EXPLICITLY SOLVABLE CASE

WE HAVE THAT THE EIGENVALUES OF THE NLEP ARE THE ROOTS OF THE QUADRATIC FOR $j=1,\dots,K-1$

$$C_2\lambda^2 + C_1\lambda + C_0 = 0$$

WHERE
$$C_2 = \frac{\tilde{\tau}_j}{3\chi_{0j}}, \quad C_1 = \hat{\tau}_j\left(\frac{3}{2} - \frac{1}{\chi_{0j}}\right) + \frac{1}{3\chi_{0j}}, \quad C_0 = \frac{3\,\mathcal{U}_0}{2w} + \frac{3}{2} - \frac{1}{\chi_{0j}} \qquad (1)$$

WHERE
$$\hat{\tau}_j \equiv \frac{\hat{\tau}_u}{D_j\,\eta_3}, \quad \frac{1}{\chi_{0j}} = 1 + \frac{\eta_3 D_j}{\alpha}, \quad \eta_3 = \frac{2\pi^2\alpha^3 K^3}{w^3}. \qquad (2)$$

WE CAN WRITE $(\lambda-\lambda_1)(\lambda-\lambda_2)=0$ SO THAT $\lambda_1\lambda_2 = C_0/C_2$, $\lambda_1+\lambda_2 = -C_1/C_2$.

WE OBSERVE $C_2 > 0$.

REMARK 1

(i) IF $C_0 < 0 \rightarrow \lambda_1,\lambda_2$ REAL WITH $\lambda_1 < 0,\ \lambda_2 > 0$

(ii) IF $C_0 > 0$ AND $C_1 > 0 \rightarrow$ stability since either $\lambda_1<0, \lambda_2<0$ REAL OR $\lambda_1 = \bar{\lambda}_2$, $\mathrm{RE}\,\lambda_1 < 0$.

(iii) IF $C_0 > 0$, $C_1 < 0 \rightarrow$ INSTABILITY SINCE EITHER $\lambda_1>0, \lambda_2>0$ OR $\lambda_1 = \bar{\lambda}_2$ WITH $\mathrm{RE}\,\lambda_1 > 0$.

A HB OCCURS WHEN $C_0 > 0$ AND $C_1 = 0$.

REMARK 2 A ZERO EIGENVALUE CROSSING OCCURS WHEN (4.6) HOLDS, I.E. WHEN

$$\frac{w}{3\,\mathcal{U}_0}\left(\frac{1}{\chi_{0j}} - \frac{3}{2}\right) = \frac{1}{2}. \quad \text{WITH} \quad \frac{1}{\chi_{0j}} = 1 + \eta_3 D_j/\alpha \quad \text{WE GET THAT}$$

$$-\frac{1}{2} + \frac{\eta_3 D_j}{\alpha} = \frac{3}{2}\frac{\mathcal{U}_0}{w} \quad \text{OR} \quad D_j = \frac{\alpha}{2\eta_3}\left(1 + 3\,\mathcal{U}_0/w\right). \quad \text{WITH } \eta_3 \text{ AS GIVEN ABOVE,}$$

$$D_j = D^*_{up} \equiv \frac{w^3}{4\pi^2 K^3 \alpha^2}\left(1 + 3\,\mathcal{U}_0/w\right) \qquad (3)_1$$

WITH
$$D_j = \frac{D_0(2K)}{S}\left(1 - \cos(\pi j/K)\right), \quad j = 1,\dots,K-1. \qquad (3)_2.$$

CLAIM I : $$C_0 = -\frac{\mathcal{H}_3}{\alpha}\left(D_j - D_{up}^*\right) \quad \Big\} \quad (4)$$

PROOF $$C_0 = \frac{3\,\Pi_0}{2w} + \frac{3}{2} - \frac{1}{\chi_{0j}} = \frac{1}{2} - \frac{\mathcal{H}_3 D_j}{\alpha} + \frac{3\,\Pi_0}{2w}.$$

WE MUST SHOW $\frac{\mathcal{H}_3}{\alpha} D_{up}^* = \frac{1}{2} + \frac{3\,\Pi_0}{2w} \longrightarrow D_{up}^* = \frac{\alpha}{2\mathcal{H}_3}\left(1 + 3\,\Pi_0/w\right)$ WHICH IS CORRECT.

THUS IF $D_j > D_{up}^* \longrightarrow j^{th}$ MODE IS UNSTABLE DUE TO A REAL POSITIVE EIGENVALUE.

NOW WE CONSIDER C_1. WE SET $C_1 = 0$ TO GET WITH $\chi_j^{-1} = 1 + \mathcal{H}_3 D_j/\alpha$ THAT:
$$\tilde{T}_j = \frac{2}{3\left(2 - 3\chi_{0j}\right)} = \frac{\left(2/\chi_{0j}\right)}{3\left(2/\chi_{0j} - 3\right)} = \frac{2}{3}\,\frac{\left(1 + \mathcal{H}_3 D_j/\alpha\right)}{\left(-1 + 2\mathcal{H}_3 D_j/\alpha\right)}$$

NOW $$\hat{T}_j = \frac{1}{3}\left(D_j + \alpha/\mathcal{H}_3\right)\Big/\left(D_j - \frac{\alpha}{2\mathcal{H}_3}\right)$$

DEFINE $$D_{low}^* = \frac{\alpha}{2\mathcal{H}_3} \equiv \frac{w^3}{4\pi^2 \mathcal{H}^3 \alpha^2}.$$

THEN $$\hat{T}_j = \frac{1}{3}\,\frac{\left(D_j + 2D_{low}^*\right)}{\left(D_j - D_{low}^*\right)} \equiv \tilde{T}_{jH} \quad \text{WITH} \quad D_{low}^* = \frac{w^3}{4\pi^2 \mathcal{H}^3 \alpha^2} \quad \Big\} \quad (5)$$

NOW WE REWRITE C_1.

CLAIM $$C_1 = \frac{1}{2}\left(1 - \frac{D_j}{D_{low}^*}\right)\left(\hat{T}_j - \hat{T}_{jH}\right)$$

PROOF $$C_1 = \left(\frac{3}{2} - \frac{1}{\chi_{0j}}\right)\left(\hat{T}_j + \frac{\left(1/3\,\chi_{0j}\right)}{\left(\frac{3}{2} - \frac{1}{\chi_{0j}}\right)}\right) = \left(\frac{1}{2} - \frac{\mathcal{H}_3 D_j}{\alpha}\right)\left(\hat{T}_j + \frac{1}{3}\,\frac{\left(1 + \mathcal{H}_3 D_j/\alpha\right)}{\left(\frac{1}{2} - \mathcal{H}_3 D_j/\alpha\right)}\right)$$

BUT $D_{low}^* = \alpha/2\mathcal{H}_3$. THUS $$C_1 = \left(\frac{1}{2} - \frac{1}{2}\frac{D_j}{D_{low}^*}\right)\left(\hat{T}_j + \frac{\frac{1}{3}\left(1 + D_j/2D_{low}^*\right)}{\left(\frac{1}{2} - \frac{1}{2}\,D_j/D_{low}^*\right)}\right)$$

THUS $$C_1 = \frac{1}{2}\left(1 - \frac{D_j}{D_{low}^*}\right)\left(\tilde{\tau}_j + \frac{1}{3}\left(2 + D_j/D_{low}^*\right)\Big/\left(1 - D_j/D_{low}^*\right)\right) \qquad \text{E3}$$

NOW $$\hat{\tau}_{jH} = \frac{1}{3}\left(2 + D_j/D_{low}^*\right)\Big/\left(D_j/D_{low}^* - 1\right), \qquad (6)_2$$

THIS YIELDS THAT $2K_3 = \alpha/D_{low}$

$$C_1 = \frac{1}{2}\left(1 - \frac{D_j}{D_{low}^*}\right)\left(\hat{\tau}_j - \hat{\tau}_{jH}\right) \qquad (6)_1$$

$$= \frac{1}{2D_j K_3}\left(1 - \frac{D_j}{D_{low}^*}\right)\left(\hat{\tau}_u - \tau_{uH,j}\right) = \frac{D_{low}}{\alpha D_j}\left(1 - \frac{D_j}{D_{low}}\right)\left(\hat{\tau}_u - \hat{\tau}_{uH,j}\right)$$

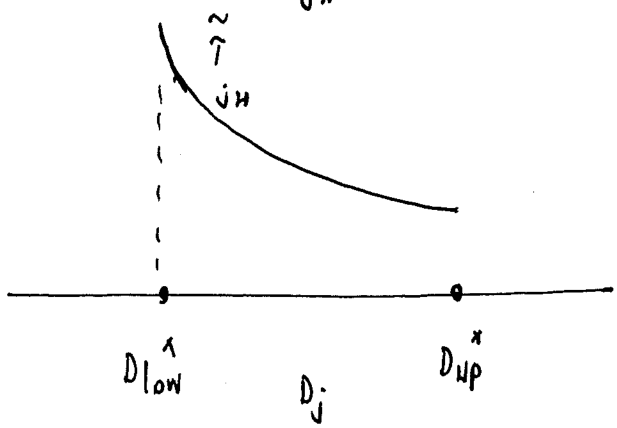
WE CONCLUDE THAT:

(i) IF $D_{low}^* < D_j < D_{up}^*$: THEN $\tilde{\tau}_{jH}$ IS REAL AND POSITIVE, FOR EACH $j = 1, .., K-1$.

MOREOVER, IF $\tilde{\tau}_j > \tilde{\tau}_{jH}$ THEN $C_1 < 0$ AND WE HAVE INSTABILITY!

(ii) IF $D_j < D_{low}^*$, THEN $\hat{\tau}_{jH} < 0$, AND $1 - D_j/D_{low}^* > 0$, SO $C_1 > 0 \rightarrow$ stability.

(iii) WE HAVE THAT $\tilde{\tau}_{jH} \rightarrow +\infty$ AS $D_j/D_{low}^* \rightarrow 1^+$.



(iv) ON $D_{low}^* < D_j < D_{up}^*$, THE HB IS AT

$$\tau_u = \tau_{uH_j} = D_j K_3 \tilde{\tau}_{jH}$$

BUT $K_3 = \dfrac{\alpha}{2D_{low}^*}$ SO $$\tau_{uH_j} = \frac{D_j \alpha}{2D_{low}^*}\tilde{\tau}_{jH}. \qquad (7).$$

NOW WE OBTAIN THAT

$$\hat{T}_{jH} = \frac{1}{3}\left(1 + \frac{3 D_{low}^*}{D_j - D_{low}^*}\right) = \frac{1}{3} + \frac{1}{\left(D_j/D_{low}^* - 1\right)}$$

WE CONCLUDE THAT

$$T_{UH_j} = \frac{D_j \alpha}{2 D_{low}^*}\left(\frac{1}{3} + \frac{1}{D_j/D_{low}^* - 1}\right). \qquad (8)$$

WHICH IS DEFINED ON $D_{low}^* < D_j < D_{up}^*$.

WE NOW DEFINE THE UNIVERSAL FUNCTION $H(B)$ BY

$$H(B) \equiv \frac{\alpha B}{2}\left(\frac{1}{3} + \frac{1}{B-1}\right) \qquad (9)_1 \qquad 1 < B < B_{up} \equiv \frac{D_{up}^*}{D_{low}^*}.$$

THEN $$\hat{T}_{UH_j} = H\left(D_j/D_{low}^*\right) \qquad (9)_2 .$$

NOW WE HAVE

$$T_{UH_j} \sim \begin{cases} \dfrac{\alpha}{2}\,\dfrac{1}{D_j/D_{low}^* - 1} & \text{AS } D_j \to D_{low}^* \text{ FROM ABOVE} \\ \dfrac{\alpha}{2}\,\dfrac{D_{up}^*}{D_{low}^*}\left(\dfrac{1}{3} + \dfrac{1}{D_{up}^*/D_{low}^* - 1}\right) & \text{AS } D_j \to D_{up}^* \text{ FROM BELOW.} \end{cases}$$

NOW WE CALCULATE SOME PROPERTIES OF $H(B)$.

<u>PROPERTIES OF $H(B)$</u> ON $1 < B < B_{up} \equiv \frac{D_{up}^*}{D_{low}^*} \equiv 1 + \frac{3 \Pi_0}{w}$, WE HAVE

$$H(B) \sim \frac{\alpha}{2}(B-1)^{-1} \text{ AS } B \to 1^+ \quad \text{AND} \quad H(B) \sim \frac{\alpha}{2} B_{up}\left(\frac{1}{3} + \frac{1}{B_{up}-1}\right) \text{ AS } B \to B_{up}.$$

(i) $$H'(B) = \frac{\alpha}{2}\left[\frac{1}{3} + \frac{1}{B-1}\right] + \frac{\alpha B}{2}\left(-\frac{1}{(B-1)^2}\right) = \frac{\alpha}{2}\left[\frac{1}{3} + \frac{1}{(B-1)^2}\left[(B-1) - B\right]\right]$$

$$\left.H'(B) = \frac{\alpha}{2}\left[\frac{1}{3} - \frac{1}{(B-1)^2}\right] = \frac{\alpha}{6(B-1)^2}\left[(B-1)^2 - 3\right]\right\} \qquad (10)_1$$

WE CONCLUDE THAT $$\left.\begin{array}{l} H'(B) < 0 \text{ IF } (B-1)^2 < 3 \\ H'(B) > 0 \text{ IF } (B-1)^2 > 3 . \end{array}\right\} \qquad (10)_2$$

THUS WE CONCLUDE THAT

$$\left.\begin{array}{ll} H'(B) < 0 & \text{IF } 1 < B < 1+\sqrt{3} \\ H'(B) > 0 & \text{IF } B > 1+\sqrt{3}. \end{array}\right\} \quad (10)_3.$$

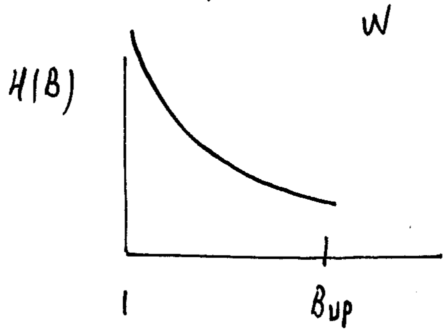
WE NOW MUST DETERMINE WHERE $B_{up} = 1 + \frac{3\mathcal{D}_0}{w}$ LIES WRT CRITICAL POINT $1+\sqrt{3}$.

WE HAVE

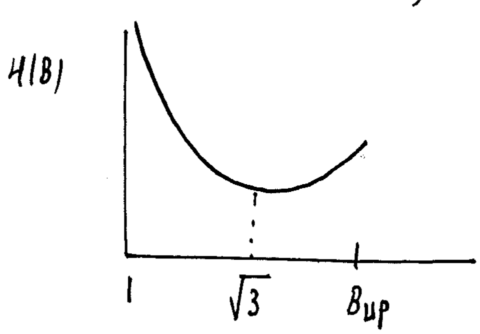
$$B_{up} < 1 + \sqrt{3} \quad \text{IFF} \quad w > \sqrt{3}\,\mathcal{D}_0 \implies S(\gamma - \alpha) > (1+\sqrt{3})\mathcal{D}_0.$$

<u>LEMMA</u> WE HAVE THE FOLLOWING RESULT ABOUT THE HOPF CURVE $H(B)$.

(i) SUPPOSE $w > \sqrt{3}\,\mathcal{D}_0$, THEN ON ITS INTERVAL OF DEFINITION GIVEN BY $1 < B < B_{up} = 1 + \frac{3\mathcal{D}_0}{w}$ WE HAVE $H'(B) < 0$.



(ii) NOW SUPPOSE $w < \sqrt{3}\,\mathcal{D}_0$, THEN WE HAVE



WE CONCLUDE THAT THE HOPF CURVE IS NOT NECESSARILY MONOTONIC IN D_0. THIS FACT WAS NOT OBSERVED IN THE INITIAL ANALYSIS OF THE HB THRESHOLD.

TWO-SPOT PATTERN

FOR ILLUSTRATION TAKE $K=2$. PLOT THE PHASE DIAGRAM AND STABILITY REGION. SET $j=1$ IN $D_j = \frac{D_0}{f}\left(1-\cos\left(\frac{\pi j}{K}\right)\right)$ WITH $f = S/2K$ AND $K=2$.

THIS GIVES $D_1 = \frac{D_0}{f} = \frac{2KD_0}{S}$. THIS GIVES $B \equiv \frac{D_1}{D_{low}^*} = \frac{2KD_0}{S D_{low}^*}$.

THEN $\tau_{H_1} = H\left(\frac{D_1}{D_{low}^*}\right)$ DEFINED ON $1 < \frac{2KD_0}{S D_{low}^*} < \frac{D_{up}^*}{D_{low}^*}$

WE CONCLUDE THAT $\tau_{H_1} = H\left(\frac{4}{S D_{low}^*} D_0\right)$ ON $\frac{S D_{low}^*}{2K} < D_0 < \frac{S D_{up}^*}{2K}$

WE CALCULATE
$$\frac{S D_{up}^*}{2K} = \frac{S}{2K}\left(\frac{w^3}{4\pi^2 K^3 \alpha^2}\right)\left(1+\frac{3\bar{U}_0}{w}\right) = \frac{w^3 S}{8\pi^2 K^4 \alpha^2}\left(1+\frac{3\bar{U}_0}{w}\right) \equiv D_{0,c}$$
$$\frac{S D_{low}^*}{2K} = \frac{S w^3}{8\pi^2 K^4 \alpha^2} \equiv \frac{D_{0,c}}{(1+3\bar{U}_0/w)}$$

WE OBSERVE THAT $\frac{S U_p^*}{2K} = D_{0,c}$ WHEN $K=2$ AS GIVEN IN (4.9).

THEREFORE, WE CONCLUDE THE FOLLOWING:

PROPOSITION ($K=2$ SPOTS)

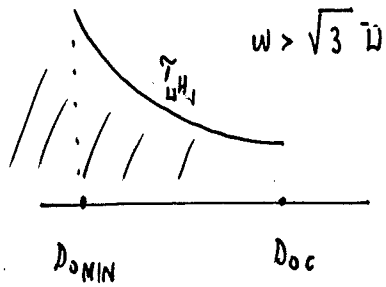
- IF $D_0 > D_{0,c} \equiv \frac{w^3 S}{128\pi^2\alpha^2}\left(1+\frac{3\bar{U}_0}{w}\right)$ WITH $w = S(\gamma-\alpha)-\bar{U}_0$

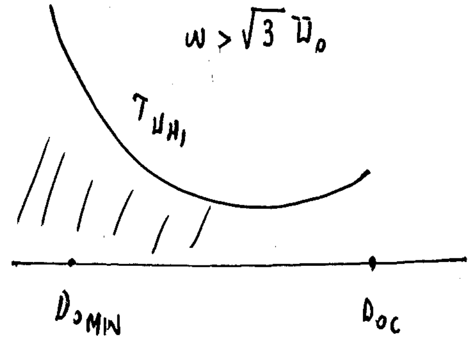
  THEN THE NLEP HAS A POSITIVE REAL EIGENVALUE $\forall \hat{\tau}_u \geq 0$.
- IF $D_0 < D_{0,c}\Big/\left(1+3\bar{U}_0/w\right)$ THEN $\mathrm{Re}\,\lambda < 0$ $\forall \hat{\tau}_u = O(1)$.
- IF $\frac{D_{0,c}}{(1+3\bar{D}_0/w)} < D_0 < D_{0,c}$, THEN $\exists$ A HB VALUE GIVEN BY

  $$\hat{\tau}_{uH_1} = \mathcal{H}\left(\frac{D_0(1+3\bar{U}_0/w)}{D_{0,c}}\right) \quad \text{WHERE } \mathcal{H}(B) \equiv \frac{\alpha B}{2}\left(\frac{1}{3}+\frac{1}{B-1}\right)$$

  WHEN $\hat{\tau}_u > \hat{\tau}_{uH_1} \rightarrow \mathrm{Re}\,\lambda > 0$ AND $\hat{\tau}_u < \hat{\tau}_{uH_1} \rightarrow \mathrm{Re}\,\lambda < 0$. (11)

THE HOPF CURVE IS MONOTONE DECREASING ON $\omega > \sqrt{3}\,\varpi_0$, OTHERWISE IT DECREASES AND THEN INCREASES





WHERE $\quad D_{0MIN} = \dfrac{D_{0,c}}{1 + 3\varpi_0/\omega}$. THE SHADED REGION IS STABLE.

HERE $K = 2$ IS SIGN-ALTERING MODE.

<u>REMARK 1</u> THE WINDOW $D_{0MIN} < D_0 < D_{0,c}$ WHERE A HOPF BIFURCATION OCCURS EXISTS ONLY BECAUSE $\varpi_0 > 0$, i.e. THE THIRD COMPONENT IN PDE.

NOW AS A PREPARATION FOR CALCULATING THE HOPF THRESHOLD AND FREQUENCY AS $D_0 \to D_{0c}^-$ WE NEED SOME SIMPLE FORMULAS.

<u>LEMMA 1</u> AT $D_0 = D_{0c}$, WE HAVE $D_1/D_{UP} = D_0/D_{0c}$ AND, (12)

$$\frac{1}{\chi_{0,1}} = \frac{3}{2}\left(1 + \frac{\varpi_0}{\omega}\right), \quad \frac{D_1}{D^*_{UPPER}} = 1 \quad , \quad \frac{D_1}{D^*_{LOWER}} = 1 + \frac{3\varpi_0}{\omega} \quad \frac{D^*_{UPPER}}{D^*_{LOWER}} = \left(1 + \frac{3\varpi_0}{\omega}\right)$$

<u>PROOF</u> WE NEED ONLY PROVE $\chi_{0,1}$ RESULT. NOTICE $K_3 = \alpha/2D_{low} \to \dfrac{K_3 D_1}{\alpha} = \dfrac{D_1}{2D_{low}} = \dfrac{1}{2}\left(1 + \dfrac{3\varpi_0}{\omega}\right)$

$$\frac{1}{\chi_{0,1}} = 1 + \frac{K_3 D_1}{\alpha} . \quad \text{BUT} \quad \frac{K_3 D_1}{\alpha} = \frac{1}{2} + \frac{3\varpi_0}{2\omega} \quad \text{AT ZERO-CROSSING.}$$

$$\text{HENCE} \quad \frac{1}{\chi_{0,1}} = \frac{3}{2}\left(1 + \varpi_0/\omega\right), \quad \text{AND} \quad K_3 D_1 \to \alpha\left(\frac{1}{2} + \frac{3\varpi_0}{2\omega}\right) = \frac{\alpha}{2}\left(1 + 3\varpi_0/\omega\right).$$

<u>LEMMA 2</u> NOW AS $D_0 \to D_{0c}^-$, THEN $\tau_{HH_1} \to \dfrac{\alpha}{2}\left(\dfrac{D_{UP}}{D_{low}}\right)\left(\dfrac{1}{3} + \dfrac{1}{D_{UP}/D_{low} - 1}\right)$

$$\hat{\tau}_{HH_1} = \frac{\alpha}{2}\left(1 + \frac{3\varpi_0}{\omega}\right)\left(\frac{1}{3} + \frac{1}{3\varpi_0/\omega}\right) = \frac{\alpha}{6}\left(1 + 3\varpi_0/\omega\right)\left(\frac{\varpi_0/\omega + 1}{\varpi_0/\omega}\right)$$

$$\text{THIS GIVES} \quad \hat{\tau}_{HH_1} \sim \frac{\omega\alpha}{6\varpi_0}\left(\frac{\varpi_0}{\omega} + 1\right)\left(\frac{3\varpi_0}{\omega} + 1\right) . \quad \text{AS} \quad D_0 \to D_{0c}^- . \quad (13)$$

NOW WE CALCULATE C_0/C_2 AT $D_0 \to D_{oc}^-$.

WE HAVE $C_0 = -\dfrac{K_3}{\alpha}\left(D_1 - D_{up}^*\right)$, $C_2 = \dfrac{\hat{T}_j}{3\chi_{oj}} = \dfrac{\hat{T}_{uHj}}{3\chi_{oj}\, K_3 D_1}$

WE SIMPLIFY C_2 AT $D_0 \to D_{oc}^-$.

$$C_2 \to \frac{w\alpha}{6\bar{U}_0}\left(\frac{\bar{U}_0}{w}+1\right)\left(\frac{3\bar{U}_0}{w}+1\right)\frac{1}{3K_3D_1\chi_{01}} = \frac{w}{18\bar{U}_0}\left(\frac{\bar{U}_0}{w}+1\right)\left(\frac{3\bar{U}_0}{w}+1\right)\left(\frac{\alpha}{K_3D_1}\right)\left(\frac{1}{\chi_{01}}\right)$$

NOW $\dfrac{K_3D_1}{\alpha} = \dfrac{1}{2}\left(1+3\bar{U}_0/w\right)$ AND $\dfrac{1}{\chi_{01}} = \dfrac{3}{2}\left(1+\bar{U}_0/w\right)$

THUS $$C_2 \to \frac{w}{18\bar{U}_0}\left(\frac{\bar{U}_0}{w}+1\right)\left(\frac{3\bar{U}_0}{w}+1\right)\frac{2}{(1+3\bar{U}_0/w)}\,\frac{3}{2}\left(1+\frac{\bar{U}_0}{w}\right)$$

$$C_2 \to \frac{w}{6\bar{U}_0}\left(\frac{\bar{U}_0}{w}+1\right)^2. \qquad (14).$$

NOW $$C_0 = -\frac{K_3D_{up}^*}{\alpha}\left(\frac{D_1}{D_{up}^*}-1\right) = -\left(\frac{K_3D_{low}^*}{\alpha}\right)\frac{D_{up}^*}{D_{low}^*}\left(\frac{D_0}{D_{oc}}-1\right)$$

NOW $\dfrac{K_3D_{low}^*}{\alpha} = \dfrac{1}{2}$ AND $D_{up}^*/D_{low}^* = 1+\dfrac{3\bar{U}_0}{w}$.

SO $$C_0 \mathrel{\hat{=}} +\frac{1}{2}\left(1+\frac{3\bar{U}_0}{w}\right)\left(1-D_0/D_{oc}\right)$$

WE GET $$\frac{C_0}{C_2} \mathrel{\hat{=}} \frac{3\bar{U}_0}{w}\left(1+\frac{3\bar{U}_0}{w}\right)\frac{\left(1-D_0/D_{oc}\right)}{\left(\bar{U}_0/w+1\right)^2}.$$

NOW FOR $D_0 \to D_{oc}^-$, $\lambda_{IH}^2 = C_0/C_2$. THUS

$$\lambda_{IH} \sim \frac{1}{(\bar{U}_0/w+1)}\sqrt{\frac{3\bar{U}_0}{w}\left(1+\frac{3\bar{U}_0}{w}\right)}\left(1-D_0/D_{oc}\right)^{1/2}. \qquad (15).$$

LEMMA NOW AS $D_0 \to D_{0,c}^-$ WE HAVE

$$\left.\begin{aligned} \hat{T}_{uH_1} &\to \frac{w\alpha}{6\bar{u}_0}\left(\frac{\bar{u}_0}{w}+1\right)\left(\frac{3\bar{u}_0}{w}+1\right) \\ \text{AND} \quad \lambda_{IH} &\to \frac{1}{(\bar{u}_0/w+1)}\sqrt{\frac{3\bar{u}_0}{w}\left(1+\frac{3\bar{u}_0}{w}\right)}\left(1-D_0/D_{0,c}\right)^{1/2}. \end{aligned}\right\} \quad (16)$$

WE NOW REMARK ON POLICE DIFFUSIVITY.

REMARK 2 RECALL $D_p = \dfrac{D_0}{\varepsilon^{5-q}\hat{T}_u}$. FOR $q = 3$, $D_p = \dfrac{D_0}{\varepsilon^2 \hat{T}_u}$, SO $D_p = O(\varepsilon^{-2})$.

OUR RESULT IS THE FOLLOWING.

SUPPOSE $D_{0MIN} < D_0 < D_{0,c}$. THEN $\exists$ A HB VALUE OF D_p FOR WHICH AN ASYNCHRONOUS OSCILLATION OCCURS. IT IS GIVEN BY

$$D_{p_H} = \frac{D_0}{\varepsilon^2 \hat{T}_{uH_1}} \qquad (17)$$

- IF $D_p > 0 \longrightarrow$ HOTSPOT PATTERN IS ALWAYS UNSTABLE WHEN $D_0 > D_{0,c}$
- IF $D_p > 0 \longrightarrow$ HOTSPOT PATTERN STABLE, WHEN $D_0 < D_{0,MIN} = \dfrac{D_{0,c}}{(1+3\bar{u}_0/w)}$

PROPOSITION SUPPOSE $q = 3$ AND $D_p = O(\varepsilon^{-2})$. THEN, FOR 2 HOTSPOTS:

- IF $D_0 > D_{0,c} \longrightarrow$ HOTSPOT PATTERN UNSTABLE $\forall D_p$.
- IF $D_0 < \dfrac{D_{0,c}}{(1+3\bar{u}_0/w)} \longrightarrow$ HOTSPOT PATTERN STABLE $\forall D_p$.
- IF $\dfrac{D_{0,c}}{(1+3\bar{u}_0/w)} < D_0 < D_{0,c}$, $\longrightarrow$ HOTSPOT PATTERN STABLE IF $D_p < \dfrac{D_0}{\varepsilon^2 \hat{T}_{uH_1}}$

  AND UNSTABLE DUE TO OSCILLATION IF $D_p > \dfrac{D_0}{\varepsilon^2 \hat{T}_{uH_1}}$.

THUS IF $D_0 < \dfrac{D_{0,c}}{(1+3\bar{u}_0/w)}$ NO POLICE INTERVENTION OR CHANGE IN DIFFUSIVITY CAN DESTABILIZE THE PATTERN.

HIGHER MODES $K \geq 2$

NOW WHEN THERE ARE MORE POSSIBLE MODES WE SEEK TO DETERMINE THE STABILITY REGION. THERE ARE IN GENERAL $K-1$ HOPF BRANCHES $\hat{\tau}_{Hj}$ FOR $j=1,\ldots,K-1$ VERSUS D_0. OUR GOAL IS

(i) DETERMINE THESE BRANCHES AND THEIR OVERLAP STRUCTURE.

(ii) IDENTIFY THE SMALLEST THRESHOLD, AND FOCUS ONLY ON REGION $D_0 < D_{0,c}$ SINCE FOR $D_0 > D_{0,c}$ WE HAVE $\lambda > 0$ DUE TO $j = K-1$ MODE FOR ANY $\hat{\tau}_H$.

NOW WE WILL INTRODUCE SOME NOTATION TO DETERMINE THE ENDS OF THE INTERVALS IN D_0 WHERE A HB OCCURS.

$$D_{0,j}^{+} \equiv D_{0,c}\left(\frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)}\right) \qquad j=1,\ldots,K-1$$

$$D_{0,j}^{-} \equiv \frac{D_{0,c}}{(1+3\bar{u}_0/w)}\left(\frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)}\right) \qquad j=1,\ldots,K-1$$

THIS IS MOTIVATED BY WE NEED $D_{low}^{*} \leq D_j \leq D_{up}^{*}$.

THEN $D_j = \frac{D_0}{f}\left(1-\cos\left(\frac{\pi j}{K}\right)\right)$ SO $f = S/2K$ YIELDS

$$D_{low}^{*} \equiv \frac{w^3}{4\pi^2 K^3 \kappa^2}, \quad D_{up}^{*} = D_{low}^{*}\left(\frac{1+3\bar{u}_0}{w}\right)$$

$$D_{low}^{*} \leq \frac{D_0\, 2K}{S}\left(1-\cos(\pi j/K)\right) \leq D_{up}^{*}$$

$$\frac{S\, D_{low}^{*}}{2K\left(1-\cos\left(\frac{\pi j}{K}\right)\right)} \leq D_0 \leq \frac{S}{2K}\,\frac{D_{up}}{1-\cos(\pi j/K)} = \frac{S}{2K\left(1-\cos\left(\frac{\pi(K-1)}{K}\right)\right)}\;\frac{1-\cos(\pi (K-1)/K)}{1-\cos(\pi j/K)}\, D_{up}$$

THUS $$\frac{D_{0,c}}{(1+3\bar{u}_0/w)}\left(\frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)}\right) \leq D_0 < D_{0,c}\left(\frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)}\right)$$

WE OBSERVE THAT FOR $j = K-1$ WE HAVE

$$D_{0,K-1}^{+} = D_{0,c}$$

$$D_{0,K-1}^{-} = D_{0,c}/(1+3\bar{u}_0/w)$$

WHICH WAS OUR THRESHOLD.

NOW WE DETERMINE RATIO

$$\frac{D_j}{D_{low}^{*}} = \frac{\frac{2KD_0}{S}\left(1-\cos(\pi j/K)\right)}{D_{low}^{*}} = \frac{D_0}{\frac{D_{low}\, S}{2K\left(1-\cos(\pi j/K)\right)}} = \frac{D_0}{\frac{S D_{up}\left(1+3\tilde{\Omega}_0/\omega\right)^{-1}\left(1-\cos(\pi(K-1)/K)\right)}{2K\left(1-\cos(\pi(K-1)/K)\right)\left(1-\cos(\pi j/K)\right)}}$$

$$= \frac{D_0}{\left(1+3\tilde{\Omega}_0/\omega\right)^{-1} D_{0c}\, \frac{\left(1-\cos(\pi(K-1)/K)\right)}{\left(1-\cos(\pi j/K)\right)}} = \frac{D_0}{D_{0,j}^{-}}.$$

WE THEN CLAIM THE FOLLOWING.

THE HB CURVES IN TERMS OF D_0 ARE

$$\hat{\tau}_{uH_j} = \mathcal{H}\left(D_0/D_{0,j}^{-}\right) \quad \text{ON} \quad D_{0,j}^{-} < D_0 < D_{0,j}^{+}$$

WHERE $D_{0,j}^{+}$ AND $D_{0,j}^{-}$ ARE DEFINED BY

$$\mathcal{H}(B) \equiv \frac{\alpha B}{2}\left(\frac{1}{3}+\frac{1}{B-1}\right)$$

$$\left.\begin{aligned} D_{0,j}^{+} &= D_{0,c}\left(\frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)}\right) \\ D_{0,j}^{-} &= \frac{D_{0,c}}{\left(1+3\tilde{\Omega}_0/\omega\right)}\left(\frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)}\right) \end{aligned}\right\} \quad (18)$$

WHERE
$$D_{0,c} \equiv \frac{\omega^3 S}{8\pi^2\alpha^2 K^4\left(1+\cos(\pi/K)\right)}\left(1+3\tilde{\Omega}_0/\omega\right)$$

<u>**PROPOSITION**</u>

WE HAVE THE FOLLOWING:

(i) IF $D_0 > D_{0,j}^{+}$ THE j^{th} MODE IS UNSTABLE DUE TO A POSITIVE REAL EIGENVALUE.

(ii) IF $D_0 < D_{0,j}^{-}$ THE j^{th} MODE IS STABLE $\forall\, \hat{\tau}_u > 0$.

(iii) IF $D_{0,j}^{-} < D_0 < D_{0,j}^{+}$, THEN $\exists$ A HB FOR j^{th} MODE WHEN $\hat{\tau}_u = \tilde{\tau}_{uH_j}$ AS DEFINED ABOVE.

IF $\hat{\tau}_u > \hat{\tau}_{uH_j} \rightarrow$ UNSTABLE, $\hat{\tau}_u < \hat{\tau}_{uH_j} \rightarrow$ STABLE

THESE GIVE THE THRESHOLDS FOR A HB.

PROPOSITION LET $\varepsilon \to 0$, $K \geq 2$, $D = D_0/\varepsilon^2$, $\hat{\tau}_u \ll O(\varepsilon^{-2})$. THEN DEFINE $D_{0,j}^+$ AND $D_{0,j}^-$ FOR $j = 1, \ldots, K-1$ AS WRITTEN.

- IF $D_0 > D_{0,j}^+ \to j^{th}$ MODE IS UNSTABLE DUE TO A POSITIVE EIGENVALUE.
- IF $D_0 < D_{0,j}^- \to j^{th}$ MODE IS ~~UN~~STABLE $\forall \hat{\tau}_u$.
- IF $D_{0,j}^- < D_0 < D_{0,j}^+$ THEN UNSTABLE IF $\hat{\tau}_u > \hat{\tau}_{uHj}$ AND STABLE IF $\hat{\tau}_u < \hat{\tau}_{uHj}$. HERE $\hat{\tau}_{uHj} = H\left(D_0/D_{0,j}^- \right)$ IS HB THRESHOLD WITH $H(B) = \dfrac{\alpha B}{2}\left(\dfrac{1}{B} + \dfrac{1}{B-1} \right)$.

NOW WE REWRITE THE COEFFICIENTS C_0 AND C_1 AND C_2 IN TERMS OF OUR NEW VARIABLES. THE QUADRATIC WAS $C_2 \lambda^2 + C_1 \lambda + C_0 = 0$

LEMMA

$$\left.\begin{aligned} C_0 &= -\frac{1}{2}\left(1 + \frac{3\overline{u}_0}{w}\right)\left(\frac{D_0}{D_{0,j}^+} - 1\right), \qquad \text{WHERE } \hat{\tau}_{uHj} = H\left(D_0/D_{0,j}^- \right) \\ C_1 &= \frac{1}{\alpha}\,\hat{\tau}_{uHj}\left(\frac{D_{0,j}^-}{D_0} - 1\right)\left(\frac{\hat{\tau}_u}{\hat{\tau}_{uHj}} - 1\right) \quad \text{AND} \quad C_2 = \frac{\hat{\tau}_{uHj}}{3\alpha}\left(\frac{2D_{0,j}^-}{D_0} + 1\right). \end{aligned}\right\} \quad (19)$$

PROOF WE HAVE $C_0 = -\dfrac{K_3 D_{up}}{\alpha}\left(\dfrac{D_j}{D_{up}} - 1\right)$. BUT $\dfrac{K_3}{\alpha} = \dfrac{1}{2D_{low}}$.

THUS $C_0 = \dfrac{-D_{up}}{2D_{low}}\left(\dfrac{D_j}{D_{up}} - 1\right)$. NOW

$$\frac{D_j}{D_{up}} = \frac{\dfrac{D_0 2K}{S}\left(1 - \cos(\pi j/K)\right)}{D_{up}} = \frac{D_0}{\dfrac{D_{up} S}{2K\left(1 - \cos\left(\frac{\pi(K-1)}{K}\right)\right)}\dfrac{(1-\cos(\frac{\pi}{K}(K-1)))}{(1-\cos(\frac{\pi j}{K}))}}$$

$$= \frac{D_0}{D_{0c}\left(\dfrac{1-\cos(\pi(K-1)/K)}{1-\cos(\pi j/K)}\right)} = \frac{D_0}{D_{0,j}^+}$$

THUS $C_0 = -\dfrac{D_{up}}{2D_{low}}\left(\dfrac{D_0}{D_{0,j}^+} - 1\right)$.

BUT $\dfrac{D_{up}}{D_{low}} = 1 + 3\overline{u}_0/w$ YIELDS $C_0 = -\dfrac{1}{2}\left(1 + \dfrac{3\overline{u}_0}{w}\right)\left(\dfrac{D_0}{D_{0,j}^+} - 1\right)$.

WE NOW CALCULATE C_1:

$$C_1 = \frac{\tilde{\tau}_{jH}}{2}\left(1 - \frac{D_j}{D_{low}^*}\right)\left(\frac{\tilde{\tau}_j}{\tilde{\tau}_{jH}} - 1\right)$$

NOW $\dfrac{\tilde{\tau}_j}{\tilde{\tau}_{jH}} = \dfrac{\hat{\tau}_W}{\hat{\tau}_{WHj}}$. $\dfrac{D_j}{D_{low}^*} = \dfrac{D_0}{D_{0,j}^-}$.

SO $$C_1 = \frac{\hat{\tau}_{WHj}}{2K_3 D_j}\left(1 - \frac{D_0}{D_{0,j}^-}\right)\left(\frac{\hat{\tau}_W}{\hat{\tau}_{WHj}} - 1\right).$$

NOW $K_3 D_j = \dfrac{\alpha}{2 D_{low}^*} D_j = \dfrac{\alpha}{2}\dfrac{D_0}{D_{0,j}^-}$. SO $2K_3 D_j = \alpha \dfrac{D_0}{D_{0,j}^-}$.

THIS YIELDS THAT $$C_1 = \frac{\hat{\tau}_{WHj}}{\alpha}\left(\frac{D_{0,j}^-}{D_0} - 1\right)\left(\frac{\hat{\tau}_W}{\hat{\tau}_{WHj}} - 1\right).$$

NOW WE CALCULATE C_2. $$C_2 = \frac{\hat{\tau}_{WHj}}{3K_j D_j \chi_{0j}} = \frac{\hat{\tau}_{WHj}}{3\left(\frac{\alpha}{2}\frac{D_0}{D_{0,j}^-}\right)}\frac{1}{\chi_{0j}} = \frac{2\hat{\tau}_{WHj}}{3\alpha}\frac{D_{0j}^-}{D_0}\frac{1}{\chi_{0,j}}.$$

THEN $$\frac{1}{\chi_{0j}} = 1 + \frac{K_3 D_j}{\alpha} = 1 + \frac{1}{2}\frac{D_0}{D_{0,j}^-}.$$

THUS $$C_2 = \frac{\hat{\tau}_{WHj}}{3\alpha}\frac{2D_{0,j}^-}{D_0}\left(1 + \frac{1}{2}\frac{D_0}{D_{0,j}^-}\right)$$

THUS $$C_2 = \frac{\hat{\tau}_{WHj}}{3\alpha}\left(\frac{2D_{0,j}^-}{D_0} + 1\right).$$

NOW WE TRY TO DETERMINE THE LIMITING FORMULA FOR THE HB AND THE HB FREQUENCY AS $D_0 \to D_{0,j}^+$. NOW AT THE HB, WE HAVE

$$\lambda_{IH}^2 = \frac{C_0}{C_2} = \frac{\frac{1}{2}\left(1 + \frac{3\bar{U}_0}{\omega}\right)\left(1 - D_0/D_{0,j}^+\right)}{\frac{\hat{\tau}_{WHj}}{3\alpha}\left(1 + \frac{2D_{0,j}^-}{D_0}\right)} = \frac{3\alpha}{2}\frac{\left(1 + 3\bar{U}_0/\omega\right)\left(1 - D_0/D_{0,j}^+\right)}{\hat{\tau}_{WHj}\left(1 + 2D_{0,j}^-/D_0\right)}$$

NOW WE CALCULATE FURTHER:

$$\hat{\tau}_{UH_j} = \frac{\alpha}{2}\frac{D_0}{D_{0,j}^-}\left(\frac{1}{3}+\frac{1}{\frac{D_0}{D_{0,j}^-}-1}\right) = \frac{\alpha}{6}\frac{D_0}{D_{0,j}^-}\left(\frac{D_0/D_{0,j}^- + 2}{D_0/D_{0j}^- - 1}\right)$$

NOW $$\frac{2}{\alpha}\hat{\tau}_{UH_j} = \frac{D_0}{3D_{0,j}^-}\left(\frac{D_0/D_{0,j}^- + 2}{D_0/D_{0,j}^- - 1}\right)$$

WE SUBSTITUTE TO GET

$$\lambda_{IH}^2 = \frac{3^2(1+3\bar{U}_0/\omega)\left(1-D_0/D_{0,j}^+\right)\left(D_0/D_{0,j}^- - 1\right)}{\left(1+2\frac{D_{0,j}^-}{D_0}\right)\frac{D_0}{D_{0,j}^-}\left(D_0/D_{0,j}^- + 2\right)} = \frac{9\left(1+\frac{3\bar{U}_0}{\omega}\right)\left(1-\frac{D_0}{D_{0,j}^+}\right)\left(\frac{D_0}{D_{0,j}^-}-1\right)}{\left(D_0/D_{0,j}^- + 2\right)^2}.$$

WE CLAIM THAT ON $D_{0,j}^- < D_0 < D_{0,j}^+$ WE HAVE

$$\left.\lambda_{IH} = \frac{3}{\left(D_0/D_{0,j}^- + 2\right)}\sqrt{\left(1+\frac{3\bar{U}_0}{\omega}\right)\left(1-\frac{D_0}{D_{0,j}^+}\right)\left(\frac{D_0}{D_{0,j}^-}-1\right)}\right\} \qquad (20)_1$$

NOW CONSIDER LIMIT $D_0 \to D_{0,j}^+$. THEN $\frac{D_0}{D_{0,j}^-} \to \frac{D_{0,j}^+}{D_{0,j}^-} = 1 + \frac{3\bar{U}_0}{\omega}$.

$$\frac{D_0}{D_{0,j}^-} + 2 \to 3\left(1+\frac{\bar{U}_0}{\omega}\right)$$

THUS $$\lambda_{IH} \longrightarrow \frac{1}{(1+\bar{U}_0/\omega)}\sqrt{\frac{3\bar{U}_0}{\omega}\left(1+\frac{3\bar{U}_0}{\omega}\right)\left(1-\frac{D_0}{D_{0,j}^+}\right)} \quad \text{AS } D_0 \to D_{0,j}^+ \qquad (20)_2$$

THIS IS SAME RESULT AS FOUND IN (15) ON PAGE E8.

NOW LET $D_0/D_{0,j}^- \to 1$. THEN $\frac{D_0}{D_{0,j}^+} \to \frac{D_{0,j}^-}{D_{0j}^+} = \frac{1}{1+3\bar{U}_0/\omega}$. HENCE

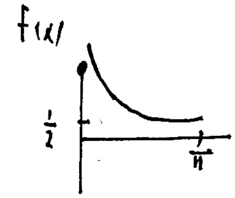
$$\lambda_{IH} \to \sqrt{(1+3\bar{U}_0/\omega)\left(1-\frac{1}{1+3\bar{U}_0/\omega}\right)\left(\frac{D_0}{D_{0,j}^-}-1\right)}$$

SO $$\lambda_{IH} \to \sqrt{\frac{3\bar{U}_0}{\omega}}\left(\frac{D_0}{D_{0,j}^-}-1\right)^{1/2} \quad \text{AS } D_0 \to D_{0,j}^-. \qquad (20)_3$$

GLOBAL BEHAVIOR OF HOPF CURVES

WE FIRST OBSERVE ORDERING PRINCIPLE THAT

$$D_{0,K-1}^{+} < D_{0,j}^{+} \quad \forall\ j=1,..,K-2 \quad \text{IF} \quad K \geq 3.$$



DEFINE $\quad f(x) = \dfrac{1}{1-\cos(x)}$

LET $\quad x_j = \pi j/K \quad$ FOR $\quad j=1,..,K-1$, SO THAT $\quad 0 < x < \pi$.

PROVE $\quad f'(x) < 0 \quad$ SO THAT $\quad f(x_{j-1}) > f(x_j) \quad \forall j$.

NOW $\quad f'(x) = -(1-\cos x)^{-2}[\sin x] = -\dfrac{\sin x}{(1-\cos x)^2} < 0 \quad$ ON $\quad 0 < x < \pi$

THUS $\quad f(x_j) < f(x_{j-1}) \quad \forall j. \quad$ WE REFER TO $D_{0,j}^{-} < D_0 < D_{0,j}^{+}$ AS THE j^{th} HOPF WINDOW.

WE HAVE

$$D_{0,j+1}^{+} > D_{0,j}^{+} \quad \forall\ j=1,..,K-2.$$

$$D_{0,j-1}^{+} > D_{0,j-1}^{-} \quad \forall\ j=1,..,K-2.$$

WE THEN HAVE THE FOLLOWING:

$$\min_j \{ D_{0,j}^{+} \} = D_{0,K-1}^{+}, \qquad \min \{ D_{0,j}^{-} \} = D_{0,K-1}^{-}.$$

WE HAVE THE FOLLOWING:

<u>PROPOSITION</u> LET $j=1,..,K-1$, THEN A K-HOTSPOT PATTERN IS LINEARLY STABLE $\forall\ \hat{\tau}_u \geq 0$ WHEN $D_0 < D_{0,K-1}^{-} = \dfrac{D_{0,c}}{(1+3\bar{u}_0/w)}$.

IT IS UNSTABLE $\forall\ \hat{\tau}_u \geq 0$ WHEN $D_0 > D_{0,K-1}^{+} = D_{0,c}$.

THEREFORE, OVER ALL $j=1,..,K-1$, THE SIGN ALTERNATING MODE $j=K-1$ SETS THE BOUNDS OF THE ~~STABILIT~~ HOPF WINDOW.

NOW ON THIS HOPF WINDOW WE WOULD LIKE TO DETERMINE THE MINIMUM OF THE HB THRESHOLDS.

E16

CASE I SUPPOSE $\omega > \sqrt{3}\,\overline{\Omega}_0$ SO THAT $H(B)$ IS MONOTONE DECREASING ON $1 \le B \le 1 + \frac{3\overline{\Omega}_0}{\omega}$.

DEFINE $B = \frac{D_0}{D^-_{0,K-1}}$ SO THAT $\hat{T}_{UHK-1} = H(B)$ ON $1 \le B \le \frac{D^+_{0,K-1}}{D^-_{0,K-1}} = 1 + \frac{3\overline{\Omega}_0}{\omega}$.

LET $K \ge 3$ AND CONSIDER j^{th} MODE IN $j = 1, .., K-2$. THEN

$$\hat{T}_{UH_j} = H\left(\frac{D_0}{D^-_{0,j}}\right) \quad \text{ON} \quad D^-_{0j} < D_0 < D^+_{0,j}.$$

WE WRITE $\hat{T}_{UH_j} = H\left(\frac{D_0}{D^-_{0,K-1}} \; \frac{D^-_{0,K-1}}{D^-_{0,j}}\right)$ ON $1 \le \frac{D_0}{D^-_{0,j}} < \frac{D_{0,j}^+}{D_{0,j}^+} \le 1 + \frac{3\overline{\Omega}_0}{\omega}$.

DEFINE $\varsigma_j \equiv \frac{D^-_{0,K-1}}{D^-_{0,j}} < 1 \quad \forall j = 1, .., K-2$. THEN

$$\hat{T}_{UH_j} = H(\varsigma_j B) \quad \text{ON} \quad \frac{1}{\varsigma_j} \le B \le \frac{1}{\varsigma_j}\left(1 + \frac{3\overline{\Omega}_0}{\omega}\right)$$

$$\hat{T}_{UHK-1} = H(B) \quad \text{ON} \quad 1 \le B \le \left(1 + \frac{3\overline{\Omega}_0}{\omega}\right).$$

NOW WE WANT TO CONSIDER THE REGION $1 \le B \le \left(1 + \frac{3\overline{\Omega}_0}{\omega}\right)$, i.e. THE INTERSECTION OF $S_B \equiv \left(1,\ 1 + \frac{3\overline{\Omega}_0}{\omega}\right) \cap \left(\frac{1}{\varsigma_j},\ \frac{1}{\varsigma_j}\left(1 + \frac{3\overline{\Omega}_0}{\omega}\right)\right)$, WITH $\varsigma_j < 0$

IF THERE IS NO INTERSECTION, i.e. IF $\frac{1}{\varsigma_j} > 1 + \frac{3\overline{\Omega}_0}{\omega}$ THEN WE ARE DONE.

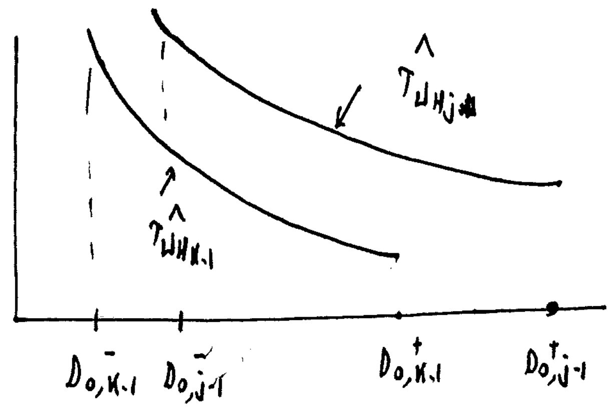
SUPPOSE THAT $\frac{1}{\varsigma_j} < 1 + \frac{3\overline{\Omega}_0}{\omega}$ SO $S_B \equiv \left(\frac{1}{\varsigma_j},\ 1 + \frac{3\overline{\Omega}_0}{\omega}\right)$.

WE WANT TO PROVE THAT $H(B) < H(\varsigma_j B)$ ON $\frac{1}{\varsigma_j} < B < 1 + \frac{3\overline{\Omega}_0}{\omega}$.

THIS FOLLOWS SINCE $H'(B) < 0$, i.e. $H(B) < H(B_1)$ WHEN $B_1 < B$.

$\Longrightarrow$ $\hat{T}_{UHK-1}$ SETS THE HOPF THRESHOLD.

THE PICTURE IS

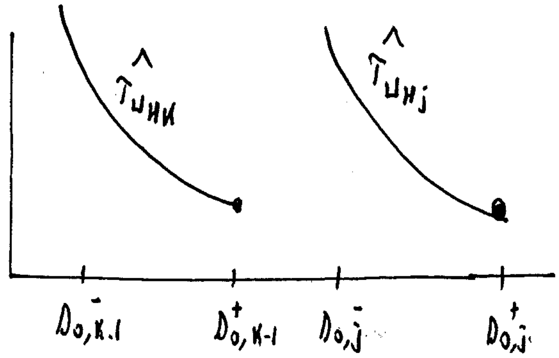


REMARK, THERE IS NO OVERLAP OF j^{th} MODE WITH K-1 ST IF

$$\frac{1}{\zeta_j} > 1 + \frac{3\bar{U}_o}{\omega} \rightarrow \frac{D_{o,j}^-}{D_{o,K-1}^-} > 1 + \frac{3\bar{U}_o}{\omega}$$

$$\frac{\dfrac{D_{oc}}{(1+3\bar{U}_o/\omega)}\dfrac{(1+\cos(\pi/K))}{(1-\cos(\pi j/K))}}{\dfrac{D_{o,c}}{(1+3\bar{U}_o/\omega)}} = \frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)} > 1 + \frac{3\bar{U}_o}{\omega}. \quad (*)$$

IN THIS CASE WE HAVE



$D_{o,j}^- > D_{o,K-1}^+$ MEANS

$$\frac{1+\cos(\pi/K)}{1-\cos(\pi j/K)} > 1 + \frac{3\bar{U}_o}{\omega}.$$

WHEN (*) OCCURS, THERE IS NOTHING TO CHECK SINCE INTERVAL WHERE MODES EXIST DO NOT OVERLAP!

PROPOSITION SUPPOSE THAT $W > \sqrt{3}\,\bar{D}_0$ (i.e. $H'(B)<0$). THEN FOR A K-HOTSPOT PATTERN WE HAVE THAT $j = K-1$ MODE SETS ALL STABILITY THRESHOLDS (WHEN CONSIDERING ALL POSSIBLE MODES), i.e.

$W > \sqrt{3}\,\bar{D}_0$

$\rightarrow \bar{D}_0 < \dfrac{S(\gamma-\alpha)}{\sqrt{3}+1}$

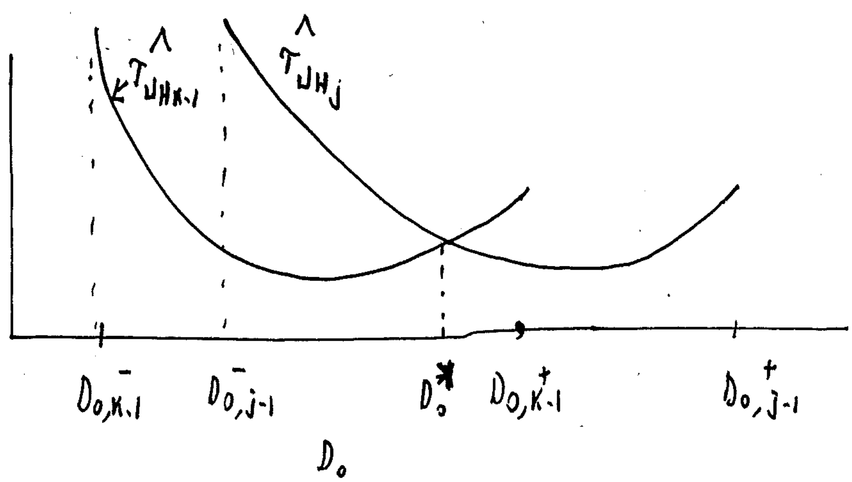
- IF $D_0 < D_{0,K-1}^-$ $\longrightarrow$ linearly stable $\forall\ \hat{\tau}_u \geq 0$.
- IF $D_0 > D_{0,K-1}^+$ $\longrightarrow$ unstable $\forall\ \hat{\tau}_u \geq 0$ due to at least one positive real axis. Could also be oscillatory instabilities if $\hat{\tau}_u > \hat{\tau}_{uH_j}$ FOR $j = 1,\ldots,K-2$ $\Rightarrow$ BUT ALWAYS UNSTABLE
- IF $D_{0,K-1}^- < D_0 < D_{0,K-1}^+$ THEN
  - INSTABILITY IF $\hat{\tau}_u > \hat{\tau}_{uHK-1} \equiv H\left(\dfrac{D_0}{D_{0,K-1}^-}\right)$

  - STABILITY IF $0 \leq \hat{\tau}_u < \hat{\tau}_{uHK-1}$.

NOTICE THAT THE K-1 ST MODE GIVES SMALLEST HB threshold when restricted to interval $D_{0,K-1}^- < D_0 < D_{0,K-1}^+$ AS PROVED ON PAGE E16.

NEXT WE CONSIDER THE MORE CHALLENGING CASE WHERE $W < \sqrt{3}\,\bar{D}_0$ SO THAT $\bar{D}_0$ SATISFIES $\dfrac{S(\gamma-\alpha)}{\sqrt{3}+1} < \bar{D}_0 < \bar{D}_{0MAX} \equiv S(\gamma-\alpha)$.

IN PARTICULAR IN THIS CASE FOR SOME j WE MIGHT HAVE A PICTURE LIKE



IF THIS PICTURE OCCURS, THEN

ON $D_{0,K-1}^{-} < D_0 < D_0^{*} \longrightarrow \hat{T}_{UH_{K-1}}$ SETS THE THRESHOLD FOR HB.

ON $D_0^{*} < D_0 < D_{0,K-1}^{+} \longrightarrow \hat{T}_{UH_j}$ SETS THE THRESHOLD FOR HB.

SO IN PRINCIPLE IN NON-MONOTONIC CASE WE COULD HAVE THIS CROSSING.

<u>REMARK</u> IN SIMON'S ORIGINAL ANALYSIS WE WORKS WITH $\hat{T}_j$ AND ONLY VERIFIES $\frac{\alpha}{2}\left(\frac{1}{3}+\frac{1}{B-1}\right)$ IS MONOTONICALLY DECREASING.

<u>HOWEVER</u> WE MUST WORK WITH $H(B) \equiv \frac{\alpha B}{2}\left(\frac{1}{3}+\frac{1}{B-1}\right)$ FOR $\hat{T}_U$ (i.e. EXTRA B FACTOR) WHICH IS NO LONGER MONOTINIC.

WE WILL SEE IF WE CAN PUSH THIS NON-MONOTONE CASE FURTHER.