-5}	(0.01141, 0.25678, 0.10719)	1.2504
	10^{-5}	4429	9.99548×10^{-6}	(0.01044, 0.25214, 0.10227)	3.7917
Shehu's Algorithm					
	10^{-3}	12681	9.99655×10^{-4}	(0.01219, 0.27165, 0.12293)	23.5480
	10^{-4}	39993	9.99653×10^{-5}	(0.01063, 0.25684, 0.10725)	73.0356
	10^{-5}	126099	9.99979×10^{-6}	(0.01000, 0.25216, 0.10230)	231.3808

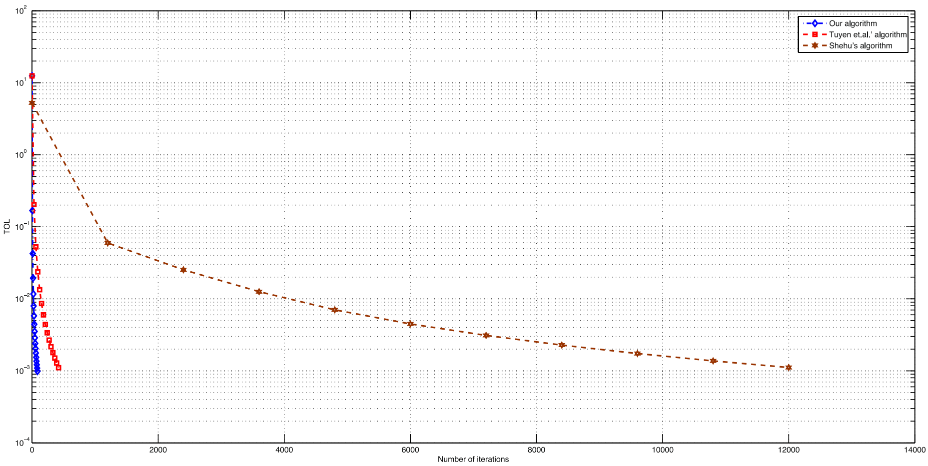

Fig. 2 The behavior of TOL_n with $\varepsilon = 10^{-3}$

Finally, we give an example in infinite dimensional Banach spaces outside Hilbert spaces.

Example 5.3 Let $p = 9/4$, $q = 9/5$, $E = F = l^p$ and $T : l^p \to l^p$ be defined by $Tx = \frac{x}{2}$ for all $x \in l^p$. In this example, we consider the two-sets split feasibility problem (MSFP (4.1) with $M = N = 1$), i.e.,

$$\text{find } z \in \Omega := C \cap T^{-1}(Q).$$

Let

$$C = \{x = (x_1, x_2, \dots) \in l^p : |x_i| \le 4 \text{ for all } i = 1, 2, \dots, 10 \text{ and } x_i = 0 \text{ for all } i \ge 11\}$$

and

$$Q = \{y = (y_1, y_2, \dots) \in l^p : |y_i| \le 1 \text{ for all } i = 1, 2, \dots, 10 \text{ and } y_i = 0 \text{ for all } i \ge 11\}.$$

Then the constant c_q in Lemma 2.1 is a unique solution of the following equation (see [50]):

$$(q-2)t^{q-1} + (q-1)t^{q-2} = 1.$$

It is easy to check that $c_q \approx 0.240368$.

We now test the convergence of the sequence $\{x_n\}$ generated by Algorithm (4.2) with $\alpha_n = 1/n$, $\theta_n = 10^{-10}$, $\rho_n = 32$, $x_1 = \{\xi_n\}$ with $\xi_i = i$ for all $i = 1, 2, \dots, 10$ and $\xi_i = 0$ for all $i \ge 11$, and $u = \{\eta_n\}$ with $\eta_i = 6 - i$ for all $i = 1, 2, \dots, 10$ and $\eta_i = 0$ for all $i \ge 11$. The solution of the problem is

$$z = (2, 2, 2, 2, 1, 0, -1, -2, -2, -2, 0, 0, \dots 0, \dots).$$

Thus we use the condition $\mathrm{TOL}_n = \|x_n - z\|^p < \varepsilon$ to stop the iterative process, where ε is a given positive real number. We obtain the following table of numerical results.

Table 3 Table of numerical results for Example 5.3

ε	n	TOL_n	x_n
10^{-4}	224	9.96995×10^{-5}	$(2.01187, 2.00833, 2.00462, 2, 1, 0, -1, -2, -2.00462, -2.00833, 0, \dots, 0, \dots)$
10^{-5}	809	9.99005×10^{-6}	$(2.00427, 2.00299, 2.00166, 2, 1, 0, -1, -2, -2.00166, -2.00299, 0, \dots, 0, \dots)$
10^{-6}	3195	9.99639×10^{-7}	$(2.00153, 2.00107, 2.00059, 2, 1, 0, -1, -2, -2.00059, -2.00107, 0, \dots, 0, \dots)$

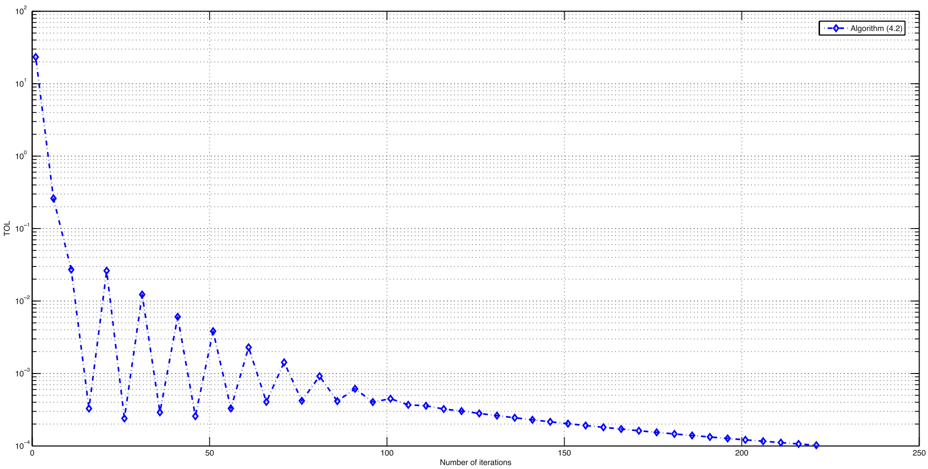

Fig. 3 The behavior of TOL_n with $\varepsilon = 10^{-4}$

The behavior of TOL_n in Table 3 is described in Fig. 3.

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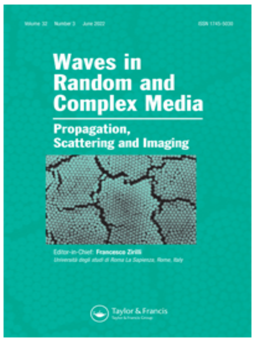
References

1. Alber, Y., Ryazantseva, I.: Nonlinear Ill-Posed Problems of Monotone Type. Springer, Dordrecht (2006)
2. Alofi, A.S., Alsulami, S.M., Takahashi, W.: Strongly convergent iterative method for the split common null point problem in Banach spaces. J. Nonlinear Convex Anal. **17**, 311–324 (2016)
3. Alber, Y.I.: Metric and generalized projection operators in banach spaces: Properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15–50. Marcel Dekker, New York (1996)
4. Aoyama, K., Kohsaka, F., Takahashi, W.: Three generalizations of firmly nonexpansive mappings: Their relations and continuity properties. J. Nonlinear Convex Anal. **10**, 131–147 (2009)
5. Bauschke, H.H., Combettes, P.L. Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edn. Springer, New York (2017)
6. Bello Cruz, J.Y., Shehu, Y.: An iterative method for split inclusion problems without prior knowledge of operator norms. J. Fixed Point Theory Appl. **19**, 2017–2036 (2017)
7. Bregman, L.M.: The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. **7**, 200–217 (1967)
8. Byrne, C.: Iterative oblique projection onto convex subsets and the split feasibility problem. Inverse Probl. **18**, 441–453 (2002)
9. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal. **13**, 759–775 (2012)
10. Cegielski, A.: Application of quasi-nonexpansive operators to an iterative method for variational inequality. SIAM J. Optim. **25**, 2165–2181 (2015)
11. Cegielski, A., Reich, S., Zalas, R.: Regular sequences of quasi-nonexpansive operators and their applications. SIAM J. Optim. **28**, 1508–1532 (2018)
12. Cegielski, A., Reich, S., Zalas, R.: Weak, strong and linear convergence of the CQ-method via the regularity of Landweber operators. Optimization **69**, 605–636 (2020)

13. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in product space. Numer. Algor. **8**, 221–239 (1994)
14. Censor, Y., Lent, A.: An iterative row-action method for interval convex programming. J. Optim. Theory Appl. **34**, 321–353 (1981)
15. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity-modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
16. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**, 2071–2084 (2005)
17. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: The Split Feasibility Model Leading to a Unified Approach for Inversion Problems in Intensity-Modulated Radiation Therapy. Technical Report. Department of Mathematics, University of Haifa, Haifa (2005)
18. Cholamjiak, P., Suantai, S., Sunthrayuth, P.: An iterative method with residual vectors for solving the fixed point and the split inclusion problems in Banach spaces. Comput. Appl. Math. **38**, 12 (2019)
19. Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic, Dordrecht (1990)
20. Combettes, P.L.: The convex feasibility problem in image recovery. Adv. Imaging Electron. Phys. **95**, 155–270 (1996)
21. Dadashi, V.: Shrinking projection algorithms for the split common null point problem. Bull. Aust. Math. Soc. **96**, 299–306 (2017)
22. Eslamian, M.: Split common fixed point and common null point problem. Math. Methods Appl. Sci. **40**, 7410–7424 (2017)
23. Eslamian, M., Eskandani, G.Z., Raeisi, M.: Split common null point and common fixed point problems between Banach spaces and Hilbert spaces. Mediter. J. Math. **14**, 119 (2017)
24. Hendrickx, J.M., Olshevsky, A.: Matrix p-norms are NP-hard to approximate if $p \neq 1, 2, \infty$. SIAM J. Matrix Anal. Appl. **31**, 2802–2812 (2010)
25. Kuo, L.-W., Sahu, D.R.: Bregman distance and strong convergence of proximal-type algorithms. Abstr. Appl. Anal. **2013**, 590519 (2013)
26. López, G., Martín-Márquez, V., Wang, F., Xu, H.-K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Probl. **28**, 085004 (2012)
27. Maingé, P.-E.: Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **325**, 469–479 (2007)
28. Maingé, P.-E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899–912 (2008)
29. Raeisi, M., Zamani Eskandani, G., Eslamian, M.: A general algorithm for multiple-sets split feasibility problem involving resolvents and Bregman mappings. Optimization **67**, 309–327 (2018)
30. Reich, S.: A weak convergence theorem for the alternating method with Bregman distances. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 313–318. Marcel Dekker, New York (1996)
31. Reich, S., Sabach, S.: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. TMA **73**, 122–135 (2010)
32. Reich, S., Sabach, S.: A projection method for solving nonlinear problems in reflexive Banach spaces. J. Fixed Point Theory Appl. **9**, 101–116 (2011)
33. Reich, S., Tuyen, T.M.: A new algorithm for solving the split common null point problem in Hilbert spaces. Numer Algor. **83**, 789–805 (2020)
34. Reich, S., Tuyen, T.M.: Two projection methods for solving the multiple-set split common null point problem in Hilbert spaces. Optimization **69**, 1913–1934 (2020)
35. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209–216 (1970)
36. Sahu, D.R., O'Regan, D., Agarwal, R.P.: Fixed Point Theory for Lipschitzian-type Mappings with Applications. Topological Fixed Point Theory and Its Applications, vol. 6. Springer, New York (2009)
37. Schöpfer, F., Schuster, T., Louis, A.K.: An iterative regularization method for the solution of the split feasibility problem in Banach spaces. Inverse Probl. **24**, 055008 (2008)
38. Shehu, Y.: Strong convergence theorem for multiple sets split feasibility problems in Banach spaces. Numer. Funct. Anal. Optim. **37**, 1021–1036 (2016)
39. Stark, H.: Image Recovery: Theory and Application. Academic Press, Orlando, FL (1987)
40. Suantai, S., Shehu, Y., Cholamjiak, P.: Nonlinear iterative methods for solving the split common null point problem in Banach spaces. Optim. Methods Softw. **34**, 853–874 (2019)
41. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
42. Takahashi, W.: The split common null point problem for generalized resolvents in two Banach spaces. Numer. Algor. **75**, 1065–1078 (2017)

43. Tang, Y.: New inertial algorithm for solving split common null point problem in Banach spaces. J. Inequal. Appl. **2019**, 17 (2019)
44. Tang, Y.: Convergence analysis of a new iterative algorithm for solving split variational inclusion problems. J. Ind. Manag. Optim. **16**, 945–964 (2020)
45. Tang, Y., Gibali, A.: New self-adaptive step size algorithms for solving split variational inclusion problems and its applications. Numer. Algor. **83**, 305–331 (2020)
46. Tang, Y., Sunthrayuth, P.: An iterative algorithm with inertial technique for solving the split common null point problem in Banach spaces. Asian-Eur. J. Math. https://doi.org/10.1142/S1793557122501200 (2021)
47. Tuyen, T.M.: A strong convergence theorem for the split common null point problem in Banach spaces. Appl. Math. Optim. **79**, 207–227 (2019)
48. Tuyen, T.M., Sunthrayuth, P., Trang, N.M.: An inertial self-adaptive algorithm for the genealized split common null point problem in Hilbert spaces. Rend. Circ. Mat. Palermo, II. Ser. https://doi.org/10.1007/s12215-021-00640-8 (2021)
49. Tuyen, T.M., Thuy, N.T.T., Trang, N.M.: A strong convergence theorem for a parallel iterative method for solving the split common null point problem in Hilbert spaces. J. Optim. Theory Appl. **183**, 271–291 (2019)
50. Xu, H.-K.: Inequalities in Banach spaces with applications. Nonlinear Anal. TMA **16**, 1127–1138 (1991)
51. Yang, Q.: On variable-step relaxed projection algorithm for variational inequalities. J. Math. Anal. Appl. **302**, 166–179 (2005)



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Phenomena of thermo-sloutal time's relaxation in mixed convection Carreau fluid with heat sink/source

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ABSTRACT

Recently, with the fast growth of new engineering technologies various researchers have presented fantastic dynamism in sightseeing the heat propagation through a wave mechanism instead of essentially by dispersion. Further studies insist that this is not a little temperature phenomenon; however, relatively one which has probable vital physical uses. This topic caught exceptional thought because of its generous uses in industrial and current built-up processes. Thus, here advanced Cattaneo–Christov heat and mass fluxes have been reported for the study of thermo-sloutal time relations. Aspects of MHD, heat sink/soure, variable properties of mass diffusivity and thermal conductivity in mixed convection Carreau fluid have been elaborated. The bvp4c numerical process has been exploited for the solutions of influential factors graphically. The mixed convection factor improves the fluid velocity of Carreau fluid. Our outcomes report the diminishing performance for thermo-relaxation factor on temperature field and sloutal-relaxation factor in concentration field. The escalating performance of mass diffusivity factor has been detected on concentration field. Tables for comparison with former studies with good agreement have been also disclosed.

Abbreviations: α: ratio of stretching rates parameter ε_1: thermal conductivity factor ε_2: mass diffusivity factor HAM: homotopy analysis method M: magnetic factor ODEs: ordinary differential equations PDEs: partial differential equations Sc: Schmidt number

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Nomenclature

$(u,\ v,\ w)$	velocity components
Γ	material constant
$(\rho c)_f$	heat capacity of fluid

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T	temperature of fluid
β_T, β_C	thermal and concentration expansion factors
$K(T)$	variable thermal conductivity
k_1	thermal conductivity of surrounding
T_∞	ambient fluid temperature
T_w	wall temperature
Γ_T, Γ_C	thermal-sloutal relaxation time
C_{fx}, C_{fy}	skin friction coefficients
$(x,\ y,\ z)$	space coordinates
ν	kinematic viscosity
n	power law index
C	concentration of fluid
g	gravity
$D(C)$	variable mass diffusivity
k_2	mass diffusivity of surrounding
C_∞	ambient fluid concentration
C_w	wall concentration
Q_1	heat sink/source coefficient
τ_{xz}, τ_{yz}	surface shear stresses
We_1, We_2	local Weissenberg numbers
λ_m	mixed convection factor
Pr	Prandtl number
N_1	buoyancy ratio factor
δ_T, δ_C	thermal and concentration relaxation time factors
δ_H	heat sink/source factor

1. Introduction

Recently, the researchers are aiming an extreme quantity of prominence on mutual forms of heat and mass configuration as mutually occurrences endorse the uniqueness in engineering and industrial tools like polymer progressions, iceboxes, plastics extrusion and heat exchangers. The heat–mass flow phenomena via a space with temperature/concentration influence of variances are described by Fourier and Fick laws, respectively. The preparation aspect to Fourier's theory modeled the parabolic expression in contradiction of temperature, representing that heat transport has unbounded propagates and speeds through medium. To overcome this dilemma of heat transport, Fourier's law needs to be reformed. Numerous efforts have been completed to resolve this enigma, but nobody is sure if it is efficacious. Addition of the thermal relaxation time factor multiplied by the time derivative of heat flux Cattaneo upgraded Fourier's law which possesses a hyperbolic kind of expressions for thermo transport occurrences and as an outcome, heat transport during the entire mediums has a limited speed. The model of Cattaneo–Maxwell is the term specified for this innovative model. Christov transforms the Cattaneo–Maxwell model by introducing the upper-convective time derivative to the time derivative with the upper-convective time derivative that combines the advanced spatial gradients, so as to attain the frame invariant. Additionally, an instance of such uses includes skin injuries and nanofluids, refrigerating of electrical plans, nutrition machinery, fissile device conserving, heat propagation in nerves and various extra. It is a greatest likely to realize that the usual occurrence of heat exchange materializes, if the temperature divergence between substances or between diverse quantities of similar substances. Thus, numerous studies in this pattern are scrutinized in Refs [1–8]. Furthermore, Chu et al. [9] reported Cattaneo–Christov double diffusion theories considering the impact of nanofluid with variable properties. They analyzed that the inverse Darcy number decayed the velocity field, whereas enhanced for second-grade factor. Waqas et al. [10] explored the thought of thermo-sloutal time relaxations on Burgers fluid. Their study examined that the thermo-sloutal time relaxation factors retard temperature and concentration fields. Magneto Reiner–Philippoff fluid with the aspect of Cattaneo–Christov heat diffusion was studied numerically by Kumara et al. [11]. The higher Prandtl number decayed the density of the thermal layer in their study. Recently, Ramana et al. [12] scrutinized Oldroyd-B fluid by exploiting the properties of Cattaneo–Christov heat flux. They noted that the temperature field has reverse influences for relaxation–retardation and time factors.

Recently, the energetic uses of non-Newtonian liquids have noteworthy prominence in built-up and microelectronics. These liquids are frequently used in all-embracing prescribing, lubricant storing, atomic productiveness, nutrition mechanisms, etc. There are insufficient occurrences of non-Newtonian fluids in splatters, plasma, ketchups, polymer clarifications, etc. Differential, integral and rate type fluids are the three most important collections of non-Newtonian fluids. In studies, these liquids are termed as non-linear liquids, non-Newtonian liquids and rheological complex liquids. The apparent viscosity is not determined in non-Newtonian fluids and is a function of shear rate and shear stress. Furthermore, non-Newtonian models arise when the most important variants in the shear rate of fluid features occur. To satisfy the performance of non-Newtonian constituents numerous rheological models had been well-thought-out. Refs [13–20] reported the numerous investigations that deal with these models considering diverse aspects.

Magneto-hydrodynamics (MHD) is the mutual outcome of hydrodynamic and electromagnetic performance of a liquid, generally plasma. Thus, plasma is a structure where particles are autonomously ionized into negative and positive charges; however, mutually unbiased. When plasma is in the impact of an exterior magnetic field (for instance, in the astral atmosphere, and plasma is fixed inside the Sun's magnetic field) the performance of plasma is directed together by equations of hydrodynamic and Maxwell's electromagnetic. The study of such structures is via MHD. Cosmological blazes, E F coatings of the Globe's atmosphere and fissile synthesis are directed by MHD. Alfven exposes firstly the waves in such structures are Alfven waves or MHD waves. Various studies concerning MHD have been explored in Ref [21–28]. Irfan [29] studied the Carreau nanofluid and analyzed thermophoretic diffusion with MHD influence numerically. He attained that larger magnetic and variable conductivity factors have reverse impact on the temperature field. Adeosun and Ukaegbu [30] exploited the spectral collocation method and examined the performance of MHD and variable aspects of electric conductivity in squeezed fluid flow. Casson nanofluid considering curved stretched surfaces with MHD and chemical reactions were elaborated by Kumar et al. [31]. Outcomes expose that the intensifying values of the curvature factor progress the velocity gradient while; reverse trend is noticed in the thermo gradient.

The main subject of current work is to disclose the influence of thermo-sloutal relaxation times considering the variable aspects of thermal conductivity and mass diffusivity in mixed convection Carrreau fluid. The existing Carreau fluid model is proficient in describing the occurrences of shear thinning/thickening fluids. The novelty of Carreau fluid occurs in blood flow analysis (blood flow via tapering veins with stenosis). The blood flow via tapering veins with stenosis has absorbed the thought of various scientists. The flows via veins position grave healthiness dangers and are a notable purpose of humanity and illness in the technically progressive area. Fall of a vein or stenosis can result from significant plaque pledge, and perhaps will object a severe weakening in blood flow. Hence, the shear thinning/thickening aspects of numerous factors are graphically studied via bvp4c numerical algorithm.

2. Development of the physical model

Here scrutinize the features of thermo-sloutal time's relaxation in magneto Carreau mixed convection fluid flow due to bidirectional stretched surface have been studied. The fluid velocities in x and y directions are reflected to be $u = ax$ and along the vertical direction $v = by$; where a, $b > 0$ and existence of flow occurs in area $z > 0$. Additionally, the variable conductivity, heat sink/source and mass diffusivity have been elaborated. These median yield the resulting Carreau fluid equations (Figure 1):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\begin{aligned} &u\tfrac{\partial u}{\partial x} + v\tfrac{\partial u}{\partial y} + w\tfrac{\partial u}{\partial z} + \tfrac{\sigma B_0^2 u}{\rho_f} - \nu\tfrac{\partial^2 u}{\partial z^2}\left[1 + \Gamma^2\left(\tfrac{\partial u}{\partial z}\right)^2\right]^{\frac{n-1}{2}} \\ &\quad + \nu\Gamma^2(n-1)\left[1 + \Gamma^2\left(\tfrac{\partial u}{\partial z}\right)^2\right]^{\frac{n-3}{2}} \left(\tfrac{\partial u}{\partial z}\right)^2 \left(\tfrac{\partial^2 u}{\partial z^2}\right) + g[\beta_t(T - T_\infty) + \beta_c(C - C_\infty)] = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} &u\tfrac{\partial v}{\partial x} + v\tfrac{\partial v}{\partial y} + w\tfrac{\partial v}{\partial z} + \tfrac{\sigma B_0^2 v}{\rho_f} - \nu\tfrac{\partial^2 v}{\partial z^2}\left[1 + \Gamma^2\left(\tfrac{\partial v}{\partial z}\right)^2\right]^{\frac{n-1}{2}} \\ &\quad + \nu\Gamma^2(n-1)\left[1 + \Gamma^2\left(\tfrac{\partial v}{\partial z}\right)^2\right]^{\frac{n-3}{2}} \left(\tfrac{\partial v}{\partial z}\right)^2 \left(\tfrac{\partial^2 v}{\partial z^2}\right) = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} &u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} - \frac{1}{(\rho c)_f}\frac{\partial}{\partial z}\left(K(T)\frac{\partial T}{\partial z}\right) + \frac{\Gamma_T Q_1}{(\rho c)_f}\left(u\frac{\partial T}{\partial x} + v\tfrac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) \\ &\quad + \Gamma_T \begin{bmatrix} u^2\tfrac{\partial^2 T}{\partial x^2} + 2uv\tfrac{\partial^2 T}{\partial x \partial y} + u\tfrac{\partial u}{\partial x}\tfrac{\partial T}{\partial x} + u\tfrac{\partial v}{\partial x}\tfrac{\partial T}{\partial y} + u\tfrac{\partial w}{\partial x}\tfrac{\partial T}{\partial z} \\ + v^2\tfrac{\partial^2 T}{\partial y^2} + 2vw\tfrac{\partial^2 T}{\partial y \partial z} + v\tfrac{\partial u}{\partial y}\tfrac{\partial T}{\partial x} + v\tfrac{\partial v}{\partial y}\tfrac{\partial T}{\partial y} + v\tfrac{\partial w}{\partial y}\tfrac{\partial T}{\partial z} \\ + w^2\tfrac{\partial^2 T}{\partial z^2} + 2uw\tfrac{\partial^2 T}{\partial x \partial z} + w\tfrac{\partial u}{\partial z}\tfrac{\partial T}{\partial x} + w\tfrac{\partial v}{\partial z}\tfrac{\partial T}{\partial y} + w\tfrac{\partial w}{\partial z}\tfrac{\partial T}{\partial z} \end{bmatrix} + \frac{Q_1(T - T_\infty)}{(\rho c)_f} = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} &u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} - \frac{\partial}{\partial z}\left(D(C)\frac{\partial C}{\partial z}\right) \\ &\quad + \Gamma_C \begin{bmatrix} u^2\tfrac{\partial^2 C}{\partial x^2} + 2uv\tfrac{\partial^2 C}{\partial x \partial y} + u\tfrac{\partial u}{\partial x}\tfrac{\partial C}{\partial x} + u\tfrac{\partial v}{\partial x}\tfrac{\partial C}{\partial y} + u\tfrac{\partial w}{\partial x}\tfrac{\partial C}{\partial z} \\ + v^2\tfrac{\partial^2 C}{\partial y^2} + 2vw\tfrac{\partial^2 C}{\partial y \partial z} + v\tfrac{\partial u}{\partial y}\tfrac{\partial C}{\partial x} + v\tfrac{\partial v}{\partial y}\tfrac{\partial C}{\partial y} + v\tfrac{\partial w}{\partial y}\tfrac{\partial C}{\partial z} \\ + w^2\tfrac{\partial^2 C}{\partial z^2} + 2uw\tfrac{\partial^2 C}{\partial x \partial z} + w\tfrac{\partial u}{\partial z}\tfrac{\partial C}{\partial x} + w\tfrac{\partial v}{\partial z}\tfrac{\partial C}{\partial y} + w\tfrac{\partial w}{\partial z}\tfrac{\partial C}{\partial z} \end{bmatrix} = 0, \end{aligned} \tag{5}$$

$$U_w(x) = u = ax,\ \ V_w(y) = v = by,\ \ w = 0,\ \ T = T_w,\ C = C_w \text{ at } z = 0, \tag{6}$$

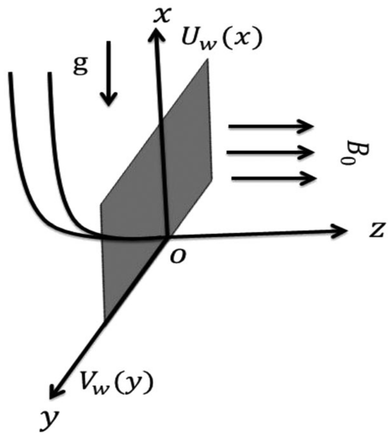

Figure 1. Systematic diagram.

$$u \to 0,\ v \to 0,\ w \to 0,\ T \to T_\infty,\ C \to C_\infty,\ \text{as}\ z \to \infty, \tag{7}$$

the variable feature of thermal conductivity $K(T)$ and mass diffusivity $D(C)$, respectively, explained as:

$$K(T) = k_1\left(1 + \varepsilon_1 \frac{T - T_\infty}{\Delta T}\right),\ D(C) = k_2\left(1 + \varepsilon_2 \frac{C - C_\infty}{\Delta C}\right), \tag{8}$$

2.1. Suitable conversions

Letting

$$u = axf'(\eta),\ v = ayg'(\eta),\ w = -\sqrt{a\nu}[f(\eta) + g(\eta)],\ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty},\ \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty},\ \eta = z\sqrt{\frac{a}{\nu}}. \tag{9}$$

Equations (8) and (9) yield the resulting ODEs:

$$f'''[1 + We_1^2 f''^2]^{\frac{n-3}{2}}[1 + nWe_1^2 f''^2] - f'^2 + f''(f+g) + M^2 f' + \lambda_m(\theta + N_1\phi) = 0, \tag{10}$$

$$g'''[1 + We_2^2 g''^2]^{\frac{n-3}{2}}[1 + nWe_2^2 g''^2] - g'^2 + g''(f+g) + M^2 g' = 0, \tag{11}$$

$$(1 + \varepsilon_1\theta)\theta'' + \varepsilon_1\theta'^2 + \Pr(f+g)\theta' - \Pr\delta_T[(f+g)(f' + g')\theta' + (f+g)^2\theta'' - \delta_H f\theta'] + \Pr\delta_H\theta = 0, \tag{12}$$

$$(1 + \varepsilon_2\phi)\phi'' + \varepsilon_2\phi'^2 + Sc(f+g)\phi' - Sc\delta_C[(f+g)(f' + g')\phi' + (f+g)^2\phi''] = 0, \tag{13}$$

$$f(0) = 0,\ g(0) = 0,\ f'(0) = 1,\ g'(0) = \alpha,\ \theta(0) = 1,\ \phi(0) = 1, \tag{14}$$

$$f' \to 0,\ g' \to 0,\ \theta \to 0,\ \phi \to 0,\ \text{as}\ \eta \to \infty. \tag{15}$$

Here, $(We_1, We_2) = \left(\sqrt{\frac{\Gamma^2 aU_w^2}{\nu}}, \sqrt{\frac{\Gamma^2 aV_w^2}{\nu}}\right)$ signify the local Weissenberg numbers, $\lambda_m \left(= \frac{g\beta_T(T_w - T_\infty)}{aU_w(x)}\right)$ the mixed convection factor, $N_1 \left(= \frac{\beta_C(C_w - C_\infty)}{\beta_T(T_w - T_\infty)}\right)$ the buoyancy ratio factor, $\alpha\left(=\frac{b}{a}\right)$ ratio of stretching rates factor, $M\left(=\frac{\sigma B_0^2}{\rho_f a}\right)$ magnetic factor, $\Pr\left(=\frac{\nu}{\alpha_1}\right)$ Prandtl number, $(\delta_T,\delta_C)=(a\Gamma_T,a\Gamma_C)$ thermal-sloutal relaxations time factors, $\delta_H\left(\frac{Q_1}{a(\rho c)_f}\right)$ and $Sc\left(=\frac{\nu}{D}\right)$ the Schmidt number.

3. Physical quantities

3.1. The coefficients of skin friction C_{fx} and C_{fy}

The quantities of this interest are

$$C_{fx}=\frac{\tau_{xz}}{\frac{1}{2}\rho U_w^2}\text{ and }C_{fy}=\frac{\tau_{yz}}{\frac{1}{2}\rho U_w^2}, \tag{16}$$

dimensionless form of the above equation.

$$\frac{1}{2}C_{fx}Re_x^{\frac{1}{2}}=f''(0)[1+We_1^2f''^2(0)]^{\frac{n-1}{2}},\ \frac{1}{2}\left(\frac{U_w}{V_w}\right)C_{fy}Re_x^{\frac{1}{2}}=g''(0)[1+We_2^2g''^2(0)]^{\frac{n-1}{2}}, \tag{17}$$

here, $Re_x=ax^2/\nu$ stands for Reynolds number.

4. Implementation of numerical approach (bvp4c)

Here the numerical process of ODEs is inferred via bvp4c approach. By the discretize process, we revise Equations (10)–(15) into first-order differential structures.

$$\begin{gathered} f=p_1,\ f'=p_2,\ f''=p_3,\ f'''=p_3',\ g=p_4,\ g'=p_5,\ g''=p_6,\ g'''=p_6',\\ \theta=p_7,\ \theta'=p_8,\ \theta''=p_8',\phi=p_9,\ \phi'=y_{10},\ \phi''=p_{10}', \end{gathered} \tag{18}$$

$$p_3'=\frac{-(p_1+p_4)p_3+p_2^2-M^2p_2-\lambda_m(p_7+N_1p_9)}{\Lambda_1};\ \Lambda_1=(1+nWe_1^2p_3^2)*(1+We_1^2p_3^2)^{\frac{n-3}{2}}, \tag{19}$$

$$p_6'=\frac{-(p_1+p_4)p_6+p_5^2-M^2p_5}{\Lambda_2};\ \Lambda_2=(1+nWe_2^2p_6^2)*(1+We_2^2p_6^2)^{\frac{n-3}{2}}, \tag{20}$$

$$p_8'=\frac{-\Pr(p_1+p_4)p_8-\varepsilon_1p_8^2+\Pr\delta_T[(p_1+p_4)(p_2+p_5)p_8-\delta_Hp_1p_8]-\Pr\delta_Hp_7}{\Lambda_3};\ \Lambda_3=(1+\varepsilon_1p_7)-\Pr\delta_T(p_1+p_4)^2, \tag{21}$$

$$p_{10}'=\frac{-Sc(p_1+p_4)p_{10}+Sc\delta_C[(p_1+p_4)(p_2+p_5)p_{10}]}{\Lambda_4};\ \Lambda_4=(1+\varepsilon_2p_9)-Sc\delta_C(p_1+p_4)^2, \tag{22}$$

$$\begin{aligned} p_1(0)&=0,\ p_2(0)=1,\ p_2(\infty)=0;\ p_4(0)=0,\ p_5(0)=\alpha,\\ p_5(\infty)&=0;\ p_7(0)=1,\ p_7(\infty)=0;\ p_9(0)=1,\ p_9(\infty)=0. \end{aligned} \tag{23}$$

4.1. Validation of bvp4c

For the confirmations of outcomes with former prose Table 1 for α and Table 2 for ε_1 and Pr are reported for $-\theta'(0)$. The higher ε_1 decrease; however, Pr enhances $-\theta'(0)$ and also compared with Ref [4]. These tables specify wonderful results related with former works.

Table 1. Outcomes of $-\theta'(0)$ for α when $We_1 = We_2 = \lambda_m = N_1 = \delta_T = \delta_C = \delta_H = \varepsilon_1 = \varepsilon_2 = Sc = 0$ and $n = 1$ are fixed.

	$-\theta'(0)$		
α	Ref. [32]	Ref. [33]	Present(bvp4c)
0.25	0.665933	0.665939	0.6659332
0.50	0.735334	0.735336	0.7353329
0.75	0.796472	0.796472	0.7964718

Table 2. Outcomes of $-\theta'(0)$ for ε_1 and Pr whe $We_1 = We_2 = \lambda_m = N_1 = \delta_T = \delta_C = \delta_H = \alpha = \varepsilon_2 = Sc = 0$ and $n = 1$ are fixed.

		$-\theta'(0)$	
ε_1	Pr	Ref. [34]	Present(bvp4c)
0.2	1.3	0.604568	0.60457302
0.3		0.569570	0.56957494
0.4		0.539040	0.53904539
0.2	1.5	0.664040	0.66404537
	1.7	0.719773	0.71978160
	2.0	0.797638	0.79765199

5. Study of outcomes

The thermo-sloutal time's relaxations considering the properties of heat sink/source in magneto mixed convection Carreau fluid have been elaborated. The variable aspects of thermal conductivity and mass diffusivity are also investigated. Here the fixed values excepting particular in figures for shear thinning $n = 0.6$ shear thickening $n = 1.6$ are $\lambda_m = 0.1, M = \delta_T = \delta_C = 0.3,\ \varepsilon_1 = \varepsilon_2 = 0.4,\ \alpha = N_1 = \delta_H = \varepsilon_1 = \varepsilon_2 = 0.2,\ \Pr = Sc = 1.3,\ We_1 = We_2 = 2.0$.

5.1. Velocity $f'(\eta)$ for λ_m

Figure 2 determines the features of velocity field for numerous values of mixed convection factor $\lambda_m(0.1, 0.3, 0.5, 0.7)$ for $n = 0.6$ and $n = 1.6$. The higher λ_m exaggerates $f'(\eta)$. Physically, buoyancy force has dominant possessions over viscid forces. Therefore, the mixed convection factor is liable to make the fluid flow quicker and hence increases $f'(\eta)$.

5.2. Temperature $\theta(\eta)$ for δ_T, ε_1 and δ_H

The properties of thermo-relaxation time factor and variable aspect of thermal conductivity ε_1 on temperature of Carreau fluid are portrayed in Figures 3(a, b) and 4(a, b). The Carrau fluid temperature diminishes for both instances, i.e. $n = 0.6$ and $n = 1.6$. The higher $\delta_T(0, 0.1, 0.2, 0.3)$ report the declining behavior because the instant wave propagation speed of medium decay and allows the transport of heat extra time. Hence $\theta(\eta)$ fall-off. Moreover, the temperature of Carreau fluid exaggerates for growing values of $\varepsilon_{1,}(0, 0.2, 0.4, 0.6)$. The thermal conductivity states the capacity of a specified material to

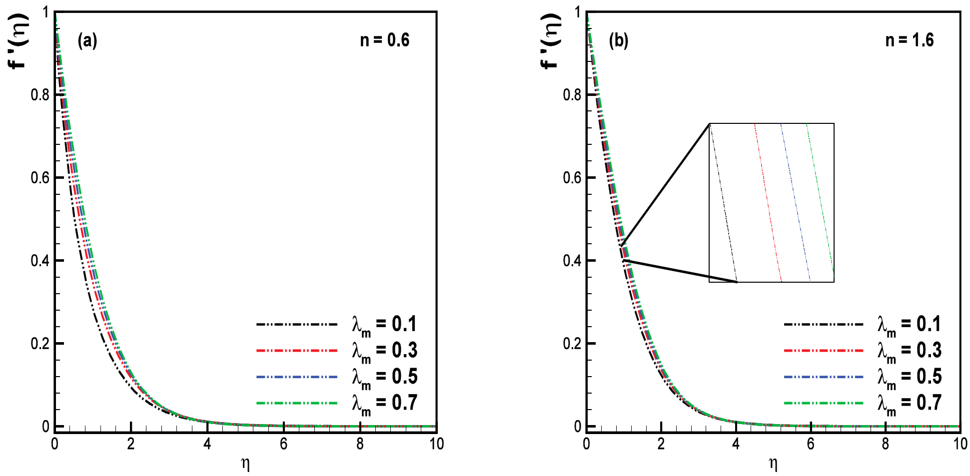

Figure 2. (a, b). Plot of η vs. $f'(\eta)$ for λ_m.

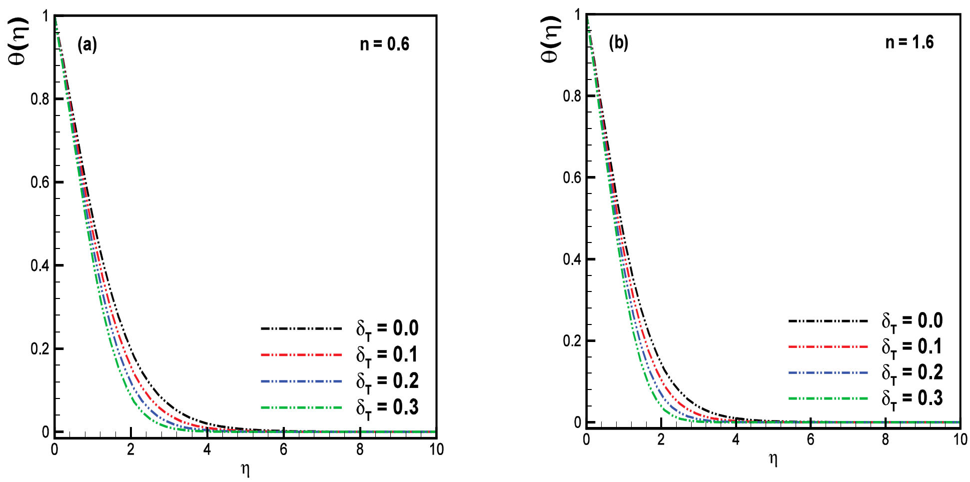

Figure 3. (a, b). Plot of η vs. $\theta(\eta)$ for δ_T.

transport heat. Considerably $\theta(\eta)$ growing for boosting values of ε_1 and massive heat transport quantity is transferred from the sheet to the solid which enhances $\theta(\eta)$. Figures 5(a, b) and 6(a, b) sightsee the characteristics of heat sink $\delta_H(-0.1, -0.3, -0.5, -0.7)$ and heat source $\delta_H(0.1, 0.3, 0.5, 0.7)$ on Carreau fluid temperature for $n = 0.6$ and $n = 1.6$. For higher values of $\delta_H < 0$ and $\delta_H > 0$, the temperature field have reverse behavior. The Carreau fluid temperature decays for $\delta_H < 0$ and exxegerates $\delta_H > 0$ for shear thinning/thickening cases. Physically, a heat sink is a reflexive heat exchanger that transports the heat produced by an electrical or a mechanical process into a coolant liquid in movement, whereas, heat source is a thing that creates or discharges heat. Furthermore, the heat is engrossed for greater $\delta_H < 0$ and the heat is transported to the liquid for $\delta_H > 0$. Hence the outcome of the $\theta(\eta)$ fall-off for $\delta_H < 0$ and intensifies for $\delta_H > 0$.

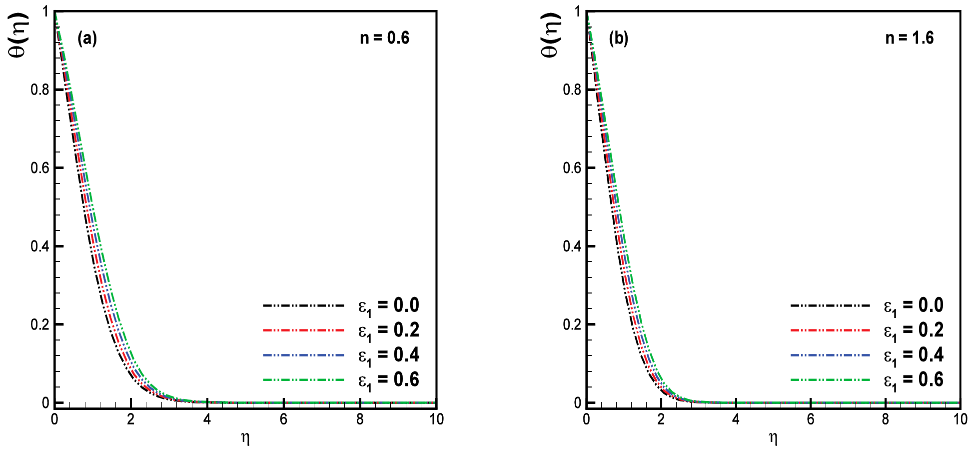

Figure 4. (a, b). Plot of η vs. $\theta(\eta)$ for ε_1.

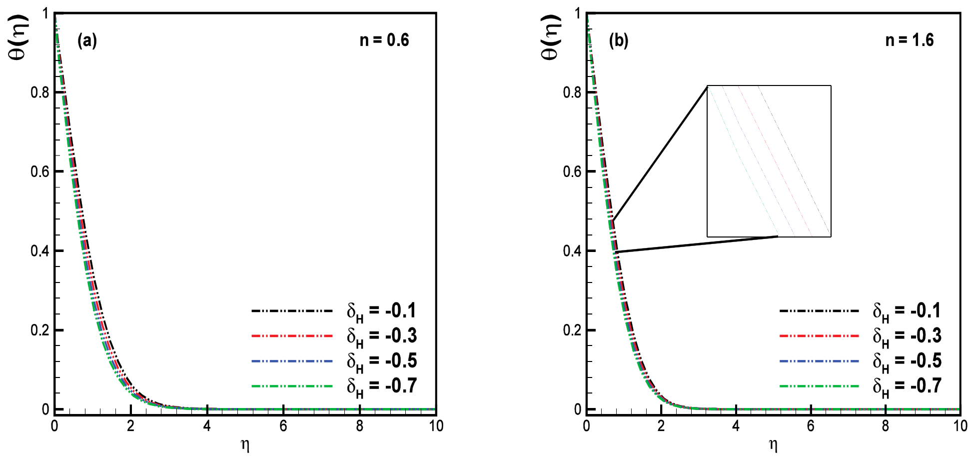

Figure 5. (a, b). Plot of η vs. $\theta(\eta)$ for $\delta_H < 0$.

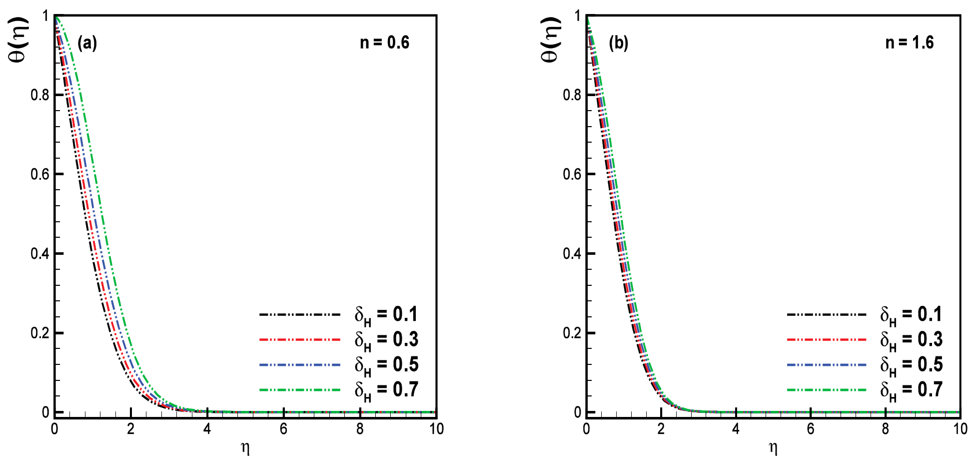

Figure 6. (a, b). Plot of η vs. $\theta(\eta)$ for $\delta_H > 0$.

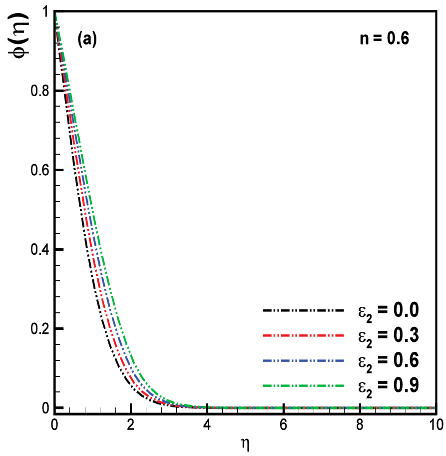

Figure 7. (a, b). Plot of η vs. $\phi(\eta)$ for ε_2.

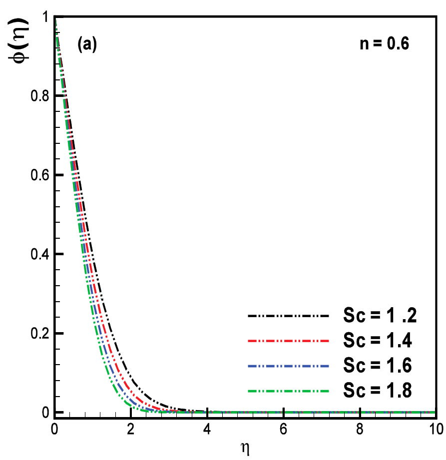

Figure 8. (a, b). Plot of η vs. $\phi(\eta)$ for Sc.

5.3. Concentration $\phi(\eta)$ for ε_2, Sc and δ_C

The depiction of concentration field for the value of ε_2 and Sc for aspects of shear thinning/thickening liquids are explored in Figures 7(a, b) and 8(a, b). The larger $\varepsilon_2(0, 0.3, 0.6, 0.9)$ exaggerates $\phi(\eta)$; however, higher $Sc(1.2, 1.4, 1.6, 1.8)$ diminishes $\phi(\eta)$. The transport of mass diffusivity increases for larger ε_2 which causes the additional mass to be transported. Hence, $\phi(\eta)$ is higher. Moreover, the enactment of larger Sc for $(n = 0.6)$ and $(n = 1.6)$ has been studied in Figure 8(a, b). Physically, Sc is the association between momentum and mass diffusivities. The mass diffusivity decreases when Sc raised. Consequently, the concentration field decays. Figure 9(a, b) explores the concentration scattering for $\delta_C(0.1, 0.2, 0.3, 0.4)$ and reported declining behavior. Physically, for higher sloutal relaxation time factor, the liquid components require huge time to diffuse which exhibit deteriorating performance of $\phi(\eta)$.

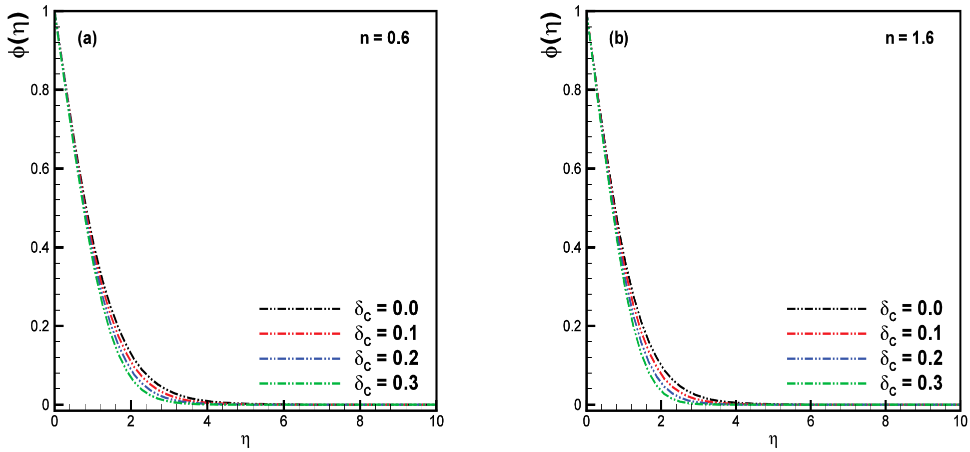

Figure 9. (a, b). Plot of η vs. $\phi(\eta)$ for δ_C.

6. Closing remarks

The properties of thermo-sloutal relaxation times and heat sink/source in mixed convection Carreau liquid considering variable aspect of conductivity and mass diffusivity have been explored.

The outstanding facts of this study are remarked as:

- The temperature field inflated for $\delta_T > 0$; however decreases for $\delta_T < 0$.
- The higher δ_T for $n = 0.6$ and $n = 1.6$ decayed $\theta(\eta)$.
- Similar effects were noted for δ_C and Sc on $\phi(\eta)$.
- Outstanding outcomes have been examined in limiting cases for $-\theta'(0)$.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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References

[1] Cattaneo C. Sulla conduzione del calore. Atti Semin Mat Fis Univ Modena Reggio Emilia. 1948;3:83–101.

[2] Christov CI. On frame indifferent formulation of the Maxwell–Cattaneo model of finite-speed heat conduction. Mech Res Commun. 2009;36:481–486.

[3] Sui J, Zheng L, Zhang X. Boundary layer heat and mass transfer with Cattaneo–Christov double-diffusion in upper-convected Maxwell nanofluid past a stretching sheet with slip velocity. Int J Ther Sci. 2016;104:461–468.

[4] Khan WA, Irfan M, Khan M. An improved heat conduction and mass diffusion models for rotating flow of an oldroyd-B fluid. Results Phy. 2017;7:3583–3589.

[5] Waqas M, Khan MI, Hayat T, et al. On Cattaneo–Christov double diffusion impact for temperature-dependent conductivity of Powell–Eyring liquid. Chinese J Phy. 2017;55:729–737.

[6] Irfan M, Khan M, Khan WA. On model for three-dimensional Carreau fluid flow with Cattaneo–Christov double diffusion and variable conductivity: a numerical approach. J Brazilian Soc Mech Sci Eng. 2018. doi:10.1007/s40430-018-1498-5.

[7] Ibrahim W, Gadisa G. Finite element solution of nonlinear convective flow of Oldroyd-B fluid with Cattaneo-Christov heat flux model over nonlinear stretching sheet with heat generation or absorption. Propuls Power Res. 2020;9:304–315.

[8] Sarojamma S, Lakshmi RV, Narayana PVS, et al. Exploration of the significance of autocatalytic chemical reaction and Cattaneo-Christov heat flux on the dynamics of a micropolar fluid. J Appl Comput Mechanics. 2020;6:77–89.

[9] Chu YM, Shah F, Khan MI, et al. Cattaneo-Christov double diffusions (CCDD) in entropy optimized magnetized second grade nanofluid with variable thermal conductivity and mass diffusivity. J Mater Res Tech. 2020;9:13977–13987.

[10] Waqas M, Hayat T, Shehzad SA, et al. Analysis of forced convective modified Burgers liquid flow considering Cattaneo-Christov double diffusion. Results Phy. 2018;8:908–913.

[11] Kumara KG, Reddy MG, Sudharani MVVNL, et al. Cattaneo-Christov heat diffusion phenomenon in Reiner-Philippoff fluid through a transverse magnetic field. Phys A Stat Mech Appl. 2020;541:123330.

[12] Ramana KV, Gangadhar K, Kannan T, et al. Cattaneo-Christov heat flux theory on transverse MHD Oldroyd-B liquid over nonlinear stretched flow. J Ther Anal Calorimetry. 2022;147:2749–2759.

[13] Khedr MEE, Chamkha AJ, Bayomi M. MHD flow of a micropolar fluid past a stretched permeable surface with heat generation or absorption. Nonlinear Anal Model Control. 2009. doi:10.15388/NA.2009.14.1.14528.

[14] Khan M, Irfan M, Khan WA, et al. A new modeling for 3D Carreau fluid flow considering nonlinear thermal radiation. Results Phys. 2017;7:2692–2704.

[15] Rasool G, Zhang T, Chamkha AJ, et al. Entropy generation and consequences of binary chemical reaction on MHD Darcy–Forchheimer Williamson nanofluid flow over non-linearly stretching surface. Entropy . 2020;22, doi:10.3390/e22010018.

[16] Chamkha AJ, Dogonchi AS, Ganji DD. Magneto-hydrodynamic flow and heat transfer of a hybrid nanofluid in a rotating system among two surfaces in the presence of thermal radiation and Joule heating. AIP Adv. 2019;9, doi:10.1063/1.5086247.

[17] Tarakaramu N, Narayana PVS. Influence of heat generation/absorption on 3D magnetohydrodynamic Casson fluid flow over a porous stretching surface. Adv Fluid Dyn. 2020: 381–392.

[18] Wakif A, Chamkha AJ, Animasaun IL, et al. Novel physical insights into the thermodynamic irreversibilities within dissipative EMHD fluid flows past over a moving horizontal Riga plate in the coexistence of wall suction and Joule heating effects: a comprehensive numerical investigation. Arabian J Sci Eng. 2020;45:9423–9438.

[19] Seyedi SH, Saray BN, Chamkha AJ. Heat and mass transfer investigation of MHD Eyring-Powell flow in a stretching channel with chemical reactions. Phys A Stat Mech Appl. 2020;544:124109.

[20] Madhukesh JK, Ramesh GK, Alsulami MD, et al. Characteristic of thermophoretic effect and convective thermal conditions on flow of hybrid nanofluid over a moving thin needle. Waves Random Comp Media. 2021. doi:10.1080/17455030.2021.2012303.

[21] Chamkha AJ. MHD-free convection from a vertical plate embedded in a thermally stratified porous medium with Hall effects. Appl Math Model. 1997;21:603–609.

[22] Waqas M, Farooq M, Khan MI, et al. Magnetohydrodynamic (MHD) mixed convection flow of micropolar liquid due to nonlinear stretched sheet with convective condition. Int J Heat Mass Transf. 2016;102:766–772.

[23] Reddy PS, Chamkha AJ. Soret and Dufour effects on MHD convective flow of Al2O3–water and TiO2–water nanofluids past a stretching sheet in porous media with heat generation/absorption. Adv Powder Tech. 2016;27:1207–1218.

[24] Irfan M, Khan M, Khan WA, et al. Magnetohydrodynamic stagnation point flow of a Maxwell nanofluid with variable conductivity. Commun Theor Phys. 2019;71:1493–1500.

[25] Lalmi AA, Afrouzi HH, Moshfegh A. Investigation of MHD effect on nanofluid heat transfer in microchannels. J Ther Anal Calorimetry. 2019;136:1959–1975.

[26] Tahir H, Khan U, Din A, et al. Heat transfer in a ferromagnetic chemically reactive species. J Thermophys Heat Transf. 2021;35:402–410.

[27] Tahir H, Khan U, Din A, et al. Hybridized two phase ferromagnetic nanofluid with NiZnFe2O4 and MnZnFe2O4. Ain Shams Eng J. 2021;12:3063–3070.

[28] Ali L, Wang Y, Ali B, et al. The function of nanoparticle's diameter and Darcy-Forchheimer flow over a cylinder with effect of magnetic field and thermal radiation. Case Studies Ther Eng. 2021;28:101392.

[29] Irfan M. Study of Brownian motion and thermophoretic diffusion on non-linear mixed convection flow of Carreau nanofluid subject to variables properties. Surfaces Interfaces. 2021. doi:10.1016/j.surfin.2021.100926.

[30] Adeosun AT, Ukaegbu JC. Effect of the variable electrical conductivity on the thermal stability of the MHD reactive squeezed fluid flow through a channel by a spectral collocation approach. Partial Diff Eq Appl Math. 2022;5:100256.

[31] Kumar RSV, Sarris IE, Sowmya G, et al. Modeling and theoretical investigation on Casson nanofluid flow over a curved stretching surface with the influence of magnetic field and chemical reaction. Heat Transf. 2022. doi:10.1002/htj.22450.

[32] Liu IC, Anderson HI. Heat transfer over a bidirectional stretching sheet with variable thermal conditions. Int J Heat Mass Transf. 2008;51:4018–4024.

[33] Munir A, Shahzad A, Khan M. Convective flow of Sisko fluid over a bidirectional stretching surface. PLOS ONE. 2015;10:e0130342.

[34] Khan WA, Irfan M, Khan M. An improved heat conduction and mass diffusion models for rotating flow of an Oldroyd-B fluid, Results Phy. 2017;7:3583–3589.

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The Modification of Generalized Mixed Equilibrium Problems for Convergence Theorem of Variational Inequality Problems and Fixed Point Problems

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Abstract The purpose of this research, we modify generalized mixed equilibrium problems and prove a strong convergence theorem for approximating a common element of the set of such a problem and variational inequality problems and the set of fixed points of infinite family of strictly pseudo contractive mappings. Utilizing our main result, we also prove a strong convergence theorem involving generalized equilibrium problems and variational inequality problems.

MSC: 47H09; 47H10; 90C33

Keywords: strictly pseudo contractive mapping; generalized mixed equilibrium problem; inverse-strongly monotone

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1. Introduction

Throughout this article, we assume that H is a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $Fix(T) := \{x \in C : Tx = x\}$.

Definition 1.1. Let $T : C \to C$ be a nonlinear mapping, then

(1) T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C,$$

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(2) T is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \forall x \in C \text{ and } \forall p \in Fix(T),$$

(3) T is said to be *κ-strictly pseudo-contractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in C.$$

Note that the class of strictly pseudo-contractive mappings includes the class of nonexpansive mappings. The mapping T is nonexpansive if and only if T is 0-strictly pseudo contractive.

A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

A mapping A is said to be *ρ-strongly monotone* if there exists a positive real number ρ such that

$$\langle Ax - Ay, x - y\rangle \geq \rho \|x - y\|^2,$$

for all $x, y \in C$.

The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, A)$. The application of the variational inequality problem has been expanded to problems from economics, finance, optimization and game theory. Many authors have studied the variational inequality problem, see for instance [1] and [2].

Let $F : C \times C \to \mathbb{R}$ be a bifunction, $A : C \to H$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *generalized mixed equilibrium problem* (see [3]), is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \tag{1.2}$$

for all $y \in C$. The set of solution of (1.2) is denoted by

$$GMEP(F, \varphi, A) = \{x \in C : F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

Generalized mixed equilibrium problem has been studied by many authors, see for example [4], [5], [6] and [7]. If $\varphi = 0$, then (1.2) reduces to *the generalized equilibrium problem*, that is,

$$EP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}. \tag{1.3}$$

If $A = 0$, then problem (1.3) reduces to *the equilibrium problem*, that is,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \tag{1.4}$$

Optimization problem, saddle point problem, variational inequality problem and Nash equilibrium problem can be applied with the equilibrium problem. Many authors have introduced iterative algorithms in order to solve the equilibrium problem, see for instance [8], [9] and [10].

In 2005, Combettes and Hirstoaga [10] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. By using the viscosity approximation method, Takahashi and Takahashi [8] introduced an iteration for finding a common element of the set $EP(A)$ and $Fix(T)$ and proved a strong convergence theorem in a Hilbert space. In 2008, Takahashi and Takahashi [11] introduced another iterative scheme for finding the common element of the set $EP(F,A)$ and $Fix(T)$.

Recently, Kangtunyakarn [12] modified the set of solutions of generalized equilibrium problem as follows:

$$\begin{aligned} &EP(F,aA+(1-a)B)\\ &\quad=\{x\in C: F(x,y)+\langle (aA+(1-a)B)x,y-x\rangle\geq 0,\forall y\in C, a\in(0,1)\}. \end{aligned}\tag{1.5}$$

He introduced an iterative scheme for finding a common element of the set of fixed points of κ-strictly pseudo-contractive mapping and the set of solution of (1.5) as follows:

$$\begin{aligned} &F(u_n,y)+\langle (aA+(1-a)B)x_n,y-x_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C,\\ &x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n P_C(I-\gamma(I-T))u_n,\forall n\geq 1, \end{aligned}\tag{1.6}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions.

Let $D_1, D_2 : C\to H$ be two nonlinear mappings. Motivated by (1.2) and (1.5), we modify the set of solution of generalized mixed equilibrium problem as follows:

$$\begin{aligned} GMEP(F,\varphi,aD_1+(1-a)D_2)=\{x\in C: F(x,y)+\varphi(y)-\varphi(x)\\ +\langle (aD_1+(1-a)D_2)x,y-x\rangle\geq 0\}, \end{aligned}\tag{1.7}$$

for all $y\in C$ and $a\in(0,1)$. If $D_1=D2$, then $GMEP(F,\varphi,aD_1+(1-a)D_2)$ is reduced to (1.2).

In this research, we modify generalized mixed equilibrium problems and prove the strong convergence theorem for approximating a common element of the set of such a problem and variational inequality problem and the set of fixed points of infinite family of a strictly pseudo contractive mappings. Based on main result, we prove a strong convergence theorem involving generalized equilibrium problems and variational inequality problems.

2. Preliminaries

In this paper, we denote weak and strong convergence by the notations " $\rightharpoonup$ " and " $\to$ ", respectively. In a real Hilbert space H, recall that the (nearest point) projection P_C from H onto C assigns to each $x\in H$ the unique point P_Cx satisfying the property

$$\|x-P_Cx\|=\min_{y\in C}\|x-y\|.$$

For a proof of the main theorem, we will use the following lemmas.

Lemma 2.1. [13] *Given $x\in H$ and $y\in C$, then $P_Cx=y$ if and only if we have the inequality*

$$\langle x-y,y-z\rangle\geq 0,\forall z\in C.$$

Lemma 2.2. [14] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1): $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2): $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$..

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *Let H be a real Hilbert space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.4. [13] *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Definition 2.5. [15] Let C be a nonempty convex subset of a real Hilbert space. Let $T_i, i = 1, 2, ...$ be mappings of C into itself. For each $j = 1, 2, ...,$ let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$. Define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,n+1} &= I \\
U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I \\
U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I \\
&\vdots \\
U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I \\
U_{n,k} &= \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I \\
&\vdots \\
U_{n,2} &= \alpha_1^2 T_2 U_{n,3} + \alpha_2^2 U_{n,3} + \alpha_3^2 I \\
S_n &= U_{n,1} = \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I.
\end{aligned}$$

Such mapping is called *S-mapping* generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$.

Lemma 2.6. [16] *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contractive mapping, then T satisfies the Lipschitz condition;*

$$\|Tx - Ty\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \forall x, y \in C.$$

For finding solutions of the equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a lower semicontinuous and convex function satisfies the following conditions:
$(A1)$ $F(x,x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F\left(tz + (1-t)x, y\right) \leq F(x,y);$$

$(A4)$ for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous;
$(B1)$ for each $x \in H$ and $r > 0$ there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

$(B2)$ C is a bounded set.

Lemma 2.7. [17] *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfies $(A1)-(A4)$, $A : C \to H$ be a continuous monotone mapping, and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$ then there exists $z \in C$ such that*

$$F(z,y) + \langle Ay, y - z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle.$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{z \in C : F(z,y) + \langle Ay, y - z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\} \tag{2.1}$$

for all $x \in H$. Then the following conclusions hold:

(1) For each $x \in H, T_r \neq \emptyset$;
(2) T_r is single-valued;
(3) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) $Fix(T_r) = GMEP(F, \varphi, A)$
(5) $GMEP(F, \varphi, A)$is closed and convex.

Lemma 2.8. [15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\kappa = sup_{i \in \mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Then, for every $x \in C$ and $k \in \mathbb{N}, \lim_{n\to\infty} U_{n,k}x$ exists.*

For every $k \in \mathbb{N}$ and $x \in C$. [15] defined mapping $U_{\infty,k}$ and $S : C \to C$ as follows:

$$\lim_{n\to\infty} U_{n,k}x = U_{\infty,k}x \tag{2.2}$$

and

$$\lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x = Sx. \tag{2.3}$$

Such a mapping S is called *S-mapping* generated by $T_n, T_{n-1}, ...$ and $\alpha_n, \alpha_{n-1}, ...$.

Remark 2.9. [15] For every $n \in \mathbb{N}$, S_n is nonexpansive and $\lim_{n\to\infty} sup_{x\in D}\|S_n x - Sx\| = 0$, for every bounded subset D of C.

Lemma 2.10. [15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I\times I\times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, \ldots$ and $\alpha_n, \alpha_{n-1}, \ldots$, respectively. Then $Fix(S) = \bigcap_{i=1}^{\infty} Fix(T_i)$.*

Lemma 2.11. [18] *Let C be a nonempty closed convex subset of a real Hilbert space H.Let A, B be α, β-inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $VI(C,A)\bigcap VI(C,B) \neq \emptyset$. Then*

$$VI(C, aA + (1-a)B) = VI(C,A)\bigcap VI(C,B), \forall a \in (0,1). \tag{2.4}$$

Furthermore if $0 \leq \gamma \leq min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is nonexpansive mapping.

Remark 2.12. From Lemma (2.4) and Lemma (2.11), we have

$$VI(C, aA+(1-a)B) = VI(C,A)\bigcap VI(C,B) = Fix(P_C(I-\gamma(aA+(1-a)B))),$$

for all $a \in (0,1)$ and $\gamma > 0$.

From (1.7), we have the following result.

Lemma 2.13. *Let C be a nonempty closed convex subset of a real Hilbert space H and F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfy $A1) - A4)$ and $F(x,z) \leq F(x,y) + F(y,z)$ for all $x, y, z \in C$. Let A, B be α, β-inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $GMEP(F,\varphi,A)\bigcap GMEP(F,\varphi,B) \neq \emptyset$. Then*

$$GMEP(F,\varphi, aA + (1-a)B) = GMEP(F,\varphi,A)\bigcap GMEP(F,\varphi,B), \forall a \in (0,1).$$

Proof. It is obvious that $GMEP(F,\varphi,A)\bigcap GMEP(F,\varphi,B) \subseteq GMEP(F,\varphi,aA + (1-a)B)$. Next, we will show that $GMEP(F,\varphi,aA+(1-a)B) \subseteq GMEP(F,\varphi,A) \bigcap GMEP(F,\varphi,B)$. Let $x_0 \in GMEP(F,\varphi,aA+(1-a)B)$ and $x^* \in GMEP(F,\varphi,A) \bigcap GMEP(F,\varphi,B)$, we have

$$F(x_0,y) + \varphi(y) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, y - x_0\rangle \geq 0, \forall y \in C, \tag{2.5}$$

$$F(x^*,y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C \tag{2.6}$$

and

$$F(x^*,y) + \varphi(y) - \varphi(x^*) + \langle Bx^*, y - x^*\rangle \geq 0, \forall y \in C. \tag{2.7}$$

For every $a \in (0,1)$, we have

$$aF(x^*,y) + a\varphi(y) - a\varphi(x^*)\langle aAx^*, y - x^*\rangle \geq 0, \forall y \in C$$

and

$$(1-a)F(x^*,y)+(1-a)\varphi(y)-(1-a)\varphi(x^*)+\langle(1-a)Bx^*, y-x^*\rangle \geq 0, \forall y \in C.$$

By the monotonicity of B and $x^*, x_0 \in C$, we have

$$\begin{aligned} aF(x_0, x^*) + a\varphi(x^*) - a\varphi(x_0) + \langle aAx_0, x^* - x_0\rangle \\ &= aF(x_0, x^*) + a\varphi(x^*) - a\varphi(x_0) + (1-a)\varphi(x^*) - (1-a)\varphi(x^*) \\ &\quad + (1-a)\varphi(x_0) - (1-a)\varphi(x_0) + (1-a)F(x_0, x^*) - (1-a)F(x_0, x^*) \\ &\quad + \langle aAx_0 + (1-a)Bx_0 - (1-a)Bx_0, x^* - x_0\rangle \\ &= F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0\rangle \\ &\quad -(1-a)F(x_0, x^*) - (1-a)\varphi(x^*) + (1-a)\varphi(x_0) - \langle (1-a)Bx_0, x^* - x_0\rangle \\ &\geq (1-a)F(x^*, x_0) + (1-a)\varphi(x_0) - (1-a)\varphi(x^*) + (1-a)\langle Bx_0, x_0 - x^*\rangle \\ &= (1-a)(F(x^*, x_0) + \varphi(x_0) - \varphi(x^*) + \langle Bx^*, x_0 - x^*\rangle + \langle Bx_0 - Bx^*, x_0 - x^*\rangle) \\ &\geq 0. \end{aligned} \tag{2.8}$$

Since $GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B) \subseteq GMEP(F, \varphi, aA + (1-a)B)$ and $x^* \in GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$, we have

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle aAx^* + (1-a)Bx^*, y - x^*\rangle \geq 0. \forall y \in C. \tag{2.9}$$

Since $x^* \in C$ and (2.5), we have

$$F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0\rangle \geq 0. \tag{2.10}$$

From (2.9) and $x_0 \in C$, we have

$$F(x^*, x_0) + \varphi(x_0) - \varphi(x^*) + \langle aAx^* + (1-a)Bx^*, x_0 - x^*\rangle \geq 0. \tag{2.11}$$

Summing up (2.10), (2.11) and (A2), we have

$$\langle a(Ax^* - Ax_0) + (1-a)(Bx^* - x_0), x_0 - x^*\rangle \geq 0. \tag{2.12}$$

Since A, B are α, β-inverse strongly monotone, respectively, and (2.12), we have

$$\begin{aligned} 0 &\leq \langle a(Ax^* - Ax_0) + (1-a)(Bx^* - Bx_0), x_0 - x^*\rangle \\ &= \langle a(Ax^* - Ax_0), x_0 - x^*\rangle + \langle (1-a)(Bx^* - Bx_0), x_0 - x^*\rangle \\ &= a\langle Ax^* - Ax_0, x_0 - x^*\rangle + (1-a)\langle Bx^* - Bx_0, x_0 - x^*\rangle \\ &\leq -a\alpha\|Ax^* - Ax_0\|^2 - (1-a)\beta\|Bx^* - Bx_0\|^2. \end{aligned}$$

This implies that

$$0 \leq -a\alpha\|Ax^* - Ax_0\|^2.$$

It follows that

$$Ax^* = Ax_0. \tag{2.13}$$

By using the same method as (2.13), we obtain

$$Bx^* = Bx_0. \tag{2.14}$$

For every $y \in C$. From (2.6), (2.8), (2.13) and $x^* \in GMEP(F, \varphi, A)$, we have

$$\begin{aligned}
F(x_0, y) &+ \varphi(y) - \varphi(x_0) + \langle Ax_0, y - x_0 \rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Ax_0, y - x^* + x^* - x_0 \rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \varphi(x^*) - \varphi(x^*) + F(x^*, y) - F(x^*, y) \\
&\quad + \langle Ax_0, y - x^* \rangle + \langle Ax_0, x^* - x_0 \rangle \\
&= F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + F(x^*, y) + \varphi(y) - \varphi(x^*) \\
&\quad + \langle Ax^*, y - x^* \rangle + \langle Ax_0, x^* - x_0 \rangle \\
&\geq F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0 \rangle \\
&\geq F(x_0, y) + F(y, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0 \rangle \\
&\geq F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0 \rangle \\
&\geq 0.
\end{aligned}$$

Then

$$x_0 \in GMEP(F, \varphi, A). \tag{2.15}$$

Since $x^*, x_0 \in C$ and (2.5), (2.13), we have

$$\begin{aligned}
(1-a)F(x_0, x^*) &+ (1-a)\varphi(x^*) - (1-a)\varphi(x_0) + \langle (1-a)Bx_0, x^* - x_0 \rangle \\
&= (1-a)F(x_0, x^*) + (1-a)\varphi(x^*) - (1-a)\varphi(x_0) + aF(x_0, x^*) \\
&\quad - aF(x_0, x^*) + \langle (1-a)Bx_0 + aAx_0 - aAx_0, x^* - x_0 \rangle \\
&= F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0 \rangle \\
&\quad - aF(x_0, x^*) + a\varphi(x_0) - a\varphi(x^*) - \langle aAx_0, x^* - x_0 \rangle \\
&\geq aF(x^*, x_0) + a\varphi(x_0) - a\varphi(x^*) + \langle aAx_0, x_0 - x^* \rangle \\
&= aF(x^*, x_0) + a\varphi(x_0) - a\varphi(x^*) + a\langle Ax^*, x_0 - x^* \rangle \\
&\geq 0.
\end{aligned} \tag{2.16}$$

For every $y \in C$, from (2.7), (2.14), (2.16) and $x^* \in GMEP(F, \varphi, B)$, we have

$$\begin{aligned}
F(x_0, y) + \varphi(y) &- \varphi(x_0) + \langle Bx_0, y - x_0 \rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Bx_0, y - x^* \rangle + \langle Bx_0, x^* - x_0 \rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \varphi(x^*) - \varphi(x^*) + F(x^*, y) - F(x^*, y) \\
&\quad + \langle Bx_0, y - x^* \rangle + \langle Bx_0, x^* - x_0 \rangle \\
&= F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + F(x^*, y) + \varphi(y) - \varphi(x^*) \\
&\quad + \langle Bx^*, y - x^* \rangle + \langle Bx_0, x^* - x_0 \rangle \\
&\geq F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0 \rangle \\
&\geq F(x_0, y) + F(y, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0 \rangle \\
&\geq F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0 \rangle \\
&\geq 0.
\end{aligned}$$

Hence

$$x_0 \in GMEP(F, \varphi, B). \tag{2.17}$$

By (2.15) and (2.17), we have $x_0 \in GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$. Then

$$GMEP(F, \varphi, aA + (1-a)B) \subseteq GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$$

■

3. Main Result

In this section, we prove a strong convergence theorem and for the set of fixed point of strictly pseudo contractive mappings and the sets of solution of generalized mixed equilibrium problems and variational inequality problems by using Lemma 2.13.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1,F_2 be bifunctions from $C \times C$ to $\mathbb{R}$ satisfy A1)-A4) and $F_i(x,z) \leq F_i(x,y) + F_i(y,z)$ for all $x, y, z \in C$ and $i = 1, 2$. Let $\varphi_1, \varphi_2 : C \to \mathbb{R}\bigcup\{+\infty\}$ be proper lower semicontinuous and convex function. Let A, B be α, β-inverse strongly monotone, respectively, and let D, E be L_D, L_E-Lipschitz continuous and μ, ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap GMEP(F_1, \varphi_1, A) \bigcap GMEP(F_1, \varphi_1, B) \bigcap GMEP(F_2, \varphi_2, A)$ $\bigcap GMEP(F_2, \varphi_2, B) \bigcap VI(C, D) \bigcap VI(C, E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ $\in I \times I \times I$ where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Assume the either B_1) or B_2) holds. Let the sequence $\{x_n\}$ generated by $x_1, u \in C$ and*

$$\begin{cases} F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \langle a_n A x_n + (1-a_n) B x_n, y - u_n\rangle \\ +\dfrac{1}{r_n^1}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ F_2(v_n, y) + \varphi_2(y) - \varphi_2(v_n) + \langle a_n A x_n + (1-a_n) B x_n, y - v_n\rangle \\ +\dfrac{1}{r_n^2}\langle y - v_n, v_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \delta_n u_n + (1-\delta_n) v_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \lambda_n S_n x_n + \eta_n P_C(I - \gamma_n(a_n D + (1-a_n)E)) y_n, \end{cases} \tag{3.1}$$

for all $n \geq 1$, where the sequences $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{\eta_n\}, \{\delta_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \lambda_n + \eta_n = 1$ for all $n \in \mathbb{N}$, $\{a_n\} \subset (0, 1)$ and $\{r_n^j\} \subseteq [b, c] \subset (0, 2min\{\alpha, \beta\})$ for all $j = 1, 2$. Suppose the following conditions hold:

(i): $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\{\beta_n\} \subseteq [d, e] \subset (0, 1)$;

(ii): $0 < \gamma_n \leq min\{\frac{2\mu}{L_D^2}, \frac{2\rho}{L_E^2}\}$;

(iii): $\lim_{n\to\infty} \delta_n = \delta \in (0, 1)$, $\sum_{n=1}^{\infty} \alpha_1^n < \infty$;

$$\textbf{(iv):} \sum_{n=1}^{\infty}\left|r_{n+1}^{j}-r_{n}^{j}\right|<\infty \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$$
$$\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty, \sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty,$$
$$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty, \sum_{n=1}^{\infty}|\eta_{n+1}-\eta_n|<\infty \textit{ for all } j=1,2.$$

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$.

Proof. First, we show that D is $\frac{\mu}{L_D^2}$-inverse strongly monotone mapping. Let $x, y \in C$, we have

$$\begin{aligned}\langle x-y, Dx-Dy\rangle &\geq \mu\|x-y\|^2\\ &\geq \frac{\mu}{L_D^2}\|Dx-Dy\|^2.\end{aligned}$$

Similarly, we get E is $\frac{\rho}{L_E^2}$-inverse-strongly monotone mapping.

Next, we show that $I-\gamma_n D$ and $I-\gamma_n E$ are nonexpansive mappings. For every $x, y \in C$, we have

$$\begin{aligned}\|(I-\gamma_n D)x-(I-\gamma_n D)y\|^2 &= \|x-y\|^2+\gamma_n^2\|Dx-Dy\|^2-2\gamma_n\langle x-y, Dx-Dy\rangle\\ &\leq \|x-y\|^2+\gamma_n^2\|Dx-Dy\|^2-\frac{2\gamma_n\mu}{L_D^2}\|Dx-Dy\|^2\\ &= \|x-y\|^2+\gamma_n\left(\gamma_n-\frac{2\mu}{L_D^2}\right)\|Dx-Dy\|^2\\ &\leq \|x-y\|^2.\end{aligned}$$

Then we obtain $I-\gamma_n D$ is a nonexpansive mapping. Similarly, we can show that $I-\gamma_n E$ is also a nonexpansive mapping.

The proof of Theorem 3.1 will be divided into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded.
From (3.1) and Lemma 2.7, we have $u_n = T_{r_n^1}(I-r_n^1(a_nA+(1-a_n)B))x_n$ and $v_n = T_{r_n^2}(I-r_n^2(a_nA+(1-a_n)B))x_n$.
From Lemma 2.7 and Lemma 2.13, we have

$$\begin{aligned}F\left(T_{r_n^1}\left(I-r_n^1\left(a_nA+(1-a_n)B\right)\right)\right) &= GMEP(F_1,\varphi_1,a_nA+(1-a_n)B)\\ &= GMEP(F_1,\varphi_1,A)\bigcap GMEP(F_1,\varphi_1,B)\end{aligned}$$

and

$$\begin{aligned}F\left(T_{r_n^2}\left(I-r_n^2\left(a_nA+(1-a_n)B\right)\right)\right) &= GMEP(F_2,\varphi_2,a_nA+(1-a_n)B)\\ &= GMEP(F_2,\varphi_2,A)\bigcap GMEP(F_2,\varphi_2,B).\end{aligned}$$

Let $z \in \mathcal{F}$. From Lemma 2.4 and Lemma 2.11, we have

$$z \in VI(C, a_nD+(1-a_n)E) = Fix(P_C(I-\gamma_n(a_nD+(1-a_n)E))).$$

From the nonexpansiveness of $T_{r_n^1}$, $T_{r_n^2}$ and Lemma 2.11, we have

$$\begin{aligned}
\|x_{n+1}-z\| &\leq \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|S_nx_n-z\| \\
&\quad + \eta_n\|P_C(I-\gamma_n(a_nDx+(1-a_n)E))y_n-z\| \\
&\leq \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| + \eta_n\|y_n-z\| \\
&= \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| \\
&\quad + \eta_n\|\delta_n(u_n-z)+(1-\delta_n)(v_n-z)\| \\
&= \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| \\
&\quad + \eta_n\|\delta_n(T_{r_n^1}(I-r_n^1(a_nA+(1-a_n)B))x_n-z) \\
&\quad + (1-\delta_n)(T_{r_n^2}(I-r_n^2(a_nA+(1-a_n)B))x_n-z)\| \\
&\leq \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| \\
&\quad + \eta_n(\delta_n\|x_n-z\|+(1-\delta_n)\|x_n-z\|) \\
&= \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| + \eta_n\|x_n-z\| \\
&= \alpha_n \|u-z\| + (1-\alpha_n)\|x_n-z\|. \qquad (3.2)
\end{aligned}$$

Put $M = max\{\|u-z\|, \|x_1-z\|\}$. From induction, we can show that $\|x_n-z\| \leq M$, for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. For every $n \in \mathbb{N}$, put $J_n = a_nD+(1-a_n)E$ and $G_n = a_nA+(1-a_n)B$. From the definition of x_n and the nonexpansivenss of $Pc(I-\gamma_nJ_n)$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + \lambda_n\|S_nx_n-S_{n-1}x_{n-1}\| + |\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
&\quad + \|\eta_nP_C(I-\gamma_nJ_n)y_n-\eta_{n-1}P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\leq |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + \lambda_n\|S_nx_n-S_nx_{n-1}\| + \lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\| \\
&\quad + |\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| + \eta_n\|P_C(I-\gamma_nJ_n)y_n-P_C(I-\gamma_nJ_n)y_{n-1}\| \\
&\quad + \eta_n\|P_C(I-\gamma_nJ_n)y_{n-1}-P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\quad + |\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\leq |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + \lambda_n\|x_n-x_{n-1}\| \\
&\quad + \lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\| + |\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
&\quad + \eta_n\|y_n-y_{n-1}\| + \eta_n\|P_C(I-\gamma_nJ_n)y_{n-1}-P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\quad + |\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|. \qquad (3.3)
\end{aligned}$$

Since $y_n = \delta_nu_n+(1-\delta_n)v_n$, we have

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|\delta_nu_n+(1-\delta_n)v_n-\delta_{n-1}u_{n-1}-(1-\delta_{n-1})v_{n-1}\| \\
&= \|\delta_n(u_n-u_{n-1})+(\delta_n-\delta_{n-1})u_{n-1}+(1-\delta_n)(v_n-v_{n-1}) \\
&\quad + (\delta_{n-1}-\delta_n)v_{n-1}\| \\
&\leq \delta_n\|u_n-u_{n-1}\| + |\delta_n-\delta_{n-1}|\|u_{n-1}\| + (1-\delta_n)\|v_n-v_{n-1}\| \\
&\quad + |\delta_n-\delta_{n-1}|\|v_{n-1}\|. \qquad (3.4)
\end{aligned}$$

From the nonexpansiveness of P_C, we have

$$\begin{aligned}\|P_C(I-\gamma_n J_n)y_{n-1}-&P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\ &\le \|(I-\gamma_n J_n)y_{n-1}-(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\ &= \|\gamma_n J_n y_{n-1}-\gamma_{n-1}J_{n-1}y_{n-1}\|\\ &\le \gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\|+a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|\\ &\quad+\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\|+(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\|.\end{aligned} \tag{3.5}$$

Substitute (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}\|x_{n+1}-x_n\| \le{}& |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|\\ &+\lambda_n\|x_n-x_{n-1}\|\\ &+\lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\|\\ &+\eta_n\delta_n\|u_n-u_{n-1}\|+\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n\|v_n-v_{n-1}\|\\ &+\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\|\\ &+\eta_n a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\|\\ &+\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\|\\ &+|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|.\end{aligned} \tag{3.6}$$

By the same method as Theorem 3.1 in [18], we have

$$\|S_nx_{n-1}-S_{n-1}x_{n-1}\| \le \alpha_1^n\frac{2}{1-\kappa}\|x_{n-1}-z\|. \tag{3.7}$$

Since $u_n=T_{r_n^1}(I-r_n^1G_n)x_n$ where $G_n=a_nA+(1-a_n)B$. From the definition of T_{r_n}, we have

$$F_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n)+\langle G_nx_n,y-u_n\rangle+\frac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle\ge 0 \tag{3.8}$$

and

$$\begin{aligned}F_1(u_{n+1},y)+\varphi_1(y)-\varphi_1(u_{n+1})+\langle G_{n+1}x_{n+1},y-u_{n+1}\rangle&\\ +\frac{1}{r_{n+1}^1}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0,&\end{aligned} \tag{3.9}$$

for all $y\in C$.
From (3.8) and (3.9), we have

$$\begin{aligned}F_1(u_n,u_{n+1})+\varphi_1(u_{n+1})-\varphi_1(u_n)+\langle G_nx_n,u_{n+1}-u_n\rangle&\\ +\frac{1}{r_n^1}\langle u_{n+1}-u_n,u_n-x_n\rangle\ge 0.&\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}F_1(u_{n+1},u_n)+\varphi_1(u_n)-\varphi_1(u_{n+1})+\langle G_{n+1}x_{n+1},u_n-u_{n+1}\rangle&\\ +\frac{1}{r_{n+1}^1}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0.&\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$F_1(u_n, u_{n+1}) + \varphi_1(u_{n+1}) - \varphi_1(u_n) + \frac{1}{r_n^1}\langle u_{n+1} - u_n, u_n - x_n + r_n^1 G_n x_n\rangle \geq 0 \tag{3.12}$$

and

$$\begin{aligned}F_1(u_{n+1}, u_n) &+ \varphi_1(u_n) - \varphi_1(u_{n+1})\\ &+ \frac{1}{r_{n+1}^1}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} + r_{n+1}^1 G_{n+1} x_{n+1}\rangle \geq 0.\end{aligned} \tag{3.13}$$

Summing up (3.12) and (3.13), we have

$$\begin{aligned}&\frac{1}{r_n^1}\langle u_{n+1} - u_n u_n - x_n + r_n^1 G_n x_n\rangle\\ &\qquad + \frac{1}{r_{n+1}^1}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} + r_{n+1}^1 G_{n+1} x_{n+1}\rangle \geq 0.\end{aligned}$$

It follows that

$$\langle u_{n+1} - u_n, \frac{u_n - (I - r_n^1 G_n)x_n}{r_n^1} - \frac{u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}}{r_{n+1}^1}\rangle \geq 0.$$

This implies that

$$\begin{aligned}0 &\leq \langle u_{n+1} - u_n, u_n - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle\\ &= \langle u_{n+1} - u_n, u_n - u_{n+1}\rangle\\ &\quad + \langle u_{n+1} - u_n, u_{n+1} - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, u_{n+1} - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle\\ &= \langle u_{n+1} - u_n, (I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_n^1 G_n)x_n\\ &\quad + \left(1 - \frac{r_n^1}{r_{n+1}^1}\right)(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle\\ &\leq \|u_{n+1} - u_n\|\big(\|(I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_n^1 G_n)x_n\|\\ &\quad + \left|1 - \frac{r_n^1}{r_{n+1}^1}\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\|\big).\end{aligned}$$

Then

$$
\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|(I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_n^1 G_n)x_n\| \\
&\quad + \frac{1}{r_{n+1}^1}\left|r_{n+1}^1 - r_n^1\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\| \\
&\leq \|(I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_{n+1}^1 G_{n+1})x_n\| \\
&\quad + \|(I - r_{n+1}^1 G_{n+1})x_n - (I - r_n^1 G_n)x_n\| \\
&\quad + \frac{1}{r_{n+1}^1}\left|r_{n+1}^1 - r_n^1\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \|r_{n+1}^1 G_{n+1}x_n - r_n^1 G_n x_n\| \\
&\quad + \frac{1}{r_{n+1}^1}\left|r_{n+1}^1 - r_n^1\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| \\
&\quad + \|r_{n+1}^1(a_{n+1}A + (1 - a_{n+1})B)x_n - r_{n+1}^1(a_n A + (1 - a_n)B)x_n\| \\
&\quad + |r_{n+1}^1 - r_n^1|\|G_n x_n\| + \frac{1}{r_{n+1}^1}\left|r_{n+1}^1 - r_n^1\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\| \\
&= \|x_{n+1} - x_n\| \\
&\quad + \|r_{n+1}^1(a_{n+1} - a_n)Ax_n + r_{n+1}^1((1 - a_{n+1}) - (1 - a_n))Bx_n\| \\
&\quad + |r_{n+1}^1 - r_n^1|\|G_n x_n\| + \frac{1}{r_{n+1}^1}\left|r_{n+1}^1 - r_n^1\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + r_{n+1}^1|a_{n+1} - a_n|\|Ax_n\| + r_{n+1}^1|a_{n+1} - a_n|\|Bx_n \\
&\quad + |r_{n+1}^1 - r_n^1|\|G_n x_n\| + \frac{1}{b}\left|r_{n+1}^1 - r_n^1\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\|.
\end{aligned} \tag{3.14}
$$

From (3.14), we have

$$
\begin{aligned}
\|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + r_n^1|a_n - a_{n-1}|\|Ax_{n-1}\| + r_n^1|a_n - a_{n-1}|\|Bx_{n-1}\| \\
&\quad + |r_n^1 - r_{n-1}^1|\|G_{n-1}x_{n-1}\| + \frac{1}{b}\left|r_n^1 - r_{n-1}^1\right| \|u_n - (I - r_n^1 G_n)x_n\|.
\end{aligned} \tag{3.15}
$$

By using the same method as (3.15), we have

$$
\begin{aligned}
\|v_n - v_{n-1}\| &\leq \|x_n - x_{n-1}\| + r_n^2|a_n - a_{n-1}|\|Ax_{n-1}\| + r_n^2|a_n - a_{n-1}|\|Bx_{n-1}\| \\
&\quad + |r_n^2 - r_{n-1}^2|\|G_{n-1}x_{n-1}\| + \frac{1}{b}\left|r_n^2 - r_{n-1}^2\right| \|v_n - (I - r_n^2 G_n)x_n\|.
\end{aligned} \tag{3.16}
$$

Substitute (3.7), (3.15) and (3.16) into (3.6), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| \le\ & |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
& +\lambda_n\|x_n-x_{n-1}\|+\lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
& +\eta_n\delta_n\|u_n-u_{n-1}\|+\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n\|v_n-v_{n-1}\| \\
& +\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\| + \eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\| \\
& +\eta_n a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\| + \eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\| \\
& +\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\| \\
& +|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
\le\ & |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\lambda_n\|x_n-x_{n-1}\| \\
& +\lambda_n\left(\alpha_1^n\frac{2}{1-\kappa}\|x_{n-1}-z\|\right)+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
& +\eta_n\delta_n(\|x_n-x_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Bx_{n-1}\| \\
& +|r_n^1-r_{n-1}^1|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^1-r_{n-1}^1\right|\|u_n-(I-r_n^1G_n)x_n\|) \\
& +\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n(\|x_n-x_{n-1}\| \\
& +r_n^2|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^2|a_n-a_{n-1}|\|Bx_{n-1}\| \\
& +|r_n^2-r_{n-1}^2|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^2-r_{n-1}^2\right|\|v_n-(I-r_n^2G_n)x_n\|) \\
& +\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\| \\
& +\eta_n a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\| \\
& +\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\| \\
& +|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
\le\ & (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+|\beta_n-\beta_{n-1}|M_1 \\
& +\alpha_1^n\frac{2}{1-\kappa}M_1+|\lambda_n-\lambda_{n-1}|M_1 \\
& +c|a_n-a_{n-1}|M_1+c|a_n-a_{n-1}|M_1+|r_n^1-r_{n-1}^1|M_1+\frac{1}{b}\left|r_n^1-r_{n-1}^1\right|M_1 \\
& +|\delta_n-\delta_{n-1}|M_1+c|a_n-a_{n-1}|M_1+c|a_n-a_{n-1}|\|M_1+|r_n^2-r_{n-1}^2|M_1 \\
& +\frac{1}{b}\left|r_n^2-r_{n-1}^2\right|M_1+|\delta_n-\delta_{n-1}|M_1+|a_n-a_{n-1}|M_1+|\gamma_n-\gamma_{n-1}|M_1 \\
& +|a_n-a_{n-1}|M_1+|\gamma_n-\gamma_{n-1}|M_1+|\eta_n-\eta_{n-1}|M_1,
\end{aligned}$$

where $M_1 := max_{n\in\mathbb{N}}\{\|u\|, \|x_n\|, \|x_n-z\|, \|S_nx_n\|, \|Ax_n\|, \|Bx_n\|, \|G_nx_n\|,$ $\|\|u_n-(I-r_n^1G_n)x_n\|, \|u_n\|, \|v_n-(I-r_n^2G_n)x_n\|, \|v_n\|, \|Dy_n\|, \|Ey_n\|,$ $\|P_C(I-\gamma_nJ_n)y_n\|\}$. From the conditions (ii), (iv) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.17}$$

Step 3. We show that $\lim_{n\to\infty}\|u_n-x_n\| = \lim_{n\to\infty}\|v_n-x_n\| = \lim_{n\to\infty}\|y_n-x_n\| = \lim_{n\to\infty}\|P_C(I-\gamma_nJ_n)y_n-x_n\| = 0$. Let $z\in\mathcal{F}$. Since $u_n = T_{r_n^1}(I-r_n^1G_n)x_n$, $v_n =$

$T_{r_n^2}(I - r_n^2 G_n)x_n$ and T_{r_n} is a firmly nonexpensive mapping, we have

$$\begin{aligned}
\|T_{r_n^1}(I - r_n^1 G_n)x_n)) - z\|^2 &= \|T_{r_n^1}(I - r_n^1 G_n)x_n - T_{r_n^1}(I - r_n^1 G_n)z\|^2 \\
&\leq \langle (I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z, u_n - z\rangle \\
&= \frac{1}{2}(\|(I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z\|^2 + \|u_n - z\|^2 \\
&\quad - \|(I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z - u_n + z\|^2) \\
&\leq \frac{1}{2}(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - r_n^1(G_n x_n - G_n z)\|^2) \\
&= \frac{1}{2}(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n)\|^2 - (r_n^1)^2\|G_n x_n - G_n z\|^2 \\
&\quad + 2r_n^1\langle x_n - T_{r_n^1}(I - r_n^1 G_n)x_n, G_n x_n - G_n z\rangle) \\
&\leq \frac{1}{2}(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 - (r_n^1)^2\|G_n x_n - G_n z\|^2 \\
&\quad + 2r_n^1\|x_n - T_{r_n^1}(I - r_n^1 G_n)x_n\|\|G_n x_n - G_n z\|).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n\|^2 - (r_n^1)^2\|G_n x_n - G_n z\|^2 \\
+ 2r_n^1\|x_n - T_{r_n^1}(I - r_n^1 G_n)x_n\|\|G_n x_n - G_n z\|.
\end{aligned} \tag{3.18}$$

Applying (3.18) and $v_n = T_{r_n^2}(I - r_n^2 G_n)x_n$, we have

$$\begin{aligned}
\|v_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - v_n\|^2 - (r_n^2)^2\|G_n x_n - G_n z\|^2 \\
+ 2r_n^2\|x_n - T_{r_n^1}(I - r_n^2 G_n)x_n\|\|G_n x_n - G_n z\|.
\end{aligned} \tag{3.19}$$

For every $x, y \in C$, we have

$$\begin{aligned}
\langle G_n x_n - G_n z, x_n - z\rangle &= \langle (a_n A + (1 - a_n)B)\, x_n - (a_n A + (1 - a_n)B)\, z, x_n - z\rangle \\
&= \langle a_n\, (Ax_n - Az) + (1 - a_n)\, (Bx_n - Bz)\, , x_n - z\rangle \\
&= a_n\, \langle Ax_n - Az, x_n - z\rangle + (1 - a_n)\, \langle Bx_n - Bz, x_n - z\rangle \\
&\geq a_n\alpha\, \|Ax_n - Az\|^2 + (1 - a_n)\beta\, \|Bx_n - Bz\|^2 .
\end{aligned} \tag{3.20}$$

From the definition of u_n and (3.20), we have

$$\begin{aligned}
\|u_n - z\|^2 &= \|T_{r_n^1}(I - r_n^1 G_n)x_n - T_{r_n^1}(I - r_n^1 G_n)z\|^2 \\
&\leq \|(I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z\|^2 \\
&= \|x_n - z\|^2 - 2r_n^1\langle x_n - z, G_n x_n - G_n z\rangle + (r_n^1)^2\|(G_n x_n - G_n z)\|^2 \\
&\leq \|x_n - z\|^2 - 2r_n^1 a_n\alpha\, \|Ax_n - Az\|^2 - 2r_n^1(1 - a_n)\beta\, \|Bx_n - Bz\|^2 \\
&\quad + (r_n^1)^2\|a_n(Ax_n - Az) + (1 - a_n)(Bx_n - Bz)\|^2 \\
&\leq \|x_n - z\|^2 - 2r_n^1 a_n\alpha\, \|Ax_n - Az\|^2 - 2r_n^1(1 - a_n)\beta\, \|Bx_n - Bz\|^2 \\
&\quad + (r_n^1)^2 a_n\|Ax_n - Az\|^2 + (1 - a_n)(r_n^1)^2\|Bx_n - B)\|^2 \\
&\leq \|x_n - z\|^2 - r_n^1 a_n(2\alpha - r_n^1)\, \|Ax_n - Az\|^2 \\
&\quad - r_n^1(1 - a_n)(2\beta - r_n^1)\, \|Bx_n - Bz\|^2 .
\end{aligned} \tag{3.21}$$

Applying (3.21) and $v_n = T_{r_n^2}(I - r_n^2 G_n)x_n$, we have

$$\|v_n - z\|^2 \le \|x_n - z\|^2 - r_n^2 a_n(2\alpha - r_n^2)\|Ax_n - Az\|^2 \\ - r_n^2(1 - a_n)(2\beta - r_n^2)\|Bx_n - Bz\|^2. \tag{3.22}$$

From the definition of x_n, (3.21) and (3.22), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|S_n x_n - z\|^2 \\ &\quad + \eta_n\|P_C(I - \gamma_n(a_n D + (1 - a_n)E))y_n - z)\|^2 \\ &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 + \eta_n\|y_n - z\|^2 \\ &= \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n\|\delta_n(u_n - z) + (1 - \delta_n)(v_n - z)\|^2 \\ &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n(\delta_n\|u_n - z\|^2 + (1 - \delta_n)\|v_n - z\|^2)\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n(\delta_n(\|x_n - z\|^2 - r_n^1 a_n(2\alpha - r_n^1)\|Ax_n - Az\|^2 \\ &\quad - r_n^1(1 - a_n)(2\beta - r_n^1)\|Bx_n - Bz\|^2) + (1 - \delta_n)(\|x_n - z\|^2 \\ &\quad - r_n^2 a_n(2\alpha - r_n^2)\|Ax_n - Az\|^2 - r_n^2(1 - a_n)(2\beta - r_n^2)\|Bx_n - Bz\|^2)) \\ &\le \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \eta_n a_n(r_n^1\delta_n(2\alpha - r_n^1) + r_n^2(1 - \delta_n)(2\alpha - r_n^2))\|Ax_n - Az\|^2 \\ &\quad - (1 - a_n)\eta_n(r_n^1\delta_n(2\beta - r_n^1) + r_n^2(1 - \delta_n)(2\beta - r_n^2))\|Bx_n - Bz\|^2.\end{aligned} \tag{3.24}$$

From (3.24), we have

$$\begin{aligned}\eta_n a_n(r_n^1\delta_n(2\alpha - r_n^1) &+ r_n^2(1 - \delta_n)(2\alpha - r_n^2))\|Ax_n - Az\|^2 \\ &\le \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\le \alpha_n\|u - z\|^2 + (\|x_n - z\|^2 + \|x_{n+1} - z\|)(\|x_{n+1} - x_n\|).\end{aligned}$$

From the condition (i) and (3.17), we have

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.25}$$

By using the same method as (3.25), we have

$$\lim_{n\to\infty}\|Bx_n - Bz\| = 0. \tag{3.26}$$

Since $G_n = a_n A + (1 - a_n)B$, we obtain

$$\|G_n x_n - G_n z\| \le a_n\alpha\|Ax_n - Az\|^2 + (1 - a_n)\beta\|Bx_n - Bz\|^2.$$

From (3.25) and (3.26), we have

$$\lim_{n\to\infty}\|G_n x_n - G_n z\| = 0. \tag{3.27}$$

From (3.21), (3.22), (3.23) and the definition of x_n, we have

$$
\begin{aligned}
\|x_{n+1}-z\|^2 \leq\ & \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2 \\
& +\eta_n(\delta_n\|u_n-z\|^2+(1-\delta_n)\|v_n-z\|^2) \\
\leq\ & \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2 \\
& +\eta_n(\delta_n(\|x_n-z\|^2-\|x_n-u_n\|^2-(r_n^1)^2\|G_nx_n-G_nz\|^2 \\
& +2r_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|) \\
& +(1-\delta_n)(\|x_n-z\|^2-\|x_n-v_n\|^2-(r_n^2)^2\|G_nx_n-G_nz\|^2 \\
& +2r_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|)) \\
\leq\ & \alpha_n\|u-z\|^2+\|x_n-z\|^2-\eta_n\delta_n\|x_n-u_n\|^2-(1-\delta_n)\eta_n\|x_n-v_n\|^2 \\
& -\eta_n(\delta_n(r_n^1)^2+(1-\delta_n)(r_n^2)^2\|G_nx_n-G_nz\|^2 \\
& +2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\| \\
& +2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\eta_n\delta_n\|u_n-x_n\|^2 \leq\ & \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2 \\
& +2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\| \\
& +2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\| \\
\leq\ & \alpha_n\|u-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)(\|x_{n+1}-x_n\|) \\
& +2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\| \\
& +2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|.
\end{aligned}
$$

From the condition (i), (3.17) and (3.27), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.28}$$

By using the same method as (3.28), we have

$$\lim_{n\to\infty}\|v_n-x_n\|=0. \tag{3.29}$$

From the definition of y_n, we have

$$
\begin{aligned}
\|y_n-x_n\| &= \|\delta_nu_n+(1-\delta_n)v_n-x_n\| \\
&\leq \delta_n\|u_n-x_n\|+(1-\delta_n)\|v_n-x_n\|.
\end{aligned}
$$

From (3.28) and (3.29), we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.30}$$

From the nonexpansiveness of P_C and $z \in \mathcal{F}$, we have

$$\begin{aligned}
\|P_C(I-\gamma_n J_n)y_n - z\|^2 &= \|P_C(I-\gamma_n J_n)y_n - P_C(I-\gamma_n J_n)z\|^2 \\
&\le \|(I-\gamma_n J_n)y_n - (I-\gamma_n J_n)z\|^2 \\
&= \|y_n - z - \gamma_n(J_n y_n - J_n z)\|^2 \\
&= \|y_n - z\|^2 - 2\gamma_n\langle y_n - z, J_n y_n - J_n z\rangle + \gamma_n^2\|J_n y_n - J_n z\|^2 \\
&\le \|x_n - z\|^2 - 2\gamma_n\langle y_n - z, J_n y_n - J_n z\rangle + \gamma_n^2\|J_n y_n - J_n z\|^2 .
\end{aligned} \tag{3.31}$$

For every $x, y \in C$, we have

$$\begin{aligned}
\langle J_n x - J_n y, x-y\rangle &= a_n\langle Dx - Dy, x-y\rangle + (1-a_n)\langle Ex - Ey, x-y\rangle \\
&\ge a_n\frac{\mu}{L_D^2}\|Dx - Dy\|^2 + (1-a_n)\frac{\rho}{L_E^2}\|Ex - Ey\|^2 .
\end{aligned} \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$\begin{aligned}
\|P_C(I-\gamma_n J_n)y_n - z\|^2 &\le \|x_n - z\|^2 - 2\gamma_n\langle y_n - z, J_n y_n - J_n z\rangle + \gamma_n^2\|J_n y_n - J_n z\|^2 \\
&\le \|x_n - z\|^2 - 2\gamma_n a_n\frac{\mu}{L_D^2}\|Dy_n - Dz\|^2 \\
&\quad - 2\gamma_n(1-a_n)\frac{\rho}{L_E^2}\|Ey_n - Ez\|^2 \\
&\quad + \gamma_n^2\|a_n(Dy_n - Dz) + (1-a_n)(Ey_n - Ez)\|^2 \\
&\le \|x_n - z\|^2 - 2\gamma_n a_n\frac{\mu}{L_D^2}\|Dy_n - Dz\|^2 \\
&\quad - 2\gamma_n(1-a_n)\frac{\rho}{L_E^2}\|Ey_n - Ez\|^2 \\
&\quad + \gamma_n^2 a_n\|Dy_n - Dz\|^2 + \gamma_n^2(1-a_n)\|Ey_n - Ez\|^2 \\
&= \|x_n - z\|^2 - \gamma_n a_n\left(\frac{2\mu}{L_D^2} - \gamma_n\right)\|Dy_n - Dz\|^2 \\
&\quad - \gamma_n(1-a_n)\left(\frac{2\rho}{L_E^2} - \gamma_n\right)\|Ey_n - Ez\|^2 .
\end{aligned} \tag{3.33}$$

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\
&\quad + \eta_n\|P_C(I-\gamma_n J_{n+1})y_n - z\|^2 \\
&\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\
&\quad + \eta_n(\|x_n - z\|^2 - \gamma_n a_n\left(\frac{2\mu}{L_D^2} - \gamma_n\right)\|Dy_n - Dz\|^2 \\
&\quad - \gamma_n(1-a_n)\left(\frac{2\rho}{L_E^2} - \gamma_n\right)\|Ey_n - Ez\|^2)
\end{aligned}$$

$$\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \eta_n \gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n \right) \|Dy_n - Dz\|^2$$
$$- \eta_n \gamma_n (1-a_n) \left(\frac{2\rho}{L_E^2} - \gamma_n \right) \|Ey_n - Ez\|^2 .$$

It implies that

$$\begin{aligned}\eta_n \gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n \right) \|Dy_n - Dz\|^2 &\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\quad - \eta_n \gamma_n (1-a_n) \left(\frac{2\rho}{L_E^2} - \gamma_n \right) \|Ey_n - Ez\|^2 \\ &\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n \|u-z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| .\end{aligned}$$

From the condition (i) and (3.17), we have

$$\lim_{n\to\infty} \|Dy_n - Dz\| = 0. \tag{3.34}$$

By using the same method as (3.35), we have

$$\lim_{n\to\infty} \|Ey_n - Ez\| = 0. \tag{3.35}$$

From the definition of J_n, we have

$$\|J_n y_n - J_n z\| \leq a_n \|Dy_n - Dz\| + (1-a_n) \|Ey_n - Ez\| . \tag{3.36}$$

From (3.34), (3.35) and (3.36), we have

$$\lim_{n\to\infty} \|J_n y_n - J_n z\| = 0. \tag{3.37}$$

From the definition of $P_C (I - \gamma_n J)$ and Lemma 2.11, it implies that

$$\begin{aligned}\|P_C (I - \gamma_n J_n) y_n - z\|^2 &= \|P_C (I - \gamma_n J_n) y_n - P_C (I - \gamma_n J_n) z\|^2 \\ &\leq \langle (I - \gamma_n J_n) y_n - (I - \gamma_n J_n) z, P_C (I - \gamma_n J_n) y_n - z \rangle \\ &= \frac{1}{2} \Big[\|(I - \gamma_n J_n) y_n - (I - \gamma_n J_n) z\|^2 + \|P_C (I - \gamma_n J_n) y_n - z\|^2 \\ &\quad - \|(I - \gamma_n J_n) y_n - (I - \gamma_n J_n) z - (P_C (I - \gamma_n J_n) y_n - z)\|^2 \Big] \\ &\leq \frac{1}{2} (\|y_n - z\|^2 + \|P_C (I - \gamma J_n) y_n - z\|^2 \\ &\quad - \|y_n - P_C (I - \gamma_n J_n) y_n - \gamma_n (J_n y_n - J_n z)\|^2) \\ &\leq \frac{1}{2} (\|x_n - z\|^2 + \|P_C (I - \gamma_n J) y_n - z\|^2 \\ &\quad - \|y_n - P_C (I - \gamma_n J_n) y_n\|^2 - \gamma_n^2 \|J_n y_n - J_n z\|^2 \\ &\quad + 2\gamma_n \langle y_n - P_C (I - \gamma_n J_n) y_n, J_n y_n - J_n z \rangle) \\ &\leq \frac{1}{2} (\|x_n - z\|^2 + \|P_C (I - \gamma_n J_n) y_n - z\|^2 \\ &\quad - \|y_n - P_C (I - \gamma_n J_n) y_n\|^2 - \gamma_n^2 \|J_n y_n - J_n z\|^2 \\ &\quad + 2\gamma_n \|y_n - P_C (I - \gamma_n J_n) y_n\| \|J_n y_n - J_n z\|).\end{aligned}$$

It follows that

$$\left\|P_C\left(I-\gamma_n J_n\right) y_n-z\right\|^2 \leq\left\|x_n-z\right\|^2-\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|^2 \\ +2 \gamma\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|\left\|J_n y_n-J_n z\right\| . \quad (3.38)$$

From the definition of x_n and (3.38), we have

$$\begin{aligned} \left\|x_{n+1}-z\right\|^2 & \leq \alpha_n\|u-z\|^2+\beta_n\left\|x_n-z\right\|^2+\lambda_n\left\|x_n-z\right\|^2+\eta_n\left\|P_C(I-\gamma_n J_n) y_n-z\right\|^2 \\ & \leq \alpha_n\|u-z\|^2+\beta_n\left\|x_n-z\right\|^2+\lambda_n\left\|x_n-z\right\|^2+\eta_n(\left\|x_n-z\right\|^2 \\ & \quad -\left\|y_n-P_C(I-\gamma_n J_n) y_n\right\|^2+2 \gamma_n\left\|y_n-P_C(I-\gamma_n J_n) y_n\right\|\left\|J_n y_n-J_n z\right\|) \\ & \leq \alpha_n\|u-z\|^2+\left\|x_n-z\right\|^2-\eta_n\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|^2 \\ & \quad +2 \eta_n \gamma_n\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|\left\|J_n y_n-J_n z\right\| . \end{aligned}$$

It implies that

$$\begin{aligned} \eta_n\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|^2 & \leq \alpha_n\|u-z\|^2+\left\|x_n-z\right\|^2-\left\|x_{n+1}-z\right\|^2 \\ & \quad +2 \eta_n \gamma_n\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|\left\|J_n y_n-J_n z\right\| \\ & \leq \alpha_n\|u-z\|^2+\left(\left\|x_n-z\right\|-\left\|x_{n+1}-z\right\|\right)\left\|x_{n+1}-x_n\right\| \\ & \quad +2 \eta_n \gamma_n\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|\left\|J_n y_n-J_n z\right\| . \end{aligned} \quad (3.39)$$

From the condition (i), (3.17), (3.37) and (3.39), we obtain

$$\lim_{n \rightarrow \infty}\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|=0. \quad (3.40)$$

Since

$$\left\|x_n-P_C\left(I-\gamma_n J_n\right) y_n\right\| \leq\left\|x_n-y_n\right\|+\left\|y_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|,$$

from (3.30) and (3.40), we have

$$\lim_{n \rightarrow \infty}\left\|x_n-P_C\left(I-\gamma_n J_n\right) y_n\right\|=0. \quad (3.41)$$

Step 4. We show that $\lim_{n \rightarrow \infty} \sup \left\langle u-z_0, x_n-z_0\right\rangle \leq 0$ where $z_0=P_{\mathcal{F}} u$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} \sup \left\langle u-z_0, x_n-z_0\right\rangle=\lim_{k \rightarrow \infty}\left\langle u-z_0, x_{n_k}-z_0\right\rangle . \quad (3.42)$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k \rightarrow \infty$ where $\omega \in C$. From (3.30), we obtain $y_{n_k} \rightharpoonup \omega$ as $k \rightarrow \infty$. From (3.28), we have $u_{n_k} \rightharpoonup \omega$ as $k \rightarrow \infty$. Assume $\omega \notin VI(C, D) \bigcap VI(C, E)$. From Lemma 2.11 and Lemma 2.4, we have

$$VI(C, D) \bigcap VI(C, E)=VI(C, J_{n_k})=Fix(P_C(I-\gamma_{n_k} J_{n_k})).$$

From the nonexpansiveness of $P_C\left(I-\gamma_{n_k} J_{n_k}\right)$, (3.41) and Opial's condition , we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \inf \left\|y_{n_k}-\omega\right\| & <\lim_{k \rightarrow \infty} \inf \left\|y_{n_k}-P_C(I-\gamma_{n_k} J_{n_k}) \omega\right\| \\ & \leq \lim_{k \rightarrow \infty} \inf \left\|y_{n_k}-P_C(I-\gamma_{n_k} J_{n_k}) y_{n_k}\right\| \\ & \quad +\lim_{k \rightarrow \infty} \inf \left\|P_C(I-\gamma_n J_{n_k}) y_{n_k}-P_C(I-\gamma_{n_k} J_{n_k}) \omega\right\| \\ & \leq \lim_{k \rightarrow \infty} \inf \left\|y_{n_k}-\omega\right\| . \end{aligned}$$

This is a contradiction. Hence

$$\omega \in VI(C,D)\bigcap VI(C,E). \tag{3.43}$$

From the definition of x_n, we have

$$x_{n+1}-x_n=\alpha_n(u-x_n)+\lambda_n(S_nx_n-x_n)+\eta_n(P_C(I-\gamma_nJ_n)y_n-x_n).$$

From the condition (i), (3.15) and (3.41), we have

$$\lim_{n\to\infty}\|S_nx_n-x_n\|=0. \tag{3.44}$$

Assume $\omega\notin\bigcap_{i=1}^{\infty}Fix(T_i)$. From Lemma 2.10 , we have $Fix(S)=\bigcap_{i=1}^{\infty}Fix(T_i)$. Then $\omega\notin Fix(S)$. From Remark 2.9, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\|x_{n_k}-S\omega\|\\ &\le \liminf_{k\to\infty}(\|x_{n_k}-S_{n_k}x_{n_k}\|+\|S_{n_k}x_{n_k}-S_{n_k}\omega\|+\|S_{n_k}\omega-S\omega\|)\\ &\le \liminf_{k\to\infty}\|x_{n_k}-\omega\|.\end{aligned}$$

This is a contradiction. Then

$$\omega\in Fix(S)=\bigcap_{i=1}^{\infty}Fix(T_i). \tag{3.45}$$

Since

$$F_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n)+\langle G_nx_n,y-u_n\rangle+\frac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle\ge 0,\forall y\in C,$$

from (A2), we have

$$\varphi_1(y)-\varphi_1(u_n)+\langle G_nx_n,y-u_n\rangle+\frac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle\ge F_1(y,u_n),\forall y\in C.$$

In particular

$$\varphi_1(y)-\varphi_1(u_{n_i})+\langle G_{n_i}x_{n_i},y-u_{n_i}\rangle+\frac{1}{r_{n_i}^1}\langle y-u_{n_i},u_{n_i}-x_{n_i}\rangle\ge F_1(y,u_{n_i}),\forall y\in C.$$

It follows that

$$\varphi_1(y)-\varphi_1(u_{n_i})+\langle G_{n_i}x_{n_i},y-u_{n_i}\rangle+\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle\ge F_1(y,u_{n_i}). \tag{3.46}$$

For $0<t\le 1$ and $y\in C$, let $y_t=ty+(1-t)\omega$. From (3.46), we have

$$\begin{aligned}&\varphi_1(y_t)-\varphi_1(u_{n_i})+\langle y_t-u_{n_i},G_{n_i}y_t\rangle\\ &\ge \langle y_t-u_{n_i},G_{n_i}y_t\rangle-\langle y_t-u_{n_i},G_{n_i}x_{n_i}\rangle\\ &\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle+F_1(y_t,u_{n_i})\\ &=\langle y_t-u_{n_i},G_{n_i}y_t-G_{n_i}u_{n_i}+G_{n_i}u_{n_i}\rangle-\langle y_t-u_{n_i},G_{n_i}x_{n_i}\rangle\\ &\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle+F_1(y_t,u_{n_i})\\ &=\langle y_t-u_{n_i},G_{n_i}y_t-G_{n_i}u_{n_i}\rangle+\langle y_t-u_{n_i},G_{n_i}u_{n_i}-G_{n_i}x_{n_i}\rangle\\ &\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle+F_1(y_t,u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|G_{n_i}u_{n_i} - G_{n_i}x_{n_i}\| \to 0$. Since $\frac{u_{n_i}-x_{n_i}}{r^1_{n_i}} \to 0$ and $\langle y_t - u_{n_i}, G_{n_i}y_t - G_{n_i}u_{n_i}\rangle \geq 0$ and (A4), we have

$$\varphi_1(y_t) - \varphi_1(\omega) + \langle y_t - \omega, G_{n_i}y_t\rangle \geq F_1(y_t, \omega). \tag{3.47}$$

Form (A1), (A4) and (3.47), we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) + \varphi_1(y_t) - \varphi_1(y_t)\\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, \omega) + t\varphi_1(y) + (1-t)\varphi_1(\omega) - \varphi_1(y_t)\\
&\leq tF_1(y_t, y) + (1-t)\varphi_1(y_t) - (1-t)\varphi_1(\omega) + (1-t)\langle y_t - \omega, G_{n_i}y_t\rangle\\
&\quad + t\varphi_1(y) + (1-t)\varphi_1(\omega) - \varphi_1(y_t)\\
&= tF_1(y_t, y) + t\varphi_1(y) - t\varphi_1(y_t) + (1-t)\langle ty + (1-t)\omega - \omega, G_{n_i}y_t\rangle\\
&= tF_1(y_t, y) + t\varphi_1(y) - t\varphi_1(y_t) + (1-t)t\langle y - \omega, G_{n_i}y_t\rangle.
\end{aligned}$$

Dividing by t, we have

$$0 \leq F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t) + (1-t)\langle y - \omega, G_{n_i}y_t\rangle.$$

Letting $t \to 0$, it follows from (A3), we have

$$0 \leq F_1(\omega, y) + \varphi_1(y) - \varphi_1(\omega) + \langle y - \omega, G_{n_i}\omega\rangle, \forall y \in C. \tag{3.48}$$

From Lemma 2.13, we have

$$\omega \in GMEP(F_1, \varphi_1, a_{n_i}A + (1-a_{n_i})B) = GMEP(F_1, \varphi_1, A) \bigcap GMEP(F_1, \varphi_1, B).$$

By using the same method as (3.48), we have

$$\omega \in GMEP(F_2, \varphi_2, A) \bigcap GMEP(F_2, \varphi_2, B).$$

Hence $\omega \in \mathcal{F}$. Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathcal{F}$, we have

$$\lim_{n\to\infty} \sup \langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle u - z_0, x_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \tag{3.49}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{\mathcal{F}}u$. From the nonexpansiveness of $P_C(I - \gamma J_n)$, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(u - z_0) + \beta_n(x_n - z_0) + \lambda_n(S_nx_n - z_0) + \eta_n(P_C(I - \gamma_nJ_n)y_n - z_0)\|^2\\
&\leq \|\beta_n(x_n - z_0) + \lambda_n(S_nx_n - z_0) + \eta_n(P_C(I - \gamma_nJ_n)y_n - z_0)\|^2\\
&\quad + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle\\
&= (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle\\
&\leq (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

Applying Lemma 2.2 and (3.49), we have the sequence $\{x_n\}$ converse strongly to $z_0 = P_{\mathcal{F}}u$. This complete the proof. ∎

Using our main theorem (Theorem 3.1), we obtain the following results.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfy $A_1) - A_4)$ and $F_i(x,z) \le F_i(x,y) + F_i(y,z)$ for all $x,y,z \in C$ and $i = 1,2$. Let A, B be α, β-inverse strongly monotone, respectively and D, E be L_D, L_E-Lipschitz continuous and μ, ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap EP(F_1,A) \bigcap EP(F_1,B) \bigcap EP(F_2,A) \bigcap EP(F_2,B) \bigcap VI(C,D) \bigcap VI(C,E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \le b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Assume the either $B_1)$ or $B_2)$ holds and let the sequence $\{x_n\}$ generated by $x_1, u \in C$ and*

$$
\begin{aligned}
&F_1(u_n,y) + \langle a_nAx_n + (1-a_n)Bx_n, y-u_n\rangle + \frac{1}{r_n^1}\langle y-u_n, u_n-x_n\rangle \ge 0,\\
&F_2(v_n,y) + \langle a_nAx_n + (1-a_n)Bx_n, y-v_n\rangle + \frac{1}{r_n^2}\langle y-v_n, v_n-x_n\rangle \ge 0,\\
&y_n = \delta_n u_n + (1-\delta_n)v_n,\\
&x_{n+1} = \alpha_n u + \beta_n x_n + \lambda_n S_n x_n + \eta_n P_C(I - \gamma_n(a_nD + (1-a_n)E))y_n, \forall n \ge 1.
\end{aligned}
$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{\eta_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \lambda_n + \eta_n = 1$ for all $n \in \mathbb{N}$, $\{a_n\} \subset (0,1)$ and $\{r_n^j\} \subseteq [b,c] \subset (0, 2min\{\alpha,\beta\})$for all $j = 1,2$. Suppose the following conditions hold:

(i): $\sum_{n=1}^{\infty}\alpha_n = \infty, \lim_{n\to\infty}\alpha_n = 0,\ \{\beta_n\} \subseteq [d,e] \subset (0,1),$

(ii): $0 < \gamma_n \le min\{\frac{2\mu}{L_D^2}, \frac{2\rho}{L_E^2}\},$

(iii): $\lim_{n\to\infty}\delta_n = \delta \in (0,1), \sum_{n=1}^{\infty}\alpha_1^n < \infty,$

(iv): $\sum_{n=1}^{\infty}\left|r_{n+1}^j - r_n^j\right| < \infty\ \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty,$

$\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| < \infty, \sum_{n=1}^{\infty}|a_{n+1}-a_n| < \infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty,$

$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty, \sum_{n=1}^{\infty}|\eta_{n+1}-\eta_n| < \infty$ *for all* $j = 1,2.$

Then, the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$.

Proof. Put $\varphi_1 \equiv \varphi_2 \equiv 0$ in Theorem 3.1. So, from Theorem 3.1, we obtain the desired result. ∎

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A, B be α, β-inverse strongly monotone, respectively and D, E be L_D, L_E-Lipschitz continuous and μ, ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap VI(C,A) \bigcap VI(C,B) \bigcap VI(C,D) \bigcap VI(C,E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \le b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Let the* *sequence* $\{x_n\}$ *generated by* $x_1, u \in C$ *and*

$$\begin{aligned} x_{n+1} =& \alpha_n u + \beta_n x_n + \gamma_n S_n x_n \\ & + \eta_n P_C(I - \gamma_n(a_n D + (1-a_n)E))P_C(I - r_n^1(a_n A + (1-a_n)B))x_n, \forall n \geq 1. \end{aligned}$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n + \eta_n = 1$ *for all* $n \in \mathbb{N}$ *,* $\{a_n\} \subset (0,1)$ *and* $\{r_n^1\} \subseteq [b,c] \subset (0, 2min\{\alpha, \beta\})$ *for all* $j = 1, 2$*. Suppose the following conditions hold:*

(i): $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \{\beta_n\} \subseteq [d,e] \subset (0,1),$

(ii): $0 < \gamma_n \leq min\{\frac{2\mu}{L_D^2}, \frac{2\rho}{L_E^2}\},$

(iii): $\sum_{n=1}^{\infty} \alpha_1^n < \infty,$

(iv): $\sum_{n=1}^{\infty} |r_{n+1}^1 - r_n^1| < \infty \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$$

$$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty.$$

Then, the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}} u$*.*

Proof. Putting $F_1 \equiv F_2 \equiv \varphi_1 \equiv \varphi_2 \equiv 0, r_n^1 = r_n^2$ and $v_n = u_n$ in Theorem 3.1, we have.

$$\langle y - u_n, x_n - r_n^1(a_n A x_n + (1-a_n)Bx_u) - u_n \rangle, \forall y \in C.$$

It implies that

$$u_n = P_C(I - r_n^1(a_n A + (1-a_n)B))x_n.$$

So, from Theorem 3.1 and Remark 2.12, we obtain the desired result. ∎

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References

[1] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese J. Math. 10 (2006) 1293–1303.

[2] L.C. Ceng, C.Y. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Methods Oper. Res. 67 (2008) 375–390.

[3] J.W. Peng, J.C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems, Taiwanese J. Math. 12 (6) (2008) 1401–1432.

[4] S. Chaiyasil, S. Suantai, A proximation method for generalized mixed equilibrium problems and fixed point problems for a countable family of nonexpansive mappings, Journal of Nonlinear Analysis and Optimization 2 (2) (2011) 337–353.

[5] I. Inchan, Extragradient method for generalized mixed equilibrium problems and fixed point problems of finite family of nonexpansive mapping, Applied Mathematical Sciences 5 (72) (2011) 3585–3606.

[6] J.W. Peng, J.C. Yao, Two extragradient methods for generalized mixed equilibrium problems, nonexpansive mappings and monotone mappings, Computers and Mathematics with Applications 58 (2009) 1287–1301.

[7] S. Yekini, Iterative method for fixed point problem, variational inequality and generalized mixed equilibrium problems with applications, J. Glob. Optim. 52 (2012) 57–77.

[8] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515.

[9] Y. Yao, Y.C. Liou, J.C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory and Applications 2007 (2007) Article ID 64363.

[10] P.L. Combettes, A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.

[11] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis: Theory, Methods and Applications 69 (3) (2008) 1025–1033.

[12] A. Kangtunyakarn, Convergence theorem of κ-strictly pseudo-contractive mapping and a modification of generalized equilibrium problems, Fixed Point Theory and Applications 2012 (2012) Article no. 89.

[13] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[14] H.K. Xu, An iterative approach to quadric optimization, J. Optim. Theory Appl. 116 (2003) 659–678.

[15] A. Kangtunyakarn, Strong convergence theorem for a generalized equilibrium problem and system of variational inequalities problem and infinite family of strict pseudo-contractions, Fixed Point Theory and Applications 2011 (2011) Article no. 23.

[16] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336–346

[17] P. Katchanga, T. Jitpeeraa, P. Kumam, Strong convergence theorems for solving generalized mixed equilibrium problems and general system of variational inequalities by the hybrid method, Nonlinear Analysis: Hybrid Systems 4 (4) (2010) 838–852.

[18] A. Kangtunyakarn, A new iterative scheme for fixed point problems of infinite family of κ_i-pseudo contractive mappings, equilibrium problem, variational inequality problems, J. Glob. Optim. 56 (4) (2013) 1543–1562.

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A Method for Solving the Variational Inequality Problem and Fixed Point Problems in Banach Spaces

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Abstract. The purpose of this research is to modify Halpern iteration's process for finding a common element of the set of solutions of a variational inequality problem and the set of fixed points of a strictly pseudo contractive mapping in q-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in q-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

1 Introduction

For the last decades, fixed point theory is a very importance tool for solving the problems in economic, computer science, physics, etc. Throughout this paper, let E be a Banach space with dual space of E^* and let C be a nonempty closed convex subset of E. We use the norm of E and E^* by the same symbol $\|\cdot\|$. We denote weak and strong convergence by notations " $\rightharpoonup$ " and " $\rightarrow$ ", respectively. Let q be a given real number with $q > 1$. The *generalized duality mapping* $J_q : E \rightarrow 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\},$$

for all $x \in E$. If $q = 2$, then $J_2 = J$ is called *normalized duality mapping.*

Remark 1. If J_q is generalized duality mapping of E into 2^{E^*}. Then the following properties are holds:

1. $J_q(tx) = t^{q-1}J_q(x)$, for all $x \in E$ and $t \in [0,\infty)$;
2. $J_q(-x) = -J_q(x)$, for all $x \in E$.

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Definition 1. Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a self-mapping. Then

1. T is called a nonexpansive mapping if
$$\|Tx - Ty\| \leq \|x - y\|,$$
for all $x, y \in C$.

2. T is called an $\eta-$strictly pseudo-contractive mapping if there exists a constant $\eta \in (0, 1)$ such that
$$\langle Tx - Ty, j_q(x - y)\rangle \leq \|x - y\|^2 - \eta\|(I - T)x - (I - T)y\|^2, \tag{1.1}$$
for every $x, y \in C$ and for some $j_q(x - y) \in J_q(x - y)$. It is clear that (1.1) is equivalent to the following
$$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq \eta\|(I - T)x - (I - T)y\|^2, \tag{1.2}$$
for every $x, y \in C$ and for some $j_q(x - y) \in J_q(x - y)$.

Definition 2. Let $C \subseteq E$ be closed convex and Q_C be a mapping of E onto C. The mapping Q_C is said to be sunny if $Q_C(Q_C x + t(x - Q_C x)) = Q_C x$, for all $x \in E$ and $t \geq 0$. A mapping Q_C is called retraction if $Q_C^2 = Q_C$. A subset C of E is called a sunny nonexpansive retraction of E if there exists a sunny nonexpansive retraction from E onto C.

For more information about (sunny) nonexpansive retraction can be found in [13].

The modulas of smootheness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by
$$\rho_E(\tau) = \{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}. \tag{1.3}$$
A Banach space E is uniformly smooth if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$. It is well known that E is q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. In a Hilbert space, $L_p(l_p)$ with $1 < p < \infty$ are q-uniformly smooth. Clearly every q-uniformly smooth Banach space is uniformly smooth. If E is smooth, then J_q is a single valued which is denoted by j_q.

An operator A of C into E is said to be *accretive* if there exists $j_q(x - y) \in J_q(x - y)$ such that
$$\langle Ax - Ay, j_q(x - y)\rangle \geq 0, \ \forall x, y \in C.$$
A mapping $A : C \to E$ is said to be *α-inverse strongly accretive* if there exists $j_q(x-y) \in J_q(x-y)$ and $\alpha > 0$ such that
$$\langle Ax - Ay, j_q(x - y)\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.4}$$

Remark 2. From (1.2) and (1.4), if T is an η-strictly pseudo-contractive mapping, then $I - T$ is η-inverse strongly accretive.

Let C be a nonemty subset of q-uniformly smooth Banach space E and $A : C \to E$ be a nonlinear operator. The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Ax^*, J_q(y - x^*)\rangle \geq 0, \ \ \forall y \in C, \tag{1.5}$$

where J_q is generalized duality mapping from E into 2^{E^*}. The set of solutions of the variational inequality in Banach space is denoted by $S_q(C, A)$. If $q = 2$, then $S_q(C, A)$ is reduced to $S(C, A)$, where $S(C, A)$ is the set of solutions of the generalized variational inequality in Banach spaces proposed by Aoyama et. al,. [1] in 2005. Many research papers have increasingly investigated variational inequality problems in Banach spaces, see, for instance, [2], [3], and the references therein.

In 1967, Halpern [4] introduced the *Halpern's iterative method* as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \forall n \geq 1,$$

where $\alpha_n \in (0, 1)$ satisfying suitable conditions, for all $n \geq 1$. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of mapping T in a real Hilbert space, where T is a nonexpansive mapping. In the last decade, many authors have studied and modified Halpern's iterative method for various nonlinear mappings, see, for instance, [5], [6], [7], [8] and the references therein.

In a uniformly convex and 2-smooth Banach space, Aoyama *et al.* [1] introduced the iterative method for finding a solution of generalized variational inequality problem for an inverse strongly accretive operator in a uniformly convex and 2-uniformly smooth Banach space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n), \forall n \geq 1,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0, 1]$, Q_C is a sunny nonexpansive retraction from E onto C, A is an α-inverse strongly accretive operator. Under suitable conditions, They also proved that the sequence generated by the proposed algorithm weakly converges to a solution of $S(C, A)$.

In 2013, Kangtunyakarn [9] introduced an iterative scheme for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contractive mappings and two sets of solutions of variational inequality problems in a uniformly convex and 2-smooth Banach space as follows:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - aA)x_n + \delta_n Q_C(I - bB)x_n + \eta_n S^A x_n, \forall n \geq 1,$$

where A, B are α and β-inverse strongly accretive mappings, respectively, Q_C is a sunny nonexpansive retraction, S^A is the S^A-mapping generated by a finite family of nonexpansive mappings and a finite family of strictly pseudo-contractive mappings and finite real numbers. He also proved a strong convergence theorem of sequence $\{x_n\}$ under suitable conditions of the parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$, and $\{\eta_n\}$.

Motivated by the results of Aoyama *et al.* [1], Kangtunyakarn [9] and by the ongoing research in this direction, we have the following question.

Question Can we prove a strong convergence theorem of two nonlinear mapping in q-uniformly smooth Banach space ?

The purpose of this manuscript is to modify Halpern iteration's process in order to answer the question above and prove a strong convergence theorem for finding a common element of the set of solutions of (1.5) and the set of fixed points of a strictly pseudo contractive mapping in q-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in q-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

2 Preliminaries

The following lemmas are important tool to prove our main results in the next section.

Lemma 2.1. *Let E be a Banach space and let $J_q : E \to 2^{E^*}$, $1 < q < \infty$ be the generalized duality mapping. Then for any $x, y \in E$, there exists $j_q(x+y) \in J_q(x+y)$ such that $\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle$.*

Lemma 2.2. *[10] Let C be a closed and convex subset of a real uniformly smooth Banach space E and $T : C \to C$ a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then there exists a unique sunny nonexpansie retraction $Q_{F(T)} : C \to F(T)$ such that*

$$\limsup_{n\to\infty}\langle u - Q_{F(T)}u, J_q(x_n - Q_{F(T)}u)\rangle \leq 0,$$

for any given $u \in C$.

Lemma 2.3. *[11] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *[12] Let $q > 1$ be a given real number and E be a real Banach space. Then the following statements are equivalent.*

(i) *E is q-uniformly smooth.*

(ii) *There is a constant $C_q > 0$ such that for all $x, y \in E$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q.$$

(iii) *There exists a constant d_q such that for all $x, y \in E$ and $t \in [0,1]$,*

$$\|(1-t)x+ty\|^q \ge (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x-y\|^q,$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^{q\cdot}$

Lemma 2.5. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be a λ-strictly pseudo contractive mapping with $F(T)\bigcap F(S) \neq \emptyset$. For every $a \in (0,1)$, defined the mapping $H : C \to C$ by $Hx = T((1-a)I + aS)x$, for all $x \in C$ and $a \in (0,\mu)$ where $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, C_q is the best q-uniformly smooth constant of E. Then $F(H) = F(T)\bigcap F(S)$.*

Proof. It is obvious that $F(T)\bigcap F(S) \subseteq F(H)$. Let $x_0 \in F(H)$ and $x^* \in F(T)\bigcap F(S)$, we have

$$\begin{aligned}
\|x_0 - x^*\|^q =& \|T((1-a)I + aS)x_0 - x^*\|^q \\
\le& \|x_0 - x^* + a(Sx_0 - x_0)\|^q \\
\le& \|x_0 - x^*\|^q + aq\langle Sx_0 - x_0, j_q(x_0 - x^*)\rangle + C_q a^q\|Sx_0 - x_0\|^q \\
=& \|x_0 - x^*\|^q + aq\langle Sx_0 - x^*, j_q(x_0 - x^*)\rangle + aq\langle x^* - x_0, j_q(x_0 - x^*)\rangle \\
& + C_q a^q\|Sx_0 - x_0\|^q \\
\le& \|x_0 - x^*\|^q + aq(\|x_0 - x^*\|^q - \lambda\|x_0 - Sx_0\|^q) - aq\|x^* - x_0\|^q \\
& + C_q a^q\|Sx_0 - x_0\|^q \\
=& \|x_0 - x^*\|^q - a(q\lambda - C_q a^{q-1})\|x_0 - Sx_0\|^q. \qquad (2.1)
\end{aligned}$$

From above it implies that $x_0 \in F(S)$. From the definition of H, we have

$$x_0 = Hx_0 = T((1-a)I + aS)x_0 = Tx_0.$$

Then $x_0 \in F(T)$. We can conclude that $x_0 \in F(S)\bigcap F(T)$. Hence $F(H) \subseteq F(S)\bigcap F(T)$. Applying (2.1), we have H is a nonexpansive mapping. □

Example 1. Let $S:\mathbb{R}^+\to\mathbb{R}^+$ defined by $Sx=\frac{x^2}{x+1}$, for all $x\in\mathbb{R}^+$ and let $T:\mathbb{R}^+\to\mathbb{R}^+$ defined by $Tx=\frac{3x}{4}$, for all $x\in[0,5]$. Define the mapping $H:\mathbb{R}^+\to\mathbb{R}^+$ by $Hx=T(\frac{9}{10}I+\frac{1}{10}S)x$ for all $x\in\mathbb{R}^+$. From Lemma 2.5, we have $F(H)=F(S)\bigcap F(T)=\{0\}$

Lemma 2.6. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let $j,j_q:E\to E^*$ be a normalized duality mapping and generalized duality mapping, respectively. Let Q_C be a retraction from E onto C. Then the following are equivalent.*

(i) *Q_C is both sunny and nonexpansive,*

(ii) *$\langle x-Q_Cx, J(y-Q_Cx)\rangle\le 0$, for all $x\in E$ and $y\in C$,*

(iii) *$\langle x-Q_Cx, J_q(y-Q_Cx)\rangle\le 0$, for all $x\in E$ and $y\in C$.*

Proof. From [13], we have $(i)\Leftrightarrow(ii)$. Then we only show that (ii) equivalent to (iii). Since $J_q(x)=\|x\|^{q-1}J(x)$, for all $x\in E$. For every $x\in E$ and $y\in C$.
If $y-Q_Cx\neq 0$, we have

$$\langle x-Q_Cx, J_q(y-Q_Cx)\rangle\le 0\Leftrightarrow\langle x-Q_Cx, J(y-Q_Cx)\rangle\le 0.$$

If $y-Q_Cx=0$, we have

$$\langle x-Q_Cx, J_q(y-Q_Cx)\rangle=\langle x-Q_Cx, J(y-Q_Cx)\rangle=0.$$

From above we can conclude the desire result. □

Remark 3. Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let $x\in E, x_0\in C$. From Lemma 2.6, we have

$$x_0=Q_Cx\Leftrightarrow\langle x-x_0, J_q(y-x_0)\rangle\le 0,\forall y\in C.$$

Lemma 2.7. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let $A:C\to E$ be a mapping. Then $S_q(C,A)=F(Q_C(I-\lambda A))$, for all $\lambda>0$, where $S_q(C,A)=\{u\in C:\langle Au, J_q(y-u)\rangle\ge 0,\forall y\in C\}$.*

Proof. Let $x^*\in F(Q_C(I-\lambda A))$, for all $\lambda>0$. Then $x^*=Q_C(I-\lambda A)x^*$. From 2.6, we have

$$\langle (I-\lambda A)x^*-x^*, J_q(y-x^*)\rangle\le 0,\forall y\in C.$$

It follows that

$$\langle Ax^*, J_q(y-x^*)\rangle\ge 0,\forall y\in C.$$

Then $x^*\in S_q(C,A)$. Hence $F(Q_C(I-\lambda A))\subseteq S_q(C,A)$. Similarly, we can conclude that $S_q(C,A)\subseteq F(Q_C(I-\lambda A))$.

□

3 Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $S : C \to C$ be λ-strictly pseudo contractive mapping and $A : C \to E$ be a α-inverse strongly accretive operator with $\mathcal{F} = F(S) \bigcap S_q(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) Q_C (I - \rho A)(aI + (1-a)S)x_n, \forall n \in \mathbb{N}, \tag{3.1}$$

where $\alpha_n \in [0,1]$, $a \in (0,1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where C_q is the q-uniformly smooth constant of E;*

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is a unique sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. First, we show that $Q_C(I - \rho A)$ is a nonexpansive mapping. Let $x, y \in C$, we have

$$\begin{aligned}
\|Q_C(I-\rho A)x - Q_C(I-\rho A)y\|^q \leq & \|x - y - \rho(Ax - Ay)\|^q \\
\leq & \|x-y\|^q - \rho q \langle Ax - Ay, j_q(x-y)\rangle + C_q \rho^q \|Ax - Ay\|^q \\
\leq & \|x-y\|^q - \rho q \alpha \|Ax - Ay\|^q + C_q \rho^q \|Ax - Ay\|^q \\
\leq & \|x-y\|^q - \rho (q\alpha - C_q \rho^{q-1}) \|Ax - Ay\|^q \\
\leq & \|x-y\|^q.
\end{aligned}$$

Then $Q_C(I - \rho A)$ is a nonexpansive mapping. Next we show that the sequence $\{x_n\}$ is bounded. Put $Wx = Q_C(I - \rho A)(aI + (1-a)S)x$, for all $x \in C$. From Lemma 2.5 and 2.7, we have

$$F(W) = F(Q_C(I-\rho A)) \bigcap F(S) = S_q(C, A) \bigcap F(S)$$

and W is a nonexpansive mapping. From (3.1), we can rewrite that

$$x_{n+1} = \alpha_n u + (1-\alpha_n) W x_n, \forall n \in \mathbb{N}. \tag{3.2}$$

Let $x^* \in \mathcal{F}$ and the definition of x_n, we have

$$\|x_{n+1} - x^*\| \leq \alpha_n \|u - x^*\| + (1-\alpha_n)\|Wx_n - x^*\|$$

$$\begin{aligned}&\leq \alpha_n\|u-x^*\|+(1-\alpha_n)\|x_n-x^*\|\\&\leq \max\{\|u-x^*\|,\|x_1-x^*\|\}.\end{aligned}$$

Applying induction, we have $\{x_n\}$ is bounded. From the definition of $\{x_n\}$, we have

$$\begin{aligned}\|x_{n+1}-x_n\| \leq&|\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|Wx_n-Wx_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Wx_{n-1}\|\\ \leq&|\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Wx_{n-1}\|.\end{aligned}$$

Since $\{x_n\}$ is bounded sequnce, the condition (iv) and Lemma 2.3, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

From (3.2), we have

$$x_{n+1}-x_n=\alpha_n(u-x_n)+(1-\alpha_n)(Wx_n-x_n). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\lim_{n\to\infty}\|Wx_n-x_n\|=0. \tag{3.5}$$

From Lemma 2.2 and (3.5), we have

$$\limsup_{n\to\infty}\langle u-z_0,j_q(x_n-z_0)\rangle\leq 0, \tag{3.6}$$

where $z_0=Q_{\mathcal{F}}u$. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0=Q_{\mathcal{F}}u$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-z_0\|^q\leq&\|\alpha_n(u-x^*)+(1-\alpha_n)(Wx_n-z_0)\|^q\\ \leq&(1-\alpha_n)\|x_n-z_0\|^q+q\alpha_n\langle u-z_0,j_q(x_{n+1}-z_0)\rangle.\end{aligned}$$

From Lemma 2.3 and (3.6), we have the sequence $\{x_n\}$ converges strongly to $z_0=Q_{\mathcal{F}}u$. □

By using the method of proof in Theorem 3.1, we have the following theorems.

Theorem 3.2. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $S:C\to C$ be be λ-strictly pseudo contractive mapping and $T:C\to E$ be a nonexpansive mapping with $\mathcal{F}=F(S)\bigcap F(T)\neq\emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1,u\in C$ and*

$$x_{n+1}=\alpha_nu+(1-\alpha_n)T(aI+(1-a)S)x_n,\forall n\in\mathbb{N}, \tag{3.7}$$

where $\alpha_n\in[0,1]$, $a\in(0,1)$ and $\rho>0$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where* C_q *is the* q*-uniformly smooth constant of* E*;*

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = Q_{\mathcal{F}}u$*, where* $Q_{\mathcal{F}}$ *is a unique sunny nonexpansive retraction of* C *onto* $\mathcal{F}$*.*

Proof. Applying the method of Theorem 3.1 and Lemma 2.5, we can conclude the desired result. □

4 Application

In this section, we use the main results to obtain fixed points theorems for a finite family of strictly pseuso contractive mappings in q-uniformly smooth Banach space. Before prove this theorems, we need the following results.

Lemma 4.1. *[14] Let* E *be a smooth Banach space and* C *be a nonempty convex subset of* E*. Given an integer* $N \geq 1$*, assume that for each* $i \in \Lambda$*,* $T_i : C \to C$ *is a* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$*. Assume that* $\{\eta_i\}_{i=1}^{N}$ *is a positive sequence such that* $\sum_{i=1}^{N} \eta_i = 1$*, then* $\sum_{i=1}^{N} \eta_i T_i : C \to C$ *is a* λ_i*-strict pseudocontraction with* $\lambda = min\{\lambda_i : 1 \leq i \leq N\}$*.*

Lemma 4.2. *[14] Let* E *be a smooth Banach space and* C *be a nonempty convex subset of* E*. Given an integer* $N \geq 1$*, assume that for each* $i \in \Lambda$*,* $\{T_i\}_{i=1}^{N} : C \to C$ *is a finite family of* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$ *such that* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$*. Assume that* $\{\eta_i\}_{i=1}^{N}$ *is a positive sequence such that* $\sum_{i=1}^{N} \eta_i = 1$*. Then* $F(\sum_{i=1}^{N} \eta_i T_i) = F$

Theorem 4.1. *Let* C *be a nonempty closed convex subset of* q*-uniformly smooth Banach space* E *and let* Q_C *be a sunny nonexpansive retraction from* E *onto* C*. Let* $T_i : C \to C$ *is a* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$ *and* $A : C \to E$ *be a* α*-inverse strongly accretive operator with* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \bigcap S_q(C, A) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by* $x_1, u \in C$ *and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Q_C (I - \rho A)(aI + (1 - a) \sum_{i=1}^{N} \eta_i T_i) x_n, \forall n \in \mathbb{N}, \tag{4.1}$$

where $\{\eta_i\}_{i=1}^{N}$ *is a positive sequence such that* $\sum_{i=1}^{N} \eta_i = 1$*,* $\alpha_n \in [0, 1]$*,* $a \in (0, 1)$ *and* $\rho > 0$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $a \in (0, \mu)$*, where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$*, where* C_q *is the* q*-uniformly smooth constant of* E*;*

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the unique sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. From Theorem 3.1, Lemma 4.1 and 4.2, we can conclude the desired result. □

Lemma 4.3. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let $S : C \to C$ be κ-strictly pseudo contractive mapping with $F(S) \neq \emptyset$. Then $F(S) = S_q(C, I - S)$.*

Proof. Obvious that $F(S) \subseteq S_q(C, I - S)$. Let $x_0 \in S_q(C, I - S)$ and $x^* \in F(S)$. Then

$$\langle (I - S)x_0, j_q(y - x_0)\rangle \geq 0, \forall y \in C.$$

Put $A = I - S$. Since $S : C \to C$ is κ-strictly pseudo contractive mapping, then there exists $j_q(x_0 - x^*)$ such that

$$\begin{aligned}
\langle Sx_0 - Sx^*, j_q(x_0 - x^*)\rangle =& \langle (I - A)x_0 - (I - A)x^*, j_q(x_0 - x^*)\rangle \\
=& \langle x_0 - x^*, j_q(x_0 - x^*)\rangle - \langle Ax_0 - Ax^*, j_q(x_0 - x^*)\rangle \\
=& \|x_0 - x^*\|^q - \langle (I - S)x_0, j_q(x_0 - x^*)\rangle \\
\leq& \|x_0 - x^*\|^q - \kappa\|(I - S)x_0\|^q.
\end{aligned}$$

It implies that

$$\kappa\|(I - S)x_0\|^q \leq \langle (I - S)x_0, j_q(x_0 - x^*)\rangle \leq 0.$$

Then $x_0 \in F(S)$. Hence $S_q(C, I - S) \subseteq F(S)$. □

Corollary 4.2. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $T_i : C \to C$ is a λ_i-strictly pseudo contractive mapping for some $0 \leq \lambda_i < 1$ and $S : C \to E$ be a α-strictly pseudo contractive mapping with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \bigcap F(S) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_C(I - \rho(I - S))(aI + (1 - a)\sum_{i=1}^{N} \eta_i T_i)x_n, \forall n \in \mathbb{N}, \quad (4.2)$$

where $\{\eta_i\}_{i=1}^{N}$ is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$, $\alpha_n \in [0, 1]$, $a \in (0, 1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where* C_q *is the* q*-uniformly smooth constant of* E;

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converses strongly to* $z_0 = Q_{\mathcal{F}}u$, *where* $Q_{\mathcal{F}}$ *is a unique sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof. From Theorem 4.1 and Lemma 4.3, we can conclude the desired result. □

5 Example and Numerical results

In this section, we give numerical results to illustrate the main theorem.

Example 2. Let $\mathbb{R}$ be a set of real number. Let $S : [0, 10] \to [0, 1]$ be a mapping defined by $Sx = \frac{2x^2}{x+2x}$, for all $x \in [0, 1]$ and let $A : [0, 10] \to \mathbb{R}$ defined by $Ax = 3x^2$ for all $x \in [0, 10]$. Suppose the sequence $\{x_n\}$ generated by (3.1), where $\alpha_n = \frac{1}{60n}$, $\rho = \frac{1}{100}$, and $a = \frac{1}{80}$. Then the sequence $\{x_n\}$ converses strongly to 0.
Solution. It is obvious that S is $\frac{1}{50}$-strictly pseudo contractive mapping and A is $\frac{1}{60}$-inverse strongly accretive operator with $F(S) \bigcap S_2(C, A) = \{0\}$. Since $\{x_n\}$ generated by (3.1), we have

$$x_{n+1} = \frac{1}{60n}u + \left(1 - \frac{1}{60n}\right) Q_{[0,10]} \left(I - \frac{1}{100}A\right) \left(\frac{1}{80}I + (1 - \frac{1}{80})S\right) x_n, \tag{5.1}$$

where $u, x_1 \in [0, 10]$. It is easy to see that α_n, for all $n \geq 1$, a, ρ satisfied all condition in Theorem 3.1. From Theorem 3.1, we have the sequence $\{x_n\}$ coonvergence strongly to 0.
Putting $u = 0.55$ and $x_1 = 0.99$ in (5.1), we have the numerical results as shown in the following Figure 1 and Table 1.

n	x_n
1	0.990000
2	0.649212
3	0.430824
4	0.287969
5	0.193551
$\vdots$	$\vdots$
46	0.000650
47	0.000635
48	0.000621
49	0.000607
50	0.000594

Table 1: The values of the sequences $\{x_n\}$ with initial values $u = 0.55$, $x_1 = 0.99$ and $n = N = 50$.

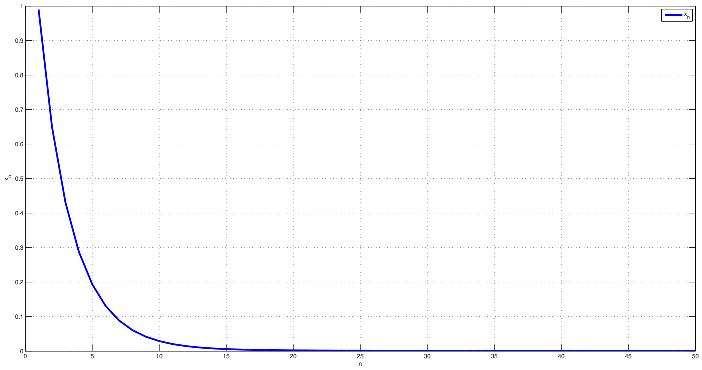

Figure 1: The behavior of the sequences $\{x_n\}$ with initial values $u = 0.55$, $x_1 = 0.99$ and $n = N = 50$.

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References

[1] K. Aoyama, H. Iiduka and W. Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces, Fixed Point Theory Appl., **2006** (2006), 35390.

[2] X. Qina, S. Y. Cho and S. M. Kang, Convergence of an iterative algorithm for systems of

variational inequalities and nonexpansive mappings with applications, J.Comput.Anal.Appl. **233** (2009), 231–240.

[3] Y. Yao, S. Maruster, Strong convergence of an iterative algorithm for variational inequalities in Banach spaces, Math. Comput. Modelling., **54** (2011), 325–329

[4] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc., **73** (1967), 957–961.

[5] Y. Li, H. Liu and K. Zheng, Halpern's iteration for Bregman strongly nonexpansive multi-valued mappings in reflexive Banach spaces with application, Fixed Point Theory Appl., **2013** (2013), 197.

[6] H. C. Wu, C. Z. Cheng, Modified Halpern-type iterative methods for relatively nonexpansive mappings and maximal monotone operators in Banach spaces, Fixed Point Theory Appl., **2014** (2014), 237.

[7] J. G. Falset, E. Llorens-Fuster, G. Marino and A. Rugiano, On Strong Convergence of Halpern's Method for Quasi-Nonexpansive Mappings in Hilbert Spaces, Math. Modell. and Anal., **21(1)** (2016), 63–82.

[8] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. **75** (1980), 287–292,

[9] A. Kangtunyakarn, A new mapping for finding a common element of the sets of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and two sets of variational inequalities in uniformly convex and 2-smooth Banach spaces, Fixed Point Theory Appl., **2013** (2013), 157.

[10] H. Zhou, Convergence theorems for λ-strict pseudocontractions in q-uniformly smooth Banach spaces, Acta Math. Sin. (Engl. Ser.), **26 (4)** (2010), 743–758.

[11] H. K. Xu, An iterative approach to quadric optimization, J. Optim. Theory Appl, **116** (2003), 659–678.

[12] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. **16** (1991), 1127–1138.

[13] S. Reich, Asymtotic behavior of contractions in Banach spaces, J. Math. Anal. Appl., **44** (1973), 57–70.

[14] H. Zhou, Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces, Nonlinear Anal. **68(10)** (2008), 2977–2983.

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The Convergence Results for an AK-Generalized Nonexpansive Mapping in Hilbert Spaces

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Abstract In this paper, we introduce a new class of nonexpansive type of mapping namely, AK-generalized nonexpansive mapping, which is more general than an α-nonexpansive mapping. Moreover, we obtain convergence results of the viscosity approximation method for an AK-generalized nonexpansive semigroups under some assumptions in Hilbert spaces. Furthermore, we prove a strong convergence theorem for a family of AK-generalized nonexpansive mapping in Hilbert spaces.

MSC: 47H05; 47H10; 47J25

Keywords: fixed point; AK-generalized nonexpansive mapping; convergence theorem; Hilbert space

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1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in C : Tx = x\}$. The mapping $T : C \to C$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for any $x, y \in C$. In 1965, Browder [1] shown that if a nonexpansive mapping $T : H \to H$ of a Hilbert space H into itself is asymptotically regular and has at least one fixed point then, for any $x \in H$, a weak limit of a weakly convergent subsequence of the sequence of successive approximations $T^n x$ is a fixed point of T.

In 2011, Aoyama and Kohsaka [2] introduced the class of α-nonexpansive mappings in Banach spaces as follows: Let E be a Banach space and let C be a nonempty subset of

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E. A mapping $T : C \to E$ is said to be α-nonexpansive for some real number $0 \leq \alpha < 1$ if

$$||Tx - Ty|| \leq \alpha||Tx - y|| + \alpha||Ty - x|| + (1 - 2\alpha)||x - y||,$$

for all $x, y \in C$. Clearly, 0-nonexpansive maps are exactly nonexpansive maps. This mapping was generalized and extanded by many authors in several directions; see for instance [3, 4] and references therein.

One of the most interesting iteration processes is the viscosity approximation method introduced by Moudafi [5]. In 2004, Xu [6] studied such method for a nonexpansive mapping in a Hilbert space and introduced an iterative scheme for finding the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$x_1 \in C, x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 1,$$

where $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, $f : C \to C$ is a contraction, and $\{\alpha_n\} \subseteq (0, 1)$. Then, they proved a strong convergence theorem under suitable conditions of the sequence $\{\alpha_n\}$.

Over the past few decades, the convergence theorem was extended and improved in many directions (see [7], [8]) due to its applications are desirable and can be used in real-world applications. So, many authors have been trying to construct new iterations to prove strong convergence theorems for nonexpansive semigroups; see for instance [9–11] and references therein. Especially, in 2008, Song and Xu [12] introduced the following implicit and explicit viscosity iterative schemes,

$$\begin{aligned} x_n &= \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, n \geq 1. \end{aligned}$$

Then they proved strong convergence theorems of a nonexpansive semigroup under suitable conditions. Very recently, Song *et al.* [13] proved a strong convergence theorem of the Halpern iteration for an α-nonexpansive semigroup in Hilbert spaces under suitable conditions as the following schemes,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, n \geq 1. \tag{1.1}$$

Moreover, they also proved some strong convergence theorems of Halperns iteration defined by a such iterative method for a family $\{T_n\}$ of α-nonexpansive mappings.

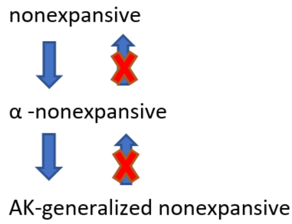
Our work improves and generalizes some of the results obtained in the above paper, we introduce a new class of nonexpansive type of mapping namely, *AK-generalized nonexpansive mapping*, which is more general than an α-nonexpansive mapping in Hilbert spaces as follow.

Definition 1.1. Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $T : C \to C$ is said to satisfy condition (AK) (or AK-generalized nonexpansive) for some real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$ if

$$\begin{aligned} ||Tx - Ty|| \leq{} & \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Ty - x|| \\ & + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||, \end{aligned} \tag{1.2}$$

for all $x, y \in C$.

Notice that the class of AK-generalized nonexpansive mappings covers several well-known mappings. For example, every α-nonexpansive mappings is an AK-generalized nonexpansive mapping and also 0-nonexpansive maps are exactly nonexpansive maps. Hence we have the following diagram.



The following example shows that the reverse implication does not hold.

Example 1.2 ([14])**.** Let $X = \{(0,0),(2,0)(0,4),(4,0),(4,5),(5,4)\}$ be a subset of $\mathbb{R}^2$ with dictionary order. Define a inner product $(X, \langle \cdot\,,\cdot \rangle = ||\cdot\,,\cdot||)$. by $||x_1, x_2|| = (|x_1| + |x_1|)^2$. Then $(X, \langle \cdot\,,\cdot \rangle)$ is a Hilbert space. Define a mapping $T : X \to X$ by

$$T(0,0) = (0,0),\ T(2,0) = (0,0),\ T(0,4) = (0,0),$$
$$T(4,0) = (2,0),\ T(4,5) = (4,0),\ T(5,4) = (0,4).$$

Then, we have T is not an α-nonexpansive mapping for any $\alpha < 1$, and $x = (4,5)$ and $y = (5,4)$, but, we consider $||T(x) - x|| = 25$ and $||T(y) - y|| = 25$. Then, we have

$$\begin{aligned} ||Tx - Ty|| = 64 < \frac{3}{4}100 + \frac{1}{4}4 & \\ \le \alpha_1 25 + \alpha_2 25 + \alpha_3 25 + \alpha_4 25 + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})4 & \\ \le \alpha_1 ||Tx - x|| + \alpha_2 ||Ty - y|| + \alpha_3 ||Tx - y|| + \alpha_4 ||Tx - y|| & \\ + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||, & \end{aligned} \tag{1.3}$$

where $\min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \frac{16}{25}$. Thus, T is an AK-generalized nonexpansive.

Example 1.3. Let $X = [0,2]$ be a nonempty closed convex subset of a Hilbert space $(H = \mathbb{R}, \langle \cdot\,,\cdot \rangle = |\cdot|)$. Suppose that $T : [0,2] \to [0,2]$ be given by $Tx = \sin x + \cos x$, for all $x \in [0,2]$. Now, we consider

$$\begin{aligned} ||Tx - Ty|| &= \frac{1}{2}|2\sin x + 2\cos x - 2\sin y - 2\cos y| \\ &\le \frac{1}{2}|\sin x + \cos x - x| + \frac{1}{2}|\sin y + \cos y - y| + \frac{1}{2}|\sin x + \cos x - y| \\ &\quad + \frac{1}{2}|\sin y + \cos y - x| \\ &\le \alpha_1|\sin x + \cos x - x| + \alpha_2|\sin y + \cos y - y| + \alpha_3|\sin x + \cos x - y| \\ &\quad + \alpha_4|\sin y + \cos y - x| + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})|x - y| \\ &= \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Tx - y|| \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||, \end{aligned} \tag{1.4}$$

where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$. Then T is an AK-generalized nonexpansive.

Our work improves and generalizes some of the results obtained in the above paper. We introduce the AK-generalized nonexpansive mapping as generalization of an α-nonexpansive mapping. We also discuss sucient and necessary conditions of some property for such mappings and obtain a convergence result of the viscosity approximation method for an AK-generalized nonexpansive semigroups under some assumptions in Hilbert spaces. Moreover, we prove a strong convergence theorem for a family of AK-generalized nonexpansive mapping in Hilbert spaces.

2. Preliminaries

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, there exists the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

For any $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

In a real Hilbert space H, it is well known that H satisfies *Opial's condition*, *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\| ,$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1 ([15]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \in \mathbb{N},$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$\text{(1)} \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ (2)} \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2. *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,$$

for all $x, y \in H$.

Now, we introduce the definitions follow on the results of Song *et al.* [13].
Let E be a Banach space. An (one-parameter) AK-generalized nonexpansive semigroup is a family $\mathcal{T} = \{T(t) : t > 0\}$ of mappings $D(\mathcal{T}) = \bigcap_{t>0} D(T(t))$ and range $R(T)$ such that

(1): $T(0)x = x$ for all $x \in D(\mathcal{T})$;
(2): $T(t + s)x = T(t)T(s)x$ for all $t, s > 0$ and $x \in D(\mathcal{T})$;
(3): for each $t > 0$, $T(t)$ is an AK-generalized nonexpansive mapping, that is,

$$\begin{aligned} ||Tx - Ty|| \leq\ & \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Ty - x|| \\ & + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||, \end{aligned} \tag{2.1}$$

for all $x, y \in C$, $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$.

Example 2.3. Let $X = [0,2]$ be a nonempty closed convex subset of a Hilbert space $(H = \mathbb{R}, \langle \cdot, \cdot \rangle = |\cdot|)$. Suppose that $T : [0,2] \to [0,2]$ be given by $Tx = 3^{-x}$, for all $x \in [0,2]$. Now, for any $t, s > 0$ and $x \in D(\mathcal{T})$;

(1) $T(0)x = 3^0 x = x$;

(2) $T(t+s)x = 3^{-(t+s)}x = 3^{-(t)}3^{-(s)}x = T(t)T(s)x$;

(3) for each $t > 0$, $T(t)$ is an AK-generalized nonexpansive mapping, that is,

$$\begin{aligned}
||Tx - Ty|| &= |3^{-x} - 3^{-y}| \\
&= \frac{1}{2}|2(3^{-x} - 3^{-y})| \\
&= \frac{1}{2}|(3^{-x} - 3^{-y}) + (3^{-x} - 3^{-y}) + x - x + y - y| \\
&= \frac{1}{2}|3^{-x} - x - 3^{-y} + y + 3^{-x} - y - 3^{-y} + x| \\
&= \frac{1}{2}|(3^{-x} - x) - (3^{-y} - y) + (3^{-x} - y) - (3^{-y} - x)| \\
&\leq \frac{1}{2}|3^{-x} - x| + \frac{1}{2}|3^{-y} - y| + \frac{1}{2}|3^{-x} - y| + \frac{1}{2}|3^{-y} - x| \\
&\leq \alpha_1|3^{-x} - x| + \alpha_2|3^{-y} - y| + \alpha_3|3^{-x} - y| \\
&\quad + \alpha_4|3^{-y} - x| + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})|x - y| \\
&= \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Tx - y|| \\
&\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||,
\end{aligned}$$

where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$.

Let $\mathcal{T} = \{T(t) : t > 0\}$ stants for one-parameter AK-generalized nonexpansive semigroup and $F(\mathcal{T}) = \bigcap_{t>0} F(T(t))$. We give the concept of the uniformly asymptotically regular as the following definitions.

Definition 2.4. An AK-generalized nonexpansive semigroup $\mathcal{T} = \{T(t) : t > 0\}$ is said to be *uniformly asymptotically regular (in short, u.a.r.)* if, for any $s \geq 0$ and any bounded subset K of $D(\mathcal{T})$,

$$\lim_{t\to\infty} \sup_{x\in K} \|T(s)(T(t)x) - T(t)x\| = 0.$$

Definition 2.5. A family $\{T_n\}$ of an AK-generalized nonexpansive mapping is said to be *uniformly asymptotically regular (in short, u.a.r.)* if, for each positive integer m and any bounded subset K of $\bigcap_n D(T_n)$,

$$\lim_{n\to\infty} \sup_{x\in K} \|T_m(T_n x) - T_n x\| = 0.$$

3. Main Results

In this section, we first study some properties of an AK-generalized nonexpansive mapping in a Hilbert space.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be an AK-generalized nonexpansive mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed convex and $||Tx - p|| \leq ||x - p||$ for all $x \in C$ and $p \in F(T)$.*

Proof. Since T is an AK-generalized nonexpansive mapping, for all $x \in C$ and $p \in F(T)$

$$\begin{aligned} ||Tx - p|| &= ||Tx - Tp|| \\ &\leq \alpha_1||Tx - x|| + \alpha_2||Tp - p|| + \alpha_3||Tx - p|| + \alpha_4||Tp - x|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - p|| \\ &\leq \alpha_1(||Tx - p|| + ||p - x||) + \alpha_3||Tx - p|| + \alpha_4||p - x|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - p||, \end{aligned} \tag{3.1}$$

and so

$$||Tx - p|| \leq \frac{1 - 2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{1 - \alpha_1 - \alpha_3}||x - p|| < ||x - p||. \tag{3.2}$$

Let $p, q \in F(T)$, $(0 \leq \lambda \leq 1)$ and set $z = \lambda p + (1 - \lambda)q$. Using the Parallelogram Law, we get

$$\begin{aligned} ||\frac{z - p}{2} - \frac{Tz - p}{2}||^2 + \frac{1}{4}||z - Tz||^2 &= \frac{1}{2}||z - p||^2 + \frac{1}{2}||Tz - p||^2 \\ &\leq ||z - p||^2, \\ ||\frac{z - q}{2} - \frac{Tz - q}{2}||^2 + \frac{1}{4}||z - Tz||^2 &= \frac{1}{2}||z - q||^2 + \frac{1}{2}||Tz - q||^2 \\ &\leq ||z - q||^2. \end{aligned}$$

(3.2) implies that

$$\begin{aligned} ||\frac{z + Tz}{2} - p||^2 = ||\frac{z - p}{2} + \frac{Tz - p}{2}||^2 &\leq ||z - p||^2 - \frac{1}{4}||z - Tz||^2 \\ &= (1 - \lambda)^2||p - q||^2 - \frac{1}{4}||z - Tz||^2, \\ ||\frac{z + Tz}{2} - q||^2 = ||\frac{z - q}{2} + \frac{Tz - q}{2}||^2 &\leq ||z - q||^2 - \frac{1}{4}||z - Tz||^2 \\ &= \lambda^2||p - q||^2 - \frac{1}{4}||z - Tz||^2. \end{aligned}$$

Suppose that $z \neq Tz$. Then, we have

$$||\frac{z + Tz}{2} - p||^2 < (1 - \lambda)^2||p - q||^2, \quad ||\frac{z + Tz}{2} - q||^2 < \lambda^2||p - q||^2.$$

So, we obtain that

$$||p - q|| \leq ||\frac{z + Tz}{2} - p|| + ||\frac{z + Tz}{2} - q|| < (1 - \lambda)||p - q|| + \lambda||p - q|| = ||p - q||,$$

which is a contradiction and so $z = Tz$. Thus $F(T)$ is convex. Now, we show $F(T)$ is closed. Suppose that $\{x_n\} \in F(T)$ with $\lim_{n\to\infty} x_n = x$, it follows from (3.3) that $||x_n - Tx|| = ||x_n - x|| \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} x_n = Tx = x$, Thus $F(T)$ is closed. ∎

Proposition 3.2. *Let H be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be an AK-generalized nonexpansive mapping. If a sequence $\{x_n\}$ in C converges weakly to $x \in C$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then $x = Tx$.*

Proof. Since $\{x_n\}$ is weakly convergent, we have $\{x_n\}$ is bounded. Since

$$||Tx_n|| \leq ||Tx_n - x_n|| + ||x_n||,$$

we get $\{Tx_n\}$ is bounded. If $0 \leq \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$, then

$$\begin{aligned}
||Tx_n - Tx|| &\leq \alpha_1||Tx_n - x_n|| + \alpha_2||Tx - x|| + \alpha_3||Tx_n - x|| + \alpha_4||Tx - x_n|| \\
&\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x|| \\
&\leq \alpha_1||Tx_n - x_n|| + \alpha_2(||Tx - Tx_n|| + ||Tx_n - x_n|| + ||x_n - x||) \\
&\quad + \alpha_3(||Tx_n - x_n|| + |x_n - x||) + \alpha_4(||Tx - Tx_n|| + ||Tx_n - x_n||) \\
&\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x||. \qquad (3.3)
\end{aligned}$$

This implies that

$$\begin{aligned}
||Tx_n - Tx|| &\leq \frac{1 - 2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{1-\alpha_2-\alpha_4}||x_n-x|| + \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{1-\alpha_2-\alpha_4}||Tx_n-x_n|| \\
&\leq ||x_n - x|| + \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n||. \qquad (3.4)
\end{aligned}$$

If $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 0$,

$$\begin{aligned}
||Tx_n - Tx|| &\leq \alpha_1||Tx_n - x_n|| + \alpha_2||Tx - x|| + \alpha_3||Tx_n - x|| + \alpha_4||Tx - x_n|| \\
&\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x|| \\
&\leq \alpha_1||Tx_n - x_n|| + \alpha_2(||Tx - Tx_n|| - ||Tx_n - x_n|| + ||x_n - x||) \\
&\quad + \alpha_3(||x_n - x|| - ||Tx_n - x_n||) + \alpha_4(||Tx - Tx_n|| - ||Tx_n - x_n||) \\
&\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x||. \qquad (3.5)
\end{aligned}$$

This implies that

$$\begin{aligned}
||Tx_n - Tx|| &\leq \frac{1 - 2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{1-\alpha_2-\alpha_4}||x_n-x|| + \frac{\alpha_1-\alpha_2-\alpha_3-\alpha_4}{1-\alpha_2-\alpha_4}||Tx_n-x_n|| \\
&\leq ||x_n - x|| + \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n||. \qquad (3.6)
\end{aligned}$$

In other cases, we obtain that

$$||Tx_n - Tx|| \leq ||x_n - x|| + \frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n||. \qquad (3.7)$$

Thus,

$$\limsup_{n\to\infty} ||Tx_n - Tx|| \leq \limsup_{n\to\infty} ||x_n - x||. \qquad (3.8)$$

Thus, by the properties of a Hilbert space H, we have

$$\begin{aligned}
||x_n - x||^2 &= ||(x_n - Tx) + (Tx - x)||^2 \\
&= ||x_n - Tx||^2 + ||Tx - x||^2 + 2\langle x_n - Tx, Tx - x\rangle \\
&\leq (||x_n - Tx_n|| + ||Tx_n - Tx||)^2 + ||Tx - x||^2 + 2\langle x_n - Tx, Tx - x\rangle.
\end{aligned}$$

Since $\{x_n\}$ weakly converges to $x \in C$, it follows from (3.8) that

$$\begin{aligned}\limsup_{n\to\infty} ||x_n - x||^2 &\leq \limsup_{n\to\infty} ||Tx_n - Tx||^2 + ||Tx - x||^2 \\ &\quad + 2\limsup_{n\to\infty}\langle x_n - Tx, Tx - x\rangle \\ &\leq \limsup_{n\to\infty} ||Tx_n - Tx||^2 + ||Tx - x||^2 + 2\langle x - Tx, Tx - x\rangle \\ &\leq \limsup_{n\to\infty} ||x_n - x||^2 - ||Tx - x||^2\end{aligned}$$

respectively, and hence $||Tx - x||^2 \leq 0$. ■

From the proof of Proposition 3.2, we have the following lemma:

Lemma 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be an AK-generalized nonexpansive mapping. Then*

$$||Tx_n - Tx|| \leq ||x_n - x|| + \frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n|| \tag{3.9}$$

for all $x, y \in C$.

Now, we prove a strong convergence theorem of the viscosity approximation method for an AK-generalized nonexpansive semigroup under some assumptions in a Hilbert space.

Theorem 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{T} = \{T(t) : t > 0\}$ be the u.a.r. semigroup of AK-generalized nonexpansive mappings from C into itself with $F(\mathcal{T}) \neq \emptyset$. Let $f : C \to C$ be a contraction mapping with coefficient $\gamma \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)T(t_n)x_n, n \geq 1, \tag{3.10}$$

where $\{\beta_n\} \subseteq (0,1)$ and $t_n > 0$ satisfy the following conditions:

$$\textit{(i)} \lim_{n\to\infty} \beta_n = 0, \textit{ (ii)} \sum_{n=1}^{\infty} \beta_n = \infty, \textit{ (iii)} \lim_{n\to\infty} t_n = +\infty.$$

Then the sequence $\{x_n\}$ converge strongly to $z_0 = P_{F(\mathcal{T})}f(z_0)$.

Proof. Firstly, we show that the sequence $\{x_n\}$ is bounded. Let $x^* = P_{F(\mathcal{T})}f(x_0)$. From Lemma 3.1, then $\|T(t)x - x^*\| \leq \|x - x^*\|$ for all $x \in C$ and $t > 0$. From the definition of x_n, we get

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \beta_n\|f(x_n) - x^*\| + (1 - \beta_n)\|T(t_n)x_n - x^*\| \\ &\leq \beta_n\|f(x_n) - x^*\| + (1 - \beta_n)\|x_n - x^*\| \\ &\leq \beta_n\|f(x_n) - f(x^*)\| + \beta_n\|f(x^*) - x^*\| + (1 - \beta_n)\|x_n - x^*\| \\ &\leq \beta_n\gamma\|x_n - x^*\| + (1 - \beta_n)\|x_n - x^*\| + \beta_n\|f(x^*) - x^*\| \\ &= (1 - \beta_n(1 - \gamma))\|x_n - x^*\| + \beta_n\|f(x^*) - x^*\|.\end{aligned}$$

By mathematical induction, we obtain that

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\right\}, \forall n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is bounded and so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$.
From the condition $\lim_{n\to\infty} \beta_n = 0$, we have

$$\lim_{n\to\infty} \|x_{n+1} - T(t_n)x_n\| = \lim_{n\to\infty} \beta_n \|f(x_n) - T(t_n)x_n\| = 0. \tag{3.11}$$

Since $\{T(t) : t > 0\}$ is the u.a.r. AK-generalized nonexpansive semigroup, then for $s > 0$,

$$\lim_{n\to\infty} \|T(s)T(t_n)x_n - T(t_n)x_n\| \le \lim_{n\to\infty} \sup_{x\in K} \|T(s)T(t_n)x - T(t_n)x\| = 0. \tag{3.12}$$

where K is any bounded subset of C containing $\{x_n\}$.
From the definition of a AK-generalized nonexpansive and Lemma 3.3, we have

$$\begin{aligned} ||T(s)(T(t_n)x_n) - T(s)x_{n+1}|| \le & ||T(t_n)x_n - x_{n+1}|| \\ & + \frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1 - \alpha_2 - \alpha_4} ||T(s)(T(t_n)x_n) - T(t_n)x_n||. \end{aligned} \tag{3.13}$$

Hence, for all $s > 0$,

$$\begin{aligned} \|x_{n+1} - T(s)x_{n+1}\| &\le \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(s)(T(t_n)x_n)\| \\ &\quad + \ \|T(s)(T(t_n)x_n) + T(s)x_{n+1}\| \\ &\le \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(s)(T(t_n)x_n)\| \\ &\quad + \ ||x_{n+1} - T(t_n)x_n|| \\ &\quad + \ \frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1 - \alpha_2 - \alpha_4} ||T(t_n)x_n - T(s)(T(t_n)x_n)|| \\ &\le 2\|x_{n+1} - T(t_n)x_n\| \\ &\quad + \left(1 + \frac{\alpha_1+|\alpha_2|+|\alpha_3|+|\alpha_4|}{1-\alpha_2-\alpha_4}\right) \|T(t_n)x_n - T(s)(T(t_n)x_n)\|. \end{aligned} \tag{3.14}$$

From (3.11), (3.12), and (3.14), we have

$$\lim_{n\to\infty} \|x_{n+1} - T(s)x_{n+1}\| = 0 \tag{3.15}$$

for all $s > 0$.
Next, we show that $\lim_{n\to\infty} \sup \langle f(z_0) - z_0, x_n - z_0\rangle \le 0$ where $z_0 = P_{F(\mathcal{T})} f(z_0)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle . \tag{3.16}$$

Without loss of generality, we may assume $\{x_{n_k}\} \rightharpoonup \omega$ for some $\omega \in C$. By Lemma 3.1 and (3.15), we have $\omega \in F(T(s))$. Since s is arbitrary, then $\omega \in F(\mathcal{T})$.
Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in F(\mathcal{T})$. By (3.16) and the properties of the metric projection, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle &= \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle \\ &= \langle f(z_0) - z_0, \omega - z_0\rangle \\ &\le 0. \end{aligned} \tag{3.17}$$

Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{F(\mathcal{T})} f(z_0)$. By Lemma 2.2, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\beta_n(f(x_n) - z_0) + (1-\beta_n)(T(t_n)x_n - z_0)\|^2 \\
&\le \|(1-\beta_n)(T(t_n)x_n - z_0)\|^2 + 2\beta_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&= (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\|f(x_n) - f(z_0)\|\|x_{n+1} - z_0\| \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\gamma\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\beta_n)^2\|x_n - z_0\|^2 + \beta_n\gamma\|x_n - z_0\|^2 + \beta_n\gamma\|x_{n+1} - z_0\|^2 \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le \frac{(1-\beta_n)^2 + \beta_n\gamma}{1-\beta_n\gamma}\|x_n - z_0\|^2 + \frac{2\beta_n}{1-\beta_n\gamma}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \left(1 - \frac{2\beta_n(1-\gamma)}{1-\beta_n\gamma}\right)\|x_n - z_0\|^2 + \frac{2\beta_n(1-\gamma)}{1-\beta_n\gamma}\left(\frac{\beta_n}{2(1-\gamma)}\|x_n - z_0\|^2\right. \\
&\quad \left. + \frac{1}{1-\gamma}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\right).
\end{aligned}$$

From the conditions (i),(ii), (3.17), and Lemma 2.1, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})} f(z_0)$. This completes the proof. ∎

By continuing in the same direction as in Theorem 3.4, we obtian the following theorem.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H and $\{T_n\}$ be a family of u.a.r. AK-generalized nonexpansive mappings from C into itself with $\mathcal{F} = F(T_n) \neq \emptyset$. Let $f : C \to C$ be a contraction mapping with coefficient $\gamma \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)T_n x_n, n \ge 1, \tag{3.18}$$

where $\{\beta_n\} \subseteq (0,1)$ and $t_n > 0$ satisfy the following conditions:

$$(i)\ \lim_{n\to\infty}\beta_n = 0,\ (ii)\ \sum_{n=1}^{\infty}\beta_n = \infty.$$

Then the sequence $\{x_n\}$ converge strongly to $z_0 = P_{\mathcal{F}} f(z_0)$.

Proof. Put the terms $T(t_n)$ and $T(s)$ with the terms T_n and T_m in Theorem 3.4. Using Definition 2.5 and the same method of proof in Theorem 3.4, we have the desired conclusion. ∎

Remark 3.6. In this work, we introduce a new class of nonexpansive type of mapping namely, AK-generalized nonexpansive mapping, which is more general than an α-nonexpansive mapping. Moreover, we obtain convergence results of the viscosity approximation method for an AK-generalized nonexpansive semigroups under some assumptions in Hilbert spaces. However, we should like remark the following:

(1) The main theorem of Song *et al.* [13] gave a strong convergence theorem of the Halpern iteration for an α-nonexpansive semigroups in Hilbert spaces by using the iterative scheme (1.1), while the main theorem of this paper give a convergence result for an AK-generalized nonexpansive semigroup by using the iterative scheme (3.10) which is improve and extend than the main results of Song *et al.* [13].

(2) The class of mappings studied in this paper is more general than the class of mappings studied in Aoyama and Kohsaka [2].

(3) We studied the AK-generalized nonexpansive mapping in Hilbert spaces as Aoyama and Kohsaka [2], Muangchoo-in *et al.* [3] and Ariza-Ruiz *et al.* [4] studied α-nonexpansive mappings in Banach space. Moreover, Xu [6] investigated a nonexpansive mapping in Hilbert spaces.

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References

[1] F.E. Browder, Fixed-point theorems for noncompact mappings in Hilbert space, Proc. Nat. Acad. Sci. U.S.A. 53 (1965) 1272–1276.

[2] K. Aoyama, F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, Nonlinear Analysis 74 (2011) 4387–4391.

[3] K. Muangchoo-in, P. Kumam, Y.J. Cho, Approximating common fixed points of two α-nonexpansive mappings, Thai Journal of Mathematics 16 (2018) 139–145.

[4] D. Ariza-Ruiz, C.H. Linares, E. Llorens-Fuster, E. Moreno-Glvez, On α-nonexpansive mappings in banach spaces, Carpathian Journal of Mathematics 32 (2016) 13–28.

[5] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46–55.

[6] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.

[7] N. Nimana, A.P. Farajzadeh, N. Petrot, Adaptive subgradient method for the split quasi-convex feasibility problems, Optimization 65 (2016) 1885–1898.

[8] P. Jailoka, S. Suantai, Split common fixed point and null point problems for demicontractive operators in Hilbert spaces, Optim. Methods Softw. 34 (2) (2019) 248–263.

[9] Y. Yao, J.L. Kang, Y.J. Cho, Y.C. Liou, Approximation of fixed points for nonexpansive semigroups in Hilbert spaces, Fixed Point Theory and Appl. 2013 (2013) Article no. 31.

[10] Y. Shehu, Strong convergence theorem for nonexpansive semigroups and systems of equilibrium problems, J. Glob. Optim. 56 (2013) 1675–1688.

[11] M. Eslamian, J. Vahidi, Split common fixed point problem of nonexpansive semigroup, Mediterr. J. Math. 13 (2016) 1177–1195

[12] Y. Song, S. Xu, Strong convergence theorems for nonexpansive semigroup in Banach spaces, J. Math. Anal. Appl. 338 (2008) 152–161.

[13] Y. Song, K. Muangchoo-in, P. Kumam, Y.J. Cho, Successive approximations for common fixed points of a family of α-nonexpansive mappings, J. Fixed Point Theory Appl. 20 (2018) Article no. 10.

[14] R. Pant, R. Shukla, Approximating fixed points of generalized α-nonexpansive mapping in Banach space, Funct. Anal. Optim. 38 (2017) 248–266.

[15] H.K. Xu, An iterative approach to quadric optimization, J. Optim. Theory Appl. 116 (2003) 659-678.

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Vieta–Fibonacci-like polynomials and some identities*

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Abstract

In this paper, we introduce a new type of the Vieta polynomial, which is Vieta–Fibonacci-like polynomial. After that, we establish the Binet formula, the generating function, the well-known identities, and the sum formula of this polynomial. Finally, we present the relationship between this polynomial and the previous well-known Vieta polynomials.

Keywords: Vieta–Fibonacci polynomial, Vieta–Lucas polynomial, Vieta–Fibonacci-like polynomial

AMS Subject Classification: 11C08, 11B39, 33C45

1. Introduction

In 2002, Horadam [1] introduced the new types of second order recursive sequences of polynomials which are called Vieta–Fibonacci and Vieta–Lucas polynomials respectively. The definition of Vieta–Fibonacci and Vieta–Lucas polynomials are defined as follows:

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Definition 1.1 ([1])**.** For any natural number n the Vieta–Fibonacci polynomials sequence $\{V_n(x)\}_{n=0}^{\infty}$ and the Vieta–Lucas polynomials sequence $\{v_n(x)\}_{n=0}^{\infty}$ are defined by

$$V_n(x) = xV_{n-1}(x) - V_{n-2}(x), \quad \text{for } n \geq 2,$$
$$v_n(x) = xv_{n-1}(x) - v_{n-2}(x), \quad \text{for } n \geq 2,$$

respectively, where $V_0(x) = 0$, $V_1(x) = 1$ and $v_0(x) = 2$, $v_1(x) = x$.

The first few terms of the Vieta–Fibonacci polynomials sequence are $0, 1, x, x^2 - 1, x^3 - 2x, x^4 - 3x^2 + 1$ and the first few terms of the Vieta–Lucas polynomials sequence are $2, x, x^2 - 2, x^3 - 3x, x^4 - 4x^2 + 2, x^5 - 5x^3 + 5x$. The Binet formulas of the Vieta–Fibonacci and Vieta–Lucas polynomials are given by

$$V_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$
$$v_n(x) = \alpha^n(x) + \beta^n(x),$$

respectively. Where $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$ are the roots the characteristic equation $r^2 - xr + 1 = 0$. We also note that $\alpha(x) + \beta(x) = x$, $\alpha(x)\beta(x) = 1$, and $\alpha(x) - \beta(x) = \sqrt{x^2 - 4}$.

Recall that the Chebyshev polynomials are a sequence of orthogonal polynomials which can be defined recursively. The n^{th} Chebyshev polynomials of the first and second kinds are denoted by $\{T_n(x)\}_{n=0}^{\infty}$ and $\{U_n(x)\}_{n=0}^{\infty}$ and are defined respectively by $T_0(x) = 1$, $T_1(x) = x, T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, for $n \geq 2$, and $U_0(x) = 1$, $U_1(x) = 2x, U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, for $n \geq 2$. These polynomials are of great importance in many areas of mathematics, particularly approximation theory. It is well known that the Chebyshev polynomials of the first kind and second kind are closely related to Vieta–Fibonacci and Vieta–Lucas polynomials. So, in [4] Vitula and Slota redefined Vieta polynomials as modified Chebyshev polynomials. The related features of Vieta and Chebyshev polynomials are given as $V_n(x) = U_n\left(\frac{1}{2}x\right)$ and $v_n(x) = 2T_n\left(\frac{1}{2}x\right)$ (see [1, 2, 5]).

In 2013, Tasci and Yalcin [6] introduced the recurrence relation of Vieta–Pell and Vieta–Pell–Lucas polynomials as follows:

Definition 1.2 ([6])**.** For $|x| > 1$ and for any natural number n the Vieta–Pell polynomials sequence $\{t_n(x)\}_{n=0}^{\infty}$ and the Vieta–Pell–Lucas polynomials sequence $\{s_n(x)\}_{n=0}^{\infty}$ are defined by

$$t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x), \quad \text{for } n \geq 2,$$
$$s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x), \quad \text{for } n \geq 2.$$

respectively, where $t_0(x) = 0$, $t_1(x) = 1$ and $s_0(x) = 2$, $s_1(x) = 2x$.

The $t_n(x)$ and $s_n(x)$ are called the n^{th} Vieta–Pell polynomial and the n^{th} Vieta–Pell–Lucas polynomial respectively. Tasci and Yalcin [6] obtained the Binet form and generating functions of Vieta–Pell and Vieta–Pell–Lucas polynomials. Also, they obtained some differentiation rules and the finite summation formulas. Moreover, the following relations are obtained

$$s_n(x) = 2T_n(x), \quad \text{and} \quad t_{n+1}(x) = U_n(x).$$

In 2015, Yalcin et al. [8], introduced and studied the Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials which defined as follows:

Definition 1.3 ([8])**.** For any natural number n the Vieta–Jacobsthal polynomials sequence $\{G_n(x)\}_{n=0}^{\infty}$ and the Vieta–Jacobsthal-Lucas polynomials sequence $\{g_n(x)\}_{n=0}^{\infty}$ are defined by

$$G_n(x) = G_{n-1}(x) - 2xG_{n-2}(x), \quad \text{for } n \geq 2,$$
$$g_n(x) = g_{n-1}(x) - 2xg_{n-2}(x), \quad \text{for } n \geq 2,$$

respectively, where $G_0(x) = 0$, $G_1(x) = 1$ and $g_0(x) = 2$, $g_1(x) = 1$.

Moreover, for any nonnegative integer k with $1-2^{k+2}x \neq 0$, Yalcin et al. [8] also considered the generalized Vieta–Jacobsthal polynomials sequences $\{G_{k,n}(x)\}_{n=0}^{\infty}$ and Vieta–Jacobsthal–Lucas polynomials sequences $\{g_{k,n}(x)\}_{n=0}^{\infty}$ by the following recurrence relations

$$G_{k,n}(x) = G_{k,n-1}(x) - 2^k xG_{k,n-2}(x), \quad \text{for } n \geq 2,$$
$$g_{k,n}(x) = g_{k,n-1}(x) - 2^k xg_{k,n-2}(x), \quad \text{for } n \geq 2,$$

respectively, where $G_{k,0}(x) = 0$, $G_{k,1}(x) = 1$ and $g_{k,0}(x) = 2$, $g_{k,1}(x) = 1$. If $k = 1$, then $G_{1,n}(x) = G_n(x)$ and $g_{1,n}(x) = g_n(x)$. In [8], the Binet form and generating functions for these polynomials are derived. Furthermore, some special cases of the results are presented.

Recently, the generalization of Vieta–Fibonacci, Vieta–Lucas, Vieta–Pell, Vieta–Pell–Lucas, Vieta–Jacobsthal, and Vieta–Jacobsthal–Lucas polynomials have been studied by many authors.

In 2016 Kocer [3], considered the bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials which are generalized of Vieta–Fibonacci, Vieta–Lucas, Vieta–Pell, Vieta–Pell–Lucas polynomials. She also gave some properties. Afterward, she obtained some identities for the bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials by using the known properties of bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials.

In 2020 Uygun et al. [7], introduced the generalized Vieta–Pell and Vieta–Pell–Lucas polynomial sequences. They also gave the Binet formula, generating functions, sum formulas, differentiation rules, and some important properties for these sequences. And then they generated a matrix whose elements are of generalized Vieta–Pell terms. By using this matrix they derived some properties for generalized Vieta–Pell and generalized Vieta–Pell–Lucas polynomial sequences.

Inspired by the research going on in this direction, in this paper, we introduce a new type of Vieta polynomial, which is called Vieta–Fibonacci-like polynomial.

We also give the Binet form, the generating function, the well-known identities, and the sum formula for this polynomial. Furthermore, the relationship between this polynomial and the previous well-known Vieta polynomials are given in this study.

2. Vieta–Fibonacci-like polynomials

In this section, we introduce a new type of Vieta polynomial, called the Vieta–Fibonacci-like polynomials, as the following definition.

Definition 2.1. For any natural number n the Vieta–Fibonacci-like polynomials sequence $\{S_n(x)\}_{n=0}^{\infty}$ is defined by

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x), \quad \text{for } n \geq 2, \tag{2.1}$$

with the initial conditions $S_0(x) = 2$ and $S_1(x) = 2x$.

The first few terms of $\{S_n(x)\}_{n=0}^{\infty}$ are 2, $2x$, $2x^2 - 2$, $2x^3 - 4x$, $2x^4 - 6x^2 + 2$, $2x^5 - 8x^3 + 6x$, $2x^6 - 10x^4 + 12x^2 - 2$, $2x^7 - 12x^5 + 20x^3 - 8x$ and so on. The n^{th} terms of this sequence are called Vieta–Fibonacci-like polynomials.

First, we give the generating function for the Vieta–Fibonacci-like polynomials as follows.

Theorem 2.2 (The generating function)**.** *The generating function of the Vieta–Fibonacci-like polynomials sequence is given by*

$$g(x,t) = \frac{2}{1 - xt + t^2}.$$

Proof. The generating function $g(x,t)$ can be written as $g(x,t) = \sum_{n=0}^{\infty} S_n(x)t^n$. Consider,

$$g(x,t) = \sum_{n=0}^{\infty} S_n(x)t^n = S_0(x) + S_1(x)t + S_2(x)t^2 + \cdots + S_n(x)t^n + \dots .$$

Then, we get

$$\begin{aligned} -xtg(x,t) &= -xS_0(x)t - xS_1(x)t^2 - xS_2(x)t^3 - \cdots - xS_{n-1}(x)t^n - \dots \\ t^2 g(x,t) &= S_0(x)t^2 + S_1(x)t^3 + S_2(x)t^4 + \cdots + S_{n-2}(x)t^n + \dots . \end{aligned}$$

Thus,

$$\begin{aligned} g(x,t)(1 - xt + t^2) &= S_0(x) + (S_1(x) - xS_0(x))t \\ &\quad + \sum_{n=2}^{\infty} (S_n(x) - xS_{n-1}(x) + S_{n-2}(x))t^n \\ &= 2, \end{aligned}$$

$$g(x,t) = \frac{2}{1 - xt + t^2}.$$

This completes the proof. □

Next, we give the explicit formula for the n^{th} Vieta–Fibonacci-like polynomials.

Theorem 2.3 (Binet's formula). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials, then*

$$S_n(x) = A\alpha^n(x) + B\beta^n(x), \tag{2.2}$$

where $A = \frac{2(x-\beta(x))}{\alpha(x)-\beta(x)}$, $B = \frac{2(\alpha(x)-x)}{\alpha(x)-\beta(x)}$ and $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$, $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$ are the roots of the characteristic equation $r^2 - xr + 1 = 0$.

Proof. The characteristic equation of the recurrence relation (2.1) is $r^2 - xr + 1 = 0$ and the roots of this equation are $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$.
It follows that

$$S_n(x) = d_1\alpha^n(x) + d_2\beta^n(x),$$

for some real numbers d_1 and d_2. Putting $n = 0$, $n = 1$, and then solving the system of linear equations, we obtain that

$$S_n(x) = \frac{2(x-\beta(x))}{\alpha(x)-\beta(x)}\alpha^n(x) + \frac{2(\alpha(x)-x)}{\alpha(x)-\beta(x)}\beta^n(x).$$

Setting $A = \frac{2(x-\beta(x))}{\alpha(x)-\beta(x)}$ and $B = \frac{2(\alpha(x)-x)}{\alpha(x)-\beta(x)}$, we get

$$S_n(x) = A\alpha^n(x) + B\beta^n(x).$$

This completes the proof. □

We note that $A + B = 2$, $AB = -\frac{4}{(\alpha(x)-\beta(x))^2}$, and $A\beta(x) + B\alpha(x) = 0$.

The other explicit forms of Vieta–Fibonacci-like polynomials are given in the following two theorems.

Theorem 2.4 (Explicit form). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n(x) = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} x^{n-2i}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 2.2, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} S_n(x)t^n &= \frac{2}{1-(xt-t^2)} \\ &= 2\sum_{n=0}^{\infty}(xt-t^2)^n\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(xt)^{n-i}(-t^2)^i \\
&= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-1)^i x^{n-i}t^{n+i} \\
&= \sum_{n=0}^{\infty}\left[2\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^i\binom{n-i}{i}x^{n-2i}\right]t^n.
\end{aligned}
$$

From the equality of both sides, the desired result is obtained. This complete the proof. □

Theorem 2.5 (Explicit form)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$
S_n(x) = 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^i\binom{n+1}{2i+1}x^{n-2i}(x^2-4)^i, \quad \textit{for } n \geq 1.
$$

Proof. Consider,

$$
\begin{aligned}
\alpha^{n+1}(x)-\beta^{n+1}(x) &= 2^{-(n+1)}[(x+\sqrt{x^2-4})^{n+1}-(x-\sqrt{x^2-4})^{n+1}] \\
&= 2^{-(n+1)}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}x^{n-i+1}(\sqrt{x^2-4})^i\right. \\
&\quad \left.-\sum_{i=0}^{n+1}\binom{n+1}{i}x^{n-i+1}(-\sqrt{x^2-4})^i\right] \\
&= 2^{-n}\left[\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2i+1}x^{n-2i}(\sqrt{x^2-4})^{2i+1}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_n(x) &= A\alpha^n(x)+B\beta^n(x) \\
&= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)} \\
&= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\sqrt{x^2-4}} \\
&= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2i+1}x^{n-2i}(x^2-4)^i.
\end{aligned}
$$

This completes the proof. □

Theorem 2.6 (Sum formula)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$\sum_{k=0}^{n-1} S_k(x) = \frac{2 - S_n(x) + S_{n-1}(x)}{2 - x}, \quad \text{for } n \geq 1.$$

Proof. By using Binet formula (2.2), we get

$$\begin{aligned}
\sum_{k=0}^{n-1} S_k(x) &= \sum_{k=0}^{n-1} \left(A\alpha^k(x) + B\beta^k(x)\right) \\
&= A\frac{1-\alpha^n(x)}{1-\alpha(x)} + B\frac{1-\beta^n(x)}{1-\beta(x)} \\
&= \frac{A + B - (A\beta(x) + B\alpha(x)) - (A\alpha^n(x) + B\beta^n(x))}{1 - x + 1} \\
&\quad + \frac{A\alpha^{n-1}(x) + B\beta^{n-1}(x)}{1 - x + 1} \\
&= \frac{2 - S_n(x) + S_{n-1}(x)}{2 - x}.
\end{aligned}$$

This completes the proof. □

Since the derivative of the polynomials is always exists, we can give the following formula.

Theorem 2.7 (Differentiation formula)**.** *The derivative of $S_n(x)$ is obtained as the follows.*

$$\frac{\mathrm{d}}{\mathrm{d}x} S_n(x) = \frac{(n+1)v_{n+1}(x) - xV_{n+1}(x)}{2(x^2 - 4)},$$

where $V_n(x)$ and $v_n(x)$ are the n^{th} Vieta–Fibonacci and Vieta–Lucas polynomials, respectively.

Proof. The result is obtained by using Binet formula (2.2). □

Again, by using Binet formula (2.2), we obtain some well-known identities as follows.

Theorem 2.8 (Catalan's identity or Simson identities)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n^2(x) - S_{n+r}(x)S_{n-r}(x) = S_{r-1}^2(x), \quad \text{for } n \geq r \geq 1. \tag{2.3}$$

Proof. By using Binet formula (2.2), we obtain

$$\begin{aligned}
&S_n^2(x) - S_{n+r}(x)S_{n-r}(x) \\
&\qquad = (A\alpha^n(x) + B\beta^n(x))^2 - \left(A\alpha^{n+r}(x) + B\beta^{n+r}(x)\right)\left(A\alpha^{n-r}(x) + B\beta^{n-r}(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -AB\left(\alpha(x)\beta(x)\right)^{n-r}\left(\alpha^{2r}(x) - 2\left(\alpha(x)\beta(x)\right)^{r} + \beta^{2r}(x)\right)\\
&= \frac{4}{(\alpha(x)-\beta(x))^2}\left(\alpha^{r}(x) - \beta^{r}(x)\right)^2\\
&= (A\alpha^{r-1}(x) + B\beta^{r-1}(x))^2\\
&= S_{r-1}^2(x).
\end{aligned}
$$

Thus,

$$S_n^2(x) - S_{n+r}(x)S_{n-r}(x) = S_{r-1}^2(x).$$

This completes the proof. □

Take $r = 1$ in Catalan's identity (2.3), then we get the following corollary.

Corollary 2.9 (Cassini's identity). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n^2(x) - S_{n+1}(x)S_{n-1}(x) = 4, \quad \textit{for } n \geq 1.$$

Theorem 2.10 (d' Ocagne's identity). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_m(x)S_{n+1}(x) - S_{m+1}(x)S_n(x) = 2S_{m-n-1}(x), \quad \textit{for } m \geq n \geq 1. \tag{2.4}$$

Proof. We will prove d' Ocagne's identity (2.4) by using Binet formula (2.2). Consider,

$$
\begin{aligned}
&S_m(x)S_{n+1}(x) - S_{m+1}(x)S_n(x)\\
&\qquad = (A\alpha^{m}(x) + B\beta^{m}(x))\left(A\alpha^{n+1}(x) + B\beta^{n+1}(x)\right)\\
&\qquad\quad - \left(A\alpha^{m+1}(x) + B\beta^{m+1}(x)\right)(A\alpha^{n}(x) + B\beta^{n}(x))\\
&\qquad = -AB\left(\alpha(x)\beta(x)\right)^{n}\left(\alpha(x) - \beta(x)\right)\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)\\
&\qquad = \frac{4}{\left(\alpha(x)-\beta(x)\right)^2}\left(\alpha(x) - \beta(x)\right)\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)\\
&\qquad = 2(A\alpha^{m-n-1}(x) + B\beta^{m-n-1}(x))\\
&\qquad = 2S_{m-n-1}(x).
\end{aligned}
$$

This completes the proof. □

Theorem 2.11 (Honsberger identity). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_{m+1}(x)S_{n+1}(x) + S_m(x)S_n(x) = \frac{4xv_{m+n+3}(x) - 8v_{m-n}(x)}{x^2-4}, \quad \textit{for } m \geq n \geq 1,$$

where $v_n(x)$ is the n^{th} Vieta–Lucas polynomials.

Proof. By using Binet formula (2.2), we obtain

$$\begin{aligned}
S_{m+1}(x)S_{n+1}(x) + S_m(x)S_n(x)& \\
&= \left(A\alpha^{m+1}(x) + B\beta^{m+1}(x)\right)\left(A\alpha^{n+1}(x) + B\beta^{n+1}(x)\right) \\
&\quad + \left(A\alpha^{m}(x) + B\beta^{m}(x)\right)\left(A\alpha^{n}(x) + B\beta^{n}(x)\right) \\
&= xA^2\alpha^{m+n+1}(x) + xB^2\beta^{m+n+1}(x) + 2AB(\alpha^{m-n}(x) + \beta^{m-n}(x)) \\
&= \frac{4x(\alpha^{m+n+3}(x) + \beta^{m+n+3}(x)) - 8(\alpha^{m-n}(x) + \beta^{m-n}(x))}{(\alpha(x) - \beta(x))^2} \\
&= \frac{4xv_{m+n+3}(x) - 8v_{m-n}(x)}{x^2 - 4}.
\end{aligned}$$

This completes the proof. □

In the next theorem, we obtain the relation between the Vieta–Fibonacci-like, Vieta–Fibonacci and the Vieta–Lucas polynomials by using Binet formula (2.2).

Theorem 2.12. *Let $\{S_n(x)\}_{n=0}^{\infty}$, $\{V_n(x)\}_{n=0}^{\infty}$ and $\{v_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Fibonacci-like, Vieta–Fibonacci and Vieta–Lucas polynomials, respectively. Then*

(1) $S_n(x) = 2V_{n+1}(x)$, *for* $n \geq 0$,

(2) $S_n(x) = v_n(x) + xV_n(x)$, *for* $n \geq 0$,

(3) $S_n(x)v_{n+1}(x) = 2V_{2n+2}(x)$, *for* $n \geq 0$,

(4) $S_{n+1}(x) + S_{n-1}(x) = 2xV_{n+1}(x)$, *for* $n \geq 1$,

(5) $S_{n+1}(x) - S_{n-1}(x) = 2v_{n+1}(x)$, *for* $n \geq 1$,

(6) $S^2_{n+2}(x) - S^2_{n-1}(x) = 4xV_{2n+2}(x)$, *for* $n \geq 1$,

(7) $2S_n(x) - xS_{n-1}(x) = 2v_n(x)$, *for* $n \geq 1$,

(8) $S_{n+2}(x) + S_{n-2}(x) = (2x^2 - 4)V_{n+1}(x)$, *for* $n \geq 2$,

(9) $S^2_{n+2}(x) - S^2_{n-2}(x) = 4x(x^2 - 2)V_{2n+2}(x)$, *for* $n \geq 2$,

(10) $v_{n+1}(x) - v_n(x) = \frac{1}{2}(x^2 - 4)S_{n-1}(x)$, *for* $n \geq 1$,

(11) $2v_{n+1}(x) - xv_n(x) = \frac{1}{2}(x^2 - 4)S_{n-1}(x)$, *for* $n \geq 1$,

(12) $4v^2_n(x) + (x^2 - 4)S^2_{n-1}(x) = 8v_n(x)$, *for* $n \geq 1$,

(13) $4v^2_n(x) - (x^2 - 4)S^2_{n-1}(x) = 16$, *for* $n \geq 1$.

Proof. The results (1)–(13) are easily obtained by using Binet formula (2.2). □

3. Matrix Form of Vieta–Fibonacci-like polynomials

In this section, we establish some identities of Vieta–Fibonacci-like and Vieta–Fibonacci polynomials by using elementary matrix methods.

Let Q_s be 2×2 matrix defined by

$$Q_S = \begin{bmatrix} 2x^2 - 2 & 2x \\ -2x & -2 \end{bmatrix}. \tag{3.1}$$

Then by using this matrix we can deduce some identities of Vieta–Fibonacci-like and Vieta–Fibonacci polynomials.

Theorem 3.1. *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials and Q_s be 2×2 matrix defined by* (3.1). *Then*

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Proof. For the proof, mathematical induction method is used. It obvious that the statement is true for $n = 1$. Suppose that the result is true for any positive integer k, then we also have the result is true for $k + 1$. Because

$$\begin{aligned} Q_S^{k+1} &= Q_S^k \cdot Q_S \\ &= 2^{k-1} \begin{bmatrix} S_{2k}(x) & S_{2k-1}(x) \\ -S_{2k-1}(x) & -S_{2k-2}(x) \end{bmatrix} \begin{bmatrix} 2x^2 - 2 & 2x \\ -2x & -2 \end{bmatrix} \\ &= 2^{(k+1)-1} \begin{bmatrix} S_{2(k+1)}(x) & S_{2(k+1)-1}(x) \\ -S_{2(k+1)-1}(x) & -S_{2(k+1)-2}(x) \end{bmatrix}. \end{aligned}$$

By Mathematical induction, we have that the result is true for each $n \in \mathbb{N}$, that is

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \qquad \square$$

Theorem 3.2. *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $2S_{2(m+n)}(x) = S_{2m}(x)S_{2n}(x) - S_{2m-1}(x)S_{2n-1}(x)$,

(2) $2S_{2(m+n)-1}(x) = S_{2m}(x)S_{2n-1}(x) - S_{2m-1}(x)S_{2n-2}(x)$,

(3) $2S_{2(m+n)-1}(x) = S_{2m-1}(x)S_{2n}(x) - S_{2m-2}(x)S_{2n-1}(x)$,

(4) $2S_{2(m+n)-2}(x) = S_{2m-1}(x)S_{2n-1}(x) - S_{2m-2}(x)S_{2n-2}(x)$.

Proof. By Theorem 3.1 and the property of power matrix $Q_s^{m+n} = Q_s^m \cdot Q_s^n$, then we obtained the results. $\square$

By Theorem 3.1 and $S_n(x) = 2V_{n+1}(x)$, we get the following Corollary.

Corollary 3.3. *Let $\{V_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci polynomials and Q_s be 2×2 matrix defined by* (3.1)*. Then*

$$Q_S^n = 2^n \begin{bmatrix} V_{2n+1}(x) & V_{2n}(x) \\ -V_{2n}(x) & -V_{2n-1}(x) \end{bmatrix}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 3.1, we get

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Since $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} Q_S^n &= 2^{n-1} \begin{bmatrix} 2V_{2n+1}(x) & 2V_{2n}(x) \\ -2V_{2n}(x) & -2V_{2n-1}(x) \end{bmatrix} \\ &= 2^n \begin{bmatrix} V_{2n+1}(x) & V_{2n}(x) \\ -V_{2n}(x) & -V_{2n-1}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \end{aligned}$$

This completes the proof. □

By Theorem 3.2 and $S_n(x) = 2V_{n+1}(x)$, we get the following Corollary.

Corollary 3.4. *Let $\{V_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci polynomials. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $V_{2(m+n)+1}(x) = V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)$,

(2) $V_{2(m+n)}(x) = V_{2m+1}(x)V_{2n}(x) - V_{2m}(x)V_{2n-1}(x)$,

(3) $V_{2(m+n)}(x) = V_{2m}(x)V_{2n+1}(x) - V_{2m-1}(x)V_{2n}(x)$,

(4) $V_{2(m+n)-1}(x) = V_{2m}(x)V_{2n}(x) - V_{2m-1}(x)V_{2n-1}(x)$.

Proof. From Theorem 3.2 and $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} V_{2(m+n)+1}(x) &= \frac{1}{2} S_{2(m+n)}(x) \\ &= \frac{1}{4}\left(S_{2m}(x)S_{2n}(x) - S_{2m-1}(x)S_{2n-1}(x)\right) \\ &= \frac{1}{4}\left(2V_{2m+1}(x)2V_{2n+1}(x) - 2V_{2m}(x)2V_{2n}(x)\right) \\ &= V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x). \end{aligned}$$

Thus, we get that (1) holds. By the same argument as above, we get that (2), (3), and (4) holds. This completes the proof. □

By Corollary 3.4 and $S_n(x) = 2V_{n+1}(x)$, we get the following corollary.

Corollary 3.5. *Let $\{S_n(x)\}_{n=0}^{\infty}$ and $\{V_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Fibonacci-like polynomials and Vieta–Fibonacci polynomials, respectively. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $S_{2(m+n)}(x) = 2\left(V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)\right),$

(2) $S_{2(m+n)-1}(x) = 2\left(V_{2m+1}(x)V_{2n}(x) - V_{2m}(x)V_{2n-1}(x)\right),$

(3) $S_{2(m+n)-1}(x) = 2\left(V_{2m}(x)V_{2n+1}(x) + V_{2m-1}(x)V_{2n}(x)\right),$

(4) $S_{2(m+n)-2}(x) = 2\left(V_{2m}(x)V_{2n}(x) + V_{2m-1}(x)V_{2n-1}(x)\right).$

Proof. From Corollary 3.4 and $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} S_{2(m+n)}(x) &= 2V_{2(m+n)+1}(x) \\ &= 2\left(V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)\right). \end{aligned}$$

Thus, we get that (1) holds. By the same argument as above, we get that (2), (3), and (4) holds. This completes the proof. □

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References

[1] A. F. Horadam: *Vieta polynomials*, Fibonacci Q 40.3 (2002), pp. 223–232.

[2] E. Jacobstha: *Uber vertauschbare polynome*, Math. Z. 63 (1955), pp. 244–276, doi: https://doi.org/10.1007/BF01187936.

[3] E. G. Kocer: *Bivariate Vieta–Fibonacci and Bivariate Vieta–Lucas Polynomials*, IOSR Journal of Mathematics 20.4 (2016), pp. 44–50, doi: https://doi.org/10.9790/5728-1204024450.

[4] *On modified Chebyshev polynomials*, J. Math. Anal. App 324.1 (2006), pp. 321–343, doi: https://doi.org/10.1016/j.jmaa.2005.12.020.

[5] N. Robbins: *Vieta's triangular array and a related family of polynomials*, Int. J. Math. Math. Sci 14 (1991), pp. 239–244, doi: https://doi.org/10.1155/S0161171291000261.

[6] D. Tasci, F. Yalcin: *Vieta–Pell and Vieta–Pell–Lucas polynomials*, Adv. Difference Equ 224 (2013), pp. 1–8, doi: https://doi.org/10.1186/1687-1847-2013-224.

[7] S. Uygun, H. Karatas, H. Aytar: *Notes on generalization of Vieta–Pell and Vieta–Pell Lucas polynomials*, International Journal of Mathematics Research 12.1 (2020), pp. 5–22, doi: https://doi.org/10.37624/IJMR/12.1.2020.5-22.

[8] F. Yalcin, D. Tasci, E. Erkus-Duman: *Generalized Vieta–Jacobsthal and Vieta–Jacobsthal Lucas Polynomials*, Mathematical Communications 20 (2015), pp. 241–251.