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On the Vieta–Jacobsthal-like polynomials

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Abstract: In this paper, we first introduce the generalization of the Vieta–Jacobsthal polynomial, which is called the Vieta–Jacobsthal-like polynomial. After that, we give the generating function, the Binet formula, and some well-known identities for this polynomial. Finally, we also present the relation between this polynomial and the previously famous Vieta-polynomials.

Keywords: Vieta–Jacobsthal polynomial, Vieta–Jacobsthal–Lucas polynomial, Generalized Vieta–Jacobsthal polynomial.
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1 Introduction

The theory of the Vieta polynomials was first introduced in 1991 by Robbins [6]. The recursive sequence of the Vieta-Fibonacci polynomial $V_n(x)$ and Vieta-Lucas polynomials $v_n(x)$ were introduced by Horadam [2]. These polynomials are defined by $V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$, and $v_n(x) = xv_{n-1}(x) - v_{n-2}(x)$ for $n \geq 2$, with the initial conditions $V_0(x) = 0, V_1(x) = 1$, and

$v_0(x) = 2, v_1(x) = x$, respectively. These polynomials are closely related to the well-known Chebyshev polynomials of the first and second kinds that are denoted by $T_n(x)$ and $U_n(x)$, respectively. The related features of Vieta and Chebyshev polynomials are given as

$$V_n(x) = U_n\left(\frac{1}{2}x\right) \quad \text{see [2]},$$

$$v_n(x) = 2T_n\left(\frac{1}{2}x\right) \quad \text{see [3,6]}$$

For more application of the Chebyshev polynomials of the first and second kinds, see [4, 5], and the references therein. In recent years, the Vieta polynomials have been much attention to many authors. In 2013, Tasci and Yalcin [7] introduced the recurrence relation of Vieta–Pell polynomials $t_n(x)$ and Vieta–Pell–Lucas polynomials $s_n(x)$ as $t_0(x) = 0$, $t_1(x) = 1$, $t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x)$, for $n \geq 2$, and $s_0(x) = 2$, $s_1(x) = 2x$, $s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x)$, for $n \geq 2$, provide that $|x| > 1$. Tasci and Yalcin [7] obtained the Binet form and generating functions of Vieta–Pell and Vieta–Pell–Lucas polynomials. Also, they received some differentiation rules and the finite summation formulas. Moreover, the following relations are obtained.

$$s_n(x) = 2T_n(x), \quad \text{and} \quad t_{n+1}(x) = U_n(x).$$

Recently, Yalcin et al. [1] introduced and studied the Vieta–Jacobsthal polynomials $G_n(x)$ and Vieta–Jacobsthal–Lucas polynomials $g_n(x)$ which defined respectively by

$$\begin{aligned} G_n(x) &= G_{n-1}(x) - 2xG_{n-2}(x), \quad \text{for } n \geq 2, \\ g_n(x) &= g_{n-1}(x) - 2xg_{n-2}(x), \quad \text{for } n \geq 2, \end{aligned}$$

where $G_0(x) = 0$, $G_1(x) = 1$ and $g_0(x) = 2$, $g_1(x) = 1$. The first few terms of $\{G_n(x)\}_{n=0}^{\infty}$ are $0,\ 1,\ 1,\ -2x+1,\ -4x+1,\ 4x^2-6x+1,\ 12x^2-8x+1, \dots$ and the first few terms of $\{g_n(x)\}_{n=0}^{\infty}$ are $2,\ 1,\ -4x+1,\ -6x+1,\ 8x^2-8x+1,\ 20x^2-10x+1$. The n-th terms of these polynomials sequences are called the Vieta-Jcobsthal and Vieta–Jacobsthal–Lucas polynomials, respectively. The Binet formulas for Vieta–Jacobsthal polynomials $G_n(x)$ and Vieta–Jacobsthal-Lucas polynomials $g_n(x)$ are given by

$$G_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha_3(x) - \beta(x)},$$

$$g_n(x) = \alpha^n(x) + \beta^n(x),$$

for all $n \geq 0$, where $\alpha(x) = \frac{1+\sqrt{1-8x}}{2}$ and $\beta(x) = \frac{1-\sqrt{1-8x}}{2}$ are the roots of the characteristic equation $r^2 - r + 2x = 0$. Moreover, they also introduced the generalization of the Vieta–Jacobsthal and Vieta–Jacobsthal polynomials, and many identities for these polynomials are derived.

In this paper, we investigate the generalization of the Vieta–Jacobsthal polynomial, which is called Vieta-Jacobthal-like polynomial. We give the generating function, the Binet formula, and some well-known identities for this polynomial. Moreover the relation between this polynomial and the Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials are also presented.

2 Vieta–Jacobsthal-like polynomials and some identities

In this section, we investigate some new generalization of the Vieta–Jacobsthal polynomials sequence that has the same recurrence relation as the Vieta–Jacobsthal polynomials sequence and the initial conditions are the combination of the Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials as follows:

Definition 2.1. *For any natural number n the Vieta–Jacobsthal-like polynomials sequence $\{W_n(x)\}_{n=0}^{\infty}$ is defined by*

$$W_n(x) = W_{n-1}(x) - 2xW_{n-2}(x), \quad \textit{for } n \geq 2, \tag{1}$$

with the initial conditions $W_0(x) = 2$ and $W_1(x) = 2$.

From Difinition 2.1, it easy to verify that $W_n(x) = G_n(x) + g_n(x)$, for all n. The first few terms of $\{W_n(x)\}_{n=0}^{\infty}$ are as follows:

$$\begin{aligned}
W_0(x) &= 2, \\
W_1(x) &= 2, \\
W_2(x) &= -4x + 2, \\
W_3(x) &= -8x + 2, \\
W_4(x) &= 8x^2 - 12x + 2, \\
W_5(x) &= 24x^2 - 16x + 2, \\
W_6(x) &= -16x^3 + 48x^2 - 20x + 2, \\
W_7(x) &= -64x^3 + 80x^2 - 24x + 2. \\
&\vdots
\end{aligned}$$

The characteristic equation of (1) is defined by

$$r^2 - r + 2x = 0, \tag{2}$$

and the roots of equation (2) are $\alpha(x) = \frac{1+\sqrt{1-8x}}{2}$ and $\beta(x) = \frac{1-\sqrt{1-8x}}{2}$. We note that $\alpha(x) + \beta(x) = 1$, $\alpha(x) - \beta(x) = \sqrt{1-8x}$, and $\alpha(x)\beta(x) = 2x$.

We first give the generating function for this polynomials sequence as the following Theorem.

Theorem 2.1 (The generating function)**.** *Let $g(x,t) = \sum_{n=0}^{\infty} W_n(x)t^n$ be the generating function of Vieta–Jacobsthal-like polynomials sequence. Then*

$$g(x,t) = \frac{2}{1 - t + 2xt^2}.$$

Proof. Consider,

$$\begin{aligned}
g_W(x,t) &= \sum_{n=0}^{\infty} W_n(x)t^n \\
&= W_0(x) + W_1(x)t + W_2(x)t^2 + \cdots + W_n(x)t^n + \ldots .
\end{aligned}$$

Then, we get

$$-tg_W(x,t) = -W_0(x)t - W_1(x)t^2 - W_2(x)t^3 - ... - W_{n-1}(x)t^n - ...$$
$$2xt^2 g_W(x,t) = 2xW_0(x)t^2 + 2xW_1(x)t^3 + 2xW_4(x)t^2 + ... + 2xW_{n-2}(x)t^n + ...$$

Thus,

$$\begin{aligned}
g(x,t)(1-t+2xt^2) &= W_0(x) + (W_1(x) - W_0(x))\, t \\
&\quad + \sum_{n=2}^{\infty} (W_n(x) - W_{n-1} + 2xW_{n-2}(x))\, t^n \\
&= W_0(x) + (W_1(x) - W_0(x))\, t \\
&= 2 + (2-2)t \\
&= 2 \\
g(x,t) &= \frac{2}{(1-t+2xt^2)}.
\end{aligned}$$

This completes the proof. □

In the following theorem, we give the Binet form for the Vieta–Jacobsthal-like polynomials.

Theorem 2.2 (Binet's formula)**.** *Let* $\{W_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta–Jacobsthal-like polynomials, then*

$$W_n(x) = A\alpha^n(x) + B\beta^n(x), \tag{3}$$

where $A = \frac{2(1-\beta(x))}{\alpha(x)-\beta(x)}$, $B = \frac{2(\alpha(x)-1)}{\alpha(x)-\beta(x)}$ *and* $\alpha(x)$, $\beta(x)$ *are the roots of the characteristic equation* (2)*.*

Proof. Since the roots of the characteristic equation (2) are distinct, we get that

$$W_n(x) = d_1\alpha^n(x) + d_2\beta^n(x), \text{ for all } n \geq 0,$$

for some real numbers d_1 and d_2. Take $n = 0, n = 1$ and then by solving the system of linear equations, we obtain

$$W_n(x) = \frac{2(1-\beta(x))}{\alpha(x)-\beta(x)}\alpha^n(x) + \frac{2(\alpha(x)-1)}{\alpha(x)-\beta(x)}\beta^n(x).$$

Setting $A = \frac{2(1-\beta(x))}{\alpha(x)-\beta(x)}$ and $B = \frac{2(\alpha(x)-1)}{\alpha(x)-\beta(x)}$, we get

$$W_n(x) = A\alpha^n(x) + B_3\beta^n(x).$$

This completes the proof. □

We note that $A + B = 2$, $AB = \frac{8x}{(\alpha(x)-\beta(x))^2}$, and $A - 1 = \frac{1}{\alpha(x)-\beta(x)} = 1 - B$. The other explicit forms of this polynomial are given in Theorem 2.3 and Theorem 2.4

Theorem 2.3 (Explicit closed form)**.** *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$W_n(x) = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}(-2x)^i, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 2.1, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n(x)t^n &= \frac{2}{1-(t-2xt^2)} \\
&= 2\sum_{n=0}^{\infty}(t-2xt^2)^n \\
&= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}t^{n-i}(-2xt^2)^i \\
&= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-2x)^i t^{n+i} \\
&= \sum_{n=0}^{\infty}\left[2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-i}{i}(-2x)^i\right] t^n.
\end{aligned}$$

From the equality of both sides, the desired result is obtained. This complete the proof. □

Theorem 2.4 (Explicit closed form)**.** *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$W_n(x) = 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2i+1}(1-8x)^i, \quad \textit{for } n \geq 1.$$

Proof. Consider,

$$\begin{aligned}
\alpha^{n+1}(x) - \beta^{n+1}(x) &= 2^{-(n+1)}[(1+\sqrt{1-8x})^{n+1} - (1-\sqrt{1-8x})^{n+1}] \\
&= 2^{-(n+1)}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}(\sqrt{1-8x})^i - \sum_{i=0}^{n+1}\binom{n+1}{i}(-\sqrt{1-8x})^i\right] \\
&= 2^{-n}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n+1}{2i+1}(\sqrt{1-8x})^{2i+1}
\end{aligned}$$

Thus,

$$\begin{aligned}
W_n(x) &= A\alpha^n(x) + B\beta^n(x) \\
&= 2\frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} \\
&= 2\frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\sqrt{1-8x}} \\
&= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n+1}{2i+1}(1-8x)^i.
\end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$\sum_{k=0}^{n-1} W_k(x) = \frac{2 - W_{n+1}(x)}{2x}.$$

Proof. By using Binet formula (3), we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} W_k(x) &= \sum_{k=0}^{n-1} \left(A\alpha^k(x) + B\beta^k(x)\right) \\
&= A\frac{1-\alpha^n(x)}{1-\alpha(x)} + B\frac{1-\beta^n(x)}{1-\beta(x)} \\
&= \frac{(A\alpha(x) + B\beta(x)) - (A\alpha^{n+1}(x) + B\beta^{n+1}(x))}{\alpha(x)\beta(x)} \\
&= \frac{W_1(x) - W_{n+1}(x)}{2x} \\
&= \frac{2 - W_{n+1}(x)}{2x}.
\end{aligned}$$

Thus

$$\sum_{k=0}^{n-1} W_k(x) = \frac{2 - W_{n+1}(x)}{2x}.$$

This completes the proof. □

Since the derivative of the polynomials are alway exists, we can give the following formula.

Theorem 2.6 (Differentiation formula)**.** *The derivative of $W_n(x)$ is obtained as the following.*

$$\frac{d}{dx}W_n(x) = \frac{-4(n+1)g_n(x) + 8G_{n+1}(x)}{1-8x}, \tag{4}$$

where $G_n(x)$ and $g_n(x)$ are the n-th Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials, respectively.

Proof. The result is obtained by using Binet formula (3). □

Again, by using Binet formula (3), we obtain some well-known identities as follows.

Theorem 2.7 (Catalan's identity)**.** *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta-Jacobsthak-like polynomials. Then*

$$W_n^2(x) - W_{n+r}(x)W_{n-r}(x) = (2x)^{n-r+1}W_{r-1}^2(x), \quad \textit{for } n \geq r \geq 1. \tag{5}$$

Proof. Consider,

$$\begin{aligned}
&W_n^2(x) - W_{n+r}(x)W_{n-r}(x) \\
&\qquad = (A\alpha^n(x) + B\beta^n(x))^2 - \left(A\alpha^{n+r}(x) + B\beta^{n+r}(x)\right)\left(A\alpha^{n-r}(x) + B\beta^{n-r}(x)\right) \\
&\qquad = -AB\,(\alpha(x)\beta(x))^{n-r}\,(\alpha^r(x) - \beta^r(x))^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{8x}{(\alpha(x)-\beta(x))^2}(2x)^{n-r}\left(\alpha^r(x)-\beta^r(x)\right)^2 \\
&= (2x)^{n-r+1}\left(2\frac{\alpha^r(x)-\beta^r(x)}{\alpha(x)-\beta(x)}\right)^2 \\
&= (2x)^{n-r+1}\left(A\alpha^{r-1}(x)+B\beta^{r-1}(x)\right)^2 \\
&= (2x)^{n-r+1}W_{r-1}^2(x).
\end{aligned}
$$

This completes the proof. □

Take $r=1$ in Catalan's identity (5), we obtain Cassini's identity as the following corollary.

Corollary 2.1 (Cassini's identity). *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$W_n^2(x) - W_{n+1}(x)W_{n-1}(x) = 4(2x)^n, \quad \textit{for } n \geq 1.$$

Proof. Take $r=1$ in Catalan's identity (5), we obtian

$$W_n^2(x) - W_{n+1}(x)W_{n-1}(x) = (2x)^n W_0^2(x) = (2x)^n 2^2 = 4(2x)^n.$$

Thus

$$W_n^2(x) - W_{n+1}(x)W_{n-1}(x) = 4(2x)^n.$$

This completes the proof. □

Theorem 2.8 (d'Ocagne's identity). *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$W_m(x)W_{n+1}(x) - W_{m+1}(x)W_n(x) = 2(2x)^{n+1}W_{m-n-1}(x), \quad \textit{for } m > n \geq 1. \tag{6}$$

Proof. We will prove d'Ocagne's identity (6) by using Binet formula (3).
Consider,

$$
\begin{aligned}
W_m(x)&W_{n+1}(x) - W_{m+1}(x)W_n(x) \\
&= \left(A\alpha^m(x)+B\beta^m(x)\right)\left(A\alpha^{n+1}(x)+B\beta^{n+1}(x)\right) \\
&\quad - \left(A\alpha^{m+1}(x)+B\beta^{m+1}(x)\right)\left(A\alpha^n(x)+B\beta^n(x)\right) \\
&= -AB\left(\alpha(x)\beta(x)\right)^n\left(\alpha(x)-\beta(x)\right)\left(\alpha^{m-n}(x)-\beta^{m-n}(x)\right) \\
&= \frac{4}{(\alpha(x)-\beta(x))^2}(2x)^{n+1}\left(\alpha(x)-\beta(x)\right)\left(\alpha^{m-n}(x)-\beta^{m-n}(x)\right) \\
&= 2(2x)^{n+1}\left(2\frac{\alpha^{m-n}(x)-\beta^{m-n}(x)}{\alpha(x)-\beta(x)}\right) \\
&= 2(2x)^{n+1}\left(A\alpha^{m-n-1}(x)+B\beta^{m-n-1}(x)\right) \\
&= 2(2x)^{n+1}W_{m-n-1}(x).
\end{aligned}
$$

This completes the proof. □

Next, by using Binet's formula, we derive the relation between the Vieta–Jacobsthal-like polynomials, Vieta–Jacobsthal polynomials, and Vieta–Jacobsthal–Lucas polynomials.

Theorem 2.9. *Let $\{W_n(x)\}_{n=0}^{\infty}$, $\{G_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Jacobsthal-like, Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials respectively. Then*

(1) $W_n(x) = 2G_{n+1}(x) = G_n(x) + g_n(x), \quad \text{for } n \geq 0,$

(2) $W_{n+1}(x) - 2xW_{n-1}(x) = 2g_{n+1}(x), \quad \text{for } n \geq 1,$

(3) $W_n(x) + 2g_{n+1}(x) = 4G_{n+2}(x), \quad \text{for } n \geq 0,$

(4) $(1-8x)W_n(x) + 2g_{n+1}(x) = 4g_{n+2}(x), \quad \text{for } n \geq 0,$

(5) $g_{n+1}(x) - 2xg_{n-1}(x) = \dfrac{1}{2}(1-8x)W_{n-1}(x), \quad \text{for } n \geq 1,$

(6) $W_n(x)g_n(x) - W_{2n}(x) = 2(2x)^n, \quad \text{for } n \geq 0,$

(7) $W_m(x)g_n(x) - (2x)^n W_{m-n}(x) = 2G_{(m+n)+1}(x), \quad \text{for } m \geq n \geq 0,$

(8) $4g_n^2(x) - (1-8x)W_{n-1}^2(x) = 16(2x)^n, \quad \text{for } n \geq 1,$

(9) $W_{n-1}(x)g_n(x) = 2G_{2n}(x), \quad \text{for } n \geq 1,$

(10) $G_m(x)g_n(x) + G_n(x)g_m(x) = W_{(m+n)-1}(x), \quad \text{for } m, n \geq 1,$

(11) $4g_m(x)g_n(x) + (1-8x)W_{m-1}(x)W_{n-1}(x) = 8g_{m+n}(x), \quad \text{for } m, n \geq 1,$

(12) $W_n(x)g_n(x) - g_n^2(x) = G_{2n}(x), \quad \text{for } n \geq 0,$

(13) $W_{n-1}(x)g_n(x) + W_n(x)g_{n-1}(x) = 4G_{2n}(x) + 2(2x)^{n-1}, \quad \text{for } n \geq 1.$

Proof. The results (1)–(13) are obtained by using Binet's formula (3). □

3 Matrix form of Vieta–Jacobsthal-like polynomials

In this section, we establish some identities of Vieta–Jacobsthal-like and Vieta–Jacobsthal polynomials by using elementary matrix methods. Let Q_w be 2×2 matrix defined by

$$Q_w = \begin{bmatrix} -4x+2 & -4x \\ 2 & -4x \end{bmatrix}. \tag{7}$$

Then by using this matrix, we can deduce some identities of Vieta–Jacobsthal-like and Vieta–Jacobsthal polynomials.

Theorem 3.1. *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials and let Q_w be the 2×2 matrix defined by* (7)*. Then*

$$Q_w^n = 2^{n-1} \begin{bmatrix} W_{2n}(x) & -2xW_{2n-1}(x) \\ W_{2n-1}(x) & -2xW_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Proof. For the explanation, the Mathematical induction method is used. It easy to see that the statement is true for $n = 1$. Suppose that the result is true for any positive integer k, then we also have the result is true for $k + 1$. Because

$$\begin{aligned} Q_w^{k+1} &= Q_w^k \cdot Q_w \\ &= 2^{k-1} \begin{bmatrix} W_{2k}(x) & -2xW_{2k-1}(x) \\ W_{2k-1}(x) & -2xW_{2k-2}(x) \end{bmatrix} \cdot \begin{bmatrix} -4x+2 & -4x \\ 2 & -4x \end{bmatrix} \\ &= 2^{(k+1)-1} \begin{bmatrix} W_{2k+2}(x) & -2xW_{2k+1}(x) \\ W_{2k+1}(x) & -2xW_{2k}(x) \end{bmatrix}. \end{aligned}$$

By Mathematical induction, we have that the result is true for each $n \in \mathbb{N}$, that is

$$Q_w^n = 2^{n-1} \begin{bmatrix} W_{2n}(x) & -2xW_{2n-1}(x) \\ W_{2n-1}(x) & -2xW_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \qquad \square$$

Theorem 3.2. *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials and let Q_w be the 2×2 matrix defined by* (7)*. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $2W_{2(m+n)}(x) = W_{2m}(x)W_{2n}(x) - 2xW_{2m-1}(x)W_{2n-1}(x)$,

(2) $2W_{2(m+n)-1}(x) = W_{2m}(x)W_{2n-1}(x) - 2xW_{2m-1}(x)W_{2n-2}(x)$,

(3) $2W_{2(m+n)-1}(x) = W_{2m-1}(x)W_{2n}(x) - 2xW_{2m-2}(x)W_{2n-1}(x)$,

(4) $2W_{2(m+n)-2}(x) = W_{2m-1}(x)W_{2n-1}(x) - 2xW_{2m-2}(x)W_{2n-2}(x)$.

Proof. By Theorem 3.1 and the property of power matrix $Q_w^{m+n} = Q_w^m Q_w^n$, we obtain the result. $\square$

By Theorem 3.1 and $W_n(x) = 2G_{n+1}(x)$, we get the following corollary.

Corollary 3.1. *Let $\{G_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal polynomials and let Q_w be the 2×2 matrix defined by* (7)*. Then*

$$Q_w^n = 2^n \begin{bmatrix} G_{2n+1}(x) & -2xG_{2n}(x) \\ G_{2n}(x) & -2xG_{2n-1}(x) \end{bmatrix}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 3.1, we get

$$Q_w^n = 2^{n-1} \begin{bmatrix} W_{2n}(x) & -2xW_{2n-1}(x) \\ W_{2n-1}(x) & -2xW_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Since $W_n(x) = 2G_{n+1}(x)$, we get that

$$\begin{aligned} Q_w^n &= 2^{n-1} \begin{bmatrix} 2G_{2n+1}(x) & -4xG_{2n}(x) \\ 2G_{2n}(x) & -4xG_{2n-1}(x) \end{bmatrix} \\ &= 2^n \begin{bmatrix} G_{2n+1}(x) & -2xG_{2n}(x) \\ G_{2n}(x) & -2xG_{2n-1}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \end{aligned}$$

This completes the proof. $\square$

By Theorem 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get the following corollary.

Corollary 3.2. *Let $\{G_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal polynomials. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $G_{2(m+n)+1}(x) = G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x)$,

(2) $G_{2(m+n)}(x) = G_{2m+1}(x)G_{2n}(x) - 2xG_{2m}(x)G_{2n-1}(x)$,

(3) $G_{2(m+n)}(x) = G_{2m}(x)G_{2n+1}(x) - 2xG_{2m-1}(x)G_{2n}(x)$,

(4) $G_{2(m+n)-1}(x) = G_{2m}(x)G_{2n}(x) - 2xG_{2m-1}(x)G_{2n-1}(x)$.

Proof. From Theorem 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get that

$$\begin{aligned} G_{2(m+n)+1}(x) &= \frac{1}{2}W_{2(m+n)}(x) \\ &= \frac{1}{4}\left(W_{2m}(x)W_{2n}(x) - 2xW_{2m-1}(x)W_{2n-1}(x)\right) \\ &= \frac{1}{4}\left(2G_{2m+1}(x)2G_{2n+1}(x) - 4xG_{2m}(x)2G_{2n}(x)\right) \\ &= G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x). \end{aligned}$$

Thus, we get that (1) holds.
By the same argument as above, we get that (2), (3) and (4) hold. This completes the proof. □

By Corollary 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get the following corollary.

Corollary 3.3. *Let $\{W_n(x)\}_{n=0}^{\infty}$ and $\{G_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Jacobsthal-like and Vieta–Jacobsthal polynomials respectively. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $W_{2(m+n)}(x) = 2\left(G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x)\right)$,

(2) $W_{2(m+n)-1}(x) = 2\left(G_{2m+1}(x)G_{2n}(x) - 2xG_{2m}(x)G_{2n-1}(x)\right)$,

(3) $W_{2(m+n)-1}(x) = 2\left(G_{2m}(x)G_{2n+1}(x) - 2xG_{2m-1}(x)G_{2n}(x)\right)$,

(4) $W_{2(m+n)-2}(x) = 2\left(G_{2m}(x)G_{2n}(x) - 2xG_{2m-1}(x)G_{2n-1}(x)\right)$.

Proof. From Corollary 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get that

$$\begin{aligned} W_{2(m+n)}(x) &= 2G_{2(m+n)+1}(x) \\ &= 2\left(G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x)\right). \end{aligned}$$

Thus, we get that (1) holds.
By the same argument as above, we get that (2), (3) and (4) hold. This completes the proof. □

4 Conclusion

In this work, we defined the Vieta–Jacobsthal-like polynomial. Then we gave the generating function, the Binet formula, and well-known identities for this polynomial. Moreover, we also presented the relation between this polynomial and the previously famous Vieta polynomials.

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Research Article

An Iterative Method for Solving Split Monotone Variational Inclusion Problems and Finite Family of Variational Inequality Problems in Hilbert Spaces

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The purpose of this paper is to study the convergence analysis of an intermixed algorithm for finding the common element of the set of solutions of split monotone variational inclusion problem (SMIV) and the set of a finite family of variational inequality problems. Under the suitable assumption, a strong convergence theorem has been proved in the framework of a real Hilbert space. In addition, by using our result, we obtain some additional results involving split convex minimization problems (SCMPs) and split feasibility problems (SFPs). Also, we give some numerical examples for supporting our main theorem.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. For a mapping $S\colon C \longrightarrow C$, we denoted by $F(S)$ the set of fixed points of S (i.e., $F(S) = \{x \in C\colon Sx = x\}$). Let $A\colon C \longrightarrow H$ be a nonlinear mapping. The variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1}$$

and the solution set of problem (1) is denoted by $VI(C, A)$. It is known that the variational inequality, as a strong and great tool, has already been investigated for an extensive class of optimization problems in economics and equilibrium problems arising in physics and many other branches of pure and applied sciences. Recall that a mapping $A\colon C \longrightarrow C$ is said to be α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2}$$

A multivalued mapping $M\colon H_1 \longrightarrow 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $\langle x - y, u - v \rangle \geq 0$, for any $u \in Mx$ and $v \in My$. A monotone mapping $M\colon H_1 \longrightarrow 2^{H_1}$ is maximal if the graph $G(M)$ for M is not properly contained in the graph of any other monotone mapping. It is generally known that M is maximal if and only if for $(x, u) \in H_1 \times H_1, \langle x - y, u - v \rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in Mx$. Let $M\colon H_1 \longrightarrow 2^{H_1}$ be a multivalued maximal monotone mapping. The resolvent mapping $J_\lambda^M\colon H_1 \longrightarrow H_1$ associated with M is defined by

$$J_\lambda^M(x) \coloneqq (I + \lambda M)^{-1}(x), \quad \forall x \in H_1, \lambda > 0, \tag{3}$$

where I stands for the identity operator on H_1. We note that for all $\lambda > 0$, the resolvent J_λ^M is single-valued, nonexpansive, and firmly nonexpansive.

In 2011, Moudafi [1] introduced the following split monotone variational inclusion problem (SMVI):

$$\text{find } x^* \in H_1 \text{ such that } \theta \in A_1(x^*) + M_1(x^*) \tag{4}$$

$$\text{and such that } y^* = Tx^* \in H_2 \text{ solves } \theta \in A_2(y^*) + M_2(y^*), \tag{5}$$

where θ is the zero vector in H_1 and H_2, M_1: $H_1 \longrightarrow 2^{H_1}$ and M_2: $H_2 \longrightarrow 2^{H_2}$ are multivalued mappings on H_1 and H_2, A_1: $H_1 \longrightarrow H_1$ and A_2: $H_2 \longrightarrow H_2$ are two given single-valued mappings, and T: $H_1 \longrightarrow H_2$ is a bounded linear operator with adjoint T^* of T. We note that if (4) and (5) are considered separately, we have that (4) is a variational inclusion problem with its solution set $VI(H_1, A_1, M_1)$ and (5) is a variational inclusion problem with its solution set $VI(H_1, A_2, M_2)$. We denoted the set of all solutions of (SMVI) by $\Omega = \{x^* \in VI(H_1, A_1, M_1)\text{: } Tx^* \in VI(H_2, A_2, M_2)\}$.

It is worth noticing that by taking $M_1 = N_C$ and $M_2 = N_Q$ normal cones to closed convex sets C and Q, then (SMVI) (4) and (5) reduce to the split variational inequality problem (SVIP) that was introduced by Censor et at. [2]. In [1], they mentioned that (SMVI) (4) and (5) contain many special cases, such as split minimization problem (SMP), split minimax problem (SMMP), and split equilibrium problem (SEP). Some related works can be found in [1, 3–10].

For solving (SMVI) (4) and (5), Modafi [1] proposed the following algorithm.

Algorithm 1. Let $\lambda > 0$, $x_0 \in H_1$, and the sequence $\{x_n\}$ be generated by

$$x_{n+1} = J_\lambda^{M_1}(1 - \lambda f)\left(x_n + \gamma T^*\left(J_\lambda^{M_2}(I - \lambda g) - I\right)Tx_n\right), \quad n \in \mathbb{N}, \tag{6}$$

where $\gamma \in (1, 1/L)$ with L being the spectral radius of the operator T^*T.

He obtained the following weak convergence theorem for algorithm (6).

Theorem 1 (see [1]). *Let H_1, H_2 be real Hilbert spaces. Let T: $H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^*. For $i = 1, 2$, let A_i: $H_i \longrightarrow H_i$ be α_i-inverse strongly monotone with $\alpha = \min\{\alpha_1, \alpha_2\}$ and let M_i: $H_i \longrightarrow 2^{M_i}$ be two maximal monotone operators. Then, the sequence generated by (6) converges weakly to an element $x^* \in \Omega$ provided that $\Omega \neq \varnothing$, $\lambda \in (0, 2\alpha)$, and $\gamma \in (1, 1/L)$ with L being the spectral radius of the operator T^*T.*

Since then, because of a lot of applications of (SMVI), it receives much attention from many authors. They presented many approximation methods for solving (SMVI) (4) and (5). Also the iterative methods for solving (SMVIP) (4) and (5) and fixed-point problems of some nonlinear mappings have been investigated (see [11–19]).

On the other hand, Yao et al. [20] presented an intermixed Algorithm 1.3 for two strict pseudo-contractions in real Hilbert spaces. They also showed that the suggested algorithms converge strongly to the fixed points of two strict pseudo-contractions, independently. As a special case, they can find the common fixed points of two strict pseudo-contractions in Hilbert spaces (i.e., a mapping S: $C \longrightarrow C$ is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0, 1)$ such that $\|Sx - Sy\|^2 \leq \|x - y\|^2 + k \|(I - S)x - (I - S)\|$, 0.3 cm $\forall x, y \in C$).

Algorithm 2. For arbitrarily given $x_0, y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n], & n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C[\alpha_n f(x_n) + (1 - k - \alpha_n)y_n + kSy_n], & n \geq 0, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of the real number in $(0, 1)$, T, S: $C \longrightarrow C$ are λ-strictly pseudo-contractions, f: $C \longrightarrow H$ is a ρ_1-contraction, g: $C \longrightarrow H$ is a ρ_2-contraction, and $k \in (0, 1 - \lambda)$ is a constant.

Under some control conditions, they proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}f(y^*)$ and $\{y_n\}$ converges strongly to $P_{F(S)}f(x^*)$, respectively, where $x^* \in F(T)$, $y^* \in F(S)$, and $P_{F(T)}$ and $P_{F(S)}$ are the metric projection of H onto $F(T)$ and $F(S)$, respectively. After that, many authors have developed and used this algorithm to solve the fixed-point problems of many nonlinear operators in real Hilbert spaces (see for example [21–27]). Question: can we prove the strong convergence theorem of two sequences of split monotone variational inclusion problems and fixed-point problems of nonlinear mappings in real Hilbert spaces?

The purpose of this paper is to modify an intermixed algorithm to answer the question above and prove a strong convergence theorem of two sequences for finding a common element of the set of solutions of (SMVI) (4) and (5) and the set of solutions of a finite family of variational inequality problems in real Hilbert spaces. Furthermore, by applying our main result, we obtain some additional results involving split convex minimization problems (SCMPs) and split feasibility problems (SFPs). Finally, we give some numerical examples for supporting our main theorem.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote the strong convergence of $\{x_n\}$ to x and the weak convergence of $\{x_n\}$ to x by notations "$x_n \longrightarrow x$ as $n \longrightarrow \infty$" and "$x_n \rightharpoonup x$ as $n \longrightarrow \infty$," respectively. For each $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\begin{aligned}\|x+y\|^2 &\leq \|x\|^2 + 2\langle y, x+y\rangle,\\ \|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|x\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.\end{aligned} \tag{8}$$

Definition 1. Let H be a real Hilbert space and C be a closed convex subset of H. Let $S\colon C \longrightarrow C$ be a mapping. Then, S is said to be

(1) Monotone, if $\langle Sx - Sy, x - y\rangle \geq 0, \forall x, y \in H$
(2) Firmly nonexpansive, if $\langle Sx - Sy, x - y\rangle \geq \|Sx - Sy\|^2, \forall x, y \in H$
(3) Lipschitz continuous, if there exists a constant $L > 0$ such that $\|Sx - Sy\| \leq L\|x - y\|, \forall x, y \in H$
(4) Nonexpansive, if $=$ $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in H$

It is well known that if S is α-inverse strongly monotone, then it is $1/\alpha$-Lipschitz continuous and every nonexpansive mapping S is 1-Lipschitz continuous. We note that if $S\colon H \longrightarrow H$ is a nonexpansive mapping, then it satisfies the following inequality (see Theorem 3 in [28] and Theorem 1 in [29]):

$$\langle Sy - Sx, (I-S)x - (I-S)y\rangle \leq \frac{1}{2}\|(I-S)x - (I-S)y\|^2,\quad \forall x, y \in H. \tag{9}$$

Particularly, for every $x \in H$ and $y \in F(S)$, we have

$$\langle y - Sx, (I-S)x\rangle \leq \frac{1}{2}\|(I-S)x\|^2. \tag{10}$$

For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\left\|x - P_C x\right\| \leq \|x - y\|, \quad \forall y \in C. \tag{11}$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 1 (see [30]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \quad \forall v \in C. \tag{12}$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\left\|P_C x - P_C y\right\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{13}$$

Moreover, we also have the following lemma.

Lemma 2 (see [31]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u, \tag{14}$$

where P_C is the metric projection of H onto C.

Lemma 3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let $A_i\colon C \longrightarrow H$ be the α_i-inverse strongly monotone with $\overline{\alpha} = \min_{i=1,2,\ldots,N}\{\alpha_i\}$. If $\cap_{i=1}^N VI(C, A_i) \neq \emptyset$, then*

$$VI\left(C, \sum_{i=1}^N a_i A_i\right) = \bigcap_{i=1}^N VI(C, A_i), \tag{15}$$

where $0 < a_i < 1$ for all $i = 1, 2, \ldots, N$ and $\sum_{i=1}^N a_i = 1$. Moreover, $I - \lambda\sum_{i=1}^N a_i A_i$ is a nonexpansive mapping for all $\lambda \in (0, 2\overline{\alpha})$.

Proof. By Lemma 4.3 of [32], we have that $VI(C, \sum_{i=1}^N a_i A_i) = \cap_{i=1}^N VI(C, A_i)$. Let $\lambda \in (0, 2\overline{\alpha})$ and let $x, y \in C$. As the same argument as in the proof of Lemma 8 in [16], we have $I - \lambda\sum_{i=1}^N a_i A_i$ as nonexpansive. □

Lemma 4 (see [33]). *Let H be a real Hilbert space, $A\colon H \longrightarrow H$ be a single-valued nonlinear mapping, and $M\colon H \longrightarrow 2^H$ be a set-valued mapping. Then, a point $u \in H$ is a solution of variational inclusion problem if and only if $u = J_\lambda^M(I - \lambda A)u, \forall \lambda > 0$, i.e.,*

$$VI(H, A, M) = F\left(J_\lambda^M(I - \lambda A)\right), \quad \forall \lambda > 0. \tag{16}$$

Furthermore, if A is α-inverse strongly monotone and $\lambda \in (0, 2\alpha]$, then $VI(H, A, M)$ is a closed convex subset of H.

Lemma 5 (see [33]). *ie resolvent operator J_λ^M associated with M is single-valued, nonexpansive, and 1-inverse strongly monotone for all $\lambda > 0$.*

The following two lemmas are the particular case of Lemmas 7 and 8 in [16].

Lemma 6 (see [16]). *For every $i = 1, 2$, let H_i be real Hilbert spaces, let $M_i\colon H_i \longrightarrow 2^{H_i}$ be a multivalued maximal monotone mapping, and let $A_i\colon H_i \longrightarrow H_i$ be an α_i-inverse strongly monotone mapping. Let $T\colon H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^* of T, and let $\widetilde{G}\colon H_1 \longrightarrow H_1$ be a mapping defined by $\widetilde{G}(x) = J_{\lambda_1}^{M_1}(I - \lambda_1 A_1)(x - \gamma T^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2))Tx)$, for all $x \in H_1$. Then, $\|\widetilde{G}x - \widetilde{G}y\|^2 \leq \|x - y\|^2 - \gamma\ (1 - \gamma L)\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2))Tx - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2))Ty\|^2$, for all $x, y \in H_1$, where L is the spectral radius of the operator T^*T, $\lambda_1 \in (0, 2\alpha_1)$, $\lambda_2 \in (0, 2\alpha_2)$, and $\gamma > 0$. Furthermore, if $0 < \gamma < 1/L$, then $\widetilde{G}$ is a nonexpansive mapping.*

Lemma 7 (see [16]). *Let H_1 and H_2 be Hilbert spaces. For $i = 1, 2$, let M_i: $H_i \longrightarrow 2^{H_i}$ be a multivalued maximal monotone mapping and let A_i: $H_i \longrightarrow H_i$ be an α_i-inverse strongly monotone mapping. Let T: $H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^*. Assume that $\Omega \neq \phi$. Then, $x^* \in \Omega$ if and only if $x^* = \widetilde{G}(x^*)$, where $\widetilde{G}$: $H_1 \longrightarrow H_1$ is a mapping defined by*

$$\widetilde{G}(x) = J_{\lambda_1}^{M_1}(I - \lambda_1 A_1)\Big(x - \gamma T^*\Big(I - J_{\lambda_2}^{M_2}(I - \lambda_2 A_2)\Big)Tx\Big), \tag{17}$$

for all $x \in H_1$, $\lambda_1 \in (0, 2\alpha_1)$, $\lambda_2 \in (0, 2\alpha_2)$, and $0 < \gamma < 1/L$, where L is the spectral radius of the operator T^*T.

Next, we give an example to support Lemma 7.

Example 1. Let $\mathbb{R}$ be a set of real number and $H_1 = H_2 = \mathbb{R}^2$, and let $\langle \cdot, \cdot \rangle$: $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be inner product defined by $\langle x, y\rangle = x \cdot y = x_1 y_1 + x_2 y_2$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and the usual norm $\|\cdot\|$: $\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\|x\| = \sqrt{x_1^2 + x_2^2}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let T: $H_1 \longrightarrow H_2$ be defined by $Tx = (2x_1, 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and T^*: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T^* z = (2z_1, 2z_2)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let M_1, M_2: $\mathbb{R}^2 \longrightarrow 2^{\mathbb{R}^2}$ be defined by $M_1 x = \{(3x_1 - 5, 3x_2 - 5)\}$ and $M_2 x = \{(x_1/3 - 2, x_2/3 - 2)\}$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping A_1, A_2: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $A_1 x = ((x_1 - 4)/2, (x_2 - 4)/2)$ and $A_2 x = ((x_1 - 2)/3, (x_2 - 2)/3)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then, $(2, 2)$ is a fixed point of $\widetilde{G}$. That is, $(2, 2) \in F(\widetilde{G})$.

Proof. It is obvious to see that $\Omega = \{(2, 2)\}$, A_1 is 2-inverse strongly monotone, and A_2 is 3-inverse strongly monotone. Choose $\lambda_1 = 1/3$. Since $M_1 x = \{(3x_1 - 5, 3x_2 - 5)\}$ and the resolvent of $M_1, J_{\lambda_1}^{M_1} x = (I + \lambda_1 M_1)^{-1} x$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, we obtain that

$$J_{\lambda_1}^{M_1} x = \frac{x}{2} + \frac{5}{6}, \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{18}$$

Choose $\lambda_2 = 1$. Since $M_2 x = \{(x_1/3 - 2, x_2/3 - 2)\}$ and the resolvent of $M_2, J_{\lambda_2}^{M_2} x = (I + \lambda_2 M_2)^{-1} x$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, we obtain that

$$J_{\lambda_2}^{M_2} x = \frac{3x}{4} + \frac{3}{2}, \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{19}$$

Since the spectral radius of the operator T^*T is 4, we choose $\gamma = 0.1$. Then, from (18) and (19), we get that

$$\begin{aligned} \widetilde{G}(x) &= J_{1/3}^{M_1}\Big(I - \frac{1}{3}A_1\Big)\Big(x - 0.1T^*\big(I - J_1^{M_2}(I - A_2)\big)Tx\Big), \\ &= \frac{x}{3} + \frac{4}{3}, \end{aligned} \tag{20}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then, by Lemma 7, we have that $(2, 2) \in F(\widetilde{G})$. □

Lemma 8 (see [34]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying $s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0$ where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(1) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(2) $\limsup_{n\longrightarrow\infty} \alpha_n/\delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$. Then $\lim_{n\longrightarrow\infty} s_n = 0$.

3. Main Results

In this section, we introduce an iterative algorithm of two sequences which depend on each other by using the intermixed method. Then, we prove a strong convergence theorem for solving two split monotone variational inclusion problems and a finite family of variational inequality problems.

Theorem 2. *Let H_1 and H_2 be Hilbert spaces, and let C be a nonempty closed convex subset of H_1. Let T: $H_1 \longrightarrow H_2$ be a bounded linear operator, and let f, g: $H_1 \longrightarrow H_1$ be ρ_f, ρ_g-contraction mappings with $\rho = \max\{\rho_f, \rho_g\}$. For $i = 1, 2$, let M_i^x, M_i^y: $H_i \longrightarrow 2^{H_i}$ be multivalued maximal monotone mappings and let A_i^x, A_i^y: $H_i \longrightarrow H_i$ be α_i^x, α_i^y–inverse strongly monotone mappings, respectively. For $i = 1, 2, \ldots, N$, let B_i^x, B_i^y: $H_1 \longrightarrow H_1$ be β_i^x, β_i^y–inverse strongly monotone mappings, respectively, $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$, and $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. Let $\widetilde{G}^x, \widetilde{G}^y$: $H_1 \longrightarrow H_1$ be defined by $\widetilde{G}^x x = J_{\lambda_1^x}^{M_1^x}(I - \lambda_1^x A_1^x)(x - \gamma^x T^*(I - J_{\lambda_2^x}^{M_2^x}(I - \lambda_2^x A_2^x))Tx), \forall x \in H_1$, and $\widetilde{G}^y y = J_{\lambda_1^y}^{M_1^y}(I - \lambda_1^y A_1^y)(y - \gamma^y T^*(I - J_{\lambda_2^y}^{M_2^y}(I - \lambda_2^y A_2^y))Ty)$, $\forall\ y \in H_1$, respectively, where $\lambda_i^x \in (0, 2\alpha_i^x), \lambda_i^y \in (0, 2\alpha_i^y)$, and $0 < \gamma^x, \gamma^y < 1/L$ with L being a spectral radius of T^*T. Assume that $\mathcal{F}^x = \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x)) \neq \varnothing$ and $\mathcal{F}^y = \Omega^y \cap (\cap_{i=1}^N VI(C, B_i^y)) \neq \varnothing$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1, y_1 \in H_1$ and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \displaystyle\sum_{i=1}^N a_i^x B_i^x\right)x_n + \eta_n\big(\alpha_n f(y_n) + (1 - \alpha_n)\widetilde{G}^x x_n\big), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^y \displaystyle\sum_{i=1}^N a_i^y B_i^y\right)y_n + \eta_n\big(\alpha_n g(x_n) + (1 - \alpha_n)\widetilde{G}^y y_n\big), \end{cases} \tag{21}$$

for all $n \geq 1$ *where* $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots, a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, *and* $\{\mu_n^x\}, \{\mu_n^y\} \subset (0,\infty)$. *Assume the following condition holds:*

(1) $\sum_{n=1}^{\infty} \mu_n^x < \infty, \sum_{n=1}^{\infty} \mu_n^y < \infty$, *and* $0 < a < \mu_n^x \leq 2\beta_x, 0 < b < \mu_n^y \leq 2\overline{\beta_y}$, *for some* $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty, lim_{n\longrightarrow\infty} \alpha_n = 0$.

(3) $\sum_{i=1}^{N} a_i^x = \sum_{n=1}^{N} a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, *for all* $n \in \mathbb{N}$, *for some* $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n|, \sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ *and* $\{y_n\}$ *converges strongly to* $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$.

Proof. We divided the proof into five steps. □

Step 1. We will show that $\{x_n\}$ and $\{y_n\}$ are bounded. Let $x^* \in \mathcal{F}^x$ and $y^* \in \mathcal{F}^y$. Then, from Lemma 7 and Lemma 6, we get

$$\begin{aligned}\left\|\widetilde{G}^x x_n - x^*\right\| &= \left\|J_{\lambda_1^x}^{M_1^x}(I - \lambda_1^x A_1^x)\left(x_n - \gamma^x T^*\left(I - J_{\lambda_2^x}^{M_2^x}(I - \lambda_2^x A_2^x)\right)T x_n\right) - x^*\right\|, \\ &\leq \left\|x_n - x^*\right\|.\end{aligned} \tag{22}$$

From (21), Lemma 3, and (22), we have

$$\begin{aligned}\left\|x_{n+1} - x^*\right\| &= \left\|\delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x\right)x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right) - x^*\right\|, \\ &\leq \delta_n\left\|x_n - x^*\right\| + \sigma_n\left\|P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x\right)x_n - x^*\right\| + \eta_n\left\|\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right) - x^*\right\| \\ &\leq \delta_n\left\|x_n - x^*\right\| + \sigma_n\left\|x_n - x^*\right\| + \eta_n\left\|\alpha_n\left(f(y_n) - x^*\right) + (1-\alpha_n)\left(\widetilde{G}^x x_n - x^*\right)\right\| \\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\left[\alpha_n\left\|f(y_n) - x^*\right\| + (1-\alpha_n)\left\|\widetilde{G}^x x_n - x^*\right\|\right] \\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\left[\alpha_n\left\|f(y_n) - f(y^*)\right\| + \alpha_n\left\|f(y^*) - x^*\right\| + (1-\alpha_n)\left\|x_n - x^*\right\|\right] \\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\left[\alpha_n\rho_f\left\|y_n - y^*\right\| + \alpha_n\left\|f(y^*) - x^*\right\| + (1-\alpha_n)\left\|x_n - x^*\right\|\right] \\ &\leq (1-\eta_n)\left\|x_n - x^*\right\| + \eta_n\alpha_n\rho\left\|y_n - y^*\right\| + \eta_n\alpha_n\left\|f(y^*) - x^*\right\| + \eta_n(1-\alpha_n)\left\|x_n - x^*\right\| \\ &= (1-\eta_n\alpha_n)\left\|x_n - x^*\right\| + \eta_n\alpha_n\rho\left\|y_n - y^*\right\| + \eta_n\alpha_n\left\|f(y^*) - x^*\right\|.\end{aligned} \tag{23}$$

Similarly, from definition of y_n, we have

$$\left\|y_{n+1} - y^*\right\| \leq (1-\eta_n\alpha_n)\left\|y_n - y^*\right\| + \eta_n\alpha_n\rho\left\|x_n - x^*\right\| + \eta_n\alpha_n\left\|g(x^*) - y^*\right\|. \tag{24}$$

Hence, from (23) and (24), we obtain

$$\begin{aligned}\left\|x_{n+1} - x^*\right\| + \left\|y_{n+1} - y^*\right\| &\leq (1-\eta_n\alpha_n)\left(\left\|x_n - x^*\right\| + \left\|y_n - y^*\right\|\right) + \eta_n\alpha_n\rho\left(\left\|x_n - x^*\right\| + \left\|y_n - y^*\right\|\right) \\ &\quad + \eta_n\alpha_n\left(\left\|f(y^*) - x^*\right\| + \left\|g(x^*) - y^*\right\|\right) \\ &= (1-(1-\rho)\eta_n\alpha_n)\left(\left\|x_n - x^*\right\| + \left\|y_n - y^*\right\|\right) + \eta_n\alpha_n\left(\left\|f(y^*) - x^*\right\| + \left\|g(x^*) - y^*\right\|\right).\end{aligned} \tag{25}$$

By induction, we have

$$\|x_n - x^*\| + \|y_n - y^*\| \le \max\left\{\|x_1 - x^*\| + \|y_1 - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1-\rho}\right\}, \tag{26}$$

for every $n \in \mathbb{N}$. Thus, $\{x_n\}$ and $\{y_n\}$ are bounded.

Step 2. We will show that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}\|y_{n+1} - y_n\| = 0$. Put $u_n = P_C(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x)x_n$, $v_n = P_C(I - \mu_n^y \sum_{i=1}^N a_i^y B_i^y)y_n$, $z_n = \alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n$, and $w_n = \alpha_n f(x_n) + (1-\alpha_n)\widetilde{G}^y y_n$, for all $n \ge 1$. From Lemma 6, we have

$$\begin{aligned}\|z_n - z_{n-1}\| &= \left\|\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right) - \left(\alpha_{n-1} f(y_{n-1}) + (1-\alpha_{n-1})\widetilde{G}^x x_{n-1}\right)\right\|,\\ &\le \alpha_n\|f(y_n) - f(y_{n-1})\| + (1-\alpha_n)\|\widetilde{G}^x x_n - \widetilde{G}^x x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|\widetilde{G}^x x_{n-1}\|\\ &\le \alpha_n\rho_f\|y_n - y_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|\widetilde{G}^x x_{n-1}\|\\ &\le \alpha_n\rho\|y_n - y_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|\widetilde{G}^x x_{n-1}\|\right).\end{aligned} \tag{27}$$

By applying Lemma 3, we get that

$$\begin{aligned}\|u_n - u_{n-1}\| &= \left\|P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n - P_C\left(I - \mu_{n-1}^x\sum_{i=1}^N a_i^x B_i^x\right)x_{n-1}\right\|,\\ &\le \|x_n - x_{n-1}\| + \left\|P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_{n-1} - P_C\left(I - \mu_{n-1}^x\sum_{i=1}^N a_i^x B_i^x\right)x_{n-1}\right\| a_i^x\|B_i^x x_{n-1}\|.\\ &\le \|x_n - x_{n-1}\| + |\mu_n^x - \mu_{n-1}^x|\sum_{i=1}^N\end{aligned} \tag{28}$$

From the definition of $\{x_n\}$, (27), and (28), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(\delta_n x_n + \sigma_n u_n + \eta_n z_n) - (\delta_{n-1}x_{n-1} + \sigma_{n-1}u_{n-1} + \eta_{n-1}z_{n-1})\|,\\ &\le \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + \sigma_n\|u_n - u_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|u_{n-1}\| + \eta_n\|z_n - z_{n-1}\| + |\eta_n - \eta_{n-1}|\|z_{n-1}\|\\ &\le \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + \sigma_n\left[\|x_n - x_{n-1}\| + |\mu_n^x - \mu_{n-1}^x|\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\|\right] + |\sigma_n - \sigma_{n-1}|\|u_{n-1}\|\\ &\quad + \eta_n\left[\alpha_n\rho\|y_n - y_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|\widetilde{G}^x x_{n-1}\|\right)\right] + |\eta_n - \eta_{n-1}|\|z_{n-1}\|\\ &= (1 - \eta_n\alpha_n)\|x_n - x_{n-1}\| + \eta_n\alpha_n\rho\|y_n - y_{n-1}\| + \sigma_n|\mu_n^x - \mu_{n-1}^x|\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\|\\ &\quad + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|u_{n-1}\| + \eta_n|\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|\widetilde{G}^x x_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\|z_{n-1}\|.\end{aligned} \tag{29}$$

By the same argument as in (27) and (29), we also have

$$\|w_n - w_{n-1}\| \le \alpha_n\rho\|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\|g(x_{n-1})\| + \|\widetilde{G}^y y_{n-1}\|\right). \tag{30}$$

$$\begin{aligned}\|y_{n+1} - y_n\| \le{}& (1-\eta_n\alpha_n)\|y_n - y_{n-1}\| + \eta_n\alpha_n\rho\|x_n - x_{n-1}\| + \sigma_n|\mu_n^y - \mu_{n-1}^y|\sum_{i=1}^N a_i^y\|B_i^y y_{n-1}\| \\ &+ |\delta_n - \delta_{n-1}|\|y_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|v_{n-1}\| + \eta_n|\alpha_n - \alpha_{n-1}|\left(\|g(x_{n-1})\| + \|\widetilde{G}^y y_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\|w_{n-1}\|.\end{aligned} \tag{31}$$

From (29) and (31), we obtain that

$$\begin{aligned}\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \le{}& (1-(1-\rho)\eta_n\alpha_n)\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right) \\ &+ \sigma_n|\mu_n^x - \mu_{n-1}^x|\left(\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\| + \sum_{i=1}^N a_i^y\|B_i^y y_{n-1}\|\right) + |\delta_n - \delta_{n-1}|\left(\|x_{n-1}\| + \|y_{n-1}\|\right) + |\sigma_n - \sigma_{n-1}|\left(\|u_{n-1}\| + \|v_{n-1}\|\right) \\ &+ \eta_n|\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|g(x_{n-1})\| + \|\widetilde{G}^x x_{n-1}\| + \|\widetilde{G}^y y_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\left(\|z_{n-1}\| + \|w_{n-1}\|\right) \\ \le{}& (1-(1-\rho)\overline{a}\alpha_n)\left(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\right) + \overline{b}|\mu_n^x - \mu_{n-1}^x|\left(\sum_{i=1}^N a_i^x\|B_i^x x_{n-1}\| + \sum_{i=1}^N a_i^y\|B_i^y y_{n-1}\|\right) \\ &+ |\delta_n - \delta_{n-1}|\left(\|x_{n-1}\| + \|y_{n-1}\|\right) + |\sigma_n - \sigma_{n-1}|\left(\|u_{n-1}\| + \|v_{n-1}\|\right) \\ &+ \overline{b}|\alpha_n - \alpha_{n-1}|\left(\|f(y_{n-1})\| + \|g(x_{n-1})\| + \|\widetilde{G}^x x_{n-1}\| + \|\widetilde{G}^y y_{n-1}\|\right) + |\eta_n - \eta_{n-1}|\left(\|z_{n-1}\| + \|w_{n-1}\|\right).\end{aligned} \tag{32}$$

From (32), conditions (1), (2), and (5), and Lemma 8, we obtain that

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0, \tag{33}$$

$$\lim_{n\longrightarrow\infty}\|y_{n+1} - y_n\| = 0. \tag{34}$$

Step 3. We show that $\lim_{n\longrightarrow\infty}\|z_n - P_C(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x)z_n\| = \lim_{n\longrightarrow\infty}\|z_n - \widetilde{G}^x z_n\| = 0$. From (21), we have that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 ={}& \left\|\delta_n(x_n - x^*) + \sigma_n\left(P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n - x^*\right) + \eta_n(z_n - x^*)\right\|^2, \\ \le{}& \delta_n\|x_n - x^*\|^2 + \sigma_n\left\|P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n - x^*\right\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + \eta_n\|z_n - x^*\|^2 \\ \le{}& (1-\eta_n)\|x_n - x^*\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + \eta_n\left\|\alpha_n\left(f\left(y_n - \widetilde{G}^x x_n\right) + \left(\widetilde{G}^x x_n - x^*\right)\right)\right\|^2 \\ \le{}& (1-\eta_n)\|x_n - x^*\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + \eta_n\left[\|\widetilde{G}^x x_n - x^*\|^2 + 2\alpha_n\langle f(y_n) - \widetilde{G}^x x_n, z_n - x^*\rangle\right] \\ \le{}& \|x_n - x^*\|^2 - \delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 + 2\eta_n\alpha_n\|f(y_n) - \widetilde{G}^x x_n\|\|z_n - x^*\|,\end{aligned} \tag{35}$$

$$\begin{aligned}\delta_n\sigma_n\left\|x_n - P_C\left(I - \mu_n^x\sum_{i=1}^N a_i^x B_i^x\right)x_n\right\|^2 \le{}& \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\eta_n\alpha_n\|f(y_n) - \widetilde{G}^x x_n\|\|z_n - x^*\|, \\ \le{}& \|x_n - x_{n+1}\|\left[\|x_n - x^*\| + \|x_{n+1} - x^*\|\right] + 2\eta_n\alpha_n\|f(y_n) - \widetilde{G}^x x_n\|\|z_n - x^*\|.\end{aligned} \tag{36}$$

Then, we have

$$\left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{37}$$

Observe that

$$x_{n+1} - x_n = \sigma_n\left(P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - x_n \right) + \eta_n (z_n - x_n). \tag{38}$$

From (33) and (37), we have

$$\| z_n - x_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{39}$$

By the same argument as above, we also have that

$$\| w_n - x_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{40}$$

Note that

$$\begin{aligned} \left\| z_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\| &\le \| z_n - x_n \| + \left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| \\ &\quad + \left\| P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\|, \\ &\le \| z_n - x_n \| + \left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| + \| x_n - z_n \| \\ &= 2\| z_n - x_n \| + \left\| x_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\|, \end{aligned} \tag{41}$$

By (37) and (39), we get that

$$\left\| z_n - P_C\left(I - \mu_n^x \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{42}$$

By the same argument as (41), we also obtain

$$\left\| w_n - P_C\left(I - \mu_n^y \sum_{i=1}^{N} a_i^y B_i^y \right) w_n \right\| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{43}$$

Consider

$$\| x_{n+1} - z_n \| \le \| x_{n+1} - x_n \| + \| x_n - z_n \|. \tag{44}$$

By (33) and (39), we get that

$$\| x_{n+1} - z_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{45}$$

However,

$$\begin{aligned} \| x_n - \widetilde{G}^x x_n \| &\le \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| + \| z_n - \widetilde{G}^x x_n \|, \\ &= \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| + \| \alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n - \widetilde{G}^x x_n \| \\ &= \| x_n - x_{n+1} \| + \| x_{n+1} - z_n \| + \alpha_n \| f(y_n) - \widetilde{G}^x x_n \|, \end{aligned} \tag{46}$$

It follows from (33) and (45) that

$$\| x_n - \widetilde{G}^x x_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{47}$$

Consider

$$\begin{aligned} \| z_n - \widetilde{G}^x z_n \| &\le \| z_n - x_n \| + \| x_n - \widetilde{G}^x x_n \| + \| \widetilde{G}^x x_n - \widetilde{G}^x z_n \|, \\ &\le 2\| z_n - x_n \| + \| x_n - \widetilde{G}^x x_n \|. \end{aligned} \tag{48}$$

From (39) and (47), we obtain

$$\| z_n - \widetilde{G}^x z_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{49}$$

Applying the same method as (48), we also have

$$\| w_n - \widetilde{G}^y w_n \| \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{50}$$

Step 4. We will show that $\limsup_{n\longrightarrow\infty} \langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x} \rangle \le 0$ and $\limsup_{n\longrightarrow\infty} \langle g(\widetilde{x}) - \widetilde{y}, z_n - \widetilde{y} \rangle \le 0$, where $\widetilde{x} = P_{\mathscr{F}^x} f(\widetilde{y})$ and $\widetilde{y} = P_{\mathscr{F}^y} f(\widetilde{x})$. First, we take a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\longrightarrow\infty} \langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x} \rangle = \lim_{k\longrightarrow\infty} \langle f(\widetilde{y}) - \widetilde{x}, z_{n_k} - \widetilde{x} \rangle. \tag{51}$$

Since $\{x_n\}$ is bounded, there exists a subsequence x_{n_k} of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q_1$ as $k \longrightarrow \infty$. From (39), we get that $z_{n_k} \rightharpoonup q_1$. Next, we need to show that $q_1 \in \mathscr{F}^x = \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x))$. Assume that $q_1 \notin \Omega^x$. By Lemma 7, we get that $q_1 \neq \widetilde{G}^x q_1$. Applying Opial's condition and (49), we get that

$$\begin{aligned}\liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\| &< \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - \widetilde{G}^x q_1\right\|,\\ &\leq \liminf_{n\longrightarrow\infty}\left\|z_{n_k} - \widetilde{G}^x z_{n_k}\right\| + \liminf_{k\longrightarrow\infty}\left\|\widetilde{G}^x z_{n_k} - \widetilde{G}^x q_1\right\|\\ &\leq \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\|.\end{aligned} \tag{52}$$

This is a contradiction. Thus, $q_1 \in \Omega^x$.

Assume that $q_1 \notin \cap_{i=1}^N VI(C, B_i^x)$. Then, from Lemma 3 and Lemma 2, we have $q_1 \notin F(P_C(I - \mu_n^x \sum_{i=1}^N a_i B_i^x))$. From Opial's condition and (42), we obtain

$$\begin{aligned}\liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\| &< \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - P_C\left(I - \mu_{n_k}^x \sum_{i=1}^N a_i^x B_i^x\right) q_1\right\|,\\ &\leq \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - P_C\left(I - \mu_{n_k}^x \sum_{i=1}^N a_i^x B_i^x\right) z_{n_k}\right\| + \liminf_{k\longrightarrow\infty}\left\|P_C\left(I - \mu_{n_k}^x \sum_{i=1}^N a_i^x B_i^x\right) z_{n_k} - P_C\left(I - \mu_{n_k} \sum_{i=1}^N a_i^x B_i^x\right) q_1\right\|\\ &\leq \liminf_{k\longrightarrow\infty}\left\|z_{n_k} - q_1\right\|.\end{aligned} \tag{53}$$

This is a contradiction. Thus, $q_1 \in \cap_{i=1}^N VI(C, B_i^x)$, and so,

$$q_1 \in \mathscr{F}^x = \Omega^x \cap \left(\bigcap_{i=1}^N VI\left(C, B_i^x\right)\right). \tag{54}$$

However, $z_{n_k} \rightharpoonup q_1$. From (54) and Lemma 1, we can derive that

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle &= \lim_{k\longrightarrow\infty}\langle f(\widetilde{y}) - \widetilde{x}, z_{n_k} - \widetilde{x}\rangle,\\ &= \langle f(\widetilde{y}) - \widetilde{x}, q_1 - \widetilde{x}\rangle\\ &\leq 0.\end{aligned} \tag{55}$$

By the same method as (55), we also obtain that

$$\limsup_{n\longrightarrow\infty}\langle g(\widetilde{x}) - \widetilde{y}, z_n - \widetilde{y}\rangle \leq 0. \tag{56}$$

Step 5. Finally, we show that the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to $\widetilde{x} = P_{\mathscr{F}^x} f(\widetilde{y})$ and $\widetilde{y} = P_{\mathscr{F}^y} f(\widetilde{x})$, respectively. From the definition of z_n, we have

$$\begin{aligned}\left\|z_n - \widetilde{x}\right\|^2 &= \langle \alpha_n (f(y_n) - \widetilde{x}) + (1 - \alpha_n)\left(\widetilde{G}^x x_n - \widetilde{x}\right), z_n - \widetilde{x}\rangle,\\ &= \alpha_n \langle f(y_n) - \widetilde{x}, z_n - \widetilde{x}\rangle + (1 - \alpha_n)\langle \widetilde{G}^x x_n - \widetilde{x}, z_n - \widetilde{x}\rangle\\ &\leq \alpha_n \langle f(y_n) - f(\widetilde{y}), z_n - \widetilde{x}\rangle + \alpha_n \langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle + (1 - \alpha_n)\left\|\widetilde{G}^x x_n - \widetilde{x}\right\|\left\|z_n - \widetilde{x}\right\|\\ &\leq \alpha_n \rho \left\|y_n - \widetilde{y}\right\|\left\|z_n - \widetilde{x}\right\| + \alpha_n \langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle + (1 - \alpha_n)\left\|x_n - \widetilde{x}\right\|\left\|z_n - \widetilde{x}\right\|\\ &\leq \frac{\alpha_n \rho}{2}\left[\left\|y_n - \widetilde{y}\right\|^2 + \left\|z_n - \widetilde{x}\right\|^2\right] + \alpha_n \langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\rangle + \frac{(1 - \alpha_n)}{2}\left[\left\|x_n - \widetilde{x}\right\|^2 + \left\|z_n - \widetilde{x}\right\|^2\right],\end{aligned} \tag{57}$$

which implies that

$$\left\|z_n - \widetilde{x}\right\|^2 \le \frac{\alpha_n \rho}{1 + \alpha_n (1-\rho)}\left\|y_n - \widetilde{y}\right\|^2 + \frac{(1-\alpha_n)}{1 + \alpha_n (1-\rho)}\left\|x_n - \widetilde{x}\right\|^2 + \frac{2\alpha_n}{1 + \alpha_n (1-\rho)}\left\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\right\rangle. \tag{58}$$

From the definition of $\{x_n\}$ and (58), we get

$$\begin{aligned}
\left\|x_{n+1} - \widetilde{x}\right\|^2 &\le \delta_n \left\|x_n - \widetilde{x}\right\|^2 + \sigma_n \left\|P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right)x_n - \widetilde{x}\right\|^2 + \eta_n \left\|z_n - \widetilde{x}\right\|^2, \\
&\le (1-\eta_n)\left\|x_n - \widetilde{x}\right\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1-\rho)}\left\|y_n - \widetilde{y}\right\|^2 + \frac{(1-\alpha_n)\eta_n}{1 + \alpha_n (1-\rho)}\left\|x_n - \widetilde{x}\right\|^2 + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1-\rho)}\left\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\right\rangle \\
&= \left(1 - \frac{\alpha_n \eta_n (2-\rho)}{1 + \alpha_n (1-\rho)}\right)\left\|x_n - \widetilde{x}\right\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1-\rho)}\left\|y_n - \widetilde{y}\right\|^2 + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1-\rho)}\left\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\right\rangle.
\end{aligned} \tag{59}$$

Applying the same argument as in (58) and (59), we get

$$\left\|y_{n+1} - \widetilde{y}\right\|^2 \le \left(1 - \frac{\alpha_n \eta_n (2-\rho)}{1 + \alpha_n (1-\rho)}\right)\left\|y_n - \widetilde{y}\right\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1-\rho)}\left\|x_n - \widetilde{x}\right\|^2 + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1-\rho)}\left\langle g(\widetilde{x}) - \widetilde{y}, z_n - \widetilde{y}\right\rangle. \tag{60}$$

From (58) and (59), we have

$$\begin{aligned}
\left\|x_{n+1} - \widetilde{x}\right\|^2 + \left\|y_{n+1} - \widetilde{y}\right\|^2 &\le \left(1 - \frac{\alpha_n \eta_n (2-\rho)}{1 + \alpha_n (1-\rho)}\right)\left[\left\|x_n - \widetilde{x}\right\|^2 + \left\|y_n - \widetilde{y}\right\|^2\right] \\
&\quad + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1-\rho)}\left[\left\|x_n - \widetilde{x}\right\|^2 + \left\|y_n - \widetilde{y}\right\|^2\right] + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1-\rho)}\left[\left\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\right\rangle + \left\langle g(\widetilde{x}) - \widetilde{y}, z_n - \widetilde{y}\right\rangle\right], \\
&\le \left(1 - \frac{2\alpha_n \eta_n (1-\rho)}{1 + \alpha_n (1-\rho)}\right)\left[\left\|x_n - \widetilde{x}\right\|^2 + \left\|y_n - \widetilde{y}\right\|^2\right] + \frac{2\alpha_n \eta_n}{1 + \alpha_n (1-\rho)}\left[\left\langle f(\widetilde{y}) - \widetilde{x}, z_n - \widetilde{x}\right\rangle + \left\langle g(\widetilde{x}) - \widetilde{y}, z_n - \widetilde{y}\right\rangle\right].
\end{aligned} \tag{61}$$

According to condition (2) and (4), (61), and Lemma 8, we can conclude that $\{x_n\}$ and $\{y_n\}$ converge strongly to $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ and $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$, respectively. Furthermore, from (39) and (40), we get that $\{z_n\}$ and $\{w_n\}$ converge strongly to $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ and $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$, respectively. This completes the proof. □

One of the great special cases of the SMVIP is the split variational inclusion problem that has a wide variety of application backgrounds, such as split minimization problems and split feasibility problems.

If we set $A_i^x = 0$ and $A_i^y = 0$ in Theorem 2, for all $i = 1, 2$, then we get the strong convergence theorem for the split variational inclusion problem and the finite families of the variational inequality problems as follows:

Corollary 1. *Let H_1 and H_2 be Hilbert spaces, and let C be a nonempty closed convex subset of H_1. Let $T\colon H_1 \longrightarrow H_2$ be a bounded linear operator, and let $f, g\colon H_1 \longrightarrow H_1$ be ρ_f, ρ_g-contraction mappings with $\rho = \max\{\rho_f, \rho_g\}$. For every $i = 1, 2$, let $M_i^x, M_i^y\colon H_i \longrightarrow 2^{H_i}$ be multivalued maximal monotone mappings. For $i = 1, 2, \ldots, N$, let B_i^x, B_i^y:*

$H_1 \longrightarrow H_1$ *be* β_i^x, β_i^y*−inverse strongly monotone with* $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$ *and* $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. *Let* $\mathcal{S}^x = \{x^* \in H_1\colon 0 \in M_1^x x^*, \widetilde{x} = Tx^* \in H_2\colon 0 \in M_2^x \widetilde{x}\}$ *and* $\mathcal{S}^y = \{y^* \in H_1\colon 0 \in M_1^y y^*, \widetilde{y} = Ty^* \in H_2\colon 0 \in M_2^y \widetilde{y}\}$. *Assume that* $\mathcal{F}^x = \mathcal{S}^x \cap (\cap_{i=1}^N VI(C, B_i^x)) \neq \emptyset$ *and* $\mathcal{F}^y = \mathcal{S}^y \cap (\cap_{i=1}^N VI(C, B_i^y)) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_1, y_1 \in H_1$ *and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right) x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n) J_{\lambda_1^x}^{M_1^x}\left(x - \gamma^x T^*\left(I - J_{\lambda_2^x}^{M_2^x}\right) T x_n\right)\right), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^y B_i^y\right) y_n + \eta_n\left(\alpha_n g(x_n) + (1-\alpha_n) J_{\lambda_1^y}^{M_1^y}\left(y - \gamma^y T^*\left(I - J_{\lambda_2^y}^{M_2^y}\right) T y_n\right)\right), \end{cases} \tag{62}$$

for all $n \geq 1$, *where* $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, $\lambda_i^x, \lambda_i^y \in (0,\infty)$ *for all* $i = 1,2$, *and* $0 < \gamma^x, \gamma^y < 1/L$ *with* L *being a spectral radius of* T^*T. *Assume the following conditions hold:*

(1) $\sum_{n=1}^\infty \mu_n^x < \infty, \sum_{n=1}^\infty \mu_n^y < \infty$, *and* $0 < a < \mu_n^x \leq 2\overline{\beta_x}$, $0 < b < \mu_n^y \leq 2\overline{\beta}_y$, *for some* $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^\infty \alpha_n = \infty, \lim_{n\longrightarrow\infty}\alpha_n = 0$.

(3) $\sum_{n=1}^N a_i^x = \sum_{n=1}^N a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, *for all* $n \in \mathbb{N}$, *for some* $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n|, \sum_{n=1}^\infty |\sigma_{n+1} - \sigma_n|$, *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ *and* $\{y_n\}$ *converges strongly to* $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$.

4. Applications

In this section, by applying our main result in Theorem 2, we can prove strong convergence theorems for approximating the solution of the split convex minimization problems and split feasibility problems.

4.1. Split Convex Minimization Problems. Let $\varphi\colon H \longrightarrow \mathbb{R}$ be a convex and differentiable function and $\psi\colon H \longrightarrow (-\infty,\infty]$ be a proper convex and lower semicontinuous function. It is well known that if $\nabla\varphi$ is $1/\alpha$-Lipschitz continuous, then it is α-inverse strongly monotone, where $\nabla\varphi$ is the gradient of φ (see [10]). It is also known that the subdifferential $\partial\psi$ of ψ is maximal monotone (see [35]). Moreover,

$$\varphi(x^*) + \psi(x^*) = \min_{x\in H}[\varphi(x) + \psi(x)] \Leftrightarrow 0 \in \nabla\varphi(x^*) + \partial\varphi(x^*). \tag{63}$$

Next, we consider the following the split convex minimization problem (SCMP): find

$$x^* \in H_1, \quad \text{such that}\ \varphi_1(x^*) + \psi_1(x^*) = \min_{x\in H_1}[\varphi_1(x) + \psi_1(x)] \tag{64}$$

and such that $y^* = Tx^* \in H_2$ solves

$$\varphi_2(y^*) + \psi_2(y^*) = \min_{y\in H_2}[\varphi_2(y) + \psi_2(y)], \tag{65}$$

where $T\colon H_1 \longrightarrow H_2$ is a bounded linear operator with adjoint T^*, φ_i, and ψ_i defined as above, for $i = 1,2$. We denoted the set of all solutions of (64) and (65) by Θ. That is, $\Theta = \{x^*$ which solves (64): Tx^* solves (65)$\}$.

If we set $A_i^x = \nabla\varphi_i^x, A_i^y = \nabla\varphi_i^y$, and $M_i^x = \partial\psi_i^x, M_i^y = \partial\psi_i^y$, for $i = 1,2$, in Theorem 2, then we get the strong convergence theorem for finding the common solution of the split convex minimization problems and the finite families of the variational inequality problems as follows.

Theorem 3. *Let* H_1 *and* H_2 *be Hilbert spaces, and let* C *be a nonempty closed convex subset of* H_1. *Let* $T\colon H_1 \longrightarrow H_2$ *be a bounded linear operator, and let* $f, g\colon H_1 \longrightarrow H_1$ *be* ρ_f, ρ_g*-contraction mappings with* $\rho = \max\{\rho_f, \rho_g\}$. *For* $i = 1,2$, *let* $\psi_i^x, \psi_i^y\colon H_i \longrightarrow (-\infty,\infty]$ *be proper convex and lower semicontinuous functions, and let* $\varphi_i^x, \varphi_i^y\colon H_i \longrightarrow \mathbb{R}$ *be convex and differentiable function such that* $\nabla\varphi_i^x$ *and* $\nabla\varphi_i^y$ *be* $1/\alpha_i^x$*-Lipschitz continuous and* $1/\alpha_i^y$*-Lipschitz continuous, respectively. For* $i = 1,2,\ldots,N$, *let* $B_i^x, B_i^y\colon H_1 \longrightarrow H_1$ *be* β_i^x, β_i^y*−inverse strongly monotone with* $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$ *and* $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. *Let* $\widetilde{G}^x, \widetilde{G}^y\colon H_1 \longrightarrow H_1$ *be defined by* $\widetilde{G}^x x = J_{\lambda_1^x}^{\partial\psi_1^x}(I - \lambda_1^x\nabla\varphi_1^x)(x - \gamma^x T^*(I - J_{\lambda_2^x}^{\partial\psi_2^x}(I - \lambda_2^x\nabla\varphi_2^x))Tx), \forall x \in H_1$, *and* $\widetilde{G}^y y = J_{\lambda_1^y}^{\partial\psi_1^y}(I - \lambda_1^y\nabla\varphi_1^y)(y - \gamma^y T^*(I - J_{\lambda_2^y}^{\partial\psi_2^y}(I - \lambda_2^y\nabla\varphi_2^y))Ty), \forall y \in H_1$, *respectively, where* $\lambda_i^x \in (0, 2\alpha_i^x), \lambda_i^y \in (0, 2\alpha_i^y)$ *and* $0 < \gamma^x, \gamma^y < 1/L$ *with* L *is a spectral radius of* T^*T. *Assume that* $\mathcal{F}^x = \Theta^x \cap (\cap_{i=1}^N$

$VI(C,B_i^x)) \neq \emptyset$ and $\mathcal{F}^y = \Theta^y \cap (\cap_{i=1}^N VI(C,B_i^y)) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1, y_1 \in H_1$ and

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right)x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n)\widetilde{G}^x x_n\right), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^y \sum_{i=1}^N a_i^y B_i^y\right)y_n + \eta_n\left(\alpha_n g(x_n) + (1-\alpha_n)\widetilde{G}^y y_n\right), \end{cases} \tag{66}$$

for all $n \geq 1$ where $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, and $\{\mu_n^x\}, \{\mu_n^y\} \subset (0,\infty)$. Assume the following condition holds:

(1) $\sum_{n=1}^\infty \mu_n^x < \infty, \sum_{n=1}^\infty \mu_n^y < \infty$, and $0 < a < \mu_n^x \leq 2\overline{\beta}_x$, $0 < b < \mu_n^y \leq 2\overline{\beta}_y$, for some $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^\infty \alpha_n = \infty$, $lim_{n\longrightarrow\infty}\alpha_n = 0$.

(3) $\sum_{n=1}^N a_i^x = \sum_{n=1}^N a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, for all $n \in \mathbb{N}$, for some $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n|$, $\sum_{n=1}^\infty |\sigma_{n+1} - \sigma_n|$, and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $\widetilde{x} = P_{\mathcal{F}^x} f(\widetilde{y})$ and $\{y_n\}$ converges strongly to $\widetilde{y} = P_{\mathcal{F}^y} g(\widetilde{x})$.

4.2. The Split Feasibility Problem. Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be the nonempty closed convex subset of H_1 and H_2, respectively. The split feasibility problem (SFP) is to find

$$\text{a point } x \in C, \quad \text{such that } Ax \in Q. \tag{67}$$

The set of all solutions (SFP) is denoted by $\Psi = \{x \in C: Ax \in Q\}$. This problem was introduced by Censor and Elfving [8] in 1994. The split feasibility problem was investigated extensively as a widely important tool in many fields such as signal processing, intensity-modulated radiation therapy problems, and computer tomography (see [36–38] and the references therein).

Let H be a real Hilbert space, and let h be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. The subdifferential ∂h of h is defined by $\partial h(x) = \{z \in H: h(x) + \langle z, u - x\rangle \leq h(u), \forall u \in H\}$ for all $x \in H$. Then, ∂h is a maximal monotone operator [39]. Let C be a nonempty closed convex subset of H, and let i_C be the indicator function of C, i.e., $i_C(x) = 0$ if $x \in C$ and $i_C(x) = \infty$ if $x \notin C$. Then, i_C is a proper, lower semicontinuous and convex function on H, and so the subdifferential ∂i_C of i_C is a maximal monotone operator. Then, we can define the resolvent operator $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$, by $J_\lambda^{\partial i_C}(x) = (I + \lambda\, \partial i_C)^{-1}(x)$, for all $x \in H$.

Recall that the normal cone $N_C(u)$ of C at a point u in H is defined by $N_C(u) = \{z \in H: \langle z, u - v \leq 0\rangle, \forall v \in C\}$ if $u \in C$ and $N_C(u) = \emptyset$ if $u \notin C$. We note that $\partial i_C = N_C$, and for $\lambda > 0$, we have that $u = J_\lambda^{\partial i_C} x$ if and only if $u = P_C x$ (see [31]).

Setting $M_1 = \partial i_C$, $M_2 = \partial i_Q$, and in (SMVI) (4) and (5), then (SMVI) (4) and (5) are reduced to the split feasibility problem (SFP) (67)

Now, by applying Theorem 2, we get the following strong convergence theorem to approximate a common solution of SFP (67) and a finite family of variational inequality problems.

Theorem 4. *Let H_1 and H_2 be Hilbert spaces, and let C and Q be the nonempty closed convex subset of H_1 and H_2, respectively. Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint T^*, and let $f, g: H_1 \longrightarrow H_1$ be ρ_f, ρ_g-contraction mappings with $\rho = \max\{\rho_f, \rho_g\}$. For $i = 1, 2, \ldots, N$, let $B_i^x, B_i^y: H_1 \longrightarrow H_1$ be β_i^x, β_i^y−inverse strongly monotone with $\overline{\beta}_x = \min_{i=1,2,\ldots,N}\{\beta_i^x\}$ and $\overline{\beta}_y = \min_{i=1,2,\ldots,N}\{\beta_i^y\}$. Assume that $\mathcal{F}^x = \Psi^x \cap (\cap_{i=1}^N VI(C,B_i^x)) \neq \emptyset$ and $\mathcal{F}^y = \Psi^y \cap (\cap_{i=1}^N VI(C,B_i^y)) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1, y_1 \in H_1$ and*

$$\begin{cases} x_{n+1} = \delta_n x_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^x B_i^x\right)x_n + \eta_n\left(\alpha_n f(y_n) + (1-\alpha_n)P_C\left(x - \gamma^x T^*\left(I - P_Q\right)T x_n\right)\right), \\ y_{n+1} = \delta_n y_n + \sigma_n P_C\left(I - \mu_n^x \sum_{i=1}^N a_i^y B_i^y\right)y_n + \eta_n\left(\alpha_n g(x_n) + (1-\alpha_n)P_C\left(y - \gamma^y T^*\left(I - P_Q\right)T y_n\right), \end{cases} \tag{68}$$

for all $n \geq 1$, *where* $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^x, a_2^x, \ldots, a_N^x\}, \{a_1^y, a_2^y, \ldots, a_N^y\} \subset (0,1)$, $\{\mu_n^x\}, \{\mu_n^y\} \subset (0,\infty)$, $\lambda_i^x, \lambda_i^y \in (0,\infty)$ *for all* $i = 1,2$, *and* $0 < \gamma^x, \gamma^y < 1/L$ *with* L *being a spectral radius of* T^*T. *Assume the following condition holds:*

(1) $\sum_{n=1}^{\infty} \mu_n^x < \infty, \sum_{n=1}^{\infty} \mu_n^y < \infty$, *and* $0 < a < \mu_n^x \leq 2\overline{\beta_x}, 0 < b < \mu_n^y \leq 2\beta_y$, *for some* $a, b \in \mathbb{R}$.

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty, lim_{n\longrightarrow\infty} \alpha_n = 0$.

(3) $\sum_{n=1}^{N} a_i^x = \sum_{n=1}^{N} a_i^y = 1$.

(4) $0 < \overline{a} \leq \delta_n, \sigma_n, \eta_n \leq \overline{b} < 1$, *for all* $n \in \mathbb{N}$, *for some* $\overline{a}, \overline{b} > 0$.

(5) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n|$, $\sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $\tilde{x} = P_{\mathcal{F}^x} f(\tilde{y})$ *and* $\{y_n\}$ *converges strongly to* $\tilde{y} = P_{\mathcal{F}^y} g(\tilde{x})$.

Proof. Set $M_1^x = \partial i_C, M_1^y = \partial i_C$, $M_2^x = \partial i_Q, M_2^y = \partial i_Q$, and $A_i^x = 0$ and $A_i^y = 0$ in Theorem 2. Then, we get the result. □

The split feasibility problem is a significant part of the split monotone variational inclusion problem. It is extensively used to solve practical problems in numerous situations. Many excellent results have been obtained. In what follows, an example of a signal recovery problem is introduced.

Example 2. In signal recovery, compressed sensing can be modeled as the following under-determined linear equation system:

$$y = Ax + \delta, \tag{69}$$

where $x \in \mathbb{R}^N$ is a vector with m non-zero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy δ, and $A: \mathbb{R}^N \longrightarrow \mathbb{R}^M\,(M < N)$ is a bounded linear observation operator. An essential point of this problem is that the signal x is sparse; that is, the number of nonzero elements in the signal x is much smaller than the dimension of the signal x. To solve this situation, a classical model, convex constraint minimization problem, is used to describe the above problem. It is known that problem (69) can be seen as solving the following LASSO problem [40]:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y - Ax\|_2^2, \quad \text{subject to } \|x\|_1 \leq t, \tag{70}$$

where $t > 0$ is a given constant and $\|\cdot\|_1$ is ℓ_1 norm. In particular, LASSO problem (70) is equivalent to the split feasibility problem (SFP) (67) when $C = \{x \in \mathbb{R}^N: \|x\|_1 \leq t\}$ and $Q = \{y\}$.

5. Numerical Examples

In this section, we give some examples for supporting Theorem 2. In example 3, we give the computer programming to support our main result.

Example 3. Let $\mathbb{R}$ be a set of real number and $H_1 = H_2 = \mathbb{R}^2$. Let $C = [-20,20] \times [-20,20]$, and let $\langle\cdot,\cdot\rangle: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be inner product defined by $\langle x, y\rangle = x \cdot y = x_1 y_1 + x_2 y_2$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and the usual norm $\|\cdot\|: \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\|x\| = \sqrt{x_1^2 + x_2^2}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $Tx = (2x_1, 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $T^*: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T^*z = (2z_1, 2z_2)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let $M_1^x, M_1^y, M_2^x, M_2^y: \mathbb{R}^2 \longrightarrow 2^{\mathbb{R}^2}$ be defined by $M_1^x x = \{(3x_1 - 2, 3x_2 - 2)\}, M_1^y x = \{(2x_1, 2x_2)\}, M_2^x = \{((x_1/3) + 2, (x_2/3) + 2)\}$, and $M_2^y x = \{((x_1/3) + 3, (x_2/3) + 3)\}$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A_1^x, A_1^y, A_2^x, A_2^y: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $A_1^x x = ((x_1 - 3)/2, (x_2 - 3)/2), A_1^y x = (x_1 + 6, x_2 + 6)$, $A_2^x x = ((x_1 - 2)/3, (x_2 - 2)/3)$, and $A_2^y = ((x_1 - 1)/3, (x_2 - 1)/3)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. For every $i = 1, 2, \ldots, N$, let the mappings $B_i^x, B_i^y: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $B_i^x x = ((x_1 - 1)/3i, (x_2 - 1)/3i)$ and $B_i^y x = ((x_1 + 2)/5i, (x_2 + 2)/5i)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$, and let $a_i^x = (2/3^i + 1/N3^N)$ and $a_i^y = (4/5^i + 1/N5^N)$. Let the mappings $f, g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $f(x) = (x_1/7, x_2/7)$ and $g(x) = (x_1/9, x_2/9)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Choose γ^x and $\gamma^y = 0.1$, $\lambda_1^x = 2, \lambda_1^y = 1.2, \lambda_2^x = 0.1$, and $\lambda_2^y = 1.9$. Setting $\{\delta_n\} = \{n/(9n + 3)\}$, $\{\sigma_n\} = \{(4n + (2/3))/(9n + 3)\}$, $\{\eta_n\} = \{(4n + (7/3))/(9n + 3)\}$, $\{\alpha_n\} = \{1/20n\}$, $\{\mu_n^x\} = \{1/7n^2\}$, and $\{\mu_n^y\} = \{1/5n^2\}$. Let $x_1 = (x_1^1, x_1^2)$ and $y_1 = (y_1^1, y_1^2) \in \mathbb{R}^2$, and let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (21) as follows:

TABLE 1: Values of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-10, 10)$, $y_1 = (-10, 10)$, and $n = N = 50$.

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−10.000000, 10.000000)	(−10.000000, 10.000000)
2	(−4.093342, 5.126154)	(−5.481385, 3.338182)
3	(−1.544910, 3.037093)	(−3.639877, 0.575101)
4	(−0.307347, 2.029037)	(−2.789106, −0.719370)
⋮	⋮	⋮
30	(0.998745, 0.996821)	(−1.996611, −1.998243)
⋮	⋮	⋮
47	(0.998973, 0.998235)	(−1.998394, −1.999022)
48	(0.998986, 0.998280)	(−1.998454, −1.999055)
49	(0.999000, 0.998324)	(−1.998511, −1.999087)
50	(0.999013, 0.998365)	(−1.998566, −1.999118)

$$\begin{cases} x_{n+1} = \dfrac{n}{9n+3}x_n + \dfrac{4n+(2/3)}{9n+3}P_C\left(I - \dfrac{1}{7n^2}\displaystyle\sum_{i=1}^{N}\left(\dfrac{2}{3^i} + \dfrac{1}{N3^N}\right)B_i^x\right)x_n + \dfrac{4n+7/3}{9n+3}\left(\dfrac{1}{20n}f(y_n) + \dfrac{20n-1}{20n}\widetilde{G}^x x_n\right), \\ y_{n+1} = \dfrac{n}{9n+3}y_n + \dfrac{4n+(2/3)}{9n+3}P_C\left(I - \dfrac{1}{5n^2}\displaystyle\sum_{i=1}^{N}\left(\dfrac{4}{5^i} + \dfrac{1}{N5^N}\right)B_i^y\right)y_n + \dfrac{4n+7/3}{9n+3}\left(\dfrac{1}{20n}g(x_n) + \dfrac{20n-1}{20n}\widetilde{G}^y y_n\right), \\ \widetilde{G}^x x_n = J_2^{M_1^x}(I - 2A_1^x)\left(x - 0.1T^*\left(I - J_{0.1}^{M_2^x}(I - 0.1A_2^x)\right)Tx_n\right), \\ \widetilde{G}^y y_n = J_{1.2}^{M_1^y}(I - 1.2A_1^y)\left(y - 0.1T^*\left(I - J_{1.9}^{M_2^y}(I - 1.9A_2^y)\right)Ty_n\right), \end{cases} \tag{71}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2)$ and $y_n = (y_n^1, y_n^2)$. By the definition of M_i^x, M_i^y, A_i^x, and A_i^y, for all $i = 1, 2$, B_i^x and B_i^y, for all $i = 1, 2, \ldots, N$, and f and g, we have that $(1, 1) \in \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x))$ and $(-2, -2) \in \Omega^y \cap (\cap_{i=1}^N VI(C, B_i^y))$. Also, it is easy to see that all parameters satisfy all conditions in Theorem 2. Then, by Theorem 2, we can conclude that the sequence $\{x_n\}$ converges strongly to $(1, 1)$ and $\{y_n\}$ converges strongly to $(-2, -2)$.

Table 1 and Figure 1 show the numerical results of $\{x_n\}$ and $\{y_n\}$ where $x_1 = (-10, 10)$, $y_1 = (-10, 10)$, and $n = N = 50$.

Next, in Example 4, we only show an example in infinite-dimensional Hilbert space for supporting Theorem 2. We omit the computer programming.

Example 4. Let $H_1 = H_2 = C = \ell_2$ be the linear space whose elements consist of all 2-summable sequence $(x_1, x_2, \ldots, x_j, \ldots)$ of scalars, i.e.,

$$\ell_2 = \left\{x\colon x = (x_1, x_2, \ldots, x_j, \ldots) \text{ and } \sum_{j=1}^{\infty}\left|x_j\right|^2 < \infty\right\}, \tag{72}$$

with an inner product $\langle\cdot,\cdot\rangle\colon \ell_2 \times \ell_2 \longrightarrow \mathbb{R}$ defined by $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j$, where $x = \{x_j\}_{j=1}^{\infty}$ and $y = \{y_j\}_{j=1}^{\infty} \in \ell_2$, and a norm $\|\cdot\|\colon \ell_2 \longrightarrow \mathbb{R}$ defined by $\|x\|_2 = (\sum_{j=1}^{\infty}|x_j|^2)^{1/2}$ where $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let $T\colon \ell_2 \longrightarrow \ell_2$ be defined by $Tx = (x_1/2, x_2/2, \ldots, x_j/2, \ldots)$ for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. And, $T^*\colon \ell_2 \longrightarrow \ell_2$ be defined by $Tx = (z_1/2, z_2/2, \ldots, z_j/2, \ldots)$ for all $x = \{z_j\}_{j=1}^{\infty} \in \ell_2$. Let $M_1^x, M_1^y, M_2^x, M_2^y\colon \ell_2 \longrightarrow \ell_2$ be defined by $M_1^x x = \{(2x_1, 2x_2, \ldots, 2x_j, \ldots)\}$, $M_1^y x = \{(x_1 - 1, x_2 - 1, \ldots, x_j - 1, \ldots)\}$, $M_2^x = \{3x_1, 3x_2, \ldots, 3x_j, \ldots\}$, and $M_2^y x = \{(2x_1 - 1, 2x_2 - 1, \ldots, 2x_j - 1, \ldots)\}$, respectively, for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let the mapping $A_1^x, A_1^y, A_2^x, A_2^y\colon \ell_2 \longrightarrow \ell_2$ be defined by $A_1^x x = (x_1/3, x_2/3, \ldots, x_j/3, \ldots)$, $A_1^y x = ((x_1 - 1)/2, (x_2 - 1)/2, \ldots, (x_j - 1)/2, \ldots)$, $A_2^x x = (x_1/4, x_2/4, \ldots, x_j/4, \ldots)$, and $A_2^y = ((2x_1 - 1)/3, (2x_2 - 1)/3, \ldots, (2x_j - 1)/3)$ respectively, for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. For every $i = 1, 2, \ldots, N$, let the mappings $B_i^x, B_i^y\colon \ell_2 \longrightarrow \ell_2$ be defined by $B_i^x x = (2x_1/3i, 2x_2/3i, \ldots, 2x_j/3i, \ldots)$ and $B_i^y x = ((2x_1 - 1)/4i, (2x_2 - 1)/4i, \ldots, (2x_j - 1)/4i)$, respectively,

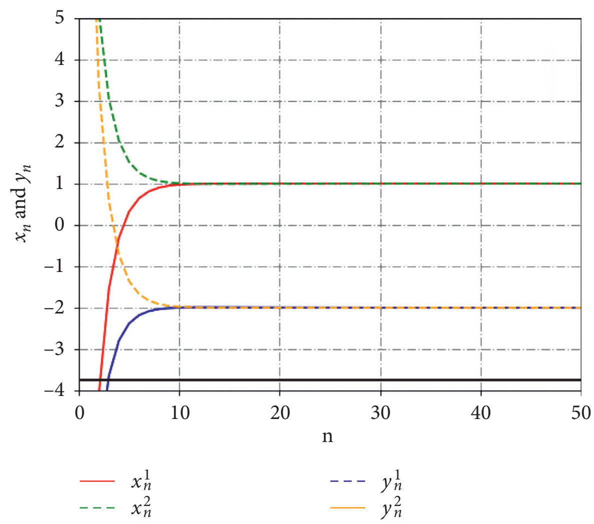

Figure 1: Convergence of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-10, 10)$, $y_1 = (-10, 10)$, and $n = N = 50$.

for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$, and let $a_i^x = (5/6^i + 1/N6^N)$ and $a_i^y = (7/8^i + 1/N8^N)$. Let the mappings $f, g\colon \ell_2 \longrightarrow \ell_2$ be defined by $f(x) = (x_1/5, x_2/5, \ldots, x_j/5, \ldots)$, $g(x) = (x_1/4, x_2/4, \ldots, x_j/4, \ldots)$, respectively, for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$.

Let $\lambda_1^x = 1, \lambda_1^y = 1, \lambda_2^x = 0.5$, and $\lambda_2^y = 2$. Since $L = 1/4$, we choose γ^x and $\gamma^y = 0.5$. Let $x_1 = (x_1^1, x_1^2, \ldots, x_1^j, \ldots)$ and $y_1 = (y_1^1, y_1^2, \ldots, y_1^j, \ldots) \in \ell_2$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by (2) as follows:

$$\begin{cases} x_{n+1} = \dfrac{n}{5n+2} x_n + \dfrac{2n+2/3}{5n+2} P_C\left(I - \dfrac{1}{3n^2} \displaystyle\sum_{i=1}^{N}\left(\dfrac{5}{6^i} + \dfrac{1}{N6^N}\right) B_i^x\right) x_n + \dfrac{2n+4/3}{5n+3}\left(\dfrac{1}{10n} f(y_n) + \dfrac{10n-1}{20n} \widetilde{G}^x x_n\right), \\ y_{n+1} = \dfrac{n}{5n+2} y_n + \dfrac{2n+2/3}{5n+2} P_C\left(I - \dfrac{1}{4n^2} \displaystyle\sum_{i=1}^{N}\left(\dfrac{7}{8^i} + \dfrac{1}{N8^N}\right) B_i^y\right) y_n + \dfrac{2n+4/3}{5n+2}\left(\dfrac{1}{10n} g(x_n) + \dfrac{10n-1}{20n} \widetilde{G}^y y_n\right), \\ \widetilde{G}^x x_n = J_1^{M_1^x}(I - A_1^x)\left(x - 0.5T^*\left(I - J_{0.5}^{M_2^x}(I - 0.5A_2^x)\right) T x_n\right), \\ \widetilde{G}^y y_n = J_1^{M_1^y}(I - A_1^y)\left(y - 0.5T^*\left(I - J_2^{M_2^y}(I - 2A_2^y)\right) T y_n\right), \end{cases} \tag{73}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2, \ldots, x_n^j, \ldots)$ and $y_n = (y_n^1, y_n^2, \ldots, y_n^j, \ldots)$. It is easy to see that M_i^x, M_i^y, A_i^x, and A_i^y, $\forall i = 1, 2$, B_i^x and B_i^y, $\forall i = 1, 2, \ldots, N$, T, f, g, and all parameters satisfy Theorem 2. Furthermore, we have that $0 \in \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x))$ and $1 \in \Omega^y \cap (\cap_{i=1}^N VI(C, B_i^y))$. Then, by Theorem 2, we can conclude that the sequence $\{x_n\}$ converges strongly to 0 and $\{y_n\}$ converges strongly to 1.

6. Conclusion

(1). Table 1 and Figure 1 in Example 3 show that the sequence $\{x_n\}$ converges to $(1, 1) \in \Omega^x \cap (\cap_{i=1}^N VI(C, B_i^x))$ and $\{y_n\}$ converges to $(-2, -2) \in \Omega^y \cap (\cap_{i=1}^N VI(C, B_i^y))$

(2) Example 4 is an example in infinite-dimensional Hilbert space for supporting Theorem 2

(3) Theorem 2 guarantees the convergence of $\{x_n\}$ and $\{y_n\}$ in Example 3 and Example 4

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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ORIGINAL PAPER

VIETA-PELL-LIKE POLYNOMIALS AND SOME IDENTITIES

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Abstract. *In this paper, we introduce some new generalizations of the Vieta-Pell polynomial, which is called the Vieta-Pell-Like polynomial. We also give the generating function, the Binet's formula, the sum formula, and some well-known identities for this Vieta polynomial. Furthermore, the relations between the Vieta-Pell-Like polynomial and the previously well-known identities are presented.*

Keywords: *Vieta-Pell polynomials; Vieta-Pell-Lucas polynomials; Vieta-Pell-Like polynomial.*

1. INTRODUCTION

The Vieta polynomials were first introduced in 1991 by Robbins [1]. After that, in 2002, Horadam [2] introduced and studied the Vieta-Fibonacci polynomial $V_n(x)$ and Vieta-Lucas polynomials $v_n(x)$. These polynomials are defined respectively by

$$V_0(x) = 0, V_1(x) = 1, V_n(x) = xV_{n-1}(x) - V_{n-2}(x), \text{ for } n \geq 2$$

and

$$v_0(x) = 2, v_1(x) = x, V_n(x) = xv_{n-1}(x) - v_{n-2}(x), for n \geq 2.$$

The Vieta-Pell polynomials $t_n(x)$ and Vieta-Pell-Lucas polynomials $s_n(x)$ were studied in 2013 by Tasci and Yalcin [3]. They defined these polynomials for $|x| > 1$ by

$$t_0(x) = 0, t_1(x) = 1, t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x), \text{ for } n \geq 2$$

and

$$s_0(x) = 2, s_1(x) = 2x, s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x), \text{ for } n \geq 2.$$

They obtained the Binet form and generating functions of Vieta-Pell and Vieta-Pell-Lucas polynomials. Also, they received some differentiation rules and the finite summation formulas. Moreover, they show that Vieta-Pell and Vieta-Pell-Lucas polynomials are closely related to the well-known Chebyshev polynomials of the first kinds $T_n(x)$ and the second kinds $U_n(x)$. The related features of Vieta-Pell, Vieta-Pell-Lucas polynomials, and Chebyshev polynomials are given as

$$s_n(x) = 2T_n(x),$$

and

$$t_{n+1}(x) = U_n(x).$$

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Vieta–Fibonacci-like polynomials and some identities*

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Abstract

In this paper, we introduce a new type of the Vieta polynomial, which is Vieta–Fibonacci-like polynomial. After that, we establish the Binet formula, the generating function, the well-known identities, and the sum formula of this polynomial. Finally, we present the relationship between this polynomial and the previous well-known Vieta polynomials.

Keywords: Vieta–Fibonacci polynomial, Vieta–Lucas polynomial, Vieta–Fibonacci-like polynomial

AMS Subject Classification: 11C08, 11B39, 33C45

1. Introduction

In 2002, Horadam [1] introduced the new types of second order recursive sequences of polynomials which are called Vieta–Fibonacci and Vieta–Lucas polynomials respectively. The definition of Vieta–Fibonacci and Vieta–Lucas polynomials are defined as follows:

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Definition 1.1 ([1])**.** For any natural number n the Vieta–Fibonacci polynomials sequence $\{V_n(x)\}_{n=0}^{\infty}$ and the Vieta–Lucas polynomials sequence $\{v_n(x)\}_{n=0}^{\infty}$ are defined by

$$V_n(x) = xV_{n-1}(x) - V_{n-2}(x), \quad \text{for } n \geq 2,$$
$$v_n(x) = xv_{n-1}(x) - v_{n-2}(x), \quad \text{for } n \geq 2,$$

respectively, where $V_0(x) = 0$, $V_1(x) = 1$ and $v_0(x) = 2$, $v_1(x) = x$.

The first few terms of the Vieta–Fibonacci polynomials sequence are $0, 1, x, x^2 - 1, x^3 - 2x, x^4 - 3x^2 + 1$ and the first few terms of the Vieta–Lucas polynomials sequence are $2, x, x^2 - 2, x^3 - 3x, x^4 - 4x^2 + 2, x^5 - 5x^3 + 5x$. The Binet formulas of the Vieta–Fibonacci and Vieta–Lucas polynomials are given by

$$V_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$
$$v_n(x) = \alpha^n(x) + \beta^n(x),$$

respectively. Where $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$ are the roots the characteristic equation $r^2 - xr + 1 = 0$. We also note that $\alpha(x) + \beta(x) = x$, $\alpha(x)\beta(x) = 1$, and $\alpha(x) - \beta(x) = \sqrt{x^2 - 4}$.

Recall that the Chebyshev polynomials are a sequence of orthogonal polynomials which can be defined recursively. The n^{th} Chebyshev polynomials of the first and second kinds are denoted by $\{T_n(x)\}_{n=0}^{\infty}$ and $\{U_n(x)\}_{n=0}^{\infty}$ and are defined respectively by $T_0(x) = 1$, $T_1(x) = x, T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, for $n \geq 2$, and $U_0(x) = 1$, $U_1(x) = 2x, U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, for $n \geq 2$. These polynomials are of great importance in many areas of mathematics, particularly approximation theory. It is well known that the Chebyshev polynomials of the first kind and second kind are closely related to Vieta–Fibonacci and Vieta–Lucas polynomials. So, in [4] Vitula and Slota redefined Vieta polynomials as modified Chebyshev polynomials. The related features of Vieta and Chebyshev polynomials are given as $V_n(x) = U_n\left(\frac{1}{2}x\right)$ and $v_n(x) = 2T_n\left(\frac{1}{2}x\right)$ (see [1, 2, 5]).

In 2013, Tasci and Yalcin [6] introduced the recurrence relation of Vieta–Pell and Vieta–Pell–Lucas polynomials as follows:

Definition 1.2 ([6])**.** For $|x| > 1$ and for any natural number n the Vieta–Pell polynomials sequence $\{t_n(x)\}_{n=0}^{\infty}$ and the Vieta–Pell–Lucas polynomials sequence $\{s_n(x)\}_{n=0}^{\infty}$ are defined by

$$t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x), \quad \text{for } n \geq 2,$$
$$s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x), \quad \text{for } n \geq 2.$$

respectively, where $t_0(x) = 0$, $t_1(x) = 1$ and $s_0(x) = 2$, $s_1(x) = 2x$.

The $t_n(x)$ and $s_n(x)$ are called the n^{th} Vieta–Pell polynomial and the n^{th} Vieta–Pell–Lucas polynomial respectively. Tasci and Yalcin [6] obtained the Binet form and generating functions of Vieta–Pell and Vieta–Pell–Lucas polynomials. Also, they obtained some differentiation rules and the finite summation formulas. Moreover, the following relations are obtained

$$s_n(x) = 2T_n(x), \quad \text{and} \quad t_{n+1}(x) = U_n(x).$$

In 2015, Yalcin et al. [8], introduced and studied the Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials which defined as follows:

Definition 1.3 ([8])**.** For any natural number n the Vieta–Jacobsthal polynomials sequence $\{G_n(x)\}_{n=0}^{\infty}$ and the Vieta–Jacobsthal-Lucas polynomials sequence $\{g_n(x)\}_{n=0}^{\infty}$ are defined by

$$\begin{aligned} G_n(x) &= G_{n-1}(x) - 2xG_{n-2}(x), \quad \text{for } n \geq 2, \\ g_n(x) &= g_{n-1}(x) - 2xg_{n-2}(x), \quad \text{for } n \geq 2, \end{aligned}$$

respectively, where $G_0(x) = 0$, $G_1(x) = 1$ and $g_0(x) = 2$, $g_1(x) = 1$.

Moreover, for any nonnegative integer k with $1-2^{k+2}x \neq 0$, Yalcin et al. [8] also considered the generalized Vieta–Jacobsthal polynomials sequences $\{G_{k,n}(x)\}_{n=0}^{\infty}$ and Vieta–Jacobsthal–Lucas polynomials sequences $\{g_{k,n}(x)\}_{n=0}^{\infty}$ by the following recurrence relations

$$\begin{aligned} G_{k,n}(x) &= G_{k,n-1}(x) - 2^k xG_{k,n-2}(x), \quad \text{for } n \geq 2, \\ g_{k,n}(x) &= g_{k,n-1}(x) - 2^k xg_{k,n-2}(x), \quad \text{for } n \geq 2, \end{aligned}$$

respectively, where $G_{k,0}(x) = 0$, $G_{k,1}(x) = 1$ and $g_{k,0}(x) = 2$, $g_{k,1}(x) = 1$. If $k = 1$, then $G_{1,n}(x) = G_n(x)$ and $g_{1,n}(x) = g_n(x)$. In [8], the Binet form and generating functions for these polynomials are derived. Furthermore, some special cases of the results are presented.

Recently, the generalization of Vieta–Fibonacci, Vieta–Lucas, Vieta–Pell, Vieta–Pell–Lucas, Vieta–Jacobsthal, and Vieta–Jacobsthal–Lucas polynomials have been studied by many authors.

In 2016 Kocer [3], considered the bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials which are generalized of Vieta–Fibonacci, Vieta–Lucas, Vieta–Pell, Vieta–Pell–Lucas polynomials. She also gave some properties. Afterward, she obtained some identities for the bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials by using the known properties of bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials.

In 2020 Uygun et al. [7], introduced the generalized Vieta–Pell and Vieta–Pell–Lucas polynomial sequences. They also gave the Binet formula, generating functions, sum formulas, differentiation rules, and some important properties for these sequences. And then they generated a matrix whose elements are of generalized Vieta–Pell terms. By using this matrix they derived some properties for generalized Vieta–Pell and generalized Vieta–Pell–Lucas polynomial sequences.

Inspired by the research going on in this direction, in this paper, we introduce a new type of Vieta polynomial, which is called Vieta–Fibonacci-like polynomial.

We also give the Binet form, the generating function, the well-known identities, and the sum formula for this polynomial. Furthermore, the relationship between this polynomial and the previous well-known Vieta polynomials are given in this study.

2. Vieta–Fibonacci-like polynomials

In this section, we introduce a new type of Vieta polynomial, called the Vieta–Fibonacci-like polynomials, as the following definition.

Definition 2.1. For any natural number n the Vieta–Fibonacci-like polynomials sequence $\{S_n(x)\}_{n=0}^{\infty}$ is defined by

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x), \quad \text{for } n \geq 2, \tag{2.1}$$

with the initial conditions $S_0(x) = 2$ and $S_1(x) = 2x$.

The first few terms of $\{S_n(x)\}_{n=0}^{\infty}$ are 2, $2x$, $2x^2 - 2$, $2x^3 - 4x$, $2x^4 - 6x^2 + 2$, $2x^5 - 8x^3 + 6x$, $2x^6 - 10x^4 + 12x^2 - 2$, $2x^7 - 12x^5 + 20x^3 - 8x$ and so on. The n^{th} terms of this sequence are called Vieta–Fibonacci-like polynomials.

First, we give the generating function for the Vieta–Fibonacci-like polynomials as follows.

Theorem 2.2 (The generating function)**.** *The generating function of the Vieta–Fibonacci-like polynomials sequence is given by*

$$g(x,t) = \frac{2}{1 - xt + t^2}.$$

Proof. The generating function $g(x,t)$ can be written as $g(x,t) = \sum_{n=0}^{\infty} S_n(x)t^n$. Consider,

$$g(x,t) = \sum_{n=0}^{\infty} S_n(x)t^n = S_0(x) + S_1(x)t + S_2(x)t^2 + \cdots + S_n(x)t^n + \ldots .$$

Then, we get

$$\begin{aligned} -xtg(x,t) &= -xS_0(x)t - xS_1(x)t^2 - xS_2(x)t^3 - \cdots - xS_{n-1}(x)t^n - \ldots \\ t^2 g(x,t) &= S_0(x)t^2 + S_1(x)t^3 + S_2(x)t^4 + \cdots + S_{n-2}(x)t^n + \ldots . \end{aligned}$$

Thus,

$$\begin{aligned} g(x,t)(1 - xt + t^2) &= S_0(x) + (S_1(x) - xS_0(x))t \\ &\quad + \sum_{n=2}^{\infty}(S_n(x) - xS_{n-1}(x) + S_{n-2}(x))t^n \\ &= 2, \end{aligned}$$

$$g(x,t) = \frac{2}{1 - xt + t^2}.$$

This completes the proof. □

Next, we give the explicit formula for the n^{th} Vieta–Fibonacci-like polynomials.

Theorem 2.3 (Binet's formula). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials, then*

$$S_n(x) = A\alpha^n(x) + B\beta^n(x), \tag{2.2}$$

where $A = \frac{2(x-\beta(x))}{\alpha(x)-\beta(x)}$, $B = \frac{2(\alpha(x)-x)}{\alpha(x)-\beta(x)}$ and $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$, $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$ are the roots of the characteristic equation $r^2 - xr + 1 = 0$.

Proof. The characteristic equation of the recurrence relation (2.1) is $r^2 - xr + 1 = 0$ and the roots of this equation are $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$.
It follows that

$$S_n(x) = d_1\alpha^n(x) + d_2\beta^n(x),$$

for some real numbers d_1 and d_2. Putting $n = 0$, $n = 1$, and then solving the system of linear equations, we obtain that

$$S_n(x) = \frac{2(x - \beta(x))}{\alpha(x) - \beta(x)}\alpha^n(x) + \frac{2(\alpha(x) - x)}{\alpha(x) - \beta(x)}\beta^n(x).$$

Setting $A = \frac{2(x-\beta(x))}{\alpha(x)-\beta(x)}$ and $B = \frac{2(\alpha(x)-x)}{\alpha(x)-\beta(x)}$, we get

$$S_n(x) = A\alpha^n(x) + B\beta^n(x).$$

This completes the proof. □

We note that $A + B = 2$, $AB = -\frac{4}{(\alpha(x)-\beta(x))^2}$, and $A\beta(x) + B\alpha(x) = 0$.

The other explicit forms of Vieta–Fibonacci-like polynomials are given in the following two theorems.

Theorem 2.4 (Explicit form). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n(x) = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i\binom{n-i}{i}x^{n-2i}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 2.2, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} S_n(x)t^n &= \frac{2}{1 - (xt - t^2)} \\ &= 2\sum_{n=0}^{\infty}(xt - t^2)^n\end{aligned}$$

$$= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(xt)^{n-i}(-t^2)^i$$
$$= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-1)^i x^{n-i}t^{n+i}$$
$$= \sum_{n=0}^{\infty}\left[2\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n-i}{i}x^{n-2i}\right]t^n.$$

From the equality of both sides, the desired result is obtained. This complete the proof. □

Theorem 2.5 (Explicit form)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n(x) = 2^{-n+1}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n+1}{2i+1}x^{n-2i}(x^2-4)^i, \quad \textit{for } n \geq 1.$$

Proof. Consider,

$$\begin{aligned}
\alpha^{n+1}(x) - \beta^{n+1}(x) &= 2^{-(n+1)}[(x+\sqrt{x^2-4})^{n+1} - (x-\sqrt{x^2-4})^{n+1}] \\
&= 2^{-(n+1)}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}x^{n-i+1}(\sqrt{x^2-4})^i\right. \\
&\quad \left. - \sum_{i=0}^{n+1}\binom{n+1}{i}x^{n-i+1}(-\sqrt{x^2-4})^i\right] \\
&= 2^{-n}\left[\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2i+1}x^{n-2i}(\sqrt{x^2-4})^{2i+1}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
S_n(x) &= A\alpha^n(x) + B\beta^n(x) \\
&= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)} \\
&= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\sqrt{x^2-4}} \\
&= 2^{-n+1}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2i+1}x^{n-2i}(x^2-4)^i.
\end{aligned}$$

This completes the proof. □

Theorem 2.6 (Sum formula)**.** *Let* $\{S_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$\sum_{k=0}^{n-1} S_k(x) = \frac{2 - S_n(x) + S_{n-1}(x)}{2-x}, \quad \textit{for } n \geq 1.$$

Proof. By using Binet formula (2.2), we get

$$\begin{aligned}
\sum_{k=0}^{n-1} S_k(x) &= \sum_{k=0}^{n-1} \left(A\alpha^k(x) + B\beta^k(x)\right) \\
&= A\frac{1-\alpha^n(x)}{1-\alpha(x)} + B\frac{1-\beta^n(x)}{1-\beta(x)} \\
&= \frac{A + B - (A\beta(x) + B\alpha(x)) - (A\alpha^n(x) + B\beta^n(x))}{1-x+1} \\
&\quad + \frac{A\alpha^{n-1}(x) + B\beta^{n-1}(x)}{1-x+1} \\
&= \frac{2 - S_n(x) + S_{n-1}(x)}{2-x}.
\end{aligned}$$

This completes the proof. □

Since the derivative of the polynomials is always exists, we can give the following formula.

Theorem 2.7 (Differentiation formula)**.** *The derivative of* $S_n(x)$ *is obtained as the follows.*

$$\frac{\mathrm{d}}{\mathrm{d}x} S_n(x) = \frac{(n+1)v_{n+1}(x) - xV_{n+1}(x)}{2(x^2-4)},$$

where $V_n(x)$ *and* $v_n(x)$ *are the* n^{th} *Vieta–Fibonacci and Vieta–Lucas polynomials, respectively.*

Proof. The result is obtained by using Binet formula (2.2). □

Again, by using Binet formula (2.2), we obtain some well-known identities as follows.

Theorem 2.8 (Catalan's identity or Simson identities)**.** *Let* $\{S_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n^2(x) - S_{n+r}(x)S_{n-r}(x) = S_{r-1}^2(x), \quad \textit{for } n \geq r \geq 1. \tag{2.3}$$

Proof. By using Binet formula (2.2), we obtain

$$\begin{aligned}
&S_n^2(x) - S_{n+r}(x)S_{n-r}(x) \\
&\qquad = (A\alpha^n(x) + B\beta^n(x))^2 - \left(A\alpha^{n+r}(x) + B\beta^{n+r}(x)\right)\left(A\alpha^{n-r}(x) + B\beta^{n-r}(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -AB\left(\alpha(x)\beta(x)\right)^{n-r}\left(\alpha^{2r}(x) - 2\left(\alpha(x)\beta(x)\right)^{r} + \beta^{2r}(x)\right)\\
&= \frac{4}{(\alpha(x)-\beta(x))^2}\left(\alpha^{r}(x) - \beta^{r}(x)\right)^2\\
&= (A\alpha^{r-1}(x) + B\beta^{r-1}(x))^2\\
&= S_{r-1}^2(x).
\end{aligned}
$$

Thus,

$$S_n^2(x) - S_{n+r}(x)S_{n-r}(x) = S_{r-1}^2(x).$$

This completes the proof. □

Take $r = 1$ in Catalan's identity (2.3), then we get the following corollary.

Corollary 2.9 (Cassini's identity)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n^2(x) - S_{n+1}(x)S_{n-1}(x) = 4, \quad \text{for } n \geq 1.$$

Theorem 2.10 (d' Ocagne's identity)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_m(x)S_{n+1}(x) - S_{m+1}(x)S_n(x) = 2S_{m-n-1}(x), \quad \text{for } m \geq n \geq 1. \tag{2.4}$$

Proof. We will prove d' Ocagne's identity (2.4) by using Binet formula (2.2). Consider,

$$
\begin{aligned}
S_m(x)&S_{n+1}(x) - S_{m+1}(x)S_n(x)\\
&= (A\alpha^{m}(x) + B\beta^{m}(x))\left(A\alpha^{n+1}(x) + B\beta^{n+1}(x)\right)\\
&\quad - \left(A\alpha^{m+1}(x) + B\beta^{m+1}(x)\right)(A\alpha^{n}(x) + B\beta^{n}(x))\\
&= -AB\left(\alpha(x)\beta(x)\right)^{n}(\alpha(x)-\beta(x))\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)\\
&= \frac{4}{\left(\alpha(x)-\beta(x)\right)^2}(\alpha(x)-\beta(x))\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)\\
&= 2(A\alpha^{m-n-1}(x) + B\beta^{m-n-1}(x))\\
&= 2S_{m-n-1}(x).
\end{aligned}
$$

This completes the proof. □

Theorem 2.11 (Honsberger identity)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_{m+1}(x)S_{n+1}(x) + S_m(x)S_n(x) = \frac{4xv_{m+n+3}(x) - 8v_{m-n}(x)}{x^2-4}, \quad \text{for } m \geq n \geq 1,$$

where $v_n(x)$ is the n^{th} Vieta–Lucas polynomials.

Proof. By using Binet formula (2.2), we obtain

$$\begin{aligned}
&S_{m+1}(x)S_{n+1}(x) + S_m(x)S_n(x)\\
&\qquad = \left(A\alpha^{m+1}(x) + B\beta^{m+1}(x)\right)\left(A\alpha^{n+1}(x) + B\beta^{n+1}(x)\right)\\
&\qquad\quad + \left(A\alpha^{m}(x) + B\beta^{m}(x)\right)\left(A\alpha^{n}(x) + B\beta^{n}(x)\right)\\
&\qquad = xA^2\alpha^{m+n+1}(x) + xB^2\beta^{m+n+1}(x) + 2AB(\alpha^{m-n}(x) + \beta^{m-n}(x))\\
&\qquad = \frac{4x(\alpha^{m+n+3}(x) + \beta^{m+n+3}(x)) - 8(\alpha^{m-n}(x) + \beta^{m-n}(x))}{(\alpha(x) - \beta(x))^2}\\
&\qquad = \frac{4xv_{m+n+3}(x) - 8v_{m-n}(x)}{x^2 - 4}.
\end{aligned}$$

This completes the proof. □

In the next theorem, we obtain the relation between the Vieta–Fibonacci-like, Vieta–Fibonacci and the Vieta–Lucas polynomials by using Binet formula (2.2).

Theorem 2.12. *Let $\{S_n(x)\}_{n=0}^{\infty}$, $\{V_n(x)\}_{n=0}^{\infty}$ and $\{v_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Fibonacci-like, Vieta–Fibonacci and Vieta–Lucas polynomials, respectively. Then*

(1) $S_n(x) = 2V_{n+1}(x)$, *for* $n \geq 0$,

(2) $S_n(x) = v_n(x) + xV_n(x)$, *for* $n \geq 0$,

(3) $S_n(x)v_{n+1}(x) = 2V_{2n+2}(x)$, *for* $n \geq 0$,

(4) $S_{n+1}(x) + S_{n-1}(x) = 2xV_{n+1}(x)$, *for* $n \geq 1$,

(5) $S_{n+1}(x) - S_{n-1}(x) = 2v_{n+1}(x)$, *for* $n \geq 1$,

(6) $S_{n+2}^2(x) - S_{n-1}^2(x) = 4xV_{2n+2}(x)$, *for* $n \geq 1$,

(7) $2S_n(x) - xS_{n-1}(x) = 2v_n(x)$, *for* $n \geq 1$,

(8) $S_{n+2}(x) + S_{n-2}(x) = (2x^2 - 4)V_{n+1}(x)$, *for* $n \geq 2$,

(9) $S_{n+2}^2(x) - S_{n-2}^2(x) = 4x(x^2 - 2)V_{2n+2}(x)$, *for* $n \geq 2$,

(10) $v_{n+1}(x) - v_n(x) = \frac{1}{2}(x^2 - 4)S_{n-1}(x)$, *for* $n \geq 1$,

(11) $2v_{n+1}(x) - xv_n(x) = \frac{1}{2}(x^2 - 4)S_{n-1}(x)$, *for* $n \geq 1$,

(12) $4v_n^2(x) + (x^2 - 4)S_{n-1}^2(x) = 8v_n(x)$, *for* $n \geq 1$,

(13) $4v_n^2(x) - (x^2 - 4)S_{n-1}^2(x) = 16$, *for* $n \geq 1$.

Proof. The results (1)–(13) are easily obtained by using Binet formula (2.2). □

3. Matrix Form of Vieta–Fibonacci-like polynomials

In this section, we establish some identities of Vieta–Fibonacci-like and Vieta–Fibonacci polynomials by using elementary matrix methods.

Let Q_s be 2×2 matrix defined by

$$Q_S = \begin{bmatrix} 2x^2 - 2 & 2x \\ -2x & -2 \end{bmatrix}. \tag{3.1}$$

Then by using this matrix we can deduce some identities of Vieta–Fibonacci-like and Vieta–Fibonacci polynomials.

Theorem 3.1. *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials and Q_s be 2×2 matrix defined by* (3.1). *Then*

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \textit{for } n \geq 1.$$

Proof. For the proof, mathematical induction method is used. It obvious that the statement is true for $n = 1$. Suppose that the result is true for any positive integer k, then we also have the result is true for $k + 1$. Because

$$\begin{aligned} Q_S^{k+1} &= Q_S^k \cdot Q_S \\ &= 2^{k-1} \begin{bmatrix} S_{2k}(x) & S_{2k-1}(x) \\ -S_{2k-1}(x) & -S_{2k-2}(x) \end{bmatrix} \begin{bmatrix} 2x^2 - 2 & 2x \\ -2x & -2 \end{bmatrix} \\ &= 2^{(k+1)-1} \begin{bmatrix} S_{2(k+1)}(x) & S_{2(k+1)-1}(x) \\ -S_{2(k+1)-1}(x) & -S_{2(k+1)-2}(x) \end{bmatrix}. \end{aligned}$$

By Mathematical induction, we have that the result is true for each $n \in \mathbb{N}$, that is

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$ □

Theorem 3.2. *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $2S_{2(m+n)}(x) = S_{2m}(x)S_{2n}(x) - S_{2m-1}(x)S_{2n-1}(x)$,

(2) $2S_{2(m+n)-1}(x) = S_{2m}(x)S_{2n-1}(x) - S_{2m-1}(x)S_{2n-2}(x)$,

(3) $2S_{2(m+n)-1}(x) = S_{2m-1}(x)S_{2n}(x) - S_{2m-2}(x)S_{2n-1}(x)$,

(4) $2S_{2(m+n)-2}(x) = S_{2m-1}(x)S_{2n-1}(x) - S_{2m-2}(x)S_{2n-2}(x)$.

Proof. By Theorem 3.1 and the property of power matrix $Q_s^{m+n} = Q_s^m \cdot Q_s^n$, then we obtained the results. □

By Theorem 3.1 and $S_n(x) = 2V_{n+1}(x)$, we get the following Corollary.

Corollary 3.3. *Let $\{V_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci polynomials and Q_s be 2×2 matrix defined by (3.1). Then*

$$Q_S^n = 2^n \begin{bmatrix} V_{2n+1}(x) & V_{2n}(x) \\ -V_{2n}(x) & -V_{2n-1}(x) \end{bmatrix}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 3.1, we get

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Since $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} Q_S^n &= 2^{n-1} \begin{bmatrix} 2V_{2n+1}(x) & 2V_{2n}(x) \\ -2V_{2n}(x) & -2V_{2n-1}(x) \end{bmatrix} \\ &= 2^n \begin{bmatrix} V_{2n+1}(x) & V_{2n}(x) \\ -V_{2n}(x) & -V_{2n-1}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \end{aligned}$$

This completes the proof. □

By Theorem 3.2 and $S_n(x) = 2V_{n+1}(x)$, we get the following Corollary.

Corollary 3.4. *Let $\{V_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci polynomials. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $V_{2(m+n)+1}(x) = V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)$,

(2) $V_{2(m+n)}(x) = V_{2m+1}(x)V_{2n}(x) - V_{2m}(x)V_{2n-1}(x)$,

(3) $V_{2(m+n)}(x) = V_{2m}(x)V_{2n+1}(x) - V_{2m-1}(x)V_{2n}(x)$,

(4) $V_{2(m+n)-1}(x) = V_{2m}(x)V_{2n}(x) - V_{2m-1}(x)V_{2n-1}(x)$.

Proof. From Theorem 3.2 and $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} V_{2(m+n)+1}(x) &= \frac{1}{2} S_{2(m+n)}(x) \\ &= \frac{1}{4}\left(S_{2m}(x)S_{2n}(x) - S_{2m-1}(x)S_{2n-1}(x)\right) \\ &= \frac{1}{4}\left(2V_{2m+1}(x)2V_{2n+1}(x) - 2V_{2m}(x)2V_{2n}(x)\right) \\ &= V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x). \end{aligned}$$

Thus, we get that (1) holds. By the same argument as above, we get that (2), (3), and (4) holds. This completes the proof. □

By Corollary 3.4 and $S_n(x) = 2V_{n+1}(x)$, we get the following corollary.

Corollary 3.5. *Let $\{S_n(x)\}_{n=0}^{\infty}$ and $\{V_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Fibonacci-like polynomials and Vieta–Fibonacci polynomials, respectively. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $S_{2(m+n)}(x) = 2\left(V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)\right),$

(2) $S_{2(m+n)-1}(x) = 2\left(V_{2m+1}(x)V_{2n}(x) - V_{2m}(x)V_{2n-1}(x)\right),$

(3) $S_{2(m+n)-1}(x) = 2\left(V_{2m}(x)V_{2n+1}(x) + V_{2m-1}(x)V_{2n}(x)\right),$

(4) $S_{2(m+n)-2}(x) = 2\left(V_{2m}(x)V_{2n}(x) + V_{2m-1}(x)V_{2n-1}(x)\right).$

Proof. From Corollary 3.4 and $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} S_{2(m+n)}(x) &= 2V_{2(m+n)+1}(x) \\ &= 2\left(V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)\right). \end{aligned}$$

Thus, we get that (1) holds. By the same argument as above, we get that (2), (3), and (4) holds. This completes the proof. □

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On the (s,t)-Pell and (s,t)-Pell-Lucas Polynomials

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Abstract

In this paper, we introduced the generalizations of Pell and Pell-Lucas polynomials, which are called (s,t)-Pell and (s,t)- Pell-Lucas polynomials. We also give the Binet formula and the generating function for these polynomials. Finally, we obtain some identities by using the Binet formulas.

Keywords: (s,t)-Pell Number, (s,t)-Pell-Lucas Number, (s,t)-Pell Polynomial, (s,t)-Pell-Lucas Polynomial.

1. Introduction

For over years, many recursive sequences have been studied in the literature. The famous examples of these sequences are Fibonacci, Lucas, Pell, and Pell-Lucas. They are used in many research areas such as Engineering, Architecture, Nature, and Art (for examples, see: (1-6)). The classical Fibonacci $\{F_n\}_{n=0}^{+\infty}$, Lucas $\{L_n\}_{n=0}^{+\infty}$, Pell $\{P_n\}_{n=0}^{+\infty}$, and Pell-Lucas $\{Q_n\}_{n=0}^{+\infty}$ sequences are defined by

$$F_0=0,\ F_1=1 \text{ and } F_n=F_{n-1}+F_{n-2},$$
$$L_0=2,\ L_1=1 \text{ and } L_n=L_{n-1}+L_{n-2},$$
$$P_0=0,\ P_1=1 \text{ and } P_n=2P_{n-1}+P_{n-2},$$
$$Q_0=2,\ Q_1=2 \text{ and } Q_n=2Q_{n-1}+Q_{n-2},$$

for $n\geq 2$, respectively. For more detailed information on the Fibonacci, Lucas, Pell, and Pell-Lucas sequences can be found in (1-2).

Recently, Fibonacci, Lucas, Pell, and Pell-Lucas sequences were generalized and studied by many authors in different ways to derive many identities. In 2012, Gulec and Taskara (7) introduced new generalizations of Pell and Pell-Lucas sequences, which are called (s,t)-Pell and (s,t)-Pell-Lucas sequences as in the following definition:

Definition 1.1 Let s,t be any real numbers with $s^2+t>0, s>0$, and $t\neq 0$. Then the (s,t)-Pell sequence $\{P_n(s,t)\}_{n=0}^{+\infty}$ and the (s,t)-Pell-Lucas sequence $\{Q_n(s,t)\}_{n=0}^{+\infty}$ are defined respectively by

$$P_n(s,t)=2sP_{n-1}(s,t)+tP_{n-2}(s,t),$$
$$Q_n(s,t)=2sQ_{n-1}(s,t)+tQ_{n-2}(s,t)$$

for $n\geq 2$, with initial conditions $P_0(s,t)=0$, $P_1(s,t)=1$ and $Q_0(s,t)=2$, $Q_1(s,t)=2s$.
The terms of these sequences are called (s,t)-Pell and (s,t)-Pell-Lucas numbers, respectively. Also, they introduced the matrices sequences, which have elements of (s,t)-Pell and (s,t)-Pell-Lucas sequences. Further, they obtained some identities of (s,t)-Pell and (s,t)-Pell-Lucas matrices sequences by using elementary matrix algebra. After that, the (s,t)-Pell and (s,t)-Pell-Lucas numbers were studied in different ways to obtain many identities of these numbers. (See: (8-10))

On the other hand, the theory of the second-order recursive sequence of the polynomials has been studied in the literature. In 1883 E.C. Catalan and E. Jacobsthal introduced and studied the polynomials, which are defined by Fibonacci-like recurrence relations. Such polynomials, called the Fibonacci polynomials.

The Fibonacci polynomials studied by Catalan are defined by the recurrence relation.

$$F_n(x)=xF_{n-1}(x)+F_{n-2}(x),\ n\geq 3$$

with initial conditions $F_1(x)=1, F_2(x)=x$.

The Fibonacci polynomials studied by Jacobsthal are defined by the recurrence relation.

$$J_n(x)=J_{n-1}(x)+xJ_{n-2}(x),\ n\geq 3$$

with initial conditions $J_1(x)=J_2(x)=1$.

In 1965, V.E. Hoggatt (11) introduced Lucas polynomials defined by the recurrence relation.

$$L_n(x)=xL_{n-1}(x)+L_{n-2}(x),\ n\geq 2$$

with initial conditions $L_0(x)=2$, $L_1(x)=x$.

Pell polynomials sequence $\{P_n(x)\}_{n=0}^{+\infty}$ and Pell-Lucas polynomials sequence $\{P_n(x)\}_{n=0}^{+\infty}$ were studied in 1985 by A.F. Horadam and J.M. Mahon (12), and these polynomials are defined by

$$P_n(x)=2xP_{n-1}(x)+P_{n-2}(x),\ n\geq 2$$
$$Q_n(x)=2xQ_{n-1}(x)+Q_{n-2}(x)\ ,\ n\geq 2$$

with initial conditions $P_0(x)=0$, $P_1(x)=1$ and $Q_0(x)=2$, $Q_1(x)=2x$.

We note that $P_n(1) = P_n$, $Q_n(1) = Q_n$, $P_n(\frac{1}{2}) = F_n$, and $Q_n(\frac{1}{2}) = L_n$ where P_n, Q_n, F_n, and L_n are the n^{th} Pell, Pell-Lucas, Fibonacci, and Lucas numbers, respectively. Moreover, $P_n(\frac{1}{2}x) = F_n(x)$ and $Q_n(\frac{1}{2}x) = L_n(x)$ where $F_n(x)$ and $L_n(x)$ are the n^{th} Fibonacci and Lucas polynomials, respectively (see (12)). For more detailed information on Fibonacci, Lucas, Pell, and Pell-Lucas polynomials can be found in (1-2). In the last decade, Fibonacci, Lucas, Pell, and Pell-Lucas polynomials have been generalized and studied by many authors (see (13-14)).

In this paper, we introduced the generalizations of Pell and Pell-Lucas polynomials, which are called (s,t)-Pell and (s,t)- Pell-Lucas polynomials. We also give the Binet formula and the generating function for these polynomials and then some identities are obtained by using the Binet formulas.

2. Main results

We begin this section with the following definition.

Definition 2.1 Let s,t be any real numbers with $s^2 + t > 0, s > 0$, and $t \neq 0$. Then the (s,t)-Pell polynomial sequence $\{P_n(s,t)(x)\}_{n=0}^{+\infty}$ and the (s,t)-Pell-Lucas polynomial sequence $\{Q_n(s,t)(x)\}_{n=0}^{+\infty}$ are defined respectively by

$$P_n(s,t)(x) = 2sxP_{n-1}(s,t)(x) + tP_{n-2}(s,t)(x),$$
$$Q_n(s,t)(x) = 2sxQ_{n-1}(s,t)(x) + tQ_{n-2}(s,t)(x)$$

for $n \geq 2$, with initial conditions $P_0(s,t)(x) = 0$, $P_1(s,t)(x) = 1$ and $Q_0(s,t)(x) = 2$, $Q_1(s,t)(x) = 2sx$.

The first few terms of the (s,t)-Pell polynomial sequence are $0, 1, 2sx, 4s^2x^2 + t, 8s^3x^3 + 4stx$. Also, the first few terms of the (s,t)-Pell-Lucas polynomial sequence are $2, 2sx$, $4s^2x^2 + 2t$, $8s^3x^3 + 6stx, 16s^4x^4 + 16s^2tx^2 + 2t^2$. The terms of the (s,t)-Pell and the (s,t)-Pell-Lucas polynomial sequences are called (s,t)-Pell and (s,t)-Pell-Lucas polynomials, respectively.

In particular, if $s = \frac{1}{2}$, $t = 1$, then the classical Fibonacci and Lucas polynomial sequences are obtained, if $s = t = 1$, then the classical Pell and Pell-Lucas polynomial sequences are obtained, and if $s = 1$, then the k-Pell and k-Pell-Lucas polynomial sequences are obtained.

Throughout this paper, for convenience, we will use the symbol $\mathcal{P}_n(x)$ and $\mathcal{Q}_n(x)$ instead of $P_n(s,t)(x)$ and $Q_n(s,t)(x)$, respectively.

First, we give the explicit formula for the n^{th} (s,t)-Pell and (s,t)-Pell-Lucas polynomials.

Theorem 2.2 (Binet's formulas) *The n^{th} (s,t)-Pell and n^{th} (s,t)-Pell-Lucas polynomials are given by*

$$\mathcal{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 0 \tag{2.1}$$
$$\mathcal{Q}_n(x) = \alpha^n + \beta^n, n \geq 0 \tag{2.2}$$

respectively, where α, β are the roots of the characteristic equation $r^2 - 2sxr - t = 0$ and $\alpha > \beta$.

Proof. The characteristic equation for the recurrence relations in Definition 2.1 is

$$r^2 - 2sxr - t = 0 \tag{2.3}$$

Let α, β be the roots of equation (2.1) and $\alpha > \beta$, we have $\alpha = sx + \sqrt{s^2x^2 + t}$ and $\beta = sx - \sqrt{s^2x^2 + t}$. Note that

$$\alpha + \beta = 2sx, \alpha - \beta = 2\sqrt{s^2x^2 + t} \text{ and } \alpha\beta = -t.$$

Since equation (2.3) has two distinct roots, then

$$\mathcal{P}_n(x) = a_1\alpha^n + a_2\beta^n$$

and

$$\mathcal{Q}_n(x) = b_1\alpha^n + b_2\beta^n$$

are the solutions for the recurrence relations in Definition 2.1, respectively, for some real numbers a_1, a_2, b_1, b_2. Giving to n the values $n = 0$ and $n = 1$, then solving system of linear equations, we obtain a unique solution $a_1 = \frac{1}{\alpha - \beta}, a_2 = -\frac{1}{\alpha - \beta}$ and $b_1 = b_2 = 1$. So,

$$\mathcal{P}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$\mathcal{Q}_n(x) = \alpha^n + \beta^n \quad \square$$

Theorem 2.3 (Generating Functions) *The generating functions for the (s,t)-Pell and (s,t)-Pell-Lucas polynomials are given by*

$$f(y) = \frac{y}{1 - 2sxy - ty^2},$$
$$g(y) = \frac{2 - 2sxy}{1 - 2sxy - ty^2}$$

respectively.

Proof. The generating function for the (s,t)-Pell polynomial sequence is defined by

$$f(y) = \sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n.$$

Since
$$\sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n = y + 2sx\sum_{n=2}^{+\infty} \mathcal{P}_{n-1}(x)y^n + t\sum_{n=2}^{+\infty} \mathcal{P}_{n-2}(x)y^n, = y + 2sxy\sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n + ty^2\sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n,$$

we get that

$$f(y) = \sum_{n=0}^{+\infty} \mathcal{P}_n(x)y^n = \frac{y}{1 - 2sxy - ty^2}.$$

Similarly, we obtain the generating function for the (s,t)-Pell-Lucas polynomial sequence as follows

$$g(y) = \frac{2 - 2sxy}{1 - 2sxy - ty^2} \quad \square$$

Next, using the Binet's formulas (2.1) and (2.2), we obtain Catalan's identities, Cassini's identities, and d'Ocagne's identities for the (s,t)-Pell and (s,t)-Pell-Lucas polynomials, which are stated in the following theorems.

Theorem 2.4 (Catalan's Identities) *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*

(i) $\mathcal{P}_{n+r}(x)\mathcal{P}_{n-r}(x) - \mathcal{P}_n^2(x) = -(-t)^{n-r}\mathcal{P}_r^2(x)$,
(ii) $\mathcal{Q}_{n+r}(x)\mathcal{Q}_{n-r}(x) - \mathcal{Q}_n^2(x) = (-t)^{n-r}(\mathcal{Q}_r^2(x) - 4(-t)^r)$,
for all $n \geq 0$, $r \geq 0$, *and* $n \geq r$.

Proof. Using Binet's formula (2.1), we have

$$\begin{aligned}\mathcal{P}_{n+r}(x)\mathcal{P}_{n-r}(x) - \mathcal{P}_n^2(x) &= \frac{-\alpha^{n+r}\beta^{n-r}-\alpha^{n-r}\beta^{n+r}+2\alpha^n\beta^n}{(\alpha-\beta)^2}\\ &= \frac{-(\alpha\beta)^{n-r}(\alpha^r-\beta^r)^2}{(\alpha-\beta)^2}\\ &= -(-t)^{n-r}\mathcal{P}_r^2(x),\end{aligned}$$

and using Binet's formula (2.2), we obtain

$$\begin{aligned}\mathcal{Q}_{n+r}(x)\mathcal{Q}_{n-r}(x) - \mathcal{Q}_n^2(x) &= \alpha^{n-r}\beta^{n-r}(\alpha^{2r} + \beta^{2r} - 2\alpha^r\beta^r)\\ &= (\alpha\beta)^{n-r}((\alpha^r + \beta^r)^2 - 4(\alpha\beta)^r)\\ &= (-t)^{n-r}(\mathcal{Q}_r^2(x) - 4(-t)^r).\end{aligned}$$

□

Theorem 2.5 (Cassini's Identities) *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*

(i) $\mathcal{P}_{n+1}(x)\mathcal{P}_{n-1}(x) - \mathcal{P}_n^2(x) = -(-t)^{n-1}$
(ii) $\mathcal{Q}_{n+1}(x)\mathcal{Q}_{n-1}(x) - \mathcal{Q}_n^2(x) = 4(-t)^{n-1}(s^2x^2 + t)$,
for all $n \geq 1$.

Proof. Take $r = 1$ in Theorem 2.4 and using the initial conditions $\mathcal{P}_1(x) = 1$, $\mathcal{Q}_1(x) = 2sx$. Then we get the results. □

Theorem 2.6 (d'Ocagne's Identities) *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*
(i) $\mathcal{P}_m(x)\mathcal{P}_{n+1}(x) - \mathcal{P}_{m+1}(x)\mathcal{P}_n(x) = (-t)^n\mathcal{P}_{m-n}(x)$,
(ii) $\mathcal{Q}_m(x)\mathcal{Q}_{n+1}(x) - \mathcal{Q}_{m+1}(x)\mathcal{Q}_n(x) = 2(-t)^n\sqrt{s^2x^2+t}(\mathcal{Q}_{m-n}(x) - 2\left(sx + \sqrt{s^2x^2+t}\right)^{m-n})$,
for all $m \geq 1$, $n \geq 1$, *and* $m \geq n$.

Proof. Using Binet's formula (2.1), we have

$$\begin{aligned}\mathcal{P}_m(x)\mathcal{P}_{n+1}(x) - \mathcal{P}_{m+1}(x)\mathcal{P}_n(x) &= \frac{(\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n}-\beta^{m-n})}{(\alpha-\beta)^2}\\ &= (\alpha\beta)^n \cdot \frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\\ &= (-t)^n\mathcal{P}_{m-n}(x),\end{aligned}$$

and using Binet's formula (2.2), we obtain

$$\begin{aligned}\mathcal{Q}_m(x)\mathcal{Q}_{n+1}(x) - \mathcal{Q}_{m+1}(x)\mathcal{Q}_n(x) &= \alpha^n\beta^n(\alpha^{m-n}\beta + \alpha\beta^{m-n} - \alpha^{m-n+1} - \beta^{m-n+1})\\ &= (\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n} + \beta^{m-n} - 2\alpha^{m-n})\\ &= 2(-t)^n\sqrt{s^2x^2+t}\left(\mathcal{Q}_{m-n}(x) - 2\left(sx + \sqrt{s^2x^2+t}\right)^{m-n}\right).\end{aligned}$$

□

Again, in the following theorems, we obtain important elementary relationships involving $\mathcal{P}_n(x)$ and $\mathcal{Q}_n(x)$ by using Binet's formulas (2.1) and (2.2).

Theorem 2.7 *For all* $n \geq 1$, *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then*

(i) $\mathcal{P}_{n+1}(x) + t\mathcal{P}_{n-1}(x) = \mathcal{Q}_n(x)$,
(ii) $\mathcal{Q}_{n+1}(x) + t\mathcal{Q}_{n-1}(x) = 4(s^2x^2 + t)\mathcal{P}_n(x)$,
(iii) $2sx\mathcal{P}_n(x) + 2t\mathcal{P}_{n-1}(x) = \mathcal{Q}_n(x)$.

Proof. Using Binet's formulas (2.1) and (2.2), we have

$$\begin{aligned}\mathcal{P}_{n+1}(x) + t\mathcal{P}_{n-1}(x) &= \frac{\alpha^{n+1}-\beta^{n+1}-\alpha\beta(\alpha^{n-1}-\beta^{n-1})}{\alpha-\beta}\\ &= \frac{(\alpha-\beta)(\alpha^n+\beta^n)}{\alpha-\beta}\\ &= \mathcal{Q}_n(x).\end{aligned}$$

Similarly, we obtain the result (ii). By using (i) and $\mathcal{P}_{n+1}(x) = 2sx\mathcal{P}_n(x) + t\mathcal{P}_{n-1}(x)$, we obtain the result (iii). □

Finally, we obtain summations involving $\mathcal{P}_n(x)$ and $\mathcal{Q}_n(x)$, which are stated in the following theorem.

Theorem 2.8 *Let* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *and* $\{\mathcal{Q}_n(x)\}_{n=0}^{\infty}$ *be the* (s,t)*-Pell and* (s,t)*-Pell-Lucas polynomials, respectively. Then for all* $m \geq 1, n \geq 1, r \geq 0$ *and* $\mathcal{Q}_m(x) \neq (-t)^m + 1$, *the following statements hold.*
(i) $\sum_{k=1}^{n}\mathcal{P}_{mk+r}(x) = \frac{\mathcal{P}_{mn+m+r}(x)-\mathcal{P}_{m+r}(x)-(-t)^m(\mathcal{P}_{mn+r}(x)-\mathcal{P}_r(x))}{\mathcal{Q}_m(x)-(-t)^m-1}$,
(ii) $\sum_{k=1}^{n}\mathcal{Q}_{mk+r}(x) = \frac{\mathcal{Q}_{mn+m+r}(x)-\mathcal{Q}_{m+r}(x)-(-t)^m(\mathcal{Q}_{mn+r}(x)-\mathcal{Q}_r(x))}{\mathcal{Q}_m(x)-(-t)^m-1}$.

Proof. Using Binet's formulas (2.1) and (2.2), we have

$$\begin{aligned}\sum_{k=1}^{n}\mathcal{P}_{mk+r}(x) &= \sum_{k=1}^{n}\frac{1}{\alpha-\beta}(\alpha^{mk+r} - \beta^{mk+r})\\ &= \frac{1}{\alpha-\beta}\left(\frac{\alpha^{m+r}(1-\alpha^{mn})}{1-\alpha^m} - \frac{\beta^{m+r}(1-\beta^{mn})}{1-\beta^m}\right)\\ &= \frac{\alpha^{m+r}(1-\alpha^{mn})(1-\beta^m)-\beta^{m+r}(1-\beta^{mn})(1-\alpha^m)}{(\alpha-\beta)(1-\alpha^m)(1-\beta^m)}\\ &= \frac{\mathcal{P}_{mn+m+r}(x)-\mathcal{P}_{m+r}(x)-(-t)^m(\mathcal{P}_{mn+r}(x)-\mathcal{P}_r(x))}{\mathcal{Q}_m(x)-(-t)^m-1}.\end{aligned}$$

Similarly, by the same argument as above, we obtain the result (ii). □

3. Conclusions

In this paper, the generalizations of Pell and Pell-Lucas polynomials, are introduced and the Binet formulas and the generating functions for these polynomials are obtained. Furthermore, some identities are given by using the Binet formulas.

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Weak and Strong Convergence of Hybrid Subgradient Method for Pseudomonotone Equilibrium Problems and Nonspreading-Type Mappings in Hilbert Spaces

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Abstract. In this paper, we introduce a hybrid subgradient method for finding an element common to both the solution set of a class of pseudomonotone equilibrium problems, and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space. We establish some weak and strong convergence theorems of the sequences generated by our iterative method under some suitable conditions. These convergence theorems are investigated without the Lipschitz condition for bifunctions. Our results complement many known recent results in the literature.

1. Introduction

Let H be a real Hilbert space in which the inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$ and we denote the set of fixed points of T by $F(T)$. Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C,$$

and it is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x \in C, \text{ and } y \in F(T).$$

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A mapping $T : C \to C$ is said to be *a strict pseudocontraction* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C,$$

where I is the identity mapping on H. If $k = 0$, then T is nonexpansive on C.

In 2008, Kohsaka and Takahashi [15] defined a mapping T in a in Hilbert spaces H to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \text{ for all } x, y \in C.$$

Following the terminology of Browder-Petryshyn [10], Osilike and Isiogugu [17] called a mapping T of C into itself *κ-strictly pseudononspreading* if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle + \kappa\|x - Tx - (y - Ty)\|^2, \text{ for all } x, y \in C.$$

Clearly, every nonspreading mapping is κ-strictly pseudononspreading but the converse is not true; see [17]. We note that the class of strict pseudocontraction mappings and the class of κ-strictly pseudononspreading mappings are independent.

In 2010, Kurokawa and Takahashi [16] obtained a weak mean ergodic theorem of Baillon's type [7] for nonspreading mappings in Hilbert spaces. Furthermore, using the idea of mean convergence in Hilbert spaces, they also proved a strong convergence theorem of Halpern's type [12] for this class of mappings. After that, in 2011, Osilike and Isiogugu [17] introduced the concept of κ-strictly pseudononspreading mappings and they proved a weak mean convergence theorem of Baillon's type similar to [16]. They further proved a strong convergence theorem using the idea of mean convergence. This theorem extended and improved the main theorems of [16] and gave an affirmative answer to an open problem posed by Kurokawa and Takahashi [16] for the case when the mapping T is averaged. In 2013 Kangtunyakarn [14] proposed a new technique, using the projection method, for κ-strictly pseudononspreading mappings. He obtained a strong convergence theorem for finding the common element of the set of solutions of a variational inequality, and the set of fixed points of κ-strictly pseudononspreading mappings in a real Hilbert space.

On the other hand, let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F, C)$. It is well known that there are several problems, such as complementarity problems, minimax problems, the Nash equilibrium problem in noncooperative games, fixed point problems, optimization problems, that can be written in the form of an EP. In other words, the EP is a unifying model for several problems arising in physics, engineering, science, optimization, economics, etc.; see [6, 8, 11] and the references therein.

In recent years the problem of finding an element common to the set of solutions of a equilibrium problems, and the set of fixed points of nonlinear mappings, has become a fascinating subject, and various methods have been developed by many authors for solving this problem (see [1, 4, 5, 20]). Most of all the existing algorithms for this problem are based on applying the proximal point method to the equilibrium problem $EP(F,C)$, and using a Mann's iteration to the fixed point problems of nonexpansive mappings. The convergence analysis has been considered when the bifunction F is monotone. This is because the proximal point method is not valid when the underlying operator F is pseudomonotone.

Recently, Anh [2] introduced a new hybrid extragradient iteration method for finding a element common to the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a pseudomonotone bifunctions. In this algorithm the equilibrium bifunction is not required to satisfy any monotonicity property, but it must satisfy a Lipschitz-type continuous bifunction i.e. there are two Lipschitz constants $c_1 > 0$ and $c_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2,\ \forall x,y,z \in C. \tag{1.2}$$

They obtained strongly convergent theorems for the sequences generated by these processes in a real Hilbert space.

Anh and Muu [3] reiterated that the Lipschitz-type condition (1.2) is not in general satisfied, and if it is, that finding the constants c_1 and c_2 is not easy. They further observed that solving strongly convex programs is also difficult except in special cases when C has a simple structure. They introduced and studied a new algorithm, which is called a hybrid subgradient algorithm for finding a common point in the set of fixed points of nonexpansive mappings and the solution set of a class of pseudomonotone equilibrium problems in a real Hilbert space. The proposed algorithm is a combination of the well-known Mann's iterative scheme for fixed point and the projection method for equilibrium problems. Furthermore, the proposed algorithm uses only one projection and does not require any Lipschitz condition for the bifunctions. To be more precise, they proposed the following iterative method:

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n,\cdot)x_n, \\ u_n = P_C(x_n - \gamma_n w_n),\ \gamma_n = \frac{\beta_n}{\max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tu_n,\ \text{for each } n = 1,2,3,..., \end{cases} \tag{1.3}$$

where $\partial_\epsilon F(x,\cdot)(x)$ stands for ϵ-subdifferential of the convex function $F(x,\cdot)$ at x and $\{\epsilon_n\}$, $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\}$, and $\{\alpha_n\}$ were chosen appropriately. Under certain conditions, they prove that $\{x_n\}$ converges strongly to a common point in the set of a class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mapping. Using the idea of Anh and Muu [3], Thailert et al. [21] proposed a new algorithm for finding a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of common fixed points of a family of strict pseudocontraction mappings in a real Hilbert space. Then Thailert et al. [22] introduced new general iterative methods for finding a common element in the solution set of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings which is a solution of a certain optimization problem related to a strongly positive linear operator. Under suitable control conditions, They proved the strong convergence theorems of such iterative schemes in a real Hilbert space.

In this paper, motivated by Anh and Muu [3], Kangtunyakarn [14], and other research going on in this direction, we proposed a hybrid subgradient method for the pseudomonotone equilibrium problem and the finite family of κ-strictly pseudononspreading mapping in a real Hilbert space. The weak and strong convergence of the proposed methods is investigated under certain assumptions. Our results improve and extend many recent results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. It is well-known that for all $x, y, z \in H$ and α, β, $\gamma \in [0, 1]$, with $\alpha+\beta+\gamma = 1$ there holds

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{2.1}$$

and

$$\| \alpha x + \beta y + \gamma z \|^2 = \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \|z\|^2 - \alpha\beta \| x - y \|^2 - \beta\gamma \|y - z\|^2. \tag{2.2}$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\| x - P_C x \| \leq \| x - y \|$ for all $y \in C$. Such a P_C is called the metric projection from H into C. We know that P_C is nonexpansive. It is also known that, $P_C x \in C$ and

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \text{for all } x \in H \text{ and } z \in C. \tag{2.3}$$

It is easy to see that (2.3) equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2, \quad \text{for all } x \in H \text{ and } z \in C. \tag{2.4}$$

Lemma 2.1.([19]) *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u,$$

where P_C is the metric projection of H onto C.

Recall that a bifunction $F : C \times C \to \mathbb{R}$ is said to be

(i) *η-strongly monotone* if there exists a number $\eta > 0$ such that

$$F(x, y) + F(y, x) \leq -\eta\|x - y\|^2, \text{ for all } x, y \in C,$$

(ii) *monotone* on C if

$$F(x, y) + F(y, x) \leq 0, \text{ for all } x, y \in C,$$

(iii) *pseudomonotone* on C with respect to $x \in C$ if

$$F(x, y) \geq 0 \text{ implies } F(y, x) \leq 0, \text{ for all } y \in C.$$

It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), for every $x \in C$. Moreover, F is said to be *pseudomonotone* on C with respect to $A \subseteq C$, if it is pseudomonotone on C with respect to every $x \in A$. When $A \equiv C$, F is called pseudomonotone on C.

The following example, taken from [18], shows that a bifunction may not be pseudomonotone on C, but yet is pseudomonotone on C with respect to the solution set of the equilibrium problem defined by F and C:

$$F(x, y) := 2y|x|(y - x) + xy|y - x|, \text{ for all } x, y \in \mathbb{R},\ C := [-1, 1].$$

Clearly, $EP(F) = \{0\}$. Since $F(y, 0) = 0$ for every $y \in C$, this bifunction is pseudomonotone on C with respect to the solution $x^* = 0$, However, F is not pseudomonotone on C. In fact, both $F(-0.5, 0.5) = 0.25 > 0$ and $F(0.5, -0.5) = 0.25 > 0$.

For solving the equilibrium problem (1.1), let us assume that Δ is an open convex set containing C and the bifunction $F : \Delta \times \Delta \to \mathbb{R}$ satisfies the following assumptions:

(A1) $F(x, x) = 0$ for all $x \in C$ and $F(x, \cdot)$ is convex and lower semicontinuous on C;

(A2) for each $y \in C$, $F(\cdot, y)$ is weakly upper semicontinuous on the open set Δ;

(A3) F is pseudomonotone on C with respect to $EP(F, C)$ and satisfies the strict paramonotonicity property, i.e., $F(y, x) = 0$ for $x \in EP(F, C)$ and $y \in C$ implies $y \in EP(F, C)$;

(A4) if $\{x_n\} \subseteq C$ is bounded and $\epsilon_n \to 0$ as $n \to \infty$, then the sequence $\{w_n\}$ with $w_n \in \partial_n F(x_n, \cdot)x_n$ is bounded, where $\partial_\epsilon F(x, \cdot)x$ stands for the ϵ-subdifferential of the convex function $F(x, \cdot)$ at x.

The following idea of the ϵ-subdierential of convex functions can be found in the work of Bronsted and Rockafellar [9] but the theory of ϵ-subdierential calculus was given by Hiriart-Urruty [13].

Definition 2.2. Consider a proper convex function $\phi : C \to \overline{\mathbb{R}}$. For a given $\epsilon > 0$, the ϵ-subdierential of ϕ at $x_0 \in Dom\phi$ is given by

$$\partial_\epsilon \phi(x_0) = \{x \in C : \phi(y) - \phi(x_0) \geq \langle x, y - x_0\rangle - \epsilon,\ \forall y \in C\}.$$

Remark 2.3. It is known that if the function ϕ is proper lower semicontinuous convex, then for every $x \in Dom\phi$, the ϵ-subdierential $\partial_\epsilon \phi(x)$ is a nonempty closed convex set (see [13]).

Next, throughout this paper, weak and strong convergence of a sequence $\{x_n\}$ in H to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. In order to prove our main results, we need the following lemmas.

Lemma 2.4.([17]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a κ-strictly pseudonospreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Remark 2.5. If $T : C \to C$ is a κ-strictly pseudononspreading mapping with $F(T) \neq \emptyset$, then from Lemma 2.8 in [14] and Lemma 2.1, we have $F(T) = VI(C,(I-T)) = F(P_C(I - \lambda(I - T)))$, for all $\lambda > 0$.

Lemma 2.6. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For every $i = 1, 2, ..., N$, let $T_i : C \to C$ be a finite family of κ_i-strictly pseudononspreading mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{a_1, a_2, ..., a_n\} \subset (0, 1)$ with $\Sigma_{i=1}^N a_i = 1$, let $\bar{\kappa} = max\{\kappa_1, \kappa_2, ..., \kappa_N\}$ and let $\lambda \in (0, 1 - \bar{\kappa})$. Then*

(i) $\bigcap_{i=1}^N F(T_i) = F(\Sigma_{i=1}^N a_i P_C(I - \lambda(I - T_i)))$.

(ii) $\|\Sigma_{i=1}^N a_i P_C(I-\lambda(I-T_i))x-y\|^2 \leq \|x-y\|^2$, *for all* $x \in C$ *and* $y \in \bigcap_{i=1}^N F(T_i)$, *i.e.* $\Sigma_{i=1}^N a_i P_C(I - \lambda(I - T_i))$ *is quasi-nonexpansive.*

Proof. (i) It easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq F(\Sigma_{i=1}^N a_i P_C(I - \lambda(I - T_i)))$. Let $x \in F(\Sigma_{i=1}^N a_i P_C(I-\lambda(I-T_i)))$ and let $x^* \in \bigcap_{i=1}^N F(T_i) \subseteq F(\Sigma_{i=1}^N a_i P_C(I-\lambda(I-T_i)))$. Note that for every $i = 1, 2, 3, ..., N$ we have

$$\begin{aligned}\|P_C(I - \lambda(I - T_i))x - x^*\|^2 &\leq \|x - x^* - \lambda(I - T_i)\|^2 \\ &= \|x - x^*\|^2 - 2\lambda\langle x - x^*, (I - T_i)x\rangle \\ &\quad + \lambda^2\|(I - T_i)x\|^2. \end{aligned} \tag{2.5}$$

Put $A_i = I - T_i$, for all $i = 1, 2, ..., N$, we have $T_i = I - A_i$ and

$$\begin{aligned}\|T_i x - T_i x^*\|^2 &= \|(I - A_i)x - (I - A_i)x^*\|^2 \\ &= \|(x - x^*) - A_i x\|^2 \\ &= \|x - x^*\|^2 - 2\langle x - x^*, A_i x\rangle + \|A_i x\|^2 \\ &\leq \|x - x^*\|^2 + \kappa_i\|(I - T_i)x - (I - T_i)x^*\|^2 + 2\langle x - T_i x, x^* - T_i x^*\rangle \\ &= \|x - x^*\|^2 + \kappa_i\|(I - T_i)x\|^2, \end{aligned} \tag{2.6}$$

which implies that

$$(1 - \kappa_i)\|(I - T_i)x\|^2 \leq 2\langle x - x^*, A_i x\rangle, \quad \text{for all } i = 1, 2, 3, ..., N$$

From (2.5) and (2.6), we have

$$\begin{aligned}\|P_C(I-\lambda(I-T_i))x-x^*\|^2 &\leq \|x-x^*\|^2-2\lambda\langle x-x^*,(I-T_i)x\rangle\\ &\quad+\lambda^2\|(I-T_i)x\|^2\\ &\leq \|x-x^*\|^2-\lambda(1-\kappa_i)\|(I-T_i)x\|^2\\ &\quad+\lambda^2\|(I-T_i)x\|^2\\ &= \|x-x^*\|^2-\lambda[(1-\kappa_i)-\lambda]\|(I-T_i)x\|^2\\ &\leq \|x-x^*\|^2, \end{aligned} \tag{2.7}$$

for all $i=1,2,3,\ldots,N$.
From the definition of x and (2.7), we have

$$\begin{aligned}\|x-x^*\|^2 &= \|\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i))x-x^*\|^2\\ &= a_1\|P_C(I-\lambda(I-T_1))x-x^*\|^2+a_2\|P_C(I-\lambda(I-T_2))x-x^*\|^2+\cdots\\ &\quad+a_N\|P_C(I-\lambda(I-T_N))x-x^*\|^2-a_1a_2\|P_C(I-\lambda(I-T_1))x\\ &\quad-P_C(I-\lambda(I-T_2))x\|^2-a_2a_3\|P_C(I-\lambda(I-T_2))x-\\ &\quad P_C(I-\lambda(I-T_3))x\|^2-\cdots-a_{N-1}a_N\|P_C(I-\lambda(I-T_{N-1}))x-\\ &\quad P_C(I-\lambda(I-T_N))x\|^2\\ &\leq \|x-x^*\|^2-a_1a_2\|P_C(I-\lambda(I-T_1))x-P_C(I-\lambda(I-T_2))x\|^2\\ &\quad-a_2a_3\|P_C(I-\lambda(I-T_2))x-P_C(I-\lambda(I-T_3))x\|^2-\cdots\\ &\quad-a_{N-1}a_N\|P_C(I-\lambda(I-T_{N-1}))x-P_C(I-\lambda(I-T_N))x\|^2.\end{aligned}$$

This implies that

$$P_C(I-\lambda(I-T_1))x=P_C(I-\lambda(I-T_2))x=\cdots=P_C(I-\lambda(I-T_N))x$$

Since $x\in F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))$, we get that $x=P_C(I-\lambda(I-T_i))x$, for all $i=1,2,3,...,N$ From Remark 2.5, we have $x\in F(T_i)$, for all $i=1,2,3,...,N$. That is $x\in\bigcap_{i=1}^N F(T_i)$. Hence $F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))\subseteq\bigcap_{i=1}^N F(T_i)$.
(ii) Let $x\in C$ and $y\in\bigcap_{i=1}^N F(T_i)=F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))$
As the same argument as in (i), we can show that

$$\|P_C(I-\lambda(I-T_i))x-y\|^2\leq\|x-y\|^2, \tag{2.8}$$

for all $i=1,2,3,...,N$. Thus

$$\begin{aligned}\|\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i))x-y\|^2 &\leq a_1\|P_C(I-\lambda(I-T_1))x-y\|^2\\ &\quad+a_2\|P_C(I-\lambda(I-T_2))x-y\|^2+\cdots\\ &\quad+a_N\|P_C(I-\lambda(I-T_N))x-y\|^2\\ &\leq \Sigma_{i=1}^N a_i\|x-y\|^2=\|x-y\|^2. \end{aligned} \tag{2.9}$$

□

Lemma 2.7.([23]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \ \ n \geq 1,$$

where $\sum_{n=0}^{\infty} b_n < \infty$. Then the sequence $\{a_n\}$ is convergent.

3. Weak Convergence Theorem

In this section, we prove weak convergence theorem for finding a common element in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{\kappa_1, \kappa_2, ..., \kappa_N\} \subset [0, 1)$ and $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

- (i) *$0 < \lambda \leq \lambda_n^i \leq \min\{1 - \kappa_1, 1 - \kappa_2, ..., 1 - \kappa_N\}$ and $\Sigma_{n=1}^{\infty} \lambda_n^i < \infty$ for all $i = 1, 2, ..., N$;*
- (ii) *$0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$;*
- (iii) *$\sum_{n=0}^{\infty} \delta_n = \infty$, $\sum_{n=0}^{\infty} \delta_n^2 < \infty$, and $\sum_{n=0}^{\infty} \delta_n \epsilon_n < \infty$.*

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

Proof. First, we will show that $\{x_n\}$ is bounded. Let $p \in \Omega$. Then we have

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\langle x_n - u_n, p - u_n\rangle \\ &\leq \|x_n - p\|^2 + 2\langle x_n - u_n, p - u_n\rangle. \end{aligned} \tag{3.2}$$

Since $u_n = P_C(x_n - \rho_n w_n)$ and $p \in C$, we get that

$$\langle x_n - u_n, p - u_n\rangle \leq \rho_n \langle w_n, p - u_n\rangle. \tag{3.3}$$

Substuting (3.3) into (3.2), we have

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - u_n\rangle \\ &= \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\rho_n\langle w_n, x_n - u_n\rangle \\ &\leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\rho_n\|w_n\|\|x_n - u_n\| \\ &\leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\delta_n\|x_n - u_n\|. \end{aligned} \tag{3.4}$$

By using $u_n = P_C(x_n - \rho_n w_n)$ and $x_n \in C$ again, we get

$$\begin{aligned}\|x_n - u_n\|^2 &= \langle x_n - u_n, x_n - u_n\rangle \\ &\leq \rho_n\langle w_n, x_n - u_n\rangle \\ &\leq \rho_n\|w_n\|\|x_n - u_n\| \\ &\leq \delta_n\|x_n - u_n\|, \end{aligned} \tag{3.5}$$

which implies that

$$\|x_n - u_n\| \leq \delta_n. \tag{3.6}$$

By condition (iii), we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.7}$$

Combining (3.4) and (3.6), we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\delta_n^2. \tag{3.8}$$

Since $w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n$, $p \in C$ and $F(x, x) = 0$ for each $x \in C$, we obtain that

$$\begin{aligned}\langle w_n, p - x_n\rangle &\leq F(x_n, p) - F(x_n, x_n) + \epsilon_n \\ &= F(x_n, p) + \epsilon_n. \end{aligned} \tag{3.9}$$

Thus, it follows from (3.8) and (3.9) that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + 2\rho_n F(x_n, p) + 2\rho_n\epsilon_n + 2\delta_n^2. \tag{3.10}$$

Form Lemma 2.6 (ii), we have

$$\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - p\|^2 \leq \|x_n - p\|^2. \tag{3.11}$$

From (3.1), (3.10) and (3.11), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + \beta_n\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + \beta_n\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - p\|^2 \\ &\quad + \gamma_n\|u_n - p\|^2 - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\Big(\|x_n - p\|^2 + 2\rho_n F(x_n, p) \\ &\quad + 2\rho_n\epsilon_n + 2\delta_n^2\Big) - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\ &= \|x_n - p\|^2 + 2\gamma_n\rho_n F(x_n, p) + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\quad - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2. \end{aligned} \tag{3.12}$$

Since $p \in EP(F, C)$ and F is pseudomonotone on F with respect to p, we get that $F(x_n, p) \leq 0$ for all $n \in \mathbb{N}$. Then from (3.12) it follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\quad -\alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\ &\leq \|x_n - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2. \end{aligned} \tag{3.13}$$

Let $\eta_n = 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2$ for all $n \geq 0$. From condition (ii) and (iii), we get that

$$\Sigma_{n=0}^{\infty}\eta_n = \Sigma_{n=0}^{\infty}(2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2) \leq 2b\Sigma_{n=0}^{\infty}\rho_n\epsilon_n + 2b\Sigma_{n=0}^{\infty}\delta_n^2 < +\infty$$

Now applying Lemma 2.7 to (3.13), we obtain that the $\lim_{n\to\infty}\|x_n - p\|$ exists, i.e. $\lim_{n\to\infty}\|x_n - p\| = \bar{a}$ for some $\bar{a} \in C$. Thus $\{x_n\}$ is bounded. Also, it easy to verify that $\{u_n\}$ and $\{\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\}$ are also bounded.

Next, we will show that $\limsup_{n\to\infty} F(x_n, p) = 0$ for any $p \in \Omega$. Since F is pseudomonotone on C and $F(p, x_n) \geq 0$, we have $-F(x_n, p) \geq 0$. From (3.12) and condition (ii), we have

$$\begin{aligned} 2\gamma_n\rho_n[-F(x_n, p)] &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2b\rho_n\epsilon_n + 2b\delta_n^2. \end{aligned} \tag{3.14}$$

Summing up (3.14) for every n, we obtain

$$\begin{aligned} 0 &\leq 2\sum_{n=0}^{\infty}\gamma_n\rho_n[-F(x_n, p)] \\ &\leq \|x_0 - p\|^2 + 2b\sum_{n=0}^{\infty}\rho_n\epsilon_n + 2b\sum_{n=0}^{\infty}\delta_n^2 < +\infty. \end{aligned} \tag{3.15}$$

By the assumption (A_4), we can find a real number w such that $\|w_n\| \leq w$ for every n. Setting $\Gamma := \max\{\sigma, w\}$, where σ is a real number such that $0 < \sigma_n < \sigma$ for every n, it follows from (ii) that

$$0 \leq \frac{2a}{\Gamma}\sum_{n=0}^{\infty}\delta_n[-F(x_n, p)] \tag{3.16}$$

$$\leq 2\sum_{n=0}^{\infty}\gamma_n\rho_n[-F(x_n, p)] < +\infty, \tag{3.17}$$

which implies that

$$0 \leq \sum_{n=0}^{\infty}\delta_n[-F(x_n, p)] < +\infty. \tag{3.18}$$

Combining with $-F(x_n, p) \geq 0$ and $\sum_{n=0}^{\infty} \delta_n = \infty$, we can deduced that $\limsup_{n\to\infty} F(x_n, p) = 0$ as desired.

Next, we will show that $\omega_\omega(x_n) \subset \Omega$, where $\omega_\omega(x_n) = \{x \in H : x_{n_i} \rightharpoonup x$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}\}$. In deed since $\{x_n\}$ is bounded and H is reflexive, $\omega_\omega(x_n)$ is nonempty. Let $\bar{x} \in \omega_\omega(x_n)$. Then there exists subsequence $\{x_{n_i}\}$ of $\{x_n\}$. converging weakly to $\bar{x}$, that is $x_{n_i} \rightharpoonup \bar{x}$ as $i \to \infty$. By the convexity, C is weakly closed and hence $\bar{x} \in C$. Since $F(\cdot, p)$ is weakly upper semicontinuous for $p \in \Omega$, we obtain

$$
\begin{aligned}
F(\bar{x}, p) &\geq \limsup_{i\to\infty} F(x_n, p) \\
&= \lim_{i\to\infty} F(x_{n_i}, p) \\
&= \limsup_{n\to\infty} F(x_n, p) \\
&= 0. \qquad (3.19)
\end{aligned}
$$

Since F is pseudomontone with respect to p and $F(p, \bar{x}) \geq 0$, we obtain $F(\bar{x}, p) \leq 0$. Thus $F(\bar{x}, p) = 0$. Furthermore, by assumption (A_3), we get that $\bar{x} \in EP(F, C)$. On the other hand, from (3.13) and conditions (ii)–(iii), we have

$$
\begin{aligned}
&\alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\
&\qquad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\
&\qquad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2b\rho_n\epsilon_n + 2b\delta_n^2 \qquad (3.20)
\end{aligned}
$$

taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\| = 0. \qquad (3.21)$$

Now, we will show that $\bar{x} \in \bigcap_{i=1}^N F(T_i)$. Assume that $\bar{x} \notin \bigcap_{i=1}^N F(T_i)$. By Lemma 2.6, we have $\bar{x} \notin F(\Sigma_{i=1}^N a_i P_C(I - \lambda_n(I - T_i)))$. From the Opial's condition, (3.21) and condition (i), we can write

$$
\begin{aligned}
\liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| &< \liminf_{i\to\infty} \|x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))\bar{x}\| \\
&\leq \liminf_{i\to\infty} \Big(\|x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_{n_i}\| \\
&\quad + \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))\bar{x}\| \Big) \\
&\leq \liminf_{i\to\infty} \Big(\|x_{n_i} - \bar{x}\| + \Sigma_{i=1}^N a_i \lambda_n^i \|(I - T_i)x_{n_i} - (I - T_i)\bar{x}\| \Big) \\
&\leq \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\|.
\end{aligned}
$$

This is a contradiction. Then $\bar{x} \in \bigcap_{i=1}^N F(T_i)$. Thus $\bar{x} \in EP(F, C) \cap F(T) = \Omega$ and so $\omega_\omega(x_n) \subset \Omega$.

Finally, we prove that $\{x_n\}$ converge weakly to an element of Ω. It's sufficient to show that $\omega_\omega(x_n)$ is a single point set. Taking $z_1, z_2 \in \omega_\omega(x_n)$ arbitrarily, and let $\{x_{n_k}\}$ and $\{x_{n_m}\}$ be subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z_1$ and $x_{n_m} \rightharpoonup z_2$ respectively. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega$ and z_1, $z_2 \in \Omega$, we get that $\lim_{n\to\infty} \|x_n - z_1\|$ and $\lim_{n\to\infty} \|x_n - z_2\|$ exist. Now, assume that $z_1 \neq z_2$, then by the Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - z_1\| &= \lim_{k\to\infty} \|x_{n_k} - z_1\| \\ &< \lim_{k\to\infty} \|x_{n_k} - z_2\| \\ &= \lim_{n\to\infty} \|x_n - z_2\| \\ &= \lim_{m\to\infty} \|x_{n_m} - z_2\| \\ &< \lim_{m\to\infty} \|x_{n_m} - z_1\| \\ &= \lim_{n\to\infty} \|x_n - z_1\|, \end{aligned} \tag{3.22}$$

which is a contradiction. Thus $z_1 = z_2$. This show that $\omega_\omega(x_n)$ is single point set. i.e. $x_n \rightharpoonup \bar{x}$. This completes the proof. □

If we set $\kappa_i = 0$ for all $i = 1, 2, ..., N$ then we get the following Corollary.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.23}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i < 1$ *and* $\Sigma_{n=1}^\infty \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$;

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$;

(ii) $\sum_{n=0}^\infty \delta_n = \infty$, $\sum_{n=0}^\infty \delta_n^2 < \infty$, *and* $\sum_{n=0}^\infty \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

4. Strong Convergence Theorem

In this section, to obtain strong convergence result, we add the control condition $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$, and then we get the strong convergence theorem for finding a common element in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{\kappa_1, \kappa_2, ..., \kappa_N\} \subset [0, 1)$ and $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i \leq \min\{1 - \kappa_1, 1 - \kappa_2, ..., 1 - \kappa_N\}$ *and* $\Sigma_{n=1}^\infty \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$;

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$ *and* $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$;

(iii) $\sum_{n=0}^\infty \delta_n = \infty$, $\sum_{n=0}^\infty \delta_n^2 < \infty$, *and* $\sum_{n=0}^\infty \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Omega$.

Proof. By a similar argument to the proof of Theorem 3.1 and (2.4), we have

$$\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - P_\Omega(x_n)\|^2 \leq \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - x_n\|^2 \\ - \|x_n - P_\Omega(x_n)\|^2$$

and

$$\|u_n - P_\Omega(x_n)\|^2 \leq \|u_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2. \tag{4.2}$$

It follows from (4.2) and condition (ii) that

$$
\begin{aligned}
&\|x_{n+1} - P_\Omega(x_{n+1})\|^2 \\
&\quad \leq \quad \|\alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n + \gamma_n u_n - P_\Omega(x_n)\|^2 \\
&\quad \leq \quad \alpha_n \|x_n - P_\Omega(x_n)\|^2 + \beta_n \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - P_\Omega(x_n))\|^2 \\
&\qquad\quad +\gamma_n \|u_n - P_\Omega(x_n)\|^2 \\
&\quad \leq \quad \alpha_n \|x_n - P_\Omega(x_n)\|^2 + \beta_n \Big(\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - x_n\|^2 \\
&\qquad\quad - \|x_n - P_\Omega(x_n)\|^2 \Big) + \gamma_n \Big(\|u_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2 \Big) \\
&\quad = \quad (\alpha_n - (\beta_n + \gamma_n))\|x_n - P_\Omega(x_n)\|^2 + \beta_n \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - x_n\|^2 \\
&\qquad\quad +\gamma_n \|u_n - x_n\|^2. \\
&\quad \leq \quad (2\alpha_n - 1)\|x_n - P_\Omega(x_n)\|^2 + b\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - x_n\|^2 \\
&\qquad\quad +b\|u_n - x_n\|^2.
\end{aligned}
$$

Combining (3.7), (3.21), conditions (ii)–(iii), and the boundedness of the sequence $\{x_n - P_\Omega(x_n)\}$, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - P_\Omega(x_{n+1})\| = 0 \tag{4.3}$$

Since Ω is convex, for all $m > n$, we have $\frac{1}{2}(P_\Omega(x_m) + P_\Omega(x_n)) \in \Omega$, and therefore

$$
\begin{aligned}
\|P_\Omega(x_m) - P_\Omega(x_n)\|^2 &= 2\|x_m - P_\Omega(x_m)\|^2 + 2\|x_m - P_\Omega(x_n)\|^2 \\
&\quad -4\|x_m - \frac{1}{2}(P_\Omega(x_m) + P_\Omega(x_n))\|^2 \\
&\leq 2\|x_m - P_\Omega(x_m)\|^2 + 2\|x_m - P_\Omega(x_n)\|^2 \\
&\quad -4\|x_m - P_\Omega(x_m)\|^2 \\
&= 2\|x_m - P_\Omega(x_n)\|^2 - 2\|x_m - P_\Omega(x_m)\|^2. \qquad (4.4)
\end{aligned}
$$

Using (3.13) with $p = P_\Omega(x_n)$, we have

$$
\begin{aligned}
\|x_m - P_\Omega(x_n)\|^2 &\leq \|x_{m-1} - P_\Omega(x_n)\|^2 + \eta_{m-1} \\
&\leq \|x_{m-2} - P_\Omega(x_n)\|^2 + \eta_{m-1} + \eta_{m-2} \\
&\leq \ldots \\
&\leq \|x_n - P_\Omega(x_n)\|^2 + \sum_{j=n}^{m-1} \eta_j, \qquad (4.5)
\end{aligned}
$$

where $\eta_j = 2\gamma_j \rho_j \epsilon_j + 2\gamma_j \delta_j^2$. It follows from (4.4) and (4.5) that

$$\|P_\Omega(x_m) - P_\Omega(x_n)\|^2 \leq 2\|x_n - P_\Omega(x_n)\|^2 + 2\sum_{j=n}^{m-1} \eta_j - 2\|x_m - P_\Omega(x_m)\|^2. \tag{4.6}$$

Together with (4.3) and $\sum_{j=0}^{\infty}\eta_j < +\infty$, this implies that $\{P_\Omega(x_n)\}$ is a Cauchy sequence, Hence $\{P_\Omega(x_n)\}$ strongly converges to some point $x^* \in \Omega$. Moreover, we obtain

$$x^* = \lim_{i\to\infty} P_\Omega(x_{n_i}) = P_\Omega(\bar{x}) = \bar{x}, \tag{4.7}$$

which implies that $P_\Omega(x_i) \to x^* = \bar{x} \in \Omega$. Then from (4.3) and (4.7), we can conclude that $x_n \to \bar{x}$. This completes the proof. □

If we set $\kappa_i = 0$ for all $i = 1, 2, ..., N$ then we get the following Corollary.

Corollary 4.2. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.8}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i < 1$ *and* $\Sigma_{n=1}^{\infty}\lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$*;*

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$ *and* $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$*;*

(iii) $\sum_{n=0}^{\infty}\delta_n = \infty$, $\sum_{n=0}^{\infty}\delta_n^2 < \infty$, *and* $\sum_{n=0}^{\infty}\delta_n\epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

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