Towards a Universe of Linear Syntaxes with Binding

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Context

$$
\begin{tikzcd}
 & \text{McB05}^{1} \arrow[dl, "\text{linearity}"'] \arrow[dr, "\text{semantics}"] & \\
\text{WA20}^{2} \arrow[dr] \arrow[rr, phantom, "\wedge" near end, shift right=6] & & \text{AACMM18/20}^{3} \arrow[dl] \\
 & \text{WIP} &
\end{tikzcd}
$$

- Linearity independent of binding
- Only one traversal over the syntax

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[1] Conor McBride. *Type-preserving renaming and substitution*. 2005. URL: http://www.strictlypositive.org/ren-sub.pdf.

[2] James Wood and Robert Atkey. “A Linear Algebra Approach to Linear Metatheory”. Accepted for Linearity/TLLA. 2020.

[3] Guillaume Allais et al. “A Type and Scope Safe Universe of Syntaxes with Binding: Their Semantics and Proofs”. Accepted for JFP. 2020.

Idea — stability under weakening[4]

Two parts:

1. Consolidate all traversals over syntax (e.g, simultaneous renaming, simultaneous substitution, NbE, printing) into a single generic traversal (*kits/semantics*).
2. Build typing rules from small building blocks, so that they admit a generic semantics by construction.

[4] Guillaume Allais et al. "A Type and Scope Safe Universe of Syntaxes with Binding: Their Semantics and Proofs". Accepted for JFP. 2020.

Generic notion of semantics

- In the intuitionistic case, we have the following fundamental lemma of semantics:

$$\frac{\text{Semantics } \mathcal{V}\ \mathcal{C} \qquad \overbrace{\Gamma \ni A \to \mathcal{V}\ A\ \Delta}^{\text{environment}}}{\underbrace{\Gamma \vdash A \to \mathcal{C}\ A\ \Delta}_{\text{traversal}}}$$

- $\text{Semantics } \mathcal{V}\ \mathcal{C}$ contains:
  - Proof that $\mathcal{V}$ is stable under weakening
  - Ways to interpret term constructors semantically, generic in context:

$$[\![\text{var}]\!] : \forall[\ \mathcal{V}\ A \mathrel{\dot{\to}} \mathcal{C}\ A\ ] \qquad [\![\text{app}]\!] : \forall[\ \mathcal{C}\ (A \to B) \mathrel{\dot{\times}} \mathcal{C}\ A \mathrel{\dot{\to}} \mathcal{C}\ B\ ]$$

$$[\![\text{lam}]\!] : \forall[\ \Box(\mathcal{V}\ A \mathrel{\dot{\to}} \mathcal{C}\ B) \mathrel{\dot{\to}} \mathcal{C}\ (A \to B)\ ] \qquad \ldots$$

Generic notion of syntax

- A type system can:
  1. Offer a multitude of term formers. e.g, $\textsc{App}$, $\textsc{Lam}$, ...
  2. For each term former, require 0 or more premises. $\dot{\times}$, $\dot{1}$
  3. For each premise, maybe bind variables. $\square$, $\mathcal{V}$
- Variables are a special case.
- Example descriptions:
  - $\textsc{App}$: $(A \to B) \times A \Rightarrow B$
  - $\textsc{Lam}$: $(A \vdash B) \Rightarrow (A \to B)$

Example derivation

$$P = (A \multimap B) \otimes A$$

$$\dfrac{\begin{array}{c}\dfrac{}{p : P \vdash p : P} \\ \vdots \end{array} \qquad \dfrac{\dfrac{}{\begin{array}{c} p : P,\ \ f : A \multimap B, \\ x : A \vdash f : A \multimap B \end{array}} \qquad \dfrac{}{\begin{array}{c} p : P,\ \ f : A \multimap B, \\ x : A \vdash x : A \end{array}}}{p : P,\ \ f : A \multimap B,\ \ x : A \vdash f\ x : B}}{\dfrac{p : P \vdash \text{let } (f \otimes x) = p \text{ in } f\ x : B}{\vdash \lambda p.\ \text{let } (f \otimes x) = p \text{ in } f\ x : \underbrace{(A \multimap B) \otimes A}_{P} \multimap B}}$$

Example derivation

$$P = (A \multimap B) \otimes A$$

$$
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\dfrac{
\begin{array}{c}\dfrac{}{1p : P \vdash p : P} \\ \vdots \end{array}
\qquad
\dfrac{
\dfrac{}{\begin{array}{c} 0p : P, 1f : A \multimap B, \\ 0x : A \vdash f : A \multimap B \end{array}}
\qquad
\dfrac{}{\begin{array}{c} 0p : P, 0f : A \multimap B, \\ 1x : A \vdash x : A \end{array}}
}{0p : P, 1f : A \multimap B, 1x : A \vdash f\ x : B}
}{1p : P \vdash \mathrm{let}\ (f \otimes x) = p\ \mathrm{in}\ f\ x : B}
}{\vdash \lambda p.\ \mathrm{let}\ (f \otimes x) = p\ \mathrm{in}\ f\ x : \underbrace{(A \multimap B) \otimes A}_{P} \multimap B}
$$

Vectors over semirings — addition

Semiring operations (operating on annotations on individual variables) are lifted to vector operations (operating on contexts-worth of variables).

$$\frac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{R} \trianglelefteq \mathcal{P} + \mathcal{Q}}{\mathcal{R}\Gamma \vdash (M \mathbin{{}_{\otimes}} N) : A \otimes B}$$

$$\frac{\mathcal{R} \trianglelefteq 0}{\mathcal{R}\Gamma \vdash ({}_{\otimes}) : 1}$$

identity, associativity, commutativity $\sim$ contexts are essentially multisets

Vectors over semirings — multiplication

$$\frac{\Gamma \ni (x : A) \qquad \mathcal{R} \trianglelefteq \langle x|}{\mathcal{R}\Gamma \vdash x : A}$$

$$\frac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{R} \trianglelefteq r\mathcal{P}}{\mathcal{R}\Gamma \vdash [M] : !_r A}$$

- $\langle x|$ — basis vector. The variable x can be used once, and every other variable can be discarded.
- 'M' for "Multiplication", also for "Modality"

Vectors over semirings

$$\frac{\mathcal{R} \trianglelefteq 0}{\mathcal{R}\Gamma \vdash (_\otimes) : 1} \qquad \frac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{Q}\Gamma \vdash N : B \qquad \mathcal{R} \trianglelefteq \mathcal{P} + \mathcal{Q}}{\mathcal{R}\Gamma \vdash (M \,_\otimes\, N) : A \otimes B}$$

$$\frac{\Gamma \ni (x : A) \qquad \mathcal{R} \trianglelefteq \langle x|}{\mathcal{R}\Gamma \vdash x : A} \qquad \frac{\mathcal{P}\Gamma \vdash M : A \qquad \mathcal{R} \trianglelefteq r\mathcal{P}}{\mathcal{R}\Gamma \vdash [M] : !_r A}$$

- These four are the basic operations of linear algebra, three of them preserved by linear transformations.
- Notice: a variable which is syntactically absent may be given annotation 0.

Generic notion of linear[5] semantics

- Fundamental lemma of semantics:

$$\frac{\text{Semantics } \mathcal{V}\ \mathcal{C} \qquad \overbrace{(i : \Gamma \ni A) \to \mathcal{V}\ A\ (\Psi_{i-}\Delta)}^{\text{environment}} \qquad \mathcal{Q} \trianglelefteq \mathcal{P}\Psi}{\underbrace{\mathcal{P}\Gamma \vdash A \to \mathcal{C}\ A\ \mathcal{Q}\Delta}_{\text{traversal}}}$$

- Semantics $\mathcal{V}\ \mathcal{C}$ contains:
  - Proof that $\mathcal{V}$ is stable under weakening *by 0-use variables*
  - Ways to interpret term constructors semantically, generic in context:

$$[\![\mathrm{var}]\!] : \forall[\ \mathcal{V}\ A \dot\to \mathcal{C}\ A\ ] \qquad [\![\mathrm{app}]\!] : \forall[\ \mathcal{C}\ (A \multimap B) * \mathcal{C}\ A \dot\to \mathcal{C}\ B\ ]$$

$$[\![\mathrm{lam}]\!] : \forall[\ \Box(\mathcal{V}\ A \mathrel{-\!*} \mathcal{C}\ B) \dot\to \mathcal{C}\ (A \multimap B)\ ] \qquad \ldots$$

[5] for a generic notion of linearity

Generic notion of linear syntax

Multiple premises are handled by bunched implications.[6]

- $I\ \mathcal{R}\Gamma := \mathcal{R} \trianglelefteq 0$
- $(\mathcal{A} * \mathcal{B})\ \mathcal{R}\Gamma := \Sigma\mathcal{P}, \mathcal{Q}.\ (\mathcal{R} \trianglelefteq \mathcal{P} + \mathcal{Q}) \times \mathcal{A}\ \mathcal{P}\Gamma \times \mathcal{B}\ \mathcal{Q}\Gamma$
- $(\mathcal{A} \twoheadrightarrow\!\!* \mathcal{B})\ \mathcal{P}\Gamma := \Pi\mathcal{Q}, \mathcal{R}.\ (\mathcal{R} \trianglelefteq \mathcal{P} + \mathcal{Q}) \to \mathcal{A}\ \mathcal{Q}\Gamma \to \mathcal{B}\ \mathcal{R}\Gamma$
- $(!_r\mathcal{A})\ \mathcal{R}\Gamma := \Sigma\mathcal{P}.\ (\mathcal{R} \trianglelefteq r\mathcal{P}) \times \mathcal{A}\ \mathcal{P}\Gamma$

Example description: $(!_r A * (rA \vdash B)) \Rightarrow B$

- $$\frac{\mathcal{P}\Gamma \vdash\ !_r A \qquad \mathcal{Q}\Gamma, rx : A \vdash B \qquad \mathcal{R} \trianglelefteq \mathcal{P} + \mathcal{Q}}{\mathcal{R}\Gamma \vdash B}$$
- $[\![\mathrm{bam}]\!] : \forall[\ \mathcal{C}\ (!_r A) * \Box(!_r(\mathcal{V}\ A) \twoheadrightarrow\!\!* \mathcal{C}\ B) \dot{\to} \mathcal{C}\ B\ ]$

[6] Arjen Rouvoet et al. "Intrinsically-Typed Definitional Interpreters for Linear, Session-Typed Languages". In: *CPP 2020*. New Orleans, LA, USA, 2020, pp. 284–298. ISBN: 9781450370974. DOI: 10.1145/3372885.3373818.

Conclusion

- Past:
  - Intuitionistic syntax-to-syntax traversals using kits.
  - Intuitionistic syntax-to-semantics traversals.
  - A generic notion of intuitionistic syntax with binding.
- Present:
  - Linear syntax-to-syntax traversals.
  - Using matrices/linear maps to describe compatibility of environments and traversals.
- Future:
  - Linear syntax-to-semantics traversals using matrices.
  - Semantics is stable under 0-use weakening.
  - A generic notion of linear syntax written in bunched implications style.
  - `https://github.com/laMudri/generic-lr`